On the interpretation of the quantum wave function

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Preface

The reason I choose to study physics was twofold. On the one hand I was looking for a challenge and on the other hand I wanted to understand the world truly. The kind of understanding I had in mind back then is the one that answers the "why questions" such as: 'Why is the sky blue?' and 'Why does an apple fall down from a tree?'. Of course these questions are answered some time ago by Rayleigh and Newton respectively, but in my opinion there will be a replacement in no time: 'Why is Rayleigh scattering happening in all directions?' and 'Why is there gravity?' What I have learned about physics is that it answers "why questions" with a description of "how" things work and there will come into mind another (more specific) "why question". We might find questions that could not be answered eventually, but I do not think this is very disturbing. Rather we would use this kind of questions to motivate ourselves to come closer to the bigger picture, step by step. With that in mind I went looking for a topic for my master's thesis. During the course 'Philosophy for physicists' I encountered my interest in the topics of quantum foundations[1]. I found out that this research field has great challenges which seemed extremely interesting to me and therefore I started my research on it.

Acknowledgements: I started this research initially under the supervision of Dr M. Seevinck. I would like to thank him for our inspiring conversations and his enthusiasm. Moreover my supervisors Prof. R. Kleiss and H. Donker deserve my deepest gratitude. They needed a lot of patience with me and without their support I would not have succeeded in finishing this project. Also I have learned a lot from the discussions we had and therefore I am very grateful. Furthermore I would like to thank Dr F. Filthaut for his willingness to work through my thesis on very short notice. Last but not least, I would like to thank my friends and family, who were there for me when I needed them the most.

Introduction

Since the birth of quantum mechanics the theory has proven to generate reliable experimental predictions. But although the theory has great value from an operational point of view, it is still not understood fully on a conceptual level. Our classical intuition can hardly handle the concepts of quantum mechanics, like entanglement, superpositions of states and the uncertainty principle. Even the status of the wave function is still at question: is it a real property of nature or is it the knowledge one has of nature? If we consider this question in an ontological model then the wave function would be called ψ-ontic and ψ-epistemic respectively. Here an ontological model is a model in which is assumed that there exist ontic states with fixed properties. These ontic states are real physical states.

The interpretation problems of quantum mechanics were open for debate for half a century. The several interpretations were not experimentally verifiable and the foundations of quantum theory were mostly seen as 'metaphysics'. This changed in 1964 when John Bell introduced an ontological model in which he was able to derive the famous Bell inequalities. With this achievement he approached the interpretation problems as physical problems. The 'metaphysics' became even experimentally verifiable. In the Bell Test the 'spooky action at a distance' of entangled states is put to the test. In 1982 the Bell Test was performed successfully and the experimental data confirmed that the action at a distance had taken place[2]. Just recently there was made an improvement on the experiment by which the major loopholes were closed[3].

Even 50 years after Bell's theorem ontological models are still an important tool to make progress in quantum foundations. An important result of the last years is the Pussy-Barrett-Rudolph (PBR) no-go theorem[4] which is an argument in favour of the ψ-ontic view of quantum states and makes use of an ontological model. A version of the PBR no-go theorem is presented in this thesis, but the focus will lie on the area of ψ-epistemic models, more specifically Spekkens' toy theory[5]. With this toy model it is possible to reproduce part of the phenomenology that is typical for quantum theory. The model is built upon epistemic states and therefore this model is seen as an argument for the ψ-epistemic view of quantum states. One of the objectives of my research was to find a way to describe the phenomenology of the Mach-Zehnder interferometer in terms of Spekkens' toy theory.

This brings us to the main research questions addressed in this thesis.

- What do ontological, ψ-epistemic models say about quantum mechanics, especially Spekkens' toy theory?
- How should we interpret the PBR no-go theorem?

And on the experimental side of the field the following question is formulated:

- Can we mimic the phenomenology of the Mach-Zehnder interferometer completely in a version of Spekkens' toy theory?

To answer these questions I will introduce the interpretation problem of the wave function in chapter 1. Here the ontic-epistemic distinction is made more clear. In chapter 2 ontological models are introduced and the PBR no-go theorem is presented. In chapter 3 we will study Spekkens' toy model in depth. Chapter 4 presents the realisation of the Mach-Zehnder interferometer in Spekkens model. Finally, chapter 5 provides the final remarks on the PBR theorem and Spekkens' toy theory followed by the conclusions.

Chapter 1

The wave function and its relation to the real world

Ontic and epistemic interpretations of the wave function

This chapter considers the relation the quantum wave function has to the real physical world. The concepts ontic and epistemic states form the basis of our classification system. In philosophy there are two branches of research that are directly related to ontic and epistemic states, namely ontology and epistemology. *Ontology* is the study of being or reality. Questions like "What is a thing?" belong to the ontological research field. The study of knowledge or *epistemology* comprises questions of a fundamental other kind, for example: "What is the nature of our knowledge of things?". Applying it to the case of the quantum wave function the question becomes: "Is the quantum wave function a real property of nature or is it only the knowledge we have gained from nature?" In the first case the quantum wave function would be 'ontic', in the second it would be 'epistemic'[6].

First we look at classical mechanics and what it means for a classical state to be ontic or epistemic. The example of a classical particle in phase space will be illustrative due to its simplicity. Secondly the concepts are explained in relation to a quantum mechanical system.

1.1 Classical states

In classical physics the interpretation of terms like property, observable and state is in line with common sense and therefore we do not have to worry about them a lot, these "innocuous terms"[6]. When adopted in quantum theory they will demand a more profound approach than is necessary in the classical perspective.

Taking the realistic interpretation into account, particles are seen as classical objects that bear determinative properties. This is again the common sense way of thinking about the world. Here we define realism as follows: physical quantities have values at any time. So the state of a system is determined independently of any observation. Because measurement is not really an issue in classical mechanics, the observer can get the real values of the quantities of the physical system by performing a measurement on it. After measuring the whole set of values that describe a classical state of the system, all future states of the system can be predicted by the information of this set. After all, classical mechanics is assumed to be a deterministic theory and thereby all future states of the system can be derived from the here and now if there is sufficient knowledge of the present state of the system. If something seems to be unpredictable in classical theory then there is a lack of knowledge of the system.

In classical physics we would speak of 'ontic states' and 'epistemic states' when concerning the states of a physical system. Ontic states are those that give a real value of a quantity of the system. In this way there is a one-to-one correspondence between the quantities (such as position and momentum) of the system and the ontic state. An epistemic state on the other hand, does not contain precise information about the system. While the ontic state can be seen as a real state of the physical system, the epistemic state is an imperfect description of it and can be seen only as knowledge of the system.

1.2 A classical particle in phase space

Imagine a classical particle, following the laws of Newtonian physics and located at a certain position in space. All the properties of the particle can be given by a list of values of the physical quantities of the particle: Its position, momentum, mass and charge. This list is the *ontic state* of the particle. Suppose the quantities that specify the properties of the particle are, apart from mass or charge, its position $\mathbf{x(t)}$ and momentum $\mathbf{p(t)}$ and the ontic state of the particle can be illustrated as a point in phase space for any moment in time. In figure 1.1 this 'ontic phase space' is indicated and the ontic state for time t_0 is marked. This example stays close to the approach of Matt Leifer[7, 8].

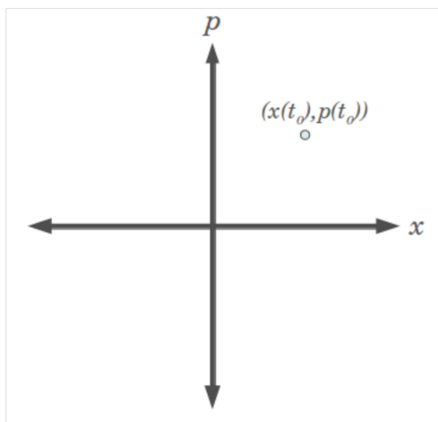

Fig. 1.1: The ontic phase space with the ontic state marked for t_0.[1]

The ontic state, $(x(t_0), p(t_0))$ gives the real values of the quantities of the physical system. In this way there is a one-to-one correspondence between the quantities of the system and the ontic state. This one-to-one correspondence should be understood by looking at the ontic state as an adequate description that could be seen as a probability density function over the ontic state space that takes the form of a Dirac delta function pictured in figure 1.2a. It is impossible to create an overlap between two of these functions (1.2b) and therefore the functions correspond one-to-one with the real states of the system.

[1]Figures 1.1, 1.3, 1.4, 1.7 and 1.8 are originally used in the online article of Matt Leifer[7].

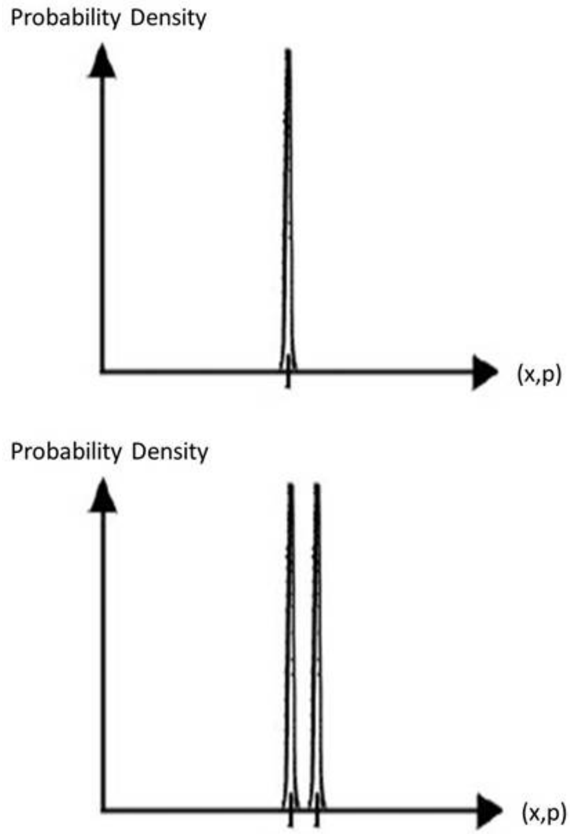

Fig. 1.2a,b: The probability distribution over an ontic state in classical physics. (Both coordinates of the ontic phase space are projected on the x-axis.)

An epistemic state does not contain exact information about the system and represents only the knowledge one has of the system. The corresponding probability densities are far wider than the Dirac delta function type densities that represent the ontic states. It is possible that two epistemic states fit one ontic state of the system as is pointed out in figure 1.3.

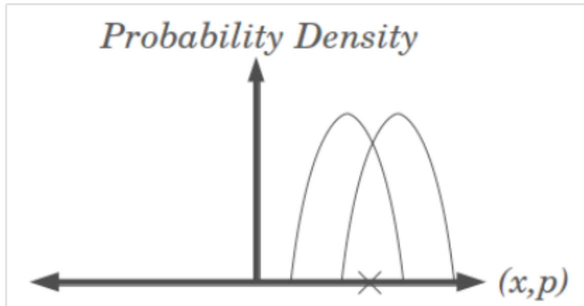

Fig. 1.3: Epistemic states overlap when describing the same ontic state, indicated by a cross.

A well-known example of a classical epistemic state can be found in statistical mechanics. Here probability distributions over the micro states form the best descriptions for macroscopic systems, because the knowledge is incomplete with respect to the underlying micro states. These micro states have a certain ontic state, although the known probability distribution does not provide the exact information to determine which one. Thus, probability distributions in statistical mechanics are epistemic states. In figure 1.4 a probability distribution of a single particle is shown that could be related to the ontic state of figure 1.1. This probability distribution is one example of many that fit the underlying real state.

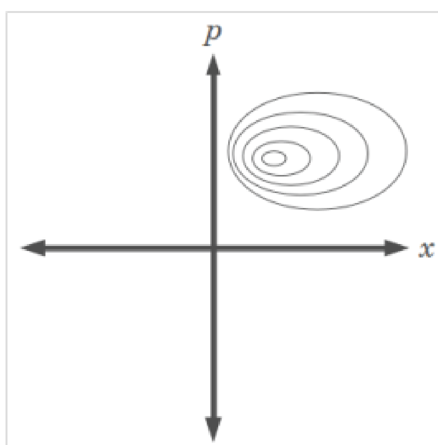

Fig. 1.4: A possible epistemic state of the position and momentum of a classical particle. The picture shows lines of constant probability in the ontic phase space.

A probability distribution over the ontic phase space can include several possible ontic states and therefore it is not possible to determine which specific ontic state is the real one. However as long as the probability distribution has no overlap with another distribution (they do not have any common ontic state) we can still speak about a property of the system. It is incomplete knowledge that we have of the system, but it is related uniquely to a set of ontic states. Let me show this with an example.

1.3 Example of an incomplete ontic state

Suppose you have a cup with a die in it. You shake it and throw the die on the table. The number that lies on top is the number of interest. You do not have any attention for the rest of the die, but you can see the sides and can predict which number is at the bottom. You can even pick it up if you want to and check your prediction. When you do not throw the die on the table, but shake it and let it rest on the bottom of the cup afterwards you can see the top of the die without seeing the other sides. Suppose you see four dots on top, this can be named state 4, or S4. Is it an ontic state? Or an epistemic one? Not all of the information of the system can be listed by the observer, but the most important information can. There are still several possible orientations the die can take, see figure 1.5. Any other state in which we might find the die (S1, S2, S3, S5 or S6) cannot be compatible with any of the states the die has in S4. Therefore we can speak of an incomplete ontic state or supplemented state of the system[9].

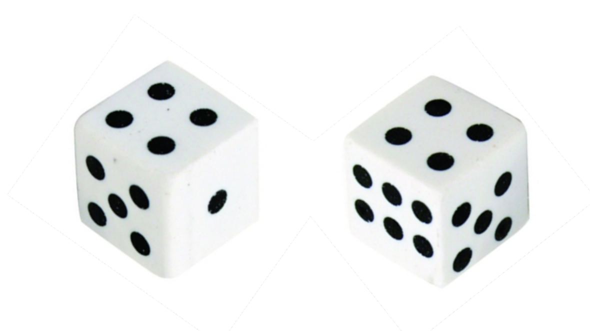

Fig. 1.5: Two dice positions.

1.4 Quantum states

In the case of the wave function, which describes the state of a system, it is not equally evident whether if we have to see it as an ontic or an epistemic entity. According to the Born rule, $|\psi|^2$ is interpreted as a probability function that prescribes the probabilities that a system will be found in a certain state[10]. It does this for several states in which the system could be found when a measurement is performed. The wave function can be seen as the description of the system and is therefore a representation of the quantum state of the system. But what does this imply for the system itself? We will distinguish three possible interpretations. The first is to think of the system as being in one of the specific states, without the observer knowing which one. Then the wave function would be merely a description of the knowledge the observer has. We call this ψ-epistemism. The second is to think of the system as being in all the states at the same time. When the observer measures some observable of the system, the system collapses into the specific state that the observer measures.[2] Thus the wave function can actually be interpreted as a state of reality that corresponds directly with the system. This is called ψ-ontism. Thirdly we can ignore the question completely and use the wave function as a practical tool to predict credible solutions for quantum systems without worrying about the nature of reality. We call this epistemism. The first and second scenario are realist interpretations of quantum theory, the third is the instrumentalist interpretation. Instrumentalists do not reserve a special status to reality as do the realists. They build their theories to predict measurements and use the theory as a practical tool without bothering about the nature of reality itself. In table 1.1 below the interpretations are summarized:

Table 1.1: The three interpretations of the wave function

Status of a measurement outcome	Ψ	**Interpretation**
Quantity of reality	ψ-epistemic	Classic realism
Extorted quantity of reality	ψ-ontic	Quantum realism
Not related to external reality	Epistemic	Instrumentalism

[2] This is called the ‘Collapse of the wave function’. In a way the measurement was extorted by the observer and the interaction between the observer and the system deserves some attention. This dilemma is called the ‘Measurement problem’ and demands an extensive study on its own. For now we just accept that this is one way to look at the world. An interpretation of quantum theory that upholds this idea is the ‘Copenhagen interpretation’.

The realist interpretations are named classic and quantum realism. The first resembles the realist interpretation we know from classical mechanics. In the classical perspective it was the natural way to look at reality as something that is outside and independent of our own observations. In this view the world itself is still classical. Quantum realism on the other hand dictates that the world itself is quantum-like. The ontic-epistemic distinction does not play a role for the instrumentalists, because they see all models as knowledge.

1.4.1 Epistemic and ontic states in quantum theory

The statistical character of quantum theory is the main distinction between quantum and classical theory. The statistical interpretation of the quantum wave function is due to Born[10]. For a wave function that specifies a quantum system we need to keep in mind that this single wave function, from a frequentist point of view, describes an ensemble of independent, identical systems. If we perform identical measurements on all of these systems we find the complete probability distribution and thus the wave function that provides us with an incomplete probabilistic description of the whole ensemble. The prediction that the wave function gives for a measurement outcome of a single system is not one specific value that the system should have, but rather a list of values that could be selected with a certain probability.

The ψ-ontic interpretation of the wave function can be divided again in two. This is due to the fact that the wave function could describe complete states but also incomplete states of physical reality. For the ψ-epistemic interpretation this is not the case, because these states are per definition incomplete. In table 1.2 below the three types of realist interpretations are listed.

Table 1.2: Classes of ontological models for quantum theory

	ψ-incomplete	ψ-complete
ψ-ontic	**ψ-supplemented**	**ψ-complete**
ψ-epistemic	**ψ-epistemic**	

In the ψ-ontic view ontic states are always corresponding to non-overlapping probability distributions and it should be possible to determine the quantum state uniquely when the ontic state is known. In this case the representation of the probability distributions looks very similar to the probability distributions of classical mechanics in figure 1.4. Here the probability distributions $\mu_i(\lambda)$ are functions over the ontic state space Λ that are related to the quantum state of a system, ψ_i. Λ is a measurable space and $\mu_i(\lambda)$ works on the elements $\lambda \in \Lambda$, satisfying: $\mu_i(\Lambda)$=1. In figure 1.7 an example is given of two distributions that represent two quantum states in a ψ-ontic model.

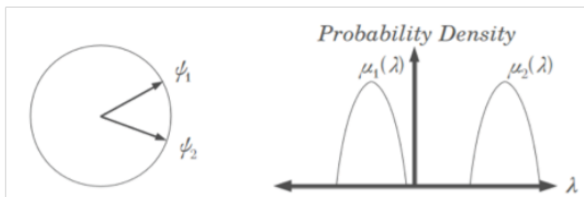

Fig. 1.7: Two quantum states in a ψ-ontic model and their representation in the Bloch sphere on the left.

The ψ-epistemic view requires that at least one ontic state corresponds to more than one quantum state. Such states are represented in figure 1.8.

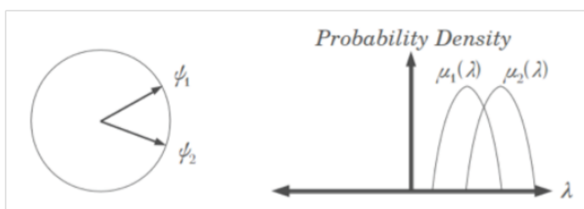

Fig. 1.8: Two quantum states in a ψ-epistemic model

Chapter 2

Ontological models and the PBR no-go theorem

Pusey, Barrett and Rudolph (PBR) published an article about the interpretation of the quantum state in Nature Physics in June 2012[4]. According to several researchers in the field of quantum foundations it presented the most important result in years[11]. In the article a no-go theorem is presented by which the ψ-epistemic view seems to be ruled out by the assumption of realism. The article was originally titled *'The quantum state cannot be interpreted statistically'*[12] but this led to confusion concerning the Born rule, which was never rejected by PBR. Therefore it was publiced in Nature Physics under the title of *'On the reality of the quantum state'*[4].

The no-go theorem makes use of an ontological model. Ontological models are built on the assumption that physical systems described by the model are ontic. This does not say their ontic state could be known. There are ψ-epistemic models as well as ψ-ontic models. These kind of models are in general equivalent to Bell's model that was used to prove his famous equations and are most often called hidden variable theories. Because in most cases the wave function is assumed to represent physical reality when it comes to hidden variable theories we follow Matt Leifer and use the term ontological models[13].

Ontological models can be described in the context of the operational formulation of any theory. This formulation has as primary goal to enable physicists to predict the values of the quantities they measure in experiment. For quantum theory this means that the probabilities for specific outcomes are prescribed by the model when certain measurement and preparation procedures are followed. Therefore the primitives on which the theory is built are preparation and measurement procedures. The following assumption is intrinsic to this formulation of quantum theory: "preparation procedures prepare a system with certain properties and the measurement procedures reveal something about those properties"[9].

The argument and proof of the no-go theorem are presented here in line with Matt Leifer, who published his thoughts about the PBR-result in the newsletter of the *American Physical Society Topical Group on Quantum Information*[8] as well as on his blog[7]. An extensive review article on ψ-ontological theorems is written by the same author[13].

In this chapter an description of an ontological model is given in section 2.1. Secondly, the no-go theorem is introduced by giving a sketch of the argumentation structure in section 2.2. Thirdly, the assumptions are defined in 2.3. Finally a restricted version of the proof is given in 2.4.

2.1 Ontological models

The most general notion of an ontological model will consist of the description of some preparation and measurement procedure with respect to the ontic state space. As we have seen in section 1.4.1, the ontic state space is a measurable space on which the probability distributions $\mu_i(\lambda)$ are defined. The ontic states are denoted with $\lambda \in \Lambda$ and $\mu(\Lambda)$=1. In an ontological model of quantum theory the preparation of a system in a state ψ implies that λ is sampled from the probability distribution $\mu_\psi(\lambda)$ that depends on ψ[7]. So the model describes states of reality that are not necessarily described directly in quantum theory. Measurements on the quantum system ψ will typically be represented by a set of functions $\xi_k^M(\lambda)$ that describe the probabilities of finding certain outcomes (k) in the case of a measurement M at the time that the system is in the ontic state λ. A logical consequence is that $\sum_k \xi_k^M(\lambda) = 1$ for all $\lambda \in \Lambda$. This means that the measurement could be stochastic.[1]

[1]On the other hand, if we were to restrict ourselves to the case for which there is a k$_i$ for each λ_i such that $\xi_{k_i}^M(\lambda_i) = 1$ for all i then there would be a direct relation between the ontic state space Λ and the set of measurement outcomes. This situation would be deterministic, but we will not restrict our model to this.

Putting together the prepare and measurement elements of our system we will find the outcome k with probability P(k|M,$|\psi_i\rangle$)=$\int_\Lambda \xi_k^M(\lambda)\mu_i(\lambda)d\lambda$ when the system is prepared in the quantum state $|\psi_i\rangle$. This probability has to be equal to the one that is given by the Born rule, assuming that we want to let our model coincide with quantum theory.

As we have seen in chapter 1, there are three types of ontological models. ψ-complete models, ψ-supplemented models and ψ-epistemic models. For these three cases the ontic state space is different. In table 2.1 below is shown how it is related to the projective Hilbert space, $\mathcal{PH}$, of the described system. Examples of each type of ontological model are also listed. These models are used throughout the literature and are presented very well elsewhere[9, 13]. Spekkens' toy theory, that is described in chapter 3 is another example of an ψ-epistemic model.

Table 2.1: The type of state space of ontological models

Type of model:	Relation $\mathcal{PH}$ to Λ	Examples
ψ-complete	$\mathcal{PH} = \Lambda$	Beltrametti-Bugajski model
ψ-supplemented	$\mathcal{PH} \subset \Lambda$	Bell-Mermin model
ψ-epistemic	$\mathcal{PH} \nsubseteq \Lambda$	Kochen-Specker model

2.2 The structure of PBR's no-go theorem

At this point we will consider the PBR no-go theorem. In this section we focus entirely on the structure of the proof. In the next paragraph the assumptions that are mentioned here are worked out in greater detail.

The no-go theorem states that any realist model in which a quantum state merely represents knowledge about a physical state of a system will predict a contradiction with quantum theory under the assumption that independent prepared systems have independent physical states.

If we work through this rather compact formulation of the no-go theorem we come across several assumptions PBR make to derive their result. The first assumption that is mentioned in the formulation above is *Realism*. The second is clearly *ψ-epistemism* which prescribes that the

quantum state only can be interpreted as knowledge of the physical state of the system. An additional *Preparation independence* assumption is required to complete the proof of the theorem.
Moreover the theorem is a no-go theorem, which means that it leads to a contradiction. In accordance with this PBR conclude that one of their assumptions is false. Their specific choice is to reject the ψ-epistemic view of quantum states.

Putting all the elements of the proof together we find the following structure:[2]

$$\boxed{\text{Realism}} \wedge \boxed{\Psi\text{-epistemism}} \wedge \boxed{\begin{array}{l}\text{Preparation}\\ \text{independence}\end{array}}$$

$$\lightning \ \text{CONTRADICTION} \ \lightning$$

$$\Downarrow$$

$$\underline{\text{Rejection of } \Psi\text{-epistemism}}$$

Fig. 2.1: The structure of the proof of the PBR no-go theorem.

2.3 Assumptions

This section gives the definitions of the required assumptions for the PBR proof. After the definitions follow their mathematical equivalents which will be used to derive the theorem in the next section.

Definition Q1:
Born's rule states that the probability of finding a specific eigenvalue is equal to the square of the amplitude for that eigenvalue. These probabilities will be normalized to one.

Born's rule could also be marked as one of the assumptions. It is often treated as a postulate of quantum theory, although some physicists try to derive it from more general statements[14]. Either way, Born's rule is

[2]In this chapter the following logical symbols are used: $\wedge$, $\vee$, $\lightning$ and $\neg$. They stand for logical conjunction, disjunction, contradiction and negation respectively.

seen as a certainty of quantum theory. Therefore PBR's acceptance of it as a first principle of quantum theory is reasonable.

The assumptions that are made by PBR are the following:

- **Assumption A1: Realism**
  Isolated, unentangled systems have a real physical state, λ.
- **Assumption A2: Ψ-epistemism**
  The quantum state of a system is a state of knowledge.
- **Assumption A3: Preparation independence**
  Systems that are prepared independently have independent physical states.

Now the assumptions are labelled we can schematically present the PBR argumentation as follows:

Assumptions & quantum theory → Contradiction → Rejection of one of the assumptions

A1 ∧ A2 ∧ A3 ∧ Q1 $\longrightarrow$ ↯ $\longrightarrow$ ¬A1 ∨ ¬A2 ∨ ¬A3

2.3.1 Mathematical equivalents

The next step towards the proof of PBR is making the definitions of the assumptions more applicable. We have to make the translation to the mathematical language used in the proof.

The Born rule says that the probability to find k as the outcome of measurement M given state $|\psi\rangle$ is P(k|M,$|\psi\rangle$) ≡ $\langle\psi| M_k |\psi\rangle$ with M_k the projector that represents the measurement on $|\psi\rangle$ with result k. These probabilities will be normalised to 1.

A1: The realist assumption is the main assumption that PBR make and it includes the realist framework in which their theory is embedded. It says that there is an underlying physical state for every system. PBR demand this specific assumption only for isolated (i) and unentangled (ii) systems, which makes the proof slightly more applicable to general systems.
In their words: "a system has a 'real physical state' - not necessarily

completely described by quantum theory, but objective and independent of the observer."[4]

A2: The ψ-epistemic assumption means that for distinct quantum states $|\psi_0\rangle$ and $|\psi_1\rangle$ the related probability functions $\mu_0(\lambda)$ and $\mu_1(\lambda)$ will overlap. "Either quantum state results in a λ from the overlap region Δ with probability at least q."[4] Here $q > 0$.

A3: The preparation assumption says that the probability distributions that represent the quantum states can be written as $\mu_0(\lambda)\mu_0(\lambda')$, $\mu_0(\lambda)\mu_1(\lambda')$, $\mu_1(\lambda)\mu_0(\lambda')$ and $\mu_1(\lambda)\mu_1(\lambda')$. Thus the distributions of both systems are independent of one another.

2.4 The proof

The contradiction with Born's normalisation condition can be derived from a specific set of quantum states. To illustrate the proof of PBR in a transparent way we will confine ourselves to a restricted version of the proof. The full proof can be found in the original article[4].

In the simplified version of the proof we start with a Hilbert space with basis $|0\rangle, |1\rangle$ and define the following states:

$$|\psi_0\rangle = |0\rangle, \tag{2.1}$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle, \tag{2.2}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle, \tag{2.3}$$

such that $|\langle\psi_0|\psi_1\rangle| = \frac{1}{\sqrt{2}}$.

Here the assumption of ψ-epistemism (A3) is made and says in this case that the probability distributions related to $|\psi_0\rangle$ and $|\psi_1\rangle$ will overlap. Thus for the distributions $\mu_0(\lambda)$ and $\mu_1(\lambda)$ there is an overlap region Δ that contains at least one common ontic state $\lambda_i \in \Delta$.

The argument goes as follows: suppose there is a measurement M working on a quantum system in a state $|\psi_A\rangle$ and there is an outcome k with zero probability. Or in terms of the Born rule: P(k|M,$|\psi_A\rangle$)=0.

Then $\xi_k^M(\lambda_i) = 0$ for all λ_i for which $\mu_A(\lambda_i) \neq 0$. This is easy to see if we realise that:

$$P(k|M\,|\psi_A\rangle) = \int_\Lambda \xi_k^M(\lambda)\mu_A(\lambda)d\lambda. \tag{2.4}$$

For completeness the probability distributions are shown in figure 2.2. The overlap region is indicated, Δ.

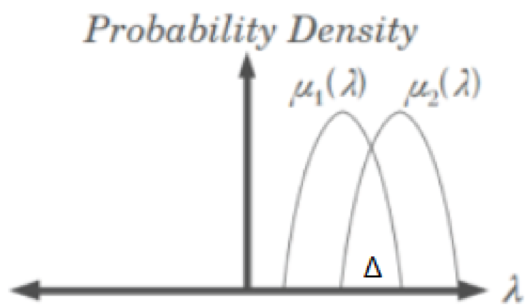

Fig. 2.2: Two probability functions that represent quantum states

Suppose now that $|\psi_A\rangle$ is prepared in $|\psi_0\rangle$ or $|\psi_1\rangle$ with equal probability. If we then can find a measurement on $|\psi_A\rangle$ with two outcomes, 1 and 2, that assign 0 to $|\psi_0\rangle$ and $|\psi_1\rangle$ respectively, then $\xi_1^M(\lambda) + \xi_2^M(\lambda) = 0$ and we have derived the contradiction with the normalisation condition $\sum \xi_k^M(\lambda) = 1$.

To find such a measurement we need another duplicate of the quantum system, $|\psi_B\rangle$. The two systems, $|\psi_A\rangle$ and $|\psi_B\rangle$ have to be prepared independently of each other in one of the states $|\psi_0\rangle$ and $|\psi_1\rangle$. This means we are dealing with two identical systems, with similar ontic state spaces. For both we assume that the ontic states λ_A and λ_B are from the overlap region of the corresponding probability distributions, $\mu_0(\lambda)$ and $\mu_1(\lambda)$. Then the combined system will be in one of the four[3] quantum states:

$$|\chi_1\rangle = |0\rangle \otimes |0\rangle \tag{2.5}$$

$$|\chi_2\rangle = |0\rangle \otimes |+\rangle \tag{2.6}$$

$$|\chi_3\rangle = |+\rangle \otimes |0\rangle \tag{2.7}$$

$$|\chi_4\rangle = |+\rangle \otimes |+\rangle \tag{2.8}$$

[3]In contrast with the four state case it is possible that the combined system is in one of two quantum states. This happens when one of the ontic states is not from the overlap region. For example the combined system will be found in $|+\rangle \otimes |0\rangle$ or $|+\rangle \otimes |+\rangle$ if $|\psi_A\rangle = |+\rangle$.

The preparation of both subsystems and a possible measurement on the combined system are visualised in figure 2.3. There are two identical devices that prepare two quantum systems independently. Both systems will be in $|0\rangle$ or in $|+\rangle$ after the preparation.

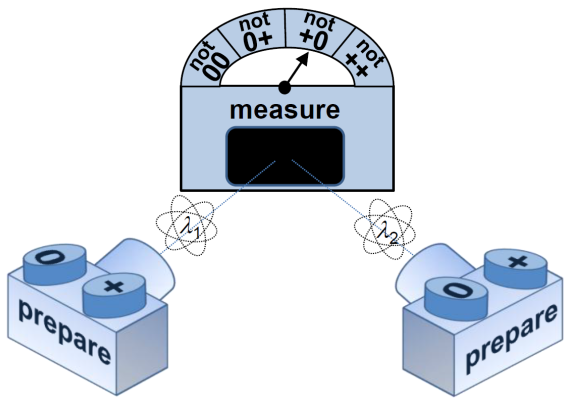

Fig. 2.3: The preparation of the system.[4]

Now we can perform a measurement that projects the two quantum states on the following basis:

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle) \tag{2.9}$$

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |-\rangle + |1\rangle \otimes |+\rangle) \tag{2.10}$$

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}(|+\rangle \otimes |1\rangle + |-\rangle \otimes |0\rangle) \tag{2.11}$$

$$|\phi_4\rangle = \frac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle) \tag{2.12}$$

Performing this measurement on the set of states $|\chi_i\rangle$ we find the following amplitudes:

$$\langle\phi_1|\chi_1\rangle = 0, \qquad \langle\phi_1|\chi_2\rangle = \tfrac{1}{2}, \qquad \langle\phi_1|\chi_3\rangle = \tfrac{1}{2}, \qquad \langle\phi_1|\chi_4\rangle = \tfrac{1}{\sqrt{2}} \tag{2.13}$$

$$\langle\phi_2|\chi_1\rangle = \tfrac{1}{2}, \qquad \langle\phi_2|\chi_2\rangle = 0, \qquad \langle\phi_2|\chi_3\rangle = \tfrac{1}{\sqrt{2}}, \qquad \langle\phi_2|\chi_4\rangle = \tfrac{1}{2} \tag{2.14}$$

$$\langle\phi_3|\chi_1\rangle = \tfrac{1}{2}, \qquad \langle\phi_3|\chi_2\rangle = \tfrac{1}{\sqrt{2}}, \qquad \langle\phi_3|\chi_3\rangle = 0, \qquad \langle\phi_3|\chi_4\rangle = \tfrac{1}{2} \tag{2.15}$$

$$\langle\phi_4|\chi_1\rangle = \tfrac{1}{\sqrt{2}}, \qquad \langle\phi_4|\chi_2\rangle = \tfrac{1}{2}, \qquad \langle\phi_4|\chi_3\rangle = \tfrac{1}{2}, \qquad \langle\phi_4|\chi_4\rangle = 0 \tag{2.16}$$

[4]This picture is taken from the PBR article[4].

Thus we see that $|\phi_i\rangle$ is orthogonal to $|\chi_i\rangle$ for i=1,2,3,4 and thus if the two quantum systems have both an ontic state from the overlap region of their probability distributions there will be a contradiction with Born's rule. In order to maintain realism it is necessary to make a compromise. Looking back at our schematics of the theorem, it is obvious one assumption has to go:

$$\text{A1} \wedge \text{A2} \wedge \text{A3} \wedge \text{Q1} \longrightarrow \lightning \longrightarrow \neg\text{A1} \vee \neg\text{A2} \vee \neg\text{A3}$$

PBR choose to renounce the ψ-epistemic assumption (A3). Thus according to them there can not be any overlap between the probability distributions representing $|0\rangle$ and $|+\rangle$ in the model after all.

Chapter 3

Spekkens Toy Theory

The toy theory that R.W. Spekkens proposed in 2007 reproduces several quantum phenomena[5]. It is a toy theory in the sense that it does not completely coincide with quantum theory. It is a non-contextual theory, where quantum theory is contextual as was pointed out by Kochen and Specker[15]. A non-contextual theory is one for which measured values of a system are independent of any measurement context. Furthermore it is a local theory and does not coincide with the Bell inequalities[16]. An important motivation for the investigation of such theories is to get a better understanding of quantum theory, which could be attained by investigating theories that do not have to fulfill all the restrictions of quantum theory. The toy theory is completely derived from the so called *knowledge balance principle* and all the states described in the theory are epistemic.[1] Therefore it is an argument in favour of the epistemic view. Because a lot of quantum phenomena have an equivalent component in the toy theory it could be seen as a first step towards a quantum theory built on epistemic states. Moreover it is an attempt to find general principles from which quantum theory can be derived.

First, the description of the knowledge balance principle is given from which the theory can be derived. The elementary system that is the main building block of Spekkens' toy theory and will follow in a natural way from the principle. Secondly, we will see how the model can be extended by taking multiple copies of the elementary system. Finally is shown how

[1] As is discussed in chapter one, an epistemic state is a "state of knowledge" in the sense that it gives information about the state of a system, but does not provide the specific ontic state. An ontic state is a "state of reality" which means that it is a complete specification of the properties of a system.[9]

the toy theory reproduces some of the behaviour of quantum states.

3.1 The knowledge balance principle

The knowledge balance principle states that at most half of the knowledge of a system can be obtained at any time. This means that an observer with maximal knowledge of the ontic state of a system will lack the same amount of knowledge with respect to this ontic state. It is introduced here as a rule of Spekkens' toy model and from the principle the model can be derived.

The following definition is the one used by Spekkens[5]:
"If one has maximal knowledge, then for every system, at every time, the amount of knowledge one possesses about the ontic state of the system at that time must equal the amount of knowledge one lacks."

Coin flipping

We start with an easy example that does not necessarily satisfy the knowledge balance principle, but it is illustrative to get a feeling of the concepts that are involved: Suppose we flip a coin. The possible outcomes are heads and tails. The system will be definitely in a determined state shortly after the preparation is performed on the system. We can look to the outcome, asking ourselves if it is heads? Or if the outcome is tails? The answer on one of the questions specifies the result completely. If we got an answer on one of the questions we already know all there is to know about the ontic state of the system. Thus the knowledge we have of the system is maximal and equals the ontic state of the system. This example is therefore not compatible with the knowledge balance principle.

Suppose we extend our system with a second coin. Now we get four possible ontic states in total, which is shown in table 3.1.

Table 3.1: Ontic states of a system with two coins

	T_1	H_1
H_2	**TH**	**HH**
T_2	**TT**	**HT**

The ontic state of the system can be found by answering the following set of questions:

A) Is the outcome of coin 1 heads?

B) Is the outcome of coin 2 heads?

If we demand that the knowledge balance principle holds for this specific system the direct implication of getting the answer on one of these questions is that it will be impossible to get an answer on the other one. It is hard to imagine a classical mechanism for these two coins that exactly fulfils this requirement.

It is also possible to use an alternative set of questions that does not always specify the state of one of the coins completely:

A') Are the outcomes of coins 1 and 2 identical?

B) Is the outcome of coin 2 heads?

If the answer on A' is answered positively, then the answer of question B will stay unknown. Consequently the ontic state of the two-coin system is HH or TT. There is no information available of the individual systems, but one still has maximal knowledge of the complete system, if one adheres the knowledge balance principle.

So far we have seen that we need a system with more than two ontic states if it has to satisfy the knowledge balance principle. This is because the minimal set of yes/no questions that can determine the ontic state of the system completely then has at least two elements. This minimal set of questions will be called a *canonical set*. If the principle holds, an observer with maximal knowledge of the system will have the answers on half of the questions in a canonical set. A logical consequence is that the canonical set of a system should consist of an even number of yes/no questions, N, and the number of ontic states of the system, X, will be 2^N.

One might think a canonical set could work for systems with less ontic states than 2^N. We need a minimum of ontic states to ensure that our set of yes/no questions is indeed a canonical set. For N questions in a canonical set we need $X \geq 2^{N-1}+1$ ontic states otherwise there will be a set with N−1 questions that can function as our canonical set.
As an example we look to N=2, X=3. The problem that now emerges is that specific answers on the first question will specify the answer on the second question, because we already know that the amount of ontic

states is limited relatively to the amount of knowledge that our canonical set can provide. (State 4 does not exist in our system.)

In table 3.2 it is shown how this works out for the canonical set $\{(1 \vee 2), (2 \vee 3)\}$.[2] A negative answer on one of the questions automatically answers both questions, because there is no state associated with double negation.

Table 3.2: Answers on a canonical set for system with 3 ontic states.

	1	*2*	*3*
$1 \vee 2$	**Y**	**Y**	**N**
$2 \vee 3$	**N**	**Y**	**Y**

So to fulfil the knowledge balance principle completely the following requirements have to be satisfied:

1. the canonical set of questions has an even amount of elements, N
2. the system has 2^N ontic states

3.2 Spekkens' Toy Bit

Here I will introduce the elementary system used in Spekkens' toy theory. Suppose we take a system with two degrees of freedom each containing two states, then we have four ontic states in total. We can label the states of the system as (x,y) with x, y '+1' or '−1', for short: '+' or '−'. This gives us the ontic states shown in figure 3.1[13].

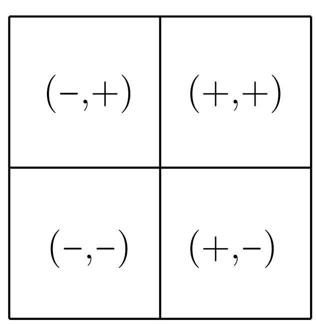

Fig. 3.1: The ontic states of Spekkens' toy bit

[2]Here $\vee$ stands for logical disjunction.

As we have seen in section 3.1 this is the smallest system that can obey the knowledge balance principle. Therefore the system with 4 ontic states is called an *elementary system*.[3]
This elementary system has 6 epistemic states:
$|x+) : (+,+) \vee (+,-)$, $|x-) : (-,+) \vee (-,-)$,
$|y+) : (-,+) \vee (+,+)$, $|y-) : (-,-) \vee (+,-)$,
$|z+) : (+,+) \vee (-,-)$, $|z-) : (-,+) \vee (+,-)$.
In figure 3.2 these epistemic states are indicated on the left by the gray areas, which are nothing more than probability distributions over the ontic state space. The gray squares denote a chance that adds up to one for each state. These states are states of maximal knowledge and can not be written as convex combinations of other epistemic states, so we call them *pure states*. Here a convex combination of two states $|\alpha)$ and $|\beta)$ is defined as $|\alpha) \vee |\beta)$.

When an observer has no information about the system at all, the best description that the model gives of the system is the *maximally mixed state*: $|I/2) \equiv (-,-)\vee(-,+)\vee(+,-)\vee(+,+)$. This is the initial state of the system after preparation by a random selection of an ontic state and is shown on the right in figure 3.2. In the example of the two coin system this simply means that we flip both coins without looking at the outcomes. The system is then in a maximally mixed state and the system has a probability of 1/4 to be in one of the four ontic states. To get the system in another state we have to prepare it by taking a measurement.

[3] If we compare this elementary system with the example of two coins we see it is in fact the same. Just relabel the ontic states from the coin example in the following way: H becomes a plus sign and T a minus sign. The (x,y) notation makes clear we can see this as the x/y coordinates of our system.

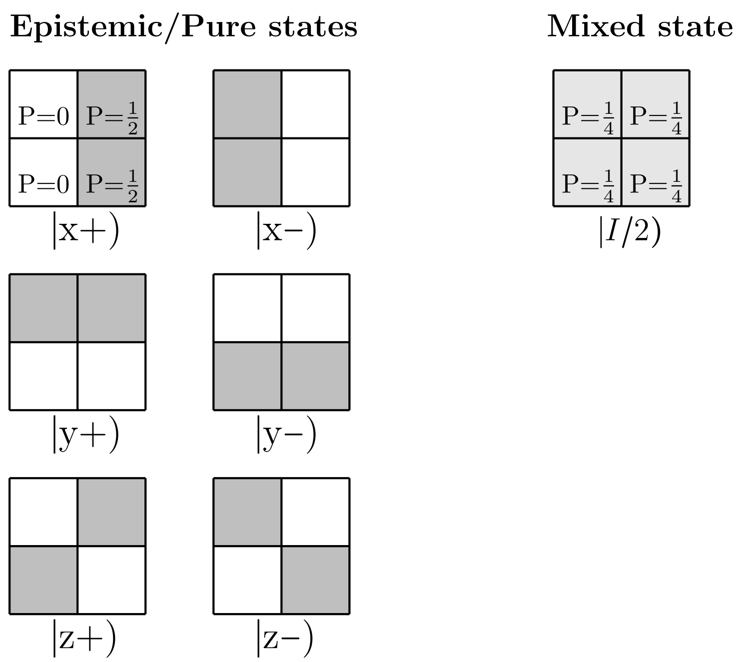

Fig. 3.2: The states in Spekkens' toy bit model

Measurements

The measurements we can perform on the elementary system are illustrated on the right in figure 3.3. After a measurement the system will be found in one of the two epistemic states on the left in the figure of the specific measurement.

The X measurement observes the x-coordinate of the system. Is it positive ($|x+)$) or negative ($|x-)$)? Equally we can look to the Y measurement as the observation of the y-coordinate. For the Z measurement this is somewhat different because our system has only two dimensions. The measurement in the Z direction measures the correlation between X and Y: Z = XY. So for x = y, z = + and for x ≠ y, z = −.[4] The measurements that can be performed on the system have to be repeatable and have to respect the knowledge balance principle. X, Y and Z are the only measurements that satisfy these conditions.

[4]Remember that with +/− is meant +/−1. Then the values of a Z measurement follow immediately for the various ontic states.

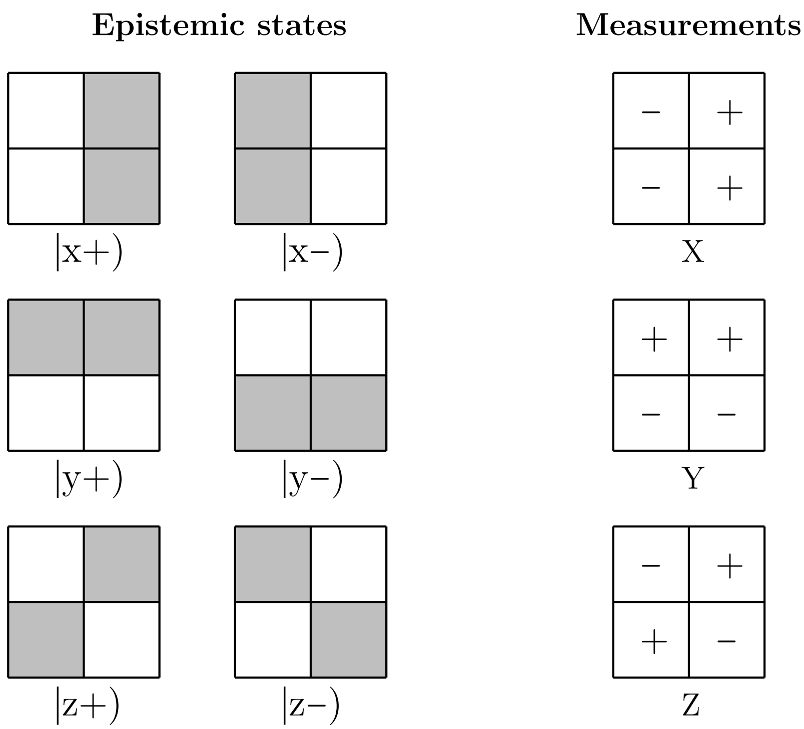

Fig. 3.3: The states and measurements in Spekkens' toy bit model

Permutations of the ontic states

When we perform a second measurement in another direction onto the system there needs to be a disturbance of the ontic states of the system. Suppose we measure in the X direction and we find the system in |x−). We could take a second measurement in this direction and it will give the same result. But if we perform a measurement in the Y direction we find the system in $|y+)$ or in |y−). If it is $|y+)$, then we know that the ontic state of the system was $(-,+)$ at the moment we just made the X measurement. This seems to violate the knowledge balance principle, but in fact it does not. The specific ontic state we found is the state the system had at the moment that the previous measurement was performed. It does not mean we know the ontic state in the present. To ensure we can not find the ontic state a permutation of the ontic states is needed during every measurement in such a way that the probability of finding the system in a specific ontic state never exceeds one half. We can think of four types of permutations: (123)(4), (12)(3)(4), (12)(34) and (1234) where the numbers 1-4 denote the ontic states $(-,+)$, $(+,+)$, $(-,-)$ and $(-,+)$ respectivily. With (123)(4) is meant the permutation of state 1 to state 2, state 2 to state 3 and state 3 to state 1. The cyclic notation that is used here is a standard way of writing the elements of the permutation group S_n[17].

At least the disturbance should work on state 1. But also on state 3, because the disturbance has to be the same every time the Y measurement is made and the initial state of the system might be $|x+)$, $|z+)$, $|z-)$ as well. Thus for every measurement outcome the ontic states of the system will be disturbed in such a way that the probabilities of finding the system in one of the ontic states that are described by the resulting epistemic state are each one half. This can be done by selecting one of the permutations (13)(2)(4) and (1)(2)(3)(4) randomly. Alternatively the permutations (13)(24) and (1)(3)(24) could be selected because here only the permutations on the ontic states 1 and 3 are of importance.

3.3 Multiple bits

One of the assumptions of Spekkens' toy theory is that "every system is built of elementary systems." So we can extend our model by adding more elementary systems. Here we start by looking at a system of two elementary systems.

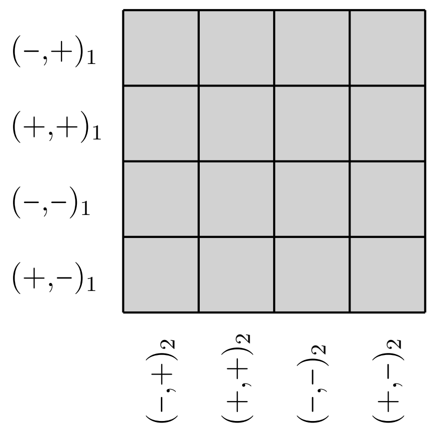

Fig. 3.3: The ontic states of a product of two toy bits

Both systems have four ontic states $(x, y)_{1/2}$, with x/y + or −. When we combine this, $(x, y)_1 \otimes (x, y)_2$, we have got sixteen states in total. In figure 3.3 these ontic states are shown. The gray squares indicate the maximally mixed state of the system if there is no measurement performed on one of the subsystems. Then the system has a probability of 1/16 to be in one of the ontic states.

The pure epistemic states of this system can be divided in two groups. One group of uncorrelated epistemic states and one of correlated epistemic states. On the left in figure 3.4 is an uncorrelated pure epistemic state ($|x+)_1 \otimes |x-)_2$) visualized and on the right a correlated one ($|I/2) \otimes |I/2)$). If you look to one of the subsystems of the first you have maximal knowledge of it. In the second group of states this is not the case. You have maximal knowledge of the combined system, but there is a lack of knowledge relative to the individual systems.

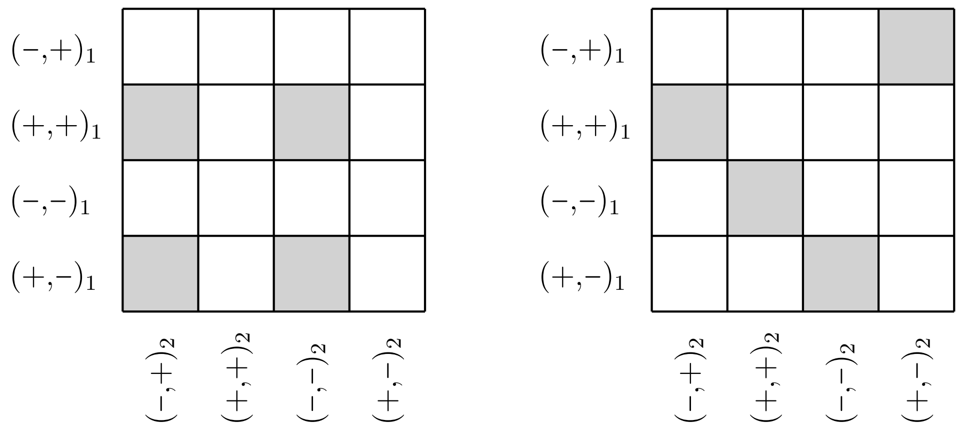

Fig. 3.4: An uncorrelated and a correlated pure epistemic state

Configurations that are not permutations of the uncorrelated and correlated examples above do not respect the knowledge balance principle. The epistemic states in figure 3.5 are two examples of states that violate the principle in some sense.[5]

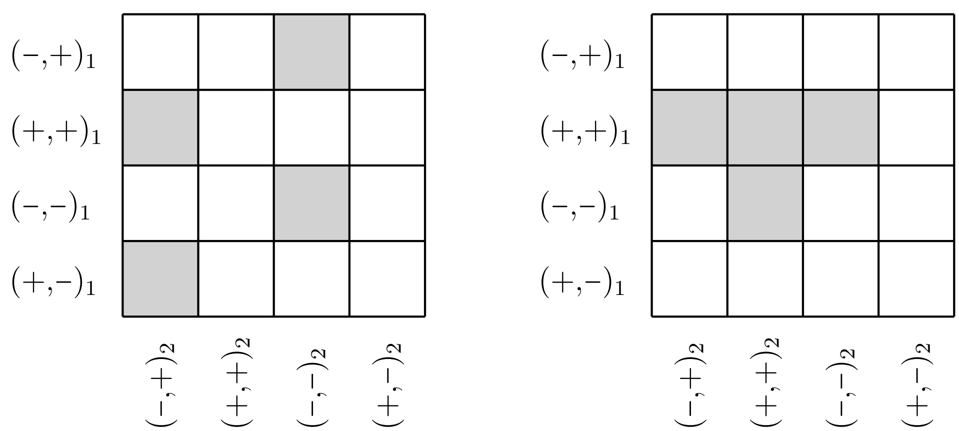

Fig. 3.5: Two states that disobey the knowledge balance principle.

Furthermore, we can combine more elementary systems to create a larger phase space. For example the combination of three elementary systems. It works roughly the same as in the extension made above. But an important distinction is that we will find more types of epistemic states than in the two-dimensional example above. We still have uncorrelated states and correlated states, but the latter can be divided in

[5]In Spekkens' article[5] on pages 12 till 15 this part of the model is worked out completely.

two categories: the triplet-correlated states and pair-correlated states. The first type means all the bits are correlated with each other. For the second type only two of the three bits are correlated. So this can be seen as a product state of a two-dimensional correlated state and a one-dimensional epistemic state.

3.4 Parallels with quantum theory

The epistemic states in the toy theory resemble quantum states.

The system of a spin-1/2 particle can be described in the model as follows: $|x\pm\rangle$, $|y\pm\rangle$, $|z\pm\rangle$ are identified with the states $|x\pm)$, $|y\pm)$, $|z\pm)$ in our model and the Pauli observables σ_x, σ_y, σ_z are represented by the observables X, Y and Z[13, 5].

3.4.1 Convex combinations

A pair of epistemic states can form a convex combination (or incoherent superposition) and is the union of two epistemic states. We will denote it as '$+_c$'. The pair of epistemic states should obey the following demands:
I: The epistemic states must be disjoint, which means that there are no ontic states in the support of both epistemic states.
II: The union of their ontic basis must form the ontic set of a valid epistemic state.
For the four state phase space this union can then only be the completely mixed state $|I/2)$ and there are only three pairs of epistemic states that can be combined with each other:

$$|I/2) = |x+) +_c |x-) = |y+) +_c |y-) = |z+) +_c |z-)$$

This is similar to the convex decompositions of the completely mixed state (I/2) of a qubit in quantum theory:

$$\mathrm{I}/2 = \tfrac{1}{2}|0\rangle\langle 0| + \tfrac{1}{2}|1\rangle\langle 1| = \tfrac{1}{2}|+\rangle\langle +| + \tfrac{1}{2}|-\rangle\langle -| = \tfrac{1}{2}|i\rangle\langle i| + \tfrac{1}{2}|-i\rangle\langle -i|$$

3.4.2 Coherent superpositions

It is also possible to formulate superpositions in the toy theory of the following form:

$$(a \vee b) +_1 (c \vee d) \equiv a \vee c \tag{3.1}$$

These superpositions select one of the ontic states of both epistemic states which forms a new epistemic state. Some conventions are necessary to let it work properly. It is important to use only epistemic states that are written in a specific order of ontic states. The order that we use here is 1:$(-,+)$, 2:$(+,+)$, 3:$(-,-)$, 4:$(+,-)$. The epistemic states should be written down in this order for all coherent superpositions. Thus $((+,+) \vee (-,+))$ is not correct, but should be written as $((-,+) \vee (+,+))$. Moreover it is only possible to take the superposition of disjoint epistemic states, because then the result will be an epistemic state as well. Logically, there are four types of these superpositions. The other three are:

$$(a \vee b) +_2 (c \vee d) \equiv b \vee d \tag{3.2}$$

$$(a \vee b) +_3 (c \vee d) \equiv b \vee c \tag{3.3}$$

$$(a \vee b) +_4 (c \vee d) \equiv a \vee d \tag{3.4}$$

For the states |y+) and $|y-)$ the four operations give the following outcomes:

$$|y+) +_1 |y-) = |x-) \tag{3.5}$$

$$|y+) +_2 |y-) = |x+) \tag{3.6}$$

$$|y+) +_3 |y-) = |z+) \tag{3.7}$$

$$|y+) +_4 |y-) = |z-) \tag{3.8}$$

The coherent superposition $+_1$ is also given in figure 3.6. This is a visualisation of equation 3.5.

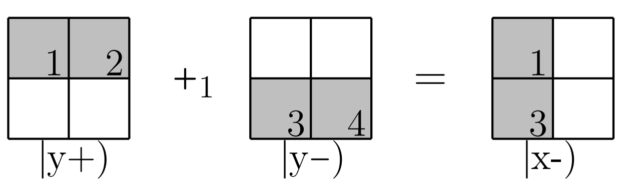

Fig. 3.6: Coherent superposition $+_1$ of the states $|y+)$ and $|y-)$

The four coherent superpositions given above are very similar to the relations of quantum states:

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle \tag{3.9}$$

$$\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle \tag{3.10}$$

$$\frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle) = |i\rangle \tag{3.11}$$

$$\frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle) = |-i\rangle \tag{3.12}$$

The first two operations in the toy theory are commutative, while $+_3$ and $+_4$ are not. For example $|y-)+_3|y+)=|z-)$. This is also the case for the quantum states for which $\frac{1}{\sqrt{2}}(|1\rangle + i\,|0\rangle)$ equals $i\,|-i\rangle$ and not $|i\rangle$.

The resemblance does not work entirely. In the toy theory there are only four coherent operations and in quantum theory there is a continuum of superpositions. Also there is no restriction that says which pair of quantum states may be used for the superpositions. Any pair will give a valid superposition of the initial states. In Spekkens' toy theory this is not the case and we only can combine disjoint epistemic states.

3.4.3 Interference

In quantum theory:
Suppose we prepare an ensemble of systems in the states: $|0\rangle$, $|1\rangle$ and $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$ and then perform the measurement $\{|+\rangle,|-\rangle\}$ on these ensembles. Then we will find that the probability measures for the measurements $\{|+\rangle,|-\rangle\}$ are (1/2,1/2), (1/2,1/2), (1,0) respectively.

In the toy theory:
Prepare $|y+)$, $|y-)$ and $|x-)$ and measure X for all the three states. Again the probability measures we will find are (1/2,1/2), (1/2,1/2), (1,0).

Chapter 4

Mach-Zehnder interferometer in Spekkens toy theory

The Mach-Zehnder interferometer (MZI) is an important experiment that illustrates typical quantum behaviour. For studies on quantum entanglement, quantum information and quantum cryptography, the MZI is a useful tool. We would like to create an equivalent in terms of Spekkens' toy theory. Interference is possible and is already mentioned in the original article by Spekkens. (Section 3.4.1)
It may seem a bit peculiar that an optical setup plays a key role in the understanding of quantum theory. But actually this is the case for many important steps made in foundational quantum research.

4.1 Setup of the Mach-Zehnder interferometer

The MZI consists of a light source, two beam splitters, two mirrors and two detectors that observe the light at the end of both paths. In addition a phase shifter can be placed at one of the beam paths. The first beam splitter can reflect a photon or let it go through. Both paths come across a mirror that brings them together in a second beam splitter. The recombination of both tracks results in the observation of a photon in detector A or B.

The beam splitter in figure 4.1 is the main building block of the Mach-Zehnder interferometer. Mathematically it is described by a matrix M_H which is in fact related to the orientation of the beam splitter in the figure. The orientation is important because of the reflective coating that is added on the lower side of the transparent material. Only reflected light from the lower side will pick up a phase change at the beam splitter because it reflects from a medium with a low refractive index on a medium with a high index. This result can be derived from the Fresnel equations[18] which describe the reflection and transmission behaviour of light at a dielectric.

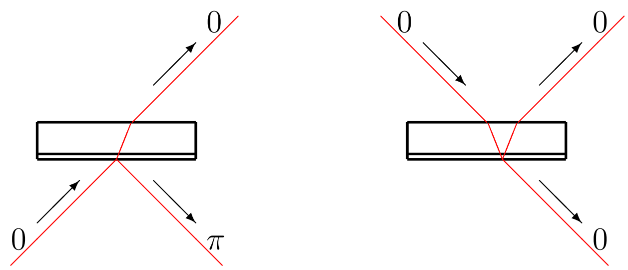

Fig. 4.1: The relative phase shifts at a beam splitter.

The complete Mach-Zehnder interferometer is schematically displayed in figure 4.2 for a photon coming in from the left. We consider the upper trajectory of the light in this case. At the first beam splitter (BS_1) the phase of the photon is unchanged. At the mirror (M_1) the photon picks up a phase shift of π and on the second beam splitter (BS_2) the phase is again unchanged for both the reflection as the transmission case. Thus we find a total phase shift of π for the light that reaches detector A or B relative to the light that entered the MZI.

For the lower trajectory we have a phase change ϕ due to the phase shifter and again π at the mirror (M_2). Following the path to detector A we get an additional phase shift of π which we do not have for the path to detector B.[1]

[1] We should remember that both paths should have exactly the same length. If this is not the case an additional phase shift should taken into account.

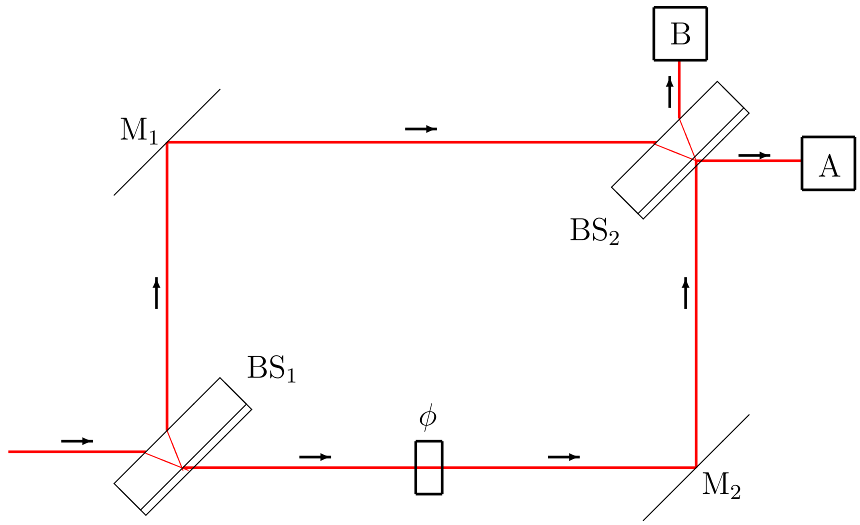

Fig. 4.2: The Mach-Zehnder interferometer

In the lower trajectory a phase shifter is installed. By use of this the constructive interference at detector A may become destructive interference and vice versa at detector B.

If we remove the second beam splitter we will detect the photon with probability 1/2 at detector A and with 1/2 at detector B. This behaviour is exactly what we would expect from a single beam splitter. The state of the photon is in a superposition of the upper and the lower path in the sense that it will take both trajectories 50% of the time. In the case of the full Mach-Zehnder setup we find the photon (for $\phi = 0$) with certainty at detector B. If the photon only moves along one of the trajectories we would again expect to find the photon 50% of the time at detector A.

4.2 Quantum behaviour

The quantum description of the experiment is presented here. We will provide a specific description for the experiment in figure 4.2, with the same orientation of the beam splitters. A photon travelling horizontally will be labelled $|0\rangle$ and a photon moving vertically $|1\rangle$:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4.1}$$

The beam splitter is described by the matrix M_H, the mirrors by M_m and the phase shifter by M_ϕ:

$$M_H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, M_m = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, M_\phi = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.2}$$

The beam splitter brings the system in a superposition of both states $|0\rangle$ and $|1\rangle$:

$$M_H |0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{4.3}$$

$$M_H |1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{4.4}$$

These superpositions are states for which the probability to find the photon in one of the trajectories is 1/2. The minus sign of the second superposition is a phase factor (π) of the horizontal path relative to the vertical path.

As we would expect the mirror just sends $|0\rangle$ to $|1\rangle$ and vice versa. It is an involutory matrix, which means that it is its own inverse. Here the complete behaviour of the mirrors is shown:

$$M_m |0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle, \tag{4.5}$$

$$M_m |1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle \tag{4.6}$$

The phase shifter adds a phase to the lower horizontal path. The order of multiplication is important here to describe the specific setup of figure 4.2. The complete description of the interferometer is given by the matrix $M_{MZI} = M_H M_m M_\phi M_H$

$$M_{MZI} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 - e^{i\phi} & 1 + e^{i\phi} \\ 1 + e^{i\phi} & 1 - e^{i\phi} \end{bmatrix} \tag{4.7}$$

The most general case of the MZI is one in which two photons approach the first beam splitter. One coming from the left and the other one from the lower side. They each have a phase (α and β respectively).

If we let the MZI scheme work on this initial state we will get the following final state out of it:

$$M_{MZI}\begin{bmatrix} e^{i\alpha} \\ e^{i\beta} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{i\alpha}(1-e^{i\phi}) + e^{i\beta}(1+e^{i\phi}) \\ e^{i\alpha}(1+e^{i\phi}) + e^{i\beta}(1-e^{i\phi}) \end{bmatrix} \tag{4.8}$$

From here on we will assume that there is only one photon initially, approaching from the left. The system is then in state $|0\rangle$. In this case the final state of the system in 4.8 will be reduced to:

$$M_{MZI}e^{i\alpha}|0\rangle = \frac{1}{2}e^{i\alpha}\begin{bmatrix} 1-e^{i\phi} \\ 1+e^{i\phi} \end{bmatrix} \tag{4.9}$$

For $\phi = 0$ this will be $|1\rangle$. For $\phi = \pi$ this will be $|0\rangle$ again. In the last case the photon is detected at detector A.

More generally, the phase shift can take the value of any rational number in the interval $[0,2\pi)$ given by the natural numbers k and l: $\phi{=}\frac{2\pi}{k}l$. To describe the general case of the Mach-Zehnder interferometer it is convenient to define the following states:

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, |-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, |i\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}, |-i\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} \tag{4.10}$$

Together with the states $|0\rangle$ and $|1\rangle$ these four states are the major vectors on the three axis of the Bloch sphere. Thus we can describe all possible states of the MZI in terms of these six states.

The states of the Mach-Zehnder interferometer can be divided into five stages:

1. The initial state
2. The transformed state after the first beam splitter
3. The phase shifted state
4. The reflected state
5. The final state

Here the states of the five stages are given in order:

$$|0\rangle \to |+\rangle \to \tfrac{1}{\sqrt{2}}(e^{i\phi}|0\rangle + |1\rangle) \to \tfrac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle) \to \tfrac{1}{2}((1-e^{i\phi})|0\rangle + (1+e^{i\phi})|1\rangle)$$

4.3 States of the MZI in Spekkens' toy theory

To describe the MZI completely in terms of Spekkens' toy theory we need a description of the different physical states and of all the components of the MZI in terms of the theory. The beam splitters, the mirrors and the phase shifter should have an equivalent in the toy model. The description we give here is restricted to the initial and final states of the MZI and to the discrete phases ϕ=0, $1/2\pi$, π and $3/2\pi$.

Table 4.1: States of the MZI for each of the five stages with different values of ϕ. The normalisation factors $1/\sqrt{2}$ (stages 2, 3, 4) and 1/2 (stage 5) are not written down.

	0	$1/2\pi$	π	$3/2\pi$
1	$\vert 0\rangle$	$\vert 0\rangle$	$\vert 0\rangle$	$\vert 0\rangle$
2	$\vert +\rangle$	$\vert +\rangle$	$\vert +\rangle$	$\vert +\rangle$
3	$\vert +\rangle$	$i\vert -i\rangle$	$-\vert -\rangle$	$-i\vert i\rangle$
4	$\vert +\rangle$	$\vert i\rangle$	$\vert -\rangle$	$\vert -i\rangle$
5	$\vert 1\rangle$	$\frac{\vert i\rangle}{1+i}$	$\vert 0\rangle$	$\frac{\vert -i\rangle}{1-i}$

To describe these states in Spekkens' toy theory we will use the six epistemic states that we introduced in chapter 3, figure 3.2:

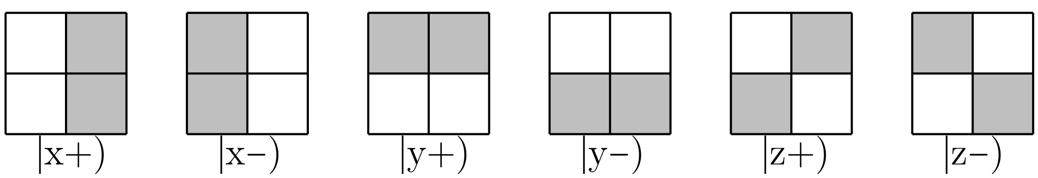

Fig. 4.3: The epistemic states in Spekkens' toy bit model.

We link each of the states from the MZI to an epistemic state in Spekkens' toy model. For example:

$$|x+) \Leftrightarrow |-\rangle \tag{4.11}$$

$$|x-) \Leftrightarrow |+\rangle \tag{4.12}$$

$$|y+) \Leftrightarrow |0\rangle \tag{4.13}$$

$$|y-) \Leftrightarrow |1\rangle \tag{4.14}$$

$$|z+) \Leftrightarrow |i\rangle \tag{4.15}$$

$$|z-) \Leftrightarrow |-i\rangle \tag{4.16}$$

If we consider the possible permutations of ontic states that switch between valid epistemic states, we find the four categories that we have seen already in section 3.2 of the permutation group S_n: (12)(3)(4), (12)(34), (1234) and (123)(4). In figure 4.4 below an example of the permutation (13)(24) is given for which the ontic states 1 and 3 are interchanged and 2 and 4 simultaneously.

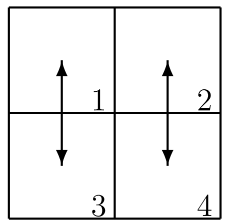

Fig. 4.4: The permutation of the ontic states

This permutation of the ontic states works on the epistemic states in the following way:

$$
\begin{aligned}
|x+) &\rightarrow |x+) \\
|x-) &\rightarrow |x-) \\
|y+) &\leftrightarrow |y-) \\
|z+) &\leftrightarrow |z-)
\end{aligned}
$$

These permutations are equivalent to the following transformations on the states of the MZI and follow directly from equations 5.11 till 5.16:

$$
\begin{aligned}
|-\rangle &\rightarrow |-\rangle \\
|+\rangle &\rightarrow |+\rangle \\
|0\rangle &\leftrightarrow |1\rangle \\
|i\rangle &\leftrightarrow |-i\rangle
\end{aligned}
$$

And this is exactly the behaviour of the MZI for ϕ=0. This means that we have an equivalent in de model for the MZI without considering the phase shift. For a discrete phase shift of π this simple model works also. The related permutation is (1)(2)(3)(4) as is M_{MZI} described by the identity matrix.

For $\phi = 1/2\pi$ the behaviour of the MZI is as follows:

$$
\begin{gathered}
|0\rangle \rightarrow |i\rangle \rightarrow |1\rangle \rightarrow |-i\rangle \rightarrow |0\rangle \\
|+\rangle \rightarrow |+\rangle \\
|-\rangle \rightarrow |-\rangle
\end{gathered}
$$

In terms of Spekkens' toy model version of MZI:

$$|y+) \to |z+) \to |y-) \to |z-) \to |y+)$$
$$|\mathrm{x}+) \to |\mathrm{x}+)$$
$$|\mathrm{x}-) \to |\mathrm{x}-)$$

It is easy to check that there is not a single permutation that covers these state transitions. If we do want to include this phase shift (and others) in our model we need an extension.

The model will be extended with a second elementary system. The total ontic state space of this system has 16 ontic states, which was shown already in figure 3.3. First we need to relate the states of the MZI to epistemic states in the model:

$$|z+) \bigotimes |z+) \Leftrightarrow |+\rangle \tag{4.17}$$

$$|z-) \bigotimes |z-) \Leftrightarrow |-\rangle \tag{4.18}$$

$$|x+) \bigotimes |y-) \Leftrightarrow |0\rangle \tag{4.19}$$

$$|y-) \bigotimes |x-) \Leftrightarrow |i\rangle \tag{4.20}$$

$$|x-) \bigotimes |y+) \Leftrightarrow |1\rangle \tag{4.21}$$

$$|y+) \bigotimes |x+) \Leftrightarrow |-i\rangle \tag{4.22}$$

Then the permutation of figure 4.5 describes exactly the permutation of the ontic states such that the behaviour of the MZI for $\phi = 1/2\pi$ is reproduced. We call this permutation therefore $P_{1/2\pi}$.

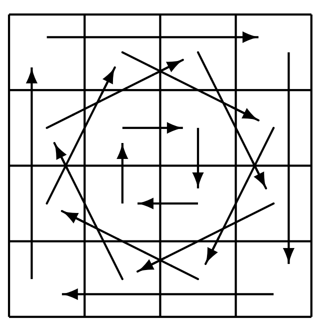

Fig. 4.5: The permutation of the ontic states, $P_{1/2\pi}$.

To complete the model for k=4, we need to include values of ϕ: 0, $1/2\pi$, π or $3/2\pi$. For ϕ = 0 we can take the permutation $P_0{=}{P_{1/2\pi}}^2$. $\phi = \pi$ is again the identity matrix, thus all the ontic states are projected to themselves. For $\phi = 3/2\pi$, $P_{3/2\pi}{=}{P_{1/2\pi}}^3$ will model the transitions of the MZI.

4.4 Future prospects

For other values of ϕ a larger toy state space is needed. For each toy bit that we add to our description we expect that we can describe ϕ more generally. If it is possible to complete the Mach-Zehnder phenomenology completely in terms of Spekkens' toy theory then that would be definitely a good argument to invest more research in ψ-epistemic theories. For such a model an infinite amount of elementary systems will be needed if we want to describe the phase shift in a continuous way. If n is the number of elementary systems we use in our model, then there are 2^n ontic states in the support of an epistemic state that represents the maximal knowledge an observer can have. The total of ontic states in the state space will be 4^n in this case. Therefore the relative amount of knowledge an observer can have of a system will be larger for higher n.

Chapter 5

Conclusion and discussion

In this chapter the final conclusions of this thesis are presented whereafter they are discussed. The research questions that are given in the introduction will be used as starting points:

What do ontological, ψ-epistemic models say about quantum mechanics, especially Spekkens' toy theory?
The fact that the toy theory resembles a great deal of quantum phenomenology could be a hint that quantum states should be interpreted epistemically. It still is a toy theory, but it might pave the way for another ψ-epistemic theory that does actually fulfils all our demands.
The importance of the 'Knowledge balance principle' on which the theory is built seems also interesting. Perhaps there is some general principle that restricts our knowledge of the physical world. Of course we immediately think about Heisenberg's uncertainty principle that does seem like the principle used in Spekkens' theory. But unlike the knowledge balance principle, the uncertainty principle has a physical status.

How should we interpret the PBR no-go theorem?
The PBR theorem has to be seen as a restriction on ontological models. The conclusion that PBR derive from it is not the only way to deal with the theorem. Instead of abandoning the ψ-epistemic assumption one might abandon the preparation independence assumption. In fact this assumption is the one on which PBR received the most criticism[13]. A third and last remedy to deal with the PBR theorem is to abandon realism, but most physicists will not consider this as a serious option.

Can we mimic the phenomenology of the Mach-Zehnder interferometer completely in a version of Spekkens' toy theory?
As we have seen in chapter 4 it is possible to model part of the phenomenology of the MZI in Spekkens' toy theory. To complete the description and include the continuous phase shift further research is needed. This can be done by extending the ontic state space that we use. For every toy bit that we add, we expect to be able to describe the phase shift more precise. For a continuous description we will need to take the limit to infinity. Furthermore it would be interesting to see if it is possible to describe the internal states of the Mach-Zehnder interferometer as well in terms of the toy model.

Halfway my research I was attracted more and more to the ψ-epistemic view. The reason for this preference is for a great part just a matter of intuition. I must say that I like the idea of a theory that is built from basic principles and can be derived completely from them. That is what I appreciate the most of Spekkens' toy theory. But I am also not completely convinced there will exist a theory based only on ψ-epistemic states. However, I think it is appropriate to mention this preference here, because it might have some influence on the way in which I present this section.

It is clear that quantum foundations is still an active field of research. There is progress on both the theoretical as the experimental side of the field. Naturally physical theories need some time to be confirmed by experiments. As we have seen in the introduction the Bell Test was only recently successfully performed. Eventually, these discoveries might be used in practical applications like a quantum computer. It is hard to imagine what kind of applications could be realised with the theories that are developed today, but the research on the interpretation problems of quantum mechanics could definitely lead to new insights comparable with Bell's inequalities. Therefore it is worthwhile to keep trying to answer the open questions in quantum foundations.

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