Lattice Isomorphisms between Riesz Spaces

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Abstract. In this thesis we will look into lattice isomorphisms between Riesz spaces. Firstly, we wonder what lattice isomorphisms look like and what Riesz structures they preserve. i.e. in how far they are similar to Riesz isomorphisms, which we know to preserve units, Archimedeanity, Riesz ideals and bands. We also wonder which extra properties might help the preservation by the lattice homomorphisms.

Secondly, we study the theory of Kaplansky, about the lattice structure of $C(X)$ characterizing $X$ up to homeomorphism, for certain topological spaces $X$. We build up his theory for compact Hausdorff spaces $X$. And broaden $C(X)$ to $C(X \to [-\infty, \infty])$.

Thirdly, we investigate lattice homomorphisms with the extra property that they preserve scalar multiplication. We see that such lattice homomorphisms are actually Riesz homomorphisms. With the help of Kaplansky we also research other assumptions on our lattice homomorphisms which can indicate Riesz homomorphisms. This will lead us to a certain kind of lattice homomorphisms. Moreover, we study whether or not a lattice isomorphism between $C(X)$ and $C(Y)$ indicate a specific homeomorphism between $X$ and $Y$, and see that our newly defined lattice isomorphism do.

Fourthly, we will try to get rid of the compactness assumption in the theory of Kaplansky. This leads to a study of the Stone-Čech compactification $\beta X$ of a Hausdorff space $X$. We can prove nearly the same results for $\beta X$ as we had for $X$, with the help of lattice homomorphisms to the space $\{0, 1\}$. Furthermore, in the special case that our spaces are metrisable we see again that $C(\hat{X})$ characterises $X$ up to homeomorphism.

Fifthly, last we look at extremally disconnected compact Hausdorff spaces $X$ and define $C^\infty(X)$. We see that every Archimedean Riesz space is Riesz isomorphic to a order dense subspace of a $C^\infty(X)$ for some extremally disconnected compact Hausdorff space $X$. From which we can prove that any lattice isomorphism between two Archimedean Riesz spaces supplies us with a lattice isomorphism between two spaces $C^\infty(X), C^\infty(Y)$ for extremally disconnected compact Hausdorff space $X, Y$. Since lattice isomorphisms between $C^\infty(X), C^\infty(Y)$ for extremally disconnected compact Hausdorff space $X, Y$ can easily be described, we get a way to describe lattice isomorphism between Archimedean Riesz spaces.

Finally we investigate lattice isomorphisms between sequence spaces.
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CHAPTER 1

Introduction

In lecture series of professor van Rooij, we have been studying Riesz spaces and $C(X)$, the set of continuous functions on a space $X$. Here we have seen that the Riesz structure of the space $C(X)$ characterises a realcompact space $X$ up to homeomorphism. Hence if two spaces $C(X), C(Y)$ are Riesz isomorphic and $X, Y$ are realcompact, then $X$ and $Y$ are homeomorphic. We started to wonder if it is necessary to know the complete Riesz structure of the space or if it might perhaps be enough to know the lattice structure of $C(X)$ to characterise $X$. This is what we decided to investigate thoroughly. We will take you by the hand on a journey through our study of lattice isomorphism on Riesz spaces.

As our topic is inspired by the courses of Riesz spaces and $C(X)$ it can hardly come as a surprise that their lecture notes [vR11],[vR14] form a basis to our study. Furthermore, we used a lot from [DJR77], which also played an important role in both lecture series. As last, we study [Kap47]. We assume the reader has a basic knowledge concerning Riesz spaces and topology. Understanding Riesz spaces is most important since the whole thesis concerns them. Topology is used through the theorems of Urysohn and Tychonov in the beginning. Chapter 5 assumes a basic understanding of the Stone-ˇCech compactification. In chapter 6 more topology is needed, but all relevant definitions will be given.

In the second chapter, we look at what Riesz structures such as units, Archimedeanity, Riesz ideals or bands, are preserved by lattice homomorphisms. Here we find that lattice homomorphisms are far weaker than Riesz homomorphisms since they do not preserve everything. We see that when we assume our Riesz spaces to be of the form $C(X)$ for some $X$, then spaces with a unit are mapped to spaces with a unit. Furthermore we prove a space realcompact space $X$ is compact if and only if $C(X)$ has a unit. Next we prove there cannot be an isomorphism from a non-Archimedean Riesz spaces to any SOS space, while the set of SOS spaces is a subset of the set of Archimedean spaces. Unfortunately we are not able to prove much for Riesz ideals, which is why we look at bands instead. For bands we prove that in Archimedean Riesz spaces bands are preserved by all lattice homomorphisms, which we prove to be Boolean Algebras.

The third chapter is due to the theory of Kaplansky telling us that the lattice structure of $C(X)$ characterises compact Hausdorff spaces $X$ up to homeomorphism. Next we broaden the theory to $C(X \to [-\infty, \infty])$ and $C(X \to [0, 1])$, and show that they also characterise $X$ when $X$ is compact Hausdorff. Finally we note that we have implicitly proven that for compact Hausdorff spaces $X$ any lattice homomorphism from $C(X), C(X \to [-\infty, \infty])$ or $C(X \to [0, 1])$ to $\mathbb{R}, \{0, 1\}$ or similar spaces is associated with a point of $X$. This will be used and researched more in the next chapters.

In the fourth chapter, we study lattice homomorphisms with the extra property that they preserve scalar multiplication. In this study the Yoshida Representation Theorem is of great help. We see that this means that the lattice homomorphisms are actually Riesz homomorphisms. Then together with the help of the third chapter we investigate an other assumption on lattice homomorphisms starting in $C(X)$, which we will call lattice $\alpha$-homomorphisms $T$, where $0 < \alpha \in \mathbb{R}$ is s.t. $\phi(\lambda f) = \text{sgn}(\lambda)|\lambda|^{\alpha}\phi(f)$ for all $\lambda \in \mathbb{R}$ and all $f \in C(X)$. When we assume these lattice $\alpha$-homomorphism $\phi$ to also have $\phi(1) = 1$ we prove that they are a special kind of evaluations at a point. Furthermore, we see that with a lattice $\alpha$-isomorphism between $C(X), C(Y)$ we can construct a homeomorphism between $X$ and $Y$. When $T: C(X) \to E$ is a lattice $\alpha$-isomorphism to any Riesz space $E$, that $E$ must be Archimedean with a unit, and $E$ is Riesz isomorphic to $C(X)$.

The fifth chapter, we will try to get rid of the assumption that our space is compact while we still want the lattice structure of $C(X)$ to characterise $X$. This leads us to a study of the Stone-ˇCech compactification $\beta X$ of $X$. Since $\beta X$ is compact we can use our theory from chapter 3 again, but then applied to $\beta X$. We see that this works perfectly, and that in the special case where $X$ is metrisable we get $C(X)$ characterising $X$. 

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again. Finally we will build up the theory of Kaplansky again, but then with the help of lattice isomorphisms \( \phi : C(X \to [0,1]) \to \{0,1\} \) instead of the prime ideals. This will also give us the results which we had before.

In the sixth chapter, we look at extremally disconnected compact Hausdorff spaces \( X \) and \( C^\infty(X) \). We see that every Archimedean Riesz space is Riesz isomorphic to a subspace of a \( C^\infty(X) \) for some extremally disconnected compact Hausdorff space \( X \). Furthermore any lattice isomorphism between two Archimedean Riesz spaces supplies us with a lattice isomorphism between two spaces \( C^\infty(X), C^\infty(Y) \) for extremally disconnected compact Hausdorff space \( X,Y \). Since lattice isomorphisms between \( C^\infty(X), C^\infty(Y) \) for extremally disconnected compact Hausdorff space \( X,Y \) can easily be described, we get a way to describe lattice isomorphism between Archimedean Riesz spaces.

Finally we look at every concrete lattice isomorphisms between sequence spaces. We will see that there are a lot of non-isomorphic spaces, but that all the \( l^p \) spaces are isomorphic to one another.
CHAPTER 2

Lattice isomorphisms and Riesz structures

We are interested in lattice isomorphisms between Riesz spaces. Since we work on Riesz spaces we look at properties which can depend heavily on the vector space structure. We investigate which properties are preserved by (certain) lattice isomorphisms. One might wonder why we think lattice isomorphisms could have anything to do with properties of Riesz spaces, but it is certainly worth the try and we might actually surprise you. We start by looking into units, but not before we defined Riesz spaces.

Definition 2.1. A Riesz space, or ”vector lattice”, is an ordered vector space that is a lattice.

Lattice isomorphisms and units

We will start out with the definition, just to refresh our memory.

Definition 2.2. An element $e$ of a Riesz space $E$ is said to be a unit if for all $a \in E$ there is a $n \in \mathbb{N}$ s.t. $a \leq ne$.

Now we can wonder: do lattice isomorphisms always send spaces with a unit to spaces with a unit? Unfortunately not, as the next example will show.

Example 2.3. Let $E = BC[2, \infty)$, which has unit $1$. Let $F = C[2, \infty)$, which has no unit. Define a lattice isomorphism $T$ from $E$ to a subspace $F'$ of $F$ by $T(f)(x) = \text{sgn}(f(x))|f(x)|^x$, this has inverse $T^{-1}(g)(x) = \text{sgn}(g(x))|g(x)|^x$. It is easy to see that the space $F'$ has no unit either. Hence a space with a unit can be lattice isomorphic to a space without a unit.

This motivates us to investigate a closely related property, which is preserved by the lattice isomorphisms.

Theorem 2.4. Let $E$ be a Riesz space with unit $u$. Then for all $n \in \mathbb{N}$ there are $a_n, b_n \in E$ s.t. $E = \bigcup[a_n, b_n]$.

Proof. Since $a_n, b_n \in E$ for all $n \in \mathbb{N}$ it is trivial that $\bigcup[a_n, b_n] \subseteq E$. Furthermore, take $a_n = -nu, b_n = nu$. Then since $u$ is a unit, for all $a \in E$ there is an $m \in \mathbb{N}$ s.t. $a, -a$ are smaller than $mu$ so that $a \in [-mu, mu] = [a_n, b_n]$. Hence $E \subseteq \bigcup[a_n, b_n]$. Thus $E = \bigcup[a_n, b_n]$. $\square$

One might wonder if it works the other way around too. Can we conclude from the existence of elements with the above introduced property that our space has a unit? To answer this question we need to investigate our new property better. We start by noting that for $E = C(\mathbb{R})$ no such elements exist.

Theorem 2.5. There are no functions $f_n, g_n \in C(\mathbb{R})$ for all $n \in \mathbb{N}$ s.t. $C(\mathbb{R}) = \bigcup[f_n, g_n]$.

Proof. Assume for all $n \in \mathbb{N}$ there are $f_n, g_n \in C(\mathbb{R})$ s.t. $C(\mathbb{R}) = \bigcup[f_n, g_n]$. Take a function $g \in C(\mathbb{R})$ s.t. $g(n) = g_n(n) + 1$. (We could define it to be 0 in all negative integers, and affine in-between successive integers.) Then $g \not\in [f_n, g_n]$, since $g(n) > g_n(n)$ for all $n \in \mathbb{N}$. Hence, there are no functions $f_n, g_n \in C(\mathbb{R})$ for all $n \in \mathbb{N}$ s.t. $C(\mathbb{R}) = \bigcup[f_n, g_n]$. $\square$

This isn’t the end of the world, because we know that the space $C(\mathbb{R})$ has no unit either. As we announced our latest property is preserved by lattice isomorphisms.

Theorem 2.6. Let $T: E \to F$ be a lattice isomorphism, let $E = \bigcup[a_n, b_n]$ where $a_n, b_n \in E$ for all $n \in \mathbb{N}$. Then $F = \bigcup[T(a_n), T(b_n)]$.

Proof. Take $c \in F$. We know $T^{-1}(c) \in E$. Hence there is a $n \in \mathbb{N}$ s.t. $a_n \leq T^{-1}(c) \leq b_n$. Since $T$ is a lattice isomorphism we know $T(a_n) \leq T(T^{-1}(c)) = c \leq T(b_n)$. So $c \in \bigcup[T(a_n), T(b_n)]$ for every $c \in F$. Thus $F = \bigcup[T(a_n), T(b_n)]$. $\square$
This is a great start, since the above theorem also tells us that $F$ can be written $F = \bigcup [a_n, b_n]$ with $a_n, b_n \in F$ for all $n \in \mathbb{N}$. In other words, our new property is preserved by lattice isomorphisms.

However, since we saw that units are not always preserved we should be worried that the converse of theorem 2.4 does not hold. An easy example proves us right on this.

**Example 2.7.** Let $E = \{ f \in C(\mathbb{R}) : \text{ there is a polynomial } P \text{ s.t. } |f| \leq P \}$. Take $g_n(x) = n + n|x| + n|x|^2 + \ldots + n|x|^n$. Then, we easily see that $E = [-g_n, g_n]$. We will now prove that $E$ has no unit.

For the sake of contradiction, assume there is a unit $e \in E$. Then by definition of our space there is a $g_n$ s.t. $|e| \leq g_n$. Since $e$ is a unit there is a $m \in \mathbb{N}$ s.t. $g_{n+1} \leq me \leq mg_n$. But $g_{n+1}(x) > mg_n(x)$ for large values of $x$, because $g_{n+1}$ contains a higher power of $x$. Thus we have the desired contradiction. Hence, $E$ has no unit.

The above example shows there exists a space $E$ s.t. $E = \bigcup [a_n, b_n]$ with $a_n, b_n \in E$ for all $n \in \mathbb{N}$, while $E$ has no unit. So as we already thought, the converse of theorem 2.4 does not hold.

In order to keep our hopes up, we will look at $C(X)$ for some Hausdorff space $X$ instead of just any Riesz space $E$. This gives us more positive results.

**Theorem 2.8.** Let $X$ be a Hausdorff space. Then $C(X)$ has a unit if and only if there are functions $f_n, g_n \in C(X)$ for all $n \in \mathbb{N}$ s.t. $C(X) = \bigcup [f_n, g_n]$. In this case $\mathbb{1}_X$ is a unit of $C(X)$.

**Proof.** ($\Rightarrow$) This side is already proved at theorem 2.4.

($\Leftarrow$) We will in fact prove that $\mathbb{1}_X$ is a unit of $C(X)$. Assume $C(X)$ has no unit, or more generally, that $\mathbb{1}_X$ is not a unit of $C(X)$. Then there exists an unbounded function $f \in C(X)^+$. Now take an arbitrary $x_1$ s.t. $f(x_1) > 1$. Construct a sequence $x_1, x_2, \ldots$ in such a way that $f(x_{n+1}) > f(x_n) + 1$ for all $n \in \mathbb{N}$.

Let $f_n, g_n \in C(X)$ be given for all $n \in \mathbb{N}$ s.t. $C(X) = \bigcup [f_n, g_n]$. Take a strictly increasing continuous bijection $\tau : \mathbb{R} \to \mathbb{R}$ s.t. $\tau(n) \geq g_n(x_n) + 1$. If we define $\tau \circ f$ as the composition of $\tau$ after $f$, then $\tau \circ f \in C(X)$ since it is the composition of two continuous functions. Furthermore, $\tau \circ f(x_n) \geq g_n(x_n) + 1$. Hence $\tau \circ f \notin \bigcup [f_n, g_n] = C(X)$, which contradicts our assumption. Thus, $\mathbb{1}_X$ is a unit of $C(X)$. □

This new and useful theorem immediately brings us a corollary.

**Corollary 2.9.** Let $T : C(X) \to E$ be a lattice isomorphism. Then $C(X)$ has a unit if and only if there exist $a_n, b_n \in E$ for all $n \in \mathbb{N}$ s.t. $E = \bigcup [a_n, b_n]$.

**Proof.** ($\Rightarrow$) If $C(X)$ has a unit, theorem 2.8 tells us $C(X) = \bigcup [-n\mathbb{1}_X, n\mathbb{1}_X]$. So according to theorem 2.6, $E = \bigcup [T(-n\mathbb{1}_X), T(n\mathbb{1}_X)]$.

($\Leftarrow$) If $E = \bigcup [a_n, b_n]$ for $a_n, b_n \in E$ for all $n \in \mathbb{N}$, then theorem 2.6 says $C(X) = \bigcup [T^{-1}(a_n), T^{-1}(b_n)]$. Hence, theorem 2.8 proves $C(X)$ has a unit. □

This corollary might not look impressive, but when $E = C(Y)$ for some compact Hausdorff space $Y$ we get what we wanted all along.

**Theorem 2.10.** Let $T : C(X) \to C(Y)$ be a lattice isomorphism. Then $C(X)$ has a unit if and only if $C(Y)$ has a unit.

**Proof.** If $C(X)$ has a unit, for all $n \in \mathbb{N}$ there are $f_n, g_n \in C(X)$ s.t. $C(Y) = \bigcup [f_n, g_n]$ according to corollary 2.9. Thus theorem 2.8 says $C(Y)$ has a unit. The other way works just the same. □

There might be some questions coming up, like: When does a set $C(X)$ have a unit? What kind of assumptions on $X$ do we have to make, apart from being Hausdorff?

It is not difficult to see that theorem 2.5 and theorem 2.8 together prove that $C(\mathbb{R})$ has no unit, since there are no functions $f_n, g_n \in C(\mathbb{R})$ for all $n \in \mathbb{N}$ s.t. $C(\mathbb{R}) = \bigcup_{n \in \mathbb{N}} [f_n, g_n]$. While on the other hand, we can prove for compact Hausdorff spaces $X$, that $C(X)$ has a unit.

**Theorem 2.11.** Let $X$ be compact Hausdorff. Then $C(X)$ has a unit.

**Proof.** Continuous functions $f$ on a compact Hausdorff are always bounded. So there is an $n \in \mathbb{N}$ s.t. $|f| \leq n\mathbb{1}_X$. Hence $\mathbb{1}_X$ is a unit. □

It would be great if we could prove that for non-compact spaces $X$, $C(X)$ has no unit. Unfortunately this is not true for just any Hausdorff space $X$. As we will show next, we can construct a rather special space as counterexample.
Definition 2.13. A space $X$ is said to be \textit{pseudocompact} if any function in $C(X)$ is bounded.

It is not difficult to note that all functions in $C(X)$ are bounded iff $1_X$ is a unit. Hence pseudocompactness really is the property we have been investigating all along. Now we introduce realcompactness.

Definition 2.14. A space $X$ is said to be \textit{realcompact} if $X$ is homeomorphic to a closed subset of $\mathbb{R}^S$ for some set $S$.

Now if we assume that our Hausdorff space $X$ is realcompact we will see that $C(X)$ has a unit, hence is pseudocompact, iff $X$ is compact.

Theorem 2.15. Let $X$ be realcompact. Then $X$ is compact iff $C(X)$ has a unit.

Proof. $(\Rightarrow)$ We have already proven this for any compact $X$ at theorem 2.8.

$(\Leftarrow)$ Let $X \subseteq \mathbb{R}^S$ be closed. Look at the coordinate functions $x \mapsto x_s$ for $s \in S$, note that they are continuous. Hence, there are $a_s, b_s \in \mathbb{R}$ s.t. $X$ is a closed subset of $\Pi_{s \in S}[a_s, b_s]$. Now Tychonoff gives us that $X$ is compact.

The research in the literature showed also that: "$X$ is compact iff $X$ is pseudocompact" holds precisely when $X$ is realcompact. Hence we also see that: "$X$ is compact iff $C(X)$ has a unit" holds precisely when $X$ is realcompact. This will not be proved here, an interested reader could take a look at page 153 of [Eng68].

Lattice isomorphisms and Archimedeanity

The next Riesz space property we investigate is being Archimedean, this we will refer to as "Archimedeanity". Will lattice isomorphisms send Archimedean Riesz spaces only to Archimedean Riesz spaces? Or can Archimedean Riesz spaces be lattice isomorphic to non-Archimedean Riesz spaces too? Again we start by remembering the definition.

Definition 2.16. We call an element $x$ of a Riesz space $E$ \textit{infinitesimal} if the set $\{nx : n \in \mathbb{Z}\}$ has an upper bound $b \in E$. Then we also call $x$ infinitesimal with respect to $b$. A Riesz space $E$ is said to be Archimedean if $0$ is the only infinitesimal element of $E$.

To study lattice isomorphisms from non-Archimedean Riesz spaces to Archimedean Riesz spaces, we first consider the easiest non-Archimedean Riesz space we know, the lexicographic plane.

Theorem 2.17. There is no lattice isomorphism from the lexicographic plane to a Archimedean Riesz space.

Proof. For the sake of contradiction, assume that $T : E \to F$ is a lattice isomorphism, that $E$ is the lexicographic plane and $F$ an Archimedean Riesz space. Since $E$ is totally ordered, $F$ must be totally ordered. So $F$ is a totally ordered Archimedean Riesz space, hence Riesz isomorphic to $\mathbb{R}$. This is proven in lemma 3.5 of [vR11]. Thus we may assume that $T : E \to \mathbb{R}$.

Now we see that for every $t \in \mathbb{R}$ the set $\{(t, x) : x \in \mathbb{R}\}$ is sent to an interval of $\mathbb{R}$, by $T$. For $s, t \in \mathbb{R}$ with $s \neq t$ we see that the images of $T$ working on $\{(t, x) : x \in \mathbb{R}\}, \{(s, x) : x \in \mathbb{R}\}$ are disjoint. All intervals in $\mathbb{R}$ contain at least one $q \in \mathbb{Q}$, hence we can prove that $\mathbb{Q}$ is uncountable which of course is a contradiction. Hence no such lattice isomorphism can exist.

Unfortunately we are not able to prove that Archimedeanity is preserved by all lattice homomorphism. Since we are unable to provide counterexamples we post the following conjecture.

Conjecture 2.18. Let $E$ be an Archimedean Riesz space, let $T : E \to F$ be a lattice isomorphism. Then $F$ is an Archimedean Riesz space.
To further study the preservation of Archimedeanity by lattice isomorphisms for a certain set of Archimedean Riesz space, we start looking at another property of Riesz spaces which, as we will soon see implies Archimedean-

**Definition 2.19.** A Riesz space $E$ is called **SOS** or strong order separable, if $E^+$ has a countable subset $X$ s.t. every element of $E^+$ is the supremum of a subset of $X$.

We start by proving that a SOS Riesz space is indeed Archimedean.

**Theorem 2.20.** If a Riesz space $E$ is SOS, then it is Archimedean.

**Proof.** We suppose $E$ is SOS, with $X$ a countable subset which satisfies the condition in the definition. For the sake of contradiction, let $a, b \in E^+$ with $0 < a \leq n^{-1}b$ for all $n \in \mathbb{N}$. Then for every $\lambda \in (0, \infty)$ there is a $x_\lambda \in X$ s.t. $x_\lambda \leq \lambda b$, $x_\lambda \leq \lambda b + a$. If $\lambda < \mu$ then $x_\lambda \leq \lambda b + a \leq \mu b$ while $x_\mu \not\leq \lambda b$, hence $x_\mu \not\geq x_\lambda$. So all $x_\lambda$ are different. Thus we have an uncountable subset of $X$ which contradicts the assumptions. So there exist no $a, b \in E$ with $0 < a \leq n^{-1}b$ for all $n \in \mathbb{N}$. Hence $E$ is Archimedean. 

Next we note that suprema are preserved by lattice isomorphisms.

**Theorem 2.21.** Let $E, F$ be Riesz spaces, $D \subset E$ and let $T : E \to F$ be a lattice isomorphism. Then $T(\sup D) = \sup\{T(d) : d \in D\}$, hence $T$ preserves the suprema.

**Proof.** Let $a = \sup D$. Then $d \leq a$ for all $d \in D$, thus $T(a) \vee T(d) = T(a \vee d) = T(a)$ for all $d \in D$. Hence $T(d) \leq T(a)$ for all $d \in D$. So $T(a)$ is an upper bound of $\{T(d) : d \in D\}$. Let $b \in F$ be a upper bound of $\{T(d) : d \in D\}$. Then by the previous argument we see that $T^{-1}(b) \in E$ and $T^{-1}(b)$ is an upper bound of $D$. Hence $a \leq T^{-1}(b)$. Thus $T(a) \leq b$, which completes the proof.

As the word "order" might already hint, strongly order separability is preserved by lattice isomorphisms which preserve 0, since only order plays a role in the definition of SOS spaces.

**Theorem 2.22.** Let $T : E \to F$ be a lattice isomorphism, with $T(0) = 0$. Let $E$ be SOS. Then $F$ is SOS.

**Proof.** Let $X$ be the countable set from the definition. We claim that $T(X)$ is a countable set such that every element of $F^+$ is the supremum of a subset of $T(X)$. Obviously $T(X)$ is countable, since $X$ is countable and $T$ a bijection. Furthermore, note that since $T(0) = 0$, $E^+$ is send to $F^+$. Now, let $a \in F^+$ be given. Then $a = T(a')$ for some $a' \in E^+$. Let $Y \subseteq X$ be the subset of $X$ such that $a' = \sup Y$. Then $a = T(\sup Y) = \sup T(Y)$, since theorem 2.21 says that the supremum is preserved by lattice isomorphisms. So, the claim is proved. Thus $F$ is SOS.

This theorem shows us that we have found a class of Archimedean spaces on which the Archimedeanity is preserved by lattice isomorphisms, hence they cannot be lattice isomorphic to a non-Archimedean space. We give some examples of SOS spaces.

**Example 2.23.** It is easy to see that the spaces $l^\infty(\mathbb{N})$ and $\mathbb{R}^\mathbb{N}$ are SOS spaces. For $\mathbb{R}^\mathbb{N}$ we can take $X = (\mathbb{Q}^+)^\mathbb{N}$. For $l^\infty(\mathbb{N})$ we can take $X$ as the set of elements of $l^\infty(\mathbb{N})$ which exclusively consist of rational numbers and at most have finitely many non-zero coordinates.

Finally we note that non-Archimedean space are not extendable to any SOS space. Hence there is no isomorphism from a non-Archimedean Riesz space to a subspace of the above mentioned spaces either.

**Lattice isomorphisms and Riesz ideals**

Thirdly, we look at Riesz ideals and wonder if they perhaps are preserved by lattice isomorphisms. Unfortunately we will not be able to prove a lot about Riesz ideals, but nevertheless we start by defining them.

**Definition 2.24.** A Riesz ideal of a Riesz space $E$ is a linear subspace $D$ of $E$ s.t. for all $a \in D$, $x \in E$, $|x| \leq |a|$ implies $x \in D$.

For ideals the 0-element of the space is very important since $\{0\}$ is a Riesz ideal. Hence we can make easy counterexamples if we do not assume that $T(0) = 0$. This we can of course settle with the extra demand that $T(0) = 0$. Then still we are not able to prove or disprove the preservation of Riesz ideals by lattice isomorphisms. Because we were unable to think up a counterexample we post the following conjecture.
Conjecture 2.25. Riesz ideals are preserved by lattice isomorphisms.

To still get some results about the preservation of Riesz ideals by lattice isomorphisms we start looking at a special kind of Riesz ideals, namely bands, just like we did for the Archimedean space. For bands we will prove preservation by lattice isomorphisms.

Lattice isomorphisms and bands

As we just said, we will now take a look at bands. As always, we start with the definition.

Definition 2.26. A band \( B \) is a Riesz ideal with the extra property that: If \( X \) is a non-empty subset of \( B \) and \( a \in E^+ \) with \( a = \sup X \) then \( a \in B \).

So bands are Riesz ideals which contain all their suprema. Let’s start by giving some examples which will become more important later on.

Example 2.27. Let \( E \) be \( C[0, 1] \). Then \( B_1 = \{ f \in E : f(\frac{1}{2}) = 0 \} \) is an ideal but not a band, since \( \mathbb{1}_{[0,1]} = \sup \{ f \in B_1 : f \leq 1 \} \) while \( \mathbb{1}_{[0,1]} \notin \).

On the other hand, \( B_2 = \{ f \in E : f([0, \frac{1}{2}]) = 0 \} \) is a band. To prove this we note that \( B_2 \) is an ideal and take any non-empty subset \( X \) of \( B_2 \). If \( X \) has supremum \( g \), then we need to prove that \( g \in B_2 \). To do that, take any \( x \in (0, \frac{1}{2}) \). Suppose \( g(x) > 0 \). Define a function \( h \in C[0, 1] \) with \( h([0, x]) = 0 \), \( h(\frac{1}{2}, 1] = 1 \) and for \( y \in (\frac{1}{2}, 1] \) we have \( h(y) = \frac{x - y}{x - \frac{1}{2}} \). Since \( g \) is bounded there is an \( n \in \mathbb{N} \) s.t. \( nh > g \) on \( [\frac{1}{2}, 1] \). Now we see that \( g \notin g \wedge nh \) while \( g \wedge nh \) is an upper bound of \( X \) too, since all \( f \in X \) are 0 on \( [0, \frac{1}{2}] \) anyway. This contradicts the assumption of \( g \) being the supremum. Hence, for all \( x \in (0, \frac{1}{2}) \) : \( g(x) = 0 \) and since \( g \) is continuous also \( g(\frac{1}{2}, 1] = 0 \). Thus the supremum of any non-empty subset of \( X \) is an element of \( B_2 \). Moreover, \( B \) is a band.

We meet a completely different example if we look at measure spaces.

Example 2.28. \( E = \mathcal{L}^1(X, A, \mu) \). Let \( Y \in A \). Then \( B = \{ f \in E : f(Y) = 0 \} \) is a band.

Before we further investigate bands, we introduce a new property: orthogonality of elements. This will allow us to define bands in a different way, when we are working in a Archimedean Riesz space.

Definition 2.29. Let \( E \) be a Riesz space. Then \( a, b \in E \) are orthogonal, \( a \perp b \), if \( |a| \wedge |b| = 0 \). Furthermore, for any subset \( D \) of \( E \) we can define the orthogonal complement \( D^\perp \) of \( D \) by:

\[
D^\perp = \{ a \in E : a \perp d : \forall d \in D \}.
\]

We start by proving some elementary facts about orthogonal elements.

Lemma 2.30 (Elementary facts). For all \( a, b, c \in E \) and \( \lambda \in \mathbb{R} \) we have:

\begin{enumerate}
  \item \( a \perp b \) and \( |c| \leq |a| \Rightarrow c \perp b \).
  \item \( a \perp b \Rightarrow \lambda a \perp b \).
  \item \( a \perp b \) and \( c \perp b \Rightarrow (a + c) \perp b \).
  \item \( a \perp b \Leftrightarrow a^+ \perp b \) and \( a^- \perp b \).
  \item Let \( D \) be a subset of \( E \). Let \( a = \sup D, b \in D^\perp \). Then \( a \perp b \).
\end{enumerate}

Proof.

\begin{enumerate}
  \item \( 0 \leq |c| \wedge |b| \leq |a| \wedge |b| = 0 \). Hence \( c \perp b \).
  \item \( \lambda |a| > 1 \). Then \( 0 \leq \lambda |a| \wedge |b| \leq |a| \wedge |b| \leq |\lambda|(|a| \wedge |b|) = |\lambda||0| = 0 \). If \( \lambda \leq 1 \) then \( 0 \leq |a| \wedge |b| \leq |a| \wedge |b| \leq (|a| \wedge |b|) = 0 \). Hence, \( \lambda a \perp b \).
  \item \( 0 \leq |a + c| \wedge |b| \leq (|a| + |c|) \wedge |b| \leq (|a| \wedge |b|) + (|c| \wedge |b|) \leq 0 + 0 = 0 \). Hence, \( a \perp b \) and \( c \perp b \Rightarrow (a + c) \perp b \).
  \item \( \Rightarrow \) Trivial, since we know that \( |a^+|, |a^-| \leq a \). Hence (i) gives us that \( a^+ \perp b \) and \( a^- \perp b \).
  \item Let \( d^+ \perp b \) and \( a^- \perp b \), then \( |a| \wedge |b| = (a^+ + a^-) \wedge |b| = 0 \). Hence \( |a| \perp b \) and thus \( a \perp b \).
\end{enumerate}

We have \( d^+ \perp b \) and \( d^- \perp b \) for all \( d \in D \). So \( a^+ = \sup \{ d^+ : d \in D \} \) and \( a^- = \inf \{ d^- : d \in D \} \). Thus \( a^+ \wedge |b| = \sup \{ d^+ \wedge |b| : d \in D \} = 0 \) and \( 0 \leq a^- \wedge |b| \leq d^+ \wedge |b| = 0 \) for all \( d \in D \). Hence \( a^+ \perp b \) and \( a^- \perp b \) so (iv) gives us \( a \perp b \).

This leads to an easy corollary.

Corollary 2.31. Let \( D \) be any subset of \( E \). Then \( D^\perp \) is a band.
Proof. In the above lemma all the properties of a band are proven 1 by 1. □

This is of course a nice beginning but won’t help us enough yet. Luckily there is more! If we assume \( E \) to be Archimedean we can characterise the bands by the sets which are equal to their double orthogonal complement.

**Theorem 2.32.** Let \( E \) be an Archimedean Riesz space. Then the bands of \( E \) are precisely the subsets \( B \) with \( B = B_{\perp\perp} = (B_{\perp})_{\perp} \).

Proof. First note that corollary 2.31 tells us that \( B_{\perp\perp} \) is a band, since it is the orthogonal complement of the set \( B_{\perp} \). So we just need to prove that for every band \( B \) in \( E \) we have \( B = B_{\perp\perp} \). For the sake of contradiction we will assume that there is a band \( B \) with \( B \neq B_{\perp\perp} \). Since it is trivial that \( B \subseteq B_{\perp\perp} \) we can take an element \( a \in B_{\perp\perp} \) with \( a \notin B \) and define the set \( M_a \) by:

\[
M_a = \{ b \in B : 0 < b < a \}.
\]

Note \( a \in B_{\perp\perp}, a \notin B \) so \( a \notin B \). Hence \( M_a \neq 0 \). Furthermore, \( a \) majorizes \( M_a \) but \( a \) is not the supremum of \( M_a \), since \( M_a \subseteq B \) and \( B \) is a band. Thus \( M_a \) has a upper bound \( c \in E \) with \( 0 < c < a \). Thus, \( 0 < b < c \) and define the set \( M_{a-c} \) in the same way as \( M_a \) was defined above. By similar argument we see that \( M_{a-c} \) is non-empty. Hence there is a \( b \in B^+ \) s.t. \( 0 < b < a-c \). Then \( 0 < b < a-c < a \), hence \( b \in M_a \). Thus \( b < c \), so \( 0 < b + b < (a-c) + c < a \), hence \( b + b \in M_a \). Thus \( b + b < c \), hence we can repeat this argument and note that for all \( n \in \mathbb{N} \) we have \( nb < c \) while \( b \neq 0 \), which contradicts the Archimedeanity of our space \( E \). Hence for all band \( B \) we have \( B = B_{\perp\perp} \). □

It can actually be proven that the bands \( B \) of \( E \) are precisely the subsets with \( B = B_{\perp\perp} \) if and only if \( E \) is Archimedean. But since we won’t need that and it would be a lot more work, it will not be proven here. An interested reader can find the proof in [DJR77].

Now we know so much about the properties of bands and have found this new characterisation, we will use it to prove that lattice isomorphisms \( T : E \rightarrow F \) with \( T(0) = 0 \) preserve bands. The following lemma will help us to achieve this.

**Lemma 2.33.** Let \( E,F \) be Riesz spaces. Let \( T : E \rightarrow F \) be a lattice isomorphism with \( T(0) = 0 \). Then:

1. For all \( x \in E \): \( (Tx)^+ = T(x^+) \), \( (Tx)^- = -T(-(x^-)) \), \( (-T(-x))^+ = T(x^-) \).
2. If \( x,y \in E^+ \) and \( x \perp y \) then \( Tx \perp Ty \), \( T(x+y) = Tx + Ty \), \( T(x-y) = Tx + T(-y) \).
3. If \( x,y \in E \) and \( x \perp y \), then \( Tx \perp Ty \), \( T(x+y) = Tx + Ty \).
4. If \( X \subseteq E \), then \( T \) maps \( X^+ \) onto \( (T(X))^+ \).

Proof. (1) Let \( x \in E \). \( (Tx)^+ = (Tx) \lor (T0) = T(x \lor 0) = T(x^+) \) and \( (Tx)^- = -(T(x) \land (T0)) = -T(x \land 0) = -T(-(x \lor 0)) = -T(-x^-) \). Also, \( T(x^-) = T((-x)^+) = (T(-x))^+ = -T(-x^-) \).
(2) Let \( x,y \in E^+ \), \( x \perp y \). Then \( x \land y = 0 \), \( Tx \land Ty = (x \land y) = 0 \), so \( Tx \perp Ty \). Hence, \( T(x+y) = T(x \lor y) = (Tx) \lor (Ty) = Tx + Ty \). Furthermore, \( T(x-y) = (T(x) - (T(y))) = T(x) + (T(y)) = T(x-y)^+ + T(-(x-y)) = Tx + T(-y) \).
(3) Let \( x,y \in E \), \( x \perp y \). Then \( (T(x+y))^+ = T((x+y)^+) = T(x^+ + y^+) = T(x^+) + T(y^+) \) and \( (T(x+y))^+ = T((x+y)^+) = T(x^+ + y^+) = T(x^+) + T(y^+) \). And \( (T(x+y))^+ = T((x+y)^+) = T(x^+ + y^+) = T(x^+) + T(y^+) \). Thus, the elements \( T(x^+), T(y^+) \), \( T(-(x^-)), T(y^-) \) of \( F^+ \) are pairwise disjoint. In particular, \( T(x^+ + T(-(x^-))) \perp T(y^+) + T(-(y^-)) \), i.e., \( (Tx)^+ - (Tx^- \perp (Ty)^+ - (Ty^-), \) i.e, \( Tx \perp Ty \). Furthermore, \( T(x+y) = (T(x+y)^+) + T(-(x^-)) = (Tx^+ + Ty^+)(T(x^-) + (Ty^-)) = T(x^+ + T(y^+) + T(y^-) - (Ty^-)) = Tx + Ty \).
(4) By applying (3) to \( T \) and to \( T^{-1} \) we see that, for \( x,y \in E \), \( x \perp y \) if and only if \( Tx \perp Ty \). (4) follows readily. □

Our patience is rewarded, with a nice theorem!

**Theorem 2.34.** Let \( E,F \) be Archimedean Riesz spaces. Let \( T : E \rightarrow F \) be a lattice isomorphism with \( T(0) = 0 \). Then \( T \) preserves bands.

Proof. Since \( E \) and \( F \) are Archimedean, we know from theorem 2.32 that the bands are precisely the sets \( B \) s.t. \( B = B_{\perp\perp} \). Furthermore we know from lemma 2.33(4) that for all \( X \subseteq E \), \( T \) maps \( X^+ \) onto \( (T(X))^+ \). Hence, \( T \) maps \( B_{\perp\perp} \) onto \( (T(B))_{\perp\perp} \). So \( T \) preserves bands. □
When we study the structure of bands on an Archimedean space $E$, we can construct a boolean algebra.

**Theorem 2.35.** Let $E$ be an Archimedean Riesz space. Let $\mathcal{B}(E)$ be the collection of bands of $E$. Then $\mathcal{B}(E)$ can be seen as a boolean algebra under inclusion.

**Proof.** For $B_1, B_2 \in \mathcal{B}(E)$ we define $B_1 < B_2$ iff $B_1 \subsetneq B_2$. And $B_1 \wedge B_2 = B_1 \cap B_2$ and $B_1 \vee B_2 = (B_1^\perp \cap B_2^\perp)^\perp$. For the proof that this is actually a Boolean Algebra we refer to theorem 4.6 of [DJR77]. □

Note that lattice isomorphisms actually preserve the boolean structure of the bands, and thus supplies us with a boolean isomorphism.

**Theorem 2.36.** Let $E, F$ be Archimedean Riesz spaces. Let $T : E \to F$ be a lattice isomorphism with $T(0) = 0$. Then $T$ implies a boolean isomorphism $T^* : \mathcal{B}(E) \to \mathcal{B}(F)$.

**Proof.** Define $T^* : \mathcal{B}(E) \to \mathcal{B}(F)$ by:

$$ T^*(B) = \{ T(b) : b \in B \} \quad (B \in \mathcal{B}(E)). $$

We know from theorem 2.34 that $T^*$ indeed sends bands to bands. Furthermore, it is obvious that a lattice isomorphism preserves the (distributive) lattice structure. Note that $T^*(\{0\}) = \{0\}$ and $T^*(E) = F$. Furthermore lemma 2.33(4) gives us that the orthogonal complements are preserved. Hence $T^*$ is a boolean isomorphism from $\mathcal{B}(E)$ to $\mathcal{B}(F)$.
CHAPTER 3

Kaplansky

In this chapter, we research the theory of Irving Kaplansky, from his article [Kap47]. In this article Kaplansky proves that, under some conditions on $X$, the space $C(X)$ as a lattice characterises $X$. One might already know that as a Riesz space $C(X)$ characterises $X$, but since Riesz spaces have a lot more structure than lattices this is a much stronger claim. To study this, we assume all spaces to be compact Hausdorff. We firstly prove Kaplansky for compact Hausdorff spaces. Secondly, we will show that instead of $X$ we can also take $C(X \to [0,1])$ or $C(X \to [-\infty,\infty])$ with some adjustments to the proofs. As last we will show what this means for lattice homomorphism $\phi : C(X) \to \mathbb{R}$, and remark that these isomorphisms could be associated with the points of $X$ in a similar way too.

**Kaplansky for $C(X)$**

As we announced, we are going to prove that as a lattice $C(X)$ characterizes $X$. In order to do this we start by giving some definitions and then prove a lot of lemmas.

**Definition 3.1.** A (lattice) prime ideal $P$ in $C(X)$ is a proper sub-lattice containing with any element all smaller ones, and whose complement $C(X) - P$ has the dual property.

An easy example will make the idea of a prime ideal more clear.

**Example 3.2.** For all $x \in X$ and all $t \in \mathbb{R}$ we can construct prime ideals $P_1, P_2$ in $C(X)$ by defining: $P_1 := \{ f \in C(X) : f(x) \leq t \}$ and $P_2 := \{ f \in C(X) : f(x) < t \}$. These prime ideals are related to the point $x$ as follows: $g \in C(X), f \in P$ with $g(x) < f(x)$ implies $g \in P$.

Based on the above introduced relation we are going to define associatedness of prime ideals of $C(X)$ with points of $X$.

**Definition 3.3.** A prime ideal $P$ in $C(X)$ is associated with a point $x \in X$ iff $g \in C(X), f \in P$ with $g(x) < f(x)$ implies $g \in P$.

This definition wouldn’t make a lot of sense if prime ideals could be associated with more than one point of $X$. And it would be very nice if every prime ideal can be associated with a point of $X$. This will immediately show us why we are interested in compact Hausdorff spaces.

**Lemma 3.4.** Any prime ideal $P$ in $C(X)$ is associated with a unique point $x \in X$.

**Proof.** Assume there is no point associated with $P$, then for all $x \in X$ there are $f_x, g_x \in C(X)$ s.t. $g_x(x) < f_x(x)$ while $f_x \in P$ and $g_x \notin P$. Thus for every $x \in X$ there is an open neighbourhood $U_x$ with $x \in U_x$ and $g_x < f_x$ on $U_x$. This gives us an open covering $\{U_x : x \in X\}$ of $X$. Since $X$ is compact, there must be a finite sub-cover. Let $U_{x_1}, ..., U_{x_n}$ be such a finite sub-cover. Then construct $h, k \in C(X)$ by $h := f_{x_1} \lor ... \lor f_{x_n}$ and $k := g_{x_1} \land ... \land g_{x_n}$. Note $h > k$, while $h \in P$ and $k \notin P$ which contradicts that $P$ is a prime ideal. Hence, there must be at least one point $x \in X$ associated with $P$.

Assume there is a prime ideal $P$ which is associated with two different points $x_1, x_2 \in X$. Since $X$ is compact Hausdorff, we know from our topology course that it must be Normal. Thus for $f, g \in C(X)$ with $f \in P$ and $g \notin P$ we can use Urysohn to construct a $h \in C(X)$ s.t. $h(x_1) < f(x_1)$ and $h(x_2) > g(x_2)$, hence $h \in P$ while also $h \notin P$. This is the desired contradiction, which shows that a point $x \in X$ associated with a prime ideal $P$ is unique. Combining the first and the second part proves that any prime ideal $P$ is associated with a unique $x \in X$. □

This lemma allows us to associate every prime ideal with a point of $X$. In order to let $C(X)$ characterise $X$ we want to make an equivalence relation between prime ideals, with the property that two prime ideals
are equivalent iff they are associated with the same point. From this we hope to be able to deduce properties of the points of $X$.

**Lemma 3.5.** Two prime ideals are associated with the same $x \in X$ iff their intersection contains a prime ideal.

**Proof.** \((\Rightarrow)\) Let $P_1, P_2$ be different prime ideals associated with the same $x \in X$. Take $f_1, f_2 \in C(X)$ s.t. $f_1 \in P_1, f_2 \in P_2$. Let $\lambda \in \mathbb{R}$ be s.t. $\lambda < f_1(x), \lambda < f_2(x)$. Then $P := \{ f \in C(X) : f(x) \leq \lambda \}$ is a prime ideal in the intersection of $P_1, P_2$.

\((\Leftarrow)\) For the sake of contractions let us assume that $P_1, P_2$ are two prime ideals associated with different points of $X$, while $P$ is a prime ideal in their intersection. Then at least one of the two prime ideals is not associated with the same point as $P$. So we may assume that $P_1$ is associated with $x_1$, $P$ is associated with $x_2$ and $x_1 \neq x_2$. Let $f, g \in C(X)$ with $f \in P, g \notin P_1$. Then again with the help of Urysohn, we can construct $h \in C(X)$ s.t. $h(x_2) < f(x_2)$ and $g(x_1) < h(x_1)$. Thus $h \in P$ while $h \notin P_1$ which contradicts the assumption that $P$ is in the intersection of $P_1, P_2$. \(\square\)

This last lemma makes sure that we have constructed an equivalence relation. We will later use this relation to make a homeomorphism from $X$ to $Y$ for a given lattice isomorphism $T : C(X) \to C(Y)$. Before we can do this, we first need to make a bijection. Of course we will use our above defined association of the points of $X$ with prime ideals to construct a bijection between $X$ and $Y$. But, to do this we first need that lattice isomorphism preserve prime ideals.

**Lemma 3.6.** Let $T : C(X) \to C(Y)$ be a lattice isomorphism, let $P$ be a prime ideal in $C(X)$. Then $T(P)$ is a prime ideal in $C(Y)$.

**Proof.** It is trivial that a lattice isomorphism preserves sub-lattices. Furthermore, let $f, g \in C(X)$ be s.t. $T(f) \in T(P), T(g) \leq T(f)$. Then $T(f) = T(g) \lor T(f) = T(g \lor f)$, so $g \leq f$, hence $g \in P$ and thus $T(g) \in T(P)$. Similarly, let $f, g \in C(X)$ be s.t. $T(f) \notin T(P)$ and $T(g) \geq T(f)$. Then $T(g) = T(g) \lor T(f) = T(g \lor f)$, so $g \geq f$, hence $g \notin P$ and thus $T(g) \notin T(P)$. \(\square\)

Next, we need that the property of being associated with the same point is preserved too. Then equivalence classes of prime ideals, associated with points in $X$ can be mapped to equivalence classes of prime ideals, associated with points in $Y$. That would allow us to construct a bijection from $X$ to $Y$.

**Lemma 3.7.** Let $T : C(X) \to C(Y)$ be a lattice isomorphism, let $P_1, P_2$ be prime ideals in $C(X)$ both associated with $x \in X$. Then $T(P_1), T(P_2)$ are prime ideals in $C(Y)$ associated with the same $y \in Y$.

**Proof.** From lemma 3.6 we know that $T(P_1), T(P_2)$ are prime ideals, so it remains to prove that they are associated with the same $y \in Y$. Lemma 3.5 says there is a prime ideal $P$ in the intersection of $P_1, P_2$. According to lemma 3.6 $T(P)$ is a prime ideal in $C(Y)$, obviously in the intersection of $T(P_1), T(P_2)$, hence lemma 3.5 proves that $T(P_1), T(P_2)$ are associated with the same point $y \in Y$. \(\square\)

Now we can actually construct our bijection from $X$ to $Y$. However, we shall first prove one more lemma which will guide us later in the prove that our bijection is continuous, hence a homeomorphism.

**Lemma 3.8.** For any subset $D$ of $X$ define $I(D)$ to be the intersection of all prime ideals associated with a point of $D$ which contain $\mathbb{1}_X$. Then a point $x \in X$ is in the closure of $D$ iff $I(D)$ is contained in a prime ideal associated with $x$.

**Proof.** \((\Rightarrow)\) For $x \in \overline{D}$ define $P_x := \{ f \in C(X) : f(x) \leq 1 \}$. Note that $I(D) \subseteq P_y$ for all $y \in D$. For $g \in I(D)$ we see $g \in P_y$ and thus $g(y) \leq 1$ for all $y \in D$. So $g \leq 1$ on $D$ and $g \in C(X)$. Thus $g \leq 1$ on $\overline{D}$ and hence $g(x) \leq 1$. Hence, $I(D) \subseteq P_x$.

\((\Leftarrow)\) Suppose $P$ is any prime ideal associated with $x \in X$ for some $x \notin \overline{D}$. For $f \in C(X), f \notin P$ we can use Urysohn to construct a $g \in C(X)$ s.t. $g < 1$ on $D$ but $g(x) > f(x)$. Hence $g \in I(D)$ while $g \notin P$, hence $I(D) \notin P_x$. \(\square\)

This lemma gives us a characterisation of closed sets, which only depends on prime ideals. Namely we have, $D$ is closed if and only if the points $x \in D$ are precisely the points $x \in X$ with an associated prime ideal $P_x$ which contains $I(D)$. This gives us enough to prove that as a lattice $C(X)$ characterises $X$.

**Theorem 3.9.** If $C(X), C(Y)$ are lattice isomorphic, then $X$ and $Y$ are homeomorphic.
Proof. Let $T : C(X) \to C(Y)$ be a lattice isomorphism. Construct a function $\tau : X \to Y$ by defining $\tau(x) = y$ where $y$ is the point associated with the image, which is a prime ideal, of any prime ideal associated with $x$. From lemma 3.7 it is clear that $\tau$ is a bijection. It remains to prove that $\tau$ is continuous, hence a homeomorphism.

Note that since $\tau$ is a bijection it is enough to prove that $\tau$ is closed. So, let $D$ be a closed set in $X$. Then lemma 3.8 gives us that for every $x \in D$, $I(D)$ is contained in a prime ideal $P_x$ associated with $x$. Hence $T(I(D))$ is contained in the prime ideal $T(P_x)$ which is associated with $\tau(x)$. So for every $y \in \tau(X)$ there is a prime ideal $P_y = T(P_x)$ s.t. $T(I(D)) \subseteq P_y$. On the other hand, let $P_y$ be a prime ideal associated with $y \notin \tau(D)$. Then there is no prime ideal $P_y$ associated with $y$ which contains $T(I(D))$, because then $T^{-1}(P_y)$ associated with $\tau^{-1}(y) \notin D$ would contain $I(D)$ which according to lemma 3.8 can’t happen for closed sets $D$. So we see that $\tau(D)$ must be closed too.

We now have proved that we have a bijection $\tau$ is closed, hence continuous. Thus $\tau$ is a homeomorphism. \[\square\]

Kaplansky for $[-\infty, \infty]$ or $[0, 1]$

We have seen that as a lattice $C(X)$ characterises $X$. Now we will show the same for the lattice $C(X \to [-\infty, \infty])$ or similarly $C(X \to [0, 1])$. The proofs will mostly be very similar, but since the problems which arise are very subtle it will be good to really go through most proofs again and focus on the nasty complications.

Lemma 3.10. Any prime ideal $P$ in $C(X \to [-\infty, \infty])$ is associated with a unique point.

Proof. The first part of this proof is the same as for $C(X)$ in lemma 3.4, hence it will not be repeated here.

Assume there is a prime ideal $P$ associated with two different points $x_1, x_2 \in X$. Since it is impossible for a function with value $-\infty$ to construct a function which is strict smaller, the first complication arises here. We can fix it, but it will require a lot more work. Note that, just like for $C(X)$ we can use Urysohn for our proof.

Assume there are $f, g \in C(X \to [-\infty, \infty])$ s.t. $f \in P$ and $g \notin P$ and $f(x_1) > -\infty$ while $g(x_2) < \infty$. Make $h \in C(X \to [-\infty, \infty])$ s.t. $h(x_1) < f(x_1)$ while $h(x_2) > g(x_2)$. Then $h \in P$ and also $h \notin P$, which leads to the desired contradiction. Hence there are no such functions $f, g \in C(X \to [-\infty, \infty])$.

So for $f, g \in C(X \to [-\infty, \infty])$ with $f \in P$ and $g \notin P$, we may assume that either $\forall f \in P : f(x_1) = -\infty = f(x_2)$ or $\exists g \in P : g(x_1) = \infty = g(x_2)$. In the first case, we can take open sets $U, V$ in $X$ s.t. $U \cap V = \emptyset$, $x_1 \in U$, $x_2 \in V$. With the help of Urysohn, make functions $h_1, h_2 \in C(X \to [-\infty, \infty])$ s.t. $h_1(x_1) = 0, h_2(x_2) = 0$ and $h_1(X \setminus U) = -\infty, h_2(X \setminus V) = -\infty$. We denote $\tilde{0}, \tilde{1}$ for the functions which are constantly 0 respectively 1. Then $h_1, h_2 \notin P$, but $\tilde{0} = h_1 \land h_2 \in P$. Hence $C(X \to [-\infty, \infty]) \land P$ is not a sub-lattice, which contradicts the assumption. For the second case we can do something similar with functions now being $\infty$ instead of $-\infty$ and the maximum of both functions. So there are no functions $f, g \in C(X \to [-\infty, \infty])$ s.t. $f \in P$ and $g \notin P$. \[\square\]

In all most lemmas problems will arise when the proof for $C(X)$ constructs a function which is smaller in a specific point. Just like we saw in the proof above we get complications when our function might be $-\infty$ in this point. Of course this is very problematic and usually a lot of work to fix. In order to make it more easy to handle in the coming lemmas we will prove that it is enough to make a functions which is less than or equal to the other function on an open subset around our given point.

Lemma 3.11. Let $P$ be a prime ideal associated with $x$, let $f, g \in C(X \to [-\infty, \infty])$ with $f \in P, g \leq f$ on an open set $U$ containing $x$. Then $g \in P$.

Proof. Suppose that $g \notin P$. Note that lemma 3.10 says that $P$ cannot be associated with a point $y \in X \setminus U = V$. Hence for all $y \in V$ there are $h_y, k_y \in C(X \to [-\infty, \infty])$ s.t. $h_y(y) > k_y(y)$ while $h_y \in P$ and $k_y \notin P$. Thus for any $y \in V$ there is an open subset $V_y$ s.t. $h_y > k_y$ on $V_y$. This gives rise to an open covering $(V_y)_{y \in Y}$ of $V$. Furthermore, $V$ is a closed subspace of a compact Hausdorff space, hence compact Hausdorff. So there is a finite sub-cover $V_{y_1}, \ldots, V_{y_n}$ for some $n \in \mathbb{N}$. Hence $h := f \lor h_{y_1} \lor \ldots \lor h_{y_n} \in P$ and $k := g \land k_{y_1} \land \ldots \land k_{y_n} \notin P$, while $h > k$. Which is a contradiction, thus $g \in P$. \[\square\]
Note that for prime ideals the dual assumptions hold on their duals. Hence lemma 3.11 also gives us that for any prime ideal $P$ associated with $x \in X$ and $g \notin P$ if $f \geq g$ on an open set $U$ containing $x$ then $f \notin P$. This will make sure most proofs can stay similar to the old proofs.

When we try to copy lemma 3.5 some more problems arise. This time, we have difficulties taking a $\lambda \in \mathbb{R}$ s.t. $\lambda < f_1(x)$ and $\lambda < f_2(x)$, because $f_1(x)$ nor $f_2(x)$ is assumed to be bigger than $-\infty$. In order to fix this, we adjust the lemma slightly.

**Lemma 3.12.** Two prime ideals are associated with the same $x \in X$ iff their intersection contains a prime ideal, or the intersection of their complements contains the dual of a prime ideal.

**Proof.** ($\Rightarrow$) Let $P_1$ and $P_2$ be prime ideals associated with $x \in X$. Let $f_1, f_2 \in C(X \rightarrow [-\infty, \infty])$ be s.t. $f_1 \in P_1, f_2 \in P_2$. If we can take $\lambda \in \mathbb{R}$ s.t. $\lambda < f_1(x), \lambda < f_2(x)$, then $P := \{ f \in C(x) : f(x) \leq \lambda \}$ is a prime ideal in the intersection of $P_1, P_2$.

If we can't take such a $\lambda \in \mathbb{R}$ then we may assume $f_1(x) = -\infty$ for all $f_1 \in P_1$. If there is a $f_2 \in P_2$ s.t. $f_2(x) > -\infty$ then $P_1 \subset P_2$, which means that $P_1$ is a prime ideal contained in the intersection of $P_1, P_2$.

It remains to prove for the case that $f_1(x) = -\infty$ for all $f_1 \in P_1$ and also $f_2(x) = -\infty$ for all $f_2 \in P_2$. Take any $\lambda \in \mathbb{R}$. Then $P := \{ f \in C(X) : f(x) \leq \lambda \}$ is a prime ideal, with dual $Q := \{ f \in C(X) : f(x) > \lambda \}$ which is contained in the intersection of the duals of $P_1, P_2$ since their duals contain any function $f \in C(X \rightarrow [-\infty, \infty])$ with $f(x) > -\infty$.

($\Leftarrow$) We again prove this side by contradiction. Assume $P_1, P_2$ are prime ideals associated with different points of $X$ while $P$ is a prime ideal in the intersection of $P_1, P_2$. Then at least one of the two prime ideals is not associated with the same point as $P$. So we may assume that $P_1$ is associated with $x_1$, $P$ is associated with $x_2$ and $x_1 \neq x_2$. Let $f, g \in C(X \rightarrow [-\infty, \infty])$ with $f \in P, g \notin P_1$. Let $U_1, U_2$ be disjoint open sets in $X$ with $x_1 \in U_1, x_2 \in U_2$. Then with the help of Urysohn, we construct $h \in C(X \rightarrow [-\infty, \infty])$ s.t. $h \leq f$ on $U_2$ and $g \leq h$ on $U_1$. Thus, according to lemma 3.11, $h \in P$ while $h \notin P_1$ which contradicts the assumption that $P$ is in the intersection of $P_1, P_2$. Hence the intersection of two prime ideals can only contain a prime ideal when the two prime ideals are associated with the same $x \in X$. Which completes the proof.

The equivalent of lemma 3.6 has the same proof as lemma 3.6 itself, hence the proof will be omitted. We will still write down the lemma.

**Lemma 3.13.** Let $T : C(X \rightarrow [-\infty, \infty]) \rightarrow C(Y \rightarrow [-\infty, \infty])$ be a lattice isomorphism, let $P$ be a prime ideal in $C(X \rightarrow [-\infty, \infty])$. Then $T(P)$ is a prime ideal in $C(Y \rightarrow [-\infty, \infty])$.

The next lemma is the equivalent of lemma 3.7. We have to change the proof a bit, but the idea stays the same.

**Lemma 3.14.** Let $T : C(X \rightarrow [-\infty, \infty]) \rightarrow C(Y \rightarrow [-\infty, \infty])$ be a lattice isomorphism, let $P_1, P_2$ be prime ideals in $C(X \rightarrow [-\infty, \infty])$ associated with the same $x \in X$. Then $T(P_1), T(P_2)$ are prime ideals in $C(Y \rightarrow [-\infty, \infty])$ associated with the same $y \in Y$.

**Proof.** From lemma 3.13 we know that $T(P_1), T(P_2)$ are prime ideals. So it remains to prove that they are associated with the same $y \in Y$. Lemma 3.12 says there is a prime ideal $P$ in the intersection of $P_1, P_2$, or the intersection of their duals contains the dual of a prime ideal $P$. According to lemma 3.6 $T(P)$ is a prime ideal in $C(Y \rightarrow [-\infty, \infty])$, obviously in the intersection of $T(P_1), T(P_2)$ if $P$ was in the intersection of $P_1, P_2$ or the dual is in the intersection of their duals if the dual of $P$ was in the intersection of the duals of $P_1, P_2$. Hence lemma 3.12 proves that $T(P_1), T(P_2)$ are associated with the same point $y \in Y$.

Also the last lemma needs a little adjustment to enable us to prove the theorem.

**Lemma 3.15.** For any subset $D$ of $X$ define $I(D)$ to be the intersection of all prime ideals associated with a point in $D$ which contain $1_X$. Then a point $x \in X$ is in the closure of $D$ iff $I(D)$ is contained in a prime ideal associated with $x$ or the complement of $I(D)$ is contained in the complement of a prime ideal associated with $x$.

**Proof.** ($\Rightarrow$) For $x \in \overline{D}$ define $P_x := \{ f \in C(X \rightarrow [-\infty, \infty]) : f(x) \leq 1 \}$. Note that $I(D) \subseteq P_y$ for all $y \in D$. For $g \in I(D)$ we see $g \in P_y$ and thus $g(y) \leq 1$ for all $y \in D$. So $g \leq 1$ on $D$ and $g \in C(X)$. Thus $g \leq 1$ on $\overline{D}$ and hence $g(x) \leq 1$. Hence, $I(D) \subseteq P_x$.

($\Leftarrow$) Suppose $P$ is any prime ideal associated with an $x \in X$ for some $x \notin \overline{D}$. For $f \in C(X \rightarrow [-\infty, \infty]), f \notin P$ we take open sets $U, V$ with $U \cap V = \emptyset$ and $x \in U, \overline{D} \in V$. Then Urysohn enables us to
construct a \( g \in C(X \to [-\infty, \infty]) \) s.t. \( g < 1 \) on \( V \) but \( g \geq f \) on \( U \). Hence \( g \in I(D) \) while \( g \notin P \), hence \( I(D) \notin P_x \).

Urysohn also enables us to construct a \( g \in C(X \to [-\infty, \infty]) \) s.t. \( g > 1 \) on \( V \) but \( g \leq f \) on \( U \). Hence \( g \notin I(D) \) while \( g \in P \), hence the complement of \( I(D) \) is not contained in the complement of \( P_x \). \( \square \)

This lemma gives us a characterisation of closed sets which only depends on prime ideals. Moreover, it enables us to prove that as a lattice \( C(X \to [-\infty, \infty]) \) characterises \( X \).

**Theorem 3.16.** Let \( C(X \to [-\infty, \infty]), C(Y \to [-\infty, \infty]) \) be a lattice isomorphic. Then \( X \) and \( Y \) are homeomorphic.

**Proof.** Let \( T : C(X \to [-\infty, \infty]) \to C(Y \to [-\infty, \infty]) \) be a lattice isomorphism. Construct a function \( \tau : X \to Y \) by defining \( \tau(x) = y \) where \( y \) is the point associated with any image, which is a prime ideal, of a prime ideal associated with \( x \). From lemma 3.14 it is clear that \( \tau \) is a bijection. It remains to prove that \( \tau \) is continuous, hence a homeomorphism.

Note that since \( \tau \) is a bijection it is enough to prove that \( \tau \) is closed. So, let \( D \) be a closed set in \( X \). Then lemma 3.15 gave us that for every \( x \in D \), \( I(D) \) is contained in a prime ideal \( P_x \) associated with \( x \), or the complement of \( I(D) \) is contained in the complement of a prime ideal associated with \( x \). Hence \( T(I(D)) \) is contained in the prime ideal \( T(P_x) \) which is associated with \( \tau(x) \) or the complement of \( I(D) \) is contained in the complement of \( T(P_x) \). So for every \( y \in \tau(X) \) there is a prime ideal \( P_y = T(P_x) \) s.t. \( T(I(D)) \subseteq P_y \) or the opposite for the complements. On the other hand, let \( P_y \) be a prime ideal associated with \( y \) which contains \( T(I(D)) \), or whose complement contains the complement of \( T(I(D)) \). Because then \( T^{-1}(P_y) \) associated with \( \tau^{-1}(y) \notin D \) would contain \( I(D) \) or the complement would contain the complement of \( I(D) \) which according to lemma 3.15 can’t happen for closed sets \( D \). So we see that \( \tau(D) \) must be closed too.

We now have proved that we have a bijection \( \tau \) is closed, hence continuous. Thus \( \tau \) is a homeomorphism. \( \square \)

**Implications of Kaplansky**

It is now time to study the implications of the theory of Kaplansky for lattice homomorphisms \( \phi \) from \( C(X) \) to either \( \mathbb{R}, [0, 1] \) or \([-\infty, \infty]\). In an example we show how it is related. It will look a lot like what we did in example 3.2.

**Example 3.17.** For any lattice isomorphism \( \phi : C(X) \to \mathbb{R} \) and \( t \in \mathbb{R} \), we can construct prime ideals \( P_t, P'_t \) in \( C(X) \) by defining: \( P_t := \{ f \in C(X) : \phi(f) \leq t \} \) and \( P'_t := \{ f \in C(X) : \phi(f) < t \} \).

That \( P_t, P'_t \) are prime ideals can easily be checked. First of all we know that lattice homomorphisms preserve sub-lattices and order. So it contains with any element all smaller ones, its complement has the dual property.

So lattice isomorphisms are related to prime ideals. This might not look very special, but we proved at lemma 3.4 that every prime ideal is associated to a unique point \( x \in X \). Thus for all \( f, g \in C(X), f \in P_t \) and \( g(x) < f(x) \) implies \( g \in P_t \). If we add one and one together, this leads to the next important theorem.

**Theorem 3.18.** Let \( \phi : C(X) \to \mathbb{R} \) be a non-zero lattice homomorphism. Then there is an \( x \in X \) s.t. for all \( f, g \in C(X), f(x) > g(x) \) implies \( \phi(f) \geq \phi(g) \).

**Proof.** Let \( P_t \) be defined as in example 3.17. Then for any \( t, s \in \mathbb{R} \) with \( t < s \) we have \( P_t \subset P_s \). Hence for any \( t, s \in \mathbb{R} \) the prime ideals \( P_t, P_s \) contain a prime ideal, namely \( P_{s\wedge t} \). Thus \( P_t, P_s \) are associated with the same point \( x \in X \). This allows us to associate a lattice homomorphism with an \( x \in X \). Now it remains to prove that for all \( f, g \in C(X), f(x) > g(x) \) implies \( \phi(f) \geq \phi(g) \).

Let \( f, g \in C(X), f(x) > g(x) \) be given. Then \( f(x) > g(x) \) and \( f \in P_{\phi(f)} \). Thus \( g \in P_{\phi(f)} \). Hence \( \phi(g) \leq \phi(f) \). Which proves the theorem. \( \square \)

We proved it for \( C(X) \) but we never use that it is \( C(X) \), so the theory also holds for \( C(X \to [-\infty, \infty]) \) and \( C(X \to [0, 1]) \). As we will see in chapter 5, it also doesn’t matter what the image space of the lattice homomorphisms is. Since in chapter 5 we will look at \( \phi : C(X) \to \{0, 1\} \).
CHAPTER 4

Lattice Homomorphisms being Riesz Homomorphisms

In this chapter we will investigate which extra assumptions a lattice homomorphism needs to be a Riesz homomorphism. We assume spaces $X,Y$ to be compact Hausdorff. We will use the generally well known Yoshida Representation Theorem. We give the theorem before we start applying it.

**Theorem 4.1** (Yoshida). Let $E$ be an Archimedean Riesz space with a unit $e$. Let $\Phi$ be the set of all Riesz homomorphisms $\phi : E \to \mathbb{R}$ with $\phi(e) = 1$, topologized as a subset of $\mathbb{R}^E$ (product topology). For $x \in E$ define a function $\hat{x} : \Phi \to \mathbb{R}$ by:

$$\hat{x}(\phi) = \phi(x) \quad (\phi \in \Phi),$$

(so that $\hat{e} = 1$), and put $\hat{E} := \{ \hat{x} : x \in X \}$. Then:

$\hat{\Phi}$ is a compact Hausdorff space, $\hat{E}$ is a Riesz subspace of $C(\hat{\Phi})$, dense in the sense of $\| \cdot \|_\infty$. The map $x \mapsto \hat{x}$ is a Riesz isomorphism from $E$ onto $\hat{E}$. The space $\hat{\Phi}$ is called the spectrum of the Riesz space $E$.

**PROOF.** This is proved in theorem 3.16 of [vR11].

We immediately start with a promising theorem.

**Theorem 4.2.** Let $E$ be an Archimedean Riesz space with unit $e$, let $\phi : E \to \mathbb{R}$ be a lattice homomorphism with $\phi(\lambda a) = \lambda \phi(a)$ for all $\lambda \in \mathbb{R}$ and all $a \in E$. Then $\phi$ is a Riesz homomorphism.

**Proof.** Since our assumption is that the scalar multiplication is preserved, we need to prove that $\phi(a + b) = \phi(a) + \phi(b)$ for all $a,b \in E$. This is done with the help of the Yoshida Representation Theorem. Theorem 4.1 shows us that $E$ is Riesz isomorphic with $\hat{E}$ which is dense in $C(\hat{\Phi})$, under $\| \cdot \|_\infty$, for some compact Hausdorff space $\Phi$.

Define $\phi^* : C(\Phi) \to \mathbb{R}$ by:

$$\phi^*(f) = \sup_{\{g \in \hat{E} : g \leq f\}} \phi(g) \quad (f \in C(\Phi)).$$

First, note $\phi^*$ coincides with $\phi$ on $\hat{E}$, because $f \leq f$ implies $\sup_{\{g \in \hat{E} : g \leq f\}} \phi(g) \geq \phi(f)$ and $\phi(g) \leq \phi(f)$ for all $g \in \hat{E}, g \leq f$ implies $\sup_{\{g \in \hat{E} : g \leq f\}} \phi(g) \leq \phi(f)$. Thus $\sup_{\{g \in \hat{E} : g \leq f\}} \phi(g) = \phi(f)$. Secondly, $\phi^*$ is a lattice homomorphism on $C(\Phi)$, because for $f,g \in C(\Phi)$ we have:

$$\phi^*(f \vee g) = \sup_{\{h \in \hat{E} : h \leq f \vee g\}} \phi(h) = \sup_{\{h \in \hat{E} : h \leq f\}} \phi((h \wedge f) \vee (h \wedge g))$$

$$= \sup_{\{h,k \in \hat{E} : h \leq f, k \leq g\}} \phi(h \wedge k) = \sup_{\{h \in \hat{E} : h \leq f\}} \phi(h) \vee \sup_{\{k \in \hat{E} : k \leq g\}} \phi(k)$$

$$= \phi^*(h) \vee \phi^*(k).$$

And:

$$\phi^*(f \wedge g) = \sup_{\{h \in \hat{E} : h \leq f \wedge g\}} \phi(h) = \sup_{\{h \in \hat{E} : h \leq f\}} \phi((h \wedge f) \wedge (h \wedge g))$$

$$= \sup_{\{h,k \in \hat{E} : h \leq f, k \leq g\}} \phi(h \wedge k) = \sup_{\{h \in \hat{E} : h \leq f\}} \phi(h) \wedge \sup_{\{k \in \hat{E} : k \leq g\}} \phi(k)$$

$$= \phi^*(h) \wedge \phi^*(k).$$

So $\phi^*$ is a lattice homomorphism on $C(\Phi)$, for some compact Hausdorff space $\Phi$. Theorem 3.18 says there is a $x \in \Phi$ s.t. $f(x) > g(x) \Rightarrow \phi^*(f) > \phi^*(g)$.

This enables us to proof $\phi^*$ is linear: For $f \in C(\Phi)$, $t \in \mathbb{R}, t < f(x)$ holds:

$$f(x) > (t \mathbf{1})(x) \Rightarrow \phi^*(f) \geq \phi^*(t \mathbf{1}) = t \phi^*(\mathbf{1}),$$

where $\mathbf{1}$ is the constant function with value 1.

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while for \( f \in C(\Phi), t \in \mathbb{R}, t > f(x) \) holds:
\[
f(x) < (t \mathbb{1})(x) \Rightarrow \phi^*(f) \leq \phi^*(t \mathbb{1}) = t\phi^*(\mathbb{1}).
\]
Hence, \( \phi^*(f) = f(x)\phi^*(\mathbb{1}) \). Thus \( \phi^*(f) + \phi^*(g) = f(x)\phi^*(\mathbb{1}) + g(x)\phi^*(\mathbb{1}) = (f(x) + g(x))\phi^*(\mathbb{1}) \). So \( \phi^* \) is linear. Hence, \( \phi(a) + \phi(b) = \phi^*(a) + \phi^*(b) = \phi^*(a + b) = \phi(a + b) \). Which proves the theorem.

We began with a nice theorem, but we made a lot of assumptions about our Riesz space. Luckily we will soon find out, that those extra assumptions are unnecessary. We start by introducing the principal ideal spanned by a single element.

**Definition 4.3.** Let \( a \in E^+ \). The principal (Riesz) ideal generated by \( a \) is the intersection of all Riesz ideals that contain \( a \). It is the set \( E_a \) defined by:
\[
E_a := \bigcup_{n \in \mathbb{N}} [-na, na].
\]
We note, that \( a \) is a unit of \( E_a \). Hence \( E_a \) is unitary. Furthermore, if \( E \) is Archimedean we see that \( E_a \) must be Archimedean since it is a subspace.

This will immediately help us to prove that we can drop the assumption of a unit in the above theorem.

**Theorem 4.4.** Let \( E \) be an Archimedean Riesz space, let \( \phi : E \to \mathbb{R} \) be a lattice homomorphism with \( \phi(\lambda a) = \lambda \phi(a) \) for all \( \lambda \in \mathbb{R} \) and all \( a \in E \). Then \( \phi \) is a Riesz homomorphism.

**Proof.** Again we need \( \phi(a + b) = \phi(a) + \phi(b) \) for all \( a, b \in E \). Given \( a, b \in E \), consider \( E_{|a|+|b|} \), the principal ideal generated by \( |a| + |b| \). We know that \( E_{|a|+|b|} \) is unitary and Archimedean, since \( E \) is Archimedean. Note further that \( a, b \in E_{|a|+|b|} \). Since a restriction of a lattice homomorphism is a lattice homomorphism on the smaller space, theorem 4.2 proves that \( \phi \) is linear on \( E_{|a|+|b|} \). Hence \( \phi(a + b) = \phi(a) + \phi(b) \), which proves the theorem.

Theorem worked fine without the assumption of a unit. Can we drop the assumption that \( E \) is Archimedean instead of dropping the assumption that it has a unit?

**Theorem 4.5.** Let \( E \) be a Riesz space with a unit \( e \), let \( \phi : E \to \mathbb{R} \) be a lattice homomorphism with \( \phi(\lambda a) = \lambda \phi(a) \) for all \( \lambda \in \mathbb{R} \) and all \( a \in E \). Then \( \phi \) is a Riesz homomorphism.

**Proof.** Since our space has a unit \( e \), every infinitesimal element \( b \in E \) is infinitesimal with respect to the unit. Define \( B := \{ b \in E : b \) is infinitesimal \( \} \) and note that \( B \) is a Riesz subspace, hence a directed set. Define \( \phi^* : E/B \to \mathbb{R} \) by:
\[
\phi^*(\{a\}) = \lim_{b \to \infty} \phi(a + b) \quad (a \in E/B).
\]
Note that this is allowed, since \( \phi(a + b)_{b \in B} \) is increasing and \( \phi(a + b) < \phi(a + e) \) for all \( b \in B \). Then we have for \( \lambda > 0 \) and \( a, c \in E \):
\[
\lambda\phi^*(\{a\}) = \lambda \lim_{b \to \infty} \phi(a + b) = \lim_{b \to \infty} \lambda \phi(a + b) = \lim_{b \to \infty} \phi(\lambda a + \lambda b) = \lim_{b \to \infty} \phi(\lambda(a + b)) = \phi^*(\{\lambda a\}).
\]
And,
\[
\phi^*(\{a \lor c\}) = \lim_{b \to \infty} \phi((a \lor c)+b) = \lim_{b \to \infty} \phi((a+b) \lor (c+b)) = \lim_{b \to \infty} \phi(a+b) \lor \phi(c+b) = \lim_{b \to \infty} \phi(a+b) \lor \lim_{b \to \infty} \phi(c+b).
\]
And,
\[
\phi^*(\{a \land c\}) = \lim_{b \to \infty} \phi((a \land c)+b) = \lim_{b \to \infty} \phi((a+b) \land (c+b)) = \lim_{b \to \infty} \phi(a+b) \land \phi(c+b) = \lim_{b \to \infty} \phi(a+b) \land \lim_{b \to \infty} \phi(c+b).
\]
So \( \phi^* \) is a lattice homomorphism on \( E/B \) which is an Archimedean Riesz space with unit \( e \). Hence, theorem 4.2 proves that \( \phi^* \) is linear. Thus, \( \phi^*(\{a + c\}) = \phi^*(\{a\}) + \phi^*(\{c\}) \) for all \( a, c \in E \).

It remains to prove that for all \( a \in E \), holds \( \phi(a) = \phi^*(\{a\}) \). We know \( \phi(a) \leq \lim_{b \to \infty} \phi(a + b) = \phi^*(\{a\}) \), while on the other hand \( b \in 0 \) for \( e \in B \) and \( \epsilon > 0 \). Thus:
\[
\phi^*(\{a\}) - \epsilon \phi^*(\{b\}) \leq \phi^*(\{a - \epsilon e\}) = \lim_{b \to \infty} \phi(a - (\epsilon e + b)) \leq \phi(a).
\]
To conclude, we proved \( \phi \) to be equal to \( \phi^* \) on \( E \) while \( \phi^* \) is a Riesz homomorphism. Hence \( \phi \) is a Riesz homomorphism.
Now we have seen that we can drop either the assumption of being Archimedean or of having a unit. We will prove that we can also drop both assumptions at the same time.

**Theorem 4.6.** Let \( E \) be a Riesz space, let \( \phi : E \to \mathbb{R} \) be a lattice homomorphism with \( \phi(\lambda a) = \lambda \phi(a) \) for all \( \lambda \in \mathbb{R} \) and all \( a \in E \). Then \( \phi \) is a Riesz homomorphism.

**Proof.** Again we need \( \phi(a + b) = \phi(a) + \phi(b) \) for all \( a, b \in E \). Given \( a, b \in E \) we again consider \( E_\|a|+|b| \), the principal ideal generated by \( |a| + |b| \). \( E_\|a|+|b| \) has a unit. This time we do not need that \( E_\|a|+|b| \) is Archimedean, since theorem 4.5 proves that \( \phi \) is a Riesz homomorphism on \( E_\|a|+|b| \) anyway. Thus \( \phi \) is linear on \( E_\|a|+|b| \), hence for all \( a, b \in E \) we can prove \( \phi(a + b) = \phi(a) + \phi(b) \) which proves the theorem. \( \square \)

First, we show an application of what we already have, which shows how we got to researching this particular property. Then we will define a new related property and see what that gives us.

**Example 4.7.** Let \( E \) be an Archimedean Riesz space, \( D \) and ideal in \( E \). Let \( \phi : D \to \mathbb{R} \) be a Riesz homomorphism. If we want to extend \( \phi \) to a Riesz homomorphism \( \phi^* : E \to \mathbb{R} \), it would be convenient to be able to define \( \phi^*(a) = \sup\{b \in D : b \leq a\} \phi(b) \) for all \( a \in E \). Unfortunately this is not possible when \( \sup\{b \in D : b \leq a\} \phi(b) = \infty \).

In order to extend our homomorphism to a bigger subspace of \( E \), we can define the ideal \( E_0 = \{a \in E : \sup\{b \in D : b \leq a\} \phi(b) < \infty\} \). We know \( D \subseteq E_0 \subseteq E \). Define \( \overline{\phi} : E_0 \to \mathbb{R} \) by defining \( \overline{\phi}(a) = \sup\{b \in D : b \leq a\} \phi(b) \) for all \( a \in E_0 \). It is easy to see that \( \overline{\phi} \) is a lattice homomorphism, but it might be more difficult to see that it is actually a Riesz homomorphism again. Luckily it is easy to see that \( \overline{\phi}(\lambda a) = \lambda \overline{\phi}(a) \) for all \( \lambda \in \mathbb{R} \) and all \( a \in E \). Hence, theorem 4.6 tells us that \( \overline{\phi} \) is a Riesz homomorphism.

We now investigate a new, weaker, assumption on the homomorphisms. This is: There is a strictly increasing bijection \( \omega : \mathbb{R} \to \mathbb{R} \) s.t. \( \phi(\lambda a) = \omega(\lambda) \phi(a) \) for all \( \lambda \in \mathbb{R} \) and all \( a \in E \). It is clear that such a lattice homomorphism \( \phi \) isn’t always a Riesz homomorphism, since we do not assume that scalar multiplication is preserved. We may wonder if \( \phi \) is some kind of evaluation: Is there an \( a \in E \) s.t. \( \phi(f) = \omega(f(a)) \)? Similarly we can wonder if the existence of such a lattice homomorphism between two spaces implies the existence of a Riesz homomorphism between the spaces too. If so, does it also supply us with a certain related Riesz homomorphism? We can answer the latter questions by looking at the spaces \( L^1(\mathbb{N}) \) and \( L^2(\mathbb{N}) \).

**Example 4.8.** Define a lattice isomorphism \( T : l^2(\mathbb{N}) \to l^1(\mathbb{N}) \) by \( T(f) = f|f| \). For all \( \lambda \in \mathbb{R} \) and all \( f \in l^2(\mathbb{N}) \) we have \( T(\lambda f) = \lambda|\lambda|T(f) \). Hence we have a strictly increasing bijection \( \omega : \mathbb{R} \to \mathbb{R} \), namely \( \omega(\lambda) = \lambda|\lambda| \) s.t. \( \omega(\lambda)f = \omega(T(\lambda f)) \). As noticed before we can see that this lattice isomorphism is not a Riesz isomorphism, because it doesn’t preserve scalar multiplication.

But there is more, we can actually prove that there exists no Riesz isomorphism \( T : l^2 \to l^1 \) exists. This is done at theorem 7.4.

With this example, we showed that the existence of a lattice isomorphism with the new property doesn’t imply the existence of a Riesz isomorphism. Let alone that it would imply a specific one.

In order to keep our hopes up we will assume our lattice homomorphism to work on \( C(X) \) for some compact Hausdorff space \( X \). To get a head start in this subject, we first examine the properties of strictly increasing bijections \( \omega : \mathbb{R} \to \mathbb{R} \) with \( \phi(\lambda f) = \omega(\lambda)\phi(f) \) for all \( \lambda \in \mathbb{R} \) and all \( f \in C(X) \). We will see that there is only a specific group of functions which satisfy this condition.

**Lemma 4.9.** Let \( \phi : C(X) \to \mathbb{R} \) be a lattice homomorphism, let \( \omega : \mathbb{R} \to \mathbb{R} \) be a strictly increasing bijection with \( \phi(\lambda f) = \omega(\lambda)\phi(f) \) for all \( \lambda \in \mathbb{R} \) and all \( f \in C(X) \). Then \( \omega \) is multiplicative.

**Proof.** Given \( \lambda, \mu \in \mathbb{R} \) and \( f \in C(X) \), we have \( \omega(\mu\lambda)\phi(f) = \phi(\mu\lambda f) = \omega(\mu)\phi(\lambda f) = \omega(\mu)\omega(\lambda)\phi(f) \). Thus \( \omega(\mu\lambda) = \omega(\mu)\omega(\lambda) \). Hence \( \omega \) is multiplicative. \( \square \)

This characterisation is very powerful. To make it more easy to work with, we note there is a \( 0 < \alpha \in \mathbb{R} \) s.t. \( \omega(\lambda) = \text{sgn}(\lambda)|\lambda|^\alpha \). This is true, because this holds for all increasing multiplicative bijections from \( \mathbb{R} \) to \( \mathbb{R} \). We will use the notation \( \lambda^{(\alpha)} \) for \( \text{sgn}(\lambda)|\lambda|^\alpha \).

**Definition 4.10.** Let \( \phi : C(X) \to \mathbb{R} \) be a lattice homomorphism. Then we call \( \phi \) a lattice \( \alpha \)-homomorphism for some \( 0 < \alpha \in \mathbb{R} \) if \( \phi(\lambda f) = \lambda^{(\alpha)}\phi(\lambda f) \) for all \( \lambda \in \mathbb{R} \) and all \( f \in C(X) \).

With the help of this new definition we start exploring lattice \( \alpha \)-homomorphisms.
Theorem 4.11. Let $\phi : C(X) \to \mathbb{R}$ be a lattice $\alpha$-homomorphism. Then $\phi^* : C(X) \to \mathbb{R}$ defined by $\phi^*(f) = \phi(f)^{\frac{1}{\alpha}}$ for all $f \in C(X)$ is a Riesz homomorphism.

Proof. The above defined $\phi^*$ clearly is a lattice homomorphism from $C(X)$ to $\mathbb{R}$, since it is a homomorphism composed with a strictly increasing bijection. Furthermore, for all $\lambda \in \mathbb{R}$ and all $f \in C(X)$ we have:

$$\phi^*(\lambda f) = \phi(\lambda f)^{\frac{1}{\alpha}} = (\lambda^{\alpha} \phi(f))^{\frac{1}{\alpha}} = \lambda^{\alpha} \phi(f)^{\frac{1}{\alpha}} = \lambda \phi^*(f).$$

So according to theorem 4.6 $\phi^*$ is a Riesz homomorphism.

This theorem shows that lattice $\alpha$-homomorphisms are closely related to Riesz homomorphisms. Furthermore, theorem 4.1 and theorem 3.18 show us that there is a tight connection between lattice $\alpha$-homomorphisms and point evaluations. We wonder if our lattice $\alpha$-homomorphisms are a special kind of evaluation? And if so, how can we define such a special evaluation? We start with a definition and later proof that this is precisely what we want.

Definition 4.12. Only for this definition we can drop the assumption of $X$ being compact Hausdorff, instead it can be just any set. Every point $x \in X$ along with a $0 < \alpha \in \mathbb{R}$ determine a lattice homomorphism $\delta_{x,\alpha} : \mathbb{R}^X \to \mathbb{R}$, which we will call the special evaluation of $\alpha$ at $x$, by:

$$\delta_{x,\alpha}(f) = (f(x))^{\alpha} \quad (f \in \mathbb{R}^X).$$

We will use the same symbol $\delta_{x,\alpha}$ to denote the restriction of $\delta_{x,\alpha}$ to any Riesz subspace of $\mathbb{R}^X$.

Now we show that these special evaluations of $\alpha$ are precisely our lattice $\alpha$-homomorphisms $\phi$ with the extra property that $\phi(1) = 1$.

Theorem 4.13. Let $0 < \alpha \in \mathbb{R}$ be fixed. Then $x \mapsto \delta_{x,\alpha}$ is a one-to-one correspondence between the points of $X$ and the lattice $\alpha$-homomorphism $\phi : C(X) \to \mathbb{R}$ with $\phi(1) = 1$.

Proof. For every $x \in X$ we know that $\delta_{x,\alpha}$ is a lattice homomorphism. Furthermore, $\delta_{x,\alpha}(\lambda f) = (\lambda f(x))^{\alpha} = \lambda^{\alpha} \delta_{x,\alpha}(f)$ and $\delta_{x,\alpha}(1) = (1(x))^{\alpha} = 1$.

On the other hand, let $\phi : C(X) \to \mathbb{R}$ be a lattice $\alpha$-homomorphism with $\phi(1) = 1$. Construct $\phi^* : C(X) \to \mathbb{R}$ by $\phi^*(f) = \phi(f)^{\frac{1}{\alpha}}$, just like in theorem 4.11. Theorem 4.11 proves that $\phi^*$ is a Riesz homomorphism. Note, $\phi^*(1) = \phi(1)^{\frac{1}{\alpha}} = 1$. So, theorem 4.1 says there is a $x \in X$ s.t. $\phi^*(f) = f(x)$. So, $\delta_{x,\alpha}(f) = \phi^*(f)^{\alpha} = (f(x))^{\alpha}$ hence $\phi$ is the special evaluation of $\alpha$ at $x$. In other words, $\phi = \delta_{a,\alpha}$ for some $x \in X$.

Since, by multiplying our $\alpha$-homomorphism with the constant $\frac{1}{\phi(1)}$, we can always achieve that $\phi(1) = 1$, hence it can’t hurt to assume that $\phi(1) = 1$ in the following theory.

We now have a basic understanding of the lattice $\alpha$-homomorphism to $\mathbb{R}$, which allows us to investigate lattice $\alpha$-isomorphisms, bijective lattice $\alpha$-homomorphism, from one Riesz space to another. Working on $C(X)$ for some compact Hausdorff space $X$ gave us a lot of useful theorems. This inspires us to commence with lattice $\alpha$-isomorphisms $T : C(X) \to C(Y)$. Theorem 3.9 tells us there is a homeomorphism from $X$ to $Y$. Will a specific $\alpha$-isomorphism allow us to construct a specific homeomorphism between $X$ and $Y$?

Theorem 4.14. A lattice $\alpha$-isomorphism $T : C(X) \to C(Y)$ supplies us with a specific homeomorphism $\tau : Y \to X$.

Proof. Every point $y \in Y$ is associated one to one with a point evaluation $\delta_y$. When we compose a point evaluation $\delta_y$ with $T$, we get a lattice $\alpha$-homomorphism $S : C(X) \to \mathbb{R}$ for some $0 < \alpha \in \mathbb{R}$. Hence, theorem 4.13 enables us to construct a bijection $\tau$ from $Y$ to $X$ by sending a $y \in Y$ to the $x \in X$ which corresponds to $\delta_y \circ T$. Now it remains to prove that $\tau$ is continuous.

Since $\tau$ is a bijection it is sufficient to prove that $\tau$ is closed. We know that a set $D$ in $Y$ is closed iff for every $y \in D, f \in C(Y)$ with $f(D) = 1$ we get $\delta_y(f) = 1$ and for all $y \notin D$ this does not hold. Note that the same characterisation holds for $X$. Let $D$ be a closed subset of $Y$, then we have $y \in D$ iff $\delta_y(f) = 1$ for every $f \in C(Y)$ with $f(D) = 1$. Hence, $x \in \tau(D)$ iff $\delta_{x,\alpha}(f) = 1$ for every $f \in C(X)$ with $f(\tau(D)) = 1$. Thus $\tau(D)$ is closed. So $\tau$ is closed, hence continuous. Moreover, $\tau$ is a homeomorphism.
This attempt was very successful but we did make very harsh assumptions on our Riesz spaces. We will now loosen them up by studying lattice α-isomorphisms $T : \mathcal{C}(X) \to E$, where $E$ is just any Riesz space. What do we know about the correspondence of $\mathcal{C}(X)$ and $E$? Are they Riesz isomorphic? If yes, can we give a specific Riesz isomorphism with the help of our lattice α-isomorphism?

In order to answer all these new questions we first need some more basic knowledge about α-isomorphism from $\mathcal{C}(X)$ to just any Riesz space $E$.

**Lemma 4.15.** Let $T : \mathcal{C}(X) \to E$ be a lattice α-isomorphism. Then $E$ is an Archimedean Riesz space with $T(1)$ as a unit.

**Proof.** Let $a \in E$. Then $a = T(g)$ for some $g \in \mathcal{C}(X)$. Since $X$ is compact Hausdorff, $1$ is a unit in $\mathcal{C}(X)$. So there is a $\lambda \in \mathbb{R}$ s.t. $g \leq \lambda 1$. Hence $a = T(g) \leq \lambda \alpha(1)$. Thus $T(1)$ is a unit for $E$.

Assume there are non-zero $a, b \in E^+$ s.t. $\{na : n \in \mathbb{N}\} \leq b$ has no upper bound. Then there are $f, g \in \mathcal{C}(X)$ s.t. $a = T(f), b = T(g)$. Hence, $T^{-1}(\{na : n \in \mathbb{N}\}) = T^{-1}((na) : n \in \mathbb{N}) = \{n(\frac{a}{b})T(a) : n \in \mathbb{N}\} \leq T(g)$, since $\mathcal{C}(X)$ is Archimedean, $T^{-1}(a) = 0$. Hence $a = 0$ which contradicts the assumption. Thus $E$ is Archimedean.

The knowledge that $E$ is Archimedean with a unit, makes it irresistible to apply theorem 4.1. This leads to the following theorem.

**Theorem 4.16.** Let $T : \mathcal{C}(X) \to E$ be a lattice α-isomorphism. Then $E$ is Riesz isomorphic to $\mathcal{C}(X)$.

**Proof.** Lemma 4.15 showed us that $E$ is an Archimedean Riesz space with a unit, $T(1)$. Hence, theorem 4.1 gives us that there is a $\Phi$ s.t. $E$ is Riesz isomorphic with a $||\cdot||_\infty$ dense subspace $\tilde{E}$ of $\mathcal{C}(\Phi)$. Define a map $\tilde{T} : \mathcal{C}(X) \to \mathcal{C}(\Phi)$ by $\tilde{T}(f) = T(f)$. Then we see that $\tilde{T}$ is a injective lattice α-homomorphism since $\tilde{T}(\lambda f) = T(\lambda f) = \lambda \alpha(T(f)) = \lambda \alpha(T(f))$ for any $\lambda \in \mathbb{R}$ and any $f \in \mathcal{C}(X)$. Later we prove $\tilde{T}$ to be surjective too and thus a bijection. First, we will construct a homeomorphism $\tau : \Phi \to X$.

Every point $\omega \in \Phi$ is associated one-to-one with a point evaluation in $\omega$, $\delta_\omega$. When we compose this one-to-one association with $\tilde{T}$, we get a lattice homomorphism $S$ from $\mathcal{C}(X)$ to $\mathbb{R}$ with the extra property that $S(\lambda f) = \lambda \alpha S(f)$ for all $\lambda \in \mathbb{R}$ and all $f \in \mathcal{C}(X)$. Hence, we can construct a bijection $\tau : \Phi \to X$ by sending $\omega \in \Phi$ to $x \in X$ for which $\delta_\omega x = \delta_\omega \circ T$. It remains to prove that $\tau$ is continuous.

Now we can copy the proof in theorem 4.14 of $\tau$ being homeomorphic: Since $\tau$ is a bijection it is sufficient to prove that $\tau$ is closed. We know that a set $D$ in $\Phi$ is closed iff for every $\omega \in D, f \in C(\Phi)$ with $f(D) = 1$ we get $\delta_\omega(f) = 1$ and for all $\omega \not\in D$ this does not hold. Note that the same characterisation holds for $X$. Let $D$ be a closed subset of $\Phi$, then we have $\omega \in D$ iff $\delta_\omega(f) = 1$ for every $f \in C(\Phi)$ with $f(D) = 1$. Hence, $x \in \tau(D)$ iff $\delta_{\tau(x)}(f) = 1$ for every $f \in \mathcal{C}(X)$ with $f(\tau(D)) = 1$. Thus $\tau(D)$ is closed. Thus $\tau(D)$ is closed. So $\tau$ is closed, hence continuous. Moreover, $\tau$ is a homeomorphism.

So far we have proven that $\mathcal{C}(X)$ and $\mathcal{C}(\Phi)$ are Riesz isomorphic, since $X$ and $\Phi$ are homeomorphic, and thus $E$ is Riesz isomorphic to a dense subspace of $\mathcal{C}(X)$. We now prove that our map $\tilde{T}$ is surjective too, and thus prove that $E$ is actually Riesz isomorphic to $\mathcal{C}(X)$.

For every $g \in \mathcal{C}(\Phi)$ we define the function $g' : X \to \mathbb{R}$ by:

$$g'(x) = (g(\tau^{-1}(x)))(\frac{1}{\tau})$$

Since $g, \tau$ are continuous $g'$ is continuous too. For all $\omega \in \Phi$ we have:

$$\delta_\omega(\tilde{T}(g')) = \delta_{\tau(\omega)}(g'(\tau(\omega))) = \delta_{\tau(\omega)}(g(\tau^{-1}(\tau(\omega))))(\frac{1}{\tau}) = g(\omega).$$

Hence we must have $\tilde{T}(g') = g$. Thus $E$ is Riesz isomorphic to $\mathcal{C}(X)$.

As last we will note that in the previous prove we implicitly proved the following.

**Corollary 4.17.** Let $T : \mathcal{C}(X) \to E$ be a lattice α-isomorphism, let $E$ be a dense subset of $\mathcal{C}(X)$ for some $X$. Then $E = \mathcal{C}(Y)$. 


**CHAPTER 5**

**Stone-Čech isomorphisms**

This chapter focuses on the Stone-Čech compactification $\beta X$ of a Hausdorff space $X$. In this way we try to prove the theory of Kaplansky for non-compact spaces $X$. Since compactness was a keyword in the theory, it is not surprising that this comes at great expense. However, we get the same results for $\beta X$ as we had for our compact spaces, since $\beta X$ is compact. In one specific situation the lattice structure will completely characterise the space $X$ again. Our investigation will be in two different ways. First of all, we study what Kaplansky can already give us. Secondly, we build up the theory of Kaplansky again, this time with lattice homomorphisms instead of prime ideals. This might remind one of example 3.17.

**Kaplansky for Stone-Čech isomorphisms**

We start by showing that any lattice isomorphism $T : C(X) \to C(Y)$ induces a lattice isomorphism $S : \mathcal{R}_X, \mathcal{I}_X \to \mathcal{R}_Y, \mathcal{I}_Y$.

**Theorem 5.1.** Let $T : C(X) \to C(Y)$ be a lattice isomorphism. Then we can construct a lattice isomorphism $S : C(X) \to C(Y)$ s.t. $S(0) = 0$ and $S(1_X) = 1_Y$.

**Proof.** First of all, note that we may assume $T(0) = 0$. Otherwise, we would just look at $T - T(0)$ instead. So let $T(0) = 0$, let $u \in C(Y)$ be defined by $u = 1_Y + T(1_X)$. Then we see $u \geq 1_Y$ and $T^{-1}(u) \geq 1_X$. Define $S : C(X) \to C(Y)$ by:

$$S(f) = \frac{u}{T(f \cdot T^{-1}(u))} \quad (f \in C(X)).$$

Then $S(0) = 0$ and:

$$S(1_X) = \frac{T(1_X \cdot T^{-1}(u))}{u} = \frac{T(T^{-1}(u))}{u} = 1_Y.$$

Note, the above theorem implicitly proved that a lattice isomorphism $T : C(X) \to C(Y)$ induces a lattice isomorphism $S : \mathcal{R}_X, \mathcal{I}_X \to \mathcal{R}_Y, \mathcal{I}_Y$. Furthermore, we know that there is a one to one correspondence between $C(X \to [0, 1])$ and $C(\beta X \to [0, 1])$. If we add these ideas together with theorem 3.16, which we can use since $\beta X$ is compact, we get the next theorem.

**Theorem 5.2.** Let $T : C(X) \to C(Y)$ be a lattice isomorphism. Then $\beta X$ and $\beta Y$ are homeomorphic.

**Proof.** As announced, this rather deep theorem is just the result of combining what we have proven already. From theorem 5.1 we get a lattice isomorphism $S$ from $\mathcal{R}_X, \mathcal{I}_X$ to $\mathcal{R}_Y, \mathcal{I}_Y$. Which could also be seen as a lattice isomorphism from $C(X \to [0, 1])$ to $C(Y \to [0, 1])$. Since for any $X$, the space $C(X \to [0, 1])$ is lattice isomorphic to $C(\beta X \to [0, 1])$, $S$ can be seen as a lattice isomorphism from $C(\beta X \to [0, 1])$ to $C(\beta Y \to [0, 1])$. Because Stone-Čech compactifications are compact Hausdorff, theorem 3.16 that proves $\beta X$ and $\beta Y$ are homeomorphic.

If we make the extra assumption on $X, Y$ that they are metrizable we actually get what we want.

**Corollary 5.3.** Let $X, Y$ be metrizable spaces, $T : C(X) \to C(Y)$ a lattice isomorphism. Then $X$ and $Y$ are homeomorphic.

**Proof.** We have proven in theorem 5.2 that $\beta X$ and $\beta Y$ are isomorphic. In theorem 7.10 of [vR14] is proven that two metrizable spaces $X, Y$ with homeomorphic Stone-Čech compactifications are homeomorphic themselves, which proves this corollary.
This beautiful result is what we hoped to get, and it worked out. So we see that the Stone-Čech compactification has something to do with lattice isomorphisms. There actually is a much closer connection between $\beta X$ and the lattice homomorphisms which we will explore now.

**Lattice homomorphisms associated with the Stone-Čech compactification**

**We assume $X$ to be any Hausdorff space.** We study lattice homomorphisms $\phi : C(X \to [0,1]) \to \{0,1\}$ and the points of $\beta X$. First, we define an association between them. Then we prove this association can be used instead of the lattice prime ideals used by Kaplansky. We will write $f^\beta$ for instead of $\beta(f)$ for any $f \in C(X \to [0,1])$, where $\beta : C(X \to [0,1]) \to C(\beta X \to [0,1])$ is the lattice isomorphism. Now we can define the following.

**Definition 5.4.** A lattice homomorphism $\phi : C(X \to [0,1]) \to \{0,1\}$ is associated with a point $\alpha \in \beta X$ iff $f^\beta(\alpha) > g^\beta(\alpha)$ implies $\phi(g) \leq \phi(f)$ for all $f,g \in C(X \to [0,1])$.

Note for every $\alpha \in \beta X$ there is a lattice homomorphism $\phi : C(X \to [0,1]) \to \{0,1\}$, which is associated with $\alpha$.

**Example 5.5.** Given $\alpha \in \beta X$ and $t \in (0,1)$. We define a non-trivial lattice homomorphism $\phi_{\alpha,t}$ from $C(X \to [0,1])$ to $\{0,1\}$ by:

$$\phi_{\alpha,t}(f) = \begin{cases} 0 & f^\beta(\alpha) < t \\ 1 & f^\beta(\alpha) \geq t, \quad (f \in C(X \to [0,1])). \end{cases}$$

This clearly is a non-trivial lattice homomorphism associated with $\alpha$.

The above example shows us that for every $\alpha \in \beta X$ there is a lattice homomorphism $\phi : C(X \to [0,1]) \to \{0,1\}$ associated with $\alpha$. We note that the opposite also holds: For every lattice homomorphism $\phi : C(X \to [0,1]) \to \{0,1\}$ there is a $\alpha \in \beta X$ associated with $\phi$.

**Lemma 5.6.** Every lattice homomorphism $\phi : C(X \to [0,1]) \to \{0,1\}$ is associated with precisely one $\alpha \in \beta X$.

**Proof.** Assume that there is no point $\alpha \in \beta X$ associated with $\phi$. Then for all $\alpha \in \beta X$ there are $f_\alpha,g_\alpha \in C(X \to [0,1])$ s.t. $g_\alpha^\beta(\alpha) < f_\alpha^\beta(\alpha)$ while $\phi(f) = 0$ and $\phi(g) = 1$. Since $f,g$ are continuous, there is an open neighbourhood $U_\alpha$ in $\beta X$ around $\alpha$ s.t. $g_\alpha^\beta < f_\alpha^\beta$ on $U_\alpha$. So we have an open covering $\{U_\alpha : \alpha \in \beta X\}$ of $\beta X$. Because $\beta X$ is compact, there is a finite sub-cover. Let $U_{\alpha_1},...,U_{\alpha_n}$ be a finite sub-cover. Then for each $\alpha \in U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}$ we get $h > k$, where $\phi(h) = 0$ and $\phi(k) = 1$. So $\phi$ is not a lattice homomorphism, which is the desired contradiction. Hence $\phi$ must be associated with at least one $\alpha \in \beta X$.

Now assume there are at least two points $\alpha_1,\alpha_2$ associated with $\phi$. Since $\beta X$ is compact Hausdorff it is Normal. Thus we may use Urysohn in our prove. For the moment, assume there are $f,g \in C(X \to [0,1])$ s.t. $\phi(f) = 0$ and $\phi(g) = 1$ and $f^\beta(\alpha_1) > 0$ and $g^\beta(\alpha_2) < 1$. Make $h \in C(X \to [0,1])$ s.t. $h^\beta(\alpha_1) < f^\beta(\alpha_1)$ hence $\phi(h) = 0$, while $h^\beta(\alpha_2) > g^\beta(\alpha_2)$ hence $\phi(h) = 1$, which leads to the desired contradiction.

If our last assumption was wrong, we know that either $\forall\{f \in C(X \to [0,1]) : \phi(f) = 0\} : f^\beta(\alpha_1) = 0 = f^\beta(\alpha_2)$ or $\forall\{g \in C(X \to [0,1]) : \phi(g) = 1\} : g^\beta(\alpha_1) = 1 = g^\beta(\alpha_2)$. In the first case we take open $U,V \subset \beta X$ s.t. $U \cap V = \emptyset$, $\alpha_1 \in U$, $\alpha_2 \in V$ and with the help of Urysohn functions $h_1,h_2 \in C(X \to [0,1])$ s.t. $h_1^\beta(\alpha_1) = \frac{1}{2}$ and $h_2^\beta(\beta X \setminus U) = 0$ while $h_2^\beta(\alpha_2) = \frac{1}{2}$ and $h_2^\beta(\beta X \setminus V) = 0$. Then $\phi(h_1) = 1 = \phi(h_2)$, but $0 = \phi(0) = \phi(h_1 \wedge h_2) = \phi(h_1) \wedge \phi(h_2)$. Hence we have proven that $\phi$ is not a lattice homomorphism, which contradicts the assumption. For the second case we can do something similar with functions now being 1 instead of 0 and the maximum instead of the minimum. Which concludes the proof.

Now we would like to set up an equivalence relation on the lattice homomorphisms to $\{0,1\}$, where two lattice homomorphisms are equivalent iff they are associated with the same point. However, we do not want to use the points of $\beta X$ for this definition. Instead we want the definition in terms of our lattice homomorphisms only. We want to define something like $\phi \sim \psi$ iff $\phi(f) < \phi(g)$ implies $\psi(f) \leq \psi(g)$ for all $f,g \in C(X \to [0,1])$ and the other way around. Unfortunately, the next example shows this would not be a good definition.
Example 5.7. Let $X$ be $[0,1]$, let $I_X$ be the identity function on $X$. Define lattice homomorphisms $\phi, \psi : C(X \to [0,1]) \to \{0,1\}$ by:

\[
\phi(f) = \begin{cases} 
0 & \text{if } f \leq I_X \text{ in a neighbourhood of } \frac{1}{2} \\
1 & \text{else ,}
\end{cases}
\]

\[
\psi(f) = \begin{cases} 
0 & \text{if } f \leq 1_X - I_X \text{ in a neighbourhood of } \frac{1}{2} \\
1 & \text{else ,}
\end{cases}
\]

for all $f \in C(X \to [0,1])$.

It is clear for both $\phi$ and $\psi$ that they must be associated with $\frac{1}{2}$ since we know from the compactness of $[0,1]$ that they must be associated with an $x \in [0,1]$ and it is clear that they are not associated with any other element. Furthermore, $f(\frac{1}{2}) < g(\frac{1}{2})$ implies $\phi(f) < \phi(g)$ and $\psi(f) < \psi(g)$ for all $f, g \in C(X \to [0,1])$.

So we have two lattice homomorphisms associated with the same point $\frac{1}{2}$, while for $f = 1_X, g = 1_X - I_X$ holds:

$\phi(f) < \phi(g)$ and $\psi(g) < \psi(f)$.

First of all, note that the above example also shows us that $\phi(f)$ is not just depending on the value of $f \in C(X \to [0,1])$ in the associated point.

Now we prove a lemma similar to lemma 3.11, but then for lattice homomorphisms instead of prime ideals. This shows us that a lattice homomorphism depends on the values of $f$ in any open set containing the associated point. This inspires us to define proper equivalence class.

Lemma 5.8. Let $\phi$ be a prime ideal associated with $\alpha \in \beta X$. Let $f, g \in C(X \to [0,1])$ s.t. $\phi(f) = 0, g^\beta \leq f^\beta$ on an open set $U$ containing $\alpha$. Then $\phi(g) = 0$. Similarly, if $\phi(f) = 1, g^\beta \geq f^\beta$ on an open set $U$ containing $\alpha$, then $\phi(g) = 1$.

Proof. Suppose that $\phi(g) = 1$. Lemma 5.6 says $\phi$ cannot be associated with any point $\gamma \in \beta X \setminus U = V$. Hence, for all $\gamma \in V$ there are $h_\gamma, k_\gamma \in C(X \to [0,1])$ s.t. $h_\gamma^\beta(\gamma) > k_\gamma^\beta(\gamma)$ while $\phi(h_\gamma) = 0$ and $\phi(k_\gamma) = 1$. This gives us an open covering of $V$ by the sets $V_\gamma := \{\alpha \in V : h_\gamma^\beta > k_\gamma^\beta\}$. Note, $V$ is a closed subspace of a compact space, hence compact. So we can take a finite sub-cover $V_{\gamma_1}, ..., V_{\gamma_n}$ for some $n \in \mathbb{N}$. Then $h := f \vee h_{\gamma_1} \vee ... \vee h_{\gamma_n}$ has $\phi(h) = 0$ and $k := g \land k_{\gamma_1} \land ... \land k_{\gamma_n}$ has $\phi(k) = 1$, while $h > k$. This contradicts the assumption. Thus $\phi(g) = 0$.

The second part is proved just the same by interchanging $\land$ and $\lor$ and substituting 1 for 0. Which completes the proof. \qed

This leads to the next characterisation of the equivalence relation on lattice homomorphism.

Definition 5.9. Two non-trivial lattice homomorphisms $\phi, \psi : C(X \to [0,1]) \to \{0,1\}$ are related if there exists an $\alpha \in \beta X$ s.t. for all $f, g \in C(X \to [0,1])$ with $g^\beta \leq f^\beta$ on an open $U$ containing $\alpha$, gives us:

$\phi(g) \leq \phi(f)$ and $\psi(g) \leq \psi(f)$.

When $\phi, \psi : C(X \to [0,1]) \to \{0,1\}$ are related we will write $\phi \sim \psi$. One might complain that this is suggestive notation, since we did not prove yet that $\sim$ is a equivalence relation, but we promise to do so in lemma 5.12.

First we note that this $\alpha$ has to be unique.

Lemma 5.10. Let $\phi, \psi : C(X \to [0,1]) \to \{0,1\}$ be two non-trivial lattice homomorphisms with $\phi \sim \psi$. Then the $\alpha \in \beta X$ from definition 5.9 is unique, and must be associated with both $\phi$ and $\psi$.

Proof. Let $\phi$ be associated with an $\alpha_1 \in \beta X$, while $\alpha_1 \neq \alpha$. Take open sets $U, V \subset \beta X$, with $\alpha \in U, \alpha_1 \in V$ s.t. $U \cap V = \emptyset$ and functions $f, g \in C(X \to [0,1])$ with $f^\beta(U) = 0, f^\beta(V) = \frac{1}{2}, g^\beta(U) = 1, g^\beta(V) = \frac{1}{2}$ with the help of Urysohn. Then $\phi(f) = 0, \phi(g) = 1$ while $g^\beta \leq f^\beta$ on $V$, which contradicts the assumption. So we have shown that $\alpha$ must be associated with both $\phi, \psi$. Since associated points are unique, $\alpha$ is unique. \qed

We now check that two lattice homomorphisms are related if and only if they are associated with the same point.
Lemma 5.11. Let $\phi, \psi : C(X \to [0, 1]) \to \{0, 1\}$ be two lattice homomorphisms. Then $\phi \sim \psi$ iff they are associated with the same $\alpha \in \beta X$, which has to be the same $\alpha$ as in definition 5.9.

Proof. ($\Rightarrow$) We have proven this at lemma 5.10. ($\Leftarrow$) Let $\phi, \psi : C(X \to [0, 1]) \to \{0, 1\}$ be associated with the same $\alpha \in \beta X$. Then, lemma 5.8 tells us that for all $f, g \in C(X \to [0, 1])$ with $g^\beta \leq f^\beta$ on an open set $U$ containing $\alpha$, we know $\phi(g) \leq \phi(f)$ and $\psi(g) \leq \psi(f)$. Which is the desired property. \hfill $\square$

Now we have this settled it is time to check that $\sim$ is an equivalence relation.

Lemma 5.12. Let $\Phi$ be the set of all non-trivial lattice homomorphisms $\phi : C(X \to [0, 1]) \to \{0, 1\}$. Then $\sim$ is an equivalence relation on $\Phi$.

Proof. We prove this by going through the definition of equivalence relations:

1. Reflexivity. For all $\phi \in \Phi$ we trivially get $\phi \sim \phi$, because $\phi$ is associated with a unique point.
2. Symmetry. If $\phi, \psi \in \Phi$ s.t. $\phi \sim \psi$ then lemma 5.11 says $\phi, \psi$ are associated with the same point. Thus $\psi \sim \phi$.
3. Transitivity. If $\phi, \psi, \theta \in \Phi$ s.t. $\phi \sim \psi$ and $\psi \sim \theta$, then lemma 5.11 says both $\phi, \psi$ and $\psi, \theta$ are associated with the same point. Since lemma 5.6 proves that $\psi$ is related to a unique point, we know that $\phi, \psi, \theta$ are all related to the same point. Hence, lemma 5.11 proves that $\phi \sim \theta$.

$\sim$ satisfies all the conditions of an equivalence relation, hence it is one. \hfill $\square$

Note that for a compact Hausdorff space $X$ this matches directly with the prime ideals in chapter 3. But since this is a generalisation to spaces $X$, which do not need to be compact, we will have to build up the rest of the theory again, just like we did for the prime ideals.

Lemma 5.13. Let $\phi, \psi : C(X \to [0, 1]) \to \{0, 1\}$ be non-trivial lattice homomorphisms. Then $\phi \sim \psi$ iff there exists a non-trivial $\theta$ s.t. for all $f \in C(X \to [0, 1])$ either $\theta(f) \leq \phi(f)$ and $\theta(f) \leq \psi(f)$ or $\theta(f) \geq \phi(f)$ and $\theta(f) \geq \psi(f)$.

Proof. ($\Rightarrow$) From lemma 5.11 we know that $\phi, \psi$ are associated with the same $\alpha$. If there exists a function $f \in C(X \to [0, 1])$ s.t. $f^\beta(\alpha) > 0$ and $\phi(f) = 1 = \psi(f)$, take any $0 < t < f(\alpha)$ and define:

$$
\theta(f) := \begin{cases} 0 & f^\beta(\alpha) \leq t \\ 1 & f^\beta(\alpha) > t. \end{cases}
$$

We easily see that $\theta$ is a lattice homomorphism which is smaller than both $\phi, \psi$.

If however, there exists an $f \in C(X \to [0, 1])$ s.t. $f^\beta(\alpha) > 0$ and $\phi(f) > 0$, while no such $f$ exists for $\psi$. Then $\phi \not\sim \psi$.

As last we need to check for the case where no $f \in C(X \to [0, 1])$ exists s.t. $f^\beta(\alpha) > 0$ and $\phi(f) > 0$ or $\psi(f) > 0$. Then the previously defined $\theta$ is bigger than both $\phi, \psi$.

($\Leftarrow$) If there exists a $\theta$ s.t. for all $f \in C(X \to [0, 1])$ either $\theta(f) \leq \phi(f)$ and $\theta(f) \leq \psi(f)$ or $\theta(f) \geq \phi(f)$ and $\theta(f) \geq \psi(f)$. We assume that $\theta(f) \leq \phi(f)$ and $\theta(f) \leq \psi(f)$, the other proof is similar. Assume further that $\phi \not\sim \psi$. Lemma 5.11 says $\phi$ is associated with a $\alpha_1 \in \beta X$ and $\psi$ is associated with a $\alpha_2 \in \beta X$ and $\alpha_1 \neq \alpha_2$. Make small enough open sets $U, V \subseteq \beta X$ with $\alpha_1 \in U, \alpha_2 \in V$ s.t. there are functions $f, g \in C(X \to [0, 1])$ with $f^\beta(U) = 0, g^\beta(V) = 0$ and $f^\beta \vee g^\beta = 1$. Then $\theta(f) \leq \phi(f) = 0$ and $\theta(g) \leq \psi(g) = 0$. Thus $\theta(1) = 0$. Which contradicts $\theta$ being non-trivial. Hence, $\phi \sim \psi$. \hfill $\square$

Now we have proven this characterisation it will be easier to prove the next lemmas.

Lemma 5.14. Let $T : C(X \to [0, 1]) \to C(C(X \to [0, 1]))$ be a lattice isomorphism, let $\phi : C(Y \to [0, 1]) \to \{0, 1\}$ be a lattice homomorphism. Then $\phi \circ T$ is a lattice homomorphism on $C(X \to [0, 1])$.

Proof. This is trivial since it is the combination of two lattice homomorphisms. \hfill $\square$

We would like to have that the equivalence relation $\sim$ is preserved by lattice homomorphisms. This would imply that the equivalence class of prime ideals of a given point in $\beta X$ are mapped to an equivalence class of a given point in $\beta Y$, which will allow us to define a bijection from $\beta X$ to $\beta Y$.

Lemma 5.15. Let $T : C(X \to [0, 1]) \to C(Y \to [0, 1])$ be a lattice isomorphism, let $\phi, \psi : C(Y \to [0, 1]) \to \{0, 1\}$ be lattice homomorphism. Then $\phi \sim \psi$ implies $\phi \circ T \sim \psi \circ T$. 

Proof. This is trivial since it is the combination of two lattice homomorphisms. \hfill $\square$
PROOF. We have just seen that $\phi \circ T, \psi \circ T$ are lattice homomorphisms, so it remains to prove that $\phi \circ T \sim \psi \circ T$. By lemma 5.13 there is a lattice homomorphism $\theta$ s.t. for all $f, g \in C(Y \to [0, 1])$ either $\theta(f) \leq \phi(f)$ and $\theta(f) \leq \psi(f)$ or $\theta(f) \geq \phi(f)$ and $\theta(f) \geq \psi(f)$. We assume that $\theta(f) \leq \phi(f)$ and $\theta(f) \leq \psi(f)$ for all $f \in C(Y \to [0, 1])$, the other proof is just the same. By lemma 5.14 $\theta \circ T$ is a lattice homomorphism. Furthermore, $\theta \circ T(g) \leq \phi \circ T(g)$ and $\theta \circ T(g) \leq \psi \circ T(g)$ for all $g \in C(X \to [0, 1])$. Hence, there exists a non-trivial lattice homomorphism $\theta \circ T$ which is smaller than both $\phi \circ T, \psi \circ T$. Thus lemma 5.13 says $\phi \circ T \sim \psi \circ T$. □

Since $T$ is a lattice isomorphism the above theorem works both ways and can be combined to the next corollary.

**Corollary 5.16.** Any lattice isomorphism $T : C(X \to [0, 1]) \to C(Y \to [0, 1])$ preserves the equivalence relation $\sim$.

As last we want to find a connection between the open sets and the lattice homomorphism, in order to proof continuity of our soon to be defined bijection.

**Lemma 5.17.** For any subset $D$ of $\beta X$ define $I(D)$ to be the intersection of the sets $\{\phi^{-1}(0)\}$ for all lattice homomorphisms $\phi : C(X \to [0, 1]) \to \{0, 1\}$ associated with a point in $D$ with $\phi(\frac{1}{2} 1_X) = 0$. Then a point $\alpha \in \beta X$ is in the closure of $D$ iff $\phi(I(D)) = 0$ for a lattice homomorphism $\phi : C(X \to [0, 1]) \to \{0, 1\}$ associated with $\alpha$.

PROOF. (⇒) For $\alpha \in D$, define $\phi : C(X \to [0, 1]) \to \{0, 1\}$ by:

$$
\phi(f) := \begin{cases} 
0 & f^\beta(\alpha) \leq \frac{1}{2} \\
1 & f^\beta(\alpha) > \frac{1}{2},
\end{cases} \quad (f \in C(X \to [0, 1])).
$$

Now $g \in I(D)$ implies for all $\gamma \in D$, $g^\beta(\gamma) \leq \frac{1}{2}$ so $g^\beta \leq \frac{1}{2}$ on $D$. Since $g^\beta$ is continuous we must also have $g^\beta(\alpha) \leq \frac{1}{2}$. Hence, $\phi(I(D)) = 0$.

(⇐) Suppose $\alpha \notin D$, $\phi$ a lattice homomorphism associated with $\alpha$ s.t. $\phi(I(D)) = 0$. Take a $f \in C(X \to [0, 1])$ s.t. $\phi(f) = 1$ we know that we can take open sets $U, V \in \beta X$ with $U \cap V = \emptyset$ and $\alpha \in U, D \subseteq V$. With the help of Urysohn, construct a $g \in C(X \to [0, 1])$ s.t. $g^\beta < \frac{1}{2}$ on $V$ but $g^\beta \geq f^\beta$ on $U$. Then $\phi(g) = 1$ but $g \notin I(D)$, hence we have the desired contradiction. Thus, for $\alpha \notin D$ there does not exist such a lattice homomorphism associated with $\alpha$. □

This lemma gives us a characterisation of closed sets which is only dependant on lattice homomorphisms, which enables us to prove the following theorem.

**Theorem 5.18.** Let $C(X \to [0, 1]), C(Y \to [0, 1])$ be lattice isomorphic. Then $\beta X$ and $\beta Y$ are homeomorphic.

PROOF. Let a lattice isomorphism $T : C(X \to [0, 1]) \to C(Y \to [0, 1])$ be given. Then we can construct a bijection $\tau : \beta X \to \beta Y$ by defining: $\tau(\alpha) = \gamma$ where $\gamma$ is the point associated with any image of a prime ideal associated with $\alpha$. It is clear that this is a bijection. Now it remains to prove that this bijection is also homeomorphic.

We have seen in lemma 5.17 that we can characterise closed sets of $\beta X$ as sets $D$ for which every point has an associated lattice homomorphism $\phi : C(X \to [0, 1]) \to \{0, 1\}$ with $\phi(I(D)) = 0$. We use this to prove that the image of a closed set in $\beta X$ is closed in $\beta Y$. Let $D$ be a closed set in $\beta X$. Then for every $\alpha \in D$, there is a lattice homomorphism $\phi : C(X \to [0, 1]) \to \{0, 1\}$ associated with $\alpha$ s.t. $\phi(I(D)) = 0$. Hence, $\psi(T(I(D))) = 0$ for a lattice homomorphism $\psi : C(Y \to [0, 1]) \to \{0, 1\}$ associated with $\tau(\alpha)$ while $\psi(T(I(D))) \neq 0$ for any lattice homomorphism $\psi : C(Y \to [0, 1]) \to \{0, 1\}$ associated with $\gamma \notin \tau(D)$. Since that would imply that $T^{-1}(\psi^{-1}(0))$ contains $I(D)$, while $T^{-1}(\psi^{-1}(0))$ is associated with $\tau^{-1}(\gamma) \notin D$. Thus $\tau(D)$ is closed. So we proved that $\tau$ is a bijection which sends closed sets to closed sets. Hence $\tau$ is a homeomorphism. □
CHAPTER 6

Extremally disconnected spaces

We will now look into spaces \( C(X) \) where \( X \) is compact Hausdorff and extremally disconnected. This last property will help us a lot in finding new connections. Later we see that all Archimedean Riesz spaces can be seen as certain subsets of these spaces. In order to keep this chapter very short we will leave out all the long and dreadful proofs. For interested readers we give citations. We will assume \( X, Y \) to be compact Hausdorff throughout the whole chapter.

We will start by giving the definition of extremally disconnected.

**Definition 6.1.** A space \( X \) is called **extremally disconnected** iff for every open set \( U \subset X \) the closure \( \overline{U} \) is open and thus clopen (closed and open).

In order to have an idea about the spaces we are dealing with we will introduce zerodimensionality other property and prove that all extremally connected spaces are zerodimensional.

**Definition 6.2.** A set \( X \) is **zerodimensional** if the clopen subsets of \( X \) form a base for the topology.

As announced, we prove that all extremally connected spaces are zerodimensional.

**Theorem 6.3.** If \( X \) is extremally disconnected then \( X \) is zerodimensional.

**Proof.** Let \( X \) be extremally disconnected. Let \( a \in X \), let \( S \) be a neighbourhood of \( a \). There exists an open set \( U \subset X \) s.t. \( a \in U \subset \overline{U} \subset S \). Consequently, \( a \in \overline{U} \subset S \). Thus, the clopen subsets of \( X \) form a base for the topology. \( \square \)

The above defined properties are properties on a space \( X \), while we are more interested in the properties of \( C(X) \). This is why we will now introduce Dedekind completeness.

**Definition 6.4.** A Riesz space \( E \) is **Dedekind complete**, or D-complete, if every non-empty subset of \( E^+ \) with an upper bound has a supremum.

Now we can prove that \( X \) is extremally disconnected iff \( C(X) \) is D-complete.

**Theorem 6.5.** \( X \) is extremally disconnected iff \( C(X) \) is D-complete.

**Proof.** (\( \Leftarrow \)) Let \( U \subset X \) be open. Then \( \{ f \in C(X) : f \leq 1_U \} \) has upper bound \( 1 \) hence has a supremum, \( e_U \), in \( C(X) \). Since for every \( a \in U \) there is an \( f \in C(X)^+ \) with \( f(a) = 1 \) and \( f \leq 1_U \), we have \( 0 \leq e_U \leq 1 \) and \( e_U = 1 \) on \( U \) hence \( e_U = 1 \) on \( \overline{U} \).

If \( b \in X \setminus \overline{U} \), then there is a \( g \in C(X) \) with \( g(b) = 0 \) and \( g = 1 \) on \( \overline{U} \) so \( e_U \leq g \) hence \( e_U = 0 \) on \( X \setminus \overline{U} \). Now we have proven that \( 1_{\overline{U}} = e_U \in C(X) \). Hence \( \overline{U} \) is open.

(\( \Rightarrow \)) This result is from H.Nakano. For this proof it is useful to be familiar with lower continuous functions. Hence we will only refer to theorem 12.16 of [DJR77] for the full proof. \( \square \)

Since this chapter is about extremally disconnected compact Hausdorff spaces, we assume from now on that all our spaces \( X, Y \) are extremally disconnect compact Hausdorff spaces. We will now look at two examples to give an idea of what we are dealing with.

**Example 6.6.** \( C[0, 1] \) is not D-complete. This we can see by looking at the set \( X = \{ f \in C[0, 1] : f \leq 1_{[0, 1]} , f = 0 \text{ on } [0, \frac{1}{2}) \} \). Clearly \( 1_{[0, 1]} \) is an upper bound of the set but the set has no supremum since any continuous function \( g \) being a supremum would have \( g = 0 \) on \( [0, \frac{1}{2}) \) and \( g = 1 \) on \( [\frac{1}{2}, 1] \) which is impossible.

On the other hand, \( L^1([0, 1]) \) is a D-complete space.

Before we get to the nice results about the just defined spaces, we need to give a few more definitions. First the definition of meagre sets, which is definition 14.1 of [DJR77].
Definition 6.7. A subset $A$ of $X$ is said to be meagre if there exists a (countable) sequence $A_1, A_2, \ldots$ of subsets of $X$ s.t.

1. $A \subseteq \bigcup A_n$
2. Every $A_n$ is closed and has empty interior, $(A_n)^\circ$.

Every subset of a meagre set is meagre. A union of countably many meagre sets is meagre.

$\mathbb{Q} \cap [0,1]$ is meagre as a subset of $[0,1]$. So is $\{0\}$, although of course $\{0\}$ is not a meagre subset of $\{0\}$.

With the help of Baire's category theorem we get the following characterisation.

Theorem 6.8. A closed subset of $X$ is meagre iff its interior is empty.

Proof. For the proof we refer to theorem 14.2(ii) of [DJR77].

Now it is finally time to introduce the space we are actually interested in; the "why" to this will be answered very soon. The definition is definition 15.2 of [DJR77].

Definition 6.9. By $C^\infty(X)$ we denote the set of all continuous functions $f : C(X \to [-\infty, \infty])$ for which the closed sets $f^{-1}(\{\infty\})$ and $f^{-1}(\{-\infty\})$ have open interior, hence by theorem 6.8 are meagre.

A lemma is in its place to help us prove that we have defined a Archimedean Riesz space.

Lemma 6.10. Let $A$ be a meagre closed subset of $X$ and let $f$ be a continuous function $X \setminus A \to \mathbb{R}$ Then $f$ has exactly one extension $X \to [-\infty, \infty]$ that is an element of $C^\infty(X)$.

Proof. This is just lemma 15.3 of [DJR77].

Theorem 6.11. $C^\infty(X)$ is a Archimedean Riesz space.

Proof. For any $f, g \in C^\infty(X)$ define $W = \{ x : |f(x)| = \infty \text{ or } |g(x)| = \infty \}$. Then $W$ is closed and meagre, while on its complement the function $f + g$ is continuous. So lemma 6.10 tells us that exists exactly one element $f + g$ of $C^\infty(X)$. Nearly the same proof goes for $fg$.

Now in order to prove Archimedeanity, assume there are $f, g \in C^\infty(X)$ s.t. $ng < f$ for all $n \in \mathbb{N}$. Since $g$ must be 0 where $f \neq \infty$ we see that $\{ x : g(x) \neq 0 \} \subset f^{-1}(\{\infty\})$. While $f^{-1}(\{\infty\})$ is meagre, hence $\{ x : g(x) \neq 0 \}$ is meagre. Thus again by lemma 6.10 we see that $g = 0$. Hence $C^\infty(X)$ is Archimedean.

This gives us enough to prove the following very interesting theorem, which tells us that the Boolean algebra of clopen sets on $X$ is Boolean isomorphic to the Boolean algebra of the bands of $C^\infty(X)$. Since we looked at relations between the structure on $X$ and $C(X)$ all along this is a great characterisation.

Theorem 6.12 (Boolean algebras). We prove that $C(X), B(C^\infty(X))$ are Boolean isomorphic, with the help of the following steps.

1. The clopen sets of $X$ form a Boolean algebra, which we will denote by $\mathcal{C}(X)$.
2. The bands of $C^\infty(X)$ form a Boolean algebra $B(C^\infty(X))$.
3. The correspondence $U \mapsto \{ g \in C(X) : g = 0 \text{ on } X \setminus U \}$ is a Boolean isomorphism from $\mathcal{C}(X)$ to $B(C^\infty(X))$.

Proof.

1. It is easy to see that it is a Boolean algebra, since both the set of open subsets of $X$ and the set of closed subsets of $X$ are a Boolean algebra.
2. Since $C^\infty(X)$ is an Archimedean Riesz space, we have already seen in theorem 2.35 that the bands of $C^\infty(X)$ form a boolean algebra $B(C^\infty(X))$.
3. We first prove that for any clopen subset $U$ we have $B_U := \{ g \in C(X) : g = 0 \text{ on } X \setminus U \}$ is a band. It clearly is a linear subspace. Furthermore, for all $g \in B_U$, $f \in C^\infty(X)$, $|f| \leq |g|$ implies $f = 0$ on $X \setminus U$. Hence $f \in B_U$. As last, take any non-empty subset $D$ of $B_U$. If $D$ has supremum $g$, then we need to prove that $g \in B_U$. To do that, take any $x \in U$. Note that $g(x) = 0$, else we define $h \in C^\infty(X)$ by $h(X \setminus U) = 0$ and $h(y) = g(y)$ for all $y \in U$. Then $g \not\leq h$ while $h$ is an upper bound of $X$, which is the desired contradiction. So $g(x) = 0$ for all $x \in U$, hence $g \in B_U$. Thus $B_U$ is a band.

It remains to prove that all bands of $C^\infty(X)$ are $B_U$ for some clopen subset of $U$. We know that all Riesz ideals in a set $C^\infty(X)$ are of the form $B_A := \{ g \in C(X) : g = 0 \text{ on } A \}$ for some set $A \subseteq X$. Now it just remains to prove that $A$ has to be clopen in order for $B_A$ to be a band. Since we work
on the set of continuous functions $B_A = B_\overline{A}$, since any function which is 0 on $A$ must also be 0 on the closure of $A$. We claim that for bands $B_A$ the set $\overline{A}$ must be clopen.

Assume $B_A$ is a band and $\overline{A}$ is not clopen. Since $\overline{A}$ is a clopen subset contained in $A$ there must be a $x \in \overline{A} \setminus A$. Hence there must be a clopen set $U$ with $x \in U$ and $U \cup \overline{A} = \emptyset$. Construct, with the help of Urysohn, for any $y \in U \setminus \overline{A}$ a function $f_y \in B_A$ s.t. $f_y(y) = 1, f \leq 1_X$ and $f((X \setminus U) \cup \overline{A}) = 0$. Then sup$f_y : y \in U = 1_U$ in $C^\infty(X)$. Since $B_\overline{A}$ is a band we must have $1_U \in B_\overline{A}$. But $1_U(x) = 1$ which is the desired contradiction. Hence there is no $x \in \overline{A} \setminus A$. Thus $\overline{A} = A$, which proves that $\overline{A}$ is clopen.

Although we have this very nice characterisation, it will be helpful to prove a bit more about our space $C^\infty(X)$. We start by proving that it is D-complete.

**Theorem 6.13.** $C^\infty(X)$ is a D-complete.

**Proof.** Let $F \subset C(X)^+$, $F \neq \emptyset$ be given with an upper bound $u \in C^\infty(X)$. Then $\{\arctan \circ f : f \in F\}$ has upper bound $\arctan \circ u$ in $C(X)$ hence it has a supremum $g \in C(X)$. $g \leq \arctan \circ u$, so $|g| < \frac{\pi}{2}$ a.e. Hence $\tan \circ g$ is sup $F$ in $C^\infty(X)$. \hfill \Box

One more property of our space $C^\infty(X)$ will be needed. This will be lateral completeness. Before we can introduce it we will need to have the notion of disjointness of a subset.

**Definition 6.14.** A subset $D$ of a Riesz space $E^+$ is disjoint if $d_1, d_2 \in D$ and $d_1 \neq d_2$ implies that $d_1 \wedge d_2 = 0$.

Now we can define laterally completeness.

**Definition 6.15.** A Riesz space $E$ is laterally complete if for all disjoint $D \subseteq E$ there is a $a \in E$ s.t. $a = \sup D$.

As announced we prove that $C^\infty(X)$ is laterally complete.

**Theorem 6.16.** $C^\infty(X)$ is laterally complete.

**Proof.** Let $F$ be a non-empty disjoint subset of $C^\infty(X)^+$. For every $f \in F$, define the set $W_f := \{x \in X : 0 < f(x) < \infty\}$. For all $f \in F$ we then have $W_f$ is open and $f = 0$ on the clopen set $X \setminus W_f$. Define $W := \bigcup W_f$. Then $W$ is clopen. So there exists a continuous function $g$ on the set $(X \setminus W) \cup \bigcup W_f$ s.t. $g = 0$ on $X \setminus W$ and for every $f \in F$ we have $g = f$ on $W_f$. Since $W \setminus \bigcup W_f$ is meagre lemma 6.10 proves that this function uniquely extends to a function in $C^\infty(X)$ which clearly is the supremum of $F$. \hfill \Box

In order to post the next big theorem we need the definition of order density.

**Definition 6.17.** Let $D$ be a Riesz subspace of and Archimedean Riesz space $E$. Then $D$ is order dense in $E$ iff:

$$a \in E^+, a \neq 0 \Rightarrow \exists d \in D^+, d \neq 0, d \leq a$$

or equivalently:

$$a \in E^+ \Rightarrow a = \sup \{d \in D : d \leq a\}.$$ 

Note that when $D$ is order dense in $E$ the map $B \to B \cap D$ is a bijection from $B(E)$ to $B(D)$.

The preservation of bands will be of use later. First we post the Maeda-Ogasawara theorem.

**Theorem 6.18 (Maeda-Ogasawara).** Let $E$ be an Archimedean Riesz space. Then there exists an $X$ and an order dense Riesz subspace $\hat{E}$ of $C^\infty(X)$ with $E$ is Riesz isomorphic to $\hat{E}$.

**Proof.** For the proof we refer to theorem 15.5 from [DJR77]. \hfill \Box

This theorem allows us to get an idea of the lattice homomorphisms from an Archimedean Riesz spaces to another, by studying lattice isomorphisms between sets $C^\infty(X)$ and $C^\infty(Y)$. But this will still require a lot of work. We start by defining a Dedekind cut.
Definition 6.19. A Dedekind cut of a Riesz space $E$ is a pair $(A : B)$ of non-empty subsets $A, B$ of $E$ s.t. $A := \{a \in E : a \text{ is a lower bound of } B\}$ and $B := \{b \in E : b \text{ is an upper bound of } A\}$.

We denote by $E^\pm$ the set of all Dedekind cuts of $E$. We view $E$ as a subset of $E^\pm$ because every element $a \in E$ determines a Dedekind cut $a^\pm$ by $a^\pm := (a + E^- : a + E^+)$. The set $E^\pm$ is not just a set, it is again a Riesz space.

**Theorem 6.20.** $E^\pm$ is a Riesz space.

**Proof.** For the proof we refer to theorem 15.15 from [DJR77].

So we actually constructed a Riesz space $E^\pm$ bigger than $E$. The reason why we did that is because we can prove $E^\pm$ to be some kind of ideal which we need in a theorem later. But first we show that a lattice isomorphism $T : E \to F$ can be extended to a lattice isomorphism $T^\pm : E^\pm \to F^\pm$, which we need later.

**Theorem 6.21.** If $T : E \to F$ is a lattice isomorphism, then there is a lattice isomorphism $T^\pm : E^\pm \to F^\pm$ s.t. $T^\pm = T$ on $E$.

**Proof.** Define $T^\pm : E^\pm \to F^\pm$ by:

$$T^\pm(A : B) = ([T(a) : a \in A] : [T(b) : b \in B]) \quad ((A : B) \in L^\pm).$$

In order to see that this is a lattice isomorphism we need to prove that $([T(a) : a \in A] : [T(b) : b \in B])$ is a element of $F^\pm$. This is true, since lower and upper bounds are preserved by lattice isomorphisms.

Now we will prove that when $E$ is an order dense Riesz subspace of $M$ that $E^\pm$ then is a order dense Riesz ideal in $M$. This will soon be of use.

**Theorem 6.22.** Let $E$ be an order dense Riesz subspace of a D-complete Riesz space $M$. Then there exists a unique Riesz homomorphism $\Omega : E^\pm \to M$ s.t. $\Omega(a^\pm) = a$ for all $a \in E$. This $\Omega$ is given by the formula:

$$\Omega(A : B) = \sup A = \inf B \quad ((A : B) \in E^\pm).$$

And $\Omega(E^\pm)$ is the ideal generated by $E$.

**Proof.** For the proof we refer to theorem 15.18 from [DJR77].

So why are we so interested in getting a order dense Riesz ideal?

**Lemma 6.23.** Let $E$ be an order dense Riesz ideal in $C^\infty(X)$. Let $f \in C^\infty(X)^+$. Then $F := \{f1_U : U \in \mathcal{C}(X) ; f1_U \in E^+\}$ is a disjoint subset of $E^+$ with sup $F = f$.

**Proof.** Let $\mathcal{U} := \{U \in \mathcal{C}(X) : f1_U \in E^+\}$. Because we know from theorem 6.13 that $C^\infty(X)$ is D-complete, the set $F = \{f1_U : U \in \mathcal{U}\}$ has a supremum in $C^\infty(X)$. Note that sup $F \leq f$ and $f \leq$ sup $F$ on $\bigcup \mathcal{U}$. We want to prove that sup $F = f$.

Since $f1_{\bigcup \mathcal{U}}$ is an upper bound of $F$ and since the supremum must be continuous and bigger than $f1_U$ for all $U \in \mathcal{U}$, we see sup $F = f1_{\bigcup \mathcal{U}}$. Consider $f1_{X \setminus \bigcup \mathcal{U}}$. Note that when $f1_{X \setminus \bigcup \mathcal{U}} = 0$, we have sup $F = f$. Thus assume $f1_{X \setminus \bigcup \mathcal{U}} \neq 0$. Since $E$ is order dense in $C^\infty(X)$ there is a $g \in E^+$, $g \neq 0$ and $g \leq f1_{X \setminus \bigcup \mathcal{U}}$. Since $C^\infty(X)$ is Archimedean, there exists a smallest $n \in \mathbb{N}$ s.t. $ng \not\leq f1_{X \setminus \bigcup \mathcal{U}}$ for this $n$ take $W := \{x \in X : ng(x) > f(x)\}$. Then $W \neq \emptyset$, while $ng \geq ng1_{X \setminus \bigcup \mathcal{U}} \geq f1_{X \setminus \bigcup \mathcal{U}}$. So $f1_{X \setminus \bigcup \mathcal{U}} \in E$ because $E$ is a Riesz ideal. But then $W \in \mathcal{U}$, which contradicts our assumption. Thus sup $F = f$.

We again prove that we can extend our lattice isomorphisms to bigger spaces.

**Theorem 6.24.** Let $T : E \to F$ be a lattice isomorphism, let $E$ be an order dense Riesz ideal in $C^\infty(X)$ and $F$ and order dense Riesz ideal in $C^\infty(Y)$. Then there is a $T^* : C^\infty(X) \to C^\infty(Y)$ with $T^* = T$ on $E$.

**Proof.** Apply the previous lemma to see that for all $f \in C^\infty(X)^+$ we get a disjoint set $F \subset E^+$ with sup $F = f$. Define $T^* : C^\infty(X) \to C^\infty(Y)$ by:

$$T^*(f) = T(\sup F) = \sup TF.$$ 

This exists since $C^\infty(Y)$ is laterally complete by theorem 6.16 and suprema are carried by lattice homomorphisms, theorem 2.21.
If we combine all we have just proven we get something very interesting. For all Archimedean Riesz spaces \( E, F \) and a lattice isomorphism \( T \) between them we get:

\[
\begin{array}{c}
E \cong \hat{E} \quad \hat{E} \subseteq C^\infty(X) \\
\downarrow \hat{T} \quad \downarrow \hat{T} = \hat{T}^* \\
F \cong \hat{F} \quad \hat{F} \subseteq C^\infty(Y)
\end{array}
\]

Since \( E \) is Riesz isomorphic to \( \hat{E} \) and \( \hat{E} \) is a order dense subspace \( C^\infty(X) \), we noted in definition 6.17 the map \( B \to B \cup E \) is a bijection from \( B(C^\infty(X)) \) to \( B(E) \). Since \( \hat{T}^* = \) is an extension to \( T \) we see that \( T \) supplies us with a unique bijection between the bands \( B(C^\infty(X)) \) and \( B(C^\infty(Y)) \). Now we can finally really see the power of theorem 6.12.

**Theorem 6.25.** Let \( E, F \) be Archimedean Riesz spaces, let \( T : E \to F \) be a lattice isomorphism. Then \( E, F \) are Riesz isomorphic to order dense subsets \( \hat{E}, \hat{F} \) of a space \( C^\infty(X) \).

**Proof.** We have just noted that \( E, F \) are Riesz isomorphic to order dense subsets \( \hat{E}, \hat{F} \) of space \( C^\infty(X), C^\infty(Y) \). Furthermore, \( T : E \to F \) gives rise to a lattice isomorphism \( \hat{T} = : C^\infty(X) \to C^\infty(Y) \). We know that all extensions of order dense subspaces preserve the bands by definition 6.17. On the other hand, theorem 2.36 says that \( \hat{T}^* \) sends \( B(C^\infty(X)) \) to \( B(C^\infty(X)) \). According to theorem 6.12 this gives us a Boolean isomorphism between the clopen subsets of \( X \) to \( Y \). Theorem 6.3 tells us that \( X \) is zerodimensional, hence the clopen subsets form a base for the topology. Thus we can make a homeomorphism from \( X \) to \( Y \) by sending an \( x \in X \) to the intersection of the images of the clopen sets containing \( x \), hence we may take \( X = Y \). \( \square \)

This means we should adjust our view on what we are doing to:

\[
\begin{array}{c}
E \cong \hat{E} \quad \hat{E} \subseteq C^\infty(X) \\
\downarrow \hat{T} \quad \downarrow \hat{T}^* = \hat{T}^* \\
F \cong \hat{F} \quad \hat{F} \subseteq C^\infty(Y)
\end{array}
\]

This shows us that for Archimedean Riesz spaces \( E, F \) we can link every lattice isomorphisms \( T : E \to F \) to a lattice automorphism of \( C^\infty(X) \) for some \( X \). Thus in order to study lattice isomorphisms between Archimedean Riesz spaces, we can just study lattice automorphisms of a space \( C^\infty(X) \). First we introduce some notation.

**Definition 6.26.** For \( s \in \mathbb{R} \), define \( \pi = sI_X \in C^\infty(X) \).

Before we start proving, we note that it is harmless to assume that every band of \( C^\infty(X) \) is sent to itself by the lattice automorphism. Now we start investigating the automorphisms.

**Theorem 6.27.** Let \( T \) be a lattice automorphism of \( C^\infty(X) \), let \( U \in \mathcal{C}(X) \). Then \( T(fI_U) = (Tf)I_U \).

**Proof.** Since \( U \) is clopen we have \( B_U = \{ f : f = fI_U \} = \{ fI_U : f \in C^\infty(X) \} \) is a band and \( f = fI_U + fI_{X \setminus U} \) for all \( f \in C^\infty(X) \) while \( fI_U \perp I_{X \setminus U} \). Hence \( T(fI_U) = T(fI_U) + T(fI_{X \setminus U}) \) with \( T(fI_U) \in B_U, T(fI_{X \setminus U}) \in B_{X \setminus U} \), thus \( T(fI_U) = T(f)I_U \). \( \square \)

This allows us to proof the following theorem.

**Theorem 6.28.** Let \( T \) be a lattice automorphism of \( C^\infty(X) \). Then there is a meagre set \( X_0 \) s.t. \( x \in X \setminus X_0 \) and all \( s, t \in \mathbb{R} \) with \( s > t \) we have \( -\infty < (T(\bar{\pi}))\langle x \rangle < (T\pi)\langle x \rangle < \infty \).

**Proof.** For clopen \( U \subset X \) we have \( T\pi U \leq T\pi \) on \( U \) implies \( T(\pi U) = T(\pi U) \leq (T\pi)I_U = T\pi I_U \) which implies \( \pi I_U \leq I_U \) which can only be if \( U = \emptyset \). Hence, \( \{ x : (T\pi)\langle x \rangle \leq (T\pi)\langle x \rangle \} \) is meagre.

Thus we can define \( X_0 \subset X \) by \( X_0 := \bigcup_{s, t \in \mathbb{Q}, s > t} \{ x : (T\pi)\langle x \rangle \leq (T\pi)\langle x \rangle \} \), which is meagre by definition. So we get for all \( x \in X \setminus X_0 \) and all \( s, t \in \mathbb{R} \) with \( s > t \) that \( -\infty < (T\pi)\langle x \rangle < (T\pi)\langle x \rangle < \infty \). \( \square \)
This leads us to the next lemma which we will need to use the previous theorem.

**Lemma 6.29.** Let $T$ be a lattice automorphism of $C^\infty(X)$, let $s \in \mathbb{R}$. Then there is a meagre set $X_0$ s.t for all $x \in X \setminus X_0$ we have $(T(s))(x) = \sup\{(T(q))(x) : q \in \mathbb{Q}, q < s\}$.

**Proof.** Take $X_0$ from theorem 6.28, then we know that for all $x \in X \setminus X_0$ we have $-\infty < (T(\tilde{q}))(x) < (T(\overline{q}))(x) < \infty$. For the sake of contradiction, we assume that there exists a $\epsilon \in \mathbb{R}$ with $\epsilon > 0$ s.t. for all $q \in \mathbb{Q}$ with $t < s$ we have $(T(\overline{q}))(x) < (T(s))(x) - \epsilon$. Take $f \in C^\infty(X)$ s.t. $f(x) = (T(s))(x) - \frac{\epsilon}{2}$. Then for all $q < s$ we have $q < (T(f))(x) < s$ by theorem 6.28, which is the desired contradiction. Hence $(T(s))(x) = \sup\{(T(q))(x) : q \in \mathbb{Q}, t < s\}$. □

Note that we could have all rational elements bigger than $s$ and the infimum. Now we can prove that lattice isomorphism send functions which are equal in a certain point, to functions which are equal in the same point.

**Theorem 6.30.** Let $F : X_1 \times \mathbb{R} \to X_1 \times \mathbb{R}$ defined by $F(x, s) = (x, (T(s))(x))$ for $x \in X_1, s \in \mathbb{R}$. Then for all $f \in C^\infty(X)$ the function $F$ maps the graph of $f$ to the graph of $T(f)$ on $X \setminus X_0$ for a meagre set $X_0$.

**Proof.** Take $X_0$ from theorem 6.28, let $x \in X \setminus X_0$. Then it is enough to prove that for all $f \in C^\infty(X)$ we have $T(f)(x) = T(f(x))$. For all $q_1 \in \mathbb{Q}$ with $q_1 < f(x)$ we have $T(q_1) < (T(f))(x)$. By lemma 6.29 we know that $(T(s))(x) = \sup\{(T(q))(x) : q \in \mathbb{Q}, q < s\}$, thus $(T(s))(x) \leq (T(f))(x)$. The other relation can be proven just the same. □

This leads to a corollary, which adds up everything we did in this chapter.

**Corollary 6.31.** Let $E, F$ be Archimedean Riesz space, let $T : E \to F$ be a lattice isomorphism. Then $E, F$ can be embedded in $C^\infty(X)$ for some $X$ and $T$ can be extended to a lattice automorphism on $C^\infty(X)$. 
CHAPTER 7

Sequence Spaces

Finally we look into more concrete lattice isomorphisms. For this we chose the sequence spaces. An easy space to start with is \( l^1(\mathbb{N}) \). We can wonder if there are lattice isomorphisms from \( l^1(\mathbb{N}) \) to either \( \mathbb{R}^N, l^\infty(\mathbb{N}), C_0 \) or \( C_{00} \). Before we start trying to proof the existence or lack of lattice isomorphisms, we think about what we already know so far. This will be very helpful later.

**Lemma 7.1.** A lattice isomorphism sends minimal bands to minimal bands.

**Proof.** Since lattice isomorphism indicate a boolean isomorphism between the bands by theorem 2.36, lattice isomorphism have to preserve minimal bands. \( \square \)

This is helpful since the bands of \( l^1(\mathbb{N}) \) are precisely \( \mathbb{R} e_n \) for some \( n \in \mathbb{N} \). Furthermore, all permutations \( \sigma \) of \( \mathbb{N} \) provide us with a lattice automorphism \( T_\sigma \) of \( l^1(\mathbb{N}) \) by:

\[
T(\lambda e_n) = \lambda e_{\sigma(n)}.
\]

This shows us that we can safely assume that the lattice isomorphism we will consider sent every coordinate to itself. It quickly follows that every lattice isomorphism can be represented by a infinite sequence of strictly increasing functions \( f_n \) s.t:

\[
T(\lambda e_n) = f_n(\lambda)e_n.
\]

When we now think about what we wanted to prove or disprove, it seems a lot more possible.

**Theorem 7.2.** \( l^1(\mathbb{N}) \) is not lattice isomorphic to:

1. \( \mathbb{R}^N \)
2. \( l^\infty(\mathbb{N}) \)
3. \( c_0 \), the space of all sequences which converge to zero.
4. \( c_{00} \), the space of all sequences with finitely many non-zero elements.

**Proof.** We prove the statements one by one. We will mostly consider lattice isomorphism form the other space onto \( l^1(\mathbb{N}) \) because that is easier, of course it doesn’t matter since it is an isomorphism.

1. Consider a lattice isomorphism \( T : \mathbb{R}^N \rightarrow l^1(\mathbb{N}) \). We can look at \( T \) by thinking of the functions \( f_1, f_2, \ldots \). Now since these functions are bijections from \( \mathbb{R} \) to \( \mathbb{R} \) we can make the sequence \((f_1^{-1}(1), f_2^{-1}(2), f_3^{-1}(3), \ldots)\), which is obviously in \( \mathbb{R}^N \) while the image is obviously not part of \( l^1(\mathbb{N}) \). Which leads to the desired contradiction. Hence no such lattice isomorphism \( T \) exists. We have proven immediately too that \( \mathbb{R}^N \) is, as a lattice, different from all the other considered spaces.

2. Now consider a lattice isomorphism \( T : l^\infty(\mathbb{N}) \rightarrow l^1(\mathbb{N}) \). We look again at \( T \) by thinking of the functions \( f_1, f_2, \ldots \). Now we notice that for all \( n \in \mathbb{N} \) the set \( \{ n \in \mathbb{N} : f_n(N) < \frac{1}{n} \} \) is infinite, otherwise the set \( \{ n \in \mathbb{N} : f_n(N) < \frac{1}{n} \} \) is finite but then the set \( T \) will be send to an element not in \( l^1(\mathbb{N}) \). But now we know this we can construct \( n_1 \) s.t. \( f_{n_1}(1) < \frac{1}{n_1} \) and if we constructed \( n_1, n_2, \ldots, n_{m-1} \) we construct \( n_m \) s.t. \( n_m > 2^m \) and \( f_{n_m}(m) < \frac{1}{n_m} \). Now we make the element of \( l^1(\mathbb{N}) \) by making a sequence \((a_i)_{i \in \mathbb{N}}\) by:

\[
a_i = \begin{cases} \frac{1}{n_m} & i = n_m \\ 0 & \text{otherwise.} \end{cases}
\]

Because we defined \( n_m > 2^m \) we know that \( \sum_{i=1}^{\infty} a_i < \sum_{i=1}^{\infty} \frac{1}{2^i} < 1 \), hence \((a_i)_{i \in \mathbb{N}} \in l^1(\mathbb{N}) \). But \( f_{n_m}^{-1}(\frac{1}{n_m}) > m \) hence \( T^{-1}(a_i)_{i \in \mathbb{N}} \) is unbounded and not part of \( l^\infty(\mathbb{N}) \), which is a contradiction to our assumption. Hence no such lattice isomorphism exists. Note that we haven proven now too that there is no lattice isomorphism \( T : l^\infty(\mathbb{N}) \rightarrow c_0 \).
(3) Consider a lattice isomorphism \( T: c_0 \rightarrow l^1(\mathbb{N}) \). We claim that there exists an \( \epsilon > 0 \) s.t. \( \sum_{n \in \mathbb{N}} f_n(\epsilon) < \infty \).

Otherwise, we would have \( \sum_{n \in \mathbb{N}} f_n(\epsilon) = \infty \) for all \( \epsilon > 0 \). So we can take \( N_0 = 0 \) and \( N_1 \) s.t. \( \sum_{n \leq N} f_n(1) \geq 1, N_2 \) s.t. \( \sum_{N_1 < n \leq N_2} f_n(\frac{1}{2}) \geq 1 \), continue like this to make \( N_m \) for all \( m \in \mathbb{N} \) s.t. \( \sum_{N_{m-1} < n \leq N_m} f_n(\frac{1}{m}) \geq 1 \). If we now define \( x_n = \frac{1}{m} \) if \( N_{m-1} < n < N_m \), then we see \( \sum_{n \in \mathbb{N}} f_n(x_n) = \sum_{m \in \mathbb{N}} \sum_{N_{m-1} < n \leq N_m} f_n(x_n) = \sum_{m \in \mathbb{N}} \sum_{N_{m-1} < n \leq N_m} f_n(\frac{1}{m}) = \infty \). While \( x \in c_0 \) and \( T(x) \notin l^\infty \), hence we have a contradiction.

But if there exists an \( \epsilon > 0 \) s.t. \( \sum_{n \in \mathbb{N}} f_n(\epsilon) < \infty \). We already have an element not in \( c_0 \) being send to an element in \( l^\infty \). Hence \( T \) cannot be a isomorphism.

(4) Consider a lattice isomorphism \( T: l^1(\mathbb{N}) \rightarrow c_0 \). Now look at \( T(1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots) \) and \( T(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots) \) both sequences must be send to a sequence which has only 0’s from some point onwards. So we can find a \( n \in \mathbb{N} \) s.t. \( f_n(\frac{1}{2^p}) = 0 = f_n(\frac{1}{2^{p+1}}) \) hence \( f_n \) is not a bijection, which is a contradiction to the previous lemma. Hence no such lattice isomorphism can exist.

\[ \square \]

It might not come as a surprise that these spaces are all not lattice isomorphic because we don’t really have a clue yet if there are a lot of lattice isomorphic spaces between sets of integrable functions. But after we had so many negative results it might be surprising that all \( L^p \) spaces are isomorphic too each other.

**Theorem 7.3.** For every \( p \in \mathbb{R} \) there exists a lattice isomorphism \( T: L^1 \rightarrow L^p \).

**Proof.** Let \( p \in \mathbb{R}, p > 1 \). Define a lattice isomorphism \( T: L^p \rightarrow L^1 \) by \( T(f) = f^{(p)} \). For all \( \lambda \in \mathbb{R} \) and all \( f \in L^p \) we then have \( T(\lambda f) = \lambda^{(p)} T(f) \). So we have a strictly increasing bijection \( \omega: \mathbb{R} \rightarrow \mathbb{R} \), namely \( \omega(\lambda) = \lambda^{(p)} \) s.t. \( T(\lambda f) = \omega(\lambda) T(f) \). Thus it is easy to see that we defined a lattice isomorphism. Now it remains to show that \( f \in L^p \) iff \( T(f) \in L^1 \). We do that in the following way. For all \( \lambda \in \mathbb{R} \) and all \( f \in L^p \) we then have \( T(\lambda f) = \lambda^{(p)} T(f) \). Then \( f \in L^p \) iff \( ||f||_p < \infty \) iff \( ||f^{(p)}||_1 < \infty \) iff \( ||T(f)||_1 < \infty \) iff \( T(f) \in L^1 \). Which concludes the proof. \[ \square \]

Note that this lattice isomorphism is not a Riesz isomorphism, because it doesn’t preserve scalar multiplication. As we promised long ago we will prove that this is no Riesz isomorphism if \( p = 2 \) and we are working on the set of the natural numbers.

**Theorem 7.4.** There is no Riesz isomorphism \( T: l^1 \rightarrow l^2 \).

**Proof.** Since Riesz isomorphisms are lattice isomorphism, we may assume that every Riesz isomorphism can be represented by an infinite sequence of strictly increasing bijections \( f_n \) s.t: \( T(\lambda e_n) = f_n(\lambda) e_n \).

Furthermore for Riesz isomorphism we see that the functions \( f_n \) must be linear for all \( n \in \mathbb{N} \). Hence we may actually assume that for every \( n \in \mathbb{N} \) there is an \( \alpha_n \in \mathbb{R} \) s.t: \( T(\lambda e_n) = \lambda \alpha_n e_n \).

Hence the existence of a Riesz isomorphism would give us that for all \( x \in \mathbb{R}^{\mathbb{N}^+} \) we have:

\[ \sum_{n \in \mathbb{N}} x_n < \infty \Leftrightarrow \sum_{n \in \mathbb{N}} (\alpha_n x_n)^2 < \infty. \]

Assume that \( \alpha_n \rightarrow \infty \). Choose a sequence \( p_1 < p_2 < \ldots \) s.t. \( \alpha_{p_i} > i^2 \) for all \( i \in \mathbb{N} \). Construct a element \( x \in \mathbb{R}^{\mathbb{N}^+} \) by:

\[ x_n = \begin{cases} 
  i^{-2} & n = p_i \\
  0 & \text{otherwise.}
\end{cases} \]

Then \( \sum_{n \in \mathbb{N}} x_n = \sum_{i \in \mathbb{N}} i^{-2} < \infty \). While \( \sum_{n \in \mathbb{N}} (\alpha_n x_n)^2 = \sum_{i \in \mathbb{N}} (\alpha_i x_{p_i})^2 \leq \sum_{i \in \mathbb{N}} 1 = \infty \). Which contradicts our assumption. Hence \( \alpha_n \neq \infty \).

Thus there is a \( N \in \mathbb{N} \) and there are \( p_1 < p_2 < \ldots \) with \( \alpha_{p_i} \leq N \) for all \( i \in \mathbb{N} \). But then we construct a element \( y \in \mathbb{R}^{\mathbb{N}^+} \) by:

\[ y_n = \begin{cases} 
  i^{-1} & n = p_i \\
  0 & \text{otherwise.}
\end{cases} \]
Now $\sum_{n \in \mathbb{N}} y_n = \sum_{i \in \mathbb{N}} i^{-1} = \infty$. While $\sum_{n \in \mathbb{N}} (\alpha_n y_n)^2 = \sum_{i \in \mathbb{N}} (\alpha_i y_p)^2 \leq \sum_{i \in \mathbb{N}} A^2 i^2 < \infty$. Which again contradicts the assumption. Hence there is no Riesz isomorphism $T : l^1 \to l^2$. □

To conclude this chapter we note that it might be interesting to investigate lattice isomorphisms between Riesz subspaces of $\mathbb{R}^N$ that are invariant under permutations of $\mathbb{N}$. 

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