Non-local Josephson Effect induced by Entangled Bogoliubov-de Gennes Quasiparticles

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1 Introduction

Superpositions of electrons and holes that are called Bogoliubov-de Gennes quasiparticles can emerge as bound states at normal-metal/superconductor interfaces. In the presence of a sufficiently strong magnetic field such quasiparticles can propagate along chiral quantum-Hall edge channels. In this work a beam-splitter quantum-Hall setup is proposed which creates pairs of entangled quasiparticles. Non-local observables are computed that can be used to prove the violation of Bell inequalities in the system and observe the non-local Josephson effect.

In Section 2 and Section 3 the theory required to understand Andreev reflection in the quantum Hall regime is briefly outlined. The proposed beam-splitter setup is outlined in Section 4 together with the derivation of current correlators in the frequency domain which will be used to prove the violation of the Bell inequality later in Section 6. The cause of the violation is due to entangled quasiparticles as discussed in Section 5 and Section 7.
2 Bogoliubov-de Gennes Equations

Superconductors can be described in a mean field approximation using the Bogoliubov-de Gennes Equations [1]. In matrix form they are usually written as

\[
\begin{pmatrix}
H_0 & \Delta \\
\Delta^* & -H_0^*
\end{pmatrix}
\begin{pmatrix}
u_n(r) \\
v_n^*(r)
\end{pmatrix} = E_n
\begin{pmatrix}
u_n(r) \\
v_n^*(r)
\end{pmatrix}
\]  

which is written in electron-hole space. The coefficients \(u_n(r)\) and \(v_n(r)\) form a spinor \(\psi_n\). The single particle Hamiltonians \(H_0\) and \(-H_0^*\) describe electrons and holes respectively and \(\Delta\) is the pairing potential defined by the self-consistency relation:

\[
\Delta(r) = g \sum_n u_n(r)v_n^*(r) \tanh(\beta E_n/2)
\]

which induces a superconducting gap. The Hamiltonian has a particle-hole symmetry, where for every vector \((u_n(r), v_n(r))^\dagger\) with eigen energy \(E\) there is a vector \((-v_n^*(r), u_n^*(r))^\dagger\) with eigen energy \(-E\). The corresponding symmetry can be written as:

\[
\mathcal{H} = -\tau_x \mathcal{H}^\dagger \tau_x.
\]

The matrix \(\tau\) is a Pauli matrix which acts on electron-hole space. Solutions to the Bogoliubov-de Gennes equations are called Bogoliubov-de Gennes quasiparticles (BdG quasiparticles).

Superconductivity in combination with fermionic statistics always allows for the existence of Majorana fermions [2]. This can be seen by introducing a unitary transformation \(\chi\) which brings Equation 2.1 to a purely imaginary form:

\[
\mathcal{H} \rightarrow \tilde{\mathcal{H}} = \chi \mathcal{H} \chi^\dagger, \quad \tilde{\chi} = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}
\]

\[
\mathcal{H} \rightarrow \tilde{\mathcal{H}} = \chi \mathcal{H} \chi^\dagger
\]

\[
\varphi \rightarrow \tilde{\varphi} = \chi \varphi
\]

The direct consequence is that the wave equation

\[
\tilde{\mathcal{H}} \tilde{\varphi} = -i\hbar \frac{\partial}{\partial t} \tilde{\varphi}
\]

is now real valued. Defining a field operator \(\hat{\Psi}\), it necessarily follows that it equals its hermitian conjugate. The creation operators for quasiparticles corresponding to \(\tilde{\mathcal{H}}\) then equal its corresponding annihilation operators.
3 Andreev Reflection

Normal-metal/superconductor interfaces allow scattering processes where electrons from the normal metal enter the superconductor as Cooper pairs, leaving a hole behind. These are called Andreev reflections. The following simple toy model illustrates this. Consider Equation 2.1 with a uniform \( \Delta (\mathbf{r}) \) and no potential energy. The solutions are then easy to find. Starting on the normal metal side with the following Ansatz:

\[
\Psi^e(\mathbf{r}) = C_e \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp (i k_e x + i k_y y) \quad (3.1)
\]

\[
\Psi^h(\mathbf{r}) = C_h \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp (i k_h x + i k_y y) \quad (3.2)
\]

which lead to the following two equations:

\[
\frac{\hbar^2}{2m} (k_e^2 + k_y^2 - k_F^2) = \epsilon \quad (3.3)
\]

\[
-\frac{\hbar^2}{2m} (k_h^2 + k_y^2 - k_F^2) = \epsilon \quad (3.4)
\]

and by writing \( k_e = k_x + k \) and \( k_h = k_y - k \) we can linearize these equations making use of the Andreev approximation \( \epsilon, \Delta \ll E_F \):

\[
\epsilon = \frac{\hbar^2}{2m} (2k_F k) = \hbar v_F k \quad (3.5)
\]

with \( v_F = \hbar k_F / m \) the Fermi velocity. By transforming back into real-space coordinates via a Fourier transform, i.e. by canonical substitution \( k \rightarrow -i \partial_x \), we find a linearized BdG equation:

\[
\begin{pmatrix}
- i \hbar v_F \partial_x & 0 \\
0 & i \hbar v_F \partial_x
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
= \epsilon
\begin{pmatrix}
u \\
v
\end{pmatrix}
(3.6)
\]

which has solutions:

\[
\Psi^e(\mathbf{r}) = \frac{c_e}{\sqrt{k_F}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp (i k_y y + k x) \quad (3.7)
\]

\[
\Psi^h(\mathbf{r}) = \frac{c_h}{\sqrt{k_F}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp (i k_y y - k x) \quad (3.8)
\]

On the superconducting side we consider energies inside the superconducting gap and so we start with the following Ansatz which exponentially decays for \( x > 0 \):

\[
\Psi^5(\mathbf{r}) = \begin{pmatrix} u \\ v \end{pmatrix} \exp (i \mathbf{k} \cdot \mathbf{r} - \alpha x). \quad (3.9)
\]
Making use of the Andreev approximation again we find the following equation for the right side:

\[
\begin{pmatrix}
i \hbar v_F \alpha & \Delta \\
\Delta^* & -i \hbar v_F \alpha
\end{pmatrix}
\begin{pmatrix} u \\
v
\end{pmatrix} = \epsilon
\begin{pmatrix} u \\
v
\end{pmatrix}
\] (3.10)

which has the solution:

\[
\Psi^S(r) = \frac{c_s}{\sqrt{k_F}} \left( e^{i\phi} \sqrt{1 - i \hbar v_F \epsilon} \right) \left( e^{i\phi} \sqrt{1 + i \hbar v_F \epsilon} \right) \exp\left( ik_y y - \alpha x \right),
\] (3.11)

with \( \epsilon = \sqrt{|\Delta|^2 - \hbar^2 v_F^2 \alpha^2} \) and \( \phi = \text{Arg}(\Delta) \) is the superconducting phase. By matching the wave functions \( \Psi_{e,h} \) with \( \Psi^S \) at \( x = 0 \) we find:

\[
c_h = e^{-i\phi} e^{-i\phi_A} c_e
\] (3.12)

with \( \phi_A \) defined as:

\[
\phi_A \equiv \arccos(\epsilon/\Delta).
\] (3.13)

In the limit of \( \epsilon \to 0 \) we find \( c_h = -ic_e \). Since the electron and hole have momentum vectors \( k_e = (k_x + k)\hat{x} + k_y \hat{y} \) and \( k_h = (k_x - k)\hat{x} + k_y \hat{y} \) respectively and because the group velocity \( v_g \) of holes is in opposite direction of \( k_h \), the hole is reflected almost parallel but opposite to the incoming electron (retro-reflection) as illustrated in Figure 1a. This is in contrast with the normal case where reflection only changes the sign of \( k_x \) (specular reflection). We can write down an S-matrix which relates incoming modes with outgoing modes:

\[
\begin{pmatrix}
\Psi_{e,L}^> \\
\Psi_{e,R}^> \\
\Psi_{h,L}^> \\
\Psi_{h,R}^>
\end{pmatrix} =
\begin{pmatrix}
0 & S_{eh}
\end{pmatrix}
\begin{pmatrix}
\Psi_{e,L}^< \\
\Psi_{e,R}^< \\
\Psi_{h,L}^< \\
\Psi_{h,R}^<
\end{pmatrix}
\] (3.14)

where the matrices \( S_{eh,he} \) are given by:

\[
S_{eh} = \begin{pmatrix}
0 & e^{-i\phi} e^{-i\phi_A} \\
e^{-i\phi} e^{i\phi_A} & 0
\end{pmatrix}
\] (3.15)

\[
S_{he} = \begin{pmatrix}
0 & e^{i\phi} e^{-i\phi_A} \\
e^{i\phi} e^{i\phi_A} & 0
\end{pmatrix}
\] (3.16)

In the more general case of Andreev reflection, the interface between normal metal region and the superconductor region is not perfect. That is, electrons and holes can scatter from impurities and defects without being Andreev reflected. Such imperfect interfaces has been studied before using the BTK
approach [3], where a potential energy term in the form of a delta function, \( \Lambda \delta(\mathbf{r}) \), is added to Equation 2.1. We will use a different approach instead. The imperfect interface is modeled by placing an insulating slab between the normal metal and superconductor region forming a NIS junction. The insulating slab has an S-matrix given by:

\[
S_I = \begin{pmatrix} \hat{r} & \hat{t}' \\ \hat{t} & -\hat{r} \end{pmatrix}
\]  

(3.17)

We can use the \( S_I \) and \( S_A \) matrices to construct a transfer matrix \( M \) which relates left going modes to right going modes. The components are given by:

\[
\hat{m}_{ee} = \hat{r} + \hat{t}' \hat{r}_{eh} \hat{r}_{he} \hat{t} + \hat{t} \hat{r}_{eh} \hat{r}_{he} \hat{r}_{eh} \hat{r}_{he} \hat{t} + \cdots
\]

(3.18)

\[
\hat{m}_{eh} = \hat{t}' \hat{r}_{eh} \hat{t} + \hat{t} \hat{r}_{eh} \hat{r}_{he} \hat{r}_{eh} \hat{t} + \cdots
\]

(3.19)

\[
\hat{m}_{he} = \hat{t}' \hat{r}_{he} \hat{t} + \hat{t} \hat{r}_{he} \hat{r}_{eh} \hat{r}_{he} \hat{t} + \cdots
\]

(3.20)

\[
\hat{m}_{hh} = \hat{r} + \hat{t}' \hat{r}_{eh} \hat{r}_{he} \hat{t} + \hat{t} \hat{r}_{eh} \hat{r}_{he} \hat{r}_{eh} \hat{r}_{he} \hat{t} + \cdots
\]

(3.21)
Using the S matrix from Equation 3.17 and setting energy to zero so that \( \exp[-i\phi_A] = -i \) the components of \( \mathcal{M} \) are given by:

\[
\begin{align*}
    m_{ee} &= \frac{2r}{1 + r^2} \\
    m_{eh} &= ie^{-i\phi} \frac{t^2}{1 + r^2} \\
    m_{he} &= ie^{i\phi} \frac{t^2}{1 + r^2} \\
    m_{hh} &= \frac{2r}{1 + r^2}.
\end{align*}
\]  

(3.22)

(3.23)

Using the following substitutions:

\[
\cos \alpha = \frac{2r}{1 + r^2} \quad \sin \alpha = \frac{t^2}{1 + r^2}
\]

(3.24)

and by absorbing the factor \( i \) into \( \phi \), i.e. \( \phi \rightarrow \phi + \pi/2 \), we find the following transfer matrix:

\[
\mathcal{M} = \begin{pmatrix}
    \cos \alpha & e^{-i\phi} \sin \alpha \\
    -e^{i\phi} \sin \alpha & \cos \alpha
\end{pmatrix}
\]

(3.25)

In the presence of sufficiently strong magnetic fields one dimensional chiral edge channels can emerge so that multiple Andreev reflections occur near SN interfaces as depicted in Figure 1b. For a superconductor of width \( L \) and a magnetic length of \( l_B \) the number of Andreev reflections equals \( n \sim L/l_B \). The transfer matrix \( \mathcal{M} \) corresponding to \( n \) such Andreev reflection is given by:

\[
\mathcal{M} \rightarrow \mathcal{M}^n = \begin{pmatrix}
    \cos n\alpha & e^{-i\phi} \sin n\alpha \\
    -e^{i\phi} \sin n\alpha & \cos n\alpha
\end{pmatrix}
\]

(3.26)
Figure 2: Beam-splitter setup which produces and mixes BdG quasiparticles.

4 Colliding Bogoliubov Quasiparticles

Beenakker [4] proposed an experiment to probe the Majorana nature of BdG quasiparticles. These quasiparticles are produced using a NS interface in the quantum Hall regime. Two beams of BdG quasiparticles then enter a beam splitter and the outgoing currents are correlated. The correlation functions strongly depend on the difference of the phase carried the quasiparticles. One of these correlation functions, dubbed the collision term, becomes zero for vanishing phase difference. The result is attributed to the annihilation of Majorana fermions by Beenakker. This section provides an overview of the proposed beam-splitter setup together with a derivation of a frequency dependent correlator function $P(\omega)$ that will be used in the coming sections. The discussion on Majorana fermion annihilation will be left for Section 8.

Let us examine the system depicted in Figure 2. Two normal metal sections are connected in a small region forming a beam splitter. In the quantum Hall regime one dimensional chiral channels form along the edges of the normal metals. In each normal metal region a current is injected into the edge channel. Both currents pass the edge of a superconductor denoted by $S_{1,2}$ which mix the electron and hole degrees of freedom. The currents then enter a beam-splitter, which mixes the two incoming currents. The currents can then be measured and correlated.

We make use of the following correlation function:

$$P(\omega) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dt \exp(i\omega t) \langle I_1(0) I_2(t) \rangle$$

(4.1)

where $I_j$ is the current measured behind the beam splitter at detector j. The
current operator is expressed as

\[ I(t) = e \left( \hat{a}_e^\dagger(t) \hat{a}_e - \hat{a}_h^\dagger(t) \hat{a}_h \right) \]

where we put the creation and annihilation operators in to two vectors:

\[ \hat{a} = \begin{pmatrix} a_e \\ a_h \end{pmatrix} \quad \hat{a}^\dagger = \begin{pmatrix} a_e^\dagger \\ a_h^\dagger \end{pmatrix} \]

The time dependent creation/annihilation operators can be expressed in terms of their energy dependent forms as:

\[ \hat{a}(t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} dE \exp(-iEt) \hat{a}(E) \quad (4.4) \]

and so the correlator Equation 4.1 takes the form:

\[ P(\omega) = G_0 \int_0^\infty dE \int_0^{\infty} dE' \int_0^{\infty} dE'' \int_0^{\infty} dE''' \]

\[ e^{-i(E''' - (E + \omega))t} \langle \hat{b}^\dagger(E') \tau_z \hat{b}(E') \hat{a}^\dagger(E) \hat{a}(E) \rangle \]

\[ = \frac{1}{4} \frac{e^2}{2\pi} \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dE' \int_{-\infty}^{\infty} dE'' \times \langle \hat{b}^\dagger(E') \tau_z \hat{b}(E') \hat{a}^\dagger(E) \hat{a}(E + \omega) \rangle. \]

(4.5)

We want to rewrite this integral in terms of positive energy only. To do so we may make use of the particle-hole symmetry which is expressed as

\[ \hat{a}(E) = \tau_z \hat{a}^\dagger(-E). \]

(4.6)

The correlator then becomes:

\[ P(\omega) = G_0 \int_0^\infty dE \int_0^{\infty} dE' \int_0^{\infty} dE'' \times \langle \hat{b}^\dagger(E') \tau_z \hat{b}(E'') \rangle \hat{a}^\dagger \tau_z \hat{a}(E + \omega) \]

\[ + \frac{1}{4} \Theta(\omega - E) \hat{b}^\dagger(E') \tau_y \hat{b}^\dagger(E'') \hat{a}(\omega - E) \tau_y \hat{a}(E), \]

(4.7)

where we substituted \( e^2/2\pi \rightarrow G_0 = e^2/h \), the conductance quantum, and \( \Theta \) is the step function. The term proportional to \( \Theta(\omega - E) \) is a consequence of the particle-hole symmetry Equation 4.6.
The currents carried by \( a \) and \( b \) can be related back to currents carried by \( c_{1,2} \) entering the beam splitter using the following S-matrix:

\[
\begin{pmatrix}
\hat{a} \\
\hat{b}
\end{pmatrix} = \begin{pmatrix} r & t \\
t & -r \end{pmatrix} \begin{pmatrix}
\hat{c}_1 \\
\hat{c}_2
\end{pmatrix}.
\] (4.8)

We can make use of a transfer matrix \( M \) to undo the effects of repeated Andreev reflections so that the correlator function can be expressed in terms of injected currents carried by \( d \):

\[
\hat{d}_j = M_j \hat{c}_j, \\
\hat{d}_j^\dagger = M_j^* \hat{c}_j^\dagger.
\] (4.9)

Beenakker proposes the following form for \( M \):

\[
M = e^{iE_p t} e^{i\gamma \tau_z} \exp \left[ i \alpha \sigma_y \otimes (\tau_z \cos \phi + \tau_y \sin \phi) + i \beta \tau_z \right] e^{i\gamma' \tau_z}.
\] (4.10)

Here the parameter \( \alpha \) describes the mixing strength between electrons and holes and \( \gamma, \gamma' \), and \( \beta \) describe the phases which are acquired due to the magnetic field (for instance Zeeman precession or Aharanov-Bohm phases). \( \sigma \) and \( \tau \) are Pauli matrices acting on spin and electron-hole space respectively. Using the following parametrization the form of \( M \) can be simplified:

\[
\begin{align*}
\gamma_p & \to \delta \gamma_p, \\
\alpha & \to \bar{\alpha}, \\
\alpha^2 + \beta^2 & \to \xi^2
\end{align*}
\] (4.11)

where

\[
\sin \bar{\alpha} = (\alpha/\xi) \sin \xi \quad \tan 2\delta \gamma = (\beta/\xi) \tan \xi,
\] (4.12)

bringing \( M \) in the following form:

\[
M = e^{iE_p t} e^{i(\gamma + \delta \gamma_p) \tau_z} \exp \left[ i \bar{\alpha} \sigma_y \otimes (\tau_z \cos \phi + \tau_y \sin \phi) \right] e^{i(\gamma' + \delta \gamma_p) \tau_z}.
\] (4.13)

In the basis of \( \{a_{e\uparrow}, a_{e\downarrow}, a_{h\uparrow}, a_{h\downarrow}\} \), \( M \) has the explicit form of:

\[
M = \begin{pmatrix} \hat{R} & \hat{T} \\
-\hat{T} & \hat{R} \end{pmatrix}
\] (4.14)

where the matrices \( \hat{T} \) and \( \hat{R} \) are given by:

\[
\hat{R} = \begin{pmatrix} \exp[i(\gamma + \gamma_p)] \cos \alpha & 0 \\
0 & \exp[-i(\gamma + \gamma_p)] \cos \alpha \end{pmatrix},
\] (4.15)

\[
\hat{T} = \begin{pmatrix} \exp[-i(\gamma - \gamma_p - \phi)] \sin \alpha & 0 \\
0 & \exp[i(\gamma - \gamma_p - \phi)] \sin \alpha \end{pmatrix}.
\] (4.16)
Evaluating $P(\omega)$ will involve computing the expectation value of a product of four fermionic operators. We will make use of Wick’s theorem to relate this to expectation values of product of two operators:

$$\langle ABCD \rangle = \langle AB \rangle \langle CD \rangle - \langle AC \rangle \langle BD \rangle + \langle AD \rangle \langle BC \rangle .$$ (4.17)

All two-products are known since the contacts that inject current are uncorrelated and in thermal equilibrium, i.e.

$$\langle \hat{d}_{j,\sigma}(E)\hat{d}^{\dagger}_{j',\sigma'}(E') \rangle = \delta_{j,j'}\delta_{\tau,\tau'}\delta_{\sigma,\sigma'}\delta(E - E')f_{\tau}(E),$$ (4.18)

with $f_{\tau}$ the Fermi-Dirac distributions for electrons and holes. For the first part of Equation 4.7 we can then write:

$$\langle c_\alpha^\dagger c_\beta c_\delta^\dagger c_\gamma \rangle = \langle c_\alpha^\dagger c_\beta \rangle \langle c_\delta^\dagger c_\gamma \rangle - \langle c_\alpha^\dagger c_\gamma \rangle \langle c_\beta c_\delta \rangle + \langle c_\alpha^\dagger c_\delta \rangle \langle c_\beta^\dagger c_\gamma \rangle$$ (4.19)

and for the second part of Equation 4.7:

$$\langle c_\alpha^\dagger c_\beta^\dagger c_\gamma^\dagger c_\delta \rangle = \langle c_\alpha^\dagger (\delta_{\beta,\gamma}\delta(E_3 - E_2) - c_\gamma^\dagger c_\beta) \rangle$$

$$= \delta_{\beta,\gamma}\delta(E_3 - E_2) \langle c_\alpha^\dagger c_\delta \rangle - \langle c_\alpha^\dagger c_\gamma^\dagger c_\beta^\dagger c_\delta \rangle .$$ (4.20)

To simplify things we examine the case of zero temperature, meaning

$$f_{\tau} = \begin{cases} 
\Theta(eV - E) & \text{for } \tau = e \\
0 & \text{for } \tau = h 
\end{cases}$$ (4.21)

and we only inject electrons at leads 1 and 2.

We are now ready to write down $P(\omega)$. Using equations Equation 4.7, Equation 4.8 and Equation 4.9 we finally arrive at the following form:

$$P(\omega) = P_{11}(\omega) + P_{12}(\omega) + P_{21}(\omega) + P_{22}(\omega)$$ (4.22)

with $P_{pq}$ given by:

$$P_{pq}(\omega) = -G_0^2 t^2 (-1)^{p+q} \int_{0}^{\infty} \text{d}E f(E)$$

$$\times \text{Tr} \left[ f(E + \omega)Z_{pq}Z_{qp} + 1/2\Theta(\omega - E)f(\omega - E)Y_{pq}^*Y_{qp} \right] ,$$ (4.23)
with $Z_{pq}$ and $Y_{pq}$ given by:

$$Z_{pq} = 1/4(1 + \tau_z)\mathcal{M}_p^\dagger \tau_z \mathcal{M}_q(1 + \tau_z)$$  \hspace{1cm} (4.24)$$

$$Y_{pq} = 1/4(1 + \tau_z)\mathcal{M}_p^\dagger \tau_y \mathcal{M}_q(1 + \tau_z).$$  \hspace{1cm} (4.25)$$

Here $P_{pq}(\omega)$ should not be confused with the correlation between current measured at contacts $p$ and $q$, i.e. $\langle I_p I_q \rangle$. Instead the partial correlators $P_{pq}(\omega)$ are the contributions to the total noise originating from currents injected at leads $p$ and $q$. So for example $P_{11}(\omega)$ will correspond to the noise where no current is injected in lead 2. The trace in Equation 4.22 and the matrix $(1 + \tau_z)$ in Equation 4.24 and Equation 4.25 take into account that only electrons are injected into the system.

The evaluation of Equation 4.25 gives us:

$$P_{pq}(\omega) = -\frac{1}{2}G_0 \tau^2 t^2(-1)^{p+q} \times [2(eV - \omega)\Theta(eV - \omega) + 1/2(2eV - \omega)\Theta(2eV - \omega)(g_{pq} - 1)] \hspace{1cm} (4.26)$$

with $g_{pq}$ and $\phi_{pq}$ given by:

$$g_{pq} = \cos 2\tilde{\alpha}_p \cos 2\tilde{\alpha}_q - \cos \phi_{pq} \sin 2\tilde{\alpha}_p \sin 2\tilde{\alpha}_q$$  \hspace{1cm} (4.27)$$

$$\phi_{pq} = \phi_p - \phi_q - 2(\gamma_p + \delta\gamma_p - \gamma_q - \delta\gamma_q).$$  \hspace{1cm} (4.28)$$

In particular, the full correlator $P(\omega)$ is given by:

$$P(\omega) = -\frac{1}{2}G_0 \tau^2 t^2(2eV_b - \omega)(\Theta(2eV - \omega))(\cos 4\alpha_1 + \cos 4\alpha_2 - 2 \cos 2\alpha_1 \cos 2\alpha_2 + 2 \cos \phi_{12} \sin 2\alpha_1 \sin 2\alpha_2).$$  \hspace{1cm} (4.29)$$

When identical BdG quasiparticles, $\alpha_1 = \alpha_2$, are injected into the system the correlator reduces to:

$$P(\omega) = G_0 \tau^2 t^2(2eV_b - \omega)(\Theta(2eV - \omega)) \sin^2 2\alpha(1 - \cos \phi) \hspace{1cm} (4.30)$$

which has a maximum for $\phi = \pi$ and vanishes for $\phi = 0$. This is the non-local analogue of the Josephson effect and the origin of this non-locality will be the subject of the next sections.
5 Entanglement Basics

When the phase difference $\phi$ between the two superconductors is zero, the quantum state of two identical BdG quasiparticles before and behind the beam splitter can be written as a product:

$$|\text{out}\rangle = \cos^2 \alpha |ee\rangle + e^{-2i\phi} \sin^2 \alpha |hh\rangle + e^{-i\phi} \cos \alpha \sin \alpha |he\rangle + e^{-i\phi} \cos \alpha \sin \alpha |he\rangle,$$

(5.1)

which is the consequence of the Pauli exclusion principle as two identical fermions cannot enter the beam splitter simultaneously. So two identical BdG quasiparticles are found, one in each channel. A local operation on one quasiparticle cannot change the state of the other. In the more general case where the phases do differ, as will be shown in following sections, the outgoing state cannot be written in a product form anymore. There a local measurement on one particle does change the state of the other. Consider two electrons denoted by A and B prepared in the spin-singlet state

$$|\text{singlet}\rangle = \frac{1}{2}(|\uparrow\rangle - |\downarrow\rangle)_{A}(|\uparrow\rangle - |\downarrow\rangle)_{B}.$$

(5.2)

and let us call $+$ and $-$ the outcome of a local measurement performed on one of the particles to be $|\uparrow\rangle$ and $|\downarrow\rangle$ respectively. The measurement outcomes on particles A and B will be strongly correlated: we always find the pairs $(\pm A, \mp B)$ but never the pairs $(\pm A, \pm B)$. Let us consider now the product state

$$|\Psi\rangle = \frac{1}{2}(|\uparrow\rangle - |\downarrow\rangle)_{A}(|\uparrow\rangle - |\downarrow\rangle)_{B}.$$

(5.3)

Any local measurement performed on particle A leaves the state of particle B undisturbed, i.e. if we find particle A to be in the state $|\uparrow\rangle_{A}$, particle B will be in the state $(|\uparrow\rangle - |\downarrow\rangle)_{B}$. A subsequent measurement on particle B will result 50% of the times in a state $|\uparrow\rangle_{B}$ and 50% of the times in a state $|\downarrow\rangle_{B}$. In contrast to the previous example, there is no correlation between the outcome of local measurements performed on particles A and B, i.e. we find all pairs $(+, +), (+, -), (-, +)$ and $(-, -)$ with equal probability. States which cannot be written in a product form are called entangled and exhibit strong correlations between the outcome of measurements.

The outcome of a measurement also depends on along which axis the spin is measured. If, for example, we measure the spin along the $x$ axis, then for the product state given by Equation 5.3 we will always find pairs
(+, +) since the state \( (|↑⟩ - |↓⟩)_A B \) is an eigen state of Pauli matrix \( \sigma_x \), while for the product state we find again pairs \((\pm, \mp)\) both with equal probability. So we see that the for the product state the outcomes of measurements are completely uncorrelated when measuring the spins along the \( z \)-axis, while strongly correlated when measuring the spins along the \( x \)-axis. Measurements performed along a single axis is not enough to fully distinguish entangled from product states and it is better, as discussed in the next section, to compare data taken from measuring spins of particle A and B with respect to a range of axis \( a \) and \( b \) respectively.

Another way of looking at product and entangled states is by asking how much information about the state is obtained from single particle measurements? Suppose we have a large amount of copies of the following two-particle product state:

\[
\psi = |↑_A ↑_B⟩.
\]  

(5.4)

Each particle is given to experimentalists A and B and are asked to determine, without communicating with each other, in which state their particle was prepared in. By measuring the spin of their particles along different axis, both experimentalists should be able to find that their particle was prepared in the \( |↑⟩_A⟩_B \) state. For a product state we can therefore say that all the information of the state lies in the results of local measurements. The case is different for entangled states as the following example will show. Consider the following entangled two-particle state:

\[
\psi = \frac{1}{\sqrt{2}} \left[ |↑_A ↑_B⟩ + |↓_A ↓_B⟩ \right]
\]  

(5.5)

and suppose experimentalist A measures the spin of its particle along an angle \( a \), between the \( z \) and \( x \) axis. The corresponding observable \( \sigma_a \) is defined as:

\[
\sigma_a = \cos a \sigma_x + \sin a \sigma_z
\]

\[
= \begin{pmatrix}
\cos a & \sin a \\
\sin a & -\cos a
\end{pmatrix}.
\]  

(5.6)

The eigen vectors of \( \sigma_a \) are given by:

\[
|+⟩ = (\cos \theta/2, \sin \theta/2)^T
\]  

(5.7)

\[
|−⟩ = (\sin \theta/2, -\cos \theta/2)^T,
\]  

(5.8)

where \( ± \) corresponds to eigen values \( ±1 \). We can write the state \( \psi \) in this new measurement basis as:

\[
\psi = \frac{1}{\sqrt{2}} \left[ (\cos a/2 |+⟩ + \sin a/2 |−⟩)|↑⟩ + (\sin a/2 |+⟩ - \cos a/2 |−⟩)|↓⟩ \right].
\]  

(5.9)
The outcome of each measurement will be 50% "up" and 50% "down", independent over which angle the experimentalist measures the spin. The same holds for experimentalist B, so that local measurements alone give us no information about the state. Where then can we find the information regarding the state of the two particles? The answer lies in the correlation between the A’s and B’s data. In this sense, entanglement can be seen as the combination of how much information can be found from correlations and how little information from local measurements.
6 Bell Inequality

During the development of the theory of quantum mechanics one of the more known opposers was A. Einstein. In a joint paper with Podolsky and Rosen entitled “Can Quantum-Mechanical Description of Physical Reality be Considered Complete?” the authors argued that the outcome of measurements performed on two spatially separated entangled electrons which under the assumption of “completeness” violate relativity. Therefore the theory of quantum mechanics is incomplete and calls for additional (or hidden) variables. Later however, Bell showed that additional variable theories are incompatible with predictions made by quantum mechanics. Moreover, Bell’s theorem could be tested experimentally by showing the breaking of certain inequalities proposed by Bell. Many experiments have been performed so far using entangled photon pairs [5] and more recently with electrons [6, 7]. In the following we will show that the proposed setup can be used as a Bell test for which Bell inequalities are broken.

We first consider the correlator $C_{ab}$ defined as [8, 9]

$$C_{ab} = \frac{\langle (N_+ - N_+)(N_+ - N_+) \rangle}{\langle (N_+ + N_-)(N_+ + N_-) \rangle},$$

(6.1)

which can be rewritten as

$$C_{ab} = \frac{\langle (N_+ - N_-)(N_+ - N_-) \rangle}{\langle (N_+ + N_-)(N_+ + N_-) \rangle},$$

(6.2)

Where $N_+$ counts the number of spin “up” and $N_-$ the number of spin “down” quasiparticles. When the spins are measured along the $z$ axis, we recognize in the numerator of Equation 6.2 the (spin) current correlator:

$$N_+ - N_- = \hat{a}_{e,\uparrow} \hat{a}_{e,\uparrow} - \hat{a}_{h,\downarrow} \hat{a}_{h,\downarrow} = \hat{a}^\dagger \sigma_z \hat{a},$$

(6.3)

where the current is defined in a similar fashion to Equation 4.2 with $\sigma_z$ acting now on spin-space instead of electron-hole space. The (spin) thermal correlator is given by:

$$N_+ + N_- = \hat{a}_{e,\uparrow} \hat{a}_{e,\uparrow} + \hat{a}_{h,\downarrow} \hat{a}_{h,\downarrow} = (\hat{a}^\dagger)^\tau \hat{a},$$

(6.4)

If we let $\theta$ be the angle between the $z$ and $x$ axis, we can define a Pauli matrix $\sigma_\theta$ as

$$\sigma_\theta = \cos \theta \sigma_x + \sin \theta \sigma_z = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix},$$

(6.5)
which is the observable corresponding to a spin measurement along an angle \( \theta \). The eigen vectors of \( \sigma_\theta \) are given by:

\[
|+\rangle = \left( \cos \theta/2, \sin \theta/2 \right)^\dagger, \quad (6.6)
|\rangle = \left( \sin \theta/2, -\cos \theta/2 \right)^\dagger, \quad (6.7)
\]

where ± corresponds to eigen values ±1. We can write \( a^\dagger \) and \( a \) in this new basis by projecting \(|\uparrow\rangle, |\downarrow\rangle \) onto the eigen vectors of \( \sigma_\theta \):

\[
a \mapsto U(\theta)a, \quad U(\theta) = \begin{pmatrix} \cos \theta/2 & \sin \theta/2 \\ \sin \theta/2 & -\cos \theta/2 \end{pmatrix}, \quad (6.8)
\]

The spin current is then given by

\[
N_+(a) - N_-(a) = (\hat{a}^\dagger)^\dagger U(a) \sigma_z U(a) \hat{a} \tag{6.9}
\]

and the thermal spin current by

\[
N_+(a) + N_-(a) = (\hat{a}^\dagger)^\dagger \hat{a}. \tag{6.10}
\]

The thermal current is independent of the chosen angle due to spin conservation. \( C_{ab} \) can now be written as:

\[
C_{ab} = \frac{\langle a^\dagger U(a) \sigma_z U(a) b^\dagger U(b) \sigma_z U(b) b \rangle}{\langle a^\dagger b \rangle}. \tag{6.11}
\]

We can make use of the frequency dependent correlators from Equation 4.7 by substituting \( \tau_z \rightarrow U(a) \sigma_z U(a) \) and \( \tau_y \rightarrow U(a) \sigma_z U(a) \tau_x \) for the numerator part and \( \tau_z \rightarrow 1 \) and \( \tau_y \rightarrow \tau_x \) for the denominator part.

We will make use of the Bell-CHSH-inequality which is expressed as:

\[
|\mathcal{B}| = |C_{ab} + C_{ad} + C_{bc} - C_{cd}| \leq 2. \tag{6.12}
\]

Before calculating \( \mathcal{B} \) for our system, it is insightful to first consider two simple examples. First, let us take a look at the product state \( \psi \) given by

\[
\psi = |+\rangle \langle +|.
\]

In the basis corresponding to measuring spins at an angle \( a \), the single particle states are written as:

\[
|+\rangle \mapsto (\cos a/2) |+\rangle + \sin a/2 |\rangle \tag{6.14}
|\rangle \mapsto (\sin a/2) |+\rangle - \cos a/2 |\rangle \tag{6.15}
\]
Figure 3: $\beta_{\text{max}}$ for various phases $\phi_{12}$. Darker areas correspond to higher breaking of the Bell inequality. On the horizontal and vertical axes are the electron-hole mixing strength of the BdG quasiparticles. For vanishing phase difference the bell inequality is not broken when identical quasiparticles enter the beam splitter. For maximal phase difference we see a maximal breaking of the Bell inequality $\beta_{\text{max}} = 2\sqrt{2}$ when $\alpha_1 + \alpha_2 = \pi/2$, which corresponds to expectation value of the total charge equal to zero.

Setting the detectors at angles $a$ and $b$, $\psi$ transforms as:

$$\psi = (\cos a/2 |+\rangle + \sin a/2 |−\rangle)(\cos b/2 |+\rangle + \sin b/2 |−\rangle)$$

(6.16)

The corresponding probabilities are given by:

$$N_{++} = \cos^2 a/2 \cos^2 b/2 \quad N_{+-} = \cos^2 a/2 \sin^2 b/2$$

$$N_{−−} = \sin^2 a/2 \sin^2 b/2 \quad N_{−+} = \sin^2 a/2 \cos^2 b/2$$

(6.17) (6.18)

and the correlator $C_{ab}$ reduces to:

$$C_{ab} = \cos a \cos b.$$  

(6.19)

Maximizing $\beta$ over angles $a, a', b, b'$ gives us:

$$\beta_{\text{max}} = 2.$$  

(6.20)

The product state we considered does not violate the CSHS inequality, a fact that holds for all product states. Note that $C_{ab}$ can have any value in the interval $[-1 : +1]$. Next, let us consider a maximally entangled state $\psi'$ given by:

$$\psi' = |+\rangle |+\rangle + |−\rangle |−\rangle.$$  

(6.21)

Setting the detectors again at angles $a, b$ the state $\psi'$ transforms into

$$\psi' \mapsto \cos(a−b) |+\rangle |+\rangle − \sin(a−b) |−\rangle |+\rangle + \sin(a−b) |+\rangle |−\rangle + \cos(a−b) |−\rangle |−\rangle.$$  

(6.22)
and the corresponding probabilities are given by:

\[
N_{++} = \frac{1}{2} \cos^2(a - b) \quad N_{+-} = \frac{1}{2} \sin^2(a - b) \quad (6.23)
\]

\[
N_{--} = \frac{1}{2} \cos^2(a - b) \quad N_{-+} = \frac{1}{2} \sin^2(a - b). \quad (6.24)
\]

The correlator \( C_{ab} \) now reduces to:

\[
C_{ab} = \cos(2(a - b)). \quad (6.25)
\]

\( \beta \) can be maximized by setting the angles \( a, a', b, b' \) to \( 0, \pi/4, 3\pi/2, 7\pi/4 \) respectively, so that \( \beta_{\text{max}} = 2\sqrt{2} \).

\[
\beta_{\text{max}} = 2\sqrt{2}. \quad (6.26)
\]

The result is true for all states which are maximally entangled. Note again that the correlator \( C_{ab} \) can have any value in the interval \([-1 : 1]\) as before. The value of the correlator by itself does not tell us whether we are dealing with an entangled or a product state, while the combined result of measurements over different angles does.

In Figure 3 we plot \( \beta_{\text{max}} \) as a function of \( \alpha_{1,2} \) for various phases \( \phi_{12} \). To remind the reader again, \( \alpha = 0 \) respectively \( \alpha = \pi/2 \) correspond to pure electrons respectively holes. In the left-most plot \( \beta_{\text{max}} \) is plotted for a vanishing phase difference. On the main diagonal \( \alpha_1 - \alpha_2 = 0 \) we find \( \beta_{\text{max}} = 2 \) so that the Bell inequality is not broken and the outgoing BdG quasiparticles form a product state. When \( \alpha_{1,2} = \pi/2 \) and \( \alpha_{2,1} = 0 \) we find \( \beta_{\text{max}} = 2\sqrt{2} \) corresponding to having a pure electron and pure hole entangled.

By changing the phase \( \phi_{12} \) gradually from zero to \( \pi \) we see a dramatic change. At \( \phi_{12} = \pi \) the whole off-diagonal \( \alpha_1 + \alpha_2 = \pi/2 \) has \( \beta_{\text{max}} = 2\sqrt{2} \) indicating maximal breaking of the Bell-inequality. The electron-hole ratio of these BdG quasiparticles are reciprocal of each other indicating that the expectation value for the total charge is equal to zero. As mentioned before, maximal breaking of the Bell-inequality indicates that the BdG quasiparticles are maximally entangled. At \( \alpha_1 = \alpha_2 = 0 \) and \( \alpha_1 = \alpha_2 = \pi/2 \) we find \( \beta_{\text{max}} = 2 \) independent of the phase \( \phi_{12} \), indicating that it is not possible to entangle two BdG quasiparticles when they are both pure electrons or pure holes.
7 Concurrence and Entangled BdG Quasiparticles

By leaving the frequency domain and working at incident time it is possible to write down state functions of the BdG quasiparticles before and after the beam-splitter. Instead of working with current operators $\hat{I}$, we work with creation operators $\hat{a}^\dagger_\tau$ and $\hat{b}^\dagger_\tau$ which act on the quasiparticle Fock space $|n_{1,e}n_{1,h}n_{2,e}n_{2,h}\rangle$, where $n_{i,\tau}$ is the number of electrons or holes in channel $i$. We neglect here the spin degree of freedom for simplicity. If we inject one electron in the first channel and another electron in the second channel in our system, the quantum state can be written as:

$$|\text{in}\rangle = \hat{a}^\dagger_e \hat{b}^\dagger_e |0000\rangle = |1010\rangle$$ (7.1)

with $\hat{a}^\dagger_e$ and $\hat{b}^\dagger_e$ the creation operators which add one electron to both channels separately.

Before the electrons enter a beam splitter they first pass a superconductor. They are transformed as:

$$(\hat{a}_e \hat{a}_h)_p \rightarrow \begin{pmatrix} \cos \alpha_p e^{i\phi_p} & \sin \alpha_p \\ -e^{-i\phi_p} \sin \alpha_p & \cos \alpha_p \end{pmatrix}_{p} (\hat{a}_e \hat{a}_h)_{p}$$ (7.2)

where the subscript $p$ denotes the channel index. After passing the superconductor, the newly formed quasi-particles enter a beam splitter and exit in two channels behind the beam splitter. The effect of the beam splitter can be described by the S-matrix:

$$(\hat{a}_r \hat{b}_t)_{\text{in}} = (r \ t \ -r \ t)_{\text{out}} (\hat{c}_d)_{\text{out}}$$ (7.3)

In order to write down the quantum state after the quasiparticles went through the beam splitter, we first write $|\text{in}\rangle$ in a more convenient way:

$$a^\dagger_e b^\dagger_e |0000\rangle = (a^\dagger \ b^\dagger) |0000\rangle$$
$$\rightarrow (a^\dagger M_1^\dagger (1 0) M_2^* b^\dagger)|0000\rangle$$
$$\sum_{\tau,\tau'} M_{\tau\tau'} a^\dagger_{\tau} b^\dagger_{\tau'} |0000\rangle.$$ (7.4)

where the matrix elements of $M$ are given by

$$M_{\tau\tau'} = M_{1,\tau r}^* M_{2,\tau r'}$$

$$= \begin{pmatrix} \cos \alpha_1 \cos \alpha_2 & \cos \alpha_1 \sin \alpha_2 e^{-i\phi_2} \\ \cos \alpha_2 \sin \alpha_1 e^{-i\phi_1} & \sin \alpha_1 \sin \alpha_2 e^{-i(\phi_1 + \phi_2)} \end{pmatrix}$$ (7.5)
and \( \tau \) denoting electron or hole. Then the outgoing state can be written as:

\[
|\text{out}\rangle = \sum_{\tau\tau'} M_{\tau\tau'} (rc_\tau\dagger + td_\tau\dagger)(tc_{\tau'}\dagger - rd_{\tau'}\dagger)|0000\rangle
\]

(7.6)

Ordering the operators by \( d_\tau\dagger e, d_{\tau'}\dagger h, c_\tau\dagger e, c_{\tau'}\dagger h \), and using the anti-commutation relations we find:

\[
|\text{out}\rangle = \left[ M_{ee}d_{\tau\dagger}c_{\tau\dagger} + M_{eh} \left( r tc_{\tau\dagger}c_{\tau'}\dagger + r^2 d_{\tau\dagger}c_{\tau\dagger} + t^2 d_{\tau'}\dagger c_{\tau'}\dagger - rtd_{\tau\dagger}d_{\tau'}\dagger \right) + M_{he} \left( -rtc_{\tau\dagger}c_{\tau'}\dagger + r^2 d_{\tau'}\dagger c_{\tau\dagger} + t^2 d_{\tau\dagger}c_{\tau\dagger} - rtd_{\tau'}\dagger d_{\tau\dagger} \right) + M_{hh}d_{\tau'}\dagger d_{\tau\dagger} \right] |0000\rangle
\]

(7.7)

and after rearranging all the terms:

\[
|\text{out}\rangle = rt(M_{he} - M_{eh})|0011\rangle - rt(M_{he} - M_{eh})|1100\rangle + \left( t^2 M_{eh} + r^2 M_{he} \right)|1001\rangle + \left( t^2 M_{he} + r^2 M_{eh} \right)|0110\rangle + M_{hh}|0101\rangle + M_{ee}|1010\rangle.
\]

(7.8)

### 7.1 Concurrence

The concurrence \( C \) is a measure of entanglement between two qubits. Whenever \( C = 1 \) and \( C = 0 \) correspond to maximally respectively minimally entangled states. States which are not entangled can be written as a product state.

For a normalized two qubit state we can write:

\[
|\Psi\rangle = \sum_{i=1}^{2} \sum_{j=1}^{2} \gamma_{ij} |i\rangle |j\rangle \Pi
\]

(7.9)

In Equation 7.8 we want to remove the neutral modes \( |0011\rangle \) and \( |1100\rangle \). This can be done by acting a “charge” operator \( Q^2 \) on \( |\text{out}\rangle \). So that

\[
|\Psi\rangle_r = (t^2 M_{eh} + r^2 M_{he})|eh\rangle + (t^2 M_{he} + r^2 M_{eh})|he\rangle + M_{hh}|hh\rangle + M_{ee}|ee\rangle
\]

(7.10)

and the matrix \( \gamma \) from Equation 7.9 as:

\[
\gamma = \frac{1}{\sqrt{N}} \begin{pmatrix}
M_{ee} & t^2 M_{eh} + r^2 M_{he} \\
(t^2 M_{he} + r^2 M_{eh}) & M_{hh}
\end{pmatrix},
\]

(7.11)

with \( \sqrt{N} \) the renormalization factor:

\[
N = \text{Tr}(\gamma \gamma\dagger).
\]

(7.12)
The concurrence $C$ for a two qubit system is defined as:

$$C = 2\sqrt{\det \gamma \gamma^\dagger}$$

(7.13)

which evaluates to

$$C = r^2t^2 \frac{1 - \cos 2\alpha_1 \cos 2\alpha_2 - \cos \phi_{12} \sin 2\alpha_1 \sin 2\alpha_2}{1 - r^2t^2(1 - \cos 2\alpha_1 \cos 2\alpha_2 - \cos \phi_{12} \sin 2\alpha_1 \sin 2\alpha_2)}.$$  

(7.14)

In Figure 6 we plot $C$ together with $\beta_{\max}$. The Bell parameter $\beta_{\max}$ behaves qualitatively the same as $C$ and the extrema $C = 0, 1$ coincide with the extrema of $\beta_{\max} = 2, 2\sqrt{2}$. Just as in the previous section we see that for vanishing phase identical BdG quasiparticles do not entangle, while maximally entangled states appear at maximal phase difference $\phi_{12} = \pi$ for $\alpha_1 + \alpha_2 = \pi/2$. A third measure for entanglement, equivalent to $C$ and $B$, called the entanglement entropy $S$ can also be used and is derived and discussed in Section A.

In the following I will distinguish two types of BdG quasiparticles: Identical and mirrored corresponding to $\alpha_1 - \alpha_2 = 0$ and $\alpha_1 + \alpha_2 = \pi/2$ respectively.

### 7.2 Identical Bogoliubov quasiparticles: $\alpha_1 = \alpha_2$

When we feed two identical BdG quasiparticles into the beam splitter, we can write the initial state as $(\cos \alpha_1, \sin \alpha_1 e^{i\phi_1}) \otimes (\cos \alpha_2, \sin \alpha_2 e^{i\phi_2})$. In this case the concurrence is given by

$$C = r^2t^2 \frac{1 - \cos^2 2\alpha - \cos \phi_{12} \sin^2 2\alpha}{1 - r^2t^2(1 - \cos^2 2\alpha - \cos \phi_{12} \sin^2 2\alpha)}.$$  

(7.15)

In Figure 5a we plot Equation 7.17 for an ideal beam-splitter with $r = t = 1/\sqrt{2}$. When the phase difference vanishes, i.e. $\phi_{12} = 0$, we find that $C = 0$, corresponding to the reduced state:

$$\Psi = \left( \begin{array}{c} \cos \alpha \\
\sin \alpha e^{-i\phi} \end{array} \right) \otimes \left( \begin{array}{c} \cos \alpha \\
\sin \alpha e^{-i\phi} \end{array} \right) \mapsto \left( \begin{array}{c} \cos \alpha \\
\sin \alpha e^{-i\phi} \end{array} \right) \otimes \left( \begin{array}{c} \cos \alpha \\
\sin \alpha e^{-i\phi} \end{array} \right)$$

(7.16)

i.e. a product form.

When $\phi_1 = \phi_2 + \pi = \phi$ the concurrence evaluates to:

$$C = r^2t^2 \frac{1 - \cos 4\alpha}{1 - r^2t^2(1 - \cos 4\alpha)}$$

(7.17)

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Figure 4: Concurrence (a) and $\beta_{\text{max}}$ (b) for various phases $\phi_{12}$. The plots for $\beta_{\text{max}}$ are included to emphasize its qualitative similarity to the concurrence. For zero phase difference the concurrence vanishes for $\alpha_1 - \alpha_2 = 0$, indicating that identical BdG quasiparticles do not get entangled due to the beam splitter. For maximal phase difference $\phi_{12} = \pi$ we find maximally entangled states when $\alpha_1 + \alpha_2 = \pi/2$. 
corresponding to the state

$$\Psi \mapsto \frac{1}{2} e^{-i\phi} \sin 2\alpha (r^2 - t^2) (|eh\rangle - |he\rangle) + \cos^2 \alpha |ee\rangle + e^{-2i\phi} \sin^2 \alpha |hh\rangle .$$

(7.18)

In the case of a perfect beam splitter the state further reduces to:

$$\Psi \mapsto -e^{-2i\phi} \sin^2 \alpha |hh\rangle + \cos^2 \alpha |ee\rangle .$$

(7.19)

When $\alpha = \pi/4$ this state corresponds to a maximally entangled state with $C = 1$.

### 7.3 Mirrored Bogoliubov quasiparticles: $\alpha_1 + \alpha_2 = \pi/2$

When we feed two “mirrored” BdG quasiparticles into the beam splitter, we can write the initial state as $(\cos \alpha, \sin \alpha e^{i\phi_1}) \otimes (\sin \alpha, \cos \alpha e^{i\phi_2})$. In this case the concurrence is given by

$$C = r^2 t^2 \frac{1 + \cos^2 2\alpha - \cos \phi_{12} \sin^2 2\alpha}{1 - r^2 t^2 (1 + \cos^2 2\alpha - \cos \phi_{12} \sin^2 2\alpha)}$$

(7.20)

which is plotted in Figure 5b for a perfect beam splitter with $r = t = 1/\sqrt{2}$. $C$ reaches its maximum value of

$$C_{\text{max}} = \frac{2r^2 t^2}{1 - 2r^2 t^2} ,$$

(7.21)

when $\phi = \pi$. The state behind the beam splitter then evolves to:

$$\Psi_{\text{out}} = \sin \alpha \cos \alpha (|ee\rangle - e^{-2i\phi} |hh\rangle) + e^{-i\phi}(r^2 \sin^2 \alpha - t^2 \cos^2 \alpha) |eh\rangle + e^{-i\phi}(t^2 \sin^2 \alpha - r^2 \cos^2 \alpha) |he\rangle .$$

(7.22)
In particular for a perfect beam splitter we find $C_{\text{max}} = 1$, independent of the particular values of $\alpha$. In order to make this result more intuitive, let us write qubit B in the following basis:

$$|\bar{0}\rangle = \sin 2\alpha |0\rangle + e^{-i\phi} (\cos 2\alpha) |1\rangle$$  \hspace{1cm} (7.23)

$$|\bar{1}\rangle = -\sin 2\alpha e^{-i\phi} |1\rangle + \cos 2\alpha |0\rangle.$$  \hspace{1cm} (7.24)

In this basis Equation 7.22 is written as:

$$\Psi_{\text{out}} = |0\bar{0}\rangle + e^{-i\phi} |1\bar{1}\rangle.$$  \hspace{1cm} (7.25)

The outcome of this measurement is 100% correlated, independent of whether qubit A or B is measured first: finding $0(1)$ at A is always paired with finding $\bar{0}\bar{1}$ at B. Moreover, the two states $|\bar{0}\rangle$ and $|\bar{1}\rangle$ are orthonormal so that local measurements at B will result in one of the two states with equal probability.
8 Conclusion & Discussion

In the previous sections it was shown how to produce and entangle two BdG quasiparticles in a beam-splitter setup. By correlating the measured spins of the two currents under different angles the Bell inequality can be broken indicating non-classical correlations. At the origin of this non-classical correlation lies the entanglement of the quasiparticles, a claim supported by calculating the concurrence of the system. Two types of quasiparticles were found: identical ($\alpha_1 = \alpha_2$) and mirrored ($\alpha_1 = \pi/2 - \alpha_2$). For vanishing phase difference identical BdG quasiparticles cannot be entangled. This can be explained using the Pauli exclusion principle where two identical fermions cannot enter the beam splitter simultaneously. When the phase difference equals $\pi$ two mirrored BdG quasiparticles get maximally entangled which can again be explained using the Pauli exclusion principle. The states of two mirrored quasiparticles are orthogonal when $\phi = \pi$, i.e. the states have zero overlap so that they are allowed to maximally mix inside the beam splitter.

For identical BdG quasiparticles the electrical current correlator was given by

$$P(\omega) = G_0 e^{i 2 (2 e V - \omega)(1 - \cos \phi_12) \sin^2 2\alpha}$$

which is the non-local analogue of the Josephson effect. Note that the factor $2 \sin^2 2\alpha$ is equal to the variance of the total charge in the system, so that the correlator is proportional to the degree of charge fluctuations and proportional to a phase factor $1 - \cos \phi_{12}$. The correlator $P(\omega)$ disappears for vanishing phase difference, while is maximal when the phase difference equals $\pi$ so that $P(\omega)$ behaves similarly to the degree of entanglement in the system. Note that no supercurrent is flowing between the two superconductors in contrast with the conventional Josephson effect. One question that remains is: what is the physical origin of this phase factor?

An Andreev-reflected hole is entangled with a Cooper pair inside the superconductor. The state $\Psi$ of a single BdG quasiparticle can thus be written as

$$\Psi = |e\rangle \otimes |-\rangle + \exp(-i\phi) |h\rangle \otimes |O\rangle$$

where $|-\rangle$ and $|O\rangle$ respectively denote the absence and presence of a single Cooper pair inside the superconductor. By tracing out the electron and hole degree of freedom of the entangled BdG pair we find the following state behind the beam splitter:

$$\Psi_{\text{red}} = \begin{cases} (|O\rangle + |-\rangle)(|O\rangle + |-\rangle) & \text{for } \phi = 0 \\ |O\rangle |O\rangle + |-\rangle |-\rangle & \text{for } \phi = \pi \end{cases}$$

(8.3)
The Cooper pairs inside the two superconductors are entangled as well when the phase difference equals $\pi$. Correlating the electrical currents is then equivalent to correlating Cooper pairs present in the two superconductors. Therefore the origin of the non-local Josephson effect as given by Equation 8.1 is due to entangled Cooper pairs and $P(\omega)$ is a direct measure for the degree of entanglement in the system. Beenakker argues in [4] that the origin of the non-local Josephson effect lies in the annihilation of Majorana fermions. The alternative interpretation presented here provides a more practical and easier way of understanding the non-locality and origin of the non-local Josephson effect in terms of entanglement present in the system.

The combination of strong magnetic fields and superconductors in the proposed setup seem incompatible with each other as the coupling between the edge channels and superconductor should decrease with stronger fields. Recently however experiments such as [10] have been performed that support the idea of Andreev reflected holes in the quantum Hall regime. The authors measured the magnetoresistance of Nb-InAs interfaces and, by varying the magnetic field above and below the critical field, showed that the magnitude of Shubnikov–de Haas oscillations increase almost twofold when in the superconducting state.

In [11] an experiment similar to our setup is described, but in absence of superconductors. Single electrons are injected into two quantum Hall channels and the two channels meet at a quantum point contact. Two outgoing currents are correlated and it is shown that the zero frequency noise can be used to obtain information about the ingoing electrons.

These two experiments combined could be used to show the breaking of the Bell inequality. At finite temperatures however one should be careful [12] and perform a theoretical analysis before conclusions can be drawn. For instance, there is in general a temperature threshold above which entanglement cannot exist. Furthermore due to decoherence the BdG quasiparticles do not form pure states but mixed states so that the entanglement can be hidden. In that case an operation can be performed called entanglement purification where pure entangled quasiparticle pairs can be distilled [12].
Appendices

A Another Entanglement Measure: Entanglement Entropy

In this section one more entanglement measure is briefly discussed called the von Neumann entanglement entropy, which counts the number of Bell pairs. Let us consider the following two-qubit state

$$\psi = \sum_{ij} \gamma_{c_{ij}} |ij\rangle.$$  \hspace{1cm} (A.1)

The index $c_{ij}$ is written as such for convenience later. The density matrix $\rho$ corresponding to this (pure) state is a $4 \times 4$ matrix whose components are given by

$$\rho_{kl} = \gamma_k \gamma_l^*$$  \hspace{1cm} (A.2)

and has the following form:

$$\rho = \begin{pmatrix}
\gamma_{c_{11}} \gamma_{c_{12}}^* & \gamma_{c_{11}} \gamma_{c_{12}}^* & \gamma_{c_{12}} \gamma_{c_{12}}^* & \gamma_{c_{12}} \gamma_{c_{22}}^* \\
\gamma_{c_{12}} \gamma_{c_{12}}^* & \gamma_{c_{12}} \gamma_{c_{12}}^* & \gamma_{c_{12}} \gamma_{c_{12}}^* & \gamma_{c_{12}} \gamma_{c_{22}}^* \\
\gamma_{c_{21}} \gamma_{c_{12}}^* & \gamma_{c_{21}} \gamma_{c_{12}}^* & \gamma_{c_{21}} \gamma_{c_{12}}^* & \gamma_{c_{21}} \gamma_{c_{22}}^* \\
\gamma_{c_{22}} \gamma_{c_{11}}^* & \gamma_{c_{22}} \gamma_{c_{12}}^* & \gamma_{c_{22}} \gamma_{c_{12}}^* & \gamma_{c_{22}} \gamma_{c_{22}}^*
\end{pmatrix}. \hspace{1cm} (A.3)

For all density matrices $\rho$ the following properties hold:

$$Tr(\rho) = 1 \hspace{1cm} (A.4)$$

$$\frac{1}{d} \leq Tr(\rho^2) \leq 1 \hspace{1cm} (A.5)$$

where $d$ is the dimension of the density matrix. For a density matrix corresponding to a pure state and to a maximally mixed state the equality $Tr[\rho^2] = 1$ and $Tr[\rho^2] = 1/d$ hold respectively.

By tracing out the degrees of freedom of one of the quantum bits the density matrix can be reduced to a $2 \times 2$ matrix $\rho^{A,B}$. These are obtained by performing a partial trace $Tr_{B,A}$ on the density matrix $\rho$. The components of these reduced density matrices are given by:

$$\rho_{ij}^A = Tr_B \rho = \sum_k \gamma_{c_{ik}} \gamma_{c_{jk}}^* \hspace{1cm} (A.6)$$

$$\rho_{ij}^B = Tr_A \rho = \sum_k \gamma_{c_{ki}} \gamma_{c_{kj}}^*. \hspace{1cm} (A.7)$$
Using the density matrix $\rho$ from Equation A.3, the reduced density matrix $\rho^A$ has the form:

$$\rho^A = \begin{pmatrix}
\gamma c_{11} & \gamma^* c_{11} \\
\gamma c_{11} & \gamma^* c_{11}
\end{pmatrix},$$  
(A.8)

In general such a reduced density matrix does not correspond to a pure state which will be illustrated by the following two examples. Consider first the pure product state $\psi$ given by

$$\psi = |11\rangle.$$  
(A.9)

The reduced density matrix $\rho^A$ corresponding to this state is given by:

$$\rho^A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$  
(A.10)

for which we find

$$\text{Tr}[\rho^A] = 1,$$  
(A.11)

meaning that the reduced density matrix $\rho^A$ corresponds to a pure state. We would have arrived at the same result starting with any other pure product state in Equation A.9. In contrast, consider next the pure maximally entangled state $\psi$ given by

$$\psi = 1/2(|00\rangle + |11\rangle).$$  
(A.12)

The reduced density matrix $\rho^A$ corresponding to this state is given by

$$\rho^A = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$  
(A.13)

for which we find

$$\text{Tr}[\rho^A] = 1/2,$$  
(A.14)

meaning that $\rho^A$ corresponds to a maximally mixed state. The result would be the same had we started with any other pure maximally entangled state in Equation A.12.

Note that in general

$$\text{Tr}[(\rho^A)^2] \neq \text{Tr}[(\rho^B)^2]$$  
(A.15)

so that it is not a convenient measure for entanglement even though it can distinguish between entanglement and product states. One measure for entanglement making use of the reduced density matrix is called the *entanglement entropy* and is given by:

$$S = -\text{Tr}\rho^A \log \rho^A,$$  
(A.16)
which is independent from over which qubit the partial trace is performed. For a product state such as Equation A.9 and a maximally entangled state such as Equation A.12 we find the following entropy $S$:

$$S = \begin{cases} 0 & \text{if } \psi \text{ a product state} \\ \log 2 & \text{if } \psi \text{ a maximally entangled state} \end{cases}.$$  

(A.17)

The entanglement entropy can be interpreted as counting the number of *entangled bits* in the system.

For a general two-qubit state $\psi$ as discussed in Section 7 the entropy $S$ together with the concurrence $C$ and Bell parameter $B$ for various phases is plotted in Figure 6. Comparing the three entanglement measures we see a strong similarity. In Figure 7 the three entanglement measures are plotted for identical BdG quasiparticles in an ideal beam-splitter setup. Here the Bell parameter $B$ and entanglement entropy $S$ are rescaled via

$$B \mapsto \frac{B - 1}{2\sqrt{2}}$$  

(A.18)

$$S \mapsto \frac{S}{\log 2}$$  

(A.19)

for clarity, so that maxima and minima coincide.
Figure 6: Comparison of the three entanglement measures concurrence (a), Bell parameter (b) and the entanglement entropy (c) for various phases $\phi_{12}$. 

(a) Concurrence $C$

(b) Bell parameter $B$

(c) Entanglement entropy $S$
Figure 7: Comparison of the three entanglement measures $C$, $S$ and $B$. The measures are computed for identical BdG quasiparticles $\alpha_1 = \alpha_2$ in an ideal beam-splitter setup with $r = t = 1/\sqrt{2}$ and are rescaled for clarity.
References


