On the nucleolus and the compromise value for communication situations

MASTER THESIS MATHEMATICS

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Chapter 1

Introduction

Game theory has started from an economic point of view with von Neumann and Morgenstern (1944). Since then, it has developed as a research field within economics, using mathematical tools and models. Nowadays, game theory can be divided into two parts: cooperative game theory and non-cooperative game theory. Both parts deal with interactions between individuals or parties, called players, in economic situations. In cooperative game theory, these situations are about cooperation, leading to joint profits. The important question in cooperative game theory is how to allocate the joint profits among the players. In contrast, non-cooperative game theory is about situations of conflict and competition. How to make the best decision or determine the best strategy if you do not know what the other players are going to do?

In a cooperative game with transferable utility, players can coordinate their actions and cooperate with each other. A certain combination of players is called a coalition and every coalition can make a joint profit. It is often assumed that all players eventually will cooperate and collect their joint profits. The question then rises how to divide these joint profits among the players taking into account the profits of all coalitions. There are several allocation rules to answer this question. The first one was developed by Shapley (1953) and is nowadays known as the Shapley value. The Shapley value is an average over the so-called marginal vectors, which are based on the marginal contributions of the players. Secondly, Schmeidler (1969) introduced the nucleolus, which is a lexicographical minimum over all vectors that divide the joint profits in such a way that every player receives at least the profit he can achieve on his own. The last allocation rule which is important in this thesis is the compromise value (Tijs, 1981). The compromise value is a compromise between an upper bound and a lower bound on what the players can demand from the joint profits.

Within the field of cooperative game theory, Myerson (1977) introduced the concept of communication situations, which is an extension of the model of a cooperative game with transferable utility. A communication situation consists of a cooperative game together with a communication graph. The graph describes the possibilities (and restrictions) for the communication between players. For every communication situation, one can
define a restricted game (cf. Myerson (1977)), which is again a cooperative game. This restricted game modifies the underlying cooperative game of a communication situation, by taking the communication restrictions into account.

The restricted game enables us to ‘solve’ a communication situation by applying allocation rules on the restricted game. Considerable research has been done to the application of the Shapley value in this context. In contrast, the nucleolus and the compromise value have not been previously associated with communication situations. Therefore, this thesis will focus on applying both the nucleolus and the compromise value on the restricted game for communication situations.

Another important issue regarding communication situations is the inheritance of certain properties. Since the restricted game is again a cooperative game, one can wonder under which conditions on the communication graph certain properties of the underlying cooperative game are inherited by the restricted game. Owen (1986) was the first who studied the inheritance of superadditivity. He showed that there are no conditions on the graph needed. Subsequently, Van den Nouweland and Borm (1991) studied the inheritance of convexity and balancedness. It turned out that for the inheritance of convexity, the communication graph must be cycle-complete. Balancedness is (just as superadditivity) always inherited.

This thesis will continue the study of the inheritance of properties, but focuses on properties which are related to the nucleolus or the compromise value. For the compromise value, compromise admissibility is an important notion, since the compromise value is only defined for compromise admissible games. In this thesis it is proven that the restricted game is always (so without any conditions on the communication graph) compromise admissible if the underlying game is compromise admissible. For the nucleolus, strong compromise admissibility and compromise stability are important, since for games satisfying one of these properties the nucleolus can be easily computed. It is proven that strong compromise admissibility is inherited if the communication graph is a 2-connected graph. Moreover, it is proven that only for those graphs the inheritance is guaranteed. This means that if a graph is not 2-connected, then there is a strongly compromise admissible game such that the restricted game is not strongly compromise admissible. Contrary to this, the inheritance of compromise stability is not straightforward. Without any extra conditions, several examples show that the inheritance of compromise stability is not guaranteed for several types of communication graphs. Still, the inheritance of compromise stability can be guaranteed, by imposing extra conditions on the underlying game.

The results about the inheritance of properties are used to derive results about the nucleolus and the compromise value of the restricted game. It can be shown that the nucleolus of the restricted game is exactly the same as the nucleolus of the underlying game under certain conditions. For example, this is true if the underlying game is strongly compromise admissible and the graph is a 2-connected graph. This also holds for the compromise value. Furthermore, for a star as communication graph it is proven that the compromise value assigns the middle player of a star more in the restricted game.
than in the underlying game. For the nucleolus, a similar statement is conjectured.

This thesis not only studies general cooperative games with specific types of communication graphs, but also covers a specific type of cooperative games, namely glove games. In a glove game, every player owns either a left-hand glove or a right-hand glove. One glove is worth nothing, while a pair of gloves (a left-hand glove together with a right-hand glove) has a worth of one. It is proven that glove games are compromise admissible and even strongly compromise admissible if there are more right-hand gloves than left-hand gloves (or visa versa). This is an extension of the result that those glove games are compromise stable from Tijs and Lipperts (1982). Furthermore, for every glove game the core, the nucleolus and the compromise value are determined, providing a good overview regarding glove games.

A glove game can be used as the underlying game in a communication situation. This kind of communication situations are called glove communication situations. For a glove communication situation with as communication graph either a star or a cycle, the nucleolus of the restricted game is determined. Also the compromise value of the restricted game is determined, in case the communication graph of a glove communication situation is a star or a 2-connected graph.

A special type of a communication graph is a random star. This is a star where every player has equal probability to become the middle player of the star. There are two (natural) ways to deal with this kind of glove communication situations. First, it is possible to determine the expected values of both the nucleoli and the compromise values. That is, to compute the average over the nucleoli and the compromise values of the restricted games corresponding to glove communication situations with a fixed star as communication graph. It is proven that in this case, both expected values coincide. Secondly, for a glove communication situation with a random star as communication graph an expected restricted game is derived. Using this expected restricted game, the nucleolus and the compromise value can be determined. An example shows that the nucleolus and the compromise value of this game do not coincide. However, they do coincide in the case there are as many left-hand glove players as right-hand glove players.

The notion of a glove game can be extended to an ingredient game, in which there are multiple ingredients needed in order to make a (delicious) meal. A single ingredient is worth nothing, a complete set of ingredients (that are needed for the meal) has a worth of one. Clearly, a meal with only two ingredients can be seen as a pair of gloves. Just as with glove games, also for ingredient games the core, the nucleolus and the compromise value are determined. These three results are generalisations of the results for glove games.

Also the idea of glove communication situations can be extended to ingredient communication situations, that is communication situations with an ingredient game as underlying game. For ingredient communication situations with a star as communication graph the nucleolus and the compromise value of the restricted game are characterized. In addition, the compromise value of the restricted game for an ingredient communication
situation with a 2-connected communication graph.

Last but not least, another specific type of cooperative games is studied in this thesis. For the class of unanimity games it is proven that the nucleolus and the compromise value coincide. Interestingly, it is seen that for unanimity communication situations (so with a unanimity game as underlying game) the nucleolus and the compromise value of the restricted game coincide too. For the unanimity games as well for the unanimity communication situations the results explicitly determine both solution concepts.

This thesis is structured in the following way. Chapter 2 is devoted to an overview of the basic definitions and notions within cooperative game theory, graph theory and communication situations. In Chapter 3 the inheritance of properties in communications situations is studied. The results about the inheritance of properties obtained are used in Chapter 4 and Chapter 5. These two chapters contain several results about the nucleolus and the compromise value of the restricted game in relation to the nucleolus and the compromise value of the underlying cooperative game.

Chapter 6 is about glove games and Chapters 7 and 8 are about glove communication situations. The former chapter contains results with either a star, a cycle or a 2-connected graph as communication graph. The latter discusses glove communication situations with a random star as communication graph. Chapters 9 and 10 are about ingredient games and ingredient communication situations, respectively. For the ingredient communication situations, only a star and a 2-connected communication graph are discussed.

The last chapter is about unanimity games and unanimity communication situations. Here, the result about unanimity communication situations contains all possible communication graphs.
Chapter 2

Preliminaries

This chapter consists of three parts of preliminaries. The first part provides the basic notions within the field of cooperative game theory. That is, the basic notions which are important for our purposes. The second part is about graph theory and gives a brief overview of the notions that are needed for the third part. The third part is about communication situations, which is an extension of cooperative games. Together, these three parts provide a complete introduction to this thesis.

2.1 Cooperative games

A cooperative game, also called a transferable utility game (TU-game), is a pair \((N,v)\) where \(N\) is a non-empty, finite set of players and \(v : 2^N \to \mathbb{R}\) a function with \(v(\emptyset) = 0\). Here, \(2^N\) is the notation for the powerset of \(N\), the collection of all subsets of \(N\). The set of all TU-games for a fixed \(N\) is denoted by \(TU^N\). The function \(v\), often called the characteristic function, assigns to each subset \(S \subseteq N\) (which is called a coalition) a number \(v(S) \in \mathbb{R}\), which is called the value or the worth of the coalition. The coalition containing all players, \(N\), is referred to as the grand coalition. For convenience, the players are numbered from 1 to \(|N| = n\).

A special class of cooperative games is the class of unanimity games. For every \(T \in 2^N \setminus \{\emptyset\}\) the unanimity game \(u_T \in TU^N\) is for every \(S \in 2^N \setminus \{\emptyset\}\) defined as

\[
u_T(S) := \begin{cases} 1 & \text{if } T \subseteq S; \\ 0 & \text{otherwise.} \end{cases}
\]

The importance of unanimity games lies in the fact that every TU-game can be written in a unique way as a linear combination of unanimity games.

A game \(v \in TU^N\) is called non-negative if \(v(S) \geq 0\) for every \(S \in 2^N\), monotonic if \(v(S) \leq v(T)\) for every \(S, T \in 2^N\) with \(S \subseteq T\) and superadditive if \(v(S \cup T) \geq v(S) + v(T)\) for every \(S, T \in 2^N\) with \(S \cap T = \emptyset\). If equality holds in this last equation for every
Define two sets $S, T \in 2^N$ with $S \cap T = \emptyset$, the game $v$ is called additive. Furthermore, a game $v \in TU_N$ is called convex if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for every $S, T \in 2^N$ and zero-normalized if $v(\{i\}) = 0$ for every $i \in N$. Additive games are convex and convex games are superadditive, whence it follows that additive games are superadditive. A game that is both superadditive and non-negative is also monotonic. A game that is both superadditive and zero-normalized is also non-negative. Putting this together, superadditive and zero-normalized games are monotonic.

The following example consists of two games, which emphasizes the need of the non-negativity condition in the result above.

**Example 2.1** Consider the two three-person games $v_1$ and $v_2$ from Table 2.1. Clearly, the game $v_1$ is monotonic (the value increases when the coalition size increases) and non-negative, but it is not superadditive. To see this, take $S = \{1, 2\}$ and $T = \{3\}$ and notice that $v(S \cup T) = v(N) = 3 \geq 4 = v(S) + v(T)$.

In contrary, the game $v_2$ is superadditive. In fact, $v_2$ is even convex (check all $S, T \in 2^N$). It is clear that $v_2$ is not non-negative and monotonic. The former is due to $\{2\}$ and the latter to $\{1\} \subseteq \{1, 2\}$. Note that both games are not zero-normalized.

<table>
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<th>${3}$</th>
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</table>

Table 2.1 – Two examples of a cooperative game

An important issue in cooperative game theory is how to allocate the value of the grand coalition among the players. In order to ‘solve’ this problem, one can come up with reasonable requirements a solution should satisfy. Such a reasonable requirement is for example the stability condition. Before defining a solution concept that satisfies the stability condition (namely the core), first the imputation set $I(v)$ for a game $v \in TU_N$ is defined as

$$I(v) := \left\{ x \in \mathbb{R}^N \middle| \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for every } i \in N \right\}.$$ 

The first condition is called efficiency and the second condition individual rationality. The core can now be defined as an extension of the imputation set, namely

$$C(v) := \left\{ x \in \mathbb{R}^N \middle| \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for every } S \in 2^N \setminus \emptyset \right\}.$$ 

The latter condition is the stability condition. This means that the core consists of vectors which divide the total value of the grand coalition in such a way that each coalition is ‘happy’ in some sense. In other words, there is no coalition $S \in 2^N$ such that...
In order to define the Shapley value, let \( \Pi(\) be the set of all permutations of a player set \( N \), i.e. \( \sigma \in \Pi(N) \) denotes a function \( \sigma : \{1, 2, \ldots, |N| = n\} \rightarrow N \). Given \( v \in \mathbb{T}U^N \) and \( \sigma \in \Pi(N) \), the marginal vector \( m^\sigma(v) \in \mathbb{R}^N \) is given by

\[
m^\sigma_{\sigma(k)}(v) := v(\{\sigma(1), \ldots, \sigma(k)\}) - v(\{\sigma(1), \ldots, \sigma(k-1)\})
\]

for every \( k \in N \). Now, for a game \( v \in \mathbb{T}U^N \) the Shapley value (Shapley, 1953), \( \Phi(v) \in \mathbb{R}^N \), is defined as

\[
\Phi(v) := \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^\sigma(v),
\]

that is the average of all marginal vectors. Alternatively, for every \( i \in N \) the Shapley value can be rewritten to

\[
\Phi_i(v) := \sum_{\substack{S \in 2^N: \ i \not\in S \atop |S| < |N| - 1}} \frac{1}{|N| \cdot (|N| - 1)} \left(v(S \cup \{i\}) - v(S)\right). \tag{2.1}
\]

For a game \( v \in \mathbb{T}U^N \) with \( I(v) \neq \emptyset \), an imputation \( x \in I(v) \) and a coalition \( S \in 2^N \) the excess \( E(S, x, v) \) (or simply \( E(S, x) \)) of coalition \( S \) with respect to \( x \) is defined as \( E(S, x, v) := v(S) - \sum_{i \in S} x_i \). Together with these excesses, there is an excess vector \( \theta(x) \in \mathbb{R}^{2^N} \) defined as the vector with on every coordinate the excess of a certain coalition \( S \in 2^N \) with respect to \( x \) denoted in non-increasing order, i.e. \( \theta(x)_k \geq \theta(x)_{k+1} \) for every \( k \in \{1, \ldots, 2^N - 1\} \). Now, a one-point solution concept for a game \( v \in \mathbb{T}U^N \) with \( I(v) \neq \emptyset \) is the nucleolus (Schmeidler, 1969), \( \text{nuc}(v) \in \mathbb{R}^N \), defined as the unique imputation such that \( \theta(\text{nuc}(v)) \preceq \theta(x) \) for all \( x \in I(v) \), where \( \preceq \) denotes the lexicographical order. Note that for every \( v \in \mathbb{T}U^N \) and every \( x \in C(v) \) it holds that \( E(S, x, v) \leq 0 \) for every \( S \in 2^N \). Furthermore, if \( C(v) \neq \emptyset \), then \( \text{nuc}(v) \in C(v) \) for a game \( v \in \mathbb{T}U^N \) with \( I(v) \neq \emptyset \).

There is no direct formula for the nucleolus. Fortunately, there is a way to check whether a certain imputation is the nucleolus. This check is called the Kohlberg criterion (Kohlberg, 1971). Before stating this criterion, first call a weight function \( \lambda : 2^N \setminus \{\emptyset\} \rightarrow \mathbb{R}_+ \) balanced if for every \( i \in N \) it holds that

\[
\sum_{S \in 2^N \setminus \{\emptyset\}: i \in S} \lambda(S) = 1.
\]

A collection \( B \subseteq 2^N \setminus \{\emptyset\} \) is called balanced if there is a balanced weight function \( \lambda \) such that \( B = B(\lambda) := \{ S \in 2^N \setminus \{\emptyset\} | \lambda(S) > 0 \} \).
Now, the Kohlberg criterion (Kohlberg, 1971) states that for a balanced game \( v \in TU^N \) a certain imputation \( x \in I(v) \) is the nucleolus, if the collection \( \bigcup_{k=1}^{t(x)} B_k(x) \) is balanced for every \( 1 \leq s \leq t(x) \) with \( t(x) \) the unique number such that \( B_k(x) \neq \emptyset \) for all \( k \in \{1, \ldots, t(x)\} \) and \( B_{t(x)+1}(x) = \emptyset \). Here, \( B_k(x) \) is defined recursively by:

\[
B_1(x) := \left\{ S \in 2^N \setminus \{\emptyset, N\} \mid E(S, x, v) \geq E(T, x, v) \text{ for all } T \in 2^N \setminus \{\emptyset, N\} \right\}
\]

and for every \( 2 \leq k \leq t(x) \):

\[
B_k(x) := \left\{ S \in 2^N \setminus \{\emptyset, N\} \mid S \notin \bigcup_{r=1}^{k-1} B_r(x) \text{ and } E(S, x, v) \geq E(T, x, v)
\]

\[
\text{for all } T \in 2^N \setminus \{\emptyset, N\} \text{ with } T \notin \bigcup_{r=1}^{k-1} B_r(x) \right\}.
\]

Just like cooperative games, one-point solution concepts (like the Shapley value and the nucleolus) have certain properties they may or may not satisfy. A one-point solution concept \( \Psi \) satisfies efficiency if \( \sum_{i \in N} \Psi_i(v) = v(N) \) for every \( v \in TU^N \). Furthermore, \( \Psi \) satisfies symmetry if \( \Psi_i(v) = \Psi_j(v) \) for every \( v \in TU^N \) and every symmetric players \( i, j \in N \) in \( v \) satisfying \( v(S \cup \{i\}) = v(S \cup \{j\}) \) for every \( S \in 2^N \setminus \{\emptyset\} \) with \( i, j \notin S \). Eventually, \( \Psi \) satisfies the dummy property if \( \Psi_i(v) = v(\{i\}) \) for every \( v \in TU^N \) and every dummy player \( i \in N \) in \( v \) satisfying \( v(S \cup \{i\}) - v(S) = v(\{i\}) \) for every \( S \in 2^N \setminus \{\emptyset\} \) with \( i \notin S \). Both the Shapley value and the nucleolus satisfy efficiency, symmetry and the dummy property.

The third one-point solution we discuss in this thesis is the compromise value. This involves some more definitions, e.g. the utopia-vector and the minimum right vector. For a game \( v \in TU^N \) the utopia-vector \( M(v) \) is defined as

\[
M_i(v) := v(N) - v(N \setminus \{i\}) \quad \text{for every } i \in N
\]

and the minimum right vector as

\[
m_i(v) := \max_{S \subseteq N, i \in S} \left\{ v(S) - \sum_{j \in S, j \neq i} M_j(v) \right\} \quad \text{for every } i \in N.
\]

Driessen (1988) provides a relation between both the utopia-vector and the minimum right vector and the core: for every \( v \in TU^N \) with \( C(v) \neq \emptyset \) and every \( x \in C(v) \) it holds that \( m(v) \leq x \leq M(v) \). The core cover (cf. Tijs and Lipperts (1982)) is the set of efficient vectors that lie between the minimum right vector and the utopia-vector. More explicitly:

\[
CC(v) := \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N) \text{ and } m(v) \leq x \leq M(v) \right\}.
\]
The compromise value is only defined for games that are compromise admissible, that is for a game 
\( v \in TU^N \) with \( CC(v) \neq \emptyset \). The set of all compromise admissible games for a 
fixed \( N \) is denoted by \( CA^N \). Note that for every \( v \in TU^N \) it holds that \( C(v) \subseteq CC(v) \) and that the core cover is non-empty if and only if \( m(v) \leq M(v) \) and \( \sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v) \). Moreover, \( C(v) = CC(v) \) for every \( v \in TU^N \) with \( |N| = 3 \).

Now the compromise value (Tijs, 1981), \( \tau(v) \in \mathbb{R}^N \), can be defined for a game \( v \in CA^N \) as the unique efficient element on the line segment between \( m(v) \) and \( M(v) \), that is \( \tau(v) := \alpha M(v) + (1 - \alpha)m(v) \) with \( \alpha \in [0, 1] \) such that \( \sum_{i \in N} \tau_i(v) = v(N) \). The compromise value by definition satisfies efficiency (on the class of compromise admissible games). It also satisfies symmetry and the dummy property on \( CA^N \).

A game \( v \in TU^N \) is called compromise stable if \( v \in CA^N \) and \( C(v) = CC(v) \). Alternatively (cf. Quant et al. (2005)), a compromise admissible game is compromise stable if for every \( S \in 2^N \setminus \{\emptyset\} \) it holds that

\[
v(S) \leq \max \left\{ \sum_{i \in S} m_i(v), v(N) - \sum_{i \notin N \setminus S} M_i(v) \right\}.
\] (2.2)

The last class of games introduced here, is the class of strongly compromise admissible games. Therefore, the notion of the gap is needed, for a game \( v \in TU^N \) defined as

\[
g^v(S) := \sum_{i \in S} M_i(v) - v(S) \quad \text{for every } S \in 2^N \setminus \{\emptyset\}.
\]

The gap can be seen as a function \( g^v : 2^N \setminus \{\emptyset\} \to \mathbb{R} \). A game \( v \in TU^N \) is called strongly compromise admissible (cf. Driessen (1988)) if \( v \in CA^N \) and \( g^v(N) \leq g^v(S) \) for all \( S \in 2^N \setminus \{\emptyset\} \). The set of all strongly compromise admissible games is denoted by \( SCA^N \). Using equation (2.2) it can be proven that every strongly compromise admissible game is compromise stable.

Example 2.2 An easy (and still interesting) example of a TU-game is the so-called glove game. There are three players (so \( N = \{1, 2, 3\} \)) and they each own one glove. Players 1 and 2 own a left-hand glove, while player 3 has a right-hand glove. A pair of gloves has a worth of 1. In Table 2.2 the characteristic function \( v \) is given.

It can be checked that \( v \in SCA^N \). Furthermore, the Shapley value is given by \( \Phi(v) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \) and the nucleolus and the compromise value are both given by \( nuc(v) = \tau(v) = (0, 0, 1) \). The core consists of only one vector: \( C(v) = \{(0,0,1)\} \).

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Table 2.2 – Glove game

The last part of this section is about bankruptcy problems (O’Neill, 1982), which are very useful for the computation of the nucleolus. A bankruptcy problem is a triple \((N,A,c)\)
where \( N \) is a non-empty, finite set of claimants, \( A \in \mathbb{R} \) with \( A \geq 0 \) a certain value (e.g. an amount of money) and \( c_i \in \mathbb{R}^N \) with \( c_i \geq 0 \) for every \( i \in N \) the claims of the claimants on \( A \). The set of all bankruptcy problems for a certain \( N \) is denoted by \( BR^N \).

To justify the name, we assume that \( \sum_{i \in N} c_i \geq A \). There are different ways to ‘solve’ a bankruptcy problem. For our purposes only two of them will be relevant. A solution for \((N, A, c) \in BR^N \), also called a rule, is a vector of length \(|N|\) which divides the total value \( A \) among the claimants. Every claimant receives something between zero and their claim.

The first rule is the \textit{constrained equal awards rule}, abbreviated to \textit{CEA}, and for every bankruptcy problem \((N, A, c) \in BR^N \) defined as

\[
CEA_i(A, c) := \min \{ \alpha, c_i \} \quad \text{for every } i \in N,
\]

with \( \alpha \in \mathbb{R} \) such that \( \sum_{i \in N} \min \{ \alpha, c_i \} = A \). For a bankruptcy problem \((N, A, c)\) the \textit{Talmud rule} (cf. Aumann and Maschler (1985)), \textit{TAL}, is defined as

\[
TAL(A, c) := \begin{cases} 
CEA(A, \frac{1}{2} c) & \text{if } \sum_{i \in N} c_i \geq 2A; \\
 c - CEA(\sum_{i \in N} c_i - A, \frac{1}{2} c) & \text{if } \sum_{i \in N} c_i \leq 2A.
\end{cases}
\]

Note that if \( \sum_{i \in N} c_i = 2A \) for \((N, A, c) \in BR^N \), we have that \( \sum_{i \in N} \frac{1}{2} c_i = A \). Thus the upper-case and the lower-case both lead to the solution \( \frac{1}{2} c \).

Quant et al. (2005) developed a direct formula for the nucleolus if the game is compromise stable.

**Theorem 2.3** Let \( v \in TU^N \) be a compromise stable game. Then the nucleolus is given by

\[
nuc(v) = m(v) + TAL(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)).
\]

For strongly compromise admissible games there is an even simpler expression for the nucleolus, due to Driessen (1988). He showed that this expression also holds for the compromise value, which proves that the nucleolus and the compromise value coincide for strongly compromise admissible games.

**Theorem 2.4** Let \( v \in TU^N \) be a strongly compromise admissible game. Then the compromise value and nucleolus are given by

\[
\tau(v) = nuc(v) = M(v) - \frac{1}{|N|} g^v(N)c^N.
\]

### 2.2 Graphs

In this section, a basic introduction in the field of Graph Theory is given. This is useful for the rest of this thesis, in particular for the following section about communication...
situations. A complete introduction and much more on Graph Theory can be found e.g. in the book of Bondy and Murty (2008).

A(n) (undirected) graph \( G = (N, E) \) consists of a non-empty, finite set of nodes, \( N \), and a finite (possibly empty) set of edges, \( E \). An edge \( e \in E \) from \( x \in N \) to \( y \in N \) is denoted by an unordered pair \( e = \{x, y\} \). For \( S \subseteq N \), \( S \neq \emptyset \) the subgraph \( G_S = (S, E_S) \) is defined as the graph consisting of the set of nodes \( S \) and the set of edges \( E_S = \{\{x, y\} \in E \mid x, y \in S\} \).

Given a graph \( G = (N, E) \), a sequence of nodes \((x_0, x_1, \ldots, x_k)\) is called a path if for each \( 1 \leq i \leq k \) there is an edge \( e = \{x_{i-1}, x_i\} \in E \) and \( x_i \neq x_j \) for all \( i, j \in \{0, \ldots, k\} \) with \( i \neq j \). The number \( k \) is the length of the path. A cycle is a sequence of nodes \((x_0, x_1, \ldots, x_k, x_0)\) with \( k \geq 2 \) such that \( \{x_k, x_0\} \in E \) and \((x_0, x_1, \ldots, x_k)\) is a path. Here, the length of the cycle is \( k + 1 \).

Throughout this thesis we assume that all graphs are simple, that is no loops (i.e. for every \( x \in N \) there is no edge \( e = \{x, x\} \)) and no parallel edges (i.e. for every \( x, y \in N \) there is at most one edge \( e = \{x, y\} \in E \)). Furthermore, we will assume that all graphs are connected: there is a path between each two nodes. By convention, the trivial graph consisting of only one node (and no edges) is connected.

There are several special types of graphs which each have interesting properties and will be used later on. A complete graph is a graph having an edge between each pair of nodes, i.e. \( \{x, y\} \in E \) for every \( x, y \in N \). A graph is called 2-connected if \( |N| \geq 3 \) and for every \( x \in N \) the subgraph \( G_{N \setminus \{x\}} \) is connected. A tree is a graph without any cycles and a star is a graph with one special node, say \( x \in N \), such that \( E = \{\{x, y\} \mid y \in N \setminus \{x\}\} \).

A graph is called a cycle if there is a sequence of nodes \((x_1, x_2, \ldots, x_{|N|}, x_1)\) which is a cycle and any other cycle is also of length \( |N| + 1 \). A graph is called a street if there is a sequence of nodes \((x_1, x_2, \ldots, x_{|N|})\) which is a path and does not contain any cycles. Eventually, a graph is called cycle-complete if for every cycle \( C \) in \( G \) it holds that the subgraph \( G_C \) is a complete graph.

In Figure 2.1 some special graphs are shown. Note that a star and a street are special
cases of a tree. Both a cycle and a complete graph are 2-connected. Furthermore, complete graphs as well as trees are cycle-complete. Consequently, a star and a street are also cycle-complete.

For the next section, the notion of components is an important one. For a graph \( G = (N, E) \), a component \( C \subseteq N, C \neq \emptyset \), is a maximal (inclusion-wise) subset of nodes such that the subgraph \( (C, E_C) \) is connected. The set of all components is denoted by \( N/E \). This notion can be extended to subgraphs. This means that for a given graph \( G = (N, E) \) and \( S \subseteq V, S \neq \emptyset \), the set of all components of the subgraph \( G_S = (S, E_S) \) is (also) denoted by \( S/E \).

### 2.3 Communication situations

The main part of this thesis is about communication situations. A communication situation (Myerson, 1977) is a triple \((N,v,E)\) where \((N,v)\) is a superadditive and zero-normalized TU-game and \((N,E)\) is a connected and simple graph. The set of all communication situations is denoted by \( CS^N \). The interpretation here is that the edges of the graph \((N,E)\) reflect the possibility to communicate with each other. Players can communicate directly if they are connected via an edge or indirectly if they are connected via a path of edges in the graph. Most of the time, the set of players will be clear from the context. Therefore, the \( N \) is often omitted while referring to the graph \((N,E)\).

Throughout this thesis we will assume that each communication graph in a communication situation is a connected and simple graph, as stated in the definition. This is not always the case. Requiring a simple graph is not very limiting, since parallel edges or loops do not increase or decrease the possibility to communicate with other players. Connectivity is justified with the argument that for disconnected graphs there are in fact multiple communication situations to explore. Each component is separated from every other component, so only players within the same component can communicate with each other.

For each communication situation \((N,v,E)\) Myerson (1977) introduced a corresponding communication game \( v^E \in TU^N \), also called the (node or graph) restricted game. This game \( v^E \) is defined as

\[
v^E(S) := \sum_{C \in S/E} v(C) \quad \text{for every } S \in 2^N \setminus \{\emptyset\}.
\]

(2.3)

For a communication situation, the underlying TU-game is assumed to be superadditive and zero-normalized. The latter assumption is very common, since it is known that every game is in some way equivalent to a zero-normalized game. Superadditivity is reasonable to require in the light of the definition of the restricted game. The following example shows that the superadditivity condition implies that a communication graph with more edges results in higher values of the restricted game. This seems a reasonable assumption, since more edges increase the communication possibilities.
Example 2.5 Consider the two communication situations \((N, v, E), (N, v, \hat{E}) \in CS^N\) with \(N = \{1, 2, 3, 4, 5\}\), a few values \(v(S)\) as given in Table 2.3, the graph \((N, E)\) (without the dashed edge) and the graph \((N, \hat{E})\) (including the dashed edge) as depicted in Figure 2.2. Some values of the restricted games are calculated with equation (2.3) and also given in Table 2.3.

In the graph \(\hat{E}\), the players from the coalition \(\{1, 2, 3, 4\}\) are connected such that the value of the restricted game \(v(\hat{E}(\{1, 2, 3, 4\})) = v(\{1, 2, 3, 4\}) = 5\), while the value of the restricted game corresponding to the graph \(E\) results in \(v(E(\{1, 2, 3, 4\})) = v(\{1, 2\}) + v(\{3, 4\}) = 3\). Superadditivity requires the former value to be greater than or equal to the latter value.

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
S & |S| = 1 & \{1, 2\} & \{1, 3\} & \{3, 4\} & \{1, 3, 4\} & \{1, 2, 3, 4\} & \{2, 3, 4, 5\} & N \\
\hline
v(S) & 0 & 1 & 1 & 2 & 3 & 5 & 7 & 10 \\
v^{E}(S) & 0 & 1 & 0 & 2 & 2 & 3 & 7 & 10 \\
v^{\hat{E}}(S) & 0 & 1 & 1 & 2 & 3 & 5 & 7 & 10 \\
\hline
\end{array}
\]

Table 2.3 – Some values of \(v\), \(v^{E}\) and \(v^{\hat{E}}\)

![Figure 2.2 – A graph E (without dashed edge) and the graph \(\hat{E}\) with the dashed edge](image-url)
Chapter 3

Inheritance of properties in communication situations

During the last four decades, the inheritance of several properties of cooperative games is studied. Given a TU-game with a certain property and a special type of graph, does the restricted game also have that property? E.g., for superadditivity and balancedness there are nice results. Before proving those results, there are some immediate consequences of the definitions as given in Section 2.3.

Consider a communication situation \((N, v, E)\) and the corresponding restricted game \(v^E\). If the graph \(E\) is a complete graph, then the game \(v^E\) is just the original game \(v\). This holds because a complete graph \(E\) implies that \(S/E = \{S\}\) and thus \(v^E(S) = v(S)\) for every \(S \in 2^N\). Note that this last part also holds in general. This means that if the subgraph \(G_S\) for a certain coalition \(S \in 2^N \setminus \{\emptyset\}\) is connected, then \(S/E = \{S\}\) and again \(v^E(S) = v(S)\). In particular for the grand coalition \(N \in 2^N\) and the singletons \(\{i\} \in 2^N\) for \(i \in N\), we have \(v^E(N) = v(N)\) and \(v^E(\{i\}) = v(\{i\})\). This last equality means that the restricted game \(v^E\) is zero-normalized, since \(v\) is zero-normalized.

There is a direct relation between the values of a certain TU-game and the values of the restricted game corresponding to a communication situation. This relation is stated in Lemma 3.1 below.

**Lemma 3.1** Let \((N, v, E) \in CS^N\) be a communication situation. Then for every \(S \in 2^N\) it holds that \(v^E(S) \leq v(S)\).

**Proof:** Let \(S \in 2^N\) be a coalition. Since two components \(C, C' \in S/E\) are disjoint, the result follows by repeatedly applying the superadditivity condition of \(v\):

\[
v^E(S) = \sum_{C \in S/E} v(C) \leq v(\bigcup_{C \in S/E} C) = v(S).
\]

\(\square\)
3.1 Superadditivity and balancedness

As mentioned before, there is a strong result regarding the superadditivity of the restricted game of a certain communication situation. Since the underlying TU-game is superadditive by definition, the restricted game is also superadditive. There are no conditions needed for the graph (except for the required connectedness and simpleness). The result is due to Owen (1986).

**Proposition 3.2** Let \((N, v, E) \in CS^N\) be a communication situation. Then the restricted game \(v^E\) is superadditive.

**Proof:** To prove superadditivity, let \(S, T \in 2^N\) be two arbitrary coalitions such that \(S \cap T = \emptyset\). We want to prove that \(v^E(S \cup T) \geq v^E(S) + v^E(T)\), which is, with equation (2.3), equivalent to

\[
\sum_{C \in (S \cup T)/E} v(C) \geq \sum_{A \in S/E} v(A) + \sum_{B \in T/E} v(B). \tag{3.1}
\]

The idea is to split each component \(C \in (S \cup T)/E\) into a disjoint union of components in \(S/E\) and \(T/E\). This can be done by writing \(C = C_S \cup C_T\) where \(C_S \subseteq S\) are the nodes from \(S\) and \(C_T \subseteq T\) the nodes from \(T\). This is possible, because \(S\) and \(T\) are disjoint. An important observation is that each component of \(C_S/E\) (as well as each component of \(C_T/E\)) is also a component of \(S/E\) (respectively \(T/E\)). This holds because initially we started with a component \(C\), which is a maximal subset of \(S \cup T\) such that the subgraph on \(S \cup T\) is connected. Hence no nodes from \(S \cup T\), and in particular from \(S\), can be added to \(C\) without disconnecting the subgraph.

Now consider the disjoint union of components in \(C_S/E\) and \(C_T/E\), which is equal to \(C\), i.e.

\[
C = \bigcup_{A \in C_S/E} A \cup \bigcup_{B \in C_T/E} B. \tag{3.2}
\]

To check this equality, take an arbitrary node \(x \in C = C_S \cup C_T\). Hence \(x \in C_S\) or \(x \in C_T\) and in each case there is a component in \(C_S/E\) or \(C_T/E\) respectively such that \(x\) lies in that component. For the other direction take a node \(x\) in the disjoint union of the components. Then \(x\) lies in a certain component, which implies that \(x \in C_S\) or \(x \in C_T\) depending on the component. This means that \(x \in C_S \cup C_T = C\), which completes the check of the equality in equation (3.2).

We can now repeatedly apply the superadditivity of \(v\) to obtain

\[
v(C) \geq \sum_{A \in C_S/E} v(A) + \sum_{B \in C_T/E} v(B). \tag{3.3}
\]

Recall the important observation that each component \(A\) (and \(B\)) are also components of \(S/E\) (and \(T/E\)). By repeating the above argument for each \(C \in (S \cup T)/E\) we cover all the components in \(S/E\) and \(T/E\). Adding all the inequalities from equation (3.3) results in the required inequality from equation (3.1), which completes the proof. \(\square\)
Recall that the restricted game is zero-normalized, since the underlying game was assumed to be zero-normalized for communication situations. With Proposition 3.2 we also have superadditivity for the restricted game. The combination of superadditivity and non-negativity results in a non-negative restricted game. Furthermore, the restricted game is monotonic. No further proof is needed, since it is just a TU-game and therefore the known results hold.

If one reformulate Proposition 3.2, it turns out to be a proposition about the inheritance of superadditivity. If a game \( v \in \text{TU}_N \) is superadditive, then also the restricted game \( v^E \) is superadditive, regardless of the graph \( E \). There is a similar statement about balancedness, as first remarked by Van den Nouweland and Borm (1991).

**Proposition 3.3** Let \((N, v, E) \in \text{CS}_N\) be a communication situation with \( v \) a balanced game. Then the restricted game \( v^E \) is balanced.

**Proof:** Take an arbitrary \( x \in C(v) \), which is possible since \( C(v) \neq \emptyset \). We are going to prove that \( x \in C(v^E) \), which implies that \( C(v^E) \neq \emptyset \). Efficiency is satisfied since

\[
\sum_{i \in N} x_i = v(N) = v^E(N).
\]

In order to prove stability, let \( S \in 2^N \setminus \{\emptyset\} \) be an arbitrary coalition. Using Lemma 3.1 and stability for \( v \), we have that

\[
v^E(S) \leq v(S) \leq \sum_{i \in S} x_i.
\]

Thus \( x \in C(v^E) \), which proves the statement. \( \square \)

In the proof of this proposition it is proven that \( C(v) \subseteq C(v^E) \). The other direction is not true, as the following example shows.

<table>
<thead>
<tr>
<th>( S )</th>
<th>{1}</th>
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<th>{3}</th>
<th>{1,2}</th>
<th>{1,3}</th>
<th>{2,3}</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v(S) )</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>( v^E(S) )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

**Table 3.1** – A three-person game and its restricted game

**Example 3.4** Consider the communication situation \((N, v, E) \in \text{CS}_N\) with \( v \in \text{TU}_N \) from Table 3.1 and \( E \) the graph from Figure 3.1. Then the restricted games has the same values as the game \( v \), except for coalition \( \{2,3\} \). The values of the restricted game are also shown in Table 3.1.

Let \( x = (x_1, x_2, x_3) \in C(v) \). Because of stability it holds that \( 0 = v(\{1\}) \leq x_1 \leq 0 \), where the last inequality follows from the fact that \( x_2 + x_3 \geq v(\{2,3\}) = 3 \) and \( x_1 + x_2 + x_3 = 3 \) (efficiency). We thus conclude that \( x_1 = 0 \) for any \( x \in C(v) \).
In contrary, $(3, 0, 0) \in C(v^E)$. Efficiency holds for sure and also stability holds. This proves that $C(v^E) \notin C(v)$.

\[\begin{array}{c}
2 \\
1 \\
3
\end{array}\]

Figure 3.1 – A three-person communication graph

### 3.2 Compromise admissibility

In the previous section, Proposition 3.2 and Proposition 3.3 show that certain properties of the underlying game are inherited by the restricted game. For superadditivity and balancedness there was no extra condition on the communication graph needed. To explore whether or not compromise admissibility is inherited, we are first going to look at the minimum right vector and the utopia-vector.

**Lemma 3.5** Let $(N, v, E) \in C_S N$ be a communication situation. Then it holds that

1. $M(v^E) \geq M(v)$;
2. $m(v^E) \leq m(v)$.

**Proof:** Let $i \in N$ be an arbitrary player. To prove the first statement, use Lemma 3.1 such that it follows that

\[
M_i(v^E) = v^E(N) - v^E(N \setminus \{i\}) = v(N) - v^E(N \setminus \{i\}) \\
geq v(N) - v(N \setminus \{i\}) = M_i(v).
\]

For the second statement, let $S \in 2^N$ be an arbitrary coalition with $i \in S$. Lemma 3.1 gives $v^E(S) \leq v(S)$ and together with the first statement this results in

\[
v^E(S) - \sum_{j \in S, j \neq i} M_j(v^E) \leq v(S) - \sum_{j \in S, j \neq i} M_j(v),
\]

which implies $m_i(v^E) \leq m_i(v)$. □

It should be noted that there are (again) no conditions on the communication graph stated in the lemma above. This means that this result holds for every (connected and non-empty) graph, which makes the result very powerful. It is the basis for the following theorem, regarding the inheritance of compromise admissibility.

**Theorem 3.6** Let $(N, v, E) \in C_S N$ be a communication situation with $v \in C A^N$. Then $v^E \in C A^N$. 

3.3. STRONG COMPROMISE ADMISSIBILITY

**Proof:** Using Lemma 3.5 and the fact that \( m(v) \leq M(v) \) (since \( v \in CA^N \)) we obtain

\[
m(v^E) \leq m(v) \leq M(v) \leq M(v^E).
\]

Furthermore, with \( \sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v) \), we have

\[
\sum_{i \in N} m_i(v^E) \leq \sum_{i \in N} m_i(v) \leq v(N) = v^E(N) \leq \sum_{i \in N} M_i(v) \leq \sum_{i \in N} M_i(v^E).
\]

Consequently, \( v^E \) is compromise admissible. \( \square \)

### 3.3 Strong compromise admissibility

For strong compromise admissibility, we do not have a similar inheritance result as in the previous section. Example 3.7 shows that for communication situations \( (N,v,E) \) with \( v \in SCA^N \) and \( E \) a tree, the restricted game \( v^E \) does not need to be strongly compromise admissible.

<table>
<thead>
<tr>
<th>( S )</th>
<th>{1}</th>
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<th>{1,3}</th>
<th>{2,3}</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v(S) )</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( g^v(S) )</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 3.2** – A three-person strongly compromise admissible game and its gap function

![Figure 3.2 - A tree](image)

<table>
<thead>
<tr>
<th>( S )</th>
<th>{1}</th>
<th>{2}</th>
<th>{3}</th>
<th>{1,2}</th>
<th>{1,3}</th>
<th>{2,3}</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v^E(S) )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( g^{v^E}(S) )</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

**Table 3.3** – The restricted game \( v^E \) and its gap function

**Example 3.7** Let \( (N,v,E) \) be a communication situation with \( v \) the three-person game as defined in Table 3.2 and \( E \) the graph as depicted in Figure 3.2. It can be easily checked that \( M(v) = (1,1,2) \) and \( m(v) = (0,0,1) \) such that \( v \) is compromise admissible. As shown in Table 3.2 also \( g^v(N) \leq g^v(S) \) holds for every \( S \in 2^N \setminus \{\emptyset\} \), which implies that \( v \) is strongly compromise admissible.
In contrary, the restricted game \( v^E \), as given in Table 3.3, is not strongly compromise admissible. Using that \( M(v^E) = (1, 3, 2) \), the coalition \( S = \{1\} \) leads to a contradiction, since \( g^v(N) = 3 \not\leq 1 = g^v(S) \). △

However, when the communication graph is a 2-connected graph, strong compromise admissibility is inherited by the restricted game.

**Theorem 3.8** Let \((N, v, E)\) be a communication situation with \( v \in SCA_N \) and \( E \) a 2-connected graph. Then \( v^E \in SCA_N \).

**Proof:** Since for every \( i \in N \) the subgraph with node set \( N \setminus \{i\} \) is still connected for a 2-connected graph, we have that \( M_i(v^E) = M_i(v) \) for every \( i \in N \). From Lemma 3.1 we know that \( v^E(S) \leq v(S) \) for every \( S \in 2^N \setminus \{\emptyset\} \). Putting this together we obtain

\[
g^v(N) = \sum_{i \in N} M_i(v) - v(N) = \sum_{i \in N} M_i(v^E) - v^E(N) = g^v(N),
\]

\[
g^v(S) = \sum_{i \in S} M_i(v) - v(S) \leq \sum_{i \in S} M_i(v^E) - v^E(S) = g^v(S)
\]

for every \( S \in 2^N \setminus \{\emptyset\} \). This implies that \( g^v(N) = g^v(S) \leq g^v(S) \leq g^v(S) \) where the first inequality follows from \( v \) being strongly compromise admissible. Combining this with Theorem 3.6 to guarantee \( v^E \in CA_N \), we have that the restricted game \( v^E \) is strongly compromise admissible. □

In particular, Theorem 3.8 holds for cycles. Cycles are 2-connected, since removing one node results in a (connected) street.

It is interesting to mention that the class of 2-connected graphs is the only class of graphs that guarantees the inheritance of strong compromise admissibility. For every graph \( E \) (assumed to be simple and connected) that is not 2-connected there exists a strongly compromise admissible game \( v \in TU_N \) such that the restricted game corresponding to the communication situation \((N, v, E)\) is not strongly compromise admissible.

**Proposition 3.9** Let \((N, E)\) be a graph with \(|N| \geq 3\) which is not 2-connected. Then there is a game \( v \in SCA_N \) such that \( v^E \notin SCA_N \).

**Proof:** Because \( E \) is not 2-connected, there is a node \( k \in N \) (at least one) such that the subgraph on \( N \setminus \{k\} \) is disconnected. As usual, write \(|N| = n\) and consider the TU-game defined as

\[
v(N) = n;
v(N \setminus \{i\}) = n - 1 \quad \text{for all } i \in N;
v(S) = 0 \quad \text{for all other } S \in 2^N.
\]

Note that \( v \in TU^N \) is both zero-normalized and superadditive, such that \((N, v, E)\) is indeed a communication situation. Furthermore, for all \( i \in N \) we have that \( M_i(v) = v(N) - v(N \setminus \{i\}) = n - (n - 1) = 1 \). Thus \( M(v) = e^N \). Also \( m(v) = e^N \) holds. To see
3.4. COMPROMISE STABILITY

this, take \( i \in N \) arbitrarily and recall that \( m_i(v) = \max_{S \ni i \subseteq S} \left\{ v(S) - \sum_{j \in S, j \neq i} M_j(v) \right\} \).

Note that both \( S = N \setminus \{ j \} \) (for every \( j \in N \)) and \( S = N \) leads to 1:

\[
v(N \setminus \{ j \}) - \sum_{p \in N \setminus \{ j \}, p \neq i} M_p(v) = n - 1 - \sum_{p \in N \setminus \{ j \}, p \neq i} c_v^N(p)
= n - 1 - (n - 2)
= 1,
\]

\[
v(N) - \sum_{p \in N, p \neq i} M_p(v) = n - \sum_{p \in N, p \neq i} c_v^N(p)
= n - (n - 1)
= 1.
\]

All other coalitions \( S \) with \( i \in S \) have \( v(S) = 0 \) and thus do not exceed the maximum of 1. Since \( M(v) = e^N = m(v) \) we have that \( v \in CA^N \). Furthermore, the gap \( g^v(N) = n - n = 0 \), while \( g^v(S) \geq 0 \) for all other \( S \in 2^N \). The latter is true because for coalitions \( S \) with \( |S| \leq n - 2 \) we have that \( g^v(S) = |S| - 0 \) and for coalitions \( S \) with \( |S| = n - 1 \) that \( g^v(S) = n - 1 - v(S) = n - 1 - (n - 1) = 0 \). All of this together implies that \( v \in SCA^N \).

In contrast to this, we have that \( v^E \not\in SCA^N \). The values for the restricted game \( v^E \) are

\[
v^E(N) = n;
\]
\[
v^E(N \setminus \{ k \}) = 0;
\]
\[
v^E(N \setminus \{ i \}) = \begin{cases} n - 1 & \text{if the subgraph on } N \setminus \{ i \} \text{ is connected;} \\ 0 & \text{if the subgraph on } N \setminus \{ i \} \text{ is not connected;} \end{cases}
\]

for all \( i \in N \) with \( i \neq k \). The utopia-vector \( M(v^E) \) is strictly positive, since \( M_k(v^E) = n \) and \( M_i(v^E) \) is either 1 (in the upper-case) or \( n \) (in the lower-case). For the gap of the grand coalition we have

\[
g^v(N) = \sum_{i \in N} M_i(v^E) - v^E(N) = n + \sum_{i \in N, i \neq k} M_i(v^E) - n = \sum_{i \in N \setminus \{ k \}} M_i(v^E).
\]

Take an arbitrary \( i \in N \) with \( i \neq k \) and consider

\[
g^v(\{ i \}) = M_i(v^E) - v^E(\{ i \}) = M_i(v^E) < \sum_{i \in N \setminus \{ k \}} M_i(v^E) = g^v(N).
\]

This proves that \( v^E \not\in SCA^N \). \( \square \)

3.4 Compromise stability

This section is about the inheritance of compromise stability. It is known that strongly compromise admissible games are also compromise stable. Therefore, one could hope to
generalise Theorem 3.8 to compromise stable games. That is, is the restricted game \( v^E \) corresponding to a communication situation \((N, v, E)\) with \( v \) a compromise stable game and \( E \) a 2-connected graph also compromise stable? Unfortunately, this is not true. A counterexample is given in Example 3.10. In fact, it also provides a counterexample for \( E \) a star or a street.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
S & |S| = 1 & \{1,2\} & \{1,3\} & \{1,4\} & \ldots & \{3,4\} \\
\hline
v(S) & 0 & 3 & 3 & 0 & \ldots & 0 \\
\hline
\end{array}
\]

Table 3.4 – A TU-game

![Graphs](image_url)

(a) \( E_1 \): a cycle  \hspace{1cm} (b) \( E_2 \): a star  \hspace{1cm} (c) \( E_3 \): a street

**Figure 3.3** – Three different graphs on 4 nodes

**Example 3.10** Consider the communication situation \((N, v, E)\) with \( N = \{1,2,3,4\} \), \( v \in TU^N \) as given in Table 3.4 and the three graphs \( E_1, E_2 \) and \( E_3 \) as depicted in Figure 3.3. The game \( v \) is zero-normalized and superadditive, as can be easily checked.

Furthermore, we have \( M(v) = (1,4,2,3) \) and \( m(v) = (1,2,2,0) \). Obviously, \( m(v) \leq M(v) \) and \( \sum_{i \in N} m_i(v) = 5 \leq 7 = v(N) \leq 10 = \sum_{i \in N} M_i(v) \). Therefore, it holds that \( v \in CA^N \). The game is also compromise stable, because the inequality from equation (2.2) is satisfied for every \( S \in 2^N \setminus \{\emptyset\} \).

Now, we are going to show that the restricted games \( v^{E_1}, v^{E_2} \) and \( v^{E_3} \) are not compromise stable. The values for the three restricted games are shown in Table 3.5. For the first one, \( v^{E_1} \), the utopia-vector is the same, i.e. \( M(v^{E_1}) = M(v) = (1,4,2,3) \), but the minimum right vector changes into \( m(v^{E_1}) = (0,2,0,0) \). The game \( v^{E_1} \) is not compromise stable, since it does not satisfy the inequality from equation (2.2) for the coalition \( \{1,2\} \):

\[
v^{E_1}(\{1,2\}) = 3 \preceq 2 = \max \{2,2\} = \max \{0 + 2, 7 - 2 - 3\}.
\]

Also \( v^{E_2} \) and \( v^{E_3} \) are not compromise stable, because the inequality does not hold for the same coalition, \( \{1,2\} \). The utopia-vectors are \( M(v^{E_2}) = (1,7,2,3) \) and \( M(v^{E_3}) = \)
(1, 7, 4, 3) and the minimum right vectors are (again) \( m(v^E_2) = m(v^E_3) = (0, 2, 0, 0) \). Therefore, it holds that

\[
\begin{align*}
v^E_2(\{1, 2\}) &= 3 \leq 2 = \max \{2, 2\} = \max \{0 + 2, 7 - 2 - 3\}; \\
v^E_3(\{1, 2\}) &= 3 \leq 2 = \max \{2, 0\} = \max \{0 + 2, 7 - 4 - 3\}.
\end{align*}
\]

Therefore, it holds that

\[
\begin{align*}
v^E_2(\{1, 2\}) &= 3 \leq 2 = \max \{2, 2\} = \max \{0 + 2, 7 - 2 - 3\}; \\
v^E_3(\{1, 2\}) &= 3 \leq 2 = \max \{2, 0\} = \max \{0 + 2, 7 - 4 - 3\}.
\end{align*}
\]

\[
\Delta
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
S & |S| = 1 & \{1, 2\} & \{1, 3\} & \{1, 4\} & \ldots & \{3, 4\} \\
\hline
v^E_1(S) & 0 & 3 & 0 & 0 & \ldots & 0 \\
v^E_2(S) & 0 & 3 & 0 & 0 & \ldots & 0 \\
v^E_3(S) & 0 & 3 & 0 & 0 & \ldots & 0 \\
\hline
\end{array}
\]

Table 3.5 – Three restricted games

Apparently, for almost all graphs compromise stability is not inheritable. Of course, a complete graph ensures the restricted game to be compromise stable, since for complete graphs the restricted game is the same as the underlying game. The following example shows that even an almost complete graph does not work.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
S & \{1, 2, 3\} & \{1, 2, 4\} & \{1, 3, 4\} & \{2, 3, 4\} & N \\
v^E_1(S) & 4 & 5 & 3 & 6 & 7 \\
v^E_2(S) & 4 & 5 & 0 & 6 & 7 \\
v^E_3(S) & 4 & 3 & 0 & 6 & 7 \\
\hline
\end{array}
\]

\[
\text{Figure 3.4 – An almost complete graph } E
\]

Example 3.11  Consider the communication situation \((N, v, E)\) with \(v\) again the game from Table 3.4 and \(E\) the graph from Figure 3.4. The restricted game \(v^E\) is the same as
the game \( v^{E_1} \) shown in Table 3.5. Therefore, also this restricted game is not compromise stable. \( \triangle \)

Examples 3.10 and 3.11 provide a four-person game as a counterexample. The question rises whether there is a counterexample with three players. It turns out that the answer is no.

**Proposition 3.12** Let \((N,v,E) \in CS^N\) with \(|N| = 3\) and \(v\) compromise stable. Then the restricted game \( v^E \) is compromise stable.

**Proof:** Compromise stability for \( v \) guarantees \( C(v) \neq \emptyset \), since \( C(v) = CC(v) \) for three-person games and \( CC(v) \neq \emptyset \) \((v \in CA^N)\). Hence, by Theorem 3.3, the restricted game \( v^E \) is balanced. Since \( \emptyset \neq C(v^E) \subseteq CC(v^E) \), \( v^E \) is also compromise admissible. What remains is to prove that \( CC(v^E) \subseteq C(v^E) \). Therefore, let \( x \in CC(v^E) \). Then efficiency is already satisfied, just as stability for \( S \) with \(|S| = 1\) \((v^E(\{i\}) \leq m_i(v^E) \leq x_i)\). For \( S \) with \(|S| = 2\), consider e.g. \( S = \{1,2\} \) and observe that \( x_3 \leq M_3(v^E) = v^E(N) - v^E(\{1,2\}) \) implies that \( v^E(\{1,2\}) \leq v^E(N) - x_3 = x_1 + x_2 \). A similar proof applies for \( S = \{1,3\} \) and \( \{2,3\} \).

So for three-person games compromise stability is always inherited by the restricted game. For a game with more than three players, however, compromise stability is not inherited by the restricted game. Fortunately, this is not the end of the story. There is a way to guarantee compromise stability for the restricted game. The way to do this, is to add an extra condition on the underlying game, namely a game with the zero vector as the minimum right vector. For zero-normalized games, it is known that e.g. convex games \( v \in TU^N \) have the property that \( m(v) = 0 \).

**Theorem 3.13** Let \((N,v,E) \in CS^N\) be a communication situation with \( v \) a compromise stable game and \( m(v) = 0 \). Then the following two statements holds:

i) If \( E \) is a 2-connected graph, then \( v^E \) is compromise stable.

ii) If \( E \) is a star, then \( v^E \) is compromise stable.

**Proof:** The restricted game \( v^E \) is zero-normalized, thus \( 0 \leq m(v^E) \) \((0 = v^E(\{i\}) \leq m_i(v^E) \) for every \( i \in N)\). Using Lemma 3.5, we have that \( 0 \leq m(v^E) \leq m(v) = 0 \), which implies that \( m(v^E) = 0 \). Furthermore, note that \( v^E \in CA^N \) according to Theorem 3.6. This allows us to use equation (2.2) such that we have to prove that for every \( S \in 2^N \setminus \emptyset \) it holds that

\[
v^E(S) \leq \max \left\{ 0, v^E(N) - \sum_{i \in N \setminus S} M_i(v^E) \right\} . \tag{3.4}
\]

i) If \( E \) is a 2-connected graph, we have \( M(v^E) = M(v) \). This holds because for each \( i \in N \) the set \( N \setminus \{i\} \) induces a connected subgraph (and thus \( v^E(N \setminus \{i\}) = v(N \setminus \{i\}) \).
holds). Then it follows that
\[
v^E(S) \leq v(S) \leq \max \left\{ 0, v(N) - \sum_{j \in N \setminus S} M_j(v) \right\}
= \max \left\{ 0, v^E(N) - \sum_{j \in N \setminus S} M_j(v^E) \right\},
\]
where the first inequality follows from Lemma 3.1 and the second from \( v \) being compromise stable.

ii) For \( E \) a star, denote \( k \in N \) for the special node in the star. Then \( v^E(S) = 0 \) if \( k \notin S \) and \( v^E(S) = v(S) \) if \( k \in S \). This means that for \( i \in N \) with \( i \neq k \) we have that
\[
M_i(v^E) = w(N) - v^E(N \setminus \{i\}) = v(N) - v(N \setminus \{i\}) = M_i(v),
\]
where the second equality follows from the fact that \( k \in N \setminus \{i\} \). In order to prove the inequality (3.4) for a certain \( S \in 2^N \setminus \{\emptyset\} \), we distinguish between whether \( k \in S \) or not. If \( k \notin S \), then \( v^E(S) = 0 \) and the inequality (3.4) holds. If \( k \in S \) then clearly \( k \notin N \setminus S \), thus we have that
\[
v^E(S) = v(S) \leq \max \left\{ 0, v(N) - \sum_{i \in N \setminus S} M_i(v) \right\}
= \max \left\{ 0, v^E(N) - \sum_{i \in N \setminus S} M_i(v^E) \right\},
\]
where the inequality follows from the assumption that \( v \) is compromise stable. This concludes the proof of compromise stability, since for every \( S \in 2^N \setminus \{\emptyset\} \) the inequality (3.4) holds. \( \square \)

Theorem 3.13 does not apply for a street as communication graph, because for a street, the values for the coalitions containing all players except one change too much. This can be seen in the following example.

| \( S \) | \( |S| = 1 \) | \( \{1,2\} \) | \( \{1,3\} \) | \ldots | \( \{3,4\} \) |
|-------|-------|-------|-------|---|---|
| \( v(S) \) | 0 | 3 | 0 | \ldots | 0 |
| \( v^E(S) \) | 0 | 3 | 0 | \ldots | 0 |

<table>
<thead>
<tr>
<th>( S )</th>
<th>( {1,2,3} )</th>
<th>( {1,2,4} )</th>
<th>( {1,3,4} )</th>
<th>( {2,3,4} )</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v(S) )</td>
<td>5</td>
<td>5</td>
<td>2</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>( v^E(S) )</td>
<td>5</td>
<td>3</td>
<td>0</td>
<td>2</td>
<td>7</td>
</tr>
</tbody>
</table>

**Table 3.6** – A convex and compromise stable game \( v \) and the restricted game \( v^E \)
**Example 3.14** Consider the communication situation \((N, v, E)\) with \(N = \{1, 2, 3, 4\}\), \(v\) as in Table 3.6 and \(E\) a street as depicted in Figure 3.5. In Table 3.6 also the values for the restricted game \(v^E\) are shown.

The game \(v\) is convex (just check the condition for all pairs of coalitions) and compromise admissible, since \(M(v) = (5, 5, 2, 2)\) and \(m(v) = 0\) (by convexity). Moreover, the game is also compromise stable (using inequality (2.2) or equivalently equation (3.4)).

However, the game \(v^E\) is not compromise stable. For the restricted game we have that \(M(v^E) = (5, 7, 4, 2)\), \(m(v^E) = 0\) and thus

\[
v^E(\{1, 2\}) = 3 \not\leq 1 = 7 - 4 - 2 = v^E(N) - M_3(v^E) - M_4(v^E).
\]

\[\triangle\]

**Figure 3.5 – A street on four nodes**

In the light of Theorem 3.13 one can wonder if it is possible to change the condition that the minimum right vector must be the zero vector into another condition on the underlying game, e.g. balancedness. However, Example 3.10 immediately shows that adding balancedness as an extra condition (instead of \(m(v) = 0\)) is not going to work. For the game \(v\) discussed there (and shown in Table 3.4) it can be easily checked that \((1, 2, 2, 2) \in C(v)\). Therefore, that example also provides a counterexample for the inheritance of compromise stability.

Instead of adding an extra condition, it is also possible to require something stronger than compromise stability. As mentioned in Chapter 2, strongly compromise admissible games are compromise stable (Quant et al., 2005). The following theorem shows that this will work. A helpful lemma in proving the theorem is due to Driessen (1988):

**Lemma 3.15** Let \(v \in SCA^N\). Then

i) \(M_i(v) - m_i(v) = g^v(N)\) for all \(i \in N\);

ii) \(m_i(v) + \sum_{j \in N \setminus \{i\}} M_j(v) = v(N)\) for all \(i \in N\).

**Theorem 3.16** Let \((N, v, E) \in CS^N\) be a communication situation with \(v \in SCA^N\) and \(E\) a star with special node \(k \in N\). Then \(v^E\) is compromise stable.

**Proof:** For strongly compromise admissible games it holds that for every \(S \in 2^N \setminus \{\emptyset\}\):

\[
\sum_{i \in S} m_i(v) \leq v(N) - \sum_{j \in N \setminus S} M_j(v).
\]

(3.5)

This can be proven using Lemma 3.15. Let \(S \in 2^N \setminus \{\emptyset\}\) and choose one player \(i \in S\).
Then
\[
\sum_{j \in S} m_j(v) + \sum_{j \in N \setminus S} M_j(v) \leq m_i(v) + \sum_{j \in N \setminus S} M_j(v) = v(N),
\]
where the inequality follows from \( v \) being compromise admissible (hence \( m(v) \leq M(v) \)) and the equality from Lemma 3.15. Using equation (3.5) and the fact that \( v \) is also compromise stable (hence equation (2.2) applies to \( v \)), we have that \( v(S) \leq v(N) - \sum_{j \in N \setminus S} M_j(v) \) for every \( S \in 2^N \setminus \{\emptyset\} \). Just as in the proof of Theorem 3.13 part ii), let \( S \in 2^N \setminus \{\emptyset\} \) and distinguish between \( k \in S \) and \( k \notin S \). If \( k \notin S \), then \( v^E(S) = 0 \) and the inequality from equation (2.2) is satisfied (\( m(v^E) \geq 0 \) due to the fact that \( v^E \) is zero-normalized). If \( k \in S \), then \( k \notin N \setminus S \). Hence \( \sum_{j \in N \setminus S} M_j(v^E) = \sum_{j \in N \setminus S} M_j(v) \).

Thus
\[
v^E(S) \leq v(S) \leq v(N) - \sum_{j \in N \setminus S} M_j(v) = v^E(N) - \sum_{j \in N \setminus S} M_j(v^E),
\]
which, together with the inheritance of compromise admissibility according to Theorem 3.6, completes the proof. \( \Box \)

Similar to Theorem 3.13, the above theorem also has a second statement for 2-connected graphs. However, Theorem 3.8 already stated a stronger result. For communication situations \((N, v, E)\) with \( v \in SCA^N \) and \( E \) a 2-connected graph, the restricted game \( v^E \in SCA^N \) and is therefore compromise stable.

The conditions of Theorem 3.13 and Theorem 3.16 are independent in the sense that they do not imply the other. The following example provides a compromise stable game \( v \in TU^N \) with \( m(v) = 0 \), such that \( v \notin SCA^N \). It also provides a strongly compromise admissible game with \( m(v) \neq 0 \).

| \( S \) | \( |S| = 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( N \) |
|---|---|---|---|---|---|
| \( v(S) \) | 0 | 6 | 6 | 5 | 3 | 11 |

Table 3.7 – A compromise stable game with \( m(v) = 0 \)

| \( S \) | \( |S| = 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( N \) |
|---|---|---|---|---|---|
| \( w(S) \) | 0 | 1 | 1 | 0 | 1 |

Table 3.8 – A strongly compromise admissible game
Example 3.17 Consider the game \( v \in TU^N \) from Table 3.7. It is readily checked that \( M(v) = (8, 6, 5, 5) \) and \( m(v) = (0, 0, 0, 0) \). Moreover, \( v \) is compromise stable, but not strongly compromise admissible. To see the latter, consider \( S = \{1\} \) and the gap function. For the grand coalition we have \( g^v(N) = 24 - 11 = 13 \) and for \( S \) the gap \( g^v(\{1\}) = 8 - 0 = 8 \), whence \( g^v(N) \not\leq g^v(S) \).

The game \( w \) from Table 3.8 shows that strongly compromise admissible games can have a strictly positive minimum right vector. It holds that \( M(w) = (1, 0, 0), m(w) = (1, 0, 0) \) and \( g^w(N) = 1 - 1 = 0 \), whereas \( g^w(S) \geq 0 \) for every \( S \in 2^N \setminus \{\emptyset\} \). \( \triangle \)
Chapter 4

The nucleolus for communication situations

In this chapter, the nucleolus of the restricted game is explored. Recall from Section 2.1 that for strongly compromise admissible games and compromise stable games, the nucleolus can be easily calculated with a direct formula. In the previous chapter, there were some theorems about the inheritance of strong compromise admissibility and compromise stability. In Section 3.3 Theorem 3.8 guarantees the inheritance of strong compromise admissibility if the communication graph is 2-connected. In Section 3.4 Theorem 3.13 guarantees the inheritance of compromise stability if the communication graph is 2-connected or a star and in addition the minimum right vector is the zero vector and Theorem 3.16 guarantees the inheritance compromise stability if the graph is a star and the underlying game is strongly compromise admissible.

We are going to explore relations between the nucleolus of the underlying game and the restricted game. Recall that for strongly compromise admissible games \( v \in TU^N \) the nucleolus is given by (cf. Theorem 2.4):

\[
\text{nuc}(v) = M(v) - \frac{1}{|N|} g^v(N)e^N.
\] (4.1)

**Theorem 4.1** Let \((N, v, E) \in CS^N\) be a communication situation with \(v \in SCA^N\) and \(E\) a 2-connected graph. Then \(\text{nuc}(v^E) = \text{nuc}(v)\).

**Proof:** For 2-connected graphs we have that \(M(v^E) = M(v)\) and thus \(g^{v^E}(N) = g^v(N)\). Theorem 3.8 allows us to also use equation (4.1) for the restricted game \(v^E\):

\[
\text{nuc}(v^E) = M(v^E) - \frac{1}{|N|} g^{v^E}(N)e^N
= M(v) - \frac{1}{|N|} g^v(N)e^N
= \text{nuc}(v). \]
For compromise stability, a zero minimum right vector was needed in order to guarantee compromise stability for the restricted game. Recall that for compromise stable games \( v \in TU^N \) the nucleolus is given by (cf. Theorem 2.3):

\[
nuc(v) = m(v) + TAL(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)).
\]

(4.2)

**Theorem 4.2** Let \((N, v, E) \in CS^N\) be a communication situation with \(v\) a compromise stable game, \(m(v) = 0\) and \(E\) a 2-connected graph. Then \(nuc(v_E) = nuc(v)\).

**Proof:** Applying Lemma 3.5 we see that \(0 \leq m(v_E) \leq m(v) = 0\) implies that \(m(v_E) = 0\). Furthermore, \(M(v_E) = M(v)\), because of 2-connectedness. With Theorem 3.13 we then have that

\[
nuc(v_E) = m(v_E) + TAL(v_E(N) - \sum_{i \in N} m_i(v_E), M(v_E) - m(v_E))
\]

\[
= TAL(v_E(N), M(v_E))
\]

\[
= TAL(v(N), M(v))
\]

\[
= m(v) + TAL(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v))
\]

\[
= nuc(v).
\]

□

For a star as communication graph, the nucleolus of the restricted game is not the same as the nucleolus of the underlying game. You could say that this is no surprise, since the player in the middle of the star is much more powerful in the restricted game than in the underlying game. This should in some way be reflected by the nucleolus and in fact, it does so, as proven by Theorem 4.4.

**Lemma 4.3** Let \((N, v, E) \in CS^N\) be a communication situation with \(E\) a star with special node \(k \in N\). Then for every \(i \in N\) it holds that

\[i)\]

\[M_i(v_E) = \begin{cases} M_i(v) & \text{if } i \neq k; \\ v(N) & \text{if } i = k. \end{cases} \]

\[ii)\]

\[m_i(v_E) = \begin{cases} 0 & \text{if } i \neq k; \\ m_k(v) & \text{if } i = k. \end{cases} \]

**Proof:**

\[i)\] For the utopia-vector it follows that the subgraph on \(N \setminus \{i\}\) for \(i \in N\), \(i \neq k\), is connected, since it contains player \(k\). Thus \(M_i(v_E) = M_i(v)\) for \(i \neq k\). For player \(k\), the subgraph on \(N \setminus \{k\}\) only consists of a bunch of trivial graphs. This implies that \(v_E(N \setminus \{k\}) = 0\) and thus \(M_k(v_E) = v_E(N) - v_E(N \setminus \{k\}) = v(N)\).

\[ii)\] Regarding the minimum right vector, let \(i \in N\) with \(i \neq k\) and consider \(m_i(v_E) = \max_{S \subseteq N} \{v_E(S) - \sum_{j \in S, j \neq i} M_j(v_E)\}\). If \(k \notin S\) for a certain coalition \(S\), then \(v_E(S) = 0\).
and the expression within the maximum is negative (since $M_j(v^E) \geq 0$ holds because $v^E$ is monotonic). If $k \in S$, then write (using i)

$$v^E(S) - \sum_{j \in S, j \neq i} M_j(v^E) = v^E(S) - M_k(v^E) - \sum_{j \in S, j \neq i, k} M_j(v^E) = v^E(S) - v(N) - \sum_{j \in S, j \neq i, k} M_j(v^E),$$

such that is becomes clear that also this expression is negative ($v^E(S) - v(N) \leq 0$ due to monotonicity). Thus for $i \in N$ with $i \neq k$ we have that $m_i(v^E) = 0$. For player $k$ it holds that $m_k(v^E) = m_k(v)$, since for all $S \in 2^N$ with $k \in S$ we have that $v^E(S) = v(S)$ and $M_j(v^E) = M_j(v)$ for all $j \in S$ with $j \neq k$.

**Theorem 4.4** Let $(N, v, E) \in CS^N$ be a communication situation with $v$ a compromise stable game, $m(v) = 0$ and $E$ a star with special node $k \in N$. Then we have that

i) $\text{nuc}_i(v^E) \leq \text{nuc}_i(v)$ for every $i \in N \setminus \{k\}$;

ii) $\text{nuc}_k(v^E) \geq \text{nuc}_k(v)$.

Moreover, if also $v(N \setminus \{k\}) = 0$, then $\text{nuc}(v^E) = \text{nuc}(v)$.

**Proof:** If $v(N \setminus \{k\}) = 0$, then $M_k(v^E) = v(N) = M_k(v)$ such that $M(v^E) = M(v)$ and $m(v^E) = m(v) = 0$ (applying Lemma 4.3). From equation (4.2) it then follows that $\text{nuc}(v^E) = \text{nuc}(v)$ (just as in the proof of Theorem 4.2). Then both i) and ii) hold with equality.

If $v(N \setminus \{k\}) \neq 0$ then still $m(v^E) = m(v) = 0$ holds. With Theorem 3.13 equation (4.2) simplifies to $\text{nuc}(v^E) = \text{TAL}(v(N), M(v^E))$ (since $v^E$ is compromise stable) and $\text{nuc}(v) = \text{TAL}(v(N), M(v))$. There are two options for the Talmud rule:

Case 1: $\sum_{i \in N} M_i(v^E) \leq 2v(N)$;

Case 2: $\sum_{i \in N} M_i(v^E) \geq 2v(N)$.

We are going to prove i) for both cases separately. Because of efficiency, ii) then follows from i). This is true, because efficiency ensures $\sum_{i \in N} \text{nuc}_i(v^E) = v(N) = \sum_{i \in N} \text{nuc}_i(v)$. If i) holds, then for all $i \in N \setminus \{k\}$ it holds that $\text{nuc}_i(v^E) \leq \text{nuc}_i(v)$, hence the $k$th term must increase, i.e. $\text{nuc}_k(v^E) \geq \text{nuc}_k(v)$. 

Case 1: The first case immediately determines the Talmud rule for the game $v$:
\[
\sum_{i \in N} M_i(v) = \sum_{i \in N, i \neq k} M_i(v) + M_k(v) = \sum_{i \in N, i \neq k} M_i(v^E) + v(N) - v(N \setminus \{k\}) = \sum_{i \in N} M_i(v^E) - v(N \setminus \{k\}) \leq 2v(N) - v(N \setminus \{k\}) \leq 2v(N),
\]
where we used Lemma 4.3 for the second and third equality and the fact that $v(N \setminus \{k\}) \geq 0$ (due to the non-negativity of $v$) for the last inequality. Therefore, both $\text{nuc}(v^E)$ and $\text{nuc}(v)$ can be calculated in the same way:
\[
\text{nuc}(v^E) = M(v^E) - \text{CEA} \left( \sum_{i \in N} M_i(v^E) - v(N), \frac{1}{2}M(v^E) \right); \tag{4.3}
\]
\[
\text{nuc}(v) = M(v) - \text{CEA} \left( \sum_{i \in N} M_i(v) - v(N), \frac{1}{2}M(v) \right). \tag{4.4}
\]

Working out these expressions we obtain for every $i \in N$:
\[
\text{nuc}_i(v^E) = M_i(v^E) - \min \{ \beta, \frac{1}{2}M_i(v^E) \}; \tag{4.5}
\]
\[
\text{nuc}_i(v) = M_i(v) - \min \{ \alpha, \frac{1}{2}M_i(v) \}, \tag{4.6}
\]
with $\beta \in \mathbb{R}$ such that $\sum_{i \in N} \min \{ \beta, \frac{1}{2}M_i(v^E) \} = \sum_{i \in N} M_i(v^E) - v(N)$ and $\alpha \in \mathbb{R}$ such that $\sum_{i \in N} \min \{ \alpha, \frac{1}{2}M_i(v) \} = \sum_{i \in N} M_i(v) - v(N)$. The estate in equation (4.3) is higher than the estate in equation (4.4). The difference is exactly $v(N \setminus \{k\})$:
\[
\sum_{i \in N} M_i(v^E) - v(N) - \left( \sum_{i \in N} M_i(v) - v(N) \right) = \sum_{i \in N, i \neq k} M_i(v^E) - \sum_{i \in N, i \neq k} M_i(v) + M_k(v^E) - M_k(v) = v(N) - v(N) + v(N \setminus \{k\}) = v(N \setminus \{k\}).
\]

For the second equality, we used Lemma 4.3. In contrary, the claims only increase with $\frac{1}{2}v(N \setminus \{k\})$ (again using Lemma 4.3):
\[
\sum_{i \in N} \frac{1}{2} M_i(v^E) - \sum_{i \in N} \frac{1}{2} M_i(v) = \sum_{i \in N, i \neq k} \frac{1}{2} M_i(v^E) - \sum_{i \in N, i \neq k} \frac{1}{2} M_i(v) + \frac{1}{2} M_k(v^E) - \frac{1}{2} M_k(v) = \frac{1}{2}v(N) - \frac{1}{2}(v(N) - v(N \setminus \{k\})) = \frac{1}{2} v(N \setminus \{k\}).
\]
So the estate increases more than the claims, which implies that $\alpha \leq \beta$. Then i) follows, because all other terms in (4.5) and (4.6) are the same for every $i \in N \setminus \{k\}$.

Case 2: For the second case there are still two options for $\text{nuc}(v)$:

- Case 2.1: $\sum_{i \in N} M_i(v) \leq 2v(N)$;
- Case 2.2: $\sum_{i \in N} M_i(v) \geq 2v(N)$.

Case 2.1: The two assumptions allows us to explicitly compute the Talmud rule. For every $i \in N \setminus \{k\}$ this results in

$$\text{nuc}_i(v^E) = \text{CEA}_i(v(N), \frac{1}{2}M(v^E))$$
$$= \min \{ \beta, \frac{1}{2}M_i(v) \};$$
$$\text{nuc}_i(v) = M_i(v) - \text{CEA}_i(\sum_{i \in N} M_i(v) - v(N), \frac{1}{2}M(v))$$
$$= M_i(v) - \min \{ \alpha, \frac{1}{2}M_i(v) \},$$
with $\alpha, \beta$ in the right way according to the CEA rule. Then for every $i \in N \setminus \{k\}$:

$$\text{nuc}_i(v) = M_i(v) - \min \{ \alpha, \frac{1}{2}M_i(v) \}$$
$$\geq \frac{1}{2}M_i(v)$$
$$\geq \min \{ \beta, \frac{1}{2}M_i(v) \}$$
$$= \text{nuc}_i(v^E).$$

Hence i) is true.

Case 2.2: For this case, the Talmud rule gives:

$$\text{nuc}(v^E) = \text{CEA}(v(N), \frac{1}{2}M(v^E))$$
$$= \min \{ \beta, \frac{1}{2}M(v^E) \};$$
$$\text{nuc}(v) = \text{CEA}(v(N), \frac{1}{2}M(v))$$
$$= \min \{ \alpha, \frac{1}{2}M(v) \},$$
again with $\alpha$ and $\beta$ in the right way. The estate in (4.7) is the same as the estate in (4.8), while the claims are higher in (4.7) ($M_i(v^E) = M_i(v)$ for $i \in N \setminus \{k\}$ and $M_k(v^E) = v(N) \geq M_k(v)$). Therefore, it holds that $\beta \leq \alpha$. Together with $M_i(v^E) = M_i(v)$ for every $i \in N \setminus \{k\}$, this implies that $\text{nuc}_i(v^E) \leq \text{nuc}_i(v)$ for every $i \in N \setminus \{k\}$. \qed
The result of Theorem 4.4 does not stand alone. For \( E \) a star and \( v \in SCAN \) the nucleolus of the restricted game \( v^E \) satisfies the same relation. We can use Theorem 3.16 to guarantee compromise stability for the restricted game and simplify the computation of the nucleolus of this game.

**Theorem 4.5** Let \((N, v, E) \in CSN\) be a communication situation with \( v \in SCAN \) and \( E \) a star with special node \( k \in N \). Then

\[
\begin{align*}
  i) \quad & nuc_i(v^E) \leq nuc_i(v) \quad \text{for every } i \in N \setminus \{k\}; \\
  ii) \quad & nuc_k(v^E) \geq nuc_k(v).
\end{align*}
\]

**Proof:** If \(|N| = 1\), then \( v^E = v \). Hence both i) and ii) follow. If \(|N| \geq 2\), then we first are going to derive the \( nuc(v^E) \) via equation (4.2). Using Lemma 4.3 we obtain

\[
\begin{align*}
  nuc(v^E) &= m(v^E) + TAL(v(N) - \sum_{i \in N} m_i(v^E), M(v^E) - m(v^E)) \\
             &= m(v^E) + TAL(v(N) - m_k(v), M(v^E) - m(v^E)).
\end{align*}
\]

For the derivation of the Talmud rule, we have (again using Lemma 4.3)

\[
\begin{align*}
  \sum_{i \in N} (M_i(v^E) - m_i(v^E)) &= \sum_{i \in N, i \neq k} M_i(v) + v(N) - m_k(v) \\
                                      &= v(N) - m_k(v) + v(N) - m_k(v) \\
                                      &= 2(v(N) - m_k(v)),
\end{align*}
\]

where the second equality follows from Lemma 3.15 \((v \in SCAN)\). This means that twice the estate equals the total amount of claims. Therefore, the Talmud rule assigns every claimant half of its claim. We thus obtain for \( i \in N \setminus \{k\} \):

\[
\begin{align*}
  nuc_i(v^E) &= m_i(v^E) + TAL_i(v(N) - m_k(v), M_i(v^E) - m_i(v^E)) \\
             &= 0 + \frac{1}{2}(M_i(v) - 0) \\
             &= \frac{1}{2}M_i(v)
\end{align*}
\]

and for \( k \in N \):

\[
\begin{align*}
  nuc_k(v^E) &= m_k(v^E) + TAL_k(v(N) - m_k(v), M_k(v^E) - m_k(v^E)) \\
             &= m_k(v) + \frac{1}{2}(v(N) - m_k(v)) \\
             &= \frac{1}{2}(v(N) + m_k(v)).
\end{align*}
\]

The nucleolus of \( v \) follows from equation (4.1):

\[
nuc(v) = M(v) - \frac{1}{|N|} g^v(N)e^N.
\]
Then for $i \in N \setminus \{k\}$, we have that
\[
\text{nuc}_i(v^E) = \frac{1}{2} M_i(v) = M_i(v) - \frac{1}{2} M_i(v) \\
\leq M_i(v) - \frac{1}{|N|} M_i(v) = M_i(v) - \frac{1}{|N|} g^v(\{i\}) \\
\leq M_i(v) - \frac{1}{|N|} g^v(N) = \text{nuc}_i(v)
\]
proves i), where we used that $M_i(v) = g^v(\{i\}) \geq g^v(N)$ due to the fact that $v$ is zero-normalized and strongly compromise admissible. Consequently, $\text{nuc}_k(v^E) \geq \text{nuc}_k(v)$, due to efficiency. □

We now have seen two results regarding a relation between the nucleolus of the restricted game and the nucleolus of the underlying game for a star as communication graph. In both cases the player in the middle of the star gains more and all other players give up something. This seems to be very natural and not only applicable for restricted games which are compromise stable. However, the problem is that for games that are not compromise stable or strongly compromise admissible the nucleolus is hard to compute.

**Conjecture 4.6** Let $(N, v, E) \in CS^N$ be a communication situation with $E$ a star with special node $k \in N$. Then

i) $\text{nuc}_i(v^E) \leq \text{nuc}_i(v)$ for every $i \in N \setminus \{k\}$;

ii) $\text{nuc}_k(v^E) \geq \text{nuc}_k(v)$.

For a three-person game, the middle player indeed gains from being the middle player. The proof relies on a monotonicity property of the nucleolus. This property was first formulated by Zhou (1991), who also proved that the nucleolus satisfies this property.

**Theorem 4.7** Let $v, w \in TU^N$ be two games with $I(v) \neq \emptyset$, $I(w) \neq \emptyset$, $v(T) \leq w(T)$ for some $T \in 2^N \setminus \emptyset$ and $v(S) = w(S)$ for all $S \in 2^N \setminus \emptyset$ with $S \neq T$. Then it holds that $\sum_{i \in T} \text{nuc}_i(v) \leq \sum_{i \in T} \text{nuc}_i(w)$.

Using Theorem 4.7 it is possible to prove the second statement of the conjecture for a three-person game, i.e. $\text{nuc}_k(v^E) \geq \text{nuc}_k(v)$ for $(N, v, E) \in CS^N$ with $E$ a star with special node $k \in N$. This is also the most interesting part of the conjecture, since it expresses the essential feature of a star.

**Proposition 4.8** Let $(N, v, E) \in CS^N$ be a communication situation with $|N| = 3$ and $E$ a star with special node $k \in N$. Then

$\text{nuc}_k(v^E) \geq \text{nuc}_k(v)$.

**Proof:** For the restricted game $v^E$ we have that $v^E(N \setminus \{k\}) = 0 \leq v(N \setminus \{k\})$ and $v^E(S) = v(S)$ for every $S \in 2^N \setminus \emptyset$ with $S \neq N \setminus \{k\}$ (see Figure 4.1). Hence, we
can use Theorem 4.7 with $v^E$ and $v$ and obtain
$$\sum_{i \in N \setminus \{k\}} \text{nuc}_i(v^E) \leq \sum_{i \in N \setminus \{k\}} \text{nuc}_i(v).$$

The nucleolus also satisfies efficiency, that is
$$\sum_{i \in N} \text{nuc}_i(v^E) = v(N) = \sum_{i \in N} \text{nuc}_i(v).$$

Therefore, it follows that $\text{nuc}_k(v^E) \geq \text{nuc}_k(v)$, which completes the proof. $\square$

\begin{figure}[h]
\centering
\begin{tikzpicture}
\Vertex[x=0,y=-1]{k}
\Vertex[x=-2,y=0]{left}
\Vertex[x=2,y=0]{right}
\Edge(left)(k)
\Edge(right)(k)
\end{tikzpicture}
\caption{A star on three nodes}
\end{figure}
Chapter 5

The compromise value for communication situations

The previous chapter was about the nucleolus of the restricted game of a communication situation. Several theorems about inheritance from Chapter 3 were used in order to derive relations between the nucleolus of the restricted game and the nucleolus of the underlying game. This chapter will follow the same approach for the compromise value. Recall that the compromise value is only defined for compromise admissible games. Due to Theorem 3.6 compromise admissibility is inherited by the restricted game for which the underlying game is compromise admissible. Therefore, only communication situations with a compromise admissible game as underlying game are discussed.

The first result of this chapter is a corollary. It follows directly from combining three results: first of all, the inheritance of strong compromise admissibility for 2-connected graphs (cf. Theorem 3.8). Secondly, the fact that the nucleolus and the compromise value coincide for strongly compromise admissible games (cf. Theorem 2.4). Eventually, the result from the previous chapter regarding the equality between the nucleolus of the restricted game and the nucleolus of the underlying game (cf. Theorem 4.1).

Corollary 5.1 Let \((N, v, E) \in CS^N\) be a communication situation with \(v \in SCA^N\) and \(E\) a 2-connected graph. Then \(\tau(v^E) = \tau(v)\).

There is no easier expression known for the compromise value for compromise stable games than the definition. Consequently, theorems about the inheritance of compromise stability are not needed. Therefore, the following theorem does not include the compromise stability condition (cf. Theorem 4.2).

Theorem 5.2 Let \((N, v, E) \in CS^N\) be a communication situation with \(v \in CA^N\), \(m(v) = 0\) and \(E\) a 2-connected graph. Then \(\tau(v^E) = \tau(v)\).

Proof: The game \(v\) is zero-normalized, which together with Lemma 3.5 implies that \(0 \leq m(v^E) \leq m(v) = 0\). Therefore, \(m(v^E) = 0\). The graph \(E\) is a 2-connected graph,
thus $M(v^E) = M(v)$. Then
\[
\tau(v) = \alpha M(v) + (1 - \alpha)m(v) = \alpha M(v);
\]
\[
\tau(v^E) = \beta M(v^E) + (1 - \beta)m(v^E) = \beta M(v),
\]
with $\alpha$ and $\beta$ such that $\sum_{i \in N} \tau_i(v) = v(N) = \sum_{i \in N} \tau_i(v^E)$. This last equation implies that
\[
\alpha \sum_{i \in N \setminus \{k\}} M_i(v) + \beta \sum_{i \in N} M_i(v),
\]
which results in the fact that $\alpha = \beta$. Then $\tau(v) = \alpha M(v) = \beta M(v) = \tau(v^E)$ proves the theorem. $\square$

The next theorem is about communication situations with a star as communication graph. If the underlying game is strongly compromise admissible, then the compromise value of the restricted game coincides with the nucleolus.

**Theorem 5.3** Let $(N, v, E) \in CS^N$ be a communication situation with $v \in SCA^N$ and $E$ a star with special node $k \in N$. Then $\tau(v^E) = nuc(v^E)$.

**Proof:** From Lemma 4.3 it follows that for every $i \in N \setminus \{k\}$, $\tau_i(v^E) = \alpha M_i(v)$ and for $k \in N$ that $\tau_k(v^E) = \alpha v(N) + (1 - \alpha)m_k(v)$ with $\alpha \in [0, 1]$ such that $\sum_{i \in N} \tau_i(v^E) = v(N)$. Writing out this equation leads to
\[
\alpha \sum_{i \in N \setminus \{k\}} M_i(v) + \alpha v(N) + (1 - \alpha)m_k(v) = v(N).
\]
This implies that
\[
\alpha = \frac{v(N) - m_k(v)}{\sum_{i \in N \setminus \{k\}} M_i(v) + v(N) - m_k(v)} = \frac{\sum_{i \in N \setminus \{k\}} M_i(v)}{2 \sum_{i \in N \setminus \{k\}} M_i(v)} = \frac{1}{2},
\]
where we applied Lemma 3.15 ii) for the second equality. Then for every $i \in N \setminus \{k\}$ it holds that
\[
\tau_i(v^E) = \frac{1}{2} M_i(v) = nuc_i(v^E),
\]
where we used that $nuc_i(v^E) = \frac{1}{2} M_i(v)$ as seen in the proof of Theorem 4.5. Because of efficiency, $\tau_k(v^E) = nuc_k(v^E)$ must hold. $\square$

The equality in Theorem 5.3 is remarkable, since it shows that the nucleolus and the compromise value coincide. For strongly compromise admissible games we know that both solution concepts coincide, but the restricted game in this case is not necessary strongly compromise admissible.

Consequently, we have the following corollary:
Corollary 5.4  Let $(N, v, E) \in CS^N$ be a communication situation with $v \in SCA^N$ and $E$ a star with special node $k \in N$. Then

\begin{align*}
  i) \quad & \tau_i(v^E) \leq \tau_i(v) \quad \text{for every } i \in N \setminus \{k\}; \\
  ii) \quad & \tau_k(v^E) \geq \tau_k(v).
\end{align*}

Proof: Theorem 5.3 ensures $\tau(v^E) = nuc(v^E)$ and Theorem 4.5 ensures the inequalities from i) and ii) for the nucleolus. Combining both theorems leads to the required statement.

Corollary 5.4 looks like the conjecture formulated in the previous chapter, Conjecture 4.6. The same reasoning as on page 39 that leads to the conjecture also applies to the compromise value (and in fact to any one-point solution concept). The essential part is that the middle (special) player is much more powerful in the restricted game than in the underlying game. Therefore, it seems reasonable to assign this player more in the restricted game than in the underlying game. For the nucleolus, it was not (yet) possible to prove or disprove this statement.

Fortunately, for the compromise value it is possible to prove it. The compromise value does satisfy the property that the middle player gains in the restricted game compared to the underlying game. This can be proved using another (other than the one in Theorem 4.7) monotonicity property, formulated by Tijs and Driessen (1986). He also proved that the compromise value satisfies this property on the class of compromise admissible games, $CA^N$.

Theorem 5.5  Let $v, w \in CA^N$ be two compromise admissible games with $v(T) < w(T)$ for some $T \in 2^N \setminus \{\emptyset, N\}$ and $v(S) = w(S)$ for all $S \in 2^N \setminus \{\emptyset\}$ with $S \neq T$. Then it holds that $\tau_i(v) \geq \tau_i(w)$ for every $i \in N \setminus T$.

Now we can prove a nice result regarding the compromise value and the middle player of a star. There are also some special cases in which the gains to the middle player are at the cost of all the other players.

Theorem 5.6  Let $(N, v, E) \in CS^N$ be a communication situation with $v \in CA^N$ and $E$ a star with special node $k \in N$. Then $\tau_k(v^E) \geq \tau_k(v)$.

Moreover, if one of the following conditions is satisfied,

\begin{align*}
  i) \quad & \sum_{i \in N} M_i(v) = v(N); \\
  ii) \quad & m_i(v) = 0 \quad \text{for } i \in N \setminus \{k\};
\end{align*}

then $\tau_i(v^E) \leq \tau_i(v)$ for every $i \in N \setminus \{k\}$.

Proof: First we are going to prove that $\tau_k(v^E) \geq \tau_k(v)$. This is done by using Theorem 5.5. Note that this is not possible in one step, since there are possibly many coalitions $S \in 2^N$ with $v^E(S) < v(S)$ (namely all coalitions $S$ with $k \notin S$ and $v(S) \neq 0$). Therefore, choose coalitions $T_1, T_2, \ldots, T_s$ for a certain $s \in \mathbb{N}$ with the property that $k \notin T_i$ and
\( v(T_i) \neq 0 \) for every \( 1 \leq i \leq s \). Furthermore, define for every \( 1 \leq i \leq s \) a game \( w_i \in \text{TU}^N \) such that \( w_i(S) = v(S) \) for every \( S \in 2^N \) with \( S \neq T_j \) for every \( 1 \leq j \leq s \) and \( w_i(T_1) = \ldots = w_i(T_{i-1}) = w_i(T_i) = 0 \) and \( w_i(T_{i+1}) = v(T_{i+1}), w_i(T_{i+2}) = v(T_{i+2}), \ldots, w_i(T_s) = v(T_s) \). This means that we have a sequence of games \( v \rightarrow w_1 \rightarrow w_2 \rightarrow \ldots \rightarrow w_s = v^E \), such that in each step forwards the value of exactly one coalition \( T_i \) is reduced.

In order to use Theorem 5.5, we have to check whether \( w_i \in \text{CA}^N \) or not. This is true and follows from observing the two Lemmas 3.1 and 3.5 and Theorem 3.6. By the same reasoning as in the proof of Lemma 3.1 it follows that \( v(S) \leq w_1(S) \leq w_2(S) \leq \ldots \leq w_s(S) = v^E(S) \) for every \( S \in 2^N \). Furthermore, the proof of Lemma 3.5 only uses this fact, so in a similar way it follows that \( M(v) \leq M(w_1) \leq \ldots \leq M(w_s) = M(v^E) \) and \( m(v^E) = m(w_s) \leq \ldots \leq m(w_1) \leq m(v) \). Eventually, this provides all ingredients to guarantee compromise admissibility for every game \( w_i \) according to Theorem 3.6 (using a similar proof).

Now we can use Theorem 5.5 for the pairs of games \( v \) and \( w_1, w_2, w_s, v^E \), and so on until the last pair \( w_{s-1} \) and \( w_s = v^E \). Note that indeed all these pairs of games satisfy the conditions of the theorem. Since \( k \notin T_i \) for every \( 1 \leq i \leq s \) we have that

\[
\tau_k(v) \leq \tau_k(w_1) \leq \tau_k(w_2) \leq \ldots \leq \tau_k(w_{s-1}) \leq \tau_k(w_s) = \tau_k(v^E).
\]

Consequently, \( \tau_i(v) \leq \tau_i(v^E) \), which proves the first statement. The second statement contains two options, which are proven separately:

i) Let \( i \in N \setminus \{k\} \). Then

\[
\tau_i(v^E) = \beta M_i(v^E) + (1 - \beta)m_i(v^E) = \beta M_i(v) \leq M_i(v) = \tau_i(v),
\]

where we used \( M_i(v^E) = M_i(v) \) and \( m_i(v^E) = 0 \) from Lemma 4.3, \( \beta \leq 1 \) (since \( \beta \in [0,1] \)) and \( \tau_i(v) = M_i(v) \) because \( M(v) \) is efficient.

ii) Let \( i \in N \setminus \{k\} \) and write \( \tau(v) = \alpha M(v) + (1 - \alpha)m(v) \) and \( \tau(v^E) = \beta M(v^E) + (1 - \beta)m(v^E) \). Using efficiency and Lemma 4.3, we can write \( \alpha \) and \( \beta \) as:

\[
\alpha = \frac{v(N) - m_k(v)}{\sum_{i \in N} M_i(v) - m_k(v)} \quad \text{and} \quad \beta = \frac{v(N) - m_k(v)}{\sum_{i \in N, i \neq k} M_i(v) + v(N) - m_k(v)}.
\]

It is easy to see that \( \alpha \geq \beta \), since \( M_k(v) \leq v(N) \). We can use this in the computation of the compromise value in order to obtain \( \tau_i(v^E) \leq \tau_i(v) \):

\[
\tau_i(v^E) = \beta M_i(v^E) = \beta M_i(v) \leq \alpha M_i(v) = \tau_i(v).
\]
Chapter 6

Glove games

This chapter is about glove games, as already seen in Example 2.2. The collection of glove games is a subclass of the class of transferable utility games with interesting properties. Therefore, it is worth to look at some notions on the subclass of glove games.

A glove game is not only the three-person glove game from the example. It is possible to define a more general glove game. This chapter starts with that definition and continues computing the core, nucleolus and compromise value of a glove game. In Appendix A there is also a characterization of the Shapley value of a glove game in the form of a recursive formula (cf. Shapley (1969)).

Definition 6.1 A glove game is a game \( v \in TU^N \) where each player \( i \) has either one left-hand glove (\( i \in L \)) or a right-hand glove (\( i \in R \)). A pair of gloves has worth 1. Formally, \( N = L \cup R \) with \( L \cap R = \emptyset \) and \( v(S) = \min \{ |L \cap S|, |R \cap S| \} \) for every \( S \in 2^N \setminus \{ \emptyset \} \). The set of all glove games will be denoted by \( GG^N \).

Notation To simplify the notation for glove games, we write the player set as \( N = \{1, \ldots, \ell, \ell + 1, \ldots, \ell + r\} \) for a glove game with \( |L| = \ell \in \mathbb{N} \) and \( |R| = r \in \mathbb{N} \). The glove game with this player set \( N \) is denoted by \( v_{\ell r} \in GG^N \). Note that \( v_{\ell r}(N) = \min \{\ell, r\} \).

It is allowed to have \( \ell = 0 \) or \( r = 0 \) for a glove game \( v_{\ell r} \in GG^N \), but not both. If \( \ell = 0 \), then it follows that \( v_{\ell r}(S) = 0 \) for every \( S \in 2^N \) and every \( r \in \mathbb{N} \setminus \{0\} \). A similar statement holds if \( r = 0 \). If both \( \ell = 0 \) and \( r = 0 \), then \( |N| = 0 \), which is not allowed according to the definition of a cooperative game.

Already in 1982, Tijs and Lipperts proved that a glove game \( v_{\ell r} \in GG^N \) with \( \ell < r \) is compromise stable. In the following lemma we will see that in fact such a game is strongly compromise admissible. For a glove game \( v_{\ell r} \in GG^N \) with \( \ell = r \) this is not true, but this game is still compromise admissible.

Notation For a glove game \( v_{\ell r} \in GG^N \), denote \( e^L \in \{0,1\}^N \) for the vector that has value 1 on the first \( \ell \) places and value 0 on the others. Similar to this, \( e^R \in \{0,1\}^N \) has value 0 on the first \( \ell \) places and value 1 on the places \( \ell + 1, \ldots, \ell + r \). This generalises
to $e^X \in \{0, 1\}^X$ for a certain $X \subseteq \mathbb{N}$:

$$e^X_i = \begin{cases} 1 & \text{if } i \in X; \\ 0 & \text{if } i \notin X. \end{cases}$$

Lemma 6.2 Let $v_{tr} \in GG^N$ be a glove game. Then $v_{tr} \in CA^N$. Moreover, if $\ell \neq r$ then $v_{tr} \in SCA^N$.

Proof: If $\ell = r$, then for every $i \in N$ it holds that $M_i(v_{tr}) = v_{tr}(N) - v_{tr}(N \setminus \{i\}) = \ell - (\ell - 1) = 1$. Hence, $M(v_{tr}) = e^N$. Since each pair of gloves consists of two gloves, two players are needed to obtain a value of 1. At least one of the two players must receive his utopia-value of 1. This results in a zero minimum right for every player $i \in N$. Hence, $m(v_{tr}) = 0$.

Then $\sum_{i \in N} m_i(v_{tr}) = 0 \leq v_{tr}(N) \leq 2\ell = \sum_{i \in N} M_i(v_{tr})$ and $m_i(v_{tr}) = 0 \leq 1 = e^N_i$ for every $i \in N$ implies that $v_{tr} \in CA^N$.

If $\ell \neq r$, assume $\ell < r$. The other case, $\ell > r$, follows in a similar way. For $i \in L$ the utopia-vector $M_i(v_{tr}) = v_{tr}(N) - v_{tr}(N \setminus \{i\}) = \ell - (\ell - 1) = 1$, since $\ell - 1 < r$. For $j \in R$, $M_j(v_{tr}) = v_{tr}(N) - v_{tr}(N \setminus \{j\}) = \ell - \ell = 0$, since $\ell \leq r - 1$. Thus $M(v_{tr}) = e^L$.

Regarding the minimum right vector, take an arbitrary $i \in L$ and consider a coalition $S \in 2^N$ with $i \in S$. Writing $k := |L \cap S|$ leads to the fact that $v_{tr}(S) \leq k$. Furthermore, $\sum_{j \in S, j \neq i} M_j(v_{tr}) = k - 1$, which together implies that $v_{tr}(S) - \sum_{j \in S, j \neq i} M_j(v_{tr}) \leq k - (k - 1) = 1$. For $S = N$ we have that $v_{tr}(N) - \sum_{j \in S, j \neq i} M_j(v_{tr}) = \ell - (\ell - 1) = 1$.

Thus $m_i(v_{tr}) = 1$.

For a player $j \in R$ and $S \in 2^N$ with $j \in S$, write $k := |L \cap S|$. Then $v_{tr}(S) \leq k$ and $\sum_{i \in S, i \neq j} M_i(v_{tr}) = k$ leads to $v_{tr}(S) - \sum_{i \in S, i \neq j} M_i(v_{tr}) \leq k - k = 0$. Thus $m_j(v_{tr}) = 0$, which leads to the minimum right vector $m(v_{tr}) = e^L$.

This already proves that $v_{tr} \in CA^N$, since $M(v_{tr}) = e^L = m(v_{tr})$. In order to prove strong compromise admissibility, consider the gap function: $g_{v_{tr}}(N) = \sum_{i \in N} M_i(v_{tr}) - v_{tr}(N) = \ell - \ell = 0$. Let $S \in 2^N \setminus \{\emptyset\}$ and (again) write $k := |L \cap S|$. Then $\sum_{i \in S} M_i(v_{tr}) = k$ and $v_{tr}(S) \leq k$ implies that $g_{v_{tr}}(S) = \sum_{i \in S} M_i(v_{tr}) - v_{tr}(S) \geq k - k = 0 = g_{v_{tr}}(N)$. This proves the strong compromise admissibility of $v_{tr}$. 

\[\square\]

6.1 The core

The next three sections are about the core, the nucleolus and the compromise value of a glove game. We start with the core, because the core will be helpful in order to derive the nucleolus.

Proposition 6.3 For a game $v_{tr} \in GG^N$ the core is as follows:

$$C(v_{tr}) = \begin{cases} \{e^L\} & \text{if } \ell < r; \\ \text{Conv}\{e^L, e^R\} & \text{if } \ell = r; \\ \{e^R\} & \text{if } \ell > r. \end{cases}$$
Proof: For the first case, \( \ell < r \), consider an arbitrary \( x \in C(v_{tr}) \). From the proof of Lemma 6.2 we know that \( M(v_{tr}) = e^L = m(v_{tr}) \). The fact that \( m(v_{tr}) \leq x \leq M(v_{tr}) \) implies that \( x = e^L \). On the other hand, \( e^L \) is indeed a core-element, which completes the proof of the first case.

This also proves the third case, by exchanging the \( \ell \) and the \( r \).

For the second case, \( \ell = r \), take any \( x \in C(v_{tr}) \) and consider the stability condition for \( S = N \setminus \{i,j\} \) with \( i \in L \) and \( j \in R \) arbitrarily,

\[
\sum_{k \in N \setminus \{i,j\}} x_k = \sum_{p \in L, p \neq i} x_p + \sum_{q \in R, q \neq j} x_q \geq r - 1,
\]

which implies \( x_i + x_j \leq 1 \) by efficiency. The choice \( S = \{i,j\} \) leads to \( x_i + x_j \geq 1 \). Thus \( x_i + x_j = 1 \) for every \( i \in L \) and \( j \in R \). Furthermore, players in \( L \) (resp. \( R \)) are symmetric, which implies that \( x_i = x_p \) for every pair of \( i, p \in L \) and \( x_j = x_q \) for every \( j, q \in R \).

In other words, for a core-element it holds that all players in \( L \) (respectively \( R \)) each receive the same and two players with a different glove (i.e. one player from \( L \) and one player from \( R \)) receive exactly 1. This proves \( C(v_{tr}) \subseteq \text{Conv}\{e^L, e^R\} \). For the other direction take any \( \lambda \in [0,1] \) and consider \( \lambda e^L + (1 - \lambda)e^R \). Efficiency is just combining \( \sum_{i \in L} e^L_i = \ell \) (resp. \( R, e^R \) and \( r \)) and \( \ell = r \). For stability, let \( S \in 2^N \setminus \{\emptyset\} \). Then

\[
\sum_{i \in S} (\lambda e^L_i + (1 - \lambda)e^R_i) = \lambda \sum_{i \in S} e^L_i + (1 - \lambda) \sum_{i \in S} e^R_i = \lambda |L \cap S| + (1 - \lambda) |R \cap S| \geq (\lambda + 1 - \lambda) \cdot \min \left\{ |L \cap S|, |R \cap S| \right\} = \min \left\{ |L \cap S|, |R \cap S| \right\} = v(S)
\]

completes the other direction. \( \square \)

Proposition 6.3 ensures that every glove game is balanced. We can use this fact for the characterization of the nucleolus of a glove game. Furthermore, recall that Proposition 3.3 guarantees that for each communication situation with a glove game as underlying game, the restricted game is also balanced. This is used in Chapter 7.

### 6.2 The nucleolus

As mentioned in Chapter 2, the nucleolus lies inside the core if the core is non-empty. This fact is very useful for the computation of the nucleolus of a glove game.
Proposition 6.4  For a game \( v_{tr} \in GG^N \) the nucleolus is as follows:

\[
\text{nuc}(v_{tr}) = \begin{cases} 
  e^L & \text{if } \ell < r; \\
  \frac{1}{2}e^N & \text{if } \ell = r; \\
  e^R & \text{if } \ell > r.
\end{cases}
\]

**Proof:** The first and third part follow immediately, since for \( \ell < r \) and \( \ell > r \), the core consists of only one vector (according to Proposition 6.3). Thus the nucleolus is equal to this vector.

For the second part, use Proposition 6.3 to write the nucleolus of \( v_{tr} \in GG^N \) in one variable, i.e. \( \text{nuc}(v_{tr}) = \lambda e^L + (1 - \lambda)e^R \) for a certain \( \lambda \in [0, 1] \). Then the excesses are of the form \(-a\lambda\) and \(b(1 - \lambda)\) for \(-\ell \leq b \leq 0 \leq a \leq \ell\), depending on the difference between the number of members from \( L \) and members from \( R \):

\[
E(S, x, y_\ell) = v(S) - \sum_{i \in S} \text{nuc}_i(v_{tr})
\]

For a positive difference (i.e. \( |L \cap S| - |R \cap S| \geq 0 \)), \( a := |L \cap S| - |R \cap S| \) leads to the excesses of the form \(-a\lambda\). A negative difference (i.e. \( |L \cap S| - |R \cap S| \leq 0 \)) leads to the excesses of the form \(b(1 - \lambda)\) with \( b := |L \cap S| - |R \cap S|\).

Note that \( a_1 \leq a_2 \) implies that \(-a_1\lambda \geq -a_2\lambda\) and \(b_1 \leq b_2\) implies that \(b_1(1 - \lambda) \leq b_2(1 - \lambda)\). The highest excesses are thus the ones corresponding to \( a = 0 \) and \( b = 0 \), the second-highest the ones with \( a = 1 \) and \( b = -1 \), etcetera. Arranging these excesses from high to low leads to

\[
(0, \ldots, 0, -\lambda, \ldots, -\lambda, -(1 - \lambda), \ldots, -(1 - \lambda), \ldots) \quad (6.1)
\]

or

\[
(0, \ldots, 0, -(1 - \lambda), \ldots, -(1 - \lambda), -\lambda, \ldots, -\lambda) \quad (6.2)
\]

depending on whether \(-\lambda > -(1 - \lambda)\) or \(-\lambda < -(1 - \lambda)\). The first condition results in \( \lambda < \frac{1}{2} \) and the second in \( \lambda > \frac{1}{2} \). We are going to prove that in both cases it is possible to find another excess vector that is lexicographically less than the vector from (6.1) or (6.2).

In the first case, let \( \frac{1}{2} > \mu > \lambda \) and consider the excess vector of \( \mu e^L + (1 - \mu)e^R \). With the same reasoning as above, this leads to an excess vector similar to (6.1) (since \( \mu < \frac{1}{2} \) and thus \(-\mu > -(1 - \mu)\)). Therefore, this vector is lexicographically less than the vector in (6.1), since \(-\mu < -\lambda\).

For the second case, choose \( \frac{1}{2} < \mu < \lambda \). Using the same argument as for the first case, the excess vector for \( \mu e^L + (1 - \mu)e^R \) is lexicographically less than the vector in (6.2).

We now can conclude that \( \lambda = \frac{1}{2} \) is the optimum. For every other \( \lambda \), you can choose a \( \mu \) closer to \( \frac{1}{2} \) to improve the solution. The nucleolus is the unique vector with the (lexicographically) lowest excess vector, thus this must be the one with \( \lambda = \frac{1}{2} \). □
The nucleolus divides the value of the grand coalition equally among the players who own a left-hand glove (if \( \ell < r \)) or a right-hand glove (if \( \ell > r \)) or among all players (if \( \ell = r \)). If there are fewer left-hand gloves \( \ell < r \), then a left-hand glove is a limiting factor in order to make pairs. Therefore, players who own a glove that is a limiting factor are more ‘powerful’ than players who do not own such a glove. The nucleolus takes this difference in power into account and assigns nothing to all players who do not own a glove that is a limiting factor. The value of the grand coalition is divided equally among the players who own a glove that is a limiting factor.

In the case that there are as many left-hand gloves as right-hand gloves, there is no difference in power any more and everyone is treated equally.

6.3 The compromise value

The third and last section of this chapter is about the compromise value. The compromise value is only defined for compromise admissible games. Lemma 6.2 guarantees compromise admissibility for a glove game, which allows us to state a proposition about the compromise value for a glove game.

**Proposition 6.5** For a game \( v_{\ell r} \in GG^N \) the compromise value is as follows:

\[
\tau(v_{\ell r}) = \text{nuc}(v_{\ell r}) = \begin{cases} 
  e_L & \text{if } \ell < r; \\
  \frac{1}{2} e^N & \text{if } \ell = r; \\
  e_R & \text{if } \ell > r.
\end{cases}
\]

**Proof:** We are going to distinguish between the case whether \( \ell \neq r \) and \( \ell = r \). In the first case, use Lemma 6.2 together with Theorem 2.4 to obtain \( \tau(v_{\ell r}) = \text{nuc}(v_{\ell r}) \). The second equality then follows from Proposition 6.4.

In the second case, apply the results from the proof of Lemma 6.2 to the direct formula of the compromise value:

\[
\tau(v_{\ell \ell}) = \alpha M(v_{\ell \ell}) + (1 - \alpha) m(v_{\ell \ell}) = \alpha e^N,
\]

with \( \alpha \in [0, 1] \) such that \( \sum_{i \in N} \tau_i(v_{\ell \ell}) = v_{\ell \ell}(N) \). Solving this last equation leads to \( 2\alpha = \ell \) and thus to \( \alpha = \frac{\ell}{2} \). Enter this in (6.3) leads to \( \tau(v_{\ell \ell}) = \frac{1}{2} e^N \). Again using Proposition 6.4 proves both the first and second equality. \( \square \)

Apparently, the compromise value is exactly the same as the nucleolus. For strongly compromise admissible games we already knew this, but a glove game with as many left-hand as right-hand glove players is not strongly compromise admissible. Therefore, it is remarkable that still both solutions coincide.
Chapter 7

Glove communication situations

In this chapter, glove communication situations are discussed, that is communication situations \((N, v_{lr}, E) \in CS^N\) with \(v_{lr} \in GG^N\). Different communication graphs are explored, started with a star. Note that only graphs that are essentially different are explored. For example, if there as many left-hand as right-hand glove players, then it does not matter whether a left-hand player is in the middle of the star or a right-hand player, since the calculation is in both cases the same.

Example 7.1 To start with an easy communication situation \((N, v_{1r}, E) \in CS^N\), where there is only one person with a left-hand glove and an arbitrary number \(r \in \mathbb{N}\) of persons with a right-hand glove. There are two possible communication graphs, which both are shown in Figure 7.1.

For the first communication graph (Figure 7.1a) the restricted game \(v_{1r}^E\) is exactly the same as \(v_{1r}\). For every coalition \(S \in 2^N\), there are only two options for the values of both games depending on whether \(1 \in S\) or not. If \(1 \in S\), then \(v_{1r}^E(S) = v_{1r}(S)\), since

![Figure 7.1 - Two possible communication graphs](image-url)
the subgraph on \( S \) is connected. If \( 1 \notin S \), then \( v_{1r}^E(S) = 0 \), but also \( v_{1r}(S) = 0 \). This is due to the fact that there is no left-hand glove present in \( S \).

For the graph in Figure 7.1b this is not true. Still \( v_{1r}^E(S) = v_{1r}(S) \) for \( S \in 2^N \) with \( 1 \notin S \), since both values are 0. For \( S \in 2^N \) with \( 1 \in S \) and \( 2 \in S \), also \( v_{1r}^E(S) = v_{1r}(S) \), since the subgraph on \( S \) is connected. But for \( S \in 2^N \) with \( 1 \in S \) and \( 2 \notin S \) the value \( v_{1r}^E(S) = 0 \), while \( v_{1r}(S) \) may have value 1 (e.g. for \( S = \{1, 3\} \)). It is interesting to note that the players \( 3, \ldots, n \) are all dummy players. Furthermore player 1 and 2 are symmetric.

\[ \triangle \]

Considering stars, it is interesting to mention that Theorem 3.16 also applies to glove games. Lemma 6.2 guarantees strong compromise admissibility for \( v_{\ell r} \in GG^N \) with \( \ell \neq r \). Consequently, the games \( v_{\ell r}^E \in GG^N \) with \( \ell \neq r \) are compromise stable if \( E \) is a star.

**Corollary 7.2** Let \((N, v_{\ell r}, E) \in CS^N\) be a glove communication situation with \( \ell \neq r \) and \( E \) a star. Then \( v_{E}^r \) is compromise stable.

In contrast to this, for communication situations \((N, v_{\ell \ell}, E) \in CS^N \) with \( E \) a star, the game \( v_{\ell \ell}^E \) is not compromise stable. This is shown in the following example.

\[ \begin{align*}
3 &: R \\
2 &: L \\
1 &: L \\
4 &: R \\
\end{align*} \]

**Figure 7.2** – A star with player 1 as the special node

**Example 7.3** Let \((N, v_{22}, E) \in CS^N\) with \( v_{22} \in GG^N \) and \( E \) as depicted in Figure 7.2. For the restricted game \( v_{22}^E \) it holds that \( M(v_{22}^E) = (2, 1, 1, 1) \) and \( m(v_{22}^E) = (0, 0, 0, 0) \). It then goes wrong with coalition \( S = \{1, 3\} \):

\[ v_{22}^E(S) = 1 = 0 = \max \{0, 0\} = \max \left\{ m_1(v_{22}^E) + m_3(v_{22}^E), v_{22}(N) - M_2(v_{22}^E) - M_4(v_{22}^E) \right\}. \]

\[ \triangle \]

#### 7.1 The nucleolus

In this section, there are two different types of graphs for which we are going to derive the nucleolus: a star and a cycle. First, the nucleolus is calculated for the restricted
game of a glove communication situation with as communication graph a star with player 1 ∈ L as the special node (as in Figure 7.1a). This generalises naturally to glove communication situations with a star with an arbitrary left-hand glove player as the special node as communication graph, by relabelling the players. If the communication graph is a star with a right-hand player as the special node, then by interchanging the left-hand players with the right-hand players (and possibly relabelling) we are again in the right conditions.

It should be noted that the player who is in the middle of the star (assumed to be player 1) is a very ‘powerful’ player. Without him, each coalition has a value of zero. In order to make a pair of gloves (i.e. to gain a value of one), the middle player is needed. In Section 6.2 we have seen that in a TU-game the nucleolus divides the value of the grand coalition equally among the players who own a glove that is a limiting factor. It seems reasonable to state that the middle player is a limiting factor too, since he is responsible for the connection between a left-hand glove player and a right-hand glove player. Without the middle player, no pairs of gloves can be made. The middle player is thus a limiting factor for every pair of glove.

If both left-hand gloves and right-hand gloves are limiting factors (i.e. ℓ = r), then there are thus three different limiting factors: the left-hand gloves, the right-hand gloves and the middle player for the connection. Extending the reasoning from Section 6.2, the nucleolus divides every pair of gloves equally among the three players that are involved. The middle player will thus gain multiple times the equal share. The following theorem shows that indeed the nucleolus divides the value of the grand coalition ‘equally’ in this way.

Note that also the middle player can own a glove that is a limiting factor, which in that case yields an extra share for him. This is due to the fact that the middle player then is both responsible for the connection and a glove that is a limiting factor.

**Theorem 7.4** Let \((N,v_{\ell r},E) \in CS^N\) be a glove communication situation with \(E\) a star with special node 1 ∈ L. Then it holds that

\[
nuc(v_{\ell r}^E) = \begin{cases} 
\left(\frac{\ell+1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, 0, \ldots, 0\right) & \text{if } \ell < r; \\
\left(\frac{\ell+1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}\right) & \text{if } \ell = r; \\
\left(\frac{r}{2}, 0, \ldots, 0, \frac{1}{2}, \ldots, \frac{1}{2}\right) & \text{if } \ell > r.
\end{cases}
\]

**Proof:** Before we are going to prove the three statements, note that the nucleolus is the zero vector if \(\ell = 0\) or \(r = 0\). So we can assume that \(\ell, r \neq 0\).

For the first statement, \(\ell < r\), we can combine Lemma 6.2 and the results in the proof of Theorem 4.5 to obtain for every \(i \in N\):

\[
nuc_i(v_{\ell r}^E) = \begin{cases} 
\frac{1}{2}M_i(v_{\ell r}) & \text{if } i \neq 1; \\
\frac{1}{2}(v_{\ell r}(N) + m_1(v_{\ell r})) & \text{if } i = 1.
\end{cases}
\]

In the proof of Lemma 6.2 we have seen that \(M(v_{\ell r}) = e^L = m(v_{\ell r})\). Plugging this in,
we get
\[
\text{nuc}_i(v^E_{\ell r}) = \begin{cases} 
\frac{1}{2} & \text{if } i \in L, i \neq 1; \\
0 & \text{if } i \in R; \\
\frac{1}{2}(\ell + 1) & \text{if } i = 1,
\end{cases}
\]
resulting in the correct vector.

For the third statement, \( \ell > r \), we can again use Lemma 6.2 and Theorem 4.5 to obtain for every \( i \in N \):
\[
\text{nuc}_i(v^E_{\ell r}) = \begin{cases} 
0 & \text{if } i \in L, i \neq 1; \\
\frac{1}{2} & \text{if } i \in R; \\
\frac{1}{2}(r + 0) & \text{if } i = 1,
\end{cases}
\]
which proves the third statement.

The second statement, \( \ell = r \), is slightly more complicated. This is proven using excesses and the excess vector, in a similar way as in the proof of Proposition 6.4. Therefore, note that each player \( j \in R \) is symmetric to every other player \( q \in R \) and that each player \( i \in L, i \neq 1 \) is symmetric to every other player \( p \in L, p \neq 1 \). Using Lemma 4.3 and the fact that \( M(v_{\ell r}) = e^N \) and \( m(v_{\ell r}) = 0 \) (from the proof of Lemma 6.2), we obtain that \( 0 \leq \text{nuc}(v^E_{\ell r}) \leq (\ell, 1, \ldots, 1) \). Furthermore, since \( \text{nuc}(v^E_{\ell r}) \in C(v^E_{\ell r}) \) we can use the stability condition for \( S = N \setminus \{p, q\} \) for \( p \in L, p \neq 1 \) and \( q \in R \). This leads to \( \sum_{i \in N \setminus \{p, q\}} \text{nuc}_i(v^E_{\ell r}) \geq v^E_{\ell r}(N \setminus \{p, q\}) = \ell - 1 \), which implies that \( \text{nuc}_p(v^E_{\ell r}) + \text{nuc}_q(v^E_{\ell r}) \leq 1 \). Therefore, the nucleolus of \( v^E_{\ell r} \) can be written as
\[
\text{nuc}(v^E_{\ell r}) = (\mu, \lambda, \ldots, \lambda, \nu, \ldots, \nu) \quad (7.1)
\]
with \( \mu \in [0, \ell] \) and \( \lambda, \nu \in [0, 1] \) such that \( \lambda + \nu \leq 1 \) and \( \mu + (\ell - 1)\lambda - \ell \nu = \ell \) due to efficiency. Rewriting this last equation leads to
\[
\mu = \ell - \ell \nu - (\ell - 1)\lambda. \quad (7.2)
\]
Now the excesses can be calculated in terms of \( \mu, \lambda \) and \( \nu \). For \( S \in 2^N \), the excesses \( E(S, \text{nuc}(v^E_{\ell r})) \) are of the form \(-\lambda, -2\lambda, \ldots, -(\ell - 1)\lambda \) and \(-\nu, -2\nu, \ldots, -(\ell - 1)\nu \) if \( 1 \notin S \). If \( 1 \in S \) for \( S \in 2^N \), then \( E(S, \text{nuc}(v^E_{\ell r})) = \min \{a + 1, b\} - \mu - a\lambda - b\nu \) with \( 0 \leq a \leq \ell - 1 \) the number of members from \( L \) (other than player 1) and \( 0 \leq b \leq \ell \) the number of members from \( R \). For the excess vector \( \theta(\text{nuc}(v^E_{\ell r})) \) it is important which excess is the largest. The excesses of the first two forms are easy: \(-\lambda \) and \(-\nu \) are the largest there. For excesses of the third form this is not so obvious. For a given \( a \) satisfying \( 0 \leq a \leq \ell - 1 \) the choice \( b = a + 1 \) leads to the highest excess. To prove this, let \( 0 \leq c \leq \ell \) with \( a + 1 = b \neq c \).

If \( c < b = a + 1 \), then
\[
\min \{a + 1, c\} - \mu - a\lambda - c\nu = c - \mu - a\lambda - c\nu = c(1 - \nu) - \mu - a\lambda \\
< b(1 - \nu) - \mu - a\lambda = b - \mu - a\lambda - b\nu \\
= \min \{a + 1, b\} - \mu - a\lambda - b\nu.
\]
If $a + 1 = b < c$, then

$$\min \{a + 1, c\} - \mu - a\lambda - cv = a + 1 - \mu - a\lambda - cv$$

$$= \min \{a + 1, b\} - \mu - a\lambda - bv.$$ 

In order to determine which $a$ leads to the highest excess, let $a_1 \leq a_2$. Then (using $b = a + 1$)

$$a_1 + 1 - \mu - a_1\lambda - (a_1 + 1)\nu = a_1(1 - \lambda - \nu) + 1 - \mu - \nu$$

$$\leq a_2(1 - \lambda - \nu) + 1 - \mu - \nu$$

$$= a_2 + 1 - \mu - a_2\lambda - (a_2 + 1)\nu,$$

where we used the fact that $\lambda + \nu \leq 1$ for the inequality. The higher $a$ is, the higher the excess. The highest possible $a = \ell - 1$ leads to a zero excess and the second-highest possible $a = \ell - 2$ leads to $\nu + \lambda - 1$.

There are thus three different forms the excesses can be, resulting in three different possible excess vectors depending on which of the excesses is the largest:

$$\theta(\text{nuc}(v^E_{\ell\ell})) = (0, \ldots, 0, -\lambda, \ldots) ; \quad (7.3)$$

$$\theta(\text{nuc}(v^E_{\ell\ell})) = (0, \ldots, 0, -\nu, \ldots) ; \quad (7.4)$$

$$\theta(\text{nuc}(v^E_{\ell\ell})) = (0, \ldots, 0, \nu + \lambda - 1, \ldots) . \quad (7.5)$$

We distinguish between three cases depending on the value of $\lambda$. Each case again consists of three sub-cases depending on the value of $\nu$. In almost all cases, we shall see that there is an improvement, i.e. another assignment of values to $\lambda$ and $\nu$ (then the value of $\mu$ follows) in (7.1) which leads to another excess vector which is lexicographically less than the vector in (7.3), (7.4) or (7.5).

\[ \lambda < \frac{1}{3} : \]

\[ \nu < \frac{1}{3} : \] Either $-\lambda$ or $-\nu$ has the highest value, thus either (7.3) or (7.4) is the corresponding excess vector. If (7.3) is the corresponding excess vector, then choose $\lambda < \hat{\lambda} < \frac{1}{3}$ to improve the solution. If it is (7.4), then $\nu < \hat{\nu} < \frac{1}{3}$ improves the solution.

\[ \nu = \frac{1}{3} : \] This leads to the excess vector in (7.3), which is improved by choosing $\lambda < \hat{\lambda} < \frac{1}{3}$.

\[ \nu > \frac{1}{3} : \] Either (7.3) or (7.5) corresponds to this case, which can be improved by either $\lambda < \hat{\lambda} < \frac{1}{3}$ or $\frac{1}{3} < \hat{\nu} < \nu$.

\[ \lambda = \frac{1}{3} : \]

\[ \nu < \frac{1}{3} : \] Then (7.4) is the excess vector. Improvement is possible via $\nu < \hat{\nu} < \frac{1}{3}$.

\[ \nu = \frac{1}{3} : \] Each of the three possible highest excesses, $-\lambda, -\nu, \nu + \lambda - 1$, leads to a value of $\frac{1}{3}$, such that there is no improvement possible.

\[ \nu > \frac{1}{3} : \] The excess vector in (7.5) can be improved by $\frac{1}{3} < \hat{\nu} < \nu$. 

\[ \lambda = \frac{1}{3} : \]

\[ \nu < \frac{1}{3} : \] Then (7.4) is the excess vector. Improvement is possible via $\nu < \hat{\nu} < \frac{1}{3}$.

\[ \nu = \frac{1}{3} : \] Each of the three possible highest excesses, $-\lambda, -\nu, \nu + \lambda - 1$, leads to a value of $\frac{1}{3}$, such that there is no improvement possible.

\[ \nu > \frac{1}{3} : \] The excess vector in (7.5) can be improved by $\frac{1}{3} < \hat{\nu} < \nu$. 

\[ \lambda = \frac{1}{3} : \]

\[ \nu < \frac{1}{3} : \] Then (7.4) is the excess vector. Improvement is possible via $\nu < \hat{\nu} < \frac{1}{3}$.

\[ \nu = \frac{1}{3} : \] Each of the three possible highest excesses, $-\lambda, -\nu, \nu + \lambda - 1$, leads to a value of $\frac{1}{3}$, such that there is no improvement possible.

\[ \nu > \frac{1}{3} : \] The excess vector in (7.5) can be improved by $\frac{1}{3} < \hat{\nu} < \nu$. 

\[ \lambda = \frac{1}{3} : \]

\[ \nu < \frac{1}{3} : \] Then (7.4) is the excess vector. Improvement is possible via $\nu < \hat{\nu} < \frac{1}{3}$.

\[ \nu = \frac{1}{3} : \] Each of the three possible highest excesses, $-\lambda, -\nu, \nu + \lambda - 1$, leads to a value of $\frac{1}{3}$, such that there is no improvement possible.

\[ \nu > \frac{1}{3} : \] The excess vector in (7.5) can be improved by $\frac{1}{3} < \hat{\nu} < \nu$. 

\[ \lambda = \frac{1}{3} : \]

\[ \nu < \frac{1}{3} : \] Then (7.4) is the excess vector. Improvement is possible via $\nu < \hat{\nu} < \frac{1}{3}$.

\[ \nu = \frac{1}{3} : \] Each of the three possible highest excesses, $-\lambda, -\nu, \nu + \lambda - 1$, leads to a value of $\frac{1}{3}$, such that there is no improvement possible.

\[ \nu > \frac{1}{3} : \] The excess vector in (7.5) can be improved by $\frac{1}{3} < \hat{\nu} < \nu$. 

\[ \lambda = \frac{1}{3} : \]

\[ \nu < \frac{1}{3} : \] Then (7.4) is the excess vector. Improvement is possible via $\nu < \hat{\nu} < \frac{1}{3}$.
CHAPTER 7. GLOVE COMMUNICATION SITUATIONS

\( \lambda > \frac{1}{3} \):

- \( \nu < \frac{1}{3} \): Either (7.4) or (7.5) is the excess vector for this case. The former can be improved by \( \nu < \hat{\nu} < \frac{1}{3} \) and the latter by \( \frac{1}{3} < \hat{\lambda} < \lambda \).
- \( \nu = \frac{1}{3} \): The choice \( \frac{1}{3} < \hat{\lambda} < \lambda \) improves the excess vector in (7.5) corresponding to this case.
- \( \nu > \frac{1}{3} \): The excess vector in (7.5) corresponds to this case and is improved by choosing \( \frac{1}{3} < \hat{\nu} < \nu \) or \( \frac{1}{3} < \hat{\lambda} < \lambda \).

From the analysis above, it can be seen that in all cases, except the case where \( \lambda = \nu = \frac{1}{3} \), there is an improvement possible. Hence, the nucleolus must be the vector in (7.1) with \( \lambda = \nu = \frac{1}{3} \). Using (7.2), this leads to the required form. \( \square \)

Just like the proof of Proposition 6.4, also this proof shows how to deal with excesses en the excess vectors. However, these proofs are not straightforward, which is exactly the reason why the nucleolus in general is difficult.

Note that for the second statement we used the fact that \( C(v^E) \neq \emptyset \), which follows directly from the fact that \( v_\ell \) is balanced together with Proposition 3.3.

In Chapter 4 there is a conjecture about the nucleolus in communication situations with a star as communication graph. Theorem 7.4 explicitly states the nucleolus of the restricted game in a communication situation with a star as communication graph. Therefore, it is interesting to check whether a communication situation \( (N, v_\ell, E) \in CS^N \) with a glove game \( v_\ell \in GG^N \) as underlying game and a star \( E \) with special node \( k \in N \) as communication graph does satisfy the property that \( nuc_k(v^E) \geq nuc_k(v) \) and \( nuc_i(v^E) \leq nuc_i(v) \) for every \( i \in N \setminus \{k\} \). The following proposition shows that this is indeed the case:

**Corollary 7.5** Let \( (N, v_\ell, E) \in CS^N \) be a glove communication situation with \( E \) a star with special node \( k \in N \). Then

i) \( nuc_i(v^E) \leq nuc_i(v_\ell) \) for every \( i \in N \setminus \{k\} \);

ii) \( nuc_k(v^E) \geq nuc_k(v_\ell) \).

**Proof:** Without loss of generality, we can reduce the statement to glove communication situations \( (N, v_\ell, E) \) with \( E \) a star with special node \( 1 \in L \). Then there are three cases where we need to compare the nucleolus of the restricted game with the nucleolus of the underlying game: \( \ell < r \) or \( \ell = r \) or \( \ell > r \). Theorem 7.4 provides the nucleolus of the restricted game, whereas Proposition 6.4 provides the nucleolus of the underlying game.

If \( \ell < r \) : \( nuc(v^E) = (\frac{\ell + 1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, 0, \ldots, 0) \) versus \( nuc(v_\ell) = e^L \);

If \( \ell = r \) : \( nuc(v^E) = (\frac{\ell + 1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}) \) versus \( nuc(v_\ell) = \frac{1}{2} e^N \);

If \( \ell > r \) : \( nuc(v^E) = (\frac{r}{2}, 0, \ldots, 0, \frac{1}{2}, \ldots, \frac{1}{2}) \) versus \( nuc(v_\ell) = e^R \).
It is important to mention that $\ell \geq 1$ (otherwise, the nucleolus will be the zero vector). First, consider player 1, because he is the middle player. Player 1 receives respectively $\frac{\ell+1}{2}$, $\frac{\ell+1}{3}$ and $\frac{1}{2}$ with the nucleolus of the restricted game and 1, $\frac{1}{2}$ and 0 respectively with the nucleolus of the underlying game. In all three cases player 1 receives more with the nucleolus of the restricted game than with the nucleolus of the underlying game. Therefore, $\text{nuc}_{1}(v_{E}) \geq \text{nuc}_{1}(v_{L})$.

In order to prove the first statement, let $i \in N \setminus \{1\}$ and see whether player $i$ pays the cost of the benefits of player 1. If player $i \in L$, then the nucleolus of the restricted game assigns him $\frac{1}{2}$ if $\ell < r$, $\frac{1}{3}$ if $\ell = r$ and 0 if $\ell > r$. With the nucleolus of the underlying game he receives a value of 1 if $\ell < r$, a value of $\frac{1}{2}$ if $\ell = r$ and 0 if $\ell > r$. For all three cases it holds that $\text{nuc}_{i}(v_{E}) \leq \text{nuc}_{i}(v_{L})$. If $i \in R$, then we also have to compare 0, $\frac{1}{3}$ and $\frac{1}{2}$ with 0, $\frac{1}{2}$ and 1 respectively. Again, for all three cases it holds that $\text{nuc}_{i}(v_{E}) \leq \text{nuc}_{i}(v_{L})$. □

The next theorem is about glove communication situations with a cycle as communication graph. Before computing the nucleolus of the restricted game of such glove communication situations, first a lemma which will be very helpful for proving the next theorem.

**Lemma 7.6** Let $A, B \subseteq 2^{N}$ be two balanced collection. Then $A \cup B$ is also a balanced collection.

**Proof:** Let $\lambda_{A}$ be the balanced weight function such that $B(\lambda_{A}) = A$ and $\lambda_{B}$ be the balanced weight function such that $B(\lambda_{B}) = B$. Define $\lambda = c\lambda_{A} + (1 - c)\lambda_{B}$ for an arbitrary $c \in [0, 1]$. We are going to show that $\lambda$ is a balanced weight function such that $B(\lambda) = A \cup B$. For $X \in A \cup B$ we have that

$$\lambda(X) = c\lambda_{A}(X) + (1 - c)\lambda_{B}(X) > 0,$$

since $\lambda_{A}(X) > 0$ and $\lambda_{B}(X) > 0$. For $X \notin A \cup B$ it holds that $X \notin A$ and $X \notin B$, thus

$$\lambda(X) = c\lambda_{A}(X) + (1 - c)\lambda_{B}(X) = 0,$$

because $\lambda_{A}(X) = 0$ and $\lambda_{B}(X) = 0$. Finally, let $i \in N$ be an arbitrary player. Then

$$\sum_{S \in 2^{N}: i \in S} \lambda(S) = \sum_{S \in 2^{N}: i \in S} c\lambda_{A}(S) + (1 - c)\lambda_{B}(S)$$

$$= c \sum_{S \in 2^{N}: i \in S} \lambda_{A}(S) + (1 - c) \sum_{S \in 2^{N}: i \in S} \lambda_{B}(S)$$

$$= c + 1 - c = 1,$$

where $\sum_{S \in 2^{N}: i \in S} \lambda_{A}(S) = 1$ and $\sum_{S \in 2^{N}: i \in S} \lambda_{B}(S) = 1$ hold because $\lambda_{A}$ and $\lambda_{B}$ are balanced weight functions. □
Theorem 7.7  Let \((N, v_{tr}, E) \in CS^N\) be a glove communication situation with \(E\) a cycle. Then

\[
nuc(v_{tr}^E) = nuc(v_{tr}) = \begin{cases} 
eq & \text{if } \ell < r; \\ \frac{1}{2}e^N & \text{if } \ell = r; \\ e^R & \text{if } \ell > r. \end{cases}
\]

Proof: For the first case, note that a cycle is 2-connected. Therefore, \(M(v_{tr}^E) = M(v_{tr}) = e^L\) (using the proof of Lemma 6.2). For the minimum right vector, we have that \(m(v_{tr}^E) = e^L\). This holds because an arbitrary player \(j \in R\) needs for every pair of gloves a player \(i \in L\). Thus, he is gaining 1 from the pair of gloves, but he has to pay the left-hand player also 1 \((M(v_{tr}^E) = e^L)\). Therefore, \(m_j(v_{tr}^E) = 0\). For \(i \in L\), let \(S \in 2^N\) be a coalition with \(i \in S\) and let \(a := |L \cap S|\). Then \(v_{tr}^E(S) \leq a\) and \(\sum_{j \in S, j \neq i} M_j(v_{tr}^E) = a - 1\), which implies that the difference is lesser than or equal to 1. For \(S = N\) the difference is in fact equal to 1, which proves that \(m_i(v_{tr}^E) = 1\).

Now, we have that \(m(v_{tr}^E) = e^L = M(v_{tr}^E)\). Because \(C(v_{tr}^E) \neq \emptyset\) (due to the inheritance of balancedness according to Proposition 3.3 and the fact that glove games are balanced), we have that \(e^L = m(v_{tr}^E) \leq nuc(v_{tr}^E) \leq M(v_{tr}^E) = e^L\). This implies that \(nuc(v_{tr}^E) = e^L\), proving the first statement.

The third statement can be proven in a similar way, using that \(m(v_{tr}^E) = e^R = M(v_{tr}^E)\).

The second case is proven using the Kohlberg criterion. Notice that \(\frac{1}{2}e^N = nuc(v_{tr}) \in C(v_{tr}) \subseteq C(v_{tr}^E)\). Therefore, it holds that \(E(S, \frac{1}{2}e^N, v_{tr}) \leq 0\) for every \(S \in 2^N\). The highest (possible) excess is thus 0 and it is not easy to check that the second-highest (possible) excess is \(-\frac{1}{2}\). Denote \(B_1\) for the collection of non-trivial coalitions with the highest excess (obviously, there are coalitions with zero excess) and \(B_2\) for the collection of non-trivial coalitions with the second-highest excess.

First, the collection \(B_2\) contains every singleton coalition \(S = \{i\}\) for every \(i \in N\). We can use this in a smart way to prove that \(B_1 \cup B_2\) is a balanced collection, namely via a balanced weight function that puts very small weights on every non-singleton coalition in \(B_1 \cup B_2\). Then count for every player the total weight that is already given to this player and put exactly the difference between 1 and the total weight for this player as the weight on the singleton coalition containing this player. This ensures for every player a total weight of exactly 1. This trick also works for \(B_1 \cup B_2 \cup B_3\) etcetera.

It only remains to prove that \(B_1\) is a balanced collection. Take an arbitrary \(S \in B_1\). Then it must hold that \(v_{tr}^E(S) = 1 - |S|\) (because \(E(S, \frac{1}{2}e^N, v_{tr}) = v_{tr}^E(S) - \sum_{i \in S} \frac{1}{2}e^N = v_{tr}^E(S) - \frac{1}{2} |S| = 0\) and \(S\) is the union of pairs of left-hand and right-hand gloves, i.e. \(S = \bigcup_{i=1}^s P_i\), with \(s \in \mathbb{N}\), \(P_i \cap P_j = \emptyset\) and \(P_i = \{p, q\}\) for \(p \in L\) and \(q \in R\). The latter is true, because every player in \(S\) contributes \(-\frac{1}{2}\) to the excess, which can only be cancelled by making pairs. Now, we are going to prove the following claim.

Claim: There is a balanced collection \(C_S \subseteq 2^N\) such that \(S \in C_S\) and \(C_S \subseteq B_1\).

Proof of claim: Consider \(S/E = \{Q_1, Q_2, \ldots, Q_t\}\) for some \(t \in \mathbb{N}\) and define \(C_S := \{S, N \setminus Q_1, N \setminus Q_2, \ldots, N \setminus Q_t\}\). Obviously, \(S \in C_S\) by construction and \(N \setminus Q_j\) is
7.1. THE NUCLEOLUS

connected for every $1 \leq j \leq t$, since $Q_j$ is connected and $E$ a cycle (see also Figure 7.3). Furthermore, it must hold that every $Q_j$ contains a certain number of pairs of gloves, since otherwise it is not possible to have $v_{\ell\ell}^E(S) = \frac{1}{2}|S|$. Then it follows that $N \setminus Q_j$ also contains a certain number of pairs of gloves and thus $v_{\ell\ell}^E(N \setminus Q_j) = \frac{1}{2}|N \setminus Q_j|$. Consequently, $E(N \setminus Q_j, \frac{1}{2}e^N, v_{\ell\ell}^E) = v_{\ell\ell}^E(N \setminus Q_j) - \sum_{i \in N \setminus Q_j} \frac{1}{2}e_i^N = 0$, implying that $N \setminus Q_j \in B_1$ for every $1 \leq j \leq t$. This proves that $C_S \subseteq B_1$.

It only remains to show that $C_S$ is a balanced collection. Define the weight function $\lambda$ as $\lambda(X) := \frac{1}{t}$ for every $X \in C_S$ and $\lambda(X) := 0$ for every $X \notin C_S$. Then it is obvious that $B(\lambda) = C_S$. Let $p \in N$ an arbitrary player. If $p \in S$, then $p \in Q_j$ for some $1 \leq j \leq t$. Hence $p \notin N \setminus Q_j$. Player $p$ is contained in exactly $t$ coalitions of $C_S$ (namely in all coalitions of $C$ except $N \setminus Q_j$). If $p \notin S$, then $p \notin Q_j$ for every $1 \leq j \leq t$. Thus player $p$ is also contained in exactly $t$ coalitions of $C_S$ (namely $N \setminus Q_1, \ldots, N \setminus Q_t$). Every player is thus contained in exactly $t$ coalitions of $C_S$, which proves that $\lambda$ is a balanced weight function and thus that $C_S$ is a balanced collection. This completes the proof of the claim.

![Figure 7.3 - Q_j connected ⇒ N \setminus Q_j connected](image)

Using this claim, we can now finish the proof. For every $S \in B_1$ there is a balanced collection $C_S \subseteq B_1$ such that $S \in C_S$. This means that $B_1$ can be written as an union of those balanced collection, i.e. $B_1 = \bigcup_{S \in B_1} C_S$. Now, repeatedly applying Lemma 7.6 leads to the fact that $B_1$ is a balanced collection. Now, all conditions of the Kohlberg criterion are satisfied such that $\nu(v_{\ell\ell}^E) = \frac{1}{2}e^N$.

Note that the first equality follows from Proposition 6.4. □

The following example provides some insights into the sets $C_S$ for every $S \in B_1$.

**Example 7.8** Consider the glove communication situation $(N, v_{33}, E) \in CS^N$ with $E$ the cycle as depicted in Figure 7.4. The set $B_1$ consists of all coalitions $S$ with
$v^{E}_{fr}(S) = \frac{1}{2} |S|$. That is, 

$$B_1 = \{\{1, 6\}, \{3, 4\}, \{1, 2, 5, 6\}, \{2, 3, 4, 5\}, \{1, 3, 4, 6\}\}.$$ 

The first four coalitions of $B_1$ have relative simple corresponding collections $C_S$. For these coalitions it holds that $\bar{S}/E = \{S\}$, such that $C_S = \{S, N \setminus S\}$. However, this does not hold for the coalition $\{1, 3, 4, 6\}$. From Figure 7.4 it can be seen that $\bar{S}/E = \{\{1, 6\}, \{3, 4\}\}$. According to the proof of Theorem 7.7 the corresponding collection $C_{\{1,3,4,6\}} = \{\{1, 3, 4, 6\}, \{2, 3, 4, 5\}, \{1, 2, 5, 6\}\}$. This is indeed a balanced collection with coalitions from $B_1$ (weights of $\frac{1}{2}$ suffices).

![Figure 7.4 - A 6-cycle](image)

Since cycles are 2-connected, one could wonder whether a similar statement as in Theorem 7.7 also holds for glove communication situations with a 2-connected communication graph. That is, $\text{nuc}(v^{E}_{fr}) = \text{nuc}(v_{fr})$ for $(N, v_{fr}, E) \in CS^N$ with $E$ a 2-connected communication graph. Unfortunately, this is not true as the following example shows.

**Example 7.9** Let $(N, v_{22}, E) \in CS^N$ be the four-person glove communication situation with $v_{22} \in GG^N$ and $E$ the 2-connected graph as depicted in Figure 7.5. We are going to show that $\text{nuc}(v^{E}_{22}) \neq \frac{1}{2} e^N = \text{nuc}(v_{22})$. Consider the values of $v^{E}_{22}$ and the excesses in Table 7.1.

Using the Kohlberg criterion, let $B_1 := \{\{1, 4\}, \{2, 3\}, \{2, 4\}\}$ be the collection containing the non-trivial coalitions with the highest excess. Due to player 1, the first coalition $\{1, 4\}$ should get a weight of 1. However, this is not possible for player 4, because $\{2, 4\}$ also needs some positive weight. We conclude that $B_1$ is not balanced. Therefore, $\text{nuc}(v^{E}_{22}) \neq \frac{1}{2} e^N$. \[\triangle\]
7.2 THE COMPROMISE VALUE

Table 7.1 – The values and excesses of $v^{E}_{22}$

| $S$  | $|S| = 1$  | $\{1,2\}$  | $\{1,3\}$  | $\{1,4\}$  | $\{2,3\}$  | $\{2,4\}$  | $\{3,4\}$  |
|------|------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $v_{22}^{E}(S)$ | 0          | 0           | 0           | 1           | 1           | 1           | 0           |
| $E(S, \frac{1}{2}e^{N})$ | $-\frac{1}{2}$ | -1          | -1          | 0           | 0           | 0           | -1          |

<table>
<thead>
<tr>
<th>$S$</th>
<th>${1,2,3}$</th>
<th>${1,2,4}$</th>
<th>${1,3,4}$</th>
<th>${2,3,4}$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_{22}^{E}(S)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$E(S, \frac{1}{2}e^{N})$</td>
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<td>$-\frac{1}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 7.5 – A 2-connected graph on 4 nodes

7.2 The compromise value

This section is about the compromise value of glove communication situations. The section consists of three results. The first result is similar to the result in the previous section about the nucleolus. It explicitly states the compromise value for the restricted game of a glove communication situation with a star as communication graph. The second result is a corollary of these two results. The third result is a nice application of Theorem 5.2.

Just like Theorem 7.4, the first result only considers a star with player 1 as the special node as communication graph of a communication situation with a glove game as underlying game. This can be done without loss of generality, since by relabelling and/or interchanging the players we can reach this situation.
Theorem 7.10 Let \((N, v_{\ell r}, E) \in CS^N\) be a glove communication situation with \(E\) a star with special node \(1 \in L\). Then it holds that

\[
\tau(v_{\ell r}^E) = \begin{cases} 
\left(\frac{\ell+1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, 0, \ldots, 0\right) & \text{if } \ell < r; \\
\left(\frac{r^2}{2}, \frac{r}{2}, \ldots, \frac{r}{2}, \frac{r}{2}, \ldots, \frac{r}{2}, 0\right) & \text{if } \ell = r; \\
\left(0, \ldots, 0, \frac{1}{2}, \ldots, \frac{1}{2}\right) & \text{if } \ell > r.
\end{cases}
\]

Proof: Theorem 5.3 guarantees \(\tau(v_{\ell r}^E) = \text{nuc}(v_{\ell r}^E)\) if \(\ell \neq r\) and \(E\) a star, since \(v_{\ell r}\) is strongly compromise admissible according to Lemma 6.2. Using Theorem 7.4 this proves the first and the third statement.

For the case where \(\ell = r\), the proof explicitly calculates \(\tau(v_{\ell r}^E)\). For player 1 it holds that \(M_1(v_{\ell r}^E) = \ell - 0 = \ell\). Furthermore, \(M_i(v_{\ell r}^E) = \ell - (\ell - 1) = 1\) for all \(i \in N\) with \(i \neq 1\). Thus

\[
M(v_{\ell r}^E) = (\ell, 1, \ldots, 1, 1, \ldots, 1).
\]

For the minimum right vector, notice that each \(i \in N\), \(i \neq 1\), needs player 1 to make any pair of gloves. Since \(M_1(v_{\ell r}^E) = \ell\) we obtain that \(m_i(v_{\ell r}^E) = 0\) for every \(i \in N\), \(i \neq 1\). For the special player \(1 \in L\), notice that he needs a player with a right-hand glove for every pair of gloves. Since \(M_j(v_{\ell r}^E) = 1\) for every \(j \in R\) we obtain that also \(m_1(v_{\ell r}^E) = 0\). Thus

\[
m(v_{\ell r}^E) = (0, 0, \ldots, 0, 0, \ldots, 0).
\]

Then \(\tau(v_{\ell r}^E) = \alpha M(v_{\ell r}^E)\) with \(\alpha \in [0, 1]\) such that \(\alpha \sum_{i \in N} M_i(v_{\ell r}^E) = v_{\ell r}(N) = \ell\). The latter equation implies that \(\alpha(3\ell - 1) = \ell\), which in turn implies that \(\alpha = \frac{\ell}{3\ell - 1}\). Then the result follows.

Theorem 7.4 and Theorem 7.10 both explicitly characterize the nucleolus and the compromise value of the restricted game of a glove communication situation respectively. For glove communication situations with a glove game \(v_{\ell r} \in GG^N\) as underlying game with \(\ell \neq r\), it was already mentioned in the proof above that both solutions coincide. It is clear that this is not the case for communication situations with a glove game \(v_{\ell r} \in GG^N\) as underlying game. However, if we let the number \(\ell\) approaches infinity (i.e. involving more and more players), then the following corollary shows that both solutions coincide in the limit.

Corollary 7.11 The following two statements hold:

i) \(\tau(v_{\ell r}^E) = \text{nuc}(v_{\ell r}^E)\) for every \((N, v_{\ell r}, E) \in CS^N\) with \(\ell \neq r\) and \(E\) a star with special node \(1 \in L\);

ii) \(\lim_{\ell \to \infty} \text{nuc}(v_{\ell r}^E) = \lim_{\ell \to \infty} \tau(v_{\ell r}^E)\) where \(v_{\ell r}^E\) is the restricted game of \((N, v_{\ell r}, E) \in CS^N\) with \(E\) a star with special node \(1 \in L\).

Proof:

i) This is implied by Theorem 5.3 and Lemma 6.2.
We are going to prove this for player 1 and an arbitrary player \( i \in N \setminus \{1\} \) separately. First, for player 1 we have that the limit of the nucleoli is going to infinity as the number of left-hand glove players approaches infinity:

\[
\lim_{\ell \to \infty} \text{nuc}_1(v^{E}_{\ell\ell}) = \lim_{\ell \to \infty} \frac{\ell + 1}{3} = \infty.
\]

This is also the case for the limit of the compromise values:

\[
\lim_{\ell \to \infty} \tau_1(v^{E}_{\ell\ell}) = \lim_{\ell \to \infty} \frac{\ell^2}{3\ell - 1} = \infty.
\]

Therefore, \( \lim_{\ell \to \infty} \text{nuc}_1(v^{E}_{\ell\ell}) = \lim_{\ell \to \infty} \tau_1(v^{E}_{\ell\ell}) \).

For player \( i \in N \setminus \{1\} \) we have that \( \text{nuc}_i(v^{E}_{\ell\ell}) = \frac{1}{3} \) and \( \tau_i(v^{E}_{\ell\ell}) = \frac{\ell}{3\ell - 1} \). Obviously, \( \lim_{\ell \to \infty} \text{nuc}_i(v^{E}_{\ell\ell}) = \frac{1}{3} \), such that we need to prove that \( \lim_{\ell \to \infty} \tau_i(v^{E}_{\ell\ell}) = \frac{1}{3} \).

We have that

\[
\lim_{\ell \to \infty} \tau_i(v^{E}_{\ell\ell}) = \lim_{\ell \to \infty} \frac{\ell}{3\ell - 1} = \lim_{\ell \to \infty} \frac{1}{3 - \frac{1}{\ell}} = \frac{1}{3}.
\]

Hence, \( \lim_{\ell \to \infty} \text{nuc}_i(v^{E}_{\ell\ell}) = \lim_{\ell \to \infty} \tau_i(v^{E}_{\ell\ell}) \) for any player \( i \in N \setminus \{1\} \). We conclude that \( \lim_{\ell \to \infty} \text{nuc}(v^{E}_{\ell\ell}) = \lim_{\ell \to \infty} \tau(v^{E}_{\ell\ell}) \).

In Chapter 3 we studied the inheritance of several properties in communication situations. The results obtained there, were used in Chapter 4 to derive relations between the nucleolus of the restricted game and the nucleolus of the underlying game. However, in Chapter 5 only the property of being strongly compromise admissible was needed for the relations between the compromise value of the restricted game and the compromise value of the underlying game. In particular, Theorem 5.2 does not include the compromise stability condition for the underlying game in contrast to Theorem 4.2.

It is for this reason that we have the following result about the compromise value of the restricted game of a glove communication situation in the case the communication graph is a 2-connected graph.

**Theorem 7.12** Let \((N,v_{\ell\ell},E) \in CS^N\) be a glove communication situation with \( E \) a 2-connected graph. Then \( \tau(v^{E}_{\ell\ell}) = \tau(v_{\ell\ell}) \).

**Proof:** If \( \ell \neq r \), then the results follows from combining Lemma 6.2 and Corollary 5.1. If \( \ell = r \), then consider the minimum right vector of \( v_{\ell\ell} \). From Lemma 6.2 we know that \( m(v_{\ell\ell}) = 0 \), which implies that \( m(v^{E}_{\ell\ell}) = 0 \) (since \( 0 \leq m(v^{E}_{\ell\ell}) \leq m(v_{\ell\ell}) = 0 \), due to the fact that \( v_{\ell\ell} \) is zero-normalized and Lemma 3.5). Furthermore, Lemma 6.2 ensures compromise admissibility for \( v_{\ell\ell} \) such that also \( v^{E}_{\ell\ell} \) is compromise admissible according to Theorem 3.6. Therefore, we can use Theorem 5.2 in order to obtain \( \tau(v^{E}_{\ell\ell}) = \tau(v_{\ell\ell}) \). \( \square \)
The above theorem also involves cycles as communication graphs, since cycles are 2-connected. Recall that for the nucleolus we only had a similar theorem about cycles, Theorem 7.7. However, for the compromise value there is also equality between the compromise value of the restricted game and the compromise value of the underlying game for 2-connected graphs.
Chapter 8

Glove communication situations with a random star

This chapter continues with glove communication situations, but in a different context. Until now, every graph was a fixed graph. What happens if the graph is not a fixed graph, but a random graph? Does that change anything to for example the nucleolus? We look at glove communication situations with a random star as communication graph. A random star is a star where every player has equal probability to become the special node of the star. For a communication situation \((N, \nu_{tr}, E) \in CS^N\) with \(E\) a random star (i.e. every player has probability \(\frac{1}{|N|}\)), denote \(E_i\) for the star with player \(i \in N\) as the middle point of the star. Recall from Section 7.1 that the nucleolus of a glove communication situation with a star as communication graph is given by (cf. Theorem 7.4):

\[
\begin{align*}
\nuc(v_{E_i}^\ell r) &= \left\{ \begin{array}{ll} 
\frac{\ell+1}{2}e^i + \frac{1}{2}e^{L \backslash \{i\}} & \text{if } i \in L; \\
\frac{\ell}{2}e^i + \frac{1}{2}e^L & \text{if } i \in R; 
\end{array} \right. \\
\text{if } \ell < r; \\
\nuc(v_{E_i}^\ell L) &= \left\{ \begin{array}{ll} 
\frac{\ell+1}{3}e^i + \frac{1}{3}e^{N \backslash \{i\}} & \text{if } i \in L; \\
\frac{\ell}{3}e^i + \frac{1}{3}e^R & \text{if } i \in R; 
\end{array} \right. \\
\text{if } \ell > r; \\
\nuc(v_{E_i}^\ell r) &= \left\{ \begin{array}{ll} 
\frac{\ell+1}{2}e^i + \frac{1}{2}e^{R \backslash \{i\}} & \text{if } i \in L; \\
\frac{\ell}{2}e^i + \frac{1}{2}e^R & \text{if } i \in R; 
\end{array} \right.
\end{align*}
\]

(8.1)

(8.2)

(8.3)

In Section 7.2 the compromise value of a glove communication situation with \(E\) a star was calculated as (cf. Theorem 7.10 and Corollary 7.11):

\[
\begin{align*}
\tau(v_{E_i}^\ell r) &= \nuc(v_{E_i}^\ell r) & \text{if } \ell \neq r; \\
\tau(v_{E_i}^\ell L) &= \frac{\ell^2}{3\ell - 1}e^i + \frac{\ell}{3\ell - 1}e^{N \backslash \{i\}}.
\end{align*}
\]

(8.4)

(8.5)

We would like to characterize the nucleolus and the compromise value of the restricted game of the glove communication situation with a random star as communication graph.
CHAPTER 8. GLOVE COMMUNICATION SITUATIONS WITH A RANDOM STAR

To do this, we have to come up with a possible way to average in some sense over all the possible communication graphs. There are two (natural) ways to do this. First of all, it is possible to take the average over all nucleoli (respectively compromise values) of the restricted games for the possible communication graphs. This is denoted by $E[nuc(v^E_{\ell r})]$ (resp. $E[\tau(v^E_{\ell r})]$) and is computed in the following way:

\[
E[nuc(v^E_{\ell r})] = \sum_{i \in N} \frac{1}{|N|} nuc(v^E_i); \quad (8.6)
\]

\[
E[\tau(v^E_{\ell r})] = \sum_{i \in N} \frac{1}{|N|} \tau(v^E_i). \quad (8.7)
\]

Using this, we have the following result:

**Theorem 8.1** Let $(N, v_{\ell r}, E) \in CS^N$ be a glove communication situation with $E$ a random star. Then it holds that

\[
E[nuc(v^E_{\ell r})] = \frac{2\ell + r}{2(\ell + r)} e^L + \frac{\ell}{2(\ell + r)} e^R \quad \text{if } \ell < r;
\]

\[
\frac{1}{2} e^N \quad \text{if } \ell = r;
\]

\[
\frac{r}{2(\ell + r)} e^L + \frac{\ell + 2r}{2(\ell + r)} e^R \quad \text{if } \ell > r.
\]

**Proof:** If $\ell \neq r$ (that is, $\ell < r$ or $\ell > r$), we can use equations (8.6) and (8.7) such that it directly follows from equation (8.4) that:

\[
E[nuc(v^E_{\ell r})] = E[\tau(v^E_{\ell r})] = \frac{1}{|N|} \left( \frac{\ell + 1}{2} + 1 + \frac{1}{2} (|N| - 1) \right) e^L + \frac{1}{|N|} \cdot \frac{\ell}{2} e^R
\]

In order to prove the explicit formula for $\ell < r$, note that a left-hand glove player is the middle player only once and $|N| - 1$ times just a left-hand glove player on the end of the star. A right-hand glove player only receives some value if he is the middle player (according to equation (8.1)). This results in:

\[
E[nuc(v^E_{\ell r})] = \sum_{i \in N} \frac{1}{|N|} nuc(v^E_i) = \sum_{i \in N} \frac{1}{|N|} \tau(v^E_i) = E[\tau(v^E_{\ell r})].
\]

The explicit formula for $\ell > r$ follows from the same reasoning. A left-hand glove player only receives some value if he is the middle player and a right-hand glove player is the
middle player only once and \(|N|−1\) times just a player on the end of the star. Therefore,

\[
\mathbb{E}[\nu(v^E_{\ell r})] = \sum_{i \in N} \frac{1}{|N|} \nu(v^E_{ri}) \\
= \frac{1}{|N|} \cdot \frac{r}{2} e^L + \frac{1}{|N|} \left( \frac{r + 1}{2} \cdot \frac{1}{2} \cdot (|N| - 1) \right) e^R \\
= \frac{1}{\ell + r} \cdot \frac{r}{2} e^L + \frac{1}{\ell + r} \cdot \frac{r + 1 + \ell + r - 1}{2} e^R \\
= \frac{r}{2(\ell + r)} e^L + \frac{\ell + 2r}{2(\ell + r)} e^R.
\]

If \(\ell = r\), we are going to compute the nucleolus and the compromise value directly, using the equations above. Note that \(|N| = 2\ell\) in this case. Furthermore, it holds that every player is the middle player only once. The other \(|N|−1\) times, this player is a ‘normal’ player. Using equation (8.2) together with equation (8.6), we obtain for the nucleolus:

\[
\mathbb{E}[\nu(v^E_{\ell \ell})] = \sum_{i \in N} \frac{1}{|N|} \nu(v^E_{ri}) \\
= \frac{1}{|N|} \left( \frac{\ell + 1}{3} + \frac{1}{3} \cdot (|N| - 1) \right) e^N \\
= \frac{1}{2\ell} \cdot \frac{\ell + 1 + 2\ell - 1}{3} e^N \\
= \frac{1}{2\ell} \cdot \frac{3\ell e^N}{3} \\
= \frac{1}{2} e^N.
\]

The computation for the compromise value goes in the same way, using equations (8.5) and (8.7):

\[
\mathbb{E}[\tau(v^E_{\ell \ell})] = \sum_{i \in N} \frac{1}{|N|} \tau(v^E_{ri}) \\
= \frac{1}{|N|} \left( \frac{\ell^2}{3\ell - 1} + \frac{\ell}{3\ell - 1} \cdot (|N| - 1) \right) e^N \\
= \frac{1}{2\ell} \cdot \frac{\ell^2 + 2\ell^2 - \ell}{3\ell - 1} e^N \\
= \frac{1}{2\ell} \cdot \frac{\ell(3\ell - 1)}{3\ell - 1} e^N \\
= \frac{1}{2} e^N.
\]

This proves that \(\mathbb{E}[\nu(v^E_{\ell \ell})] = \mathbb{E}[\tau(v^E_{\ell \ell})] = \frac{1}{2} e^N\). \(\Box\)

There is also a second way to characterize the nucleolus and the compromise value of a glove communication situation with a random star as communication graph. In the
above theorem we took the average over all possible nucleoli (resp. compromise values). It is also possible to first take the average for the value of a certain coalition. That is, we first derive an expected game and then are going to characterize the nucleolus and the compromise value of this expected game. So we are going to determine for every \( S \in 2^N \) the expected value of the restricted game, \( E[v_{E}(S)] \). The expected game with these values is denoted by \( E[v_{E}^E] \).

**Lemma 8.2** Let \((N, v_{E}, E) \in CS^N\) be a glove communication situation with \( E \) a random star. Then it holds that \( E[v_{E}^E(S)] = \frac{|S|}{|N|}v_{E}(S) \).

**Proof:** Just write out the expected value of \( v_{E}^E(S) \):

\[
E[v_{E}^E(S)] = \sum_{i \in N} \frac{1}{|N|} v_{E_i}^E(S) \\
= \sum_{i \in S} \frac{1}{|N|} v_{E}(S) + \sum_{i \notin S} \frac{1}{|N|} 0 \\
= \frac{|S|}{|N|}v_{E}(S),
\]

where we used the fact that \( v_{E_i}^E(S) = v_{E}(S) \) if \( i \in S \) and \( v_{E_i}^E(S) = 0 \) if \( i \notin S \). \( \square \)

With Lemma 8.2 it is possible to characterize the nucleolus and the compromise value of the expected (restricted) game. Unfortunately, a similar statement as in Theorem 8.1 does not hold. This means that \( \text{nuc}(E[v_{E}^E]) \neq \tau(E[v_{E}^E]) \). This is shown by the following star.

**Example 8.3** Consider the five-person glove communication situation \((N, v_{23}, E) \) with \( E \) a random star. Table 8.1 shows all the values of \( v_{E_i}^E \), \( E[v_{E}^E]\) and the excesses of the imputation \( \tau(E[v_{E}^E]) \) as calculated below.

From Table 8.1 it follows that \( M(E[v_{E}^E]) = \frac{8}{9}e^L + \frac{2}{3}e^R \) and \( m(E[v_{E}^E]) = 0 \). The latter involves some calculations, but this is straightforward. Combining the utopia-vector and the minimum right vector, we obtain the compromise value of the expected restricted game:

\[
\tau(E[v_{E}^E]) = \alpha M(E[v_{E}^E]) = \frac{5}{9} M(E[v_{E}^E]) = \frac{2}{3}e^L + \frac{2}{9}e^R.
\]

In the above calculation, \( \alpha \in [0, 1] \) is calculated as follows: \( \alpha(2\frac{8}{9} + 3\frac{2}{3}) = 2 \) implies that \( \alpha \frac{18}{3} = 2 \), which results in the fact that \( \alpha = \frac{5}{9} \).

We are going to show that \( \text{nuc}(E[v_{E}^E]) \neq \tau(E[v_{E}^E]) \) using the Kohlberg criterion. The highest excess is \( -\frac{8}{35} \) such that \( B_1 = \{\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 4, 5\}\} \). This is clearly not a balanced collection (look at players 1 and 3 for example). Therefore, \( \text{nuc}(E[v_{E}^E]) \neq \tau(E[v_{E}^E]) \).

\( \triangle \)

Example 8.3 is about the situation where \( \ell \neq r \). In the other case, \( \ell = r \), it seems natural to treat every player equal. Every player has the same probability to become the middle
Therefore, random star. Then nucleolus and the compromise value are both the same and assign to every player the same? The following theorem provides a positive answer to this question.

**Theorem 8.4** Let \((N, v_{\ell}), E \in CS^N\) be a glove communication situation with \(E\) a random star. Then \(\text{nuc}(E[v_{\ell}^E]) = \tau(E[v_{\ell}^E]) = \frac{1}{2}e^N\).

**Proof:** We first compute the compromise value of the expected game, \(\tau(E[v_{\ell}^E]) = \frac{1}{2}e^N\). Afterwards, we are going to prove that the nucleolus coincides with the compromise value using the Kohlberg criterion.

Let \(i \in N\) be an arbitrary player and consider the utopia-value for this player. With Lemma 8.2 it holds that

\[
M_i(E[v_{\ell}^E]) = E[v_{\ell}^E(N)] - E[v_{\ell}^E(N \setminus \{i\})] = \ell - \frac{2\ell - 1}{2\ell} (\ell - 1) = \frac{3\ell - 1}{2\ell}
\]

Therefore, \(M(E[v_{\ell}^E]) = \frac{3\ell - 1}{2\ell} e^N\). For the minimum right vector, consider an arbitrary

| \(S\) | \{1\} | \{2\} | \{3\} | \{4\} | \{5\} | \{1, 2\} | \{1, 3\} | \{1, 4\} | \{1, 5\} | \{2, 3\} |
|---|---|---|---|---|---|---|---|---|---|
| \(v_{\ell}(S)\) | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| \(E[v_{\ell}^E(S)]\) | 0 | 0 | 0 | 0 | 0 | \(\frac{2}{3}\) | \(\frac{2}{3}\) | \(\frac{2}{3}\) | \(\frac{2}{3}\) |
| Excess | \(-\frac{2}{3}\) | \(-\frac{2}{3}\) | \(-\frac{2}{3}\) | \(-\frac{2}{3}\) | \(-\frac{4}{3}\) | \(-\frac{22}{35}\) | \(-\frac{22}{35}\) | \(-\frac{22}{35}\) | \(-\frac{22}{35}\) |

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**Table 8.1** – The values of \(v_{\ell}, E[v_{\ell}^E]\) and the excesses \(E(S, \frac{2}{3}e^L + \frac{2}{3}e^R, E[v_{\ell}^E(S)])\)
coalition \( S \in 2^N \) with \( i \in S \). Then we have that (using Lemma 8.2 in the first line)

\[
\mathbb{E}[v^E_\ell(S)] - \sum_{j \in S, j \neq i} M_j(\mathbb{E}[v^E_\ell]) = \frac{|S|}{|N|} v^E_\ell(S) - \sum_{j \in S, j \neq i} \frac{3\ell - 1}{2\ell} e^N_j
\]

\[
= \frac{|S|}{2\ell} v^E_\ell(S) - \frac{3\ell - 1}{2\ell} (|S| - 1)
\]

\[
= \frac{|S|}{2\ell} (v^E_\ell(S) - 3\ell + 1) + \frac{3\ell - 1}{2\ell}
\]

\[
\leq \frac{|S|}{2\ell} (-2\ell + 1) + \frac{3\ell - 1}{2\ell}
\]

\[
\leq -2\ell + 1 + \frac{3\ell - 1}{2\ell}
\]

\[
= \frac{4\ell^2 + 5\ell - 1}{2\ell^2} - 2\ell + \frac{5}{2} - \frac{1}{2\ell},
\]

where we used \( v^E_\ell(S) \leq \ell \) and \( \frac{|S|}{2\ell} \leq 1 \) for the first and second inequality respectively. Define \( f(\ell) := \frac{4\ell^2 + 5\ell - 1}{2\ell^2} - 2\ell + \frac{5}{2} - \frac{1}{2\ell} \) such that we want to check that \( f(\ell) \leq 0 \) for every \( \ell \geq 1 \). It is easy to see that \( f(1) = 0 \). Moreover, it holds that \( f(\ell) \geq f(\ell + 1) \) for every \( \ell \geq 1 \):

\[
f(\ell) - f(\ell + 1) = -2\ell + \frac{5}{2} - \frac{1}{2\ell} - (-2(\ell + 1) + \frac{5}{2} - \frac{1}{2(\ell + 1)})
\]

\[
= -2\ell + \frac{5}{2} - \frac{1}{2\ell} + 2\ell + 2 - \frac{5}{2} + \frac{1}{2(\ell + 1)}
\]

\[
= 2 - \frac{1}{\ell} + \frac{1}{\ell + 1}
\]

\[
= \frac{2\ell(\ell + 1)}{\ell(\ell + 1)} - \frac{1}{\ell(\ell + 1)} + \frac{1}{2(\ell + 1)}
\]

\[
= \frac{2\ell^2 + 2\ell - 1}{\ell(\ell + 1)}
\]

\[
\geq 0,
\]

since \( 2\ell^2 + 2\ell \geq 4 \) due to the fact that \( \ell \geq 1 \). Now, we can conclude that \( f(\ell) \leq 0 \) for every \( \ell \geq 1 \). This implies that

\[
\mathbb{E}[v^E_\ell(S)] - \sum_{j \in S, j \neq i} M_j(\mathbb{E}[v^E_\ell]) \leq 0
\]

for every \( S \in 2^N \) with \( i \in S \). Thus \( m_i(\mathbb{E}[v^E_\ell]) \leq 0 \). For a zero-normalized game it always holds that the minimum right vector is non-negative, implying that \( m_i(\mathbb{E}[v^E_\ell]) = 0 \). Therefore, \( m(\mathbb{E}[v^E_\ell]) = 0 \).

Now, we can calculate the compromise value of the expected game, using the utopia-
vector and the minimum right vector:
\[
\tau(\mathbb{E}[v^{E}_{\ell\ell}]) = \alpha M(\mathbb{E}[v^{E}_{\ell\ell}]) + (1 - \alpha)m(\mathbb{E}[v^{E}_{\ell\ell}]) \\
= \frac{3\ell - 1}{2\ell}e^{N} \\
= \frac{\ell}{3\ell - 1} \cdot \frac{3\ell - 1}{2\ell}e^{N} \\
= \frac{1}{2}e^{N},
\]
\[
\alpha \in [0, 1]
\]

where \(\alpha\) is computed in the following way: \(\alpha \frac{3\ell - 1}{2\ell} \cdot 2\ell = \ell\) implies that \(\alpha(3\ell - 1) = \ell\) and thus that \(\alpha = \frac{\ell}{3\ell - 1}\). This completes the proof of the first part.

Secondly, we are going to prove that \(\text{nuc}(\mathbb{E}[v^{E}_{\ell\ell}]) = \frac{1}{2}e^{N}\) using the Kohlberg criterion.

First, the excesses are of the following form:
\[
E(S, \frac{1}{2}e^{N}, \mathbb{E}[v^{E}_{\ell\ell}]) = \mathbb{E}[v^{E}_{\ell\ell}(S)] - \sum_{i \in S} \frac{1}{2}e^{N} \\
= \frac{|S|}{|N|}v^{\ell\ell}(S) - \frac{1}{2}|S|
\]
for every \(S \in 2^{N}\). Note that for every \(i \in N\) it holds that \(E(\{i\}, \frac{1}{2}e^{N}, \mathbb{E}[v^{E}_{\ell\ell}]) = -\frac{1}{2}\).

Moreover, we have the following claim:

**Claim:** For every \(S \in 2^{N} \setminus \{\emptyset, N\}\) it holds that \(E(S, \frac{1}{2}e^{N}, \mathbb{E}[v^{E}_{\ell\ell}]) \leq -\frac{1}{2}\).

**Proof of claim:** Let \(S \in 2^{N}\) be a coalition with \(S \neq \emptyset\) and \(S \neq N\). If \(S\) is a singleton, then the excess is equal to \(-\frac{1}{2}\) as proven above. If \(S = N \setminus \{i\}\) for any \(i \in N\), then it holds that \(v^{\ell\ell}(S) = \frac{1}{2}|N| - 1\) (since there is one player missing, leaving one pair of gloves incomplete). This results in an excess of:
\[
E(S, \frac{1}{2}e^{N}, \mathbb{E}[v^{E}_{\ell\ell}]) = \frac{|S|}{|N|}v^{\ell\ell}(S) - \frac{1}{2}|S| \\
= \frac{|N| - 1}{|N|} \left( \frac{1}{2}|N| - 1 \right) - \frac{1}{2} \left( |N| - 1 \right) \\
= \frac{|N| - 1}{|N|} \left( \frac{1}{2}|N| - 1 \right) - \frac{1}{2} \left( |N| - 1 \right) \\
\leq \frac{1}{2} \left( |N| - 1 \right) - \frac{1}{2} - \frac{1}{2} (|N| - 1) \\
= -\frac{1}{2}.
\]

This proves the inequality for coalitions \(S \in 2^{N}\) with \(|S| = |N| - 1\). For all other coalitions \(S \in 2^{N}\) it holds that \(v^{\ell\ell}(S) \leq \frac{1}{2}|S|\), because every pair of gloves consists of two gloves and thus involves two players. Therefore, \(E(S, \frac{1}{2}e^{N}, \mathbb{E}[v^{E}_{\ell\ell}]) \leq \frac{|S|}{|N|} \cdot \frac{1}{2}|S| - \frac{1}{2} |S|\). Denote \(s := |S|\) and define \(g(s) := \frac{1}{|N|}s^2 - \frac{s}{2} + \frac{1}{2}\), such that we need to show that \(g(s) \leq 0\) for
every $2 \leq s \leq |N| - 2$ (i.e. for every $S$ with $2 \leq |S| \leq |N| - 2$). Since $\frac{1}{|N|} > 0$ we know that $g$ is a parabola shaped like a valley. Therefore, calculating the roots of $g$ lead to an interval of non-positive values of $g$. Solving $g(s) = 0$ with the quadratic formula leads to $s = \frac{|N|}{2} \pm |N| \sqrt{\frac{1}{4} - \frac{1}{|N|}}$. Furthermore, it holds that $\frac{|N|}{2} - 2 \leq |N| \sqrt{\frac{1}{4} - \frac{1}{|N|}}$:

$$\left(\frac{|N|}{2} - 2\right)^2 - \left(|N| \sqrt{\frac{1}{4} - \frac{1}{|N|}}\right)^2 = \frac{|N|^2}{4} - 2|N| + 4 - |N|^2 \left(\frac{1}{4} - \frac{1}{|N|}\right)$$

$$= 4 - |N|$$

$$\leq 0,$$

where the inequality follows from the fact that $|N| \geq 4$. This leads to the two inequalities $\frac{|N|}{2} - |N| \sqrt{\frac{1}{4} - \frac{1}{|N|}} \leq 2$ and $|N| - 2 \leq \frac{|N|}{2} + |N| \sqrt{\frac{1}{4} - \frac{1}{|N|}}$. This means that for every $2 \leq s \leq |N| - 2$ we have that $g(s) \leq 0$, because of the valley-shape (see also Figure 8.1). This completes the proof of the claim.

Now, we have that every excess is lesser than or equal to $\frac{1}{2}$ with equality for the singletons. Using the Kohlberg criterion, this leads to the fact that $\{i\} \in B_1$ for every $i \in N$. Using the trick we already used on page 58, we see that $B_1$ is a balanced collection. Moreover, also the next collections ($B_1 \cup B_2$, $B_1 \cup B_2 \cup B_3$, etcetera) are balanced using this trick. We conclude that $\frac{1}{2} e^N$ is the nucleolus of the expected game, i.e. $\text{nuc}(E[v_{ij}]) = \frac{1}{2} e^N$. □

![Figure 8.1 – Sketch of the function $g$](image-url)
Chapter 9

Ingredient games

It is possible to generalise the concept of a glove game. The idea of making pairs of gloves therefore does not longer hold. This idea is replaced by the idea of combining ingredients in order to make a delicious meal. Every player has only one of the ingredients needed for a certain meal, but there are maybe more players with the same ingredient. A complete set of ingredients makes it possible to cook a delicious meal, which has a value of 1. This translates in the following definition.

**Definition 9.1** An ingredient game is a game \( v_m \in TU^N \) with \( N = \bigcup_{i=1}^m N_i \) for \( m \in \mathbb{N} \) such that \( |N_1| \leq |N_2| \leq \ldots \leq |N_m| \) and \( N_i \cap N_j = \emptyset \) for every \( i, j \in \{1, \ldots, m\} \) with \( i \neq j \) and for every \( S \in 2^N \setminus \{\emptyset\} \) defined as

\[
v_m(S) := \min_{1 \leq i \leq m} \{ |N_i \cap S| \}.
\]

Furthermore, define \( k \in \{1, \ldots, m\} \) as the first index with \( |N_k| < |N_{k+1}| \) (i.e. \( |N_1| = |N_2| = \ldots = |N_k| < |N_{k+1}| \leq \ldots \leq |N_m| \)). The first \( k \) ingredients (corresponding to \( N_1, N_2, \ldots, N_k \)) are the limiting factors. All players that own an ingredient that is a limiting factor are members of the set \( F := \bigcup_{i=1}^k N_i \). The set of all ingredient games is denoted by \( G^N_m \).

First of all, note that \( v_m(N) = |N_1| \) for every \( v_m \in G^N_m \), since the ingredients from \( N_1 \) are always a limiting factor. Secondly, it is immediately clear that \( G^N_2 = GG^N \). The case \( |N_1| < |N_2| \) (i.e. \( k = 1 \)) includes the games \( v_{\ell r} \in GG^N \) with \( \ell \neq r \) and the case \( |N_1| = |N_2| \) (i.e. \( k = 2 \)) includes the games \( v_{\ell r} \in GG^N \). The set \( G^N_3 \) consists of all games where there are three ingredients needed in order to cook the meal. It is also possible to extend the notion of making pairs of gloves and interpret a game \( v_3 \in G^N_3 \) as a game where every player owns either a left-hand glove (\( L \)), a right-hand glove (\( R \)) or a knit cap (\( M \)).

The assumption that \( |N_1| \leq |N_2| \leq \ldots \leq |N_m| \) can be done without loss of generality, because it is just a renumbering of the sets \( N_i \). With this assumption it is possible to note that \( |N_m| \neq 0 \), since this would imply that \( |N_i| = 0 \) for every \( i \in \{1, \ldots, m\} \) which
means that $N = \emptyset$, but that does not define a game. However, it is allowed to have $|N_1| = |N_2| = \ldots = |N_j| = 0$ for a certain $j \in \{1, \ldots, m\}$. This then results in a zero game, i.e. $v_m(S) = 0$ for every $S \in 2^N$.

It is possible to generalise almost all results from the previous two chapters about glove games and glove communication situations. The proofs are inspired by the proofs of the earlier results. Note that after proving the general results, the earlier results can be seen as special cases. The first result is about the (strongly) compromise admissibility of the ingredient game.

**Lemma 9.2** Let $v_m \in G_m^N$ be an ingredient game. Then $v_m \in CA^N$. Moreover, if $|N_1| < |N_2|$ then $v_m \in SCA^N$.

**Proof:** If $|N_1| = |N_2|$ (i.e. $k > 1$), consider for an arbitrary player $p \in F$ (that is, player $p$ owns an ingredient that is a limiting factor) the utopia-vector $M_p(v_m) = v_m(N) - v_m(N \setminus \{p\}) = |N_1| - (|N_1| - 1) = 1$, since there are only $|N_1| - 1$ members left that own the same ingredient as player $p$ and that ingredient was a limiting factor. For a player $q \in N \setminus F$ we have $M_q(v_m) = |N_1| - |N_1| = 0$, since the ingredient owned by player $q$ was not a limiting factor. Therefore, $M(v_m) = e^F$.

For the minimum right vector, note that there are at least two limiting factors ($k > 1$). Therefore, every player must pay the utopia-value to at least one member of $F$. This means that in order to complete a set of ingredients, i.e. in order to gain a value of 1, this player must pay a value of 1 (the utopia-value of a member in $F$). This results in a minimum right vector of $m(v_m) = 0$.

Then $\sum_{i \in N} m_i(v_m) = 0 \leq v_m(N) = |N_1| \leq k \cdot |N_1| = \sum_{i \in N} M_i(v_m)$ and $m_i(v_m) = 0 \leq e_i^F = M_i(v_m)$ for every player $i \in N$ implies that $v_m \in CA^N$.

If $|N_1| < |N_2|$ (i.e. $k = 1$), consider for an arbitrary player $p \in N_1$ the utopia-vector $M_p(v_m) = |N_1| - (|N_1| - 1) = 1$, since $N_1$ is the limiting factor. For $q \in N \setminus N_1$ it holds that $M_q(v_m) = |N_1| - |N_1| = 0$. Therefore, $M(v_m) = e^{N_1}$.

For the minimum right vector, let $p \in N_1$ and $S \in 2^N \setminus \{\emptyset\}$ with $p \in S$ arbitrarily. Write $a := |N_1 \cap S|$ to obtain $v_m(S) \leq a$ and $\sum_{i \in S, i \neq p} M_i(v_m) = a - 1$. Then it holds that $v_m(S) - \sum_{i \in S, i \neq p} M_i(v_m) \leq a - (a - 1) = 1$. Moreover, for $S = N$ we have equality in this last expression. Then $m_p(v_m) = \max_{S \subseteq S} \left\{ v_m(S) - \sum_{i \in S, i \neq p} M_i(v_m) \right\} = 1$.

Let $q \in N \setminus N_1$, $S \in 2^N \setminus \{\emptyset\}$ with $q \in S$ and $a := |N_1 \cap S|$. Then $v_m(S) \leq a$ and $\sum_{i \in S, i \neq q} M_i(v_m) = a$ such that it follows that $v_m(S) - \sum_{i \in S, i \neq q} M_i(v_m) \leq a - a = 0$. The choice $S = \{q\}$ leads to equality, thus $m_q(v_m) = 0$. Therefore, $m(v_m) = e^{N_1}$.

This already proves that $v_m \in CA^N$. For strong compromise admissibility note that $g^{v_m}(N) = \sum_{i \in N} M_i(v_m) - v_m(N) = |N_1| - |N_1| = 0$. Let $S \in 2^N \setminus \{\emptyset\}$ and (again) write $a := |N_1 \cap S|$. Then

$$g^{v_m}(S) = \sum_{i \in S} M_i(v_m) - v_m(S) \geq a - a = 0 = g^{v_m}(N)$$

proves that $v_m \in SCA^N$. \qed
9.1 The core

Just like Chapter 6 about glove games, this chapter consists of three section: the first one is about the core of an ingredient game, the second about the nucleolus and the third about the compromise value. The next proposition is about the core of an ingredient game.

**Proposition 9.3** For an ingredient game \( v_m \in G_m^N \) the core is as follows:

\[
C(v_m) = \text{Conv}\{e^{N_i} \mid 1 \leq i \leq k\}.
\]

**Proof:** Take an arbitrary \( x \in C(v_m) \) and let \( q \in N \setminus F \) be an arbitrary player. By the stability condition we have that \( \sum_{j \in N \setminus \{q\}} x_j \geq v_m(N \setminus \{q\}) = |N_1| \). This implies that \( x_q \leq 0 \) due to efficiency. Individual rationality ensures \( x_q \geq v_m(\{q\}) = 0 \), which together implies that \( x_q = 0 \).

For every \( i \in \{1, \ldots, m\} \) take an arbitrary \( p_i \in N_i \). Then it holds that \( \sum_{i=1}^k x_{p_i} = \sum_{i=1}^m x_{p_i} \geq v_m(\{p_1, \ldots, p_m\}) = 1 \), where we used \( x_{p_i} = 0 \) for \( i > k \). Moreover, \( \sum_{j \in N \setminus \{p_1, \ldots, p_m\}} x_j \geq v_m(N \setminus \{p_1, \ldots, p_m\}) = |N_1| - 1 \) implies that \( \sum_{i=1}^k x_{p_i} = \sum_{i=1}^m x_{p_i} \leq 1 \). Together this results in \( \sum_{i=1}^k x_{p_i} = 1 \).

Just as in the proof of Proposition 6.3 it now holds that every player in \( N_i \) (for every \( i \in \{1, \ldots, k\} \)) receives the same as each other player in \( N_i \) and together (i.e., one player from each \( N_i, 1 \leq i \leq k \)) they receive exactly 1. This proves \( C(v_m) \subseteq \text{Conv}\{e^{N_i} \mid 1 \leq i \leq k\} \). For the other direction take \( \lambda_i \in [0,1] \) for every \( 1 \leq i \leq k \) such that \( \sum_{i=1}^k \lambda_i = 1 \) and consider \( \sum_{i=1}^k \lambda_i e^{N_i} \). Efficiency is satisfied by noting that for every \( 1 \leq i \leq k \) we have that \( \sum_{p \in N} e^p_i = \sum_{p \in N_i} e^p_i = |N_i| = |N_1| \) and \( N_i \cap N_j = \emptyset \) for every \( i, j \in \{1, \ldots, m\} \) with \( i \neq j \).

In order to prove stability, let \( S \subseteq 2^N \setminus \{\emptyset\} \). Then

\[
\sum_{p \in S} \sum_{i=1}^k \lambda_i e^p_i = \sum_{i=1}^k \lambda_i \sum_{p \in S} e^p_i = \sum_{i=1}^k \lambda_i |N_i \cap S| \\
\geq \sum_{i=1}^k \lambda_i \min_{1 \leq j \leq m} \{|N_j \cap S|\} \\
= v_m(S) \sum_{i=1}^k \lambda_i = v_m(S)
\]

proves stability and thus \( \text{Conv}\{e^{N_i} \mid 1 \leq i \leq k\} \subseteq C(v_m) \). \( \square \)
In Section 6.1 it is proven that glove games are balanced games. Proposition 9.3 generalises this to ingredient games. Also ingredient games are balanced.

9.2 The nucleolus

Just like with glove games, we use the result about the core of an ingredient game for the computation of the nucleolus. Already in Section 6.2 we have seen that the nucleolus divides the total value of the grand coalition among the players who own an ingredient that is a limiting factor. This idea perfectly generalises to an ingredient game.

**Proposition 9.4** For an ingredient game $v_m \in G_m^N$ the nucleolus is as follows:

$$\text{nuc}(v_m) = \frac{1}{k}e^F.$$  

**Proof:** First of all, note that $C(v_m) \neq \emptyset$, such that $\text{nuc}(v_m) \in C(v_m)$. Therefore, the nucleolus can be written as $\text{nuc}(v_m) = \sum_{i=1}^{k} \lambda_i e^{N_i}$ with $\lambda_i \in [0, 1]$ and $\sum_{i=1}^{k} \lambda_i = 1$. We are going to look at the excesses and the excess vector. Note that in order to lexicographically minimize the excess vector, first of all the highest excess should be minimized. Therefore, we are going to look at the highest possible excess in terms of $\lambda_i$.

Of course, the highest excess equals 0, so we are interested in the highest non-zero excess. This are exactly the excesses $-\lambda_i$ for $1 \leq i \leq k$. This can be seen by observing that $\lambda_i + \lambda_j \geq \lambda_i$ and $\sum_{i=1}^{k} \lambda_i = 1$. The latter tells us that making complete sets of ingredients does not affect the excess. The former means that if there are two members remaining (after deleting all complete sets) for a certain coalition, the excess of this coalition is lower than a coalition with only one member remaining. The highest possible excesses are thus the excesses of coalitions with exactly one member remaining after deleting all complete sets.

Just as in the proof of Proposition 6.4, we consider the possible excess vectors and try to determine the optimal value for each $\lambda_i$, i.e. values for $\lambda_i$ such that the excess vector is the lexicographical minimum.

This minimum is obtained with $\lambda_i = \frac{1}{k}$ for every $1 \leq i \leq k$. For the sake of contradiction, suppose there is an $1 \leq j \leq k$ with $\lambda_j \neq \frac{1}{k}$. Then either $\lambda_j < \frac{1}{k}$ or $\lambda_j > \frac{1}{k}$. In the first case, improve the solution with $\lambda_j < \lambda_j < \frac{1}{k}$. In the second case, improve with $\lambda_j > \lambda_j > \frac{1}{k}$. Note that this indeed improves the solution, since all other excesses remains the same. The excess vector of the ‘new’ solution (with $\lambda_j$) is then lexicographically less than the ‘old’ excess vector.

Consequently, $\text{nuc}(v_m) = \sum_{i=1}^{k} \frac{1}{k} e^{N_i} = \frac{1}{k} \sum_{i=1}^{k} e^{N_i} = \frac{1}{k} e^F$. $\square$
9.3 The compromise value

In order to compute the compromise value, an ingredient game has to be compromise admissible. Fortunately, Lemma 9.2 guarantees the compromise admissibility for ingredient games. Moreover, it guarantees strong compromise admissibility for ingredient games with only one limiting factor. Recall from Section 6.3 that the compromise value of a glove game is exactly the same as the nucleolus of a glove game. Also this result generalises to ingredient games, as the following proposition shows.

**Proposition 9.5** For an ingredient game $v_m \in G^N_m$ the compromise value is as follows:

$$\tau(v_m) = \text{nuc}(v_m) = \frac{1}{k} e^F.$$

**Proof:** If $|N_1| < |N_2|$ (i.e. $k = 1$), then from Lemma 9.2 it follows that $v_m \in SCA^N$. Theorem 2.4 then ensures $\tau(v_m) = \text{nuc}(v_m)$, which together with Proposition 9.4 proves the statement.

If $|N_1| = |N_2|$ (i.e. $k > 1$), then we are going to explicitly calculate the compromise value using $M(v_m) = e^F$ and $m(v_m) = 0$ obtained from the proof of Lemma 9.2. The compromise value is given by

$$\tau(v_m) = \alpha M(v_m) + (1 - \alpha)m(v_m) = \alpha e^F$$

with $\alpha \in [0, 1]$ such that $\sum_{i \in N} \alpha e_i^F = v_m(N) = |N_1|$. This last expression leads to $|N_1| = \alpha \sum_{i \in N} e_i^F = \alpha |N_1| k$, which implies that $\alpha = \frac{1}{k}$. The compromise value is thus given by $\tau(v_m) = \frac{1}{k} e^F$ which is exactly the same as the nucleolus (cf. Proposition 9.4). \qed
Chapter 10

Ingredient communication situations

Similar to the glove communications situations from Chapter 7, ingredient communication situations are communication situations with an ingredient game as the underlying game. In this chapter, most of the results from Chapter 7 are generalised with a proof similar to the proof of the earlier results.

10.1 The nucleolus

We start with the characterization of the nucleolus for a communication situation with a star as a communication graph.

**Theorem 10.1** Let \((N, v_m, E_j) \in CS_N^N\) be an ingredient communication situation with \(E_j\) a star with special node \(z \in N_j\) for \(j \in \{1, \ldots, m\}\). Then it holds that

\[
nuc(v_{E_j}^m) = \frac{|N_1|}{k+1}e\{z\} + \frac{1}{k+1}e^F.
\]

**Proof:** We first mention that, since \(nuc(v_{E_j}^m) \in C(v_{E_j}^m) \neq \emptyset\) (non-empty due to the inheritance of balancedness cf. Proposition 3.3), we have that for every player \(q \notin F\), \(q \neq z\) it holds that \(m_q(v_{E_j}^m) = 0 \leq nuc_q(v_{E_j}^m) \leq 0 = M_q(v_{E_j}^m)\) (where the utopia-vector and the minimum right vector of \(v_{E_j}^m\) are obtained from the proof of Lemma 9.2) which implies that \(nuc_q(v_{E_j}^m) = 0\).

Secondly, notice that each player \(p \in N_i, p \neq z\) (for every \(1 \leq i \leq k\)) is symmetric to every other player \(q \in N_i, q \neq z\). Therefore, the nucleolus can be written as

\[
nuc(v_{E_j}^m) = \mu e\{z\} + \sum_{i=1}^{k} \lambda_i e^{N_i}, \tag{10.1}
\]
since every player in \(N_i\) now receives \(\lambda_i\) and player \(z\) receives either \(\mu + \lambda_j\) or \(\mu\) depending on whether \(z \in F\) (i.e. \(j \in \{1, \ldots, k\}\)) or not. Because of efficiency we have \(\mu + \sum_{i=1}^{k} \lambda_i |N_i| = |N_1|\) (due to the fact that \(|N_i| = |N_1|\) for every \(1 \leq i \leq k\)).

Furthermore, we can use the stability condition for \(S = N \setminus \{p_i \mid 1 \leq i \leq k\}\) with \(p_i \in N_i\) (\(p_j \neq z\) if \(z \in F\)) arbitrary. Then it holds that

\[
\sum_{q \in N \setminus \{p_i \mid 1 \leq i \leq k\}} nuc_q(v^E_m) \geq v^E_m(N \setminus \{p_i \mid 1 \leq i \leq k\}) = |N_1| - 1.
\]

Using efficiency, this implies that \(\sum_{i=1}^{k} nuc_{p_i}(v^E_m) \leq 1\) and thus \(\sum_{i=1}^{k} \lambda_i \leq 1\). Moreover, individual rationality implies that \(\lambda_i \geq 0\) for every \(1 \leq i \leq k\).

In order to determine the highest excess \(E(S, nuc(v^E_m))\) for \(S \in 2^N\), we distinguish between coalitions \(S \in 2^N\) with \(z \notin S\) and coalitions with \(z \in S\). If \(z \notin S\), then the highest (non-zero) excesses are of the form \(-\lambda_i\) for \(1 \leq i \leq k\). This can be seen by noting that \(v^E_m(S) = 0\) for these coalitions. Therefore, the bigger a coalition, the more negative the excess will be. The highest (non-zero) excess is thus obtained by a coalition with only one player in it. This results in excesses of the form \(-\lambda_i\).

If \(z \in S\) we distinguish between the case where \(z \notin F\) and the case where \(z \in F\). We will see that both cases lead to an excess of the same form. In the first case, there is already a contribution of \(-\mu = -|N_1| + |N_1| \sum_{i=1}^{k} \lambda_i\) to the excess. We now can add complete sets of ingredients in order to gain a value of 1 at the cost of \(-\sum_{i=1}^{k} \lambda_i\). Note that only complete sets of ingredients are helpful, since otherwise there is always another coalition with only complete sets with a higher excess. The question rises how many complete sets of ingredients we have to add in order to obtain the highest excess. The answer: as many as possible, since \(\sum_{i=1}^{k} \lambda_i \leq 1\). This means that adding a complete set leads to a positive contribution of 1 and a negative contribution of \(\sum_{i=1}^{k} \lambda_i\), resulting in a contribution of \(1 - \sum_{i=1}^{k} \lambda_i \geq 0\). Adding complete sets thus increase the excess. Therefore, the highest excess possible is to add \(|N_1|\) complete sets, leading to an excess of \(E(S, nuc(v^E_m)) = |N_1| - |N_1| + |N_1| \sum_{i=1}^{k} \lambda_i - |N_1| \sum_{i=1}^{k} \lambda_i = 0\). The highest non-zero excess is thus of the form \(|N_1| - 1 - |N_1| + |N_1| \sum_{i=1}^{k} \lambda_i - (|N_1| - 1) \sum_{i=1}^{k} \lambda_i = \sum_{i=1}^{k} \lambda_i - 1\).

Writing \(\lambda := 1 - \sum_{i=1}^{k} \lambda_i\) we thus have an excess of the form \(-\lambda\).

If \(z \in F\), then a similar argument also leads to an excess of the form \(-\lambda\). Note that in this case it is important to mention that player \(z\) contributes already \(-\mu - \lambda_j\). The first complete set of ingredients only consists of players from \(N_i\) with \(1 \leq i \leq k\) and \(i \neq j\).

We are now going to determine the optimal excess vector, that is the excess vector that is lexicographically less than or equal to every other excess vector corresponding to a certain imputation. We have \(k + 1\) options for this excess vectors, namely

\[
\theta_i(nuc(v^E_m)) = (0, \ldots, 0, -\lambda_i, \ldots);
\]

\[
\theta(nuc(v^E_m)) = (0, \ldots, 0, -\lambda, \ldots).
\]

We are going to see that the optimal excess vector is the one corresponding to \(\lambda_i = \frac{1}{k+1}\).
for every $1 \leq i \leq k$. In order to prove this, suppose for the sake of contradiction that there an index $1 \leq x \leq k$ with $\lambda_x \neq \frac{1}{k+1}$. Then either all $\lambda_i \leq \frac{1}{k+1}$ or some $\lambda_i \geq \frac{1}{k+1}$. And some $\lambda_i \geq \frac{1}{k+1}$. In the first case, let $y$ be the index corresponding to the smallest $\lambda_i$. Then $\theta_y$ from (10.2) is the corresponding excess vector. We can improve this solution by choosing $\lambda_y < \hat{\lambda}_y < \frac{1}{k+1}$. In the second case, improve either $\theta_y$ for the smallest $\lambda_y$ or $\theta_i$, depending on which of the two is the lowest (lexicographically) excess vector. Improvement is possible via $\lambda_y < \hat{\lambda}_y < \frac{1}{k+1}$ or $\frac{1}{k+1} < \hat{\lambda}_i < \lambda_i$ for some $\lambda_i > \frac{1}{k+1}$. Both cases lead to a contradiction.

Hence, $\lambda_i = \frac{1}{k+1}$ for every $1 \leq i \leq k$. Therefore, also $\lambda = \frac{1}{k+1}$ since $\lambda = 1 - \sum_{i=1}^{k} \lambda_i = \frac{k+1}{k+1} - \frac{k}{k+1} = \frac{1}{k+1}$. This results in $\mu = \frac{|N_1|}{k+1}$ and thus, using (10.1), the nucleolus is $\text{nuc}(v_m) = \frac{|N_1|}{k+1}e(z) + \sum_{i=1}^{k} \frac{1}{k+1}e_{N_i} = \frac{|N_1|}{k+1}e(z) + \frac{1}{k+1}e^F$. □

In Section 7.1, Corollary 7.5 proves the fact that according to the nucleolus the middle player is receiving more in the restricted game than in the underlying game and that all other players pay the cost of his benefit. Since ingredient games are a generalisation of glove games, the question rises whether this is also true for ingredient games. The answer turns out to be yes.

**Corollary 10.2** Let $(N, v_m, E_j) \in CS^N$ be an ingredient communication situation with $E_j$ a star with special node $z \in N_j$ for $j \in \{1, \ldots, m\}$. Then it holds that

i) $\text{nuc}_i(v_m^{E_j}) \leq \text{nuc}_i(v_m)$ for every $i \in N \setminus \{z\}$;

ii) $\text{nuc}_z(v_m^{E_j}) \geq \text{nuc}_z(v_m)$.

**Proof:** We have to compare the nucleolus of an ingredient game,

$$\text{nuc}(v_m) = \frac{1}{k}e^F$$

(according to Proposition 9.4), with the nucleolus of the restricted game of an ingredient communication situation with a star as communication graph,

$$\text{nuc}(v_m^{E_j}) = \frac{|N_1|}{k+1}e(z) + \frac{1}{k+1}e^F$$

(according to Theorem 10.1).

For the first statement, if $i \in F$, then $\text{nuc}_i(v_m^{E_j}) = \frac{1}{k+1}$ while $\text{nuc}_i(v_m) = \frac{1}{k}$. If $i \notin F$, then $\text{nuc}_i(v_m^{E_j}) = 0 = \text{nuc}_i(v_m)$. So for all other player $i \in N \setminus \{z\}$ it holds that $\text{nuc}_i(v_m^{E_j}) \leq \text{nuc}_i(v_m)$. This proves the first statement.

For the second statement, we have to compare a pay-off of 0 with a pay-off of $\frac{|N_1|}{k+1}$ if $z \notin F$ and a pay-off of $\frac{1}{k}$ with a pay-off of $\frac{|N_1|}{k+1} + \frac{1}{k+1}$ if $z \in F$. If $z \notin F$, then the pay-off of the player $z$ is (strictly) increasing from 0 to $\frac{|N_1|}{k+1}$. If $z \in F$, then it holds that $\frac{|N_1|}{k+1} + \frac{1}{k+1} \geq \frac{1}{k}$, because $|N_1| \geq 1$ and $k \geq 1$. Hence, $\text{nuc}_z(v_m^{E_j}) \geq \text{nuc}_z(v_m)$ such that
the middle player benefits from being more powerful in the ingredient communication situation.

Until now, every result about glove games from Chapter 6 and glove communication situations from Chapter 7 perfectly generalises to results about ingredient games in Chapter 9 and ingredient communication situations in this chapter. Until now. The next theorem in Section 7.1 was about glove communication situations with a cycle as communication graph. It was proven that the nucleolus of the restricted game coincides with the nucleolus of the underlying game, i.e. \( \text{nuc}(v_{E}^{\ell_r}) = \text{nuc}(v_{\ell_r}) \) for \( (N, v_{\ell_r}, E) \in CS^{N} \) with \( E \) a cycle. Unfortunately, this theorem does not generalise to ingredient games, i.e. for \( (N, v_{m}, E) \in CS^{N} \) with \( E \) a cycle \( \text{nuc}(v_{E}^{m}) \neq \text{nuc}(v_{m}) \) in general. The following example provides a counterexample.

![Figure 10.1 – A 7-cycle for an ingredient game](image)

**Example 10.3** Consider the 7-person ingredient communication situation \( (N, v_{3}, E) \in CS^{N} \) with \( N = N_{1} \cup N_{2} \cup N_{3}, N_{1} = \{1, 2\}, N_{2} = \{3, 4\}, N_{3} = \{5, 6, 7\} \) and \( E \) the cycle as depicted in Figure 10.1. Note that \( k = 2 \) and \( F = N_{1} \cup N_{2} \). We are going to show that \( \text{nuc}(v_{3}^{E}) \neq \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0\right) = \frac{1}{2}e^{F} = \text{nuc}(v_{3}) \) (cf. Proposition 9.4) using the Kohlberg criterion.

First note that because \( \frac{1}{2}e^{F} = \text{nuc}(v_{3}) \subseteq C(v_{3}) \) it holds that \( E(S, \frac{1}{2}e^{F}, v_{3}^{E}) \leq 0 \) for every \( S \in 2^{N} \). The highest (possible) excess is thus 0. The collection of coalitions with a zero excess is

\[
B_{1} = \{\{5\}, \{6\}, \{7\}, \{5, 6\}, \{5, 7\}, \{6, 7\}, \{5, 6, 7\}, \\
1, 4, 5, 6, 7\}, \{1, 2, 3, 4, 5, 7\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2, 3, 4, 6, 7\}\}.
\]
Due to player 2 and player 3, the sum of the weights of coalitions \{1,2,3,4,5,7\}, \{1,2,3,4,5,6\} and \{1,2,3,4,6,7\} must be equal to 1. This contradicts the fact that also the coalition \{1,4,5,6,7\} must get positive weight due to player 1. We conclude that \(B_1\) is not a balanced collection. Therefore, \(\text{nuc}(v_3^E) \neq \frac{1}{2}e^F = \text{nuc}(v_3)\). \(\triangle\)

Nevertheless, in the case that \(|N_1| < |N_2|\) (i.e. \(k = 1\)) the nucleoli do coincide. This is proven in the following proposition.

**Proposition 10.4** Let \((N, v_m, E) \in CS^N\) be an ingredient communication situation with \(|N_1| < |N_2|\) and \(E\) a cycle. Then \(\text{nuc}(v_m^E) = \text{nuc}(v_m) = e^{N_1}\).

**Proof:** First note that for ingredient games with \(|N_1| < |N_2|\) (i.e. \(k = 1\)) it holds that \(\text{nuc}(v_m) = \frac{1}{2}e^F = e^{N_1}\) (cf. Proposition 9.4). We are going to prove that also \(\text{nuc}(v_m^E) = e^{N_1}\). Furthermore, note that \(C(v_m^E) \neq \emptyset\) due to the inheritance of balancedness (cf. Proposition 3.3) and the fact that ingredient games are balanced (cf. Proposition 9.3). Therefore, it holds that \(m(v_m^E) \leq \text{nuc}(v_m^E) \leq M(v_m^E)\).

From the proof of Lemma 9.2 it can be seen that \(m(v_m) = e^{N_1} = M(v_m)\). A cycle is 2-connected such that it follows that \(M(v_m^E) = M(v_m) = e^{N_1}\). For the minimum right vector, exactly the same argument as in the proof of Lemma 9.2 also applies for \(v_m^E\), implying that \(m(v_m^E) = m(v_m) = e^{N_1}\).

Consequently, \(e^{N_1} = m(v_m^E) \leq \text{nuc}(v_m^E) \leq M(v_m^E) = e^{N_1}\), which implies that \(\text{nuc}(v_m^E) = e^{N_1}\). This completes the proof. \(\square\)

### 10.2 The compromise value

The second section of this chapter is about the compromise value. This is a generalisation of Section 7.2

**Theorem 10.5** Let \((N, v_m, E_j) \in CS^N\) be a communication situation with \(v_m \in G^N_m\) and \(E_j\) a star with special node \(z \in N_j\) for \(j \in \{1, \ldots, m\}\). Then it holds that

i) If \(|N_1| < |N_2|\) (i.e. \(k = 1\)), then

\[
\tau(v_m^E) = \frac{|N_1|}{2}e^z + \frac{1}{2}e^{N_1}.
\]

ii) If \(|N_1| = |N_2|\) (i.e. \(k > 1\)), then

\[
\tau(v_m^E) = \begin{cases} 
\frac{|N_1|}{k+1}e^z + \frac{1}{k+1}e^F & \text{if } z \notin F; \\
\frac{|N_1|^2}{(k+1)(N_1-N_2)}e^z + \frac{|N_1|}{(k+1)(N_1-N_2)}e^F(z) & \text{if } z \in F.
\end{cases}
\]

**Proof:** i) From Lemma 9.2 it follows that \(v_m \in \text{SCA}^N\) in this case. Therefore, we can apply Theorem 5.3 in order to obtain \(\tau(v_m^E) = \text{nuc}(v_m^E)\). Eventually, Theorem 10.1 completes the proof of i).
ii) First of all, recall from the proof of Lemma 9.2 that $M(v_m) = e^F$ and $m(v_m) = 0$ if $|N_1| = |N_2|$. For the minimum right vector of the restricted game we have that $m(v^E_{m}) = 0$ since $0 \leq m(v^E_{m}) \leq m(v_m) = 0$. In order to obtain the utopia-vector for $v^E_{m}$, Lemma 4.3 can be applied. Then the utopia-vector is as follows:

$$M_i(v^E_{m}) = \begin{cases} 1 & \text{if } i \neq z, i \in F; \\ 0 & \text{if } i \neq z, i \not\in F; \\ |N_1| & \text{if } i = z. \end{cases}$$

The compromise value is thus proportional to the utopia-vector: $\tau(v^E_{m}) = \beta M(v^E_{m})$ with $\beta \in [0, 1]$ such that $\sum_{i\in N} \beta \tau_i(v^E_{m}) = v^E_{m}(N) = |N_1|$. Recall that $|N_1| = |N_2| = \ldots = |N_k|$. We have to distinguish between $z \not\in F$ and $z \in F$. For the first option, $z \not\in F$, we can determine $\beta$ by solving the equation:

$$\beta(|N_1| + k|N_1|) = |N_1|,$$

where the first $|N_1|$ is due to player $z$ and the other $k|N_1|$ follows from the fact that there are $|F| = k|N_1|$ players with an utopia-value of 1. This results in $\beta = \frac{1}{k+1}$ and thus $\tau(v^E_{m}) = \frac{1}{k+1} M(v^E_{m}) = \frac{|N_1|}{k+1} e^z + \frac{1}{k+1} e^F$.

For the second option, $z \in F$, the calculation for $\beta$ are as follows:

$$\beta(|N_1| + k|N_1| - 1) = |N_1|,$$

where the minus one follows from the fact that $z \in F$, but his utopia-value already was taken into account (namely the first $|N_1|$). This implies that $\beta = \frac{|N_1|}{(k+1)|N_1|-1}$ and thus that $\tau(v^E_{m}) = \frac{|N_1|}{(k+1)|N_1|-1} M(v^E_{m})$, which turns out to be equal to

$$\tau(v^E_{m}) = \frac{|N_1|^2}{(k+1)|N_1|-1} e^z + \frac{|N_1|}{(k+1)|N_1|-1} e^{F\setminus\{z\}}. \square$$

The first lines of this section stated that this section is a generalisation of Section 7.2. That means that Theorem 10.5 must be a generalisation of Theorem 7.10. And indeed, this is the case. With $N_1 = L$, $N_2 = R$, $m = 2$ and $z = 1$ the special node of the star $E_1$, the compromise value of the restricted game of $(N, v_2, E_1) \in CS^N$ according to Theorem 10.5 is exactly the same as the compromise value of the restricted game of $(N, v_{tr}, E) \in CS^N$ according to Theorem 7.10:
If $|N_1| < |N_2|$ (i.e. $k = 1$), then $\ell < r$ and the compromise value:

$$
\tau(v_{m}^{E_j}) = \frac{|N_1|}{2} e^{(z)} + \frac{1}{2} e^{N_1}
$$

$$
= \frac{\ell}{2} e^{(1)} + \frac{1}{2} e^{L}
$$

$$
= \left( \frac{\ell + 1}{2}, \frac{\ell}{2}, \ldots, 0, \ldots, 0 \right).
$$

If $|N_1| = |N_2|$ (i.e. $k > 1$ thus $k = 2$), then $\ell = r$ and $z = 1 \in L \subseteq N = F$ and the compromise value:

$$
\tau(v_{m}^{E_j}) = \frac{|N_1|^2}{(k + 1)|N_1| - 1} e^{(z)} + \frac{|N_1|}{(k + 1)|N_1| - 1} e^{F\setminus\{z\}}
$$

$$
= \frac{\ell^2}{(2 + 1)\ell - 1} e^{(1)} + \frac{\ell}{(2 + 1)\ell - 1} e^{N\setminus\{1\}}
$$

$$
= \left( \frac{\ell^2}{3\ell - 1}, \frac{\ell}{3\ell - 1}, \ldots, 0, \ldots, 0 \right).
$$

For glove communication situation with a star as communication graph it was proven in Corollary 7.11 that the nucleolus and the compromise value come towards each other if there are more and more players. Moreover, in some cases, the nucleolus and the compromise value of the restricted game coincide. We can generalise this result for ingredient games, as the following corollary shows:

**Corollary 10.6** The following three statements hold:

i) $nuc(v_{m}^{E_j}) = \tau(v_{m}^{E_j})$ for every $(N, v_m, E_j) \in CS^N$ with $|N_1| < |N_2|$ and $E_j$ a star with special node $z \in N_j$ for $j \in \{1, \ldots, m\}$;

ii) $nuc(v_{m}^{E_j}) = \tau(v_{m}^{E_j})$ for every $(N, v_m, E_j) \in CS^N$ with $|N_1| = |N_2|$ and $E_j$ a star with special node $z \in N_j$ for $j \in \{k + 1, \ldots, m\}$ (i.e. $z \notin F$);

iii) $\lim_{|N_1| \to \infty} nuc(v_{m}^{E_j}) = \lim_{|N_1| \to \infty} \tau(v_{m}^{E_j})$ where $v_{m}^{E_j}$ is the restricted game of $(N, v_m, E_j) \in CS^N$ with $|N_1| = |N_2|$ and $E_j$ a star with special node $z \in N_j$ for $j \in \{1, \ldots, k\}$ (i.e. $z \in F$).

**Proof:** i) This follows directly from Lemma 9.2 and Theorem 5.3.

ii) Combine Theorem 10.1 with Theorem 10.5 to see that the nucleolus of the restricted game coincides with the compromise value of the restricted game.

iii) First, use Theorem 10.1 to derive the limit of the nucleoli of the restricted games for player $z$:

$$
\lim_{|N_1| \to \infty} nuc_z(v_{m}^{E_j}) = \lim_{|N_1| \to \infty} \frac{|N_1|}{k + 1} = \infty.
$$
Secondly, the limit of the compromise values of the restricted games is as follows, now using Theorem 10.5:

$$\lim_{|N_1| \to \infty} \tau_z(v_{m}^{E}) = \lim_{|N_1| \to \infty} \frac{|N_1|^2}{(k + 1)|N_1| - 1} = \infty.$$ 

Therefore, $$\lim_{|N_1| \to \infty} \text{nuc}_z(v_{m}^{E}) = \lim_{|N_1| \to \infty} \tau_z(v_{m}^{E}).$$

Let $$i \in F, i \neq z$$ and note that the nucleolus always assigns this player $$\frac{1}{k+1}$$ (cf. Theorem 10.1). Thus we only need to derive the limit of the compromise values of the restricted games, using Theorem 10.5:

$$\lim_{|N_1| \to \infty} \tau_i(v_{m}^{E}) = \lim_{|N_1| \to \infty} \frac{|N_1|}{(k + 1)|N_1| - 1} = \frac{1}{k + 1}.$$ 

Hence, $$\lim_{|N_1| \to \infty} \text{nuc}_i(v_{m}^{E}) = \lim_{|N_1| \to \infty} \tau_i(v_{m}^{E}).$$ For all other players $$j \notin F$$ it holds that both the nucleolus and the compromise value assign nothing to these players. We can thus conclude that $$\lim_{|N_1| \to \infty} \text{nuc}(v_{m}^{E}) = \lim_{|N_1| \to \infty} \tau(v_{m}^{E}).$$

In the proof of this corollary we make the assumption that the number $$k$$ is fixed. This means that if $$|N_1|$$ is approaching infinity, then also $$|N_{k+1}|, |N_{k+2}|, \ldots, |N_{m}|$$ approaches infinity.

For the upcoming theorem it is clear that this is a generalisation of Theorem 7.12. It shows that the compromise value does not change when there is a 2-connected communication graph.

**Theorem 10.7** Let $$(N, v_m, E) \in CS^N$$ be a communication situation with $$v_m \in G^N_m$$ and $$E$$ a 2-connected graph. Then $$\tau(v_{m}^{E}) = \tau(v_{m}).$$

**Proof:** The proof is based on Corollary 5.1 and Theorem 5.2. The first one requires $$v_m \in SCA^N$$, while the second one requires $$v_m \in CA^N$$ and $$m(v) = 0$$. All these three requirements can be found in the proof of Lemma 9.2.

If $$|N_1| < |N_2|$$, then $$v_m \in SCA^N$$. In this case, Corollary 5.1 proves that $$\tau(v_{m}^{E}) = \tau(v_{m}).$$

If $$|N_1| = |N_2|$$, then $$v_m \in CA^N$$ and $$m(v_m) = 0$$. Here, Theorem 5.2 proves that $$\tau(v_{m}^{E}) = \tau(v_{m}).$$

$$\square$$
Chapter 11

Unanimity games and unanimity communication situations

This last chapter is a short chapter about unanimity games and unanimity communication situations. Unanimity games are very important, since they form a basis for the set of all TU-games. First of all, the nucleolus and the compromise value are characterized using efficiency, symmetry and the dummy property. Secondly, it is shown that unanimity games are balanced and thus compromise admissible. Eventually, the nucleolus and the compromise value of the restricted game of unanimity communication situations are derived. This is done without specifying the communication graph.

Proposition 11.1  For a unanimity game \( u_T \in TU^N \) with \( T \in 2^N \setminus \{\emptyset\}, |T| > 1 \), the nucleolus and the compromise value are as follows:

\[
nuc(u_T) = \tau(u_T) = 1/|T|e^T.
\]

Proof: The proof is a combination of three properties: efficiency, symmetry and the dummy property. Two players \( i, j \in T \), \( i \neq j \), are symmetric, since \( j \notin S \cup \{i\} \) and \( i \notin S \cup \{j\} \) for every \( S \subseteq N \setminus \{i,j\} \). A player \( i \notin T \) is a dummy player, because he never can make a losing coalition a winning coalition. That is, \( u_T(S \cup \{i\}) - u_T(S) = 0 \) for every \( S \subseteq N \setminus \{i\} \). Due to efficiency it follows that \( nuc(u_T) = \tau(u_T) = 1/|T|e^T \). □

The next proposition is about the fact that unanimity games are balanced and thus compromise admissible.

Proposition 11.2  Let \( u_T \in TU^N \) be a unanimity game with \( T \in 2^N \setminus \{\emptyset\}, |T| > 1 \). Then it holds that \( u_T \) is balanced (i.e. \( C(u_T) \neq \emptyset \)). Moreover, \( u_T \in CA^N \).

Proof: From the fact that every player from \( T \) is needed in order to obtain a value of 1 it follows that \( C(u_T) = \text{Conv} \{e^i \mid i \in T\} \). Therefore, \( u_T \) is a balanced game. Since \( C(u_T) \subseteq CC(u_T) \) it immediately follows that \( u_T \in CA^N \). □

The last theorem of this chapter is about the nucleolus and the compromise value of
the restricted game of unanimity communication situations. We are going to show that both solutions coincide for every possible communication graph. Before proving this, we need an extra definition.

**Definition 11.3** Let \((N, u_T, E) \in CS^N\) be a unanimity communication situation with \(T \in 2^N \setminus \{\emptyset\}, |T| > 1\). Define the following two collections:

\[
\mathcal{R} := \{R \subseteq N \setminus T \mid \text{the subgraph on } R \cup T \text{ is connected}\};
\]

\[
\mathcal{R}^{\text{min}} := \{R \in \mathcal{R} \mid \text{there is no } \bar{R} \in \mathcal{R}, \bar{R} \neq R \text{ such that } \bar{R} \subseteq R\}.
\]

Furthermore, define a set of ‘important’ players as follows:

\[
I := \{i \in N \setminus T \mid i \in R \text{ for every } R \in \mathcal{R}^{\text{min}}\}.
\]

The first collection, \(\mathcal{R}\), consists of coalitions such that the players from \(T\) are getting connected. The second collection, \(\mathcal{R}^{\text{min}}\), only consists of the minimal (inclusion-wise) coalitions that satisfy this condition. The set of ‘important’ players, \(I\), consists of those players that are needed in order to connect the players from \(T\). The following examples provide some communication graphs such that the definition of \(\mathcal{R}\) and \(\mathcal{R}^{\text{min}}\) and the set \(I\) becomes clear.

**Example 11.4** This first example provides two graphs such that the set \(\mathcal{R}^{\text{min}}\) consists of a unique coalition. Let \(N = \{1, 2, 3, 4, 5, 6\}\) be a set of players and \(u_{\{4,5,6\}} \in TU^N\) be a unanimity game. Consider the two communication graphs \((N, E_1)\) and \((N, E_2)\) as depicted in Figures 11.1a and 11.1b.

For the first unanimity communication situation \((N, u_{\{4,5,6\}}, E_1) \in CS^N\) it can be checked that \(\mathcal{R} = \{\{2,3\}, \{1,2,3\}\} \text{ and } \mathcal{R}^{\text{min}} = \{\{2,3\}\}. Therefore, } I = \{2,3\}.\]
For the second unanimity communication situation \((N, u_{\{4,5,6\}}, E_2) \in CS^N\) it can be checked that \(R = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}\) and \(R^{\text{min}} = \{\emptyset\}\). So, \(I = \emptyset\).

**Example 11.5** It is also possible that the set \(R^{\text{min}}\) consists of more coalitions. This is shown by the following unanimity communication situation. Let \((N, u_T, E) \in CS^N\) be a unanimity communication situation with \(N = \{1, 2, 3, 4, 5\}, T = \{4, 5\}\) and \((N, E)\) the graph from Figure 11.2.

Here, \(R = \{\{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}\) such that \(R^{\text{min}} = \{\{1\}, \{2, 3\}\}\). Both coalitions \(\{1\}\) and \(\{2, 3\}\) connect the players from \(T\). Since there is no overlap between these two coalitions, \(I = \emptyset\). 

![Figure 11.2 – The graph E, a 5-cycle](image)

It is interesting to notice that there are two ways to have an empty set of ‘important’ players. In Example 11.4, there were no ‘important’ players due to the fact that the subgraph on \(T\) already was connected. On the other hand, in Example 11.5, there were two ways to connect the players from \(T\). This means that no player is that important. The following example shows that it is also possible to have a non-empty set of ‘important’ players, while there are more coalitions contained in the collection \(R^{\text{min}}\).

**Example 11.6** Consider the unanimity communication situation \((N, u_T, E) \in CS^N\) with \(N = \{1, 2, 3, 4, 5\}, T = \{4, 5\}\) and \(E\) the graph from Figure 11.3. It is easy to check that \(R = \{\{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}\) and \(R^{\text{min}} = \{\{1, 2\}, \{1, 3\}\}\). We thus conclude that \(I = \{1\}\). Player 1 is the only player who is really needed in order to connect the players from \(T\).

Now, we can formulate a general theorem involving all communication graphs that are possible. It even characterize explicitly the nucleolus and the compromise value of the restricted game of a unanimity communication situation.
CHAPTER 11. UNANIMITY GAMES AND UNANIMITY COMMUNICATION SITUATIONS

Figure 11.3 – The graph $E$

Theorem 11.7 Let $(N, u_T, E) \in CS^N$ be a unanimity communication situation with $T \in 2^N \setminus \{\emptyset\}$, $|T| > 1$. Then it holds that

$$nuc(u_T^E) = \tau(u_T^E) = \frac{1}{|T| + |I|}e^{T \cup J}.$$  

Proof: We start the proof by showing that every player $j \notin T \cup I$ receives nothing in both the nucleolus and the compromise value. For a player $j \notin T \cup I$ it holds that $u_T^E(N \setminus \{j\}) = 1$. This is true, because player $j$ is not necessary needed in order to connect the players from $T$ (otherwise, $j \in I$). Therefore, $M_j(u_T^E) = 0$.

Since the unanimity game $u_T$ is balanced, also $u_T^E$ is balanced (cf. Proposition 3.3). Consequently, $nuc(u_T^E) \in C(u_T^E)$ which implies that $nuc(u_T^E) \leq M(u_T^E)$. Thus we can conclude that $nuc_j(u_T^E) = 0$. For the compromise value it holds that $\tau u_T^E \leq M(u_T^E)$ by definition, such that also here it follows that $\tau_j(u_T^E) = 0$.

Next, we are going to prove that every two players $i, j \in T \cup I$, $i \neq j$, are symmetric. In order to prove this, let $i, j \in T \cup I$ with $i \neq j$ and let $S \subseteq N \setminus \{i, j\}$. Then it follows that $u_T^E(S \cup \{i\}) = 0 = u_T^E(S \cup \{j\})$. The first equality is due to the fact that $j \notin S \cup \{i\}$, because either $j \notin T$ or $j \notin I$. If $j \notin T$, then the equality is obvious. If $j \in T$, then this means that $j \in R$ for every $R \in R_{\text{min}}$. This implies that also $j \in R$ for every $R \in \mathcal{R}$. Consequently, player $j$ is definitely needed in order to connect the players from $T$ and absence of player $j$ leads the equality above. The second equality follows from the same reasoning for player $i$.

Thus every two players $i, j \in T \cup I$, $i \neq j$, are symmetric and every player $j \notin T \cup I$ receives nothing in both the nucleolus and the compromise value. Note that the nucleolus and the compromise value both satisfy symmetry and efficiency. Applying both properties completes the proof. $\Box$

Theorem 11.7 is a interesting result, because it shows that for every communication graph, the nucleolus and the compromise value coincide. This already was the case for unanimity games without communication restrictions (cf. Proposition 11.1), but is now extended to unanimity communication situations.
Example 11.8  Consider the unanimity communication situation \((N, u_T, E) \in CS^N\) with \(N = \{1, 2, 3, 4, 5\}\), \(T = \{4, 5\}\) and \(E\) the communication graph from Figure 11.3. For this unanimity communication situation we have that \(I = \{1\}\). Therefore, \(\text{nuc}(u_T^E) = \tau(u_T^E) = \frac{1}{2}e^{(1,4,5)}\).

Note that in the proof of Proposition 11.1 we also used the dummy property, while in the proof of Theorem 11.7 the utopia-vector is used. The following example shows the reason for this.

Theorem 11.7 gives \(\text{nuc}(u_T^E) = \tau(u_T^E)\). Since \(m(u_T) = 0\) according to Lemma 11.9, we can apply Theorem 5.6 to obtain the result.
Appendix A

The Shapley value of a glove game

In 1969, Shapley introduced a recursive formula for the Shapley value of a glove game. In this appendix we are going to prove this recursive formula using mathematical tools. For convenience, this appendix starts with introducing a notation for the Shapley value of a glove game.

Notation  For a game \( v_{\ell r} \in GG^N \) write the Shapley value \( \Phi(v_{\ell r}) \) as \( \Phi_{\ell r} \).

In the next proposition the Shapley value of a glove game \( v_{\ell r} \in GG^N \) is determined. This is done via a recursive formula. Therefore, observe that \( \Phi_{\ell r} \) in some way looks like the entries of an infinite matrix. The entry in the upper left corner for example can be seen as the corresponding Shapley value of \( v_{11} \), i.e. \( \Phi_{11} \). But since this is an vector in \( \mathbb{R}^N \), it is not exactly an infinite matrix. To fix this problem, we are going to look at the combined pay-offs of all members in \( L \).

Notation  For a game \( v_{\ell r} \in GG^N \) denote \( \Phi_{\ell r}(L) \) for the combined pay-offs of members in \( L \) according to the Shapley value, i.e. \( \Phi_{\ell r}(L) := \sum_{i \in L} (\Phi_{\ell r})_i \).

In this way, we have created an infinite matrix with the value of \( \Phi_{\ell r}(L) \) on the entry in the \( \ell \)th row and \( r \)th column. This matrix is denoted by \( \Phi(L) \).

At first sight, this new notation seems not very helpful, since you add an extra sum. However, this notation makes it easier to work with the marginal vectors. This will be clear from the proof of Proposition A.1. Furthermore, since the Shapley value is symmetric and every two players from \( L \) are symmetric, the combined pay-offs of members in \( L \) lead immediately to the pay-off of each one of the members from \( L \), since everyone gets the same. Note that because of this, it holds that \( \Phi_{\ell r}(L) = \ell \cdot (\Phi_{\ell r})_i \) for a certain player \( i \in L \) if \( \ell \neq 0 \).

Furthermore, note that the choice of \( L \) rather than \( R \) is an arbitrary choice. Above notations can be easily defined for \( R \), leading to \( \Phi_{\ell r}(R) \). Due to efficiency, we have the relation \( \Phi_{\ell r}(L) + \Phi_{\ell r}(R) = \min \{ \ell, r \} \), which makes the calculations for both \( L \) and \( R \) unnecessary.
Proposition A.1 For the Shapley value of a game \( v_{tr} \in GG^N \) with \( \ell, r \in \mathbb{N} \setminus \{0\} \) we have that

\[
\Phi_{\ell r}(L) = \begin{cases} 
\frac{r}{\ell + r} \Phi_{\ell, r-1}(L) + \frac{\ell}{\ell + r} \Phi_{\ell-1, r}(L) + \frac{r}{\ell + r} & \text{if } \ell < r; \\
\frac{\ell}{\ell + r} \Phi_{\ell, r-1}(L) + \frac{r}{\ell + r} \Phi_{\ell-1, r}(L) & \text{if } \ell = r; \\
\frac{r}{\ell + r} \Phi_{\ell, r-1}(L) + \frac{\ell}{\ell + r} \Phi_{\ell-1, r}(L) & \text{if } \ell > r.
\end{cases}
\]

Proof: For the second case \( (\ell = r) \), take an arbitrary \( i \in L \) and recall that the Shapley value of player \( i \) is alternatively given by (cf. equation (2.1))

\[
(\Phi_{\ell r})_i = \sum_{S \in 2^N \setminus \{i\}} \frac{1}{2\ell(2^{|S|} - 1)} (v_{\ell r}(S \cup \{i\}) - v_{\ell r}(S)).
\]

We know that every \( S \in 2^N \) with \( i \notin S \), a certain number \( p \) of members from \( L \) and a certain number \( q \) of members from \( R \) correspond to a unique \( T \in 2^N \) with a specific \( j \in R \) such that \( j \notin T \) and with \( p \) members from \( R \) and \( q \) members from \( L \). This holds for every pair of \( i \in L \) and \( j \in R \), because there is a unique connection between members from \( L \) and members from \( R \). Therefore it holds that \( (\Phi_{\ell r})_i = (\Phi_{\ell r})_j \) for every \( i \in L \) and \( j \in R \). Furthermore we have that members in \( L \) (respectively \( R \)) are symmetric, hence \( \Phi_{\ell r} = \frac{1}{2} e^N \) and thus \( \Phi_{\ell r}(L) = \frac{1}{2} \ell \).

For the first and third part, we are first going to prove that the first part is a consequence of the third part. Therefore, note that because of efficiency for every \( \ell, r \in \mathbb{N} \setminus \{0\} \) it holds that \( \Phi_{\ell r}(L) + \Phi_{r \ell}(L) = \min \{\ell, r\} \). In order to prove the first part, let \( \ell < r \). Then

\[
\Phi_{\ell r}(L) = \ell - \Phi_{r \ell}(L)
\]

\[
= \ell - \frac{\ell}{\ell + r} \Phi_{r, \ell-1}(L) - \frac{r}{\ell + r} \Phi_{r-1, \ell}(L)
\]

\[
= \ell - \frac{\ell}{\ell + r} (\ell - 1 - \Phi_{\ell-1, r}(L)) - \frac{r}{\ell + r} (\ell - \Phi_{\ell, r-1}(L))
\]

\[
= \ell - \ell \left( \frac{\ell}{\ell + r} + \frac{r}{\ell + r} \right) + \frac{\ell}{\ell + r} \Phi_{\ell-1, r}(L) + \frac{r}{\ell + r} \Phi_{\ell, r-1}(L)
\]

\[
= \frac{\ell}{\ell + r} + \frac{\ell}{\ell + r} \Phi_{\ell-1, r}(L) + \frac{r}{\ell + r} \Phi_{\ell, r-1}(L)
\]

proves the first part, where we used efficiency for the first and third equality and the third part for the second equality.

What left is the proof of the third part, \( \ell > r \). For that purpose, define a relation on the set of all permutations, \( \Pi(N) \). Let \( \sigma, \pi \in \Pi(N) \) and define

\[
\sigma \sim \pi \iff \text{for all } i \in N \text{ : either } \sigma(i), \pi(i) \in L \text{ or } \sigma(i), \pi(i) \in R.
\]

(A.1)

This relation is an equivalence relation: it is reflexive and symmetric by definition. It is also transitive: let \( \sigma, \pi, \tau \in \Pi(N) \) and consider an arbitrary \( i \in N \). Then \( \sigma(i), \pi(i) \in L \) implies that \( \tau(i) \in L \) (\( \sigma \sim \tau \)) and thus \( \sigma(i), \tau(i) \in L \) which implies \( \sigma \sim \tau \). The other case, \( \sigma(i), \pi(i) \in R \), goes in a similar way.
Thus \(\sim\) is an equivalence relation, which makes it possible to consider the quotient space \(\Pi(N)/\sim\). This is the set of all equivalence classes \([\sigma]\), reflecting all possible orders with a fixed ordering between members from \(L\) and members from \(R\). Note that \(|\Pi(N)/\sim| = \frac{(\ell+r)!}{\ell!r!}\), since there are \(\ell!\) possible ways to order the players from \(L\) on the different spots for \(L\) and \(r!\) possible ways to order the players from \(R\). Together, each equivalence class consists of \(\ell!r!\) orders. An example is given afterwards.

Now, adjust the definition of the marginal vector for the quotient space \(\Pi(N)/\sim\):

\[
m_L^{[\sigma]}(v_{\ell r}) := \sum_{i \in L} m_i^\sigma(v_{\ell r}).
\] (A.2)

This definition is independent of the choice of the representative, because \(m_i^\sigma(v_{\ell r}) = 1\) only if the set of predecessors of \(i\) contains more members from \(R\) than members from \(L\). But if that’s the case, then also \(m_{\pi^\sigma(i)}^\sigma(v_{\ell r}) = 1\) for a \(\pi \in [\sigma]\), since \(\pi(\sigma^{-1}(i)) \in L\) (this holds because \(\sigma(\sigma^{-1}(i)) = i \in L\)).

The Shapley value \(\Phi_{\ell,r}(L)\) can be expressed using the marginal vectors for the quotient space:

\[
\Phi_{\ell,r}(L) = \sum_{i \in L} (\Phi_{\ell,r})_i = \sum_{i \in L} \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m_i^\sigma(v_{\ell r})
\]
\[
= \sum_{i \in L} \frac{1}{|N|!} \sum_{[\sigma] \in \Pi(N)/\sim} \ell!r! \cdot m_i^\sigma(v_{\ell r})
\]
\[
= \frac{1}{|N|!} \sum_{[\sigma] \in \Pi(N)/\sim} \sum_{i \in L} \ell!r! \cdot m_i^\sigma(v_{\ell r})
\]
\[
= \frac{\ell!r!}{(\ell + r)!} \sum_{[\sigma] \in \Pi(N)/\sim} m_L^{[\sigma]}(v_{\ell r}).
\] (A.3)

Here, the last equality follows from equation (A.2). Moreover, the value of the marginal vectors \(m_i^\sigma(v_{\ell r})\) only depends on the sort of glove (either left-hand or right-hand) and not on the actual player owning this glove. This justifies the step from \(\Pi(N)\) to \(\Pi(N)/\sim\), since there are \(\ell!r!\) orders in each equivalence class. Note that

\[
\Phi_{\ell,r-1}(L) = \frac{\ell!(r-1)!}{(\ell + r - 1)!} \sum_{[\sigma] \in \Pi(N \setminus \{\ell + r\})/\sim} m_L^{[\sigma]}(v_{\ell,r-1});
\]
\[
\Phi_{\ell-1,r}(L) = \frac{(\ell - 1)!r!}{(\ell + r - 1)!} \sum_{[\sigma] \in \Pi(N \setminus \{\ell\})/\sim} m_L^{[\sigma]}(v_{\ell-1,r}).
\]
This implies that
\[
\frac{r}{\ell + r} \Phi_{\ell,r-1}(L) + \frac{\ell}{\ell + r} \Phi_{\ell-1,r}(L)
\]
\[
= \frac{\ell! r!}{(\ell + r)!} \left( \sum_{[\sigma] \in \Pi(N \setminus \{\ell + r\})/\sim} m_{L}^{[\sigma]}(v_{\ell,r-1}) \right)
\]
\[
+ \sum_{[\sigma] \in \Pi(N \setminus \{\ell\})/\sim} m_{L}^{[\sigma]}(v_{\ell-1,r}) \right). \tag{A.4}
\]

Using this and equation (A.3), we only need to prove that
\[
\sum_{[\sigma] \in \Pi(N)/\sim} m_{L}^{[\sigma]}(v_{\ell r}) = \sum_{[\sigma] \in \Pi(N \setminus \{\ell + r\})/\sim} m_{L}^{[\sigma]}(v_{\ell,r-1})
\]
\[
+ \sum_{[\sigma] \in \Pi(N \setminus \{\ell\})/\sim} m_{L}^{[\sigma]}(v_{\ell-1,r}). \tag{A.4}
\]

Take an arbitrary $[\sigma] \in \Pi(N)/\sim$ and recall that $\sigma : \{1, 2, \ldots, |N| = n\} \to N$. We distinguish between whether $\sigma(n) \in L$ or $\sigma(n) \in R$. By symmetry of the players in $L$ (respectively $R$), we can thus distinguish between

i) $\sigma(n) = \ell \in L$;

ii) $\sigma(n) = \ell + r \in R$.

For i), note that $m_{\sigma(n)}^{[\sigma]}(v_{\ell r}) = 0$, since $\ell - 1 \geq r$ (i.e. the last player in the order $\sigma$ can not make a pair of gloves, because there are too few right-hand gloves). So it holds that

\[
m_{L}^{[\sigma]}(v_{\ell r}) = \sum_{i \in L} m_{i}^{[\sigma]}(v_{\ell r}) = \sum_{i \in L \setminus \{\ell\}} m_{i}^{[\sigma]}(v_{\ell r})
\]
\[
= \sum_{i \in L} m_{i}^{[\sigma]}(v_{\ell-1,r}) = m_{L}^{[\sigma]}(v_{\ell-1,r}),
\]

where the $L$ of the second row correspond to the game $v_{\ell-1,r}$ and $\pi = \sigma \upharpoonright N \setminus \{\ell\}$. So, $\pi$ is exactly the same order as $\sigma$, except the last player $\sigma(n) = \ell$ is omitted.

For ii), note that the last pair of gloves is made possible by $\sigma(n) = \ell + r$, because the right-hand gloves are the limiting factor. This means that there is no contribution to $m_{L}^{[\sigma]}(v_{\ell r})$ for this last player in the order $\sigma$, because he is no member of $L$. Again,

\[
m_{L}^{[\sigma]}(v_{\ell r}) = \sum_{i \in L} m_{i}^{[\sigma]}(v_{\ell r}) = \sum_{i \in L} m_{i}^{[\sigma]}(v_{\ell,r-1}) = m_{L}^{[\sigma]}(v_{\ell,r-1}),
\]

with $\pi = \sigma \upharpoonright N \setminus \{\ell + r\}$ defined in the same way as above.
To conclude, we have found for every \([\sigma] \in \Pi(N)/\sim\) an order \([\pi] \in \Pi(N \setminus \{\ell\})/\sim\) or an order \([\pi] \in \Pi(N \setminus \{\ell + r\})/\sim\) depending on whether we are in case i) or ii). Note that

\[
\begin{align*}
|\{\sigma \in \Pi(N)/\sim \mid \sigma(n) \in L\}| &= \Pi(N \setminus \{\ell\})/\sim = \frac{(|N| - 1)!}{(\ell - 1)!r!}, \\
|\{\sigma \in \Pi(N)/\sim \mid \sigma(n) \in R\}| &= \Pi(N \setminus \{\ell + r\})/\sim = \frac{(|N| - 1)!}{\ell!(r - 1)!}.
\end{align*}
\]

This proves equation (A.4), since

\[
\sum_{[\sigma] \in \Pi(N)/\sim} m^{|\sigma|}_L(v_{\ell r}) = \sum_{[\sigma] \in \Pi(N)/\sim; \sigma(\ell + r) \in L} m^{|\sigma|}_L(v_{\ell r}) + \sum_{[\sigma] \in \Pi(N)/\sim; \sigma(\ell + r) \in R} m^{|\sigma|}_L(v_{\ell r})
= \sum_{[\sigma] \in \Pi(N \setminus \{l\})/\sim} m^{|\sigma|}_L(v_{\ell - 1,r}) + \sum_{[\sigma] \in \Pi(N \setminus \{\ell + r\})/\sim} m^{|\sigma|}_L(v_{\ell,r - 1})
\]

\(\square\)

In the proof above, some new notions are introduced. In the following example it becomes clear why this is done.

**Example A.2** We are going to calculate \(\Phi_{23}(L)\) using the adjusted marginal vectors defined in equation (A.2). In Table A.1 the different possible equivalence classes are shown in the first columns. Note that each equivalence class consists of twelve orders \(\sigma\), e.g.

\[LLRRR = \{ (1, 2, 3, 4, 5), (1, 2, 3, 5, 4), (1, 2, 4, 3, 5), (1, 2, 4, 5, 3), (1, 2, 5, 3, 4), (1, 2, 5, 4, 3), (2, 1, 3, 4, 5), (2, 1, 3, 5, 4), (2, 1, 4, 3, 5), (2, 1, 4, 5, 3), (2, 1, 5, 3, 4), (2, 1, 5, 4, 3) \}.\]

In the table, the adjusted marginal vectors are calculated and shown. Checking the first one, \(m^{LLRRR}_L(v_{23})\), using for example \(\sigma = (2, 1, 5, 4, 3) \in LLRRR\) leads to:

\[m^{LLRRR}_L(v_{23}) = m^\sigma_2(v_{23}) + m^\sigma_3(v_{23}) = 0 + 0 = 0.\]

Since all \(\pi \in [\sigma]\) leads to the same marginal contribution for \(L\), the Shapley value can be calculated using this adjusted marginal vectors:

\[
\Phi_{23}(L) = \frac{213!}{5!} \sum_{[\sigma] \in \Pi(N)/\sim} m^{|\sigma|}_L(v_{23})
= \frac{12}{120} \cdot (0 + 0 + 1 + 1 + 1 + 2 + 2 + 2 + 2 + 2 + 2)
= \frac{1}{10} \cdot 13 = \frac{13}{10}.
\]

\(\triangle\)
APPENDIX A. THE SHAPLEY VALUE OF A GLOVE GAME

<table>
<thead>
<tr>
<th>$[\sigma]$</th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>LLRRR</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>LRLRR</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>LRRRL</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>LRRRL</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>RLLRR</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$[\sigma]$</th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>RLRLR</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>RLRRL</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>RRLRL</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>RRLRL</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Table A.1 - Marginal vectors for $\Phi_{23}$

**Remark**  
Recall that for $\ell = 0$ or $r = 0$ it holds that $v_{\ell 0}(S) = 0 = v_{0r}(S)$ for every $S \in 2^N$. This implies that the Shapley value is $\Phi_{\ell r}(L) = 0$.

Using the recursive formula it is possible to calculate the entries $\Phi_{\ell r}(L)$ of the matrix $\Phi(L)$. For example, the diagonal entries are very easy. Continuing in this way, leads to the matrix $\Phi(L)$ in (A.5):

$$\Phi(L) = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
2 & 3 & 4 & 5 & 6 & 7 \\
1 & 13 & 22 & 11 & 23 \\
3 & 10 & 15 & 7 & 14 \\
1 & 7 & 3 & 67 & 121 & 65 \\
4 & 10 & 2 & 35 & 56 & 28 \\
1 & 8 & 38 & 2 & 317 & 298 \\
5 & 15 & 35 & 2 & 126 & 105 \\
1 & 3 & 47 & 187 & 5 & 718 \\
6 & 7 & 56 & 126 & 2 & 231 \\
1 & 5 & 19 & 122 & 437 & 3 \\
7 & 7 & 28 & 105 & 231 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}$$  

(A.5)


