Abstract

Supersymmetry is an attempt to relate two classes of particles, the fermions and bosons. To this end the symmetry group is replaced with a supersymmetric equivalent, usually considered on the Lie superalgebra level rather than the Lie supergroup. For Minkowski space this means the Poincaré algebra is extended to the Poincaré superalgebra, and its representations give an idea of possible particles with their supersymmetric partners. In this thesis we consider Lie superalgebras in general and partially classify the representations of the Poincaré superalgebra.
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1 Introduction

Supersymmetry is a theory within the field of particle physics that attempts to relate two classes of particles, bosons and fermions, with each other. It introduces a symmetry between those particles, turning the symmetry group into a supersymmetry group. For Minkowski space, this amounts to extending the Poincaré algebra into a Lie superalgebra called the Poincaré superalgebra. This Poincaré superalgebra is important to physics, so it is often considered from physicists point of view. As such it can be a bit challenging for a mathematician to understand. The goal of this thesis is to define the Poincaré superalgebra and find its representations as a mathematician.

The thesis is divided into two main parts. The first part deals with basic definitions concerning superalgebras in section 2, and Lie superalgebras and their representations in section 3. To practice with these notions it also contains the test case $\mathfrak{osp}(1,2)$ and its representations in section 4. The second part starts with the 'non-super' Poincaré group and algebra in section 5. The Poincaré super algebra is finally defined in section 6, and we find its representations in section 7.

Basic knowledge of groups, algebras and Lie algebras is assumed. The thesis can be read without any knowledge of physics if the reader is willing to simply accept the occasional physical motivation or interpretation.

Finally, I would like to thank my supervisor Erik Koelink. Even though both of us were a far cry from a specialist on the subject when I started, he helped find my mistakes when my calculations seemed to go nowhere and he suggested directions when I didn’t know what to try next. I also owe him a big thank you for being so patient when I just could not work as much as we both would have liked.

I want to thank my friends and family as well, for just nodding along as I clumsily tried to explain in simple term what I was working on, for listening to me complain when I once again had too many of too little complex $i$’s, but mostly for refraining from even using the word thesis (or rather, substituting it with "het s-woord") whenever I really did not feel like talking about it.
Part I
Lie superalgebras

A Lie superalgebra is an extension of a familiar structure. Before we can start on supersymmetry, we need the basic definitions and an understanding of how Lie superalgebras work.

2 Superalgebras

Many of the following definitions only need a field \( k \) of characteristic 0, but for simplicity we set either \( k = \mathbb{C} \) or \( k = \mathbb{R} \). Let \( V \) be a vector space over this field. For any ring \( \Gamma \), we define a \( \Gamma \)-gradation of \( V \) as a family of subspaces \( V_\gamma \) such that
\[
V = \bigoplus_{\gamma \in \Gamma} V_\gamma. \tag{2.1}
\]
If such a family exists, the vector space is called \( \Gamma \)-graded. An element of \( V_\gamma \) is called homogeneous of degree \( \gamma \).

Many definitions for vector spaces have a ‘graded version’. A subspace \( U \subset V \) is again \( \Gamma \)-graded if \( U = \bigoplus_{\gamma \in \Gamma} (U \cap V_\gamma) \), which just says that \( U \) contains all the homogeneous components of its elements. If we have two \( \Gamma \)-graded vector spaces \( V = \bigoplus_{\gamma \in \Gamma} V_\gamma \) and \( V' = \bigoplus_{\gamma \in \Gamma} V'_\gamma \), we can take their direct sum, product and tensor product, using the following \( \Gamma \)-gradations
\[
(V \oplus V')_\gamma = V_\gamma \oplus V'_\gamma \quad (V \times V')_\gamma = \bigoplus_{\alpha + \beta = \gamma} (V_\alpha \times V'_\beta) \quad (V \otimes V')_\gamma = \bigoplus_{\alpha + \beta = \gamma} (V_\alpha \otimes V'_\beta).
\]

If \( \Gamma = \mathbb{Z}_2 = \{\bar{0}, \bar{1}\} \), we get the special case we are interested in.

**Definition 2.1.** The vector space \( V \) is called a super vector space if it has a \( \mathbb{Z}_2 \)-gradation, meaning we have vector spaces \( V_{\bar{0}} \) and \( V_{\bar{1}} \) such that \( V \) decomposes as
\[
V = V_{\bar{0}} \oplus V_{\bar{1}}. \tag{2.2}
\]
In this case we may call the elements of \( V_{\bar{0}} \) even and those of \( V_{\bar{1}} \) odd, rather than homogeneous of degree \( \bar{0} \) and \( \bar{1} \) respectively.

A super subspace is a subspace \( W \) of the vector space \( V \) which is a super vector space \( W = W_{\bar{0}} \oplus W_{\bar{1}} \) with gradation such that \( W_{\bar{0}} = W \cap V_{\bar{0}} \) and \( W_{\bar{1}} = W \cap V_{\bar{1}} \).

**Definition 2.2.** On the homogeneous elements we define the parity function by
\[
p : V_{\bar{0}} \cup V_{\bar{1}} \to \mathbb{Z}_2, \quad p(v) = \begin{cases} 
\bar{0} & \text{v even} \\
\bar{1} & \text{v odd}
\end{cases} \tag{2.3}
\]
Remark 2.3. The parity function is only defined on homogeneous elements, so any equation involving parity only makes sense when the relevant entries are homogeneous. Whenever we use those equations with non-homogeneous entries, we implicitly use a linearly extended version of the equation.

As an example, we consider the complex vector space \( \mathbb{C}^{m+n} \), for \( m, n \in \mathbb{N} \). This can be decomposed into \( V_0 = \mathbb{C}^m \) and \( V_1 = \mathbb{C}^n \). The vector space \( \mathbb{C}^{m+n} \) together with this gradation is a super vector space, denoted by \( \mathbb{C}^{m|n} \). Of course, we could have also decomposed \( \mathbb{C}^{m+n} \) into \( \mathbb{C}^{m'} \) and \( \mathbb{C}^{n'} \) with \( m' \neq m \) and \( n' \neq n \), provided \( m' + n' = m + n \). Although the underlying vector space remains the same, this will give a different super vector space \( \mathbb{C}^{m'|n'} \).

To better understand when super vector spaces are ‘the same’ or not, we need to define what an isomorphism is.

**Definition 2.4.** A mapping \( f : V \to W \) between two super vector spaces is called a homomorphism if it is linear and preserves the grading, i.e. \( p(f(v)) = p(v) \) for all homogeneous \( v \in V \). We can also write the second requirement as \( f(V_\alpha) \subset W_\gamma \). An isomorphism is a bijective (i.e. injective and surjective) homomorphism.

We can apply this to \( \mathbb{C}^{m|n} \) and \( \mathbb{C}^{m'|n'} \). Suppose we have an isomorphism \( f : \mathbb{C}^{m'|n'} \to \mathbb{C}^{m|n} \). Because \( f \) preserves the grading, we have \( f(\mathbb{C}^m) \subset \mathbb{C}^{m'} \) and \( f(\mathbb{C}^n) \subset \mathbb{C}^{n'} \). The surjectivity of \( f \) then tells us that \( f(\mathbb{C}^m) = \mathbb{C}^{m'} \) and \( f(\mathbb{C}^n) = \mathbb{C}^{n'} \). Since \( f \) is also injective, this means the restrictions to \( \mathbb{C}^m \) and \( \mathbb{C}^n \) are vector space isomorphisms, which can only be true if \( m = m' \) and \( n = n' \). Thus, for \( m \neq m' \) and \( n \neq n' \), \( \mathbb{C}^{m|n} \) and \( \mathbb{C}^{m'|n'} \) are truly two different super vector spaces.

An algebra is a vector space equipped with a multiplication. A similar construction works for a super vector space, giving the following definition:

**Definition 2.5.** Let \( A \) be an algebra over \( k \). We call \( A \) a superalgebra if the underlying vector space is a super vector space, and we have for \( \alpha, \beta \in \mathbb{Z}_2 \)

\[
A_\alpha A_\beta \subset A_{\alpha + \beta}.
\]  
(2.4)

Equation (2.4) gives a compatibility between the gradation and multiplication.

The parity function \( p \) on a superalgebra \( A \) is the parity function on the underlying super vector space. We can ask ourselves what it does on the multiplication of two homogeneous elements \( a, b \in A \). The greek letters in equation (2.4) are basically the parity of the elements in their corresponding vector spaces, so we must have

\[
p(ab) = p(a) + p(b).
\]  
(2.5)

If this holds for all homogeneous elements, it is in fact equivalent to equation (2.4). In literature equation (2.5) is sometimes used in the definition of superalgebras instead, see e.g. [12, chap. 3].
As an example we define a superalgebra that will be useful later on. First take two super vector spaces \( V, W \). Then \( \text{Hom}(V, W) \), the vector space of all linear maps \( f : V \rightarrow W \), has a natural super vector space structure with gradation given by

\[
\text{Hom}(V, W)_\alpha = \{ f \in \text{Hom}(V, W) \mid f(V_\beta) \subset W_{\alpha + \beta} \quad \forall \beta \in \mathbb{Z}_2 \}
\]  

(2.6)

As with all other super vector spaces, elements of \( \text{Hom}(V, W)_\alpha \) are homogeneous of degree \( \alpha \) and we may call them even or odd if \( \alpha = 0, 1 \) respectively. We also note that the even functions are exactly the homomorphisms between the super vector spaces.

For \( V = W \), the space \( \text{Hom}(V, V) = \text{Hom}(V) \) is an algebra with composition as multiplication. Now suppose \( f \in \text{Hom}(V)_\alpha \) and \( g \in \text{Hom}(V)_\beta \). Then for all \( \gamma \in \mathbb{Z}_2 \) we have \( f(V_\gamma) \subset V_{\alpha + \gamma} \) and \( g(V_\gamma) \subset V_{\beta + \gamma} \). Combining this gives

\[
(fg)(V_\gamma) = f(g(V_\gamma)) \subset f(V_{\beta + \gamma}) \subset V_{\alpha + \beta + \gamma}.
\]  

(2.7)

This hold true for all \( \gamma \), hence \( fg \in V_{\alpha + \beta} \). Thus

\[
\text{Hom}(V)_\alpha \text{Hom}(V)_\beta \subset \text{Hom}(V)_{\alpha + \beta}.
\]  

(2.8)

We conclude that \( \text{Hom}(V) \) is indeed a superalgebra.

Now suppose that \( V \) is finite-dimensional. Since a super vector space is just a vector space with ‘something extra’, we can take the trace of any element in \( \text{Hom}(V) \). However, that means throwing away the extra information from the gradation. To keep that from happening, we need to define a graded version of the trace.

**Definition 2.6.** On \( V \) a finite-dimensional super vector space we define \( \gamma : V \rightarrow V \) on homogeneous elements \( x \in V \) by

\[
\gamma(x) = (-1)^p(x)x
\]  

(2.9)

and extend linearly. This is a linear map, so \( \gamma \in \text{Hom}(V) \). The supertrace, denoted by \( \text{str} \), is then defined by

\[
\text{str}(f) = \text{tr}(\gamma f)
\]  

(2.10)

where \( \text{tr} \) is the regular trace and \( \gamma f \) is the multiplication in \( \text{Hom}(V) \), i.e. composition.

There are a few familiar definitions for algebras that have a graded version too. A *graded subalgebra* of the superalgebra \( A \) is a subalgebra of the algebra \( A \) such that the underlying vector space is a graded subspace. Similarly, a *graded ideal* of the superalgebra \( A \) is an ideal of the algebra \( A \) such that the underlying vector space is a graded subspace.

**Definition 2.7.** A *superalgebra homomorphism* is an algebra homomorphism that preserves the grading. An *isomorphism* is a bijective homomorphism.
Two superalgebras may be isomorphic as algebras, but not as superalgebras. As an example consider the spaces $\text{Hom}(\mathbb{C}^{2|0})$ and $\text{Hom}(\mathbb{C}^{1|1})$. In both cases the underlying algebra is $\text{Hom}(\mathbb{C}^{2})$. A isomorphism $\varphi : \text{Hom}(\mathbb{C}^{2|0}) \to \text{Hom}(\mathbb{C}^{1|1})$ has to be bijective and preserve the grading. However, $\text{Hom}(\mathbb{C}^{2|0})_{\bar{1}} = \{0\}$ since any $f \in \text{Hom}(\mathbb{C}^{2|0})_{\bar{1}}$ should satisfy $f(\mathbb{C}^{2|0})_{\bar{1}} = (\mathbb{C}^{2|0})_{\bar{0}} \subset (\mathbb{C}^{2|0})_{\bar{1}} = \{0\}$, while $\text{Hom}(\mathbb{C}^{1|1})_{\bar{1}}$ has non-zero elements such as the map that flips the even and odd subspaces (i.e. $(a, b) \mapsto (b, a)$ for $a \in (\mathbb{C}^{1|1})_{\bar{0}}$ and $b \in (\mathbb{C}^{1|1})_{\bar{1}}$).

3 Lie superalgebras

We can also build a Lie superalgebra from a super vector space.

3.1 The definition

Definition 3.1. Let $\mathfrak{g}$ be a super vector space with a mapping $\{\ , \} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. We call $\mathfrak{g}$ a Lie superalgebra and the mapping a superbracket if $\{ \ , \} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ has the following properties:

- Bilinearity
- Compatibility with super vector space structure: for $\alpha, \beta \in \mathbb{Z}_{2}$
  \[ \mathfrak{g}_{\alpha} \cdot \mathfrak{g}_{\beta} \subset \mathfrak{g}_{\alpha + \beta} \]  \hspace{1cm} (3.1)
- Graded skew-symmetry: for all homogeneous $A, B \in \mathfrak{g}$ we have
  \[ \{A, B\} = -(-1)^{p(A)p(B)} \{B, A\} \]  \hspace{1cm} (3.2)
- Graded Jacobi identity: for all homogeneous $A, B, C \in \mathfrak{g}$ we have
  \[ 0 = (-1)^{p(C)p(A)} \{A, \{B, C\}\} + (-1)^{p(A)p(B)} \{B, \{C, A\}\} + (-1)^{p(B)p(C)} \{C, \{A, B\}\} \]  \hspace{1cm} (3.3)

Note that we can see Lie superalgebras as a special case of non-associative superalgebras, which is how they are sometimes defined in literature, see e.g. [9].

Depending on the parity, equations (3.2) and (3.3) simplify to the non-graded versions we know from Lie algebras. Specifically, if we take $A, B, C$ to be even they become

\[ \{A, B\} = -\{B, A\} \]
\[ \{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0, \]

therefore the subalgebra $\mathfrak{g}_{\bar{0}}$ of $\mathfrak{g}$ with this multiplication is a Lie algebra.
Furthermore, by the compatibility we know that \( \langle g_0, g_1 \rangle \subseteq g_1 \), so for \( A \in g_0 \) we can consider \( \langle A, \cdot \rangle \) as a map \( g_1 \rightarrow g_1 \). The bilinearity makes this map linear, and the graded skew-symmetry and graded Jacobi identity can be used to show that for all \( A, B \in g_0 \) and \( C \in g_1 \) we have

\[
\langle \langle A, B \rangle, C \rangle = \langle A, \langle B, C \rangle \rangle - \langle B, \langle A, C \rangle \rangle
\]  
(3.4)

Hence \( A \mapsto \langle A, \cdot \rangle \) with representation space \( g_1 \) is a Lie algebra representation of \( g_0 \). This is equivalent to saying that \( g_1 \) is an \( g_0 \)-module.

It is important to notice that the Lie superalgebra \( g \) is in general not a Lie algebra. We have seen that super vector spaces and superalgebras are just vector spaces and algebras with an extra structure added, but for Lie superalgebras only the even part is actually a Lie algebra. Theorem 3.5 will show how to build a Lie superalgebra starting from a Lie algebra and module.

**Definition 3.2.** A Lie superalgebra homomorphism is a map between Lie superalgebras that preserves the respective superbrackets and gradings. A Lie superalgebra isomorphism is a bijective Lie superalgebra homomorphism.

A trivial way to build a Lie superalgebra is to use \( g_0 \) any Lie algebra and \( g_1 = \{0\} \), but this doesn’t provide much insight. For a better first example we start with a superalgebra \( A \). We define a superbracket on the homogeneous elements \( a, b \in A \)

\[
\langle a, b \rangle = ab - (-1)^{p(a)p(b)}ba
\]  
(3.5)

and extend linearly.

**Lemma 3.3.** A superalgebra \( A \) with superbracket defined by equation (6.2) is a Lie superalgebra.

**Proof.** There are four things we need to show.

- **Bilinearity:** To show bilinearity, we need to show linearity in both arguments. We start with the first argument for homogeneous elements. Let \( a, a', b \) be homogeneous elements in \( A \), with \( p(a) = p(a') \). Then \( p(a + a') = p(a) = p(a') \). Using the distributivity of the algebra multiplication we can rewrite

\[
\langle a + a', b \rangle = (a + a')b - (-1)^{p(a+a')p(b)}b(a + a')
= ab + a'b - (-1)^{p(a+a')p(b)}ba - (-1)^{p(a+a')p(b)}ba'
= ab - (-1)^{p(a)p(b)}ba + a'b - (-1)^{p(a')p(b)}ba'
= \langle a, b \rangle + \langle a', b \rangle
\]

For a scalar \( \lambda \in k \), we again use the properties of the algebra multiplica-
tion, and the fact that \( p(\lambda a) = p(a) \):

\[
\langle \lambda a, b \rangle = (\lambda a)b - (-1)^{p(\lambda a)p(b)}b(\lambda a)
\]

\[
= \lambda ab - \lambda(-1)^{p(a)p(b)}ba
\]

\[
= \lambda(ab - (-1)^{p(a)p(b)}ba)
\]

\[
= \lambda(a, b)
\]

This together shows the linearity in the first argument for homogeneous elements. By definition we extended linearly for non-homogenous elements, hence we have linearity in the first argument for all elements.

Linearity in the second argument can be proven the same way. First we take homogeneous elements \( a, b, b' \) with \( p(b) = p(b') \) and a scalar \( \lambda \). The properties of the algebra multiplication allow us to rewrite:

\[
\langle a, b + b' \rangle = \langle a, b \rangle + \langle a, b' \rangle \quad \text{and} \quad \langle a, \lambda b \rangle = \lambda \langle a, b \rangle,
\]

and extending linearly lets us conclude linearity in the second argument for all elements.

- **Compatibility:** Let \( a \in A_\alpha \) and \( b \in A_\beta \). Since \( A \) is a superalgebra, we know that \( A_\alpha A_\beta \subset A_{\alpha+\beta} \) and \( A_\beta A_\alpha \subset A_{\beta+\alpha} = A_{\alpha+\beta} \) (using that \( \mathbb{Z}_2 \) is commutative for the second), hence \( ab, ba \in A_{\alpha+\beta} \). The even and odd parts \( A_0 \) and \( A_1 \) are vector spaces, so they are closed under addition and scalar multiplication. This gives us \( ab - (-1)^{p(a)p(b)}ba \subset A_{\alpha+\beta} \). Thus \( \langle A_\alpha, A_\beta \rangle \subset A_{\alpha+\beta} \).

- **Graded skew-symmetry:** Let \( a, b \) be homogeneous elements in \( A \). We note that \( -(-1)^{p(a)p(b)}(-(-1)^{p(a)p(b)}ab + ba) = 1 \) and use this to rewrite

\[
\langle a, b \rangle = ab - (-1)^{p(a)p(b)}ba
\]

\[
= -(-1)^{p(a)p(b)}(-(-1)^{p(a)p(b)}ab + ba)
\]

\[
= -(-1)^{p(a)p(b)}(b, a)
\]

which is exactly what we needed.

- **Graded Jacobi identity:** Let \( a, b, c \) be homogeneous elements in \( A \). Using the same trick as above, together with \( p(ab) = p(a) + p(b) \) (see equation
(2.5)), we can write out the different terms of the graded Jacobi equation.

\[-1^{p(c)p(a)}\langle \langle a, \langle b, c \rangle \rangle = (-1)^{p(c)p(a)}\langle a, bc - (-1)^{p(b)p(c)}cb \rangle \]
\[= (-1)^{p(c)p(a)}abc \]
\[-(-1)^{p(c)p(a)}(-1)^{p(a)p(b)}bca \]
\[-(-1)^{p(c)p(a)}(-1)^{p(b)p(c)}acb \]
\[+ (-1)^{p(c)p(a)}(-1)^{p(b)p(c)}(-1)^{p(a)p(cb)}cba \]
\[= (-1)^{p(c)p(a)}abc \]
\[-(-1)^{p(a)p(b)}bca \]
\[-(-1)^{p(c)p(a)}(-1)^{p(b)p(c)}acb \]
\[+ (-1)^{p(b)p(c)}(-1)^{p(a)p(b)}cba \]

The other two terms are the same, just with \(a, b, c\) permuted cyclicly.

\[-(-1)^{p(a)p(b)}\langle b, \langle c, a \rangle \rangle = (-1)^{p(a)p(b)}bca \]
\[-(-1)^{p(b)p(c)}cab \]
\[-(-1)^{p(a)p(b)}(-1)^{p(c)p(a)}bac \]
\[+ (-1)^{p(c)p(a)}(-1)^{p(b)p(c)}acb \]

\[-(-1)^{p(b)p(c)}\langle c, \langle a, b \rangle \rangle = (-1)^{p(b)p(c)}cab \]
\[-(-1)^{p(c)p(a)}abc \]
\[-(-1)^{p(b)p(c)}(-1)^{p(a)p(b)}cba \]
\[+ (-1)^{p(a)p(b)}(-1)^{p(c)p(a)}bac \]

We see that adding these indeed gives zero.

\[\square\]

We can apply this lemma to the superalgebra \(\text{Hom}(V)\) we defined before. The resulting Lie superalgebra is important enough to warrant the following definition.

**Definition 3.4.** Let \(V\) be a super vector space. The **general linear Lie superalgebra** of \(V\), denoted by \(\mathfrak{pl}(V)\), is the Lie superalgebra constructed from \(\text{Hom}(V)\) with gradation given by equation (2.6) and superbracket given by equation (6.2).

If \(V = V_0 \oplus V_1\) is finite-dimensional with \(\dim V_0 = m\) and \(\dim V_0 = n\), we can choose a basis \(\{e_1, \ldots, e_m\}\) of \(V_0\) and \(\{e_{m+1}, \ldots, e_{m+n}\}\) of \(V_1\), and write all elements of \(\mathfrak{pl}(V)\) as \((m+n) \times (m+n)\)-matrices. Specifically, we can write

\[X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (3.6)\]
With $A$ an $m \times m$-matrix, $B$ an $m \times n$-matrix, $C$ an $n \times m$-matrix and $D$ an $n \times n$-matrix. The even elements preserve the grading, so they must be of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

The odd elements reverse the grading, so they are of the form

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

Even without knowing the underlying vector space and its basis, this definition makes sense. We define the general linear Lie superalgebra $\mathfrak{gl}(m,n)$ as the $(m+n) \times (m+n)$-matrices $X$ together with the above gradation. Writing the elements of $\mathfrak{gl}(V)$ as matrices is an isomorphism between $\mathfrak{gl}(V)$ and $\mathfrak{gl}(m,n)$.

### 3.2 A different perspective

As we have noted before, the even part of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_\mathbb{0} \oplus \mathfrak{g}_\mathbb{1}$ is a Lie algebra $\mathfrak{g}_\mathbb{0}$ and the odd part is a Lie algebra module $\mathfrak{g}_\mathbb{1}$ of $\mathfrak{g}_\mathbb{0}$. This gives the idea to construct a Lie superalgebra from a Lie algebra and an appropriate module. The Lie bracket and the module action will give the superbracket for the fully even and the mixed terms, but they don’t provide a mapping for the fully odd terms. Here we need to make a choice, although we do have some restrictions.

To better formulate the restrictions on our choice for the odd superbracket, we note that the tensor product $\mathfrak{g}_\mathbb{1} \otimes \mathfrak{g}_\mathbb{1}$ is also a module by the action

$$a \cdot (b \otimes c) = (a \cdot b) \otimes c + b \otimes (a \cdot c) \quad (3.7)$$

with $a \in \mathfrak{g}_\mathbb{0}$ and $b, c \in \mathfrak{g}_\mathbb{1}$. The Lie algebra $\mathfrak{g}_\mathbb{0}$ itself is also an $\mathfrak{g}_\mathbb{0}$-module via the adjoint representation. Thus the superbracket of fully odd terms can be regarded as a map between $\mathfrak{g}_\mathbb{0}$-modules. We now prove the following theorem.

**Theorem 3.5.** Suppose $\mathfrak{g} = \mathfrak{g}_\mathbb{0} \oplus \mathfrak{g}_\mathbb{1}$ is a super vector space with $(\mathfrak{g}_\mathbb{0}, [\ , \ ])$ a Lie algebra, $(\mathfrak{g}_\mathbb{1}, \cdot \ )$ a $\mathfrak{g}_\mathbb{0}$-module and $(\ , \ ) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ a bilinear map such that

- The restriction of $(\ , \ )$ to the even terms gives the Lie bracket $[\ , \ ]$, i.e. $\langle a, b \rangle = [a, b]$ for all $a, b \in \mathfrak{g}_\mathbb{0}$.
- The mixed terms are given by the module action, i.e. $\langle a, b \rangle = a \cdot b$ and $\langle b, a \rangle = -\langle a, b \rangle = -a \cdot b$ for all $a \in \mathfrak{g}_\mathbb{0}$ and $b \in \mathfrak{g}_\mathbb{1}$.
- The restriction of $(\ , \ )$ to the odd terms is a symmetric map $\mathfrak{g}_\mathbb{1} \otimes \mathfrak{g}_\mathbb{1} \rightarrow \mathfrak{g}_\mathbb{0}$ that intertwines with the $\mathfrak{g}_\mathbb{0}$-module action.
- For all $a \in \mathfrak{g}_\mathbb{1}$ we have

$$\langle a, \langle a, a \rangle \rangle = 0 \quad (3.8)$$

Then $\mathfrak{g}$ together with the superbracket $(\ , \ )$ is a Lie superalgebra.
Proof. By definition $\langle \, , \rangle$ is bilinear. The compatibility with the super vector space structure and the graded skew-symmetry follow directly from the assumptions on the superbracket and the Lie algebra and module structures. The only thing that needs proving is the graded Jacobi identity. We will consider it case by case, depending on how many elements are even and odd

- If all elements are even, the graded Jacobi identity is just the Jacobi identity. Since $g_0$ is a Lie algebra, we know the identity holds on the even part.

- From the module structure we know that for $a, b$ even and $c$ odd we should have
  \[ \langle a, b \rangle \cdot c = [a, b] \cdot c = a \cdot (b \cdot c) - b \cdot (a \cdot c) \quad (3.9) \]
  We use this to rewrite
  \[ -\langle c, \langle a, b \rangle \rangle = \langle \langle a, b \rangle, c \rangle = \langle a, b \rangle \cdot c = a \cdot (b \cdot c) - b \cdot (a \cdot c) = a \cdot (b, c) + b \cdot (c, a) = \langle a, (b, c) \rangle + \langle b, (c, a) \rangle \]
  which is the graded Jacobi identity for two even and one odd element.

- Suppose $a$ even and $b, c$ odd. Since the bracket intertwines the module structure, we know that
  \[ a \cdot \langle b, c \rangle = \langle a \cdot b, c \rangle + \langle b, a \cdot c \rangle. \quad (3.10) \]
  Note that the left side of this equation uses the module action on $g_0$, while the right side uses the action on $g_1$, and by extension $g_1 \otimes g_1$. Both actions are denoted by ".", and both are interpreted as the superbracket. From there we use the symmetry of the odd terms and the anti-symmetry of the mixed terms to find
  \[ \langle a, \langle b, c \rangle \rangle = a \cdot \langle b, c \rangle = \langle a \cdot b, c \rangle + \langle b, a \cdot c \rangle = \langle c, (a, b) \rangle - \langle b, (c, a) \rangle \]
  which is the graded Jacobi identity for one even and two odd elements.

- For the case of three odd elements we will need use equation (3.8) multiple times. First on $xa + yb$ with $a, b$ odd and $x, y \in k$. Using the bilinearity of the bracket we find
  \[ 0 = \langle xa + yb, \langle xa + yb, xa + yb \rangle \rangle \]
  \[ = \langle xa, \langle xa, xa \rangle \rangle + \langle xa, \langle xa, yb \rangle \rangle + \langle xa, \langle yb, xa \rangle \rangle + \langle xa, \langle yb, yb \rangle \rangle + \langle yb, \langle xa, xa \rangle \rangle + \langle yb, \langle xa, yb \rangle \rangle + \langle yb, \langle yb, xa \rangle \rangle + \langle yb, \langle yb, yb \rangle \rangle \]
  \[ = x^2 y(\langle b, (a, a) \rangle + 2\langle a, (a, b) \rangle) + xy^2(\langle a, (b, b) \rangle + 2\langle b, (b, a) \rangle) \]

where in the final step we have used equation (3.8) and the symmetry of the superbracket on odd elements. Because this equation has to hold for all \( x, y \in k \), we may conclude that the two parts must both be zero. Hence we find for all odd \( a, b \)

\[
\langle b, \langle a, a \rangle \rangle + 2 \langle a, \langle a, b \rangle \rangle = 0 \tag{3.11}
\]

Now let \( a, b, c \) be odd. We use equation (3.8) again, this time on \( a + b + c \). Using bilinearity we can write this out to 27 terms. Some of these are zero by equation (3.8), others sum to zero by equation (3.11). Only six terms remain, and with the symmetry of the bracket for odd terms we can rewrite to get

\[
0 = \langle a + b + c, \langle a + b + c, a + b + c \rangle \rangle \\
= \ldots \\
= 2(\langle a, \langle b, c \rangle \rangle + \langle b, \langle c, a \rangle \rangle + \langle c, \langle a, b \rangle \rangle)
\]

Thus the graded Jacobi identity holds for three odd elements as well.

We conclude that \( \mathfrak{g} \) is indeed a Lie superalgebra with superbracket \( \langle , \rangle \). \( \square \)

Now that we know how to build Lie superalgebras from Lie algebras and modules, we might wonder about Lie subalgebras and submodules. In general we have the following defintion.

**Definition 3.6.** Let \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) be a Lie superalgebra with superbracket \( \langle , \rangle \). A super subspace \( \mathfrak{g}' \subset \mathfrak{g} \) is called a **Lie subsuperalgebra** if it is closed under the superbracket.

From the compatibility with the super vector space structure we know that \( \langle \mathfrak{g}_1, \mathfrak{g}_1 \rangle \subset \mathfrak{g}_0 \). If we take some Lie subalgebra \( \mathfrak{g}_0' \) there is a chance this no longer holds, so caution is warranted. Fortunately, taking a submodule is not as complicated.

**Lemma 3.7.** If \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) is a Lie superalgebra and \( \mathfrak{h} \subset \mathfrak{g}_1 \) is a \( \mathfrak{g}_0 \)-submodule, then \( \mathfrak{g}_0 \oplus \mathfrak{h} \) is a Lie subsuperalgebra of \( \mathfrak{g} \).

**Proof.** Since \( \mathfrak{h} \) is a submodule, we have \( \langle \mathfrak{h}, \mathfrak{g}_0 \rangle \subset \mathfrak{h} \) and \( \langle \mathfrak{g}_0, \mathfrak{h} \rangle \subset \mathfrak{h} \). For the fully odd terms is is obvious that \( \langle \mathfrak{h}, \mathfrak{h} \rangle \subset \langle \mathfrak{g}_1, \mathfrak{g}_1 \rangle \subset \mathfrak{g}_0 \). Thus \( \mathfrak{g}_0 \oplus \mathfrak{h} \) is closed under the superbracket. \( \square \)

### 3.3 Universal enveloping algebra

A Lie superalgebra is a super vector space with a superbracket. Unless this bracket was derived from a superalgebra, there is no ‘normal’ multiplication. Similarly to the Lie algebra case, we define the universal enveloping algebra to make sense of such a multiplication. We will only give the very basics here.
Definition 3.8. Let \( g \) be a Lie superalgebra. A **universal enveloping algebra** of \( g \) is a pair \((A, i)\) with a unital, associative superalgebra \( A \), which we can see as a Lie superalgebra by lemma 3.3, together with a Lie superalgebra homomorphism \( i : g \rightarrow A \) such that it satisfies a universal property: if we have another such pair \((B, j)\), then there exists a unique superalgebra homomorphism \( \phi : A \rightarrow B \) with \( \phi \circ i = j \).

Remember that a Lie superalgebra homomorphism respects the bracket, i.e.
\[
i((x, y)) = i(x)i(y) - (-1)^{p(x)p(y)}i(y)i(x),
\]
and preserves the grading.

Theorem 3.9. The universal enveloping algebra exists and is unique up to isomorphism.

Proof. The universal enveloping algebra is unique up to isomorphism. This follows fairly directly from the definition. Suppose we have two pairs \((A, i)\) and \((B, j)\) satisfying the universal property, then there exist unique homomorphisms \( \phi : A \rightarrow B \) and \( \psi : B \rightarrow A \) such that \( \phi \circ i = j \) and \( \psi \circ j = i \). We can also invoke the universal property on \((A, i)\) and \((A, i)\), which should give a unique map \( A \rightarrow A \) such that \( i \) is invariant under it. We have seen \( \psi \circ \phi \) satisfies this requirement, but \( \text{id}_A \) does as well. Thus \( \psi \circ \phi = \text{id}_A \), and similarly \( \phi \circ \psi = \text{id}_B \), so the universal enveloping algebras are isomorphic.

Existence can be shown by actually constructing the universal enveloping algebra. This is also useful to get a better feeling for the structure.

A Lie superalgebra \( g \) is in particular also a vector space. We use it to make a tensoralgebra
\[
T(g) = k \oplus g \oplus (g \otimes g) \oplus (g \otimes g \otimes g) \oplus \ldots
\]
(3.13)
Since \( g \) is a Lie superalgebra, we have the bilinear bracket \( \langle \cdot, \cdot \rangle : g \times g \rightarrow g \). The universal property of the tensor product then gives a bilinear \( \langle \cdot, \cdot \rangle : g \otimes g \rightarrow g \). We want to extend this map to the rest of \( T(g) \). For a commutator bracket we can find the equation \([a, bc] = b[a, c] + [a, b]c\) by writing it out. This has a super version too, which is also shown just by writing it out. Suppose \( a, b, c \in A \) are homogeneous elements, with \( A \) a superalgebra that also has superbracket derived from the supermultiplication. Remember that for a superalgebra we have \( p(ab) = p(a) + p(b) \). Then
\[
\langle a, bc \rangle = abc - (-1)^{p(a)p(bc)}bca
= abc - (-1)^{p(a)p(b)}bac + (-1)^{p(a)p(b)}bac - (-1)^{p(a)p(b)+p(a)p(c)}bca
= (ab - (-1)^{p(a)p(b)}ba)c + (-1)^{p(a)p(b)}b(ac - (-1)^{p(a)p(c)}ca)
= \langle a, b \rangle c + (-1)^{p(a)p(b)}b(a, c)
\]
and similarly

$$\langle ab, c \rangle = abc - (-1)^{p(a)p(b)p(c)} cab$$

$$= abc - (-1)^{p(b)p(c)} acb + (-1)^{p(b)p(c)} acb - (-1)^{p(a)p(c)+p(b)p(c)} cab$$

$$= a\langle b, c \rangle + (-1)^{p(b)p(c)} \langle a, c \rangle b.$$ 

We want to impart this structure on the tensor algebra too.

First we extend the parity to $T(g)$. This amounts to giving a grading for $T(g)$. Fortunately it has a natural one, inspired by the gradings for the direct sum and tensor product.

$$\langle a, b \otimes c \rangle = \langle a, b \rangle \otimes c + (-1)^{p(a)p(b)} b \otimes \langle a, c \rangle \quad (3.15)$$

$$\langle a \otimes b, c \rangle = a \otimes \langle b, c \rangle + (-1)^{p(b)p(c)} \langle a, c \rangle \otimes b \quad (3.16)$$

Now define the two-sided ideal $I$ generated by all elements of the form

$$a \otimes b - (-1)^{p(a)p(b)} b \otimes a - \langle a, b \rangle \quad (3.17)$$

and with it define

$$\mathcal{U}(g) = T(g)/I. \quad (3.18)$$

The ideal $I$ is generated by homogeneous elements of $T(g)$, making it a graded ideal. Using this we can define a grading

$$\mathcal{U}(g)_\alpha = T(g)_\alpha / I_\alpha, \quad (3.19)$$

turning $\mathcal{U}(g)$ into a superalgebra. Let $i : g \to \mathcal{U}(g)$ be the injection. Then $i$ preserves grading because $\mathcal{U}(g)$ inherits its grading from $g$, and it is forced to preserve grading by dividing out all elements as in equation (3.17). All that is left is to show that $(\mathcal{U}(g), i)$ has the appropriate universal property.

So suppose we have another pair $(B, j)$, $B$ a unital associative superalgebra and $j$ a homomorphism as in definition (3.3). We extend $j : g \to B$ to a map $\mathcal{U}(g): T(g)$ is generated by $g$, so we can define $j : T(g) \to B$ by requiring it to be a superalgebra homomorphism, i.e. $j(a+b) = j(a) + j(b)$ and $j(a \otimes b) = j(a)j(b)$. Note that $p(j(a)j(b)) = p(j(a)) + p(j(b)) = p(a) + p(b) = p(a \otimes b)$, since $j$ is already a Lie superalgebra homomorphism on $g$, so the extension does indeed preserve grading. Since $j$ satisfies equation (3.12), we know $j(I) = 0$. This means it descends to another superalgebra homomorphism on $\mathcal{U}(g)$ that shall be denoted by $\phi : \mathcal{U}(g) \to B$. Then by construction $\phi \circ i = j$. 

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Finally, suppose there exists another homomorphism \( \psi : \mathcal{U}(\mathfrak{g}) \to \mathcal{B} \) such that \( \psi \circ i = j \). Then for all \( a \in \mathfrak{g} \) (seen as a subset of \( \mathcal{U}(\mathfrak{g}) \))

\[
\psi(a) = \psi \circ i(a) = j(a) = \phi \circ i(a) = \phi(a) \tag{3.20}
\]

and since \( \psi, \phi \) are superalgebra homomorphisms, this means \( \psi(a) = \phi(a) \) for all \( a \in \mathcal{U}(\mathfrak{g}) \). This gives the uniqueness of \( \phi \). Thus \( (\mathcal{U}(\mathfrak{g}), i) \) has the universal property for universal enveloping algebras.

Note that if \( \mathfrak{g} = \mathfrak{g}_0 \), i.e. \( \mathfrak{g} \) is a Lie algebra, the construction in the proof gives the familiar universal enveloping algebra of a Lie algebra. Thus we may use the same notation \( \mathcal{U} \) in the graded and non-graded case, as there is no ambiguity.

For all Lie superalgebras the even part \( \mathfrak{g}_0 \) is a Lie algebra. Suppose \( \mathfrak{g}_0 \) has a totally ordered basis \( X = \{ x_1, \ldots, x_n \} \). The Poincaré-Birkhoff-Witt theorem then states that

\[
\{ x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} | i_1, \ldots, i_n \in \mathbb{N} \} \tag{3.21}
\]

is a basis of \( \mathcal{U}(\mathfrak{g}_0) \), called the Poincaré-Birkhoff-Witt basis, see e.g. [3, chap. V] The theorem for Lie superalgebras looks very similar. It will only be stated here, but a proof can be found in [12, chap. 7].

**Theorem 3.10.** Let \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) be a Lie superalgebra. Let \( X = \{ x_1, \ldots, x_n \} \) be a basis for \( \mathfrak{g}_0 \) and \( Y = \{ y_1, \ldots, y_m \} \) a basis for \( \mathfrak{g}_1 \). Then

\[
\{ x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} y_1^{j_1} y_2^{j_2} \cdots y_m^{j_m} | i_1, \ldots, i_n \in \mathbb{N} \text{ and } j_1, \ldots, j_m \in \mathbb{Z}_2 \} \tag{3.22}
\]

is a basis for the universal enveloping algebra \( \mathcal{U}(\mathfrak{g}) \).

It makes sense to consider \( j_1, \ldots, j_m \in \mathbb{Z}_2 \) only. The multiplication of two odd elements is even, so any added powers of odd elements can be rewritten as even elements.

**Corollary 3.11.** For \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) we have \( \mathcal{U}(\mathfrak{g}_0) \subset \mathcal{U}(\mathfrak{g}) \).

**Proof.** The result follows from comparing the bases given by equations (3.21) and (3.22). \( \square \)

Finally, the universal enveloping algebra contains an element that will prove to be useful later on.

**Definition 3.12.** A Casimir element of \( \mathfrak{g} \) is an element of the center of the universal enveloping algebra of \( \mathfrak{g} \), i.e. it is an element that commutes with all of \( \mathcal{U}(\mathfrak{g}) \).

This definition is the same for Lie algebras and Lie superalgebras, but the universal enveloping algebras look different. If \( C \) is a Casimir of the Lie algebra \( \mathfrak{g}_0 \) it might be a Casimir of \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) as well, but there is no certainty. There will be examples of both cases in later sections.
Remark 3.13. The definition of the universal enveloping algebra given here is in line with the one given in [12, chap. 7]. Specifically, the universal enveloping algebra is defined as a superalgebra. This in contrast to some literature, like [6, chap. 6], where it is defined as an algebra. To make sense of the Casimir elements either one would suffice. That is not to mean it does not matter at all which definition we use. For Lie algebras, there is a correspondence between the representations of a Lie algebra and its universal enveloping algebra. This choice of definition gives a correspondence in the Lie superalgebra case as well, provided we define representations as in definition 3.14.

3.4 Graded representations

The Lie superalgebra \( \mathfrak{p}(V) \) often plays the role \( \mathfrak{gl}(V) \) has in the theory of Lie algebras. This is also the case for representations.

**Definition 3.14.** A (graded) representation of a Lie superalgebra \( \mathfrak{g} \) in the super vector space \( V \) is a Lie superalgebra homomorphism \( \pi: \mathfrak{g} \to \mathfrak{p}(V) \).

By definition a Lie superalgebra homomorphism preserves the superbracket, so in particular it preserves the Lie bracket on the even part \( \mathfrak{g}_0 \) of \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \). Thus the restriction \( \mathfrak{g}_0 \to \text{Hom}(V) \) is a Lie algebra representation of the Lie algebra \( \mathfrak{g}_0 \) with representation space \( V \).

**Definition 3.15.** A subrepresentation of \( (\pi, V) \) is a super subspace \( W \subset V \) that is preserved by the action of \( \mathfrak{g} \), i.e. \( \pi(A)(W) \subset W \) for all \( A \in \mathfrak{g} \).

Any representation \( (\pi, V) \) has at least two subrepresentation, namely \( \{0\} \) and \( V \). These are called trivial subrepresentations.

**Definition 3.16.** If a representation has no subrepresentations other than the trivial ones, we call that representation irreducible.

An important Lie algebra representation is the adjoint representation \( \text{ad}: \mathfrak{g} \to \mathfrak{p}(\mathfrak{g}) \). This too has an obvious Lie superalgebra equivalent.

**Definition 3.17.** Let \( \mathfrak{g} \) be a Lie superalgebra. For all \( A \in \mathfrak{g} \) we define the map \( \text{ad}(A): \mathfrak{g} \to \mathfrak{g} \) by
\[
\text{ad}(A)(B) = \langle A, B \rangle \quad \forall B \in \mathfrak{g}.
\] (3.23)

Using the graded skew-symmetry and graded Jacobi identity, as well as
\[\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta \rangle \subset \mathfrak{g}_{\alpha+\beta}, \] we can show that \( \text{ad} \) is a homomorphism of \( \mathfrak{g} \) into \( \mathfrak{p}(\mathfrak{g}) \). This means \( A \to \text{ad}(A) \) is a graded representation, which we will also call the adjoint representation.

If we take as field \( k = \mathbb{C} \), we can look at a specific type of representations. Suppose \( V \) is equipped with an inner product (with linearity in the first argument) \( \langle ., . \rangle \) such that \( V_0 \) and \( V_1 \) are orthogonal. Any \( A \in \text{Hom}(V) \) then has an adjoint \( A^\dagger \) defined by
\[
\langle Ax, y \rangle = \langle x, A^\dagger y \rangle \quad \text{for all } x, y \in V.
\] (3.24)
This is the familiar definition from vector spaces and it is relevant to super vector spaces as well. However, it is not the only possible extension of the adjoint. Often when we go from ‘normal’ to ‘super’, we had a factor depending on the parity. We can do the same here: for homogeneous \( A \in \text{Hom}(V) \) and homogeneous \( x, y \in V \) we define the superadjoint \( A^\dagger \) (sometimes called graded adjoint) by
\[
(Ax \mid y) = (-1)^{p(A)p(x)}(x \mid A^\dagger y).
\] (3.25)
The parity is only defined on homogeneous element, but by extending linearly we can apply the equation to all \( x, y \in V \). Note that for \( A \) even it does reduce to the non-super adjoint.

We also need a ‘super’-equivalent to the adjoint operation on a Lie algebra. Again, there are two ways to extend to Lie superalgebras. Let \( g \) be a complex Lie superalgebra.

**Definition 3.18.** A *star or adjoint operation* in \( g \) is a mapping \( \dagger : \text{Hom}(g) \to \text{Hom}(g) \), \( A \mapsto A^\dagger \) such that

- The \( \dagger \) is even, i.e. even elements are mapped to even elements and odd to odd.

- Let the bar denote the complex conjugation. Then for all \( A, B \in g \) and \( a, b \in \mathbb{C} \)
\[
(aA + bB)^\dagger = \bar{a}A^\dagger + \bar{b}B^\dagger.
\] (3.26)

- For all homogeneous \( A, B \in g \)
\[
\langle A, B \rangle^\dagger = (B^\dagger, A^\dagger).
\] (3.27)

- For all homogeneous \( A \in g \)
\[
(A^\dagger)^\dagger = A.
\] (3.28)

The parity function is not used in this definition, so it is actually not necessary for \( A \) and \( B \) in to be homogeneous for equations (3.27) and (3.28) to make sense. However, the homogeneous elements generate the Lie superalgebra, so by equation (3.26) this is equivalent to requiring equations (3.27) and (3.28) for all \( A, B \in g \). In the next definition it is necessary to use homogeneous elements, so to emphasize their similarity we have chosen to do so in both.

**Definition 3.19.** A *superstar or superadjoint operation* (sometimes also called graded adjoint) in \( L \) is a mapping \( \ddagger : \text{Hom}(g) \to \text{Hom}(g) \), \( A \mapsto A^\ddagger \) such that

- The \( \ddagger \) is even, i.e. even elements are mapped to even elements and odd to odd.

- Let the bar denote the complex conjugation. Then for all \( A, B \in g \) and \( a, b \in \mathbb{C} \)
\[
(aA + bB)^\ddagger = \bar{a}A^\ddagger + \bar{b}B^\ddagger.
\] (3.29)
• For all homogeneous $A, B \in \mathfrak{g}$

$$
\langle A, B \rangle^\dagger = (-1)^{p(A)p(B)} \langle B^\dagger, A^\dagger \rangle
$$

which, by the first condition and equation (3.2), is actually equivalent to

$$
\langle A, B \rangle^\dagger = -\langle A^\dagger, B^\dagger \rangle.
$$

(3.30)

• For all homogeneous $A \in \mathfrak{g}$

$$
(A^\dagger)^\dagger = (-1)^{p(A)}A.
$$

(3.31)

Both the star and the superstar are truly extensions of the adjoint on the Lie algebra. When we take only even elements, all factors involving the parity disappear, leaving the familiar definition of the adjoint.

Now suppose the Lie superalgebra $\mathfrak{g}$ is equipped with a star and $V$ with an adjoint, both denoted by $\dagger$.

**Definition 3.20.** A representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is called a *star representation* if it satisfies

$$
\pi(A^\dagger) = \pi(A)^\dagger \text{ for all } A \in \mathfrak{g}.
$$

(3.32)

Similarly, if $\mathfrak{g}$ has a superstar and $V$ a superadjoint both denoted by $\dagger$, we have

**Definition 3.21.** A representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is called a *superstar representation* if it satisfies

$$
\pi(A^\dagger) = \pi(A)^\dagger \text{ for all } A \in \mathfrak{g}.
$$

(3.33)

For any Lie superalgebra $\mathfrak{g}$ we can now ask ourselves which (super)stars are permitted on $\mathfrak{g}$ and what are the associated (super)star representations.

### 4 Orthosymplectic Lie superalgebras

For this section we assume $k = \mathbb{C}$. One of the simplest classes of Lie superalgebras is the orthosymplectic Lie superalgebras $\mathfrak{osp}(m, 2r)$. We will use the simplest of these, $\mathfrak{osp}(1, 2)$, as an example. First we will make preparations to give a general definition.

**Definition 4.1.** Let $V, W$ be super vector spaces. We call a bilinear map $\varphi : V \times V \to W$ *supersymmetric* if for all homogeneous $x, y$

$$
\varphi(y, x) = (-1)^{p(x)p(y)} \varphi(x, y)
$$

(4.1)

holds, and *skew-supersymmetric* if for all homogeneous $x, y$

$$
\varphi(y, x) = -(-1)^{p(x)p(y)} \varphi(x, y).
$$

(4.2)
Now suppose $V$ is a finite-dimensional super vector space, and $\varphi : V \times V \to \mathbb{C} = \mathbb{C}^{1|0}$ is non-degenerate (i.e. $\varphi(x, y) = 0$ for all $y$ implies $x = 0$), supersymmetric and even. As always even means that the gradation is preserved. The odd part of $\mathbb{C}^{1|0}$ is $\{0\}$, so the requirement $p(x) + p(y) = p(x, y) = p(\varphi(x, y))$ for homogeneous $v, w \in V$ implies that the even and odd spaces of $V$ are orthogonal (i.e. $\varphi(V_0, V_1) = 0$). We now define the orthosymplectic Lie superalgebra defined by $\varphi$ as

$$\mathfrak{osp}(V, \varphi) = \{ f \in \mathfrak{pl}(V) \mid \varphi(f(x), y) + (-1)^{p(f)p(x)} \varphi(x, f(y)) = 0 \text{ for all homogeneous } x, y \in V \}$$

Of course $p(f)$ is only defined for homogeneous $f$, but again we extend linearly to all elements of $\mathfrak{pl}(V)$.

By definition we have $\mathfrak{osp}(V, \varphi) \subset \mathfrak{pl}(V)$. We know that for appropriate $m, n$ we have $\mathfrak{pl}(V) \cong \mathfrak{pl}(m, n)$ (see definition (3.4)). We want to look at $\mathfrak{osp}(V, \varphi)$ as a subset of $\mathfrak{pl}(m, n)$. To that end we first consider the map $\varphi$ as a matrix.

Since it is even, it has the form

$$\varphi(x, y) = x^T \begin{pmatrix} G & 0 \\ 0 & H \end{pmatrix} y$$

with $G$ an $m \times m$-matrix and $H$ an $n \times n$-matrix. We also know that $\varphi$ is supersymmetric. This means that $G$, which only works on the even part, is a symmetric matrix. On the odd part the supersymmetry gives a minus sign, so $H$ is a skew-symmetric matrix. This immediately implies that $n$ is even, so $n = 2r$ for some natural number $r$. Finally, the non-degeneracy of $\varphi$ implies that $G$ and $H$ are non-degenerate as well. Thus we can choose a basis such that

$$G = I_m$$

$$H = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$$

with $I_n$ and $I_r$ the identity matrix in dimension $n$ and $r$ respectively.

We can rewrite the requirement $\varphi(f(x), y) + (-1)^{p(f)p(x)} \varphi(x, f(y)) = 0$ in terms of matrices now. Let $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mathfrak{pl}(m, n)$ be the matrix of $f$ and $x, y \in V$

$$0 = \varphi(f(x), y) + (-1)^{p(f)p(x)} \varphi(x, f(y))$$

$$= (Xx)^t \begin{pmatrix} G & 0 \\ 0 & H \end{pmatrix} y + (-1)^{p(X)p(x)} x^t \begin{pmatrix} G & 0 \\ 0 & H \end{pmatrix} Xy$$

$$= x^t X^t \begin{pmatrix} G & 0 \\ 0 & H \end{pmatrix} y + (-1)^{p(X)p(x)} x^t \begin{pmatrix} G & 0 \\ 0 & H \end{pmatrix} Xy$$

$$= x^t \left( A^t \begin{pmatrix} G & 0 \\ 0 & H \end{pmatrix} C^t \begin{pmatrix} G & 0 \\ 0 & H \end{pmatrix} D^t \begin{pmatrix} A & B \\ C & D \end{pmatrix} y + (-1)^{p(X)p(x)} x^t \begin{pmatrix} G & 0 \\ 0 & H \end{pmatrix} Xy \right)$$
where like before \( p(X) \) is defined as the linear extension if necessary. Since this has to hold for all \( x, y \), we find the following restrictions on \( X \)

\[
A^t I_m + I_mA = 0 \tag{4.4}
\]

\[
B^t I_m - \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} C = 0 \tag{4.5}
\]

\[
D^t \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} + \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} D = 0 \tag{4.6}
\]

**Definition 4.2.** We define the orthosymplectic Lie superalgebra \( \mathfrak{osp}(m,n) \) as the matrices \( X \in \mathfrak{pl}(m,n) \) that satisfy equations (4.4), (4.5) and (4.6), with the gradation inherited from \( \mathfrak{pl}(m,n) \).

Equations (4.4) and (4.6) look familiar, as they are the defining relations for the orthogonal algebra and the symplectic algebra respectively. Hence we have

\[
\mathfrak{osp}(m,2r)_{\bar{0}} \cong \mathfrak{o}(m) \times \mathfrak{sp}(2r) \tag{4.7}
\]

Such a nice description doesn’t exist for the odd part, but we can say something about the dimension. The matrices \( B \) and \( C \) both have \( m \cdot 2r \) variables, but equation (4.5) gives us \( m \cdot 2r \) relations as well. Together this means

\[
\dim(\mathfrak{osp}(m,2r)_{\bar{1}}) = 2(m \cdot 2r) - m \cdot 2r = m \cdot 2r \tag{4.8}
\]

It gets especially nice when we take \( m = 1 \) and \( r = 1 \). We know \( \mathfrak{o}(1) = \{0\} \) and \( \mathfrak{sp}(2) \cong \mathfrak{sl}(2) \), hence \( \mathfrak{osp}(1,2)_{\bar{0}} \cong \mathfrak{sl}(2) \). This is a very well known algebra, and we will make good use of our knowledge. We take a standard \( \mathfrak{sl}(2) \)-basis \( \{l_0, l_+, l_-\} \) as basis for the even part. Since \( \dim(\mathfrak{osp}(1,2)_{\bar{1}}) = 2 \), we need only two elements to span the odd part. We pick \( q_+, q_- \in \mathfrak{osp}(1,2)_{\bar{1}} \) such that

- \( \langle l_0, l_\pm \rangle = \pm 2l_\pm \)
- \( \langle l_+, l_- \rangle = l_0 \)
- \( \langle l_0, q_\pm \rangle = \pm q_\pm \)
- \( \langle q_\pm, q_\pm \rangle = \pm 4l_\pm \)
- \( \langle q_+, q_- \rangle = 2l_0 \)

The above relations give the superbracket for all basis elements, so by linearity for the whole Lie superalgebra. Hence we could have defined \( \mathfrak{osp}(1,2) \) as basis elements \( l_0, l_\pm \) for the even part and \( q_\pm \) for the odd part, with these defining relations. Remember that on the even part the superbracket acts like a commutator, so we find \( \mathfrak{sl}(2) \) back this way as well. We could have found \( \mathfrak{osp}(1,2) \) theorem 3.5 on \( \mathfrak{sl}(2) \) and the module spanned by the odd element.

It is also good to note that \( (\mathfrak{osp}(1,2)_{\bar{1}}, \mathfrak{osp}(1,2)_{\bar{1}}) = \mathfrak{osp}(1,2)_{\bar{0}} \) and not just a subset. This means we need five basis elements to span the Lie superalgebra.
as a vector space, but only two to generate it with the superbracket.

There are two special elements in \( \mathfrak{osp}(1, 2) \) that are useful to mention, namely the Casimir elements for \( \mathfrak{sl}(2) \) and \( \mathfrak{osp}(1, 2) \). These elements commute with all other elements in their respective spaces, \([7]\). For \( \mathfrak{sl}(2) \) we have

\[
\Omega = -\frac{1}{4}(4l_-l_+ + l_0^2 + 2l_0)
\]

(4.9)

This is not yet a Casimir for \( \mathfrak{osp}(1, 2) \), but it can be extended to one:

\[
C = -4\Omega + \frac{1}{2}(q_+q_+ - q_-q_-).
\]

(4.10)

With the above superbracket relations it can be shown that the following relation holds

\[
C^2 = (1 - 4\Omega)(2C + 4\Omega).
\]

(4.11)

### 4.1 The star-structures on \( \mathfrak{osp}(1, 2) \)

Now that we have an idea of how \( \mathfrak{osp}(1, 2) \) looks, we can try to find its (super)stars. Rather than defining it out of nowhere, we would like to use the fact that we know the possible adjoints on \( \mathfrak{sl}(2) \). We have remarked before that the star and superstar should behave as an adjoint on the even part, so we might as well start building from there.

In general, suppose we have an adjoint \( \dagger \) on the even part of \( L = L_0 \oplus L_\bar{1} \).

We need to define a map \( \sigma \) on the odd part in such a way that together they make a (super)star. As intermediary we define the following:

**Definition 4.3.** A generalized adjoint on a Lie superalgebra \( L \) is a map \( L \to L \) such that

1. It is additive and maps \( L_i \to L_i \) for \( i = 0, \bar{1} \).
2. The restriction to \( L_0 \) is an adjoint \( \dagger \), i.e.
   - \((aP + bQ)\dagger = aP\dagger + bQ\dagger\)
   - \([P, Q]\dagger = [Q\dagger, P\dagger]\)
   - \((Q\dagger)\dagger = Q\)
   for \( P, Q \in L_0 \) and \( a, b \in \mathbb{C} \).
3. The restriction to \( L_\bar{1} \) is a bijective antilinear \( \sigma : L_\bar{1} \to L_\bar{1} \), i.e. \( \sigma \) is a bijective map such that
   \[
   \sigma(aU + bV) = \bar{a}\sigma(U) + \bar{b}\sigma(V)
   \]
   for \( U, V \in L_\bar{1} \) and \( a, b \in \mathbb{C} \).
4. The adjoints on $L_0$ and $L_1$ are compatible, which in this case means

$$\sigma(\langle Q, U \rangle) = -\langle Q^\dagger, \sigma(U) \rangle$$

(4.13)

for $Q \in L_0$ and $U \in L_1$.

If we want to make a (super)star, this is the least we need. The compatibility requirement might seem odd at first glance, but it is simply equation (3.27) or (3.30) for one even and one odd element. Now let $U, V \in L_1$. If furthermore

$$\langle U, V \rangle^\dagger = \langle \sigma(U), \sigma(V) \rangle \quad \text{and} \quad \sigma^2 = 1$$

(4.14)

then $\sigma$ extends the adjoint into a star operator, and if

$$\langle U, V \rangle^\dagger = -\langle \sigma(U), \sigma(V) \rangle \quad \text{and} \quad \sigma^2 = -1$$

(4.15)

then $\sigma$ extends the adjoint into a superstar operator.

Now we go back to $\mathfrak{osp}(1, 2)$. There are two possible adjoints on $\mathfrak{sl}(2)$, we start with the one that corresponds with the Lie group $\text{SU}(2)$, very suggestively denoted by $\dagger$. On the even generators it is $\hat{l}_0^1 = l_0$ and $\hat{l}_\pm^1 = l_\mp$. We define $\sigma : \mathfrak{osp}(1, 2)_{\dagger} \rightarrow \mathfrak{osp}(1, 2)_{\dagger}$ by setting

$$\sigma(q_+) = a_+q_+ + b_+q_-$$
$$\sigma(q_-) = a_-q_+ + b_-q_-$$

and extend antilinearly, with $a_\pm, b_\pm \in \mathbb{C}$. How should we choose the $a_\pm, b_\pm$ to make $\dagger$ and $\sigma$ a generalized adjoint?

1. By definition the map is additive and and $\mathfrak{osp}(1, 2)_{\dagger} \rightarrow \mathfrak{osp}(1, 2)_{\dagger}$.

2. By definition $\dagger$ is an adjoint.

3. By definition $\sigma$ is antilinear. We have bijectivity if $a_+b_- \neq a_-b_+$, so we will take care to choose our factors like that.

4. By writing out the compatibility requirement for the basis elements we get restrictions on the factors

   - We consider two equations at the same time:

   $$\sigma(\langle l_0, q_\pm \rangle) = \sigma(\pm q_\pm)$$
   $$= \pm \sigma(q_\pm)$$
   $$= \pm a_\pm q_+ + \pm b_\pm q_-$$
   $$- \langle \hat{l}_0^1, \sigma(q_\pm) \rangle = -\langle l_0, a_\pm q_+ + b_\pm q_- \rangle$$
   $$= -a_\pm \langle l_0, q_+ \rangle - b_\pm \langle l_0, q_- \rangle$$
   $$= -a_\pm q_+ + b_\pm q_-$$

   and because we require equality we must have

   $$\pm a_\pm = -a_\pm \quad \text{and} \quad \pm b_\pm = b_\pm$$

   (4.16)

   which means $a_+ = 0$ and $b_- = 0$. Using this we write $\sigma(q_\pm) = c_\pm q_\mp$. 

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• Next we want to test the compatibility for \( l_\pm \) and \( q_\pm \), but we don’t know the superbracket for these two yet. We can find it by using the graded Jacobi equation:

\[
4\langle l_\pm, q_\pm \rangle = \langle 4l_\pm, q_\pm \rangle = \langle \pm \langle q_\pm, q_\pm \rangle, q_\pm \rangle = -2\langle \pm \langle q_\pm, q_\pm \rangle, q_\pm \rangle = -8\langle l_\pm, q_\pm \rangle
\]

Hence \( \langle l_\pm, q_\pm \rangle = 0 \). Both sides of the compatibility relation are zero because of this, so we get no new information.

• Again, we don’t have the superbracket for \( l_\pm \) and \( q_\mp \) yet. Using the graded Jacobi identity we find

\[
\langle l_\pm, q_\mp \rangle = -q_\pm \quad (4.17)
\]

This does give new information:

\[
\sigma \langle l_\pm, q_\mp \rangle = \sigma (-q_\pm) = -c_\mp q_\mp
\]

\[
-\langle l_\pm^\dagger, \sigma(q_\mp) \rangle = -\langle l_\mp^\dagger, c_\pm q_\mp \rangle = -c_\pm \langle l_\mp^\dagger, q_\mp \rangle = c_\pm q_\mp
\]

and since we require equality we find \( c_+ = -c_- \).

Thus \( \sigma(q_\pm) = \pm cq_\mp \) for some \( c \in \mathbb{C} \) gives a generalized adjoint.

Now that we have a generalized adjoint, can we turn it into a (super)star? We first demand equation (4.14). This means

\[
2l_0 = 2l_0^\dagger = \langle q_+, q_- \rangle^\dagger = \langle \sigma(q_+), \sigma(q_-) \rangle = -c^2 \langle q_-, q_+ \rangle = -2c^2 l_0
\]

\[
q_\pm = \sigma^2(q_\pm) = \sigma(cq_{\mp}) = \pm \mp \bar{c}cq_\pm = -\bar{c}cq_\pm
\]

Hence \( \bar{c}c = -1 \) and \( c^2 = -1 \), which is impossible. Hence the generalized adjoint cannot be extended to a star.

Next we demand equation (4.15). This means

\[
2l_0 = 2l_0^\dagger = \langle q_+, q_- \rangle^\dagger = -\langle \sigma(q_+), \sigma(q_-) \rangle = c^2 \langle q_-, q_+ \rangle = 2c^2 l_0
\]

\[
-q_\pm = \sigma^2(q_\pm) = \sigma(cq_{\mp}) = \pm \mp \bar{c}cq_\pm = -\bar{c}cq_\pm
\]

Hence \( \bar{c}c = 1 \), which means \( c \) is on the unit circle, and \( c^2 = 1 \). Thus we find two superstars, one for \( c = 1 \), and one for \( c = -1 \).
We have another adjoint on \( \mathfrak{sl}(2) \), denoted by \( \dagger \), this one corresponding to the Lie group \( \text{SU}(1,1) \). On the even generators this is \( l_0^\dagger = l_0, l_\pm^\dagger = -l_\mp \). This is very similar to the other adjoint, and in fact we only need to change a few minus signs near the end. The final compatibility equation becomes

\[
\sigma((l_\pm, q_\mp)) = \sigma(-q_\pm) \\
= -c_\mp q_\mp \\
-\langle l_\mp^\dagger, \sigma(q_\mp) \rangle = (l_\mp, c_\pm q_\pm) \\
= c_\pm (l_\mp, q_\pm) \\
= -c_\pm q_\mp
\]

so we find \( c_+ = c_- \). Thus with the map \( \sigma(q_{\pm}) = cq_{\pm} \) for some \( c \in \mathbb{C} \) we get a generalized adjoint. This time it turns out equation (4.15) leads to a contradiction, making a superstar impossible. If we require equation (4.14), we find

\[
2l_0 = 2l_0^\dagger = \langle q_+, q_- \rangle^\dagger = \langle \sigma(q_+), \sigma(q_-) \rangle = c^2(q_-, q_+) = 2c^2l_0 \\
q_\pm = \sigma^2(q_\pm) = \sigma(cq_\mp) = \bar{c}cq_\pm = c^2q_\mp
\]

So \( \bar{c}c = 1 \) implies \( c \) is on the unit circle, and from \( c^2 = 1 \) we conclude \( c = 1 \) or \( c = -1 \). Thus we find two stars in this case.

4.2 The representations

We want to find out what representations for \( \mathfrak{osp}(1,2) \) might look like, specifically the (super)star representations. To that end we will try to build an irreducible representation \( (\pi, V) \) of \( \mathfrak{osp}(1,2) \), with the extra assumption that \( V \) has at least one even eigenvector for \( l_0 \), i.e. for some \( \mu \in \mathbb{C} \) we have

\[
\pi(l_0)v_0 = 2\mu v_0 \quad \text{(4.18)}
\]

If \( V \) is finite-dimensional, then such a \( v_0 \) certainly exists. This \( v_0 \) will be the starting point for the construction.

We know that the restriction of \( \pi \) to the even part should give a Lie algebra representation of \( \mathfrak{osp}(1,2)_0 \cong \mathfrak{sl}(2) \), but as \( \mathfrak{sl}(2) \)-representation it is not necessarily irreducible. We define

\[
v_{2m} = \pi(l_+)^m v_0, \quad v_{-2m} = \pi(l_-)^m v_0 \quad \text{for } m \in \mathbb{N}.
\]

These \( v_{2m} \) might be zero, so we take an index set \( I = \{ m \in \mathbb{Z} \mid v_{2m} \neq 0 \} \) and define \( W_0 = \text{span}\{v_{2m} \mid m \in I \} \). From Lie algebra theory we know ([3, chap. 7]) that \( (\pi, W_0) \) is an irreducible representation of \( \mathfrak{sl}(2) \) and \( I \) is an interval. Furthermore, we know the action of \( l_0 \) on \( W_0 \):

**Lemma 4.4.** For all elements \( v_{2m} \in W_0 \) we have

\[
\pi(l_0)v_{2m} = (2\mu + 2m)v_{2m}
\]

(4.20)
Proof. Starting from the superbracket relations

\[ [l_0, l_+] = \langle l_0, l_+ \rangle = 2l_+ \quad \text{and} \quad [l_0, l_-] = \langle l_0, l_- \rangle = -2l_- , \]

we use induction to find the following relations in \( \mathcal{U}(\mathfrak{sl}(2)) \):

\[ [l_0, (l_+)^m] = 2m(l_+)^m \quad \text{and} \quad [l_0, (l_-)^m] = -2m(l_-)^m . \]

With these commutation relations it is simple to write out the action of \( l_0 \) on \( W_0 \). Let \( m \in I \) be positive, then

\[
\pi(l_0)v_{2m} = \pi(l_0)\pi(l_+)^m v_0 \\
= \pi(l_0(l_+)^m)v_0 \\
= \pi((l_+)^m l_0 + 2m(l_+)^m)v_0 \\
= (2\mu + 2m)\pi(l_+)^m v_0 \\
= (2\mu + 2m)v_{2m}
\]

For negative \( m \) we can make a similar calculation with \( l_- \).

Next we define

\[ v_1 = \pi(q_+)v_0, \quad v_{-1} = \pi(q_-)v_0. \quad (4.21) \]

Using the relation \( \langle l_0, q_{\pm} \rangle = \pm q_{\pm} \), we find that \( l_0v_1 = (2\mu + 1)v_1 \) and \( l_0v_{-1} = (2\mu - 1)v_{-1} \). Thus \( v_1 \) and \( v_{-1} \) are eigenvectors of \( l_0 \), with eigenvalues \( 2\mu + 1 \) and \( 2\mu - 1 \) respectively. As we have seen, the eigenvalues for elements in \( W_0 \) are not of this form, hence \( v_{\pm 1} \notin W_0 \). As \( l_0 \)-eigenvectors, they too generate irreducible \( \mathfrak{sl}(2) \)-representation spaces, which we denote by \( W_1 \) and \( W_{-1} \).

The spaces \( W_1 \) and \( W_{-1} \) might be the same. To understand this better we first look at \( \pi(q_{\pm})v_{\mp 1} \).

Lemma 4.5. Both \( q_- q_+ \) and \( q_+ q_- \) act diagonally on \( v_0 \).

Proof. The Casimirs \( \Omega \) and \( C \) commute with \( l_0 \), hence they preserve the eigenspace of \( l_0 \) with eigenvalue \( \mu \). Thus we can find eigenvectors for \( \Omega \) and \( C \) within this space, and since \( \Omega \) and \( C \) commute we can in fact find a simultaneous eigenvector. Without loss of generality we assume that \( v_0 \) is such a simultaneous eigenvector. We will later find that the eigenvalues are non-degenerate.

\[
av_0 = \pi(C - \Omega)v_0 \\
= \frac{1}{2} \pi(q_- q_+ - q_+ q_-)v_0 \\
= \frac{1}{2} \pi(-2q_+ q_- + 2l_0)v_0 \\
= -\pi(q_+ q_-)v_0 + 2\mu v_0
\]

We used the superbracket relation \( q_+ q_- + q_- q_+ = 2l_0 \) to write this entirely in terms of \( q_+ q_- \). This gives \( \pi(q_+ q_-)v_0 = (2\mu - c)v_0 = c'v_0 \), so \( q_+ q_- \) indeed acts diagonally. This directly implies that \( q_- q_+ \) acts diagonally as well, since \( \pi(C - \Omega)v_0 = \frac{1}{2} \pi(q_- q_+ - q_+ q_-)v_0 \).
The actions of $q_+q_-$ and $q_+q_-$ on $v_0$ could still be zero, but if either one is non-zero, we have $W_1 = W_{-1}$. For example, suppose $\pi(q_+q_-)v_0 = c'v_0$ with $c'$ non-zero. Using that $2q_+q_+ = \pi(q_+q_+q_+) = 4l_+$ we find

$$\pi(l_+)v_{-1} = \frac{1}{2}\pi(q_+q_+)\pi(q_-)v_0 = \frac{1}{2}\pi(q_+)\pi(q_+)v_0 = \frac{1}{2}c'v_1 \in W_1. \quad (4.22)$$

Since $W_{-1}$ and $W_1$ are generated $v_{-1}$ and $v_1$ respectively, this would imply that $v_1$ is an element of both $W_{-1}$ and $W_1$.

We consider $\{v_{2m+1} = \pi(l_+)^m v_1 \mid m \in \mathbb{N}\} \subset W_1$ and $\{v_{-2m-1} = \pi(l_-)^m v_1 \mid m \in \mathbb{N}\} \subset W_{-1}$. Again, we can see from the superbracket relations that $2l_\pm = \pm q_\pm q_\pm$. Because $q_\pm$ certainly commutes with itself, we can rewrite the above to

$$v_{2m+1} = \pi(l_+)^m \pi(q_+)v_0 = \pi(q_+)v_{2m} \quad (4.23)$$

$$v_{-2m-1} = \pi(l_-)^m \pi(q_-)v_0 = \pi(q_-)v_{-2m}. \quad (4.24)$$

provided the elements $v_{2m+1}$ and $v_{-2m-1}$ exist in $W_1$ and $W_{-1}$ respectively. Also note that if $W_0$ has a maximal (resp. minimal) weight vector so has $W_1$ (resp. $W_{-1}$) by the same argument.

**Lemma 4.6.** The representation space is given by $V = W_0 \oplus W_{-1} \oplus W_1$ with gradation $V_0 = W_0$ and $V_1 = W_{-1} \oplus W_1$. If $W_1 = W_{-1}$ we assume $\{v_{2m+1} = \pi(l_+)^m v_1 \mid m \in \mathbb{N}\} = W_1$ and $\{v_{-2m-1} = \pi(l_-)^m v_1 \mid m \in \mathbb{N}\} = W_{-1}$.

**Proof.** The spaces $W_0$, $W_1$ and $W_{-1}$ together contain all the elements we can reach by repeatedly applying $\pi(q_+)$ and $\pi(q_-)$ to $v_0$, and all other elements of $\mathfrak{osp}(1,2)$ can be written in terms of $q_\pm$. We assumed representation is irreducible, so our representation space must be $V = W_{-1} \oplus W_0 \oplus W_1$. Since we require the representation map to be an even Lie superalgebra homomorphism, the representation space must have gradation $V_0 = W_0$ and $V_1 = W_{-1} \oplus W_1$. \hfill $\Box$

Note that the even subspace of $V$ is an irreducible $\mathfrak{sl}(2)$ representation space, but the odd subspace is the direct sum of two irreducible $\mathfrak{sl}(2)$ representation spaces. However, $W_{-1}$ and $W_1$ might be equal, or either or both might be trivial. We will later find that for the irreducible (super)star $\mathfrak{osp}(1,2)$-representations the odd subspace is also one irreducible $\mathfrak{sl}(2)$ representation space.

Now we want to know the actions of all the basis elements of $\mathfrak{osp}(1,2)$. The action of $l_0$ is simple to find.

**Lemma 4.7.** The action of $l_0$ is given by

$$\pi(l_0)v_m = (2\mu + m)v_m \quad (4.25)$$

for all $m \in \mathbb{Z}$.

**Proof.** For $W_0$ we have already shown this. We have the commutation relation $[l_0, q_\pm] = \langle l_0, q_\pm \rangle = \pm q_\pm$. Using this we find, for $m \in \mathbb{N}$

$$\pi(l_0)v_{2m+1} = \pi(l_0)\pi(q_+)v_{2m} = \pi(q_0 + q_+)v_{2m} = (2\mu + 2m + 1)v_{2m+1} \quad (4.26)$$

And similarly $\pi(l_0)v_{-2m-1} = (2\mu - 2m - 1)v_{-2m-1}$. \hfill $\Box$
Next we want to find the actions of the $q_{\pm}$ on all of $V$. Since we can write both $l_{\pm}$ in terms of $q_{\pm}$, this is enough to completely define $\pi$. We use the Casimirs to do this. First off, $C$ commutes with all elements of $\mathfrak{oosp}(1,2)$. There is a basis $\{ v_{m} \mid m \in \mathbb{Z} \}$ of eigenvectors of $l_{0}$ and according to lemma 4.7 the eigenvalues are all non-degenerate, so there must exist a scalar $\lambda$ such that

$$\pi(C)v = \lambda v \quad \text{for all } v \in V. \quad (4.27)$$

Right now we only know that $\Omega$ commutes with the even elements. Hence we can only claim it acts diagonally on the elements of $V$ we can ‘reach’ with $l_{\pm}$ from $v_{0}$. We say there exists a scalar $\delta$ such that

$$\pi(\Omega)v_{2m} = -\delta(\delta + 1)v_{2m} \quad \text{for all } m \in I \quad (4.28)$$

These two scalars are not independent. By applying both sides of equation (4.11) to an element $v_{2m}$ we get a quadratic equation with two possible solutions.

$$\lambda_{1} = 2\delta(2\delta + 1) \quad \text{and} \quad \lambda_{2} = 2(\delta + 1)(2\delta + 1). \quad (4.29)$$

We immediately note that the transformation $\delta \rightarrow -\delta - 1$ brings us from one case to the other. We simply choose $\lambda = \lambda_{1}$ for now.

We don’t know the action of $\Omega$ on $W_{\pm 1}$ yet. With the commutation relations $\Omega$ and $q_{\pm}$ we could avoid this problem. After all, the elements of $W_{\pm 1}$ can be written as the image under $q_{\pm}$ of an elements of $W_{0}$ and we know $\Omega$ on $W_{0}$. These commutation relations can be found using the definitions of the Casimirs and the superbracket relations. The calculation is rather tedious, with as result (see [7])

$$[\Omega, q_{\pm}] = \frac{1}{4}q_{\pm} - \frac{1}{2}q_{\mp}C - 2q_{\pm}\Omega. \quad (4.30)$$

From here it is simply a case of applying the operators and rewriting the result, so we state without further proof:

**Lemma 4.8.** The operator $\Omega$ on $W_{\pm 1}$ is given by

$$\pi(\Omega)v_{2m+1} = -(\delta - \frac{1}{2})(\delta + \frac{1}{2})v_{2m+1} \quad (4.31)$$

for $m \in \mathbb{Z}$.

As we have seen before, we can write $q_{\pm}q_{\mp}$ in terms of $l_{0}, C, \Omega$. Now that we know the actions of those three, we can find $\pi(q_{\pm})$. For example: for $m \geq 1$

$$\pi(q_{-})v_{2m} = \pi(q_{-})\pi(l_{+})v_{2m-2}$$

$$= \frac{1}{2}\pi(q_{-})\pi(q_{+}+q_{+})v_{2m-2}$$

$$= \frac{1}{2}\pi(q_{-}q_{+})\pi(q_{+})v_{2m-2}$$

$$= \frac{1}{2}\pi(l_{0}-C+\Omega)v_{2m-1}$$

$$= (\mu + m + \delta)v_{2m-1}$$

The other cases can be written out similarly. The result is the following lemma,
Lemma 4.9. The Lie superalgebra homomorphism $\pi$ is given by

\[
\begin{align*}
\pi(q_-)v_{-2m} &= v_{-2m-1} \\
\pi(q_-)v_{-2m-1} &= -2v_{-2m-2} \\
\pi(q_-)v_{2m} &= (\mu + m + \delta)v_{2m-1} \\
\pi(q_-)v_{2m+1} &= 2(\mu + m - \delta)v_{2m} \\
\pi(q_+)v_{2m} &= v_{2m+1} \\
\pi(q_+)v_{2m+1} &= 2v_{2m+2} \\
\pi(q_+)v_{-2m} &= -(\mu - m - \delta)v_{-2m+1} \\
\pi(q_+)v_{-2m-1} &= 2(\mu - m + \delta)v_{-2m}
\end{align*}
\]

Remember that if either $\pi(q_+q_-)v_0$ or $\pi(q_-q_+)v_0$ is non-zero, we can conclude $W_{-1} = W_1$. First note that if one of $v_{\pm 1}$ is zero, then their whole space $W_{\pm 1}$ is zero by definition. In this case the odd subspace of $V$ is $\{0\}$ (if both $v_{\pm 1} = 0$), $W_1$ (if only $v_{-1} = 0$) or $W_{-1}$ (if only $v_1 = 0$). Either way it is an irreducible $\mathfrak{sl}(2)$ representation space.

If we assume they are both non-zero, applying lemma 4.9 gives $\pi(q_+q_-)v_0 = 2(\mu + \delta)v_0$ and $\pi(q_-q_+)v_0 = 2(\mu - \delta)v_0$. We can only have $\mu + \delta = 0$ and $\mu - \delta = 0$ if $\mu = \delta = 0$. In all other cases $W_{-1} = W_1$, so the odd subspace is again an irreducible $\mathfrak{sl}(2)$ representation space. The only possible exception we are left with is the case $\mu = \delta = 0$.

All that remains is to make this representation into a (super)star representation. We consider the case for the superstar $\hat{\mathfrak{l}}_0, \hat{\mathfrak{l}}_\pm = l_\pm, q_{\pm} = \pm q_\pm$. The other cases are similar..

Lemma 4.10. Suppose we have a sesquilinear form $(\cdot | \cdot) : V \times V \to \mathbb{C}$ for which $(\pi,V)$ is a superstar representation. Then $\{v_m | m \in \mathbb{Z}, v_m \neq 0\}$ is an orthogonal basis for $V$.

Proof. By construction this set is a basis for $V$. Using lemma 4.7 we find

\[
(2\mu + m)(v_m | v_n) = (\pi(l_0)v_m | v_n) = (-1)^{l_0p(v_m)}(v_m | \pi(l_0)v_n) = (v_m | \pi(l_0)v_n) = (v_m | \pi(l_0)v_n) = (2\mu + n)(v_m | v_n)
\]

thus $(v_m | v_n) = 0$ for $m \neq n$. 

Let $J$ be the index set defined by $J = \{m \in \mathbb{Z} | v_m \neq 0\}$. The basis can now be written as $\{v_m | m \in J\}$. If we want a superstar representation, lemma 4.10 tells us $\{v_m | m \in J\}$ needs to be orthogonal. Hence we first define a general form by

\[
(v_m | v_n) = a_m \delta_{mn} \quad \text{for } m, n \in J \quad (4.32)
\]
with scalars \( a_m \) and Kronecker delta \( \delta_{mn} \). We scale this form such that \( a_0 = 1 \).

We want \( (\cdot \mid \cdot) \) to be an inner product such that \( \pi \) is a superstar representation. This places restrictions on both the representation, namely \( a_m > 0 \) for all \( m \in J \) and \( (\pi(X)v \mid w) = (-1)^{p(X)p(v)}(v \mid \pi(X)\lvert w) \) for all homogeneous \( X \in \mathfrak{osp}(1,2) \) and \( v, w \in V \). First of all we find

\[
2\mu = 2\mu a_0 = (\mu (l_0) v_0 \mid v_0) = (v_0 \mid \pi(l_0)v_0) = 2\overline{\mu} a_0 = 2\overline{\mu} \quad (4.33)
\]

hence \( \mu \) is real. A similar argument with \( C \) and \( \Omega \) shows that \( \delta \) must also be real. Next we want to require \( (\pi(q_{\pm})v \mid w) = (v \mid \pi(q_{\pm})w) \). Note that this will be enough to conclude \( (\pi(X)v \mid w) = (-1)^{p(X)p(v)}(v \mid \pi(X)\lvert w) \) for all homogeneous \( X \in \mathfrak{osp}(1,2) \), as we can use this together with \( l_\pm = \pm \frac{1}{2} q_{\pm} q_{\pm} \) to write

\[
(\pi(l_\pm)v \mid w) = \frac{1}{2}(\pi(q_{\pm})\pi(q_{\pm})v \mid w)
= \frac{1}{2}(-1)^{p(q_{\pm})p(\pi(q_{\pm})v)}(\pi(q_{\pm})v \mid \pi(q_{\pm}^\dagger)w)
= \frac{1}{2}(-1)^{p(q_{\pm})p(\pi(q_{\pm})v)}(-1)^{p(q_{\pm})p(v)}(v \mid \pi(q_{\pm}^\dagger)\pi(q_{\pm})w)
= \mp \frac{1}{2}(v \mid \pi(q_{\mp})\pi(q_{\mp})w)
= \mp \frac{1}{2}(v \mid \pi(q_{\pm} q_{\mp})w)
= (v \mid \pi(l_\mp) w)
\]

Here we also used that \( \pi(q_{\pm}) \) is odd. For homogeneous \( v \) this means we must have \( p(v) \neq p(\pi(q_{\pm})v) \), thus \((-1)^{p(q_{\pm})p(\pi(q_{\pm})v)}(-1)^{p(q_{\pm})p(v)} = -1 \).

Using the actions of \( q_{\pm} \), we find recursive relation for the \( a_m \). We look at the positive and negative indices separately, so let \( j \in \mathbb{N} \). Then

\[
a_{2j+1} = 2(\mu + j - \delta)a_{2j}
\]

\[
a_{2j+2} = -\frac{1}{2}(\mu + j + 1 + \delta)a_{2j+1}
\]

\[
a_{-2j-1} = -2(\mu - j + \delta)a_{-2j}
\]

\[
a_{-2j-2} = \frac{1}{2}(\mu - j - 1 - \delta)a_{-2j-1}
\]

Since \( \mu \) and \( \delta \) are both real and \( a_0 = 1 \), these scalars are also real. We require \( a_m > 0 \) for all \( m \in J \), which means the factors \( 2(\mu + j - \delta), -\frac{1}{2}(\mu + j + 1 + \delta), -2(\mu - j + \delta) \) and \( \frac{1}{2}(\mu - j - 1 - \delta) \) should be positive if they correspond to one of the basic elements. We see that \( 2(\mu + j - \delta) \) and \(-2(\mu - j + \delta) \) are positive for all \( j \in \mathbb{N} \), provided we have

\[
\mu - \delta > 0
\]

\[
\mu + \delta < 0
\]
This restricts the possible values for \((\delta, \mu)\) as illustrated by figure 1.

Even though we formulated the restrictions as strict inequalities, we don’t actually exclude the cases \(\mu - \delta = 0\) and \(\mu + \delta = 0\). The first implies \(a_1 = 0\), so \(1 \notin J\), and the second implies \(a_{-1} = 0\), so \(-1 \notin J\). This means \(v_0\) is a vector of maximal or minimal weight respectively. Note that for \((\delta, \mu) = (0, 0)\), \(v_0\) is simultaneously minimal and maximal. Hence \(V = V_0 = \mathbb{C}v_0\), a one-dimensional irreducible \(\mathfrak{sl}(2)\)-representation, and \(V_1 = \{0\}\), the trivial \(\mathfrak{sl}(2)\)-representation.

There are still further restrictions. For \(-\frac{1}{2}(\mu + j + 1 + \delta)\) and \(\frac{1}{2}(\mu - j - 1 - \delta)\) to be positive, we need \(\mu + j + 1 + \delta < 0\) and \(\mu - j - 1 - \delta > 0\). Obviously, these cannot hold for all \(j \in \mathbb{N}\). If we actually have \(\mu + j + 1 + \delta \geq 0\), then \(2j + 2 \notin J\). For \(j\) such that \(\mu - j - 1 - \delta \leq 0\), we must have \(-2j - 2 \notin J\). Thus the representation must have both a minimal and a maximal weight vector. This means the representation, and hence the even and odd subspaces, are finite-dimensional.

There were two possibilities for \(\lambda\). Had we chosen \(\lambda = \lambda_2\), the restrictions would be slightly different. They are easily found by using the transformation \(\delta \to -\delta - 1\):

\[
\begin{align*}
a_{2j+1} &= 2(\mu + j + \delta + 1)a_{2j} \\
a_{2j+2} &= -\frac{1}{2}(\mu + j - \delta)a_{2j+1} \\
a_{-2j-1} &= -2(\mu - j - \delta - 1)a_{-2j} \\
a_{-2j-2} &= \frac{1}{2}(\mu - j + \delta)a_{-2j-1}
\end{align*}
\]

This gives the restriction in the \((\delta, \mu)\)-space as in figure 2. Again we also necessarily have a minimal and maximal weight vector, so in this we also find that the representation is the direct sum of two finite \(\mathfrak{sl}(2)\)-representations.

For the other stars we can find similar restrictions on the general representation given by lemmas 4.6 and 4.9. See [7] for other cases. The general conclusion
is given by the following theorem.

**Theorem 4.11.** Any irreducible (super)star $\mathfrak{osp}(1, 2)$-representation is the direct sum of two $\mathfrak{sl}(2)$-representations. Specifically, an irreducible superstar representation with a superstar corresponding to an SU(2)-star is the direct sum of two finite-dimensional $\mathfrak{sl}(2)$-representations. An irreducible star representation with a star corresponding to an SU(1,1)-star is the direct sum of two positive discrete series representations or two negative discrete series representations.
Part II
Poincaré superalgebra

The basic idea of the theory of special relativity is that the laws of physics and the speed of light in vacuum are independent of the frame of reference they are observed in, provided there is no acceleration. Such a non-accelerating frame of reference is called an inertial system. This means that the laws of physics ought to be invariant under transformations from one inertial system to another, and all these transformations together form a group we can see as the symmetry group of the universe.

We can use this symmetry group to make sure a quantum mechanical description of a particle is compatible with special relativity. A particle in a given state (i.e. with given location, momentum, spin, etc.) can be described by a vector in a (projective) Hilbert space, and the symmetry transformations take one state to another. It turns out this can only be done consistently if we have a group representation. Classifying the representations of the symmetry group of the universe gives us an idea of the different particles that possibly exist.

Of course physicists already have an idea of many of the particles that exist. These are divided in two classes, the bosons and the fermions. These two types of particles behave differently and have to be considered separately. The idea behind supersymmetry is to find a relationship, a symmetry, between the bosons and the fermions. If the symmetry group is extended to a supersymmetry group, the odd elements give such a relationship.

5 The Poincaré group

In the theory of special relativity, there is no separate time (1-dimensional) and space (3-dimensional), but only the combined spacetime ((1 + 3)-dimensional). Special relativity is commonly formulated on the Minkowski spacetime, $\mathbb{R}^4$, equipped with the quadratic form

$$x^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

with $x = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4$. Here $x^0$ is the time coordinate (up to the scalar $c =$ speed of light in vacuum, which is often set to 1), while the other three are the space coordinates. Minkowski spacetime is also called flat spacetime, as it does not take into account the curvature that comes with general relativity.

First we need to find the symmetry group of Minkowski space. The group $O(1, 3)$ contains all transformations that leave the quadratic form invariant. It is often called the Lorentz group. It has four disconnected components and going from one component to another requires the transformations

$$(x^0, x^1, x^2, x^3) \mapsto (x^0, -x^1, -x^2, -x^3)$$
$$(x^0, x^1, x^2, x^3) \mapsto (-x^0, x^1, x^2, x^3)$$
or both. These are reflections in space and time respectively. Leaving out the first is called proper, leaving out the second orthochronous. The laws of physics invariant are invariant under these reflections, but they don’t map to another inertial system, as only one orientation actually exists. Thus we are often more concerned with the proper orthochronous Lorentz group, which is the component containing the identity. It is identical to $\text{SO}(1,3)^0$.

While these are not all the symmetries we want to concern ourselves with, it will be useful to consider the restricted Lorentz group in a little more detail. The group $\text{SO}(1,3)^0$ is a real $4 \times 4$-matrix group, but we can also see the Lorentz transformations as complex $2 \times 2$-matrices. To do this we first note that any point $x = (x^0, x^1, x^2, x^3)$ in Minkowski spacetime can be viewed as a Hermitian matrix by

$$x \mapsto \mathbf{x} = x^0 \sigma_0 + \sum_{k=1}^{3} x^k \sigma_k$$

where $\sigma_0 = I_2$ the identity matrix and $\sigma_k$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

(5.3)

This is a linear, one-to-one map between Minkowski space and the real linear vector space $\text{span}\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$. Furthermore, we notice that

$$\det(x) = \det \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} = (x^0 + x^3)(x^0 - x^3) - (x^1 + ix^2)(x^1 - ix^2) = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = x^2.$$  

This allows us to write the bilinear form associated with the quadratic form in equation (5.1) in terms of the determinant of matrices:

$$(x + y, x + y) = (x, x) + (y, y) + 2(x, y)$$

(5.4)

hence

$$(x, y) = \frac{1}{2}(\det(x + y) - \det(x) - \det(y)).$$

(5.5)

Now we define for any $a \in \text{SL}(2, \mathbb{C})$ the map $\Lambda(a) : \text{span}\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\} \to \text{span}\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ by

$$\Lambda(a)(x) = a \mathbf{x} a^\dagger.$$  

(5.6)

By the identification given in equation (5.2) we can see this as a linear mapping.
of the Minkowski space onto itself, also denoted by $\Lambda(a)$. Since we have
\[(\Lambda(a)x)^2 = \det(\Lambda(a)x)\]
\[= \det(axa^\dagger)\]
\[= \det(a)\det(x)\det(a^\dagger)\]
\[= \det(x)\]
\[= x^2,\]
the linear mapping is actually a Lorentz transformation. This is true for any $a$, so we have a map $\Lambda : \text{SL}(2, \mathbb{C}) \to \text{O}(1, 3)$. In fact, $\Lambda$ is a continuous group homomorphism. Since $\text{SL}(2, \mathbb{C})$ is connected its image under a continuous group homomorphism is a connected subgroup. This means $\Lambda(\text{SL}(2, \mathbb{C}))$ lies within one of the connected components of the Lorentz group, specifically the identity component, since $\Lambda(I_2) = I_4$.

We might wonder if $\Lambda : \text{SL}(2, \mathbb{C}) \to \text{SO}(1, 3)^0$ is bijective. Writing everything out in matrices reduces this question to some explicit calculations. It turns out there is a preimage for every element of $\text{SO}(1, 3)^0$, but as shown in appendix B is only uniquely determined up to sign. This means $\Lambda$ is not injective. If we divide this out we are, finally, left with an isomorphism
\[\text{SO}(1, 3)^0 \simeq \text{SL}(2, \mathbb{C})/\{\pm\}\] (5.7)

Besides the Lorentz transformations there is one more group of symmetries that should keep the laws of physics invariant, namely the spacetime translations $\mathbb{R}^4$.

**Definition 5.1.** The total symmetry group is called the *Poincaré group*, and it is given by the semidirect product
\[P = \mathbb{R}^4 \rtimes \text{SO}(1, 3)^0.\] (5.8)
Where we have used the natural action of $\text{SO}(1, 3)^0$ on $\mathbb{R}^4$ to define the semidirect product.

### 5.1 Representations

Although the Poincaré group was just defined as $\mathbb{R}^4 \rtimes \text{SO}(1, 3)^0$, we shall now try to find the representations of $\mathbb{R}^4 \rtimes \text{SL}(2, \mathbb{C})$. This is because we are ultimately interested in the Lie algebra representations and it turns out that just looking at $P = \mathbb{R}^4 \rtimes \text{SO}(1, 3)^0$ is not enough, i.e. there exist Poincaré algebra representations that are not derived from a $\mathbb{R}^4 \rtimes \text{SO}(1, 3)^0$-representation. If we use $\mathbb{R}^4 \rtimes \text{SL}(2, \mathbb{C})$ this will not be a problem.

To find the representations it is convenient to use the semidirect product structure.
**Definition 5.2.** Let $G$ be a locally compact, second countable group, with a closed subgroup $G_0$. Suppose $(\sigma, V)$ is a unitary representation of $G_0$. We define the *induced representation* as the space

$$\text{Ind}^G_{G_0} V = \{ s : G \to V \mid s(hk) = \sigma(k^{-1})s(h) \quad \forall h \in G, k \in G_0 \text{ and } s \in L^2(G/G_0) \}$$

(5.9)

together with the action $\rho : G \to \text{GL}(\text{Ind}^G_{G_0} V)$ given for all $s \in \text{Ind}^G_{G_0} V$ and $g, h \in G$ by

$$\left(\rho(g)(s)\right)(h) = s(g^{-1}h).$$

(5.10)

This is indeed a unitary representation for the whole group $G$ built from a unitary representation of the subgroup $G_0$, see for example [12, Chap. 1] for more informations.

We can now define the representations of a semidirect product. Let $G = A \rtimes H$ be a semidirect product of $A$ abelian and $H$ acting on $A$ through automorphisms. Then we have an action of $H$ on the character group $G$ as well. Suppose $O$ is an orbit of $H$ in $A$ and fix $\chi \in O$. Let $H_\chi$ be the stabilizer of $\chi$ in $H$ and suppose $(\sigma, V)$ is a unitary representation of $H_\chi$. Then we have the induced representation $(\rho, \text{Ind}^H_{H_\chi} V)$ of $H$ as defined before. Next we define for all $a \in A$ a unitary operator $U(a)$ on $\text{Ind}^H_{H_\chi} V$ by

$$\left(U(a)(s)\right)(h) = (h \cdot \chi)(a)s(h)$$

(5.11)

with $s \in \text{Ind}^H_{H_\chi} V$, $h \in H$ and $\chi$ the fixed element in $O$. This gives an action of $A$ on the induced representation space. We can combine this with $\rho$ to define for all $a \in A$ and $h \in H$ the map $L_{O,\sigma} G \to \text{Ind}^H_{H_\chi} V$ by

$$L_{O,\sigma}(a, h) = U(a)\rho(h).$$

(5.12)

See [12, Chap. 1] for more on this construction and the following theorem:

**Theorem 5.3.** The pair $(L_{O,\sigma}, \text{Ind}^H_{H_\chi} V)$ as defined above is a unitary irreducible representation of $G$. Furthermore, if all the orbits are locally closed, every irreducible unitary representation is of this form. The representation is independent of the choice of $\chi \in O$, and $L_{O,\sigma} \simeq L_{O',\sigma'} \iff O = O'$ and $\sigma \simeq \sigma'$.

Using this theorem we can classify the unitary representations of the Poincaré group. First we set $A = \mathbb{R}^4$ and $H = \text{SL}(2, \mathbb{C})$. We can identify $A$ with another copy of $\mathbb{R}^4$, often written as $\mathbb{P}^4$ and called the *momentum space*, by the map $p = (p_\mu) \mapsto \chi_p$ with $\chi_p(x) = e^{i(x \cdot p)}$ for all $x = (x_\mu) \in \mathbb{R}^4$. This bracket is defined by

$$(x, p) = x_0p_0 - x_1p_1 - x_2p_2 - x_3p_3.$$ 

(5.13)

The action of $\text{SL}(2, \mathbb{C})$ on $\mathbb{P}^4$ is the natural action of $\text{SO}(1, 3)^0$ on $\mathbb{P}^4$ via the homomorphism $\Lambda$ as in equation (5.6). The quadratic form $p_0^2 - p_1^2 - p_2^2 - p_3^2$ is an invariant. The level sets of this form are invariant under $\text{SL}(2, \mathbb{C})$ and fill up
The level sets contain two orbits: we need to split them depending on the sign of \( p_0 \). It is convenient to separate the representations by the different orbit types.

- **\( X_\pm^m \):** Let \( m > 0 \). The sets \( X_\pm^m \) are defined by

\[
X_\pm^m = \{ p \in \mathbb{P}^4 \mid p_0^2 - p_1^2 - p_2^2 - p_3^2 = m^2, p_0 \gtrless 0 \} \tag{5.14}
\]

where the + refers to the sets with \( p_0 > 0 \) and – to the sets with \( p_0 < 0 \).

We need to pick an orbit and fix an element to find a representation; we choose \((m, 0, 0, 0) \in X_+^m\). This point is the rest frame of this orbit, because the physical interpretation of \((p_1, p_2, p_3)\) is the momentum, which is zero when a massive particle is at rest. The \( m \) can be interpreted as the mass. The stabilizer is

\[
H_\chi = \{ g \in \text{SL}(2, \mathbb{C}) \mid g \cdot (m, 0, 0, 0) = (m, 0, 0, 0) \}. \tag{5.15}
\]

We identify \((m, 0, 0, 0)\) with its corresponding Hermitian matrix as in equation (5.2) and let \( g \) work through \( \Lambda \), i.e.

\[
g \cdot (m, 0, 0, 0) = g \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} g^\dagger = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} = (m, 0, 0, 0). \tag{5.16}
\]

This amounts to \( gg^\dagger = I_2 \), so \( H_\chi \simeq \text{SU}(2) \). The representations of \( \text{SU}(2) \) are well known, and they are labeled by \( j \in \frac{1}{2} \mathbb{Z} \). The \( j \) gives the spin of the particle. So we have found a representation \( L_{m,j}^+ \), which corresponds to a particle of mass \( m \) and spin \( j \). For example \( L_{m,\frac{1}{2}}^+ \) describes the electron if \( m \) takes on the right value.

Note that finding the representations of \( \mathbb{R}^4 \rtimes \text{SO}(1, 3)^0 \) rather than \( \mathbb{R}^4 \rtimes \text{SL}(2, \mathbb{C}) \) would require \( \text{SO}(3) \) rather than \( \text{SU}(2) \). However, \( \text{SO}(3) \) only has odd-dimensional representations, so we would not have found any for the even (i.e. \( j \) half-integer) representations. For more on the connection between \( \text{SO}(3) \) and \( \text{SU}(2) \) representations see appendix A.

We can do the same for \((-m, 0, 0, 0) \in X_-^m\). The stabilizer group is again \( \text{SU}(2) \) and its representations are labeled by \( j \in \frac{1}{2} \mathbb{Z} \), so we find a representation \( L_{m,j}^- \). However, the time reflection is an isomorphism between \( X_+^m \) and \( X_-^m \) and this gives an isomorphism of \( L_{m,j}^+ \) with \( L_{m,j}^- \). We could say that \( L_{m,j}^- \) corresponds to the same kind of particle travelling back in time, but the more common explanation is the antiparticle, with the same mass and spin but opposite charge.

- **\( X_0^\pm \):** These orbits are defined by

\[
X_0^\pm = \{ p \in \mathbb{P}^4 \mid p_0^2 - p_1^2 - p_2^2 - p_3^2 = 0, p_0 \gtrless 0 \} \tag{5.17}
\]
The points in these orbits represent particles which move with the speed of light (and are therefore massless), classically that’s only photons.

This time we pick \((1, 0, 0, 1) \in X_0^+\). As we have seen before it is possible to see Minkowski space and the Lorentz transformations as \(2 \times 2\)-matrices. Our fixed point becomes

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.
\] (5.18)

To find the stabilizer we look at the action of a general \(2 \times 2\)-matrix

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger = \begin{pmatrix} 2a\bar{a} & 2a\bar{c} \\ 2\bar{a}c & 2\bar{c}\bar{c} \end{pmatrix},
\] (5.19)

so the stabilizer in \(SL(2, \mathbb{C})\) is the group

\[
H_\chi = \left\{ e^{\alpha, \beta} = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} | a, b \in \mathbb{C}, |a| = 1 \right\}.
\] (5.20)

This itself is a semidirect product \(\mathbb{C} \rtimes \mathbb{T}\) of \(\{ e^{1, b} | b \in \mathbb{C} \} \simeq \mathbb{C}\) and \(\{ e^{a, 0} | a \in \mathbb{C}, |a| = 1 \}\) which is isomorphic to the circle group \(\mathbb{T}\), with the action defined by \((a, b) \mapsto a^2b\). The only finite-dimensional unitary representations of this group are (according to [12, Chap. 1]) given by

\[
\sigma_n : \left( a \begin{pmatrix} b \\ a^{-1} \end{pmatrix} \mapsto a^n \right)
\] (5.21)

with \(n \in \mathbb{Z}\). Thus we find representations \(L_{0,n}^+\) for this orbit. Again the ‘\(-\)’ orbits give the same representations but reflected in time.

- \(Y_m\): Let \(m > 0\).

\[
Y_m = \{ p \in \mathbb{P}^4 | p_0^2 - p_1^2 - p_2^2 - p_3^2 = -m^2 \}
\] (5.22)

These orbits would lead to particles with imaginary mass, so they are unphysical.

- \(\{0\}\): The orbit is only the origin of \(\mathbb{P}^4\). The trivial one-dimensional representation has this orbit, and it corresponds to vacuum. All other representations are also unphysical, as they turn out to have an infinite dimensional internal space.[12, Chap. 1]

### 5.2 The algebra

Sometimes it is easier to work with algebras than with groups. The algebra that comes from the Poincaré group is, unsurprisingly, called the Poincaré algebra. It can be given by

\[
p = t \oplus so(1, 3).
\] (5.23)
Here \( t \simeq \mathbb{R}^4 \) is the abelian Lie algebra of spacetime translations and \( \mathfrak{so}(1,3) \) the Lorentz Lie algebra, i.e. the Lie algebra of \( \text{SO}(1,3) \). The sum is actually semidirect, making \( t \) an abelian ideal of \( \mathfrak{g}_0 \).

We would like to find the generators of the Poincaré algebra. We can find the elements of \( \mathfrak{so}(1,3) \) by differentiating generators of \( \text{SO}(1,3) \). The standard choice of basis comes from physical applications and it consists of three rotations and three boosts.

We start with the rotations. A rotation around the \( x_1 \)-axis can be written as a \( 4 \times 4 \)-matrix on Minkowski space. This gives us the following algebra element:

\[
J_1 = J_{x_1} = \frac{d}{dt} \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & \cos(t) & -\sin(t) & 0 \\
0 & \sin(t) & \cos(t) & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\bigg|_{t=0} = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\] (5.24)

By considering the rotations around the \( x_2 \) - and \( x_3 \)-axis we find two more generators:

\[
J_2 = J_{x_2} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}, \quad J_3 = J_{x_3} = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\] (5.25)

The boosts transform an inertial system to one that moves at a constant speed with regard to the original system. Just like the rotations, these can be written as matrices. The boost in the \( x_1 \)-direction gives us

\[
K_1 = K_{x_1} = \frac{d}{dt} \begin{pmatrix}
\cosh(t) & \sinh(t) & 0 & 0 \\
\sinh(t) & \cosh(t) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\bigg|_{t=0} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\] (5.26)

And again, there are two additional generators for the \( x_2 \) - and \( x_3 \)-directions.

\[
K_2 = K_{x_2} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad K_3 = K_{x_3} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\] (5.27)

We would like to do this for the translation part as well, so it would be nice to write those elements as matrices too. We can write the transformation \( x \mapsto Ax + a \), which is an elements of the Poincaré group, as a \( 5 \times 5 \) matrix

\[
\begin{pmatrix}
A & a \\
0 & 1
\end{pmatrix}
\]
Multiplying the matrices corresponding to two transformations \((A, a)\) and \((B, b)\) gives the correct result as well:

\[
\begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} AB & a + Ab \\ 0 & 1 \end{pmatrix}
\]  
(5.28)

Now consider the translation in time. This gives

\[
P_0 = \frac{d}{dt} \begin{pmatrix} 1 & 1 & t \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \bigg|_{t=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]  
(5.29)

And similarly

\[
P_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, P_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\]  
(5.30)

The commutation relations are now easy to compute using matrix multiplication. (Of course when multiplying a translation with a rotation of boost generator, the matrix of the latter need some extra zeroes at the right and bottom) Using the Levi-Civita symbol for three indices, i.e.

\[
\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of } 123 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 123 \\ 0 & \text{otherwise} \end{cases}
\]  
(5.31)

and the Minkowski metric \(\eta = \text{diag}(1, -1, -1, -1)\) we can write the bracket in a concise manner.

\[
\begin{align*}
[J_i, J_j] &= \epsilon_{ijk} J_k \\
[J_i, K_j] &= \epsilon_{ijk} K_k \\
[K_i, K_j] &= -\epsilon_{ijk} J_k \\
[J_i, P_0] &= 0 \\
[J_i, P_j] &= \epsilon_{ijk} P_k \\
[K_i, P_0] &= -P_i \\
[K_i, P_j] &= \eta_{ij} P_0 \\
[P_i, P_j] &= 0
\end{align*}
\]  
(5.32–5.39)

The center of the universal enveloping algebra of the Poincaré algebra is 2-dimensional, i.e. we can find two independent Casimirs. The first one we have
already used implicitly. The invariant \( p_0^2 - p_1^2 - p_2^2 - p_3^2 \) by which we separated the orbits with different types of representations corresponds to the element

\[
C_1 = P^2 = \eta_{\mu\nu} P_\mu P_\nu
\]  

(5.40)

with summation over repeated indices and \( \eta = \text{diag}(1, -1, -1, -1) \).

The second Casimir is less simple. As a stepping stone we define another useful operator.

**Definition 5.4.** The Pauli-Lubinski pseudo vector is defined by

\[
W_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} M_{\nu\rho} P_\sigma
\]  

(5.41)

where again we sum over repeated indices. The \( \epsilon \) is the Levi-Civita symbol with four indices and takes value +1 for an even permutation of (0123) and value −1 for an odd permutation. The tensor \( M \) is called the relativistic angular momentum tensor, and its entries depend on the generators of rotation and boost. Specifically \( M_{\mu\mu} = 0 \), \( M_{\mu0} = K_j = -M_{0j} \) and \( M_{ij} = \epsilon_{ijk} J_k = -M_{ji} \) with \( \mu = 0, 1, 2, 3 \) and \( i, j, k = 1, 2, 3 \).

To give an example we will write out the 0-component

\[
W_0 = \frac{1}{2} \epsilon_{0\nu\rho\sigma} M_{\nu\rho} P_\sigma
\]

\[
= \frac{1}{2}(\epsilon_{0123} M_{12} P_3 + \epsilon_{0132} M_{13} P_2 + \epsilon_{0213} M_{21} P_3 \\
+ \epsilon_{0231} M_{23} P_1 + \epsilon_{0312} M_{31} P_2 + \epsilon_{0321} M_{32} P_1)
\]

\[
= \frac{1}{2}(M_{12} P_3 - M_{13} P_2 - M_{21} P_3 + M_{23} P_1 + M_{31} P_2 - M_{32} P_1)
\]

\[
= \frac{1}{2}(J_3 P_3 + J_2 P_2 + J_3 P_3 + J_1 P_1 + J_2 P_2 + J_1 P_1)
\]

\[
= J_1 P_1 + J_2 P_2 + J_3 P_3
\]

The second Casimir is given by [11, chap. 10]

\[
C_2 = W^2 = \eta_{\mu\nu} W_\mu W_\nu.
\]  

(5.42)

Written in terms of the generators as defined in this section gives:

\[
C_2 = (J_1 P_1 + J_2 P_2 + J_3 P_3)^2 - (-J_1 P_0 - K_2 P_3 + K_3 P_2)^2 \\
- (K_1 P_3 - J_2 P_0 - K_3 P_1)^2 - (-K_1 P_2 + K_2 P_1 + J_3 P_0)^2
\]  

(5.43)

To show it commutes with every element in the Poincaré algebra it is enough to show it commutes with all the generators. Using the previously given relations together with \([AB, C] = A[B, C] + [A, C]B\) this is reduced to writing out the equation.
Finally, we note that Lie group representations correspond to Lie algebra representations by taking the derivative. This would not have given all representations of the Poincaré algebra had we considered the group $\mathbb{R}^4 \rtimes \text{SO}(1,3)^0$, because it is not simply connected. Its universal cover $\mathbb{R}^4 \rtimes \text{SL}(2,\mathbb{C})$ has the same Lie algebra but is simply connected, so the representations $dL_{O,\sigma}$ give all irreducible unitary Lie algebra representations.

6 The Poincaré superalgebra

Minkowski spacetime can be extended to a superspacetime and its symmetry group to a supersymmetry group. Rather than the supergroup we will consider the corresponding Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

6.1 The definition

For any superspacetime it comes with at least the following assumptions.[12, Chap. 2]

- $\mathfrak{g}$ is a real Lie superalgebra.
- $\mathfrak{g}_1$ is a spinorial $\mathfrak{g}_0$-module.
- $\langle \mathfrak{g}_1, \mathfrak{g}_1 \rangle$ should contain the translation subspace of $\mathfrak{g}_0$.

The term spinorial will be explained shortly, for now we just state that it implies that $\mathfrak{g}_0$, or some quotient of it, is an orthogonal Lie algebra. Furthermore, the third assumption can only hold if the translations are a subspace of $\mathfrak{g}_0$. This places restrictions on our choice for $\mathfrak{g}_0$. Fortunately the obvious choice, the symmetry algebra of the Minkowski spacetime, satisfies these demands. From now on we take $\mathfrak{g}_0 = \mathfrak{t} \oplus \mathfrak{so}(1,3)$.

It will be extended to a Poincaré superalgebra by a spinorial $\mathfrak{g}_0$-module $\mathfrak{g}_1$. The ‘spinorial’ means that the complexification of $\mathfrak{g}_1$ is a direct sum of spin modules, which are equivalent to certain representations of the special orthogonal group called spin representations. This leaves too many possible extensions, so we make one further assumption: there exists no Lie subsuperalgebra that strictly includes $\mathfrak{g}_0$ and is strictly included in $\mathfrak{g}$. Basically this means that $\mathfrak{g}$ is a minimal Poincaré superalgebra.

First we note that if $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra and $\mathfrak{h} \subset \mathfrak{g}_1$ is a $\mathfrak{g}_0$-submodule, then by lemma (3.7) $\mathfrak{g}_0 \oplus \mathfrak{h}$ is a Lie subsuperalgebra. By minimality of $\mathfrak{g}$ the only possible submodules of $\mathfrak{g}_1$ are $\{0\}$ and $\mathfrak{g}_1$, so $\mathfrak{g}_1$ must be an irreducible $\mathfrak{g}_0$-module. Of course this is only as a real module, since $\mathfrak{g}$ is a real Lie superalgebra. If we extend the scalars to $\mathbb{C}$, the module $\mathfrak{g}_1$ might not be irreducible anymore.

Lemma 6.1. If $\mathfrak{g}$ is minimal the action of $\mathfrak{t}$ on $\mathfrak{g}_1$ is 0.
Proof. Since $\mathfrak{t}$ is abelian, it is certainly solvable. Using Lie’s theorem we can find an element of $(\mathfrak{g}_1)_{\mathbb{C}}$, the complexification of $\mathfrak{g}_1$, that is an eigenvector for every element of $\mathfrak{t}$. As such there exists a linear map $\lambda : \mathfrak{t} \to \mathbb{C}$ such that

$$V_\lambda = \{ v \in (\mathfrak{g}_1)_{\mathbb{C}} \mid \langle X, v \rangle = \lambda(X)v, \text{ for all } X \in \mathfrak{t} \}$$

(6.1)

is non-zero. The superbracket is extended linearly to the complex elements.

We want to find an action of the Lorentz group on $\mathfrak{t}$ and $(\mathfrak{g}_1)_{\mathbb{C}}$. Since $\mathfrak{t}$ is an ideal in $\mathfrak{g}_0$, we can see $\mathfrak{t}$ as an $\mathfrak{g}_0$-module. By restriction it is also an $\mathfrak{so}(1,3)$-module. Of course the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ is also an $\mathfrak{so}(1,3)$-module and we can extend this module action linearly to $(\mathfrak{g}_1)_{\mathbb{C}}$.

For a simply connected Lie group with this as Lie algebra we could integrate these actions to find a Lie group action, but the Lorentz algebra is not connected. We do know that the exponential map $\exp : \mathfrak{so}(1,3) \to SO(1,3)$ is one-to-one on a neighbourhood around zero, and fortunately that is all we need. This gives us an action for $h \in U \subset SO(1,3)$ on $\mathfrak{t}$ and $(\mathfrak{g}_1)_{\mathbb{C}}$, with $U$ some sufficiently small open subset containing the identity. In both cases we denote the action by a superscript, i.e. $X \mapsto X^h$. For $X \in \mathfrak{t}$ we have

$$\text{ad}(X^h) = h\text{ad}(X)h^{-1},$$

(6.2)

so for an element $v \in V_\lambda$ and $X \in \mathfrak{t}$ we have

$$\langle X, v^h \rangle = \langle (X, v^h)^{h^{-1}}, h \rangle$$

$$= hh^{-1}\text{ad}(X)h(v)$$

$$= \text{had}(X^{h^{-1}})(v)$$

$$= \langle (X^{h^{-1}}, v)^h, h \rangle$$

$$= \langle (\lambda(X^{h^{-1}}), v)^h, h \rangle$$

$$= \lambda(X^{h^{-1}})v^h.$$  

Thus $h$ maps $V_\lambda$ to $V_{\lambda^h}$, where $\lambda^h(X) = \lambda(X^{h^{-1}})$. We know $\mathfrak{g}_1$ is finite dimensional, so its complexification must be as well. This means there can only be finitely many $\mu$ for which $V_\mu$ is non-empty. Suppose $\lambda \neq 0$. If $V_\lambda$ is non-trivial (i.e. contains a non-zero element $v$) then $V_{\lambda^h}$ with $h \in U$ is non-trivial as well (since $v^h \in V_{\lambda^h}$). The action of the Lorentz group gives us infinitely many non-trivial $V_{\lambda^h}$, because $SO(1,3)$ is a continuous group. That means there are infinitely many linearly independent $v^h \in (\mathfrak{g}_1)_{\mathbb{C}}$, which is a contradiction. Thus $\lambda = 0$.

Now let $v \in V_0$ and $Y \in \mathfrak{so}(1,3)$. For any $X \in \mathfrak{t}$ we can use the graded Jacobi identity to find

$$\langle X, (Y, v) \rangle = -\langle Y, (v, X) \rangle - \langle v, (X, Y) \rangle = 0 + 0 = 0$$

(6.3)

where in the final step we use that $\langle X, Y \rangle \in \mathfrak{t}$, because $\mathfrak{t}$ is an ideal. This means $\langle Y, v \rangle \in V_0$. Hence $V_0$ is stable under $\mathfrak{so}(1,3)$, and since the action of $\mathfrak{t}$ is just
zero, it is in fact stable under \( g_0 \). This means \( g_1 = g_1 \cap V_0 \) is a \( g_0 \)-submodule of \( g_1 \), hence \( g_0 \oplus g_{1,0} \) is a Lie subsuperalgebra. By minimality it is then all of \( g \), so the action of \( t \) on \( g_1 = g_{1,0} \) is 0.

We want to find a module \( g_1 \) such that \( g \) is minimal. Since the action of \( t \) is zero, we can see \( g_1 \) as just an \( so(1,3) \)-module rather than a \( g_0 \)-module. To this end we first look at the complexification \( so(1,3)_\mathbb{C} \). Through a string of identifications, which will be shown later, this is actually equivalent to a representation of \( SL(2, \mathbb{C}) \) seen as a real Lie group. These are known; the irreducible representation are \( k \otimes \overline{m} \). Here \( k \) is the irreducible holomorphic representation of dimension \( k \) and \( m \) is the irreducible anti-holomorphic representation of dimension \( m \).

We also need \( g_1 \) to be a spinorial module. For \( SL(2, \mathbb{C}) \) these are \( 2 \otimes \overline{1} \) and \( 1 \otimes \overline{2} \) (see [12, Chap. 5]), by abuse of notation denoted by \( 2 \) and \( \overline{2} \) respectively. However, \( 2 \) and \( \overline{2} \) are complex, while we need a real module.

Next we note that \( 2 \oplus \overline{2} \) has a real form. If we write its representation space as \( \mathbb{C}^2 \oplus \mathbb{C}^2 \), the action of \( g \in SL(2, \mathbb{C}) \) can be written as

\[
g \cdot (u, v) = (g \cdot u, \overline{g} \cdot v), \quad u, v \in \mathbb{C}^2
\]

This commutes with the conjugation \( \sigma \) given by

\[
\sigma : (u, v) \mapsto (\overline{v}, \overline{u})
\]

We finally define a real spinorial module as

\[
m = (2 \oplus \overline{2})^\sigma.
\]

This \( m \) is irreducible under \( g_0 \), so by minimality we have \( g_1 = m \). Since applying complex conjugation twice amounts to doing nothing, we find \( m = \{ (u, \overline{u}) \mid u \in \mathbb{C}^2 \} \). We define a basis

\[
\{ u_1 = (e_1, e_1), u_2 = (ie_1, -ie_1), u_3 = (e_2, e_2), u_4 = (ie_2, -ie_2) \}
\]

where \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \).

We don’t have a Lie superalgebra structure on \( g_0 \oplus m \) just yet. We use the Lie bracket on the Poincaré algebra \( g_0 \) for two even elements and the superbracket of an even and odd element is given by the module action. As theorem 3.5 states, all that’s left is an appropriate superbracket \( \langle \ , \ \rangle : m \otimes m \to g_0 \) for two odd elements.

**Lemma 6.2.** There exists a projectively unique symmetric \( so(1,3) \)-map

\[
L : (2 \oplus \overline{2}) \otimes (2 \oplus \overline{2}) \to t_\mathbb{C}
\]

**Proof.** First of all, we define the isomorphism

\[
F : (2 \oplus \overline{2}) \otimes (2 \oplus \overline{2}) \to (2 \oplus \overline{2}) \otimes (2 \oplus \overline{2}), \quad u \otimes v \mapsto v \otimes u.
\]

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Projecting a subspace of \((2 \oplus \overline{2})\) that is symmetric under \(F\) onto \(t_C\) and sending the rest to zero gives an \(L\) as desired. We use the distributivity of the tensor product over the direct sum to find
\[
(2 \oplus \overline{2}) \otimes (2 \oplus \overline{2}) = (2 \otimes 2) \oplus (\overline{2} \otimes \overline{2}) \oplus (2 \otimes \overline{2}) \oplus (\overline{2} \otimes 2).
\]
(6.10)

We consider parts of this expression separately to find a symmetric subspace.

We know that \(2 \oplus \overline{2}\) is an \(\text{SL}(2, \mathbb{C})\) module. As noted before, \(\text{SO}(1, 3)^0 \simeq \text{SL}(2, \mathbb{C})/\{\pm\}\), so if it weren’t for the signs we could see any \(\text{SL}(2, \mathbb{C})\)-module as a \(\text{SO}(1, 3)^0\)-module. In this case, \(-1\) acts as \(-1\) on both factors of \(2 \oplus \overline{2}\), so on the whole it acts as 1. Thus \(2 \oplus \overline{2}\) descends to an \(\text{SO}(1, 3)^0\)-representation. It is an irreducible 4-dimensional representation, and from the representation theory of \(\text{SL}(2, \mathbb{C})\) we know that there is only one irreducible 4-dimensional representation.

We write \(4_v\) for this vector representation, and note that \(4_v \simeq t_C\). Using \(F\) we see
\[
2 \otimes \overline{2} \simeq \overline{2} \otimes 2 \simeq 4_v,
\]
(6.11)
so \((2 \otimes \overline{2}) \oplus (\overline{2} \otimes 2) \simeq 4_v \oplus 4_v\). This space is stable under \(F\) so it splits into a symmetric and skew-symmetric subspace with respect to \(F\), both are also submodules. The symmetric elements are of the form \((x,x)\), while the skew-symmetric ones look like \((x,-x)\). Hence we find \(((2 \otimes \overline{2}) \oplus (\overline{2} \otimes 2))^{\text{symmetric}} \simeq 4_v\).

Finally we know that \(2 \otimes 2 = 1 \oplus 3\) with \(3\) being the symmetric part, and \(\overline{2} \otimes \overline{2} = 1 \oplus 3\) with \(3\) the symmetric part.

All together this gives
\[
((2 \oplus \overline{2}) \otimes (2 \oplus \overline{2}))^{\text{symmetric}} \simeq 3 \oplus \overline{3} \oplus 4_v
\]
(6.12)
so by dividing out all the spaces we don’t need we find a map \(L\) to \(4_v \simeq t_C\) that is symmetric. Since there is only one copy of \(4_v\) in equation (6.12), it is also projectively unique. \(\square\)

Now we choose an \(L\) and set
\[
\langle a, b \rangle_C = L(a \otimes b)
\]
(6.13)
Then the extended bracket is symmetric and intertwines with the module action of the even part. Since it lands in \(t_C\), which acts trivially on the odd part by minimality, we also have
\[
\langle a, \langle a, a \rangle_C \rangle_C = 0
\]
(6.14)
By theorem 3.5 we have constructed a Lie superalgebra \((\mathfrak{g}_0)_C \oplus (2 \oplus \overline{2})\).

However, the goal was a real Lie superalgebra. We can find that by finding an \(L\) that is fixed by an appropriate conjugation and applying the same theorem. Let \(\phi\) be the conjugation such that \((t_C)^\phi \simeq t\). Then choosing \(L\) such that
\[
\phi \circ L \circ (\sigma \otimes \sigma) = L
\]
(6.15)
gives a superbracket \(\langle , \rangle = L|_{\mathfrak{m} \otimes \mathfrak{m}}\) which maps into \(t\). This extension of the Lie algebra bracket gives a real Lie superalgebra structure on \(\mathfrak{g}_0 \oplus \mathfrak{m}\).
6.2 The superbracket

We now know that there exists a Poincaré superalgebra, but the previous definition is often not very easy to work with. It is more useful to have an explicit superbracket. In fact, just like the case for Lie algebras, the Poincaré superalgebra can also be defined as a set of generators and defining relations. We already know the bracket on the even part, i.e. the Poincaré algebra, but we still need to give the module action of \( \mathfrak{so}(1,3) \) on \( m \) and the bracket on the odd part \( m \) explicitly.

We consider the action of the \( \mathfrak{so}(1,3) \)-module \( m \) first. As an SL(2, C)-module the action is easy; for \( g \in \text{SL}(2, \mathbb{C}) \) and \( (u, \bar{u}) \in m \) the action is given by \( g \cdot (u, \bar{u}) = (gu, \bar{g} \bar{u}) \), where \( gu \) and \( \bar{g} \bar{u} \) are given by simply applying the 2×2-matrix \( g \). From this we can find action of our chosen standard basis for \( \mathfrak{so}(1,3) \) by using several identification. Our first goal is to find explicitly \( \mathfrak{so}(1,3) \rightarrow \mathfrak{so}(1,3)_C \simeq \mathfrak{su}(2)_C \oplus \mathfrak{su}(2)_C \simeq (\mathfrak{sl}(2, \mathbb{C})_R)_C \) \hspace{1cm} (6.16)

As stated before, our choice of basis for \( \mathfrak{so}(1,3) \) is \( \{J_j, K_k | j, k = 1, 2, 3\} \). In the complexification \( \mathfrak{so}(1,3)_C \) we have the basis \( \{A_j, B_k | j, k = 1, 2, 3\} \), defined by

\[
A_j = \frac{1}{2}(J_j + iK_j) \hspace{1cm} (6.17)
\]
\[
B_k = \frac{1}{2}(J_k - iK_k). \hspace{1cm} (6.18)
\]

Using the Lie bracket on \( \mathfrak{so}(1,3) \) and the properties of the Levi-Civita symbol, it is not hard to find

\[
[A_j, A_k] = \frac{1}{4}[J_j + iK_j, J_k + iK_k]
= \frac{1}{4}([J_j, J_k] + i[J_j, K_k] + i[K_j, J_k] - [K_j, K_k])
= \frac{1}{4}(\epsilon_{jkl}J_l + i\epsilon_{jkl}K_l - i\epsilon_{kjl}K_l + \epsilon_{jkl}J_l)
= \frac{1}{2}\epsilon_{jkl}(J_l + iK_l)
= \epsilon_{jkl}A_l.
\]

And similarly \( [B_j, B_k] = \epsilon_{jkl}B_l \). These are equivalent to the Lie bracket for \( \mathfrak{su}(2) \). Furthermore, it can be shown that \( [A_j, B_k] = 0 \) for all \( j, k = 1, 2, 3 \). This gives \( \mathfrak{so}(1,3)_C \simeq \mathfrak{su}(2)_C \oplus \mathfrak{su}(2)_C \).

The next identification is slightly trickier. The space \( \mathfrak{sl}(2, \mathbb{C})_R \) already has complex-valued matrices, but we look at it as a real Lie algebra. We complexify it to find \( (\mathfrak{sl}(2, \mathbb{C})_R)_C \). This means we introduce a second \( i \), which should be kept away very carefully from the first \( i \).
We start with
\[
\mathfrak{sl}(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in \mathbb{C} \right\}.
\] (6.19)

This is typically a complex Lie algebra. We restrict the scalars to \( \mathbb{R} \) to get the real Lie algebra \( \mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}} \). Any complex \( a \in \mathbb{C} \) can be written as \( a = R(a) + iI(a) \), with \( R(a) \) the real part of \( a \) and \( I(a) \) the imaginary part of \( a \). Thus
\[
\mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}} = \left\{ \begin{pmatrix} a_R + ia_I & b_R + ib_I \\ c_R + ic_I & -a_R - ia_I \end{pmatrix} \mid a_R, a_I, b_R, b_I, c_R, c_I \in \mathbb{R} \right\} \quad (6.20)
\]

This suggests a basis for \( \mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}} \) given by
\[
R_{11} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad I_{11} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
\]
\[
R_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad I_{12} = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}
\]
\[
R_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad I_{21} = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}
\]

So we have
\[
\mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}} = \mathbb{R}R_{11} \oplus \mathbb{R}R_{12} \oplus \mathbb{R}R_{21} \oplus \mathbb{R}I_{11} \oplus \mathbb{R}I_{12} \oplus \mathbb{R}I_{21} \quad (6.21)
\]

With the first \( i \) neatly hidden away in the basis elements, we can safely complexify. This gives
\[
(\mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}})_{\mathbb{C}} = \mathbb{C}R_{11} \oplus \mathbb{C}R_{12} \oplus \mathbb{C}R_{21} \oplus \mathbb{C}I_{11} \oplus \mathbb{C}I_{12} \oplus \mathbb{C}I_{21} \quad (6.22)
\]

Since we have explicit \( 2 \times 2 \)-matrices for the basis, the commutation relations can be calculated. It gets somewhat repetitive to check, but with these relations it can be shown that there is an isomorphism:
\[
A_1 \mapsto -\frac{i}{4}(R_{12} + iI_{12} + R_{21} + iI_{21})
\]
\[
A_2 \mapsto -\frac{1}{4}(R_{12} + iI_{12} - R_{21} - iI_{21})
\]
\[
A_3 \mapsto -\frac{i}{4}(R_{11} + iI_{11})
\]
\[
B_1 \mapsto -\frac{i}{4}(R_{12} - iI_{12} + R_{21} - iI_{21})
\]
\[
B_2 \mapsto -\frac{1}{4}(R_{12} - iI_{12} - R_{21} + iI_{21})
\]
\[
B_3 \mapsto -\frac{i}{4}(R_{11} - iI_{11})
\]

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Using equations (6.17) and (6.18) we then find the homomorphism that corresponds to equation (6.16):

\[
\begin{align*}
J_1 & \mapsto -\frac{i}{2}(R_{12} + R_{21}) \\
J_2 & \mapsto -\frac{1}{2}(R_{12} - R_{21}) \\
J_3 & \mapsto -\frac{i}{2}R_{11} \\
K_1 & \mapsto -\frac{i}{2}(I_{12} + I_{21}) \\
K_2 & \mapsto -\frac{1}{2}(I_{12} - I_{21}) \\
K_3 & \mapsto -\frac{i}{2}I_{11}
\end{align*}
\]

It is not an isomorphism, since \(\mathfrak{so}(1, 3)\) is real and \((\mathfrak{sl}(2, \mathbb{C})_\mathbb{R})_\mathbb{C}\) is complex, but we never needed that anyway. If we know how \(R_{mn}\) and \(I_{mn}\) act on \(2 \oplus 2\), then we can use this map to find action of \(\mathfrak{so}(1, 3)\).

The Lie algebra action comes from the corresponding Lie group action which has been given before, i.e. \(g \cdot (u, v) = (g \cdot u, \bar{g} \cdot v)\) for \(u, v \in \mathbb{C}^2\). To find the Lie algebra action on our basis as in equation (6.7), we need to know the group action with respect to this basis. Let \(g \in \text{SL}(2, \mathbb{C})\), and denote its components by \(g_{mn}\). We separate the components into a real and an imaginary part, i.e.

\[
g = \begin{pmatrix}
R(g_{11}) + iI(g_{11}) & R(g_{12}) + iI(g_{12}) \\
R(g_{21}) + iI(g_{21}) & R(g_{22}) + iI(g_{22})
\end{pmatrix}
\]

and the same for \(v \in \mathbb{C}^2:\)

\[
v = \begin{pmatrix}
v_1 \\ v_2
\end{pmatrix} = \begin{pmatrix}
R(v_1) + iI(v_1) \\
R(v_2) + iI(v_2)
\end{pmatrix}
\]

We find

\[
g \cdot v = \begin{pmatrix}
g_{11}v_1 + g_{12}v_2 \\ g_{21}v_1 + g_{22}v_2
\end{pmatrix}
= \begin{pmatrix}
R(g_{11})R(v_1) - I(g_{11})I(v_1) + R(g_{12})R(v_2) - I(g_{12})I(v_2) \\
+ i(R(g_{11})I(v_1) + I(g_{11})R(v_1) + R(g_{12})I(v_2) + I(g_{12})R(v_2)) \\
R(g_{21})R(v_1) - I(g_{21})I(v_1) + R(g_{22})R(v_2) - I(g_{22})I(v_2) \\
+ i(R(g_{21})I(v_1) + I(g_{21})R(v_1) + R(g_{22})I(v_2) + I(g_{22})R(v_2))
\end{pmatrix}
\]

We also need \(\bar{g}\) working on \(\bar{v}\), but fortunately we can use \(\bar{g} \cdot \bar{v} = \bar{g}v\), so there is no need to write that out separately.

Now suppose \(v = e_1\). This is a real vector, in fact \(R(v_1) = 1\) and all other parts are zero. That simplifies the previous equation quite a bit. We find

\[
g \cdot e_1 = \begin{pmatrix}
R(g_{11}) + iI(g_{11}) \\
R(g_{21}) + iI(g_{21})
\end{pmatrix} = R(g_{11})e_1 + I(g_{11}) \cdot i e_1 + R(g_{21})e_2 + I(g_{21}) \cdot i e_2
\]
We have a similar expression for $\bar{g} \cdot \bar{e}_1 = \bar{g} \cdot e_1$, so we conclude for the basis element $u_1$ as in equation (6.7) that

$$g \cdot u_1 = R(g_{11})u_1 + I(g_{11}) \cdot u_1 + R(g_{21})u_2 + I(g_{21}) \cdot u_2 \quad (6.26)$$

In the same way $g \cdot u_2, g \cdot u_3$ and $g \cdot u_4$ can be found. With respect to this basis we find

$$g = \begin{pmatrix} R(g_{11}) & -I(g_{11}) & R(g_{12}) & -I(g_{12}) \\ I(g_{11}) & R(g_{11}) & I(g_{12}) & R(g_{12}) \\ R(g_{21}) & -I(g_{21}) & R(g_{22}) & -I(g_{21}) \\ I(g_{21}) & R(g_{21}) & I(g_{22}) & R(g_{22}) \end{pmatrix} \quad (6.27)$$

Now we can use the exponential map to send algebra elements to group elements, apply the group action, and finally take the derivative again.

- $R_{11}$:
  
  $$\exp(tR_{11}) = \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix} \quad (6.28)$$

  So with respect to $\{u_1, u_2, u_3, u_4\}$ this is

  $$R_{11} = \frac{d}{dt} \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix} \bigg|_{t=0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (6.29)$$

- $I_{11}$:
  
  $$\exp(tI_{11}) = \begin{pmatrix} e^{it} & 0 & 0 & 0 \\ 0 & e^{-it} & 0 & 0 \\ 0 & 0 & e^t & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix} \quad (6.30)$$

  So with respect to $\{u_1, u_2, u_3, u_4\}$ this is

  $$I_{11} = \frac{d}{dt} \begin{pmatrix} e^{it} & 0 & 0 & 0 \\ 0 & e^{-it} & 0 & 0 \\ 0 & 0 & e^t & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix} \bigg|_{t=0} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (6.31)$$

- $R_{12}$:
  
  $$\exp(tR_{12}) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad (6.32)$$

  So with respect to $\{u_1, u_2, u_3, u_4\}$ this is

  $$R_{12} = \frac{d}{dt} \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \bigg|_{t=0} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (6.33)$$
\[ \text{• } I_{12} : \quad \exp(t I_{12}) = \begin{pmatrix} 1 & it \\ 0 & 1 \end{pmatrix} \] (6.34)

So with respect to \( \{u_1, u_2, u_3, u_4\} \) this is

\[ I_{12} = \frac{d}{dt} \begin{pmatrix} 1 & 0 & 0 & -t \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \bigg|_{t=0} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \] (6.35)

\[ \text{• } R_{21} : \quad \exp(t R_{21}) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \] (6.36)

So with respect to \( \{u_1, u_2, u_3, u_4\} \) this is

\[ R_{21} = \frac{d}{dt} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ t & 0 & 1 & 0 \\ 0 & t & 0 & 1 \end{pmatrix} \bigg|_{t=0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \] (6.37)

\[ \text{• } I_{21} : \quad \exp(t I_{21}) = \begin{pmatrix} 1 & 0 \\ it & 1 \end{pmatrix} \] (6.38)

So with respect to \( \{u_1, u_2, u_3, u_4\} \) this is

\[ I_{21} = \frac{d}{dt} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -t & 1 & 0 \\ t & 0 & 0 & 1 \end{pmatrix} \bigg|_{t=0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \] (6.39)

We put all this together and use that \( iu_1 = u_2, iu_2 = -u_1, iu_3 = u_4 \) and \( iu_4 = u_3 \) to find:

\[ J_1 \cdot u_1 = -\frac{1}{2} u_4, \quad J_1 \cdot u_2 = \frac{1}{2} u_3, \quad J_1 \cdot u_3 = -\frac{1}{2} u_2, \quad J_1 \cdot u_4 = \frac{1}{2} u_1 \] (6.40)

\[ J_2 \cdot u_1 = \frac{1}{2} u_3, \quad J_2 \cdot u_2 = \frac{1}{2} u_4, \quad J_2 \cdot u_3 = -\frac{1}{2} u_1, \quad J_2 \cdot u_4 = \frac{1}{2} u_2 \] (6.41)

\[ J_3 \cdot u_1 = \frac{1}{2} u_2, \quad J_3 \cdot u_2 = \frac{1}{2} u_1, \quad J_3 \cdot u_3 = \frac{1}{2} u_4, \quad J_3 \cdot u_4 = -\frac{1}{2} u_3 \] (6.42)

\[ K_1 \cdot u_1 = \frac{1}{2} u_4, \quad K_1 \cdot u_2 = \frac{1}{2} u_3, \quad K_1 \cdot u_3 = \frac{1}{2} u_1, \quad K_1 \cdot u_4 = \frac{1}{2} u_2 \] (6.43)

\[ K_2 \cdot u_1 = \frac{1}{2} u_4, \quad K_2 \cdot u_2 = -\frac{1}{2} u_3, \quad K_2 \cdot u_3 = -\frac{1}{2} u_2, \quad K_2 \cdot u_4 = \frac{1}{2} u_1 \] (6.44)

\[ K_3 \cdot u_1 = \frac{1}{2} u_1, \quad K_3 \cdot u_2 = \frac{1}{2} u_2, \quad K_3 \cdot u_3 = -\frac{1}{2} u_3, \quad K_3 \cdot u_4 = -\frac{1}{2} u_4 \] (6.45)
Since \( \langle J_m, u_n \rangle = J_m \cdot u_n \) and \( \langle K_m, u_n \rangle = K_m \cdot u_n \), we now know the superbracket on one even and one odd element.

Now we just need the superbracket on two odd elements. In section 6 we showed that an extension of the bracket to the odd part exists, but we did not show what it looks like. Now as we know \( g_1 = \mathfrak{m} \), and it’s elements are of the form \((u, \bar{u})\) with \( u \in \mathbb{C}^2 \).

**Lemma 6.3.** Let \((u, \bar{u})\) and \((v, \bar{v})\) in \( \mathfrak{m} \). The mapping

\[
\langle (u, \bar{u}), (v, \bar{v}) \rangle = \frac{1}{2} (u \bar{v}^T + v \bar{u}^T)
\]

with the superscript \( T \) standing for the transpose of the 2-vector, is an extension of the Lie bracket as in section 6.

**Proof.** Note that the bracket maps into \( M_2(\mathbb{C}) \), the \( 2 \times 2 \) complex matrices. Specifically, it maps to hermitian matrices and as noted before these can be seen as elements of Minkowski space by equation (5.2). The identification with the generators of translation is then straightforward.

Now we need to show the bracket satisfies theorem 3.5, i.e. it is bilinear, symmetric, intertwines with module action and \( \langle a, \langle a, a \rangle \rangle = 0 \). The bilinearity and symmetry are immediately apparent from the definition. We also know that \( \langle a, a \rangle \in \mathfrak{t} \) for all \( a \in \mathfrak{m} \). Since the translations act trivially on the odd part, this gives \( \langle a, \langle a, a \rangle \rangle = 0 \) for all \( a \in \mathfrak{m} \).

Now let \( g \in SL(2, \mathbb{C}) \). Then

\[
\langle g \cdot (u, \bar{u}) \otimes (v, \bar{v}) \rangle = \langle (gu, \bar{g} \bar{u}), (gv, \bar{g} \bar{v}) \rangle
\]

\[
= \frac{1}{2} (gu(\bar{g} \bar{v})^T + gv(\bar{g} \bar{u})^T)
\]

\[
= \frac{1}{2} (gu \bar{v}^T \bar{g}^T + gv \bar{u}^T \bar{g}^T)
\]

\[
= g \left( \frac{1}{2} (u \bar{v}^T + v \bar{u}^T) \right) g^\dagger
\]

\[
= \Lambda(g) \cdot \frac{1}{2} (u \bar{v}^T + v \bar{u}^T)
\]

\[
= \Lambda(g) \cdot \langle (u, \bar{u}), (v, \bar{v}) \rangle
\]

Using that \( \bar{g}^T = g^\dagger \) and \( \Lambda \) as in equation (5.6). The action of \( g_0 \) comes from this group action, we conclude it is an intertwiner for the action of \( g_0 \).

Thus equation (6.46) gives the last part of the superbracket explicitly. We can now calculate the bracket of the basis elements of \( \mathfrak{m} \). For example, remember
that \( u_1 = (e_1, e_1) \). Then

\[
\langle (e_1, e_1), (e_1, e_1) \rangle = e_1 e_1^T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
= \frac{1}{2} \sigma_0 + \frac{1}{2} \sigma_3 \sim \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \approx \frac{1}{2} (P_0 + P_3)
\]

Similarly the bracket on all odd elements can be found. For the basis \( u_1 = (e_1, e_1), u_2 = (ie_1, -ie_1), u_3 = (e_2, e_2), u_4 = (ie_2, -ie_2) \) this gives:

\[
\begin{align*}
\langle u_1, u_1 \rangle &= \frac{1}{2} (P_0 + P_3) \\
\langle u_1, u_2 \rangle &= 0 \\
\langle u_1, u_3 \rangle &= \frac{1}{2} P_1 \\
\langle u_1, u_4 \rangle &= \frac{1}{2} P_2 \\
\langle u_2, u_2 \rangle &= \frac{1}{2} (P_0 + P_3) \\
\langle u_2, u_3 \rangle &= -\frac{1}{2} P_2 \\
\langle u_2, u_4 \rangle &= \frac{1}{2} P_1 \\
\langle u_3, u_3 \rangle &= \frac{1}{2} (P_0 - P_3) \\
\langle u_3, u_4 \rangle &= 0 \\
\langle u_4, u_4 \rangle &= \frac{1}{2} (P_0 + P_3)
\end{align*}
\]

In literature on the Poincaré superalgebra, for example [13], another basis is commonly used. The generators for the translations part are the same \( P_\mu \), the generators of the Lorentz algebra are denoted by \( M_{\mu\nu} \) and the generator of the odd part are written as \( \{Q_\alpha, Q_\beta\} \), with \( \mu, \nu = 0, 1, 2, 3 \) and \( \alpha, \beta = 1, 2 \). The
defining relations for this basis are
\[ \langle P_\mu, P_\nu \rangle = 0 \]
\[ \langle M_{\mu\nu}, P_\rho \rangle = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \]
\[ \langle M_{\mu\nu}, M_{\rho\sigma} \rangle = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}) \]
\[ \langle M_{\mu\nu}, Q_\alpha \rangle = \frac{1}{2}(\sigma^{\mu\nu})_{\alpha\beta}Q_\beta \]
\[ \langle M_{\mu\nu}, \overline{Q}_\alpha \rangle = \frac{1}{2}(\sigma^{\mu\nu})_{\alpha\beta}\overline{Q}_\beta \]
\[ \langle Q_\alpha, P_\mu \rangle = \langle \overline{Q}_\alpha, P_\mu \rangle = 0 \]
\[ \langle Q_\alpha, \overline{Q}_\beta \rangle = 2(\sigma^\mu)_{\alpha\beta}P_\mu \]
\[ \langle Q_\alpha, Q_\beta \rangle = 2(\sigma^\mu)_{\alpha\beta} \]

with implied summation over repeated indices, regardless whether they are high or low, the Minkowski metric \( \eta = \text{diag}(1, -1, -1, -1) \), \( \sigma^0 \) the 2 \( \times \) 2 identity matrix and \( \sigma^j \) the Pauli matrices, and the matrix \( \sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] \) with \( \{\gamma^\mu, \gamma^\nu\} = 2\eta_{\mu\nu} \). Note that from these relations we see that \( M_{\mu\mu} = 0 \) and \( M_{\mu\nu} = -M_{\nu\mu} \), so the Lorentz algebra has six generators, as it should have.

This basis is related to the one we defined earlier. The following identifications can be checked simply by writing out the brackets and seeing the relations correspond.

\[ J_j = -\frac{1}{2} \epsilon_{jmn}M_{mn} \]
\[ K_j = M_{j0} \]
\[ u_1 = \frac{1}{2}(Q_1 + \overline{Q}_1) \]
\[ u_2 = \frac{1}{2i}(Q_1 - \overline{Q}_1) \]
\[ u_3 = \frac{1}{2}(Q_2 + \overline{Q}_2) \]
\[ u_4 = \frac{1}{2i}(Q_2 - \overline{Q}_2) \]

It should be noted that complex \( i \) used here is not the same \( i \) as the one used in the definition of the \( u_j \), and they should not be mixed up.

### 7 Poincaré superalgebra representations

We want to find irreducible representations for the Poincaré superalgebra. These superrepresentations must also be representations for the Lie algebra, although not necessarily irreducible. Still, at the very least the Lie superalgebra representation is a direct sum of irreducible representations for the Lie algebra that makes up the even part. Thus we may use the irreducible Poincaré algebra representations as a starting point. As before we need to consider different cases depending on the type of orbit.
7.1 Massless representations

We will look at the orbit $X_0^+$ in $H = \text{SL}(2, \mathbb{C})$ first. This is the massless case, since $m = 0$ is identified as the mass of the corresponding particle. Take the point $\chi = (E, 0, 0, E)$ in this orbit. The little group $H_\chi \subset H$ and representation $(\sigma_n, \mathbb{C})$ are as in equations (5.20) and (5.21), and with these we can define the representation space as

$$\text{Ind}_{H_\chi}^H \mathbb{C} = \{ s : H \to \mathbb{C} \mid s(hk) = \sigma_n(k^{-1})s(h) \text{ for } k \in H_\chi, s \in L^2(H/H_\chi) \}$$

(7.1)

The action of the Poincaré group on this space was given was given by equations (5.10) and (5.11) for the Lorentz part and the translation part respectively. The action of the algebra can be found by taking its derivative. This gives the Lie algebra representation specified by the integers $m = 0$ and $n \in \mathbb{Z}$, denoted by $H(0, n)$.

The Lie superalgebra representation contains at least one Lie algebra representation. Therefore, we assume $H(0, n)$ for some $n \in \mathbb{Z}$ is part of the direct sum that makes up the Lie superalgebra representation. Furthermore, we assume its elements to be homogeneous. The Poincaré algebra elements are even, hence their action preserves the parity. Since $H(0, n)$ is an irreducible representation, this implies all elements have the same parity. Whether $H(0, n)$ is even or odd does not matter at this point.

The first step is to find useful eigenvectors for the generators of translation $P_\mu$ for the Lie algebra representation. Let $g \in H$ and define $s_g \in H(0, n) = \text{Ind}_{H_\chi}^H \mathbb{C}$ by

$$s_g(h) = \begin{cases} \sigma_n(k^{-1}) & \text{if there exists } k \in H_\chi \text{ such that } h = gk \\ 0 & \text{otherwise.} \end{cases}$$

(7.2)

This is certainly a map $H \to \mathbb{C}$ with $s(hk) = \sigma_n(k^{-1})s(h)$ for $k \in H_\chi$. Considered as a map on $H/H_\chi$ it is zero except at the point $[g] \in H/H_\chi$, so it is also integrable.

Note that if $g' \in H$ a $k \in H_\chi$ such that $g' = gk$, then for all $h \in H$ we have $s_{g'}(h) = \sigma(k^{-1})s_g$, so in a sense $s_g$ and $s_{g'}$ are equivalent. Choose a representative for each class in $H/H_\chi$ to define the set $\{ s_g \mid [g] \in H/H_\chi \}$. The $s_g$ are the ‘smallest’ elements of the representation space, they do not make basis for $H(0, n)$ since we can’t make all $s$ using linear combinations. However, we assume we can decompose any $s \in H(0, n)$ in a way that is reminiscent of a Fourier transformation. First define an inner product on $H(0, n)$ by

$$\langle s_1, s_2 \rangle = \int_H (s_1(h), s_2(h)) \, dh = \int_H s_1(h)s_2(h) \, dh$$

(7.3)

then using this inner product we can project on the $s_g$. Then for all $s \in H(0, n)$

$$s = \int_{H/H_\chi} \langle s, s_g \rangle s_g \, dg.$$  

(7.4)
Thus we can find the action of $P_\mu$ on $s$ if we know the action on all the $s_g$.

How does $P_\mu$ act on an $s_g$? We don’t have the algebra action explicitly, so we will have the work with the group action via the exponential map. First note that
\[
\exp(tP_\mu) = I_5 + tP_\mu = \begin{pmatrix} I_4 & t e_\mu \\ 0 & 1 \end{pmatrix} \approx t p_\mu
\]
Here $P_\mu$ is the Lie algebra elements and $p_\mu$ the corresponding Lie group element. Using this we find
\[
(dL_{O,\sigma}(P_\mu)(s_g))(h) = \frac{d}{dt} L_{O,\sigma}(\exp(tP_\mu)(s_g)(h)|_{t=0})
\]
\[
= \frac{d}{dt} L_{O,\sigma}(tp_\mu)(s_g)(h)|_{t=0}
\]
\[
= \frac{d}{dt} U(tp_\mu)(s_g)(h)|_{t=0}
\]
\[
= \frac{d}{dt} (h \cdot \chi)(tp_\mu)s_g(h)|_{t=0}
\]
\[
= \frac{d}{dt} e^{i(tp_\mu \cdot h \cdot \chi)}s_g(h)|_{t=0}
\]
\[
= i(p_\mu, h \cdot \chi)s_g(h)
\]
Now suppose $h = gk$ for a $k \in H_\chi$. Then $k \cdot \chi = \chi$, so
\[
(p_\mu, h \cdot \chi) = (p_\mu, g \cdot k \cdot \chi) = (p_\mu, g \cdot \chi)
\]
which is not dependent on $h$ anymore. If there exists no $k \in H_\chi$ such that $h = gk$, then $s_g(h) = 0$ anyway. Thus $dL_{O,\sigma}(P_\mu)(s_g) = i(p_\mu, g \cdot \chi)s_g$.

The next step is to compute $i(p_\mu, g \cdot \chi)$. We use the Iwasawa decomposition (see appendix B) to write $SL(2, \mathbb{C}) = KAN$, with
\[
K = SU(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\} \quad (7.5)
\]
\[
A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \mid t \in \mathbb{R} \right\} \quad (7.6)
\]
\[
N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{C} \right\} \quad (7.7)
\]
so any $g \in SL(2, \mathbb{C})$ can be decomposed as $g = u \cdot a \cdot n$ with $u \in K, a \in A, n \in N$. We will write $u, a, n$ explicitly in terms of $\alpha$ and $\beta$ , $t$, and $b$ respectively. Note that $N \subset H_\chi$, so $n \cdot \chi = \chi$. We identify the 4-vectors with $2 \times 2$-matrices again,
so \( \chi = E\sigma_0 + E\sigma_3 = \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix} \) and \( p_\mu = \sigma_\mu \). Then using equation (5.5) we find

\[
i(p_\mu, g \cdot \chi) = i(p_\mu, u \cdot a \cdot \chi)
\]

\[
= \frac{i}{2} \left( \det \left( ua \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix} (ua)^\dagger + \sigma_\mu \right) - \det \left( ua \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix} (ua)^\dagger \right) - \det(\sigma_\mu) \right)
\]

where we use that \( \det \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix} = 0 \) and \( a^\dagger = a \) in the last step. Then we calculate

\[
ua \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix} a^\dagger u = \begin{pmatrix} \alpha & -\beta \\ \beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix}
\]

\[
= \begin{pmatrix} \alpha & -\beta \\ \beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 2e^{2t}E & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & -\beta \\ \beta & \bar{\alpha} \end{pmatrix}
\]

\[
= \begin{pmatrix} \alpha & -\beta \\ \beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 2e^{2t}E\bar{\alpha} & 2e^{2t}E\bar{\beta} \\ 0 & 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} 2e^{2t}E|\alpha|^2 & 2e^{2t}E\alpha\bar{\beta} \\ 2e^{2t}E\bar{\alpha}\beta & 2e^{2t}E|\beta|^2 \end{pmatrix}
\]

This leaves four calculations, for the cases \( \mu = 0, 1, 2, 3 \):

\[\begin{itemize}
\item \( \mu = 0, \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Then:

\[
i(p_0, u \cdot a \cdot \chi) = \frac{i}{2} \left( \det \begin{pmatrix} 2e^{2t}E|\alpha|^2 + 1 & 2e^{2t}E\alpha\bar{\beta} \\ 2e^{2t}E\bar{\alpha}\beta & 2e^{2t}E|\beta|^2 + 1 \end{pmatrix} - \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)
\]

\[
= \frac{i}{2} (2e^{2t}E|\alpha|^2 + 1)(2e^{2t}E|\beta|^2 + 1) - (2e^{2t}E\alpha\bar{\beta})(2e^{2t}E\bar{\alpha}\beta) - 1
\]

\[= ie^{2t}E(|\alpha|^2 + |\beta|^2)\]

\item \( \mu = 1, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Then:

\[
i(p_1, u \cdot a \cdot \chi) = \frac{i}{2} \left( \det \begin{pmatrix} 2e^{2t}E|\alpha|^2 & 2e^{2t}E\alpha\bar{\beta} + 1 \\ 2e^{2t}E\bar{\alpha}\beta + 1 & 2e^{2t}E|\beta|^2 \end{pmatrix} - \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)
\]

\[
= \frac{i}{2} ((2e^{2t}E|\alpha|^2)(2e^{2t}E|\beta|^2) - (2e^{2t}E\alpha\bar{\beta} + 1)(2e^{2t}E\bar{\alpha}\beta + 1) + 1
\]

\[= -ie^{2t}E(\alpha\bar{\beta} + \bar{\alpha}\beta)\]
\end{itemize}\]
\( \mu = 2, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \). Then:

\[
i(p_2, u \cdot a \cdot \chi) = \frac{i}{2} \left( \det \left( \begin{array}{cc} 2e^{2t}E|\alpha|^2 & 2e^{2t}E\alpha \bar{\beta} - i \\ 2e^{2t}E\bar{\alpha} \beta + i & 2e^{2t}E|\beta|^2 \end{array} \right) - \det \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right) \right)
\]
\[
= \frac{i}{2} \left( (2e^{2t}E|\alpha|^2)(2e^{2t}E|\beta|^2) - (2e^{2t}E\alpha \bar{\beta} - i)(2e^{2t}E\bar{\alpha} \beta + i) + 1 \right)
\]
\[
= e^{2t}E(\alpha \bar{\beta} - \bar{\alpha} \beta)
\]

\( \mu = 3, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). Then:

\[
i(p_3, u \cdot a \cdot \chi) = \frac{i}{2} \left( \det \left( \begin{array}{cc} 2e^{2t}E|\alpha|^2 + 1 & 2e^{2t}E\alpha \bar{\beta} \\ 2e^{2t}E\bar{\alpha} \beta & 2e^{2t}E|\beta|^2 - 1 \end{array} \right) - \det \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \right)
\]
\[
= \frac{i}{2} \left( (2e^{2t}E|\alpha|^2 + 1)(2e^{2t}E|\beta|^2 - 1) - (2e^{2t}E\alpha \bar{\beta})(2e^{2t}E\bar{\alpha} \beta) + 1 \right)
\]
\[
= ie^{2t}E(|\beta|^2 - |\alpha|^2)
\]

We summarize the above calculation in a lemma:

**Lemma 7.1.** The \( s_g \) are simultaneous eigenvectors for the generators of translation, with the following eigenvalues with respect to \( P_\mu \):

\[
\mu = 0: \quad ie^{2t}E(|\alpha|^2 + |\beta|^2)
\]
\[
\mu = 1: \quad -ie^{2t}E(\alpha \bar{\beta} + \bar{\alpha} \beta)
\]
\[
\mu = 2: \quad e^{2t}E(\alpha \bar{\beta} - \bar{\alpha} \beta)
\]
\[
\mu = 3: \quad ie^{2t}E(|\beta|^2 - |\alpha|^2)
\]

These \( s_g \) are all elements of the irreducible Lie algebra representations space specified by \( m = 0 \) and \( n \in \mathbb{Z} \). The Lie algebra elements, i.e. the even elements of the Lie superalgebra, won’t map the \( s_g \) outside of this representation space. The same cannot be said for the odd elements. The odd elements map between the Lie algebra representation spaces, so a Lie superalgebra representation must be a direct sum of Lie algebra representation. We would like to know how many are needed to make an irreducible representation. To this end we look at the odd generators.

We calculated the superbracket of the odd generators in section 6.2. It can be written in a more compact form:

\[
\langle u_m, u_n \rangle = \frac{1}{2} \begin{pmatrix} P_0 + P_3 & 0 & P_1 & P_2 \\ 0 & P_0 + P_3 & -P_2 & P_1 \\ P_1 & -P_2 & P_0 - P_3 & 0 \\ P_2 & P_1 & 0 & P_0 - P_3 \end{pmatrix}_{mn}
\] (7.8)

i.e. \( \langle u_m, u_n \rangle \) is the \( mn \)th element of the matrix. Now we pick a convenient \( g \in H \), namely

\[
g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\] (7.9)
Then $\beta = 1$ and $\alpha = 0, t = 0, b = 0$ with regards to the Iwasawa decomposition, so $s_g$ is an eigenvector for $P_\mu$ with eigenvalues $(iE, 0, 0, iE)$. Thus

$$dL_{O, \sigma}(\langle u_m, u_n \rangle)s_g = \begin{pmatrix} iE & 0 & 0 & 0 \\ 0 & iE & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{mn}s_g. \quad (7.10)$$

We rename

$$dL_{O, \sigma}(u_1) = \sqrt{\frac{iE}{2}}a, \quad dL_{O, \sigma}(u_2) = \sqrt{\frac{iE}{2}}b \quad (7.11)$$

to map the odd elements to the algebra generated by $a, b$ with

$$a^2 = 1, \quad b^2 = 1, \quad ab + ba = 0 \quad (7.12)$$

Note that $dL_{O, \sigma}(u_3) = 0$ and $dL_{O, \sigma}(u_4) = 0$ on $s_g$, thus this algebra is isomorphic to the algebra generated by the $dL_{O, \sigma}(u_j)$.

**Lemma 7.2.** The irreducible representations of the algebra defined by equation (7.12) are 2-dimensional.

**Proof.** There are no 1-dimensional representations. We have $a^2 = 1, b^2 = 1$, so $a = \pm 1, b = \pm 1$. This would mean $ab + ba = 2ab = \pm 2$, which is in contradiction with equation (7.12).

There are 2-dimensional irreducible representations. For example

$$a \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (7.13)$$

It is also irreducible. A proper subrepresentation would have to be 1-dimesional and we have just seen those don’t exist.

Finally, we want to show there are no irreducible representations of dimension 3 or higher. Let $V$ be the representation space, then $a \in \text{End}(V)$. Any elements of $\text{End}(V)$ has a unique decomposition into a semisimple part and a nilpotent part, and these parts commute.[3] Write $a = s + n$, with $s$ semisimple and $n$ nilpotent. Then

$$1 = (s + n)^2 = s^2 + 2sn + n^2. \quad (7.14)$$

Since $s^2$ is again semisimple and $2sn + n^2$ again nilpotent, this gives the unique decomposition of $1 \in \text{End}(V)$ into a semisimple and nilpotent part. Since $1$ is already semisimple, this means $2sn + n^2 = n(2s + n) = 0$, so $n = 0$. Thus $a = s$ is semisimple.

Being semisimple implies that $a$ is diagonalizable. Because $a^2 = 1$, the eigenvalues must be $\pm 1$. We can decompose $V$ into eigenvalue spaces, i.e. $V = V_+ \oplus V_-$. Now let $v$ be an eigenvector with eigenvalue $\lambda$. Then

$$abv = -bav = -\lambda bv \quad (7.15)$$

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since $ab + ba = 0$. Hence

$$b : V_+ \to V_+$$  \hspace{1cm} (7.16)

We know that $b^2 = 1$, so $b$ is invertible. In particular it is injective, so $\dim V_+ = \dim V_-$. Hence $\dim V$ is even. The algebra does not have odd-dimensional representations.

Now suppose we have a representation space such that $\dim V = 2n$ with $n \geq 2$. We can choose a $v \in V_+$, since $\dim V_+ = \dim V_- = n \neq 0$. Then $av = v$, $abv = -bv$ and $b^2v = v$, so span$(v, bv)$ is invariant under the algebra. As the dimension of the linear span is 2, this gives a non-trivial subrepresentation. Thus any representation such that $\dim V = 2n$ with $n \geq 2$ is reducible.

Since the algebra generated by $\{dL_{O, \sigma}(u_j) \mid j = 1, 2, 3, 4\}$ is isomorphic to the one generated by (7.12), it follows from the lemma that its irreducible representations are all 2-dimensional too. Now consider the representation space created by taking $s_g$ and letting the odd elements act on it. By construction this gives an irreducible representation (since $s_g$ is non-zero), so it must be a two dimensional space.

We already know that $u_3$ and $u_4$ act as zero. We know $dL_{O, \sigma}(u_1)s_g \neq 0$, since $dL_{O, \sigma}(u_1^2)s_g = \frac{1}{2}Es_g$. Since $dL_{O, \sigma}(u_1)$ is an odd map it flips the parity, so it cannot map $s_g$ to a multiple of itself. Thus $s_g$ and $dL_{O, \sigma}(u_1)s_g$ span the representation space. Since $dL_{O, \sigma}(u_2)$ is an odd map as well, it must map $s_g$ to some multiple of $dL_{O, \sigma}(u_1)s_g$.

We know $s_g \in H(0, n)$ and we would like to know which irreducible Poincaré algebra representation $dL_{O, \sigma}(u_1)s_g$ belongs to. Letting $W_0$ as defined by equation (5.41) act on the right element gives the answer.

**Lemma 7.3.** Let $g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and let $s_g \in H(0, n)$. Then

$$W_0s_g = \frac{n}{2}Es_g.$$  \hspace{1cm} (7.17)

**Proof.** Remember that $W_0 = J_1P_1 + J_2P_2 + J_3P_3$ and $s_g$ is an eigenvector for the $P_{ij}$ with eigenvalue $(iE, 0, 0, iE)$. Thus $W_0s_g = iEj_3s_g$. All we need is the action of $J_3$ on $s_g$.

The Lie algebra action is derived from the Lie group action, i.e.

$$dL_{O, \sigma}(J_3)(s_g)(h) = \frac{d}{dt}L(\exp(tJ_3))(s_g)(h)|_{t=0}$$  \hspace{1cm} (7.18)

By definition of $J_3$ we have $\exp(tJ_3) = R_3(t)$, were $R_3(t)$ is the 4 × 4-matrix of the rotation over $t$ around the third ($x_3$) axis. The representation $L$ was defined on 2 × 2-matrices, which can be identified with 4 × 4-matrix by $\Lambda$ as in equation (5.6). In this case (the calculation can be found in appendix B.2) we identify

$$R_3(t) \sim \begin{pmatrix} \cos \left(-\frac{t}{2}\right) + i \sin \left(-\frac{t}{2}\right) & 0 \\ 0 & \cos \left(-\frac{t}{2}\right) - i \sin \left(-\frac{t}{2}\right) \end{pmatrix}.$$  \hspace{1cm} (7.19)
We will denote the \(2 \times 2\)-matrix by \(r_3(t)\). Note that \((\cos(-\frac{t}{2}) + i \sin(-\frac{t}{2}))^{-1} = \cos(-\frac{t}{2}) - i \sin(-\frac{t}{2})\) and \(|\cos(-\frac{t}{2}) + i \sin(-\frac{t}{2})| = 1\), so \(r_3(t)\) is an element of \(H_X\).

With this we can write out

\[
\frac{d}{dt} L_{\sigma}(J_3)(s_g)(h) = \frac{d}{dt} L(r_3(t))(s_g)(h)|_{t=0} = \frac{d}{dt} \rho(r_3(t))(s_g)(h)|_{t=0} = \frac{d}{dt} s_g(r_3(t)^{-1}h)|_{t=0} = \frac{d}{dt} \left\{ \begin{array}{ll} \sigma_n(k^{-1}) & \exists k \in H_X \text{ such that } r_3(t)^{-1}h = gk \\ 0 & \text{otherwise} \end{array} \right.
\]

We calculate

\[
\begin{pmatrix}
\cos(-\frac{t}{2}) + i \sin(-\frac{t}{2}) & 0 \\
0 & \cos(-\frac{t}{2}) - i \sin(-\frac{t}{2})
\end{pmatrix}
\cdot
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\cos(-\frac{t}{2}) & 0 \\
0 & -\cos(-\frac{t}{2}) - i \sin(-\frac{t}{2})
\end{pmatrix}
\cdot
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\cdot
\begin{pmatrix}
\cos(-\frac{t}{2}) & 0 \\
0 & \cos(-\frac{t}{2}) + i \sin(-\frac{t}{2})
\end{pmatrix}
\]

i.e. \(r_3(t)g = gr_3(t)^{-1}\). Of course \(r_3(t)^{-1}\) is also an element of \(H_X\), so \(r_3(t)^{-1}l \in H_X\) for all \(l \in H_X\). Similarly \(r_3(t)^{-1}g = gr_3(t)\) and \(r_3(t)l \in H_X\). There are two cases to consider:

- Suppose there does not exist an \(l \in H_X\) such that \(h = gl\), so \(s_g(h) = 0\). If there exists a \(k\) such that \(r_3(t)^{-1}h = gk\), then

\[
\frac{d}{dt} L_{\sigma}(J_3)(s_g)(h) = \frac{d}{dt} s_g(r_3(t)^{-1}h)|_{t=0} = \frac{d}{dt} \left\{ \begin{array}{ll} \sigma_n(k^{-1}) & \exists k \in H_X \text{ such that } r_3(t)^{-1}h = gk \\ 0 & \text{otherwise} \end{array} \right. (7.20)
\]

and as noted before \(r_3(t)^{-1}k \in H_X\). Setting \(l = r_3(t)^{-1}k\) would be a contradiction, hence \(s_g(r_3(t)^{-1}h) = 0\) for all \(t\). Thus \(dL_{\sigma}(J_3)(s_g))(h) = 0\).

- Suppose there exists an \(l \in H_X\) such that \(h = gl\), so \(s_g(h) = \sigma_n(l^{-1})\). Then

\[
r_3(t)^{-1}h = r_3(t)^{-1}gl = gr_3(t)l (7.21)
\]
so we set \( k = r_3(t)l \). With this we calculate

\[
\frac{d}{dt} \sigma_n(k^{-1})|_{t=0} = \frac{d}{dt} \sigma_n((r_3(t)l)^{-1})|_{t=0}
\]

\[
= \frac{d}{dt} \sigma_n(l^{-1}r_3(t)^{-1})|_{t=0}
\]

\[
= \frac{d}{dt} \sigma_n(l^{-1}) \sigma_n(r_3(t)^{-1})|_{t=0}
\]

\[
= \frac{d}{dt} s_g(h) \sigma_n(r_3(t)^{-1})|_{t=0}
\]

\[
= s_g(h) \left( \frac{d}{dt} \left( \cos \left( -\frac{t}{2} \right) - i \sin \left( -\frac{t}{2} \right) \right) \right) \Bigg|_{t=0}
\]

\[
= -\frac{in}{2} s_g(h)
\]

So for all \( h \in H \) we find \( dL_{O,\sigma}(J_3)(s_g)(h) = -\frac{in}{2} s_g(h) \). Thus

\[
W_0 s_g(h) = iE \cdot -\frac{in}{2} s_g(h) = \frac{n}{2} E s_g(h) \quad (7.22)
\]

**Corollary 7.4.** Let \( s \in H(0,n) \) be an element of the representation space specified by \( m = 0 \) and \( n \in \mathbb{Z} \) with eigenvalues \((iE,0,0,iE)\) with respect to \( P_\mu \), \( \mu = 0,1,2,3 \). Then

\[
W_0 s = \frac{n}{2} E s_g. \quad (7.23)
\]

**Proof.** Since \( s \) has eigenvalues \((iE,0,0,iE)\), it must be built up exclusively with \( s_g \) that also have eigenvalues \((iE,0,0,iE)\). From lemma 7.1 we conclude that only those \( g \) with \( t = 0, \alpha = 0 \) and \( |\beta| = 1 \) satisfy this condition. That gives many possible choices:

\[
g = \begin{pmatrix} 0 & -ie^{ix} \\ e^{ix} & 0 \end{pmatrix} \quad (7.24)
\]

with \( x \in \mathbb{R} \). Fortunately they are all equivalent to

\[
g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (7.25)
\]

in \( H/H_\chi \). Thus \( s = \lambda s_g \) with \( g \) as in equation (7.25) and \( \lambda \) some scalar in \( \mathbb{C} \). Application of the lemma gives the desired result. \( \square \)

We chose \( g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) to get an \( s_g \) with eigenvalues \((iE,0,0,iE)\). Since the \( P_\mu \) commute with the odd part of the Lie superalgebra, \( dL_{O,\sigma}(u_1)s_g \) is also an eigenvector with the same eigenvalues. According to corollary 7.4, this means applying \( W_0 \) tells us which Lie algebra representation it belongs to. From here we omit the \( dL_{O,\sigma} \) when letting Lie algebra elements act on the representation space, since it makes equations unnecessarily long and complex to read.
Lemma 7.5. Let \( s_g \in H(0,n) \) be as defined before. Then \( dL_{O,\sigma}(u_1)s_g \) is an element of \( H(0,n+1) \), i.e. belongs to the representation space specified by \( m = 0 \) and \( n + 1 \).

Proof. First we use that \( \langle P_\mu, u_1 \rangle = 0 \) and the action of \( J \) given by equations (6.40, 6.41, 6.42) to calculate

\[
\langle W_0, u_1 \rangle = \langle J_1 P_1 + J_2 P_2 + J_3 P_3, u_1 \rangle \\
= \langle J_1, u_1 \rangle P_1 + \langle J_2, u_1 \rangle P_2 + \langle J_3, u_1 \rangle P_3 \\
= \frac{1}{2} (-u_4 P_1 + u_3 P_2 - u_2 P_3)
\]

Combining this with lemma 7.3 and eigenvalues \((iE, 0, 0, iE)\) of \( s_g \) gives

\[
W_0 u_1 s_g = (\langle W_0, u_1 \rangle + u_1 W_0) s_g \\
= \frac{1}{2} (-u_4 P_1 + u_3 P_2 - u_2 P_3) s_g + u_1 W_0 s_g \\
= -\frac{iE}{2} u_2 s_g + \frac{n}{2} E u_1 s_g \\
= \frac{E}{2} u_1 s_g + \frac{n}{2} E u_1 s_g \\
= \frac{n+1}{2} E u_1 s_g.
\]

Applying corollary 7.4 concludes the proof. \( \square \)

So far we can conclude that a massless irreducible representation of the Poincaré superalgebra is the direct sum of at least two Lie algebra representations, namely those with \( n \) and \( n + 1 \) for some \( n \in \mathbb{Z} \). It turns out that \( H(0,n) \oplus H(0,n+1) \) is enough: the action of the Poincaré superalgebra elements will not ‘escape’ the direct sum.

Theorem 7.6. The irreducible massless representations of the Poincaré superalgebra are the direct sum of two irreducible Poincaré algebra representations. The representation space is \( H(0,n) \oplus H(0,n+1) \) with \( n \in \mathbb{Z} \).

Proof. First note that the Lie algebra representations are irreducible. This means that any element of \( H(0,n) \) can be written as \( A s_g \) with \( A \in \mathcal{U}(\mathfrak{p}) \), otherwise we would have a non-trivial subalgebra generated by \( s_g \). Now consider the example \( A = J_1 \):

\[
u_1 J_1 s_g = (J_1 u_1 + \frac{1}{2} u_4) s_g \\
= J_1 u_1 s_g + \frac{1}{2} u_4 s_g \\
= J_1 u_1 s_g
\]

Since \( H(0,n+1) \) is a Lie algebra representation, this implies \( u_1 J_1 s_g \in H(0,n+1) \) as well. Something similar holds true for all \( A \in \mathcal{U}(\mathfrak{p}) \) and all \( u_1 \). Using the
superbracket relations for the Poincaré superalgebra, we can flip the odd and even elements until

$$u_i A s_g = \bigoplus_{i=1}^{4} A_i u_i s_g$$

with $A_i \in \mathcal{U}(\mathfrak{p})$. Since $u_i s_g \in H(0, n+1)$ and $A_i \in \mathcal{U}(\mathfrak{p})$ won’t map outside the representation space, we conclude that $u_i A s_g \in H(0, n+1)$.

As for the elements of $H(0, n+1)$, since it is an irreducible representation all elements can be written as $A u_1 s_g$ with $A \in \mathcal{U}(\mathfrak{p})$. Then using equation (7.10) we find

$$u_i A u_1 s_g = \bigoplus_{i=1}^{4} A_i u_i u_1 s_g = (A_1 + iE A_2) s_g.$$ 

Thus $u_i A u_1 s_g \in H(0, n)$.

Now all that is left is to turn $H(0, n) \oplus H(0, n+1)$ into a super vector space (otherwise we would not satisfy definition 3.14). The odd generators need to map between subspaces and the even elements within, so we choose either $H(0, n)$ even and $H(0, n+1)$ odd or the other way around.

\[\square\]

### 7.2 Massive representations

Finding massive, i.e. $m > 0$ representations follows the same method as the massless. We will follow the same steps, but (detailed) proof will be omitted. This time we consider the orbit $X^+_m$ and take as fixed point $(m, 0, 0, 0)$. The representation space is defined just as it was for the massless case, only with $H_\chi$ as in equation (5.15) and a $H_\chi$-representation $\sigma_j$. Taking the derivative gives the Lie algebra representation, denoted by $H(m, j)$.

Again, let $g \in H$ and define $s_g \in H(m, j)$ by

$$s_g(h) = \begin{cases} 
\sigma_j(k^{-1}) & \text{if there exists } k \in H_\chi \text{ such that } h = gk \\
0 & \text{otherwise.} 
\end{cases}$$

(7.26)

Like before, these are eigenvectors with eigenvalues

$$i(p_\mu, g \cdot \chi) = \frac{i}{2} (\det(g \chi g^\dagger + \sigma_\mu) - \det(g \chi g^\dagger) - \det(\sigma_\mu))$$

(7.27)

with respect to $P_\mu$. Using $\text{SL}(2, \mathbb{C})(2, \mathbb{C}) = \text{NAK}$ (see appendix B) we write $g = nau$. Note that $K = H_\chi$, hence $nau \cdot \chi = na \cdot \chi$. For $\mu = 0$ and $a$ and $n$ as
in equations (7.6) and (7.7) this means

\[
i(p_0, g \cdot \chi) = \frac{i}{2} \left( \det \begin{pmatrix} m(e^{2t} + |b|^2 e^{-2t}) + 1 & m be^{-2t} \\ m be^{2t} & me^{-2t} + 1 \end{pmatrix} \\
- \det \begin{pmatrix} m(e^{2t} + |b|^2 e^{-2t}) & m be^{-2t} \\ m be^{2t} & me^{-2t} \end{pmatrix} - \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)
\]

\[
= \frac{i}{2} \left( m^2(1 + |b|^2 e^{-4t}) + m(e^{2t} + |b|^2 e^{-2t}) + me^{-2t} + 1 \\
- m^2|b|^2 - m^2(1 + |b|^2 e^{-4t}) + m^2|b|^2 - 1 \right)
\]

\[
= \frac{i}{2} m(e^{2t} + (1 + |b|^2)e^{-2t})
\]

The calculations for \( \mu = 1, 2, 3 \) are similar, leading to the following lemma:

**Lemma 7.7.** The \( s_g \) are simultaneous eigenvectors for the generators of translation, with the following eigenvalues with respect to \( P_\mu \):

\[
\mu = 0 : \quad \frac{i}{2} m(e^{2t} + (1 + |b|^2)e^{-2t})
\]

\[
\mu = 1 : \quad -\frac{i}{2} m(be^{2t} + be^{-2t})
\]

\[
\mu = 2 : \quad -\frac{1}{2} m(be^{2t} - be^{-2t})
\]

\[
\mu = 3 : \quad \frac{i}{2} m(-e^{2t} + (1 - |b|^2)e^{-2t})
\]

If we choose \( t = 0 \) and \( b = 0 \), i.e. \( g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), the eigenvalues of \( s_g \) are \((im, 0, 0, 0)\). Equation (7.8) holds regardless of representation, and for this representation and for this \( s_g \) it leads to

\[
dL_{O,\sigma}(\langle u_m, u_n \rangle)s_g = \begin{pmatrix} im & 0 & 0 & 0 \\ 0 & im & 0 & 0 \\ 0 & 0 & im & 0 \\ 0 & 0 & 0 & im \end{pmatrix}_{mn} s_g. \tag{7.28}
\]

Just like in the massless case we normalize:

\[
dL_{O,\sigma}u_j \mapsto \frac{1}{2} \sqrt{im} x_j \tag{7.29}
\]

This gives an algebra generated by \( \{ x_j \mid j = 1, 2, 3, 4 \} \) with the relations

\[
x_j^2 = 1, \quad x_j x_k + x_k x_j = 0 \text{ for } j \neq k. \tag{7.30}
\]

This is isomorphic to the algebra spanned by the \( dL_{O,\sigma}(u_j) \), so again we are interested in its representations.

**Lemma 7.8.** The irreducible representations of the algebra defined by equation (7.30) are 4-dimensional.
The proof to this lemma is similar to lemma 7.2. The $x_j$ are diagonalizable, with eigenvalues $\{\pm 1\}$. Since $x_jx_k + x_kx_j = 0$, the $x_j$ map $(+1)$-eigenvectors to $(-1)$-eigenvectors. So

$$x_1, x_2, x_3 : \ker(x_4 + 1) \to \ker(x_4 + 1)$$

(7.31)

and the similarly for the eigenspaces of $x_1, x_2, x_3$, thus $\dim(\ker(x_j + 1)) = \dim(\ker(x_j - 1))$. This means all representations are even-dimensional. A 2-dimensional representation turns out to be impossible: we cannot find four $2 \times 2$-matrices that satisfy equation (7.30). An irreducible 4-dimensional representation is possible, for example

$$x_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, x_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix}, x_3 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}, x_4 \mapsto \begin{pmatrix} 0 & i \\ i & -i \\ 0 & i \\ -i & 0 \end{pmatrix}$$

Any higher dimensional representations turn out to be reducible.

This means the representation space created by applying the odd generators $u_j$ to $s_g$ is four-dimensional. This time we use the Casimir element $C_2$ to discover which Lie algebra representation space the odd elements 'escape' to.

**Lemma 7.9.** Let $s \in H(m, j)$. Then

$$C_2s = m^2j(j+1)s.$$  

(7.32)

**Proof.** On $s_g$ the Casimir element $C_2$ looks much simpler if we use $P_{1,2,3}s_g = 0$. Using the eigenvalue and the SU(2) representation we find

$$C_2s_g = (J_1J_1 + J_2J_2 + J_3J_3)P_0P_0s_g = m^2j(j+1)s_g$$  

(7.33)

Since $C_2$ commutes with all elements of the Poincaré algebra, this implies $C_2s = m^2j(j+1)s$.

Since the Casimir element shows which representation an element belongs to, finding the representations becomes a matter of computation. For example

$$C_2u_1s_g = \langle C_2, u_1 \rangle s_g + u_1C_2s_g$$

$$= -m^2(j + \frac{1}{2})u_1s_g + m^2j(j+1)u_1s_g$$

$$= m^2(j - \frac{1}{2})(j - \frac{1}{2} + 1)u_1s_g$$

This results in the following lemma.
**Lemma 7.10.** Let $s_g \in H(m, j)$ as before. Then $u_1s_g, u_2s_g \in H(m, j - \frac{1}{2})$ and $u_3s_g, u_4s_g \in H(m, j + \frac{1}{2})$.

Schematically this means the odd generators map elements of the form $s_g$ as in the figure.

\[
\begin{array}{c}
\text{H}(m, j) \xrightarrow{u_1, u_2} \text{H}(m, j - \frac{1}{2}) \\
\downarrow u_3, u_4 \\
\text{H}(m, j) \xrightarrow{u_1, u_2} \text{H}(m, j + \frac{1}{2})
\end{array}
\]

As in the massless case the even elements will not leave these Lie algebra representations. Thus these four representations are enough.

**Theorem 7.11.** The irreducible massive representations of the Poincaré superalgebra are the direct sum of four irreducible Poinacaré algebra representations. The representations space is given for some $j \in \frac{1}{2}\mathbb{Z}$ by $H(m, j) \oplus H(m, j) \oplus H(m, j - \frac{1}{2}) \oplus H(m, j + \frac{1}{2})$. If $s_g$ is even (resp. odd) the grading is given by $H(m, j) \oplus H(m, j)$ even (resp. odd) and $H(m, j - \frac{1}{2}) \oplus H(m, j + \frac{1}{2})$ odd (resp. even).

## 8 Physical importance and conclusion

For the massless case, the particles corresponding to the representation $H(0, n)$ are usually denoted by $\frac{n}{2}$ rather than $n$. If $\frac{n}{2}$ is an integer the particle is a boson, if it is a half-integer we are dealing with a fermion. For the massive case, $j$ is the spin. An integer $j$ denotes a boson, a half-integer a fermion. In both cases, we see that the odd generators of the Poincaré superalgebra map a boson(-representation space) to a fermion(-representation space) and vice versa. Supersymmetry indeed provides a relationship between the two classes of particles.

The classification can tell us more. The odd generators map a boson (resp. fermion) to a fermion (resp. boson) of the same mass. If supersymmetry with this Poincaré superalgebra is an accurate description of reality, then every particle has a supersymmetric partner of equal mass. This also means there exists an equal number of bosonic and fermionic particles.

Unfortunately for this theory, so far supersymmetric partners have not been found by experimental physicists. That is a problem, because if the partners have equal mass they would have been found already. Does that mean supersymmetry is not true? In its most basic form it certainly isn’t.

There are more possibilities than the minimal extension we have looked at in this thesis. There are other extensions, with more generators for the odd subspace $\mathfrak{g}_1$, which give different representations. However, they do not solve the mass issue. If supersymmetry holds true in some way, the symmetry has to
be broken.[1, Chap. 2] This means there is some mechanism that messes up the symmetry of the system and makes the supersymmetric partners unfavorable.

So is there any point in trying to classify the Poincaré superalgebra representations? Of course there is! Even if this supersymmetry is not 'real', it is a nice starting point and a good way to understand methods to deal with Lie superalgebras.

Still, useful or not the graded representations found in this thesis look rather nice. Of course this is only a partial classification, as we did not even look at the Poincaré superalgebra representations with orbits $Y_m$ and $\{0\}$. From a physical point of view this choice makes sense, but from a mathematical point of view it is perhaps regrettable. After all, even if physicists cannot use the end result, mathematics has value in and of itself.
Part III

Appendix

A Representations and cohomology

In section 5.1 we found the representation for $\mathbb{R}^4 \rtimes \text{SL}(2, \mathbb{C})$ rather than $\mathbb{R}^4 \rtimes \text{SO}(1,3)^0$. The latter has less representations, because not all of them descend. This is because not all SU(2)-representations descend to SO(3). So why is that?

A.1 A projective representation

We know that SU(2) is a double cover of SO(3), i.e.

$$\text{SO}(3) \simeq \text{SU}(2)/\mathbb{Z}_2$$

by a continuous, surjective group homomorphism $p : \text{SU}(2) \to \text{SO}(3)$. It is surjective, so we can choose a continuous section

$$q : \text{SO}(3) \to \text{SU}(2)$$

such that $pq = 1$. Later on both maps will be given explicitly.

Now suppose we have an irreducible representation $\sigma : \text{SU}(2) \to \text{GL}(V)$. Then we have a map

$$\sigma q : \text{SO}(3) \to \text{GL}(V).$$

However, there is no guarantee this respects the multiplication, so it might not be a representation. In fact, we have no guarantee that for $g_1, g_2 \in \text{SO}(3)$

$$q(g_1)q(g_2) = q(g_1g_2)$$

holds. Supposing that $q(g_1)q(g_2) = f(g_1, g_2)q(g_1g_2)$ with $f(g_1, g_2) \in \text{SU}(3)$, we find

$$g_1g_2 = pq(g_1)pq(g_2)$$

$$= p(f(g_1, g_2)q(g_1g_2))$$

$$= p(f(g_1, g_2))pq(g_1g_2)$$

$$= p(f(g_1, g_2))g_1g_2$$

since $p$ respects multiplication. This means $p(f(g_1, g_2)) = e$. Since SU(2) is a double cover, there are only two possible values for $f(g_1, g_2)$, namely $e$ and $-e$. Because $\sigma$ is irreducible, we know $\sigma(e) = \text{Id}_V$ and $\sigma(-e) = \pm \text{Id}_V$, so

$$\sigma q(g_1)\sigma q(g_2) = \sigma(q(g_1)q(g_2))$$

$$= \sigma(f(g_1, g_2)q(g_1g_2))$$

$$= \sigma(f(g_1, g_2))\sigma q(g_1g_2)$$

$$= \pm \sigma q(g_1g_2)$$

This is not necessarily a representation, but we can try to divide out the scalars.

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Definition A.1. Let $V$ be a vector space over $\mathbb{C}$ and define the quotient map

$$
\phi : \text{GL}(V) \to \text{PGL}(V) = \text{GL}(V)/\mathbb{C}^\times.
$$

(A.5)

The group $\text{PGL}(V)$ is called the \textit{projective linear group}. A \textit{projective representation} of a group $G$ on the vector space $V$ is a group homomorphism $G \to \text{PGL}(V)$.

Since $\phi, \sigma$ and $q$ are group homomorphisms, so is their composition. Thus $\phi \sigma q : \text{SO}(3) \to \text{PGL}(V)$ is a projective representation of $\text{SO}(3)$.

We can make this more explicit. The rotations in $\text{SO}(3)$ can be described using the \textit{Euler angles}. The triple $(\alpha, \beta, \gamma)$ describes the rotation

$$
R(\alpha, \beta, \gamma) = R_z(\alpha)R_y(\beta)R_z(\gamma),
$$

(A.6)

meaning we first rotate around the $z$-axis by $\gamma$, then around the $y$-axis by $\beta$ and finally around the $z$-axis by $\alpha$. Any rotation can be written like that, with $0 \leq \alpha \leq 2\pi$, $0 \leq \beta \leq \pi$ and $0 \leq \gamma \leq 2\pi$. Next we shall state without proof (see citation):

\textbf{Lemma A.2.} The group $\text{SU}(2)$ can be parametrized by Euler angles within the range $0 \leq \alpha < 2\pi$, $0 \leq \beta \leq \pi$ and $0 \leq \gamma < 4\pi$. Furthermore, for $g \in \text{SU}(2)$ corresponding to $(\alpha, \beta, \gamma)$, the map

$$
g \mapsto D^{1/2}(\alpha, \beta, \gamma) = \begin{pmatrix}
e^{-i\alpha/2} \cos(\beta/2) e^{-i\gamma/2} & -e^{-i\alpha/2} \sin(\beta/2) e^{i\gamma/2} \
e^{i\alpha/2} \sin(\beta/2) e^{-i\gamma/2} & e^{i\alpha/2} \cos(\beta/2) e^{i\gamma/2}
\end{pmatrix}
$$

(A.7)

is an irreducible representation of $\text{SU}(2)$ on $\mathbb{C}^2$.

The covering map $p : \text{SU}(2) \to \text{SO}(3)$ can be considered in terms of Euler angles as well. We have

$$
p : D^{1/2}(\alpha, \beta, \gamma) \mapsto R(\alpha, \beta, \gamma \text{ mod } 2\pi)
$$

(A.8)

so the return map can simply be

$$
q : R(\alpha, \beta, \gamma) \mapsto D^{1/2}(\alpha, \beta, \gamma).
$$

(A.9)

So we have a group homomorphism $\lambda : \text{SO}(3) \to \text{GL}(\mathbb{C}^2)$ given by

$$
\lambda : g = R(\alpha, \beta, \gamma) \mapsto D^{1/2}(\alpha, \beta, \gamma)
$$

(A.10)

Like we stated in general, this \textit{could} be a representation, if the multiplication is respected. As it turns out, this is not the case. Indeed, using that a rotation of
2\pi over the z-axis does nothing at all, we can calculate
\[
\lambda(R(0, 0, \pi)R(0, 0, \pi)) = \lambda(R(0, 0, 2\pi)) \\
= \lambda(R(0, 0, 0)) \\
= D^{1/2}(0, 0, 0) \\
= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
\[
\lambda(R(0, 0, \pi))\lambda(R(0, 0, \pi)) = D^{1/2}(0, 0, \pi)D^{1/2}(0, 0, \pi) \\
= \begin{pmatrix} e^{-i\pi/2} & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} \begin{pmatrix} e^{-i\pi/2} & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} \\
= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\]

By applying the quotient map we at least find a projective representation of SU(3) given by
\[
g = R(\alpha, \beta, \gamma) \mapsto \phi(D^{1/2}(\alpha, \beta, \gamma)). \quad (A.11)
\]

This is not yet the end of the line, though. Simply ignoring the quotient map is not the only way to lift the projective representation to a map SU(3) \to GL(C^2). A different lift might give a different result. To understand when we can or cannot lift to a linear representation, some knowledge of group cohomology is required.

### A.2 Intermezzo: group cohomology

This section uses definitions taken from [4]. We suppose \( M \) is an abelian group (written as a multiplicative group) and \( G \) is a Lie group which acts on \( M \). Later we suppose this is a trivial action, i.e. \((g, m) \mapsto g \cdot m = m\). We can define the groups

\[
C^n(G, M) = \{ f : G \times \ldots \times G \to M \mid f(gg_0, \ldots, gg_n) = g \cdot f(g_0, \ldots, g_n) \} \quad (A.12)
\]

together with mappings \( d_n : C^n(G, M) \to C^{n+1}(G, M) \) given by

\[
(d_n f)(g_0, \ldots, g_{n+1}) = \prod_{i=0}^{n} (f(g_0, \ldots, \hat{g}_i, \ldots, g_{n+1})^{(-1)^i} \quad (A.13)
\]

where \( \hat{g}_i \) means that the \( i \)th argument is ignored. It can be shown that \( d_{n+1} \circ d_n = 1 \). This gives a sequence

\[
C^0(G, M) \xrightarrow{d_0} C^1(G, M) \xrightarrow{d_1} C^2(G, M) \xrightarrow{d_2} \ldots
\]

which is an example of a cochain complex. Since \( d_{n+1} \circ d_n = 1 \), we have \( \text{Im}(d_n) \subseteq \text{Ker}(d_{n+1}) \). This allows us to make the following definition:
Definition A.3. Let \((C^\bullet(G, M), d)\) be a cochain complex. Then we have the group of \(n\)-cocycles

\[ Z^n(G, M) = \{ f \in C^n(G, M) \mid d_n f = 1 \} \]  \hspace{1cm} (A.14)

and the group of \(n\)-coboundaries

\[ B^n(G, M) = \{ f \in C^n(G, M) \mid \exists h \in C^{n-1}(G, M), f = d_{n-1} h \}. \]  \hspace{1cm} (A.15)

By our earlier observation we have \(B^n \subseteq Z^n\), so we may define the cohomology groups as

\[ H^n(G, M) = Z^n(G, M) / B^n(G, M). \]  \hspace{1cm} (A.16)

It turns out that a lifting of a projective representation defines a class in cohomology. This is more easily seen in an alternative cochain complex. Suppose \(G\) acts trivially on \(M\).

\[ \tilde{C}^n(G, M) = \{ \tilde{f} : G \times \ldots \times G \to M \} \]  \hspace{1cm} (A.17)

where \(\tilde{f}\) takes \(n\) arguments. We have \(C^n(G, M) \cong \tilde{C}^n(G, M)\). This isomorphism is given by

\[ \tilde{f}(g_1, \ldots, g_n) = f(e, g_1, g_1^{-1} g_2, \ldots, g_1^{-1} \cdot \ldots \cdot g_n) \]

\[ f(g_0, \ldots, g_n) = g_0 \cdot f(e, g_0^{-1} g_1, \ldots, g_0^{-1} g_n) \]

\[ = g_0 \cdot \tilde{f}(g_0^{-1} g_1, g_1^{-1} g_2, \ldots, g_1^{-1} g_n) \]

\[ = \tilde{f}(g_0^{-1} g_1, g_1^{-1} g_2, \ldots, g_1^{-1} g_n) \].

Where we use the trivial action of \(G\) in the final step. We define \(\tilde{d}\) by \(\tilde{d}_{n+1} \tilde{d}_n \tilde{f} = (d_n f)\). Then

\[ \tilde{d}_{n+1} \tilde{d}_n \tilde{f} = (d_{n+1} d_n f)^\sim = \tilde{1} = 1 \]  \hspace{1cm} (A.18)

So this also a cochain complex

\[ \tilde{C}^0(G, M) \longrightarrow \tilde{C}^1(G, M) \longrightarrow \tilde{C}^2(G, M) \longrightarrow \ldots \]

with similar definitions for \(\tilde{Z}^n(G, M), \tilde{B}^n(G, M)\) and \(\tilde{H}^n(G, M)\). In fact, by definition of \(\tilde{d}\) the following diagram commutes,

\[ \begin{array}{ccccccccc}
C^0(G, M) & \xrightarrow{d_0} & C^1(G, M) & \xrightarrow{d_1} & C^2(G, M) & \xrightarrow{d_2} & \ldots \\
\sim & & \sim & & \sim & & \\
\tilde{C}^0(G, M) & \xrightarrow{\tilde{d}_0} & \tilde{C}^1(G, M) & \xrightarrow{\tilde{d}_1} & \tilde{C}^2(G, M) & \xrightarrow{\tilde{d}_2} & \ldots
\end{array} \]
meaning that no matter which way you walk through the diagram, the outcome does not change, i.e. \( \tilde{d}_n \tilde{f} = (d_n f) \sim \). A collection of maps \( C^* \to \tilde{C}^* \) that commutes with the respective boundaries is called a cochain map. It sends cochains to cochain, coboundaries to coboundaries, hence it descends to a map on the cohomology groups. This means \( \sim \) induces an isomorphism between cohomology groups, so we may conclude

\[
H^n(G, M) \cong \tilde{H}^n(G, M) \tag{A.19}
\]

Thus we can use this \( \sim \)-cochain complex and still use any prior knowledge we might have of the cohomology groups \( H^n(G, M) \).[2]

We are interested in the second cohomology group \( H^2(G, M) \). We need \( \tilde{d}_1 \) to find the elements of \( \tilde{B}^2(G, M) \) and \( d_2 \) to find the elements of \( \tilde{Z}^2(G, M) \). We use equation (A.13) and the definition \( \tilde{d}_n \tilde{f} = (d_n f) \sim \), as well as the fact that \( G \) acts trivially on \( M \), to find

- The 0-boundary operator:

\[
\tilde{d}_0 \tilde{f}(g_1) = (d \tilde{f}) \sim (g_1)
= d f(e, g_1)
= f(g_1) f(e)^{-1}
= g_1 \cdot f(e) f(e)^{-1}
= e
\]

- The 1-boundary operator:

\[
\tilde{d}_1 \tilde{f}(g_1, g_2) = (d_1 \tilde{f}) \sim (g_1, g_2)
= d_1 f(e, g_1, g_1 g_2)
= f(g_1, g_1 g_2) f(e, g_1 g_2)^{-1} f(e, g_1)
= g_1 \cdot \tilde{f}(g_2) \tilde{f}(g_1 g_2)^{-1} \tilde{f}(g_1)
= \tilde{f}(g_2) \tilde{f}(g_1 g_2)^{-1} \tilde{f}(g_1)
\]

- The 2-boundary operator:

\[
\tilde{d}_2 \tilde{f}(g_1, g_2, g_3) = (d_2 \tilde{f}) \sim (g_1, g_2, g_3)
= d_2 f(e, g_1, g_1 g_2, g_1 g_2 g_3)
= f(g_1, g_1 g_2, g_1 g_2 g_3) f(e, g_1 g_2, g_1 g_2 g_3)^{-1}
\cdot f(e, g_1, g_1 g_2 g_3) f(e, g_1, g_1 g_2)^{-1}
= g_1 \cdot \tilde{f}(g_2, g_3) \tilde{f}(g_1 g_2, g_3)^{-1} \tilde{f}(g_1, g_2 g_3) \tilde{f}(g_1, g_2)^{-1}
= \tilde{f}(g_2, g_3) \tilde{f}(g_1 g_2, g_3)^{-1} \tilde{f}(g_1, g_2 g_3) \tilde{f}(g_1, g_2)^{-1}
\]
A.3 Lifting a projective representation

Let $G$ be a group, $V$ a vector space and $\pi$ a projective representation $\pi : G \to \text{PGL}(V)$. To lift this to a linear representation, we need a lift $\text{PGL}(V) \to \text{GL}(V)$. With it we can find a map

$$
\begin{array}{c}
G \\ \pi \\
\downarrow L \\
\text{PGL}(V) \\
\downarrow \text{Lift} \\
\text{GL}(V)
\end{array}
$$

A lifted map $L : G \to \text{GL}(V)$ satisfies

$$L(gh) = c(g, h)L(g)L(h) \quad (A.20)$$

with $c(g, h) \in \mathbb{C}^\times$. If $c(g, h) = 1$ for all $g, h \in G$, then $L$ is a linear representation. Of course the $c$ depends on the chosen lift $L$, which is generally not unique. Even if $(L, c)$ is not a linear representation, a transformed lift $(L', c')$ might still be. Fortunately, equation (A.20) gives some restrictions on $c$ and a relationship between $c$ and $c'$.

Firstly, note that $c$ is a mapping $c : G \times G \to \mathbb{C}^\times$, so $c \in \tilde{Z}^2(G, \mathbb{C}^\times)$. Now let $g, h, k \in G$. Then

$$L(ghk) = c(g, hk)L(g)L(h) = c(g, h) c(h, k) L(g)L(h)L(k) = c(g, h) c(g, h) L(g)L(h)L(k).$$

So we find

$$c(g, h) c(h, k) = c(g, h) c(g, h). \quad (A.21)$$

This gives the 2-cocycle equation, i.e.

$$\tilde{d}_2 c(g, h, k) = c(h, k) c(g, h, k) c(g, h, k) = 1, \quad (A.22)$$

thus $c \in \tilde{Z}^2(G, \mathbb{C}^\times)$.

Now we consider another lift $L'$. Note that $\phi(L(g)) = \phi(L'(g))$ for all $g \in G$, where $\phi$ is the quotient map, so $L(g)$ and $L'(g)$ can only differ by a scalar factor. There must be a mapping $f : G \to \mathbb{C}^\times$ such that $L'(g) = f(g)L(g)$. Now $c'$ can be found in terms of $f$ and $c$:

$$L'(gh) = c'(g, h)L'(g)L'(h) = c'(g, h)f(g)f(h)L(g)L(h) = f(gh)L(gh) = f(gh)c(g, h)L(g)L(h)$$

Thus

$$c'(g, h) = f(gh)f(g)^{-1}f(h)^{-1}c(g, h) = \tilde{d}_1 f(g, h)c(g, h), \quad (A.23)$$

so $c$ and $c'$ are cohomologous.
This means that no matter the choice of \( L \), the corresponding \( c \) will belong to the same class in \( H^2(G, \mathbb{C}^\times) \), and thus in \( H^2(G, \mathbb{C}^\times) \). If \( c \) belongs to the trivial class, there is an \( f \in C^1(G, \mathbb{C}^\times) \) such that \( c(g, h) = f(gh) f(g)^{-1} f(h)^{-1} \).

Define \( L'(g) = f(g)^{-1} L(g) \). Then

\[
L'(gh) = f(gh)^{-1} L(gh) \\
= f(gh)^{-1} c(g, h) L(g) L(h) \\
= f(gh)^{-1} c(g, h) f(g) f(h) L'(g) L'(h) \\
= L'(g) L'(h).
\]

So \( L' \) is a linear representation. (Note that this means any projective representation can be lifted if the second cohomology group itself is trivial, i.e. contains only one element.) However, if \( c \) is in a non-trivial class there is no such \( f \), so the projective representation cannot be lifted to a linear representation.

Now to get back to \( \text{SO}(3) \) and \( \text{SU}(2) \). We already know that the lift \( \lambda \) given by equation (A.10) is not a linear representation. We can calculate the corresponding coboundary \( c(g, h) \) for all \( g, h \in \text{SO}(3) \), but the question remains whether it is trivial or not. From the literature we can learn

\[
H^2(\text{SO}(3), \mathbb{C}^\times) \cong \mathbb{Z}_2. \quad (A.24)
\]

From the literature we also know that there exists no 2-dimensional linear representation of \( \text{SO}(3) \), and indeed \( c \) defines the non-trivial cohomology class. In fact, \( \text{SO}(3) \) has no even-dimensional representations at all. Only the odd dimensional ones descend.[11, Chap. 7]

\section{The Iwasawa decomposition}

The Iwasawa decomposition of a group factors a semisimple Lie group into closed subgroups. In this thesis we use it on the specific case \( \text{SL}(2, \mathbb{C}) \) to write general matrices as a product of easier-to-handle factors. More on the Iwasawa decomposition and a proof for the general case is readily available in literature, for example in [5, Chap. VI].

\subsection{The Iwasawa decomposition for \( \text{SL}(2, \mathbb{C}) \)}

The easiest case of the Iwasawa decomposition, \( \text{SL}(m, \mathbb{C}) \), uses the Gram-Schmidt orthonormalization method to split a matrix into a unitary part and an upper triangular part. We will only consider \( m = 2 \) here.

\textbf{Lemma B.1.} \textit{The Iwasawa decomposition of \( \text{SL}(2, \mathbb{C}) \) is \( \text{SL}(2, \mathbb{C}) = KAN \) with}

\[
K = \text{SU}(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \bigg| \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\} \quad (B.1)
\]
\[ A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \mid t \in \mathbb{R} \right\} \]  
(B.2)

\[ N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{C} \right\} \]  
(B.3)

**Proof.** First we note that \( K, A, N \) contain only matrices with determinant 1, so a multiplication of these will always be an elements of \( \text{SL}(2, \mathbb{C}) \).

Next we take a \( g \in \text{SL}(2, \mathbb{C}) \). Let \( \{e_1, e_2\} \) be the standard, orthonormal basis for \( \mathbb{C}^2 \) and let \((, )\) be the standard inner product. We consider the space \( \text{span}\{ge_1, ge_2\} \) with basis \( \{ge_1, ge_2\} \). With the Gram-Schmidt method this basis can be orthonormalized to a basis \( \{v_1, v_2\} \) with

\[
v_1 = \frac{ge_1}{\|ge_1\|}
\]

\[
v_2 = \frac{ge_2 - \frac{(ge_1, ge_2)}{\|ge_1\|^2} ge_1}{\|ge_2 - \frac{(ge_1, ge_2)}{\|ge_1\|^2} ge_1\|}
\]

Now we define a matrix \( k \) such that \( k^{-1}v_j = e_j \) (this is sending one basis to another, so it is invertible). Since the \( v_j \) are an orthonormal basis this implies

\[
\delta_{ij} = (v_i, v_j) = (ke_i, ke_j) = (k^* ke_i, e_j)
\]

The standard basis is orthonormal, hence \( k^* k = I_2 \). This means \( k \in \text{U}(2) \) and specifically \( \text{det}(k) = \pm 1 \).

Next consider \( k^{-1}g \). We can write out its action on the standard basis in terms of \( g \) only:

\[
k^{-1}ge_1 = \|ge_1\| e_1
\]

\[
k^{-1}ge_2 = \|ge_2\| - \frac{(ge_1, ge_2)}{\|ge_1\|^2} ge_1 + \frac{(ge_1, ge_2)}{\|ge_1\|} e_1
\]

So with respect to \( \{e_1, e_2\} \) the matrix looks like

\[
k^{-1}g = \begin{pmatrix} \|ge_1\| & \frac{(ge_1, ge_2)}{\|ge_1\|^2} \\ 0 & \|ge_2\| - \frac{(ge_1, ge_2)}{\|ge_1\|^2} ge_1 \end{pmatrix}
\]  
(B.4)

From this we see that \( \text{det}(k^{-1}g) > 0 \). We already know \( \text{det}(g) = 1 \) and \( \text{det}(k^{-1}) = \text{det}(k)^{-1} = \pm 1 \), so we conclude that \( \text{det}(k) = 1 \). Thus \( k \in \text{SU}(2) \).

Furthermore, this implies \( \text{det}(k^{-1}g) = 1 \) so we must have

\[
\|ge_2\| - \frac{(ge_1, ge_2)}{\|ge_1\|^2} ge_1 = \|ge_1\|^{-1}
\]  
(B.5)
This lets us rewrite

\[ k^{-1}g = \begin{pmatrix} \|ge_1\| & \frac{(ge_1, ge_2)}{\|ge_1\|} \\ 0 & \|ge_1\|^{-1} \end{pmatrix} \]

\[ = \begin{pmatrix} \|ge_1\| & 0 \\ 0 & \|ge_1\|^{-1} \end{pmatrix} \begin{pmatrix} 1 & \frac{(ge_1, ge_2)}{\|ge_1\|^2} \\ 0 & 1 \end{pmatrix} \]

These are matrices in \(A\) and \(N\), thus \(k^{-1}g \in AN\). We conclude \(g = k(k^{-1}g) \in KAN\), so the map \(K \times A \times N \to \text{SL}(2, \mathbb{C})\) given by matrix multiplication is surjective.

Note that if \(g \in \text{SL}(2, \mathbb{C})\), then so is \(g^{-1}\). We can use the Iwasawa decomposition to write \(g^{-1} = uan\), with \(u \in K\), \(a \in A\) and \(n \in N\). This gives \(g = (g^{-1})^{-1} = (uan)^{-1} = n^{-1}a^{-1}u^{-1}\), and since they are groups \(n^{-1} \in N\), \(a^{-1} \in A\) and \(u^{-1} \in K\). Thus \(\text{SL}(2, \mathbb{C})/\mathbb{C}\) = \(NA\) as well.

**B.2 The Iwasawa decomposition for \(\text{SO}(1, 3)\)**

We can translate this Iwasawa decomposition to \(\text{SO}(1, 3)\) by using the identification given by equation (5.6), i.e. \(\Lambda(a)(x) = axa\). Since \(\Lambda\) is a homomorphism, we get \(\text{SO}(1, 3) = \Lambda(K)\Lambda(A)\Lambda(N)\). The \(\Lambda\) sends \(\text{SL}(2, \mathbb{C})\) matrices to 4 \(\times\) 4-matrices working on \(\text{span}\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}\), so we can find the 4 \(\times\) 4-matrices explicitly by calculating how they map the \(\sigma_\mu\). We can do this calculation with a general element of each of these groups.

We start with

\[ K = \text{SU}(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \middle| \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}. \tag{B.6} \]

Note that for a general element \(K\) we have

\[ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}^\dagger = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix} \tag{B.7} \]

The images of the basis elements are the columns, so we get

- First column:
  \[ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \sigma_0 \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix} = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix} \]
  \[ = \begin{pmatrix} |\alpha|^2 + |\beta|^2 & 0 \\ 0 & |\alpha|^2 + |\beta|^2 \end{pmatrix} \]
  \[ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma_0 \]
• Second column:

\[
\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \sigma_1 \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix} = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix} \\
= \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} -\beta & \alpha \\ \bar{\alpha} & \bar{\beta} \end{pmatrix} \\
= \begin{pmatrix} -\alpha\beta - \bar{\alpha}\beta & \alpha^2 - \bar{\beta}^2 \\ \bar{\alpha}^2 - \beta^2 & \alpha\beta + \bar{\alpha}\beta \end{pmatrix} \\
= R(\bar{\alpha}^2 - \beta^2)\sigma_1 + I(\bar{\alpha}^2 - \beta^2)\sigma_2 - 2R(\alpha\beta)\sigma_3
\]

• Third column:

\[
\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \sigma_2 \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix} = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix} \\
= \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} i\beta & -i\alpha \\ i\alpha & i\beta \end{pmatrix} \\
= \begin{pmatrix} i(\alpha\beta - \bar{\alpha}\beta) & -i(\alpha^2 + \bar{\beta}^2) \\ i(\bar{\alpha}^2 + \beta^2) & -i(\alpha\beta - \bar{\alpha}\beta) \end{pmatrix} \\
= -I(\bar{\alpha}^2 + \beta^2)\sigma_1 + R(\bar{\alpha}^2 + \beta^2)\sigma_2 - 2I(\alpha\beta)\sigma_3
\]

• Fourth column:

\[
\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \sigma_3 \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix} = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix} \\
= \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & -\alpha \end{pmatrix} \\
= \begin{pmatrix} |\alpha|^2 - |\beta|^2 & -2\alpha\bar{\beta} \\ 2\bar{\alpha}\beta & -|\alpha|^2 + |\beta|^2 \end{pmatrix} \\
= 2R(\bar{\alpha}\beta)\sigma_1 + 2I(\bar{\alpha}\beta)\sigma_2 + (|\alpha|^2 - |\beta|^2)\sigma_3
\]

We denote the corresponding $4 \times 4$-matrix by $\Lambda(\alpha, \beta)$ as a shorthand. Then:

\[
\Lambda(\alpha, \beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & R(\bar{\alpha}^2 - \beta^2) & -I(\bar{\alpha}^2 + \beta^2) & 2R(\alpha\beta) \\ 0 & I(\bar{\alpha}^2 - \beta^2) & R(\bar{\alpha}^2 + \beta^2) & 2I(\alpha\beta) \\ 0 & -2R(\alpha\beta) & -2I(\alpha\beta) & (|\alpha|^2 - |\beta|^2) \end{pmatrix} \tag{B.8}
\]

Note that $\Lambda(\alpha, \beta) = \Lambda(-\alpha, -\beta)$ because all matrix entries depend on homogeneous polynomials in $\alpha, \beta$ of degree 2. In fact $\Lambda$ is 2-to-1 rather than 1-to-1 on $K$. To show this we set $\alpha = a + ib$ and $\beta = c + id$ and rewrite to

\[
\Lambda(a+ib, c+id) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a^2 - b^2 - c^2 + d^2 & 2(ab - cd) & 2(ac - bd) \\ 0 & -2(ab + cd) & a^2 - b^2 + c^2 - d^2 & 2(ad - bc) \\ 0 & -2(ac - bd) & -2(ad + bc) & a^2 + b^2 - c^2 - d^2 \end{pmatrix} \tag{B.9}
\]
Now suppose we have a $\Lambda = (\Lambda_{mn})_{m,n=0,1,2,3} \in \Lambda(K)$ and we want to know the preimage. Comparing with equation (B.20) gives equations in $a, b, c, d$ we can solve. We find $a^2 - b^2 = \frac{1}{4}(\Lambda_{11} + \Lambda_{22})$ and $ab = \frac{1}{4}(\Lambda_{12} - \Lambda_{21})$. Combined this gives a quadratic equation in $a^2$ which can be solved with the quadratic formula:

$$a^2 = \frac{1}{4}(\Lambda_{11} + \Lambda_{22}) \pm \sqrt{(\Lambda_{11} + \Lambda_{22})^2 + (\Lambda_{12} - \Lambda_{21})^2}$$

(B.10)

We know $a$ is real, so $a^2$ must be positive. Since certainly $\sqrt{(\Lambda_{11} + \Lambda_{22})^2 + (\Lambda_{12} - \Lambda_{21})^2}$, only the $+$-option is viable. That leaves two possibilities for $a$:

$$a = \pm \sqrt{\frac{1}{4}(\Lambda_{11} + \Lambda_{22}) \pm \sqrt{(\Lambda_{11} + \Lambda_{22})^2 + (\Lambda_{12} - \Lambda_{21})^2}}$$

(B.11)

Once we fix the sign of $a$, the other variables can be found using

$$b = \frac{\Lambda_{12} - \Lambda_{21}}{4a}, \quad c = -\frac{\Lambda_{23} + \Lambda_{32}}{4b}, \quad d = \frac{\Lambda_{23} - \Lambda_{32}}{4a}.$$ 

(B.12)

Thus $\Lambda(K) \simeq K/\{\pm\}$. As we will see in a bit $\Lambda$ is 1-to-1 on the other subgroups, so in it is $K$ specifically that makes $\Lambda$ a 2-to-1 map on $SL(2, \mathbb{C})$.

Next consider

$$A = \left\{ \left( \begin{array}{cc} e^t & 0 \\ 0 & e^{-t} \end{array} \right) \mid t \in \mathbb{R} \right\}$$

(B.13)

Note that

$$\left( \begin{array}{cc} e^t & 0 \\ 0 & e^{-t} \end{array} \right) \sigma_0 = \left( \begin{array}{cc} e^t & 0 \\ 0 & e^{-t} \end{array} \right)$$

(B.14)

- First column:

$$\left( \begin{array}{cc} e^t & 0 \\ 0 & e^{-t} \end{array} \right) \sigma_0 = \left( \begin{array}{cc} e^t & 0 \\ 0 & e^{-t} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} e^{2t} & 0 \\ 0 & e^{-2t} \end{array} \right) = \cosh(2t) \sigma_0 + \sinh(2t) \sigma_3$$

- Second column:

$$\left( \begin{array}{cc} e^t & 0 \\ 0 & e^{-t} \end{array} \right) \sigma_1 = \left( \begin{array}{cc} e^t & 0 \\ 0 & e^{-t} \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & e^t \\ e^{-t} & 0 \end{array} \right) = \sigma_1$$
• Third column:
\[
\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \sigma_2 \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2
\]

• Fourth column:
\[
\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \sigma_3 \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix} = \sinh(2t)\sigma_0 + \cosh(2t)\sigma_3
\]

We denote the corresponding 4 × 4-matrix by \( \Lambda(t) \):
\[
\Lambda(t) = \begin{pmatrix} \cosh(2t) & 0 & 0 & \sinh(2t) \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\sinh(2t) & 0 & 0 & \cosh(2t) \end{pmatrix}
\]

(B.15)

Since the sinus hyperbolicus is bijective on \( \mathbb{R} \), we have \( \Lambda(A) = \{ \Lambda(t) \mid t \in \mathbb{R} \} \simeq \mathbb{R} \), and since the exponential map is bijective on \( \mathbb{R} \) we find \( A \simeq \mathbb{R} \). Thus \( \Lambda(A) \simeq A \).

Finally we have
\[
N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{C} \right\}
\]

(B.16)

Note that
\[
\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 0 \\ \bar{b} & 1 \end{pmatrix}
\]

(B.17)

• First column:
\[
\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \sigma_0 \begin{pmatrix} 1 & 0 \\ \bar{b} & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{b} & 1 \end{pmatrix} = \begin{pmatrix} 1 + |b|^2 \\ 2b \end{pmatrix} \begin{pmatrix} \sigma_0 + R(b)\sigma_1 - I(b)\sigma_2 + \frac{|b|^2}{2} \sigma_3 \\ \sigma_0 + R(b)\sigma_1 - I(b)\sigma_2 + \frac{|b|^2}{2} \sigma_3 \end{pmatrix}
\]
• Second column:

\[
\begin{pmatrix}
1 & b \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\sigma_1 \\
\sigma_1
\end{pmatrix}
= \begin{pmatrix}
1 & b \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
1 & 0
\end{pmatrix}
= \begin{pmatrix}
1 & b \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
b & 1 \\
1 & 0
\end{pmatrix}
= \begin{pmatrix}
b + \bar{b} & 1 \\
1 & 0
\end{pmatrix}
= R(b)\sigma_0 + \sigma_1 + R(b)\sigma_3
\]

• Third column:

\[
\begin{pmatrix}
1 & b \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\sigma_2 \\
\sigma_2
\end{pmatrix}
= \begin{pmatrix}
1 & b \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
1 & 0
\end{pmatrix}
= \begin{pmatrix}
1 & b \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
-i\bar{b} & -i \\
i & 0
\end{pmatrix}
= \begin{pmatrix}
i b + i\bar{b} & -i \\
i & 0
\end{pmatrix}
= -I(b)\sigma_0 + \sigma_2 - I(b)\sigma_3
\]

• Fourth column:

\[
\begin{pmatrix}
1 & b \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\sigma_3 \\
\sigma_3
\end{pmatrix}
= \begin{pmatrix}
1 & b \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\bar{b} & 1
\end{pmatrix}
= \begin{pmatrix}
1 & b \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-\bar{b} & -1
\end{pmatrix}
= \begin{pmatrix}
1 - |b|^2 & -b \\
-\bar{b} & -1
\end{pmatrix}
= -\frac{|b|^2}{2}\sigma_0 - R(b)\sigma_1 + I(b)\sigma_2 + \left(1 - \frac{|b|^2}{2}\right)\sigma_3
\]

We use the shorthand \(\Lambda(b)\):

\[
\Lambda(b) = \begin{pmatrix}
1 + \frac{|b|^2}{2} & R(b) & -I(b) & -\frac{|b|^2}{2} \\
R(b) & 1 & 0 & -R(b) \\
-I(b) & 0 & 1 & I(b) \\
\frac{|b|^2}{2} & R(b) & -I(b) & 1 - \frac{|b|^2}{2}
\end{pmatrix}
\]

(B.18)

Then \(\Lambda(N) = \{\Lambda(b) \mid b \in \mathbb{C}\}\). The map \(\Lambda|_N : N \to \Lambda(N)\) is surjective by definition of \(\Lambda(N)\). It is also injective: if \(b \neq b'\), then \(R(b) \neq R(b')\) or \(I(b) \neq I(b')\) and for the explicit form of the matrix we see that both imply \(\Lambda(b) \neq \Lambda(b')\). Thus \(\Lambda(N) \simeq N\).
Because $\Lambda$ is 2-to-1 it does not have a proper inverse. However, we would like to find a preimage for some specific elements. With explicitly knowledge of $\Lambda$ this is reduced to a puzzle. As example, consider the rotation over $t$ around the third axis:

$$R_3(t) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos(t) & -\sin(t) & 0 \\
0 & \sin(t) & \cos(t) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \quad (B.19)$$

It looks like it might be an element of $\Lambda(K)$. What should we pick for $a, b, c, d$ in equation (B.20)? Judging by the entries that ought to be 0 or 1, setting $c, d = 0$ is a reasonable guess. This leaves us with

$$\Lambda(a + ib, c + id) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & a^2 - b^2 & 2ab & 0 \\
0 & -2ab & a^2 - b^2 & 0 \\
0 & 0 & 0 & a^2 + b^2
\end{pmatrix} \quad (B.20)$$

The expressions in $a$ and $b$ look similar to familiar trigonometric equations, and indeed $a = \cos\left(-\frac{t}{2}\right)$ and $b = \sin\left(-\frac{t}{2}\right)$ give the desired matrix. Thus $R_3(t) = \Lambda \left(\cos\left(-\frac{t}{2}\right) + i \sin\left(-\frac{t}{2}\right), 0\right)$ and

$$R_3(t) \sim \begin{pmatrix}
\cos\left(-\frac{t}{2}\right) + i \sin\left(-\frac{t}{2}\right) \\
0 \\
\cos\left(-\frac{t}{2}\right) - i \sin\left(-\frac{t}{2}\right)
\end{pmatrix}. \quad (B.21)$$
References


