Magnetization dynamics in a honeycomb ferromagnet

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1 Introduction

One of the lattices of greatest interest of study is the honeycomb lattice. Since the discovery of graphene, which is an example of such a lattice, there has been a lot of interest in the properties of such lattices. An example of a typical honeycomb lattice is given by Figure 1. We will specifically look at honeycomb lattices that exhibit antiferromagnetic properties.

In this thesis we will describe several magnetic properties of a honeycomb antiferromagnet, which are the conductivity, the spin-orbit torque, and the Gilbert damping. We are specifically interested in how small perturbations to the lattice affect these properties.

In chapter 2 we will discuss the Green's functions as well as some of their applications. These functions are described as a solution to the Schrödinger equation in the many-body system and define several properties of the system. In chapter 3 we explain the concept of linear response theory and disorder averaging. In chapter 4 we discuss the concept of magnetization dynamics. The model we used for the honeycomb lattice and Hamiltonian is explained in chapter 5. The results obtained from this model are presented in chapter 6.

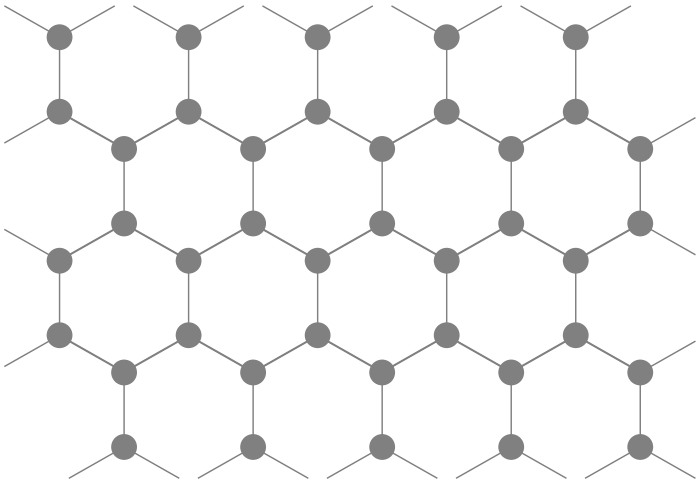

Figure 1: An example of a standard honeycomb lattice.

2 Green's functions

2.1 Derivation of the bare Green's functions

A honeycomb lattice is a many-body system, so in order to discuss such a system in detail, the concept of a propagator should be introduced. For a single particle at an initial position x' at inital time t' the propagator is the probability amplitude to observe the particle at position x at time t [1]. It turns out that in the case of quantum particles the propagator is actually a construct known as the Green's function. For a single particle in the system the Schrödinger equation is:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = H\psi(x,t) \tag{1}$$

Since we do not know the inital conditions of the system, we use the Green's function to specify these inital conditions. We describe the Green's function as $G(x,t;x',t')$ where the particle is known to be at initial position x' at initial time t' and we want to know the probability amplitude to find it at position x at time t. We would like to write our Green's functions in terms of the Hamiltonian of the system, and we follow the derivation in Rammer [2]. $G(\mathbf{x},t;\mathbf{x}',t')$ can be written as the trace of the transition operator, which can be simplified to:

$$G(\mathbf{x},t;\mathbf{x}',t') =< \mathbf{x},t|\mathbf{x}',t' >= \delta(\mathbf{x}-\mathbf{x}')(t-t') \tag{2}$$

It is also important to introduce the retarded and advanced Green's functions G^R and G^A:

$$G^R(\mathbf{x},t;\mathbf{x}',t') = -i\theta(t-t')G(\mathbf{x},t;\mathbf{x}',t') \tag{3}$$

$$G^A(\mathbf{x},t;\mathbf{x}',t') = i\theta(t'-t)G(\mathbf{x},t;\mathbf{x}',t') \tag{4}$$

where $\theta(t-t')$ and $\theta(t'-t)$ are the Heaviside step functions. Since the retarded and advanced Green's functions propagate wave functions forwards and backwards in time respectively, they are only valid for certain t. The retarded Green's function G^R is only valid for $t \geq t'$ and the advanced Green's function G^A is only valid for $t \leq t'$, hence the need for the step functions θ in their expressions.

For the Green's functions it is more convenient to write them in momentum representation instead of position representation. In momentum representation the retarded and advanced Green's functions become:

$$G^R(\mathbf{p},t;\mathbf{p}',t') = -i\theta(t-t') < \mathbf{p},t|\mathbf{p}',t' >= -i\theta(t-t')\delta(\mathbf{p}-\mathbf{p}')(t-t') \tag{5}$$

$$G^A(\mathbf{p},t;\mathbf{p}',t') = i\theta(t'-t) < \mathbf{p},t|\mathbf{p}',t' >= i\theta(t'-t)\delta(\mathbf{p}-\mathbf{p}')(t-t') \tag{6}$$

It is convenient to introduce the operator counterparts of the Green's functions:

$$\hat{G}^R(t,t') = -i\theta(t-t')\hat{U}(t,t') \tag{7}$$

$$\hat{G}^A(t,t') = i\theta(t'-t)\hat{U}(t,t') \tag{8}$$

Here $\hat{U}$ is the operator that describes the time-evolution of the system in the time interval $(t-t')$ which can usually be described in terms of the time-evolution of the Hamiltonian.

$$\hat{U}(t,t') = \exp\left(\frac{-i}{\hbar}\hat{H}(t-t')\right) \tag{9}$$

This allows the Green's functions in momentum representation to be rewritten in terms of their operator forms:

$$G^R(\mathbf{p},t;\mathbf{p}',t') =< \mathbf{p}|\hat{G}^R(t,t')|\mathbf{p}' > \tag{10}$$

$$G^A(\mathbf{p},t;\mathbf{p}',t') =< \mathbf{p}|\hat{G}^A(t,t')|\mathbf{p}' > \tag{11}$$

which can be rewritten as:

$$G^R(\mathbf{p},t;\mathbf{p}',t') = -i\theta(t-t') < \mathbf{p}|\hat{U}(t,t')|\mathbf{p}' > \tag{12}$$

$$G^A(\mathbf{p},t;\mathbf{p}',t') = i\theta(t'-t) < \mathbf{p}|\hat{U}(t,t')|\mathbf{p}' > \tag{13}$$

These equations correspond to:

$$G^R(\mathbf{p},t;\mathbf{p}',t') = -i\theta(t-t')\exp\left(\frac{-i}{\hbar}H(t-t')\right)\delta(\mathbf{p}-\mathbf{p}') \tag{14}$$

$$G^A(\mathbf{p},t;\mathbf{p}',t') = i\theta(t'-t)\exp\left(\frac{-i}{\hbar}H(t-t')\right)\delta(\mathbf{p}-\mathbf{p}') \tag{15}$$

where δ is the Dirac delta function, and $\hbar$ is the Planck constant:

$$\hbar = \frac{h}{2\pi} \tag{16}$$

We assume the particles studied are all free particles that have the following Hamiltonian

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} \tag{17}$$

where m is the electron mass. Inserting this Hamiltonian into Eq.(14) and Eq.(15) and Fourier transforming the resulting expression with respect to time yields the expression for the bare Green's functions for a free particle:

$$G^R = \frac{1}{\epsilon - H + i0} \tag{18}$$

$$G^A = \frac{1}{\epsilon - H - i0} \tag{19}$$

The complex term $i0$ arises to ensure that the Green's functions are restricted to the domain specified by the Heaviside step functions.

2.2 Applications of Green's functions

2.2.1 Density of states

Using the bare Green's functions we can derive several properties of a system depending only its Hamiltonian. One of these properties is the d-dimensional density of states ν:

$$\nu(\epsilon) = \frac{1}{\pi}\mathrm{Im}\int \frac{dp^d}{(2\pi\hbar)^d}G^R \tag{20}$$

which, for a free particle with a Hamiltonian given by Eq.(17), yield the following density of states:

$$\nu(\epsilon) = \begin{cases} \frac{1}{\pi}\sqrt{\frac{2m}{\hbar^2\epsilon}}, & d=1 \\ \frac{m}{\pi\hbar^2}, & d=2 \\ \frac{m\sqrt{2m\epsilon}}{2\pi^2\hbar^3}, & d=3 \end{cases} \tag{21}$$

if we apply the residue theorem:

$$\oint_\gamma f(z)dz = 2\pi i\sum \mathrm{Res}(f,z_0) \tag{22}$$

to the density of states function Eq.(20). Residues encountered due to the Green's functions consist solely of simple poles, as Green's functions are linearly inverse. The residues for simple poles can be calculated using

$$\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z) \tag{23}$$

This will return the results of Eq.(21) for the density of states.

2.2.2 Fermi's Golden Rule

In another application we can calculate the transition probability per unit time, also known as Fermi's Golden Rule. Since Fermi's Golden Rule applies to perturbations of a Hamiltonian, we need to consider the perturbation of the Green's functions due to a potential V. The derivation of Fermi's Golden rule will also follow the derivation used in Rammer [2]. The first order perturbation term of the retarded Green's function is given by:

$$G_1^R(\mathbf{p}, t; \mathbf{p}', t') = \frac{1}{\hbar} \int_{-\infty}^{\infty} dt_1 \int_{p_1', p_1} G_0^R(\mathbf{p}, t; \mathbf{p}_1, t_1) < \mathbf{p}_1, t_1 | \hat{V}(t_1) | \mathbf{p}_1', t_1 > G_0^R(\mathbf{p}_1', t_1; \mathbf{p}', t') \tag{24}$$

so that we can write the total Green's function as:

$$G^R(\mathbf{p}, t; \mathbf{p}', t') = G_0^R(\mathbf{p}, t; \mathbf{p}', t') + G_1^R(\mathbf{p}, t; \mathbf{p}', t') \tag{25}$$

Here G_0^R is the Green's functions in the absence of a potential and is given by Eq.(12). The term in bra-ket notation is given by:

$$< \mathbf{p}_1, t_1 | \hat{V}(t_1) | \mathbf{p}_1', t_1 > = < \mathbf{p} | \hat{V} | \mathbf{p}' > = (2\pi\hbar)^{-3} \int e^{-\frac{i}{\hbar} x(p - p')} V(\mathbf{x}, t) d\mathbf{x} \tag{26}$$

If for a constant potential V we assume that at time $t' = 0$ has a momentum $\mathbf{p}'$ then Eq.(24) simplifies to:

$$\begin{aligned} G_1^R(\mathbf{p}, t; \mathbf{p}', t' = 0) &= -\frac{< \mathbf{p} | \hat{V} | \mathbf{p}' >}{\hbar} \int_0^t dt' e^{-\frac{i}{\hbar} \epsilon_p (t - t')} e^{-\frac{i}{\hbar} \epsilon_{p'} t} \\ &= -\frac{< \mathbf{p} | \hat{V} | \mathbf{p}' >}{\hbar} e^{-\frac{i}{\hbar} \epsilon_p t} \frac{e^{\frac{i}{\hbar} t (\epsilon_p - \epsilon_{p'})} - 1}{\frac{i}{\hbar} (\epsilon_p - \epsilon_{p'})} \end{aligned} \tag{27}$$

In order to find the transition probability we need the absolute value of the square of this expression, which is given by:

$$|G_1^R(\mathbf{p}, t; \mathbf{p}', t' = 0)|^2 = 2 | < \mathbf{p} | \hat{V} | \mathbf{p}' > |^2 \frac{1 - \cos \frac{t}{\hbar} (\epsilon_p - \epsilon_{p'})}{(\epsilon_p - \epsilon_{p'})^2} \tag{28}$$

In the limit of large times the second term of this function behaves similar to a delta function and it can be simplified to:

$$|G_1^R(\mathbf{p}, t; \mathbf{p}', t' = 0)|^2 = 2 | < \mathbf{p} | \hat{V} | \mathbf{p}' > |^2 \frac{t\pi}{\hbar} \delta(\epsilon_p - \epsilon_{p'}) \tag{29}$$

from this we can determine the probability per unit time to be:

$$\Gamma_{pp'} = \frac{2\pi}{\hbar} | < \mathbf{p} | \hat{V} | \mathbf{p}' > |^2 \delta(\epsilon_p - \epsilon_{p'}) \tag{30}$$

which is Fermi's Golden Rule.

3 Linear response theory

For this thesis, we are interested in the response of electrons to certain perturbations. This will be done using linear response theory. In this thesis we will look at three quantities. The first is the conductivity, which is the response of the velocity to an external electric field. The second will be the spin-orbit torque, which is the response of the electron spin to an external magnetic field. The third and final quantity is the Gilbert damping, which is the response of the electron spin to a change in the direction of the magnetization. The calculations will be based on the Kubo formula:

$$\sigma_{\alpha\beta} = \frac{e^2\hbar}{2\pi}\int \frac{d\mathbf{p}}{(2\pi\hbar)^2}[2G^R v_\alpha G^A v_\beta - G^A v_\alpha G^A v_\beta - G^R v_\alpha G^R v_\beta] \tag{31}$$

Here $v_{\alpha,\beta}$ is the velocity operator of the system and $\sigma_{\alpha\beta}$ is the conductivity tensor. We must treat the crystal we look at in this thesis as a crystal with impurities, since in a perfect crystal momentum is conserved and as a result the conductivity becomes infinite since there is no mechanism to stop electrons from accelerating. These impurities will lead to finite conductivity, spin-orbit torque, and Gilbert damping. In order to treat this disorder accurately we will need to treat it in two steps. The first step replacing the Green's functions G^R and G^A in Eq.(31) with Green's functions in the Born approximation. The second step consists of performing vertex corrections.

3.1 Born approximation

We can include the disorder into the self-energy of the system. The self-energy is the potential due to the interaction of a many-body system, in this case the lattice, with the electrons. We are strictly interested in the complex part of the self-energy. The real part of the self-energy usually tends to infinity, but can be disregarded if we compensate for the effects of the self-energy in a process called renormalization. In the following, we assume that the self-energy is strictly imaginary. An expression for the self-energy Σ is defined in the Born approximation as [3]:

$$\Sigma^{R(A)} = 2\pi\alpha\int \frac{d\mathbf{p}}{(2\pi\hbar)^2}G^{R(A)} \tag{32}$$

where α is the parameter that represents the disorder strength. Inserting the bare Green's functions into the expression for the self-energy will return the disorder-averaged Green's functions:

$$G^R = \frac{1}{\epsilon - H - \Sigma^R} \tag{33}$$

$$G^A = \frac{1}{\epsilon - H - \Sigma^A} \tag{34}$$

As an example, we can calculate the self-energy for a free particle electron with Hamiltonian Eq.(17). The retarded Green's function is:

$$G^R = \frac{1}{\epsilon - \frac{p^2}{2m} + i0} \tag{35}$$

Inserting this into Eq.(32) gives:

$$\Sigma^R = 2\pi\alpha\int \frac{d\mathbf{p}}{(2\pi\hbar)^2}\frac{1}{\epsilon - \frac{p^2}{2m} + i0} \tag{36}$$

This integral is easier to solve in polar coordinates: $d\mathbf{p} \to pdpd\theta$. This means:

$$\Sigma^R = 2\pi\alpha\int \frac{pdp}{(2\pi\hbar)^2}\frac{1}{\epsilon - \frac{p^2}{2m} + i0}\int_0^{2\pi} d\theta \tag{37}$$

The first integral is simply 2π. Then $\hbar$ can be set to 1 for convenience resulting in the integral:

$$\Sigma^R = \alpha \int p dp \frac{1}{\epsilon - \frac{p^2}{2m} + i0} \tag{38}$$

Now we can apply the substitution:

$$q = p^2 \quad dq = 2pdp \tag{39}$$

This allows the integral to be solved with the residue method, since the integral

$$\Sigma^R = \frac{\alpha}{2} \int \frac{dq}{\epsilon - \frac{q}{2m} + i0} \tag{40}$$

now has a simple pole. The residue method gives the following self-energy:

$$\Sigma^R = -i\pi\alpha m \tag{41}$$

for a free particle electron. In a more relevant example, we can calculate the self energy of the 2D Dirac fermion. The Hamiltonian for this particle is:

$$H = v_f(p\cos(\theta)\sigma_1 + p\sin(\theta)\sigma_2) + m\sigma_3 \tag{42}$$

where σ are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{43}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{44}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{45}$$

The Green's function of this particle is:

$$G^R = (\epsilon\sigma_0 - (p\cos(\theta)\sigma_1 + p\sin(\theta)\sigma_2) - m\sigma_3)^{-1} \tag{46}$$

where we have introduced the matrix σ_0, which is the identity matrix:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{47}$$

Since we now use matrices, the Green's function becomes a matrix inverse:

$$\begin{aligned} G^R &= \begin{pmatrix} \epsilon - m & -p\cos(\theta) + ip\sin(\theta) \\ -p\cos(\theta) - ip\sin(\theta) & \epsilon + m \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -\dfrac{m+\epsilon}{m^2+p^2-\epsilon^2} & -\dfrac{p\cos(\theta) - ip\sin(\theta)}{m^2+p^2-\epsilon^2} \\ -\dfrac{p\cos(\theta) + ip\sin(\theta)}{m^2+p^2-\epsilon^2} & \dfrac{m-\epsilon}{m^2+p^2-\epsilon^2} \end{pmatrix} \end{aligned} \tag{48}$$

Inserting this expression into the equation for the self-energy in this case gives:

$$\Sigma^R = 2\pi\alpha \int \frac{pdp}{(2\pi)^2} \int_0^{2\pi} d\theta \begin{pmatrix} -\dfrac{m+\epsilon}{m^2+p^2-\epsilon^2} & -\dfrac{p\cos(\theta) - ip\sin(\theta)}{m^2+p^2-\epsilon^2} \\ -\dfrac{p\cos(\theta) + ip\sin(\theta)}{m^2+p^2-\epsilon^2} & \dfrac{m-\epsilon}{m^2+p^2-\epsilon^2} \end{pmatrix} \tag{49}$$

where, again, $\hbar$ is set to 1. Integrating over angles gives:

$$\Sigma^R = 2\pi\alpha \int \frac{pdp}{(2\pi)^2} \begin{pmatrix} -2\pi \dfrac{m+\epsilon}{m^2+p^2-\epsilon^2} & 0 \\ 0 & 2\pi \dfrac{m-\epsilon}{m^2+p^2-\epsilon^2} \end{pmatrix} \tag{50}$$

Applying substitution Eq.(39) and solving using the residue method returns the following expression for Σ^R:

$$\Sigma^R = -\frac{i\pi\alpha}{2}(\epsilon\sigma_0 + m\sigma_3) \tag{51}$$

3.2 Vertex correction

In another step of disorder averaging, we can replace the function that describes the vertices of the Feynman diagrams by correcting them for any disorder lines that might appear in the diagram. This process is called vertex correction. The vertex corrected function of a function f can be written in the ladder approximation as:

$$f^{vc} = f + f^{(1)} + f^{(2)} + \dots \tag{52}$$

where $f^{(n)}$ is the function that contains n disorder lines and can be calculated using:

$$f^{(n)} = 2\pi\alpha \int \frac{d\mathbf{p}}{(2\pi)^2} G^R f^{(n-1)} G^A \tag{53}$$

where the Green's functions G^R and G^A are the disorder-averaged Green's functions Eq.(33) and Eq.(34). This expression can also be written in terms of the basis elements of f called f_i:

$$f_0^{(1)} = 2\pi\alpha \int \frac{d\mathbf{p}}{(2\pi)^2} G^R f_0 G^A = C_{00}f_0 + C_{01}f_1 + C_{02}f_2 + \dots + C_{0n}f_n \tag{54}$$

$$f_1^{(1)} = 2\pi\alpha \int \frac{d\mathbf{p}}{(2\pi)^2} G^R f_1 G^A = C_{10}f_0 + C_{11}f_1 + C_{12}f_2 + \dots + C_{1n}f_n \tag{55}$$

$$f_2^{(1)} = 2\pi\alpha \int \frac{d\mathbf{p}}{(2\pi)^2} G^R f_2 G^A = C_{20}f_0 + C_{21}f_1 + C_{22}f_2 + \dots + C_{2n}f_n \tag{56}$$

et cetera, for all n basis elements of f. These expressions constitute a matrix multiplication:

$$\begin{pmatrix} f_0^{(1)} \\ f_1^{(1)} \\ f_2^{(1)} \\ \vdots \\ f_n^{(1)} \end{pmatrix} = \begin{pmatrix} C_{00} & C_{01} & C_{02} & \dots & C_{0n} \\ C_{10} & C_{11} & C_{12} & \dots & C_{1n} \\ C_{20} & C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{n0} & C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \tag{57}$$

From this it follows that:

$$f_i^{(1)} = \sum_j C_{ij} f_j, \quad f_i^{(2)} = \sum_j C_{ij} f_j^{(1)}, \quad \dots \tag{58}$$

so that the vertex corrected function basis elements can be written as:

$$f_i^{vc} = \sum_j [1 + C + C^2 + C^3 + ...]_{ij} f_j \tag{59}$$

This means the total vertex corrected function is simply

$$\begin{aligned} f^{vc} &= f + Cf + C^2 f + \cdots + C^n f \\ &= \frac{1}{1-C} f \end{aligned} \tag{60}$$

as long as the determinant of C is smaller than 1. For this expression to be valid two conditions need to be satisfied. The first is that the trace operation of two basis elements must be orthogonal:

$$\mathrm{Tr} f_i f_j = N \delta_{ij} \tag{61}$$

where δ_{ij} is the Kronecker delta. The second condition is that none of the eigenvalues of the matrix C can be equal to 1. This condition is necessary to let the expression $(1-C)^{-1}$ exist at all times. For the basis we use in this thesis we expand a basis of vector products of Pauli matrices:

$$\omega = [\sigma_0 \otimes \tau_0, \ldots, \sigma_0 \otimes \tau_3, \sigma_1 \otimes \tau_0, \ldots, \sigma_1 \otimes \tau_3, \sigma_2 \otimes \tau_0, \ldots, \sigma_2 \otimes \tau_3, \sigma_3 \otimes \tau_0, \ldots, \sigma_3 \otimes \tau_3] \tag{62}$$

resulting in a basis with 16 elements. We can then define a 16×16 matrix that has the property:

$$2\pi\alpha \int \frac{d\mathbf{p}}{(2\pi)^2} G^R \omega_\alpha G^A \omega_\beta = \sum_{\beta=1}^{16} F_{\alpha\beta} \omega_\beta \tag{63}$$

which can be rewritten to:

$$F_{\alpha\beta} = \frac{\pi\alpha}{2} \mathrm{Tr} \int \frac{d\mathbf{p}}{(2\pi)^2} G^R \omega_\alpha G^A \omega_\beta \tag{64}$$

which is called the response matrix. Using vertex-correction the response matrix can be replaced by the dressed response matrix:

$$\begin{aligned} \tilde{F} &= \frac{2}{\pi\alpha} \mathrm{Tr} \int \frac{d\mathbf{p}}{(2\pi)^2} [1 + F_{\alpha\beta} + F_{\alpha\beta}^2 + \ldots] G^R \omega_\alpha G^A \omega_\beta \\ &= [1 + F_{\alpha\beta} + F_{\alpha\beta}^2 + \ldots] \frac{2}{\pi\alpha} \mathrm{Tr} \int \frac{d\mathbf{p}}{(2\pi)^2} G^R \omega_\alpha G^A \omega_\beta \\ &= \frac{2}{\pi\alpha} \frac{F_{\alpha\beta}}{1 - F_{\alpha\beta}} \end{aligned} \tag{65}$$

We can now also introduce the expressions for the bare quantities we want to measure: the bare conductivity $\sigma_{\alpha\beta}$, the bare spin-orbit torque $R_{\alpha\beta}$, and the bare Gilbert damping $\Gamma_{\alpha\beta}$:

$$\sigma_{\alpha\beta} = \frac{e^2}{2\pi} \mathrm{Tr} \int \frac{d\mathbf{p}}{(2\pi)^2} G^R v_\alpha G^A v_\beta \tag{66}$$

$$R_{\alpha\beta} = \Delta_{sd} \mathrm{Tr} \int \frac{d\mathbf{p}}{(2\pi)^2} G^R s_\alpha G^A v_\beta \tag{67}$$

$$\Gamma_{\alpha\beta} = \Delta_{sd}^2 \mathrm{Tr} \int \frac{d\mathbf{p}}{(2\pi)^2} G^R s_\alpha G^A s_\beta \tag{68}$$

where v_i is the velocity operator:

$$v_i = \sigma_i \otimes \tau_0 \tag{69}$$

and s_i is the spin operator:

$$s_i = \sigma_0 \otimes \tau_i \tag{70}$$

These operators turn out to correspond to the elements of the chosen basis Eq.(62) we use in this thesis. This means that the expressions for the integral terms of the bare quantities will

be contained in the response matrix $F_{\alpha\beta}$, since we see that the velocity operators in the x- and y-direction respectively correspond to the fifth and ninth basis elements. The spin operators correspond to the second, third, and fourth basis elements. We can then also express the bare quantities in terms of the elements of the response matrix:

$$\sigma = \frac{e^2}{2\pi}\frac{2}{\pi\alpha}\begin{pmatrix} F_{55} & F_{59} \\ F_{95} & F_{99} \end{pmatrix} \tag{71}$$

$$R = \Delta_{sd}\frac{2}{\pi\alpha}\begin{pmatrix} F_{25} & F_{29} \\ F_{35} & F_{39} \\ F_{45} & F_{49} \end{pmatrix} \tag{72}$$

$$\Gamma = \Delta_{sd}^2\frac{2}{\pi\alpha}\begin{pmatrix} F_{22} & F_{23} & F_{24} \\ F_{32} & F_{33} & F_{34} \\ F_{42} & F_{43} & F_{44} \end{pmatrix} \tag{73}$$

This dressed response matrix contains the terms for the dressed conductivity tensor, spin-orbit torque and Gilbert damping in its matrix elements. In similar fashion to the bare response matrix, we can obtain expressions for the dressed quantities in terms of elements of the dressed response matrix $\tilde{F}$:

$$\tilde{\sigma} = \frac{e^2}{2\pi}\frac{2}{\pi\alpha}\begin{pmatrix} \tilde{F}_{55} & \tilde{F}_{59} \\ \tilde{F}_{95} & \tilde{F}_{99} \end{pmatrix} \tag{74}$$

$$\tilde{R} = \Delta_{sd}\frac{2}{\pi\alpha}\begin{pmatrix} \tilde{F}_{25} & \tilde{F}_{29} \\ \tilde{F}_{35} & \tilde{F}_{39} \\ \tilde{F}_{45} & \tilde{F}_{49} \end{pmatrix} \tag{75}$$

$$\tilde{\Gamma} = \Delta_{sd}^2\frac{2}{\pi\alpha}\begin{pmatrix} \tilde{F}_{22} & \tilde{F}_{23} & \tilde{F}_{24} \\ \tilde{F}_{32} & \tilde{F}_{33} & \tilde{F}_{34} \\ \tilde{F}_{42} & \tilde{F}_{43} & \tilde{F}_{44} \end{pmatrix} \tag{76}$$

4 Magnetization dynamics

In order to understand why we are deriving these quantities, it is important to understand what they are and what they can be used for. To do so, we take a look at magnetization dynamics, which is the evolution of the magnetization of a system. Since magnetization is nothing more than the magnetic moment $\mathbf{m}$ of the system per unit volume, we will focus on the time-evolution of these magnetic moments. The magnetic moments behave differently depending on the type of magnet and as such they will also behave differently when an electric current is applied to them. In a ferromagnet all magnetic moments will face the same direction when a magnetic field is applied to it, see Figure 2.

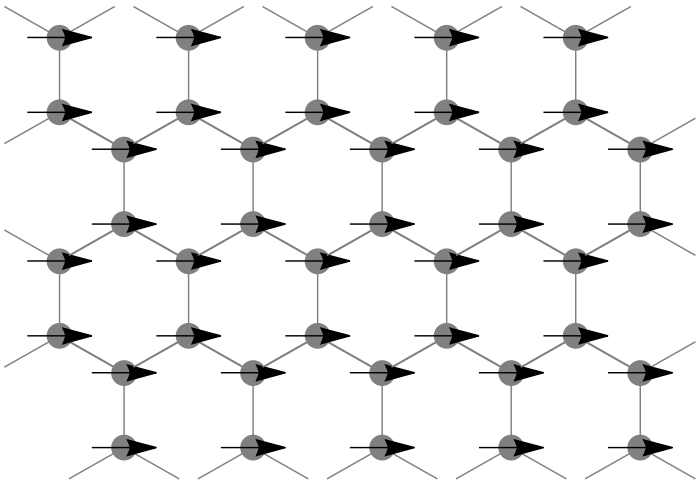

Figure 2: The alignment of magnetic moments in a honeycomb ferromagnet.

In an antiferromagnet any neighbouring magnetic moments are aligned in opposite directions, as is shown in Figure 3. This will result in a net magnetic moment of zero for the magnet. However, if an external magnetic field is applied to an antiferromagnet it can display a nonzero net magnetic moment. This phenomenon is known as ferrimagnetism.

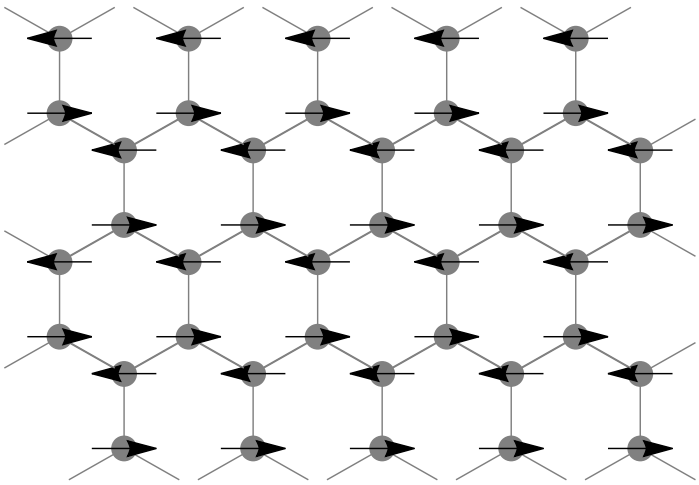

Figure 3: The alignment of magnetic moments in a honeycomb antiferromagnet.

4.1 Spin-orbit torque

If an external electric field is sent through a material electrons can exchange their spin-orbit torque with any impurities they collide with. The linear response formula is used to calculate the average spin-density $\mathbf{s}$. If in any magnet the spin of the atoms is not aligned with the local magnetic moment there will be a torque acting on both the spin and magnetic moment. The rotation of the magnetic moments due to these torques is described by the Landau-Lifshitz equation [4]:

$$\frac{d\mathbf{m}}{dt} = -\gamma(\mathbf{m} \times \mathbf{H}_{eff}) - \gamma\frac{\alpha}{m_s}\mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{eff}) \tag{77}$$

where H_{eff} is the effective magnetic field experienced by the atoms, α is a dimensionless damping parameter, and m_s is the magnetic moment of the system at the saturation magnetization, which is the point where an increase in magnetic field will not increase the magnetic moment further. γ is the gyromagnetic ratio, which for a free electron is:

$$\gamma \approx 1.761 \times 10^{11}\mathrm{s}^{-1}\mathrm{T}^{-1} \tag{78}$$

It is also possible to define the spin-orbit torque in terms of the spin-density $\mathbf{s}$:

$$\frac{d\mathbf{m}}{dt} = \mathbf{m} \times \mathbf{s} \tag{79}$$

Since we can calculate $\mathbf{s}$ microscopically, it is possible to use this equation as a substitute to the Landau-Lifshitz equation. Furthermore, since the direction of the magnetic moment is determined by the spin direction we can also substitute $\mathbf{m}$ with the direction of localized spins $\mathbf{n}$.

4.2 Gilbert damping

The first term in Eq.(77) describes the precession of the magnetic moments due to the external magnetic field. The second term represents the influence of damping on the magnetic moment, or the rate at which the magnetic moment returns to its equilibrium. In 1955 Gilbert replaced the second term of the Landau-Lifshitz equation, resulting in the so-called Landau-Lifshitz-Gilbert equation [5]:

$$\frac{d\mathbf{m}}{dt} = -\gamma'\left(\mathbf{m} \times \mathbf{H}_{eff}\right) - \frac{\alpha}{m_s}\left(\mathbf{m} \times \frac{d\mathbf{m}}{dt}\right) \tag{80}$$

where the damping term now depends on the time derivative of the magnetic moment instead of the effective magnetic field. Eq.(80) is actually equivalent to Eq.(77) under the relation:

$$\gamma' = \gamma(1 + \alpha^2) \tag{81}$$

It should be noted that both Eq.(77) and Eq.(80) originally describe the rotation of the magnetization itself. However, since magnetization is simply the magnetic moment per unit volume it is valid to substitute the magnetization with the magnetic moment. We can use the linear response formula to determine the damping constant α. It should also be noted that in reality Eq.(80) is more complex, as it will gain additional terms.

5 Model

In modeling lattices, it would be ideal to describe the lattice as a Bravais lattice. This is because Bravais lattices will look the same in each direction when viewed from any lattice point. The problem however is that a honeycomb lattice is not a Bravais lattice on a one-atom per unit cell basis. This can be circumvented however since a honeycomb lattice can actually be viewed as a Bravais lattice with a two-atom per unit cell basis instead. This will result in two sublattices, see Figure 4. This means that the model must account for any interactions between atoms from different sublattices, as an atom from the red sublattice only has neighbors from the blue sublattice and vice versa.

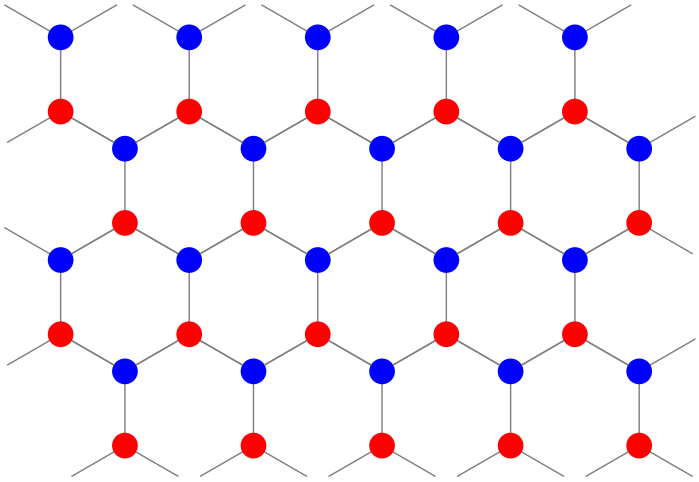

Figure 4: A representation of the two sublattices (colored red and blue) that appear when treating the honeycomb lattice as a Bravais lattice with a two-atom basis.

Ideally we would like to treat the atoms on the lattice as Dirac fermions, since we can then treat them analogously to the earlier examples. This means that the dispersion relation for the lattice needs to be linear in the wave vector k. It turns out that this is indeed the case, as the energies of this lattice are [6]:

$$E = \pm\hbar\nu k \tag{82}$$

if the assumption is that the Hamiltonian of the system is the tight-binding Hamiltonian.

5.1 Absence of spin-orbit interaction

The Hamiltonian for the honeycomb lattice in the two-atom per unit cell basis is the tight-binding Hamiltonian with only interactions between nearest neighbours. This Hamiltonian can be expressed in terms of annihilation and creation operators [7]. The Hamiltonian can be written as:

$$H = \sum_{\langle i,j\rangle} t c_i c_j^\dagger + \sum_i J_{sd} c_{i,\sigma} (\mathbf{n}\cdot\boldsymbol{\sigma})_{\sigma\sigma'} c_{i,\sigma'}^\dagger + h.c. \tag{83}$$

since this is the tight-binding Hamiltonian without any spin-orbit term [8]. The spin-orbit term will be added later as a perturbation. Here $\mathbf{n}$ is the direction of localized spins, t is the nearest neighbor hopping energy, J_{sd} is the exchange interaction between classical spins and the spins of conducting electrons, and $c_{i,\sigma}$ and $c_{i,\sigma}^\dagger$ are the annihilation and creation operators for an electron at position i with spin σ. In our case we can simplify this Hamiltonian to the continuous low-energy Hamiltonian:

$$H = v_f(\mathbf{p}\cdot\boldsymbol{\tau})\otimes\sigma_0\otimes\Lambda_0 + \Delta_{sd}\tau_0\otimes(\mathbf{n}\cdot\boldsymbol{\sigma})\otimes\Lambda_0 \tag{84}$$

where the exchange interaction is now given by Δ_{sd} and σ_i,τ_i, and Λ_i are Pauli matrices acting on spin, sublattices, and valley-space respectively. For a case of an ideal honeycomb lattice, the valley spaces can be decoupled from the Hamiltonian, even under small external influences [6]. This means we can reduce the Hamiltonian without any spin-orbit interaction to:

$$H = v_f(\mathbf{p}\cdot\boldsymbol{\tau})\otimes\sigma_0 + \Delta_{sd}\tau_0\otimes(\mathbf{n}\cdot\boldsymbol{\sigma}) \tag{85}$$

In this thesis, we will choose the coordinate system so that $\mathbf{n}$ lies in the x-z plane.

$$\mathbf{n} = (n_x, 0, n_z) \tag{86}$$

while we take the crystal surface to be in the x-y plane. With the direction of localized spins given by Eq.(86) the full Hamiltonian that describes the honeycomb lattice without any spin-orbit interaction is:

$$H = v_f(p\cos(\theta)\tau_1 + p\sin(\theta)\tau_2)\otimes\sigma_0 + \Delta_{sd}\tau_0\otimes(n_x\sigma_1 + n_z\sigma_3) \tag{87}$$

where θ is the angle between the x- and y-components of the momentum vector. Furthermore, since n_x and n_z are components of a direction vector we have the condition:

$$n_x^2 + n_z^2 = 1 \tag{88}$$

to ensure that $\mathbf{n}$ is a normalized vector. In order to quantify the effect of a spin-orbit interaction we need to first introduce a spin-orbit interaction. For this thesis, we take the Rashba spin-orbit interaction:

$$\begin{aligned} H' &= H + \lambda(\boldsymbol{\sigma}\times\boldsymbol{\tau})_z \\ &= H + \lambda(\sigma_1\otimes\tau_2 - \sigma_2\otimes\tau_1) \end{aligned} \tag{89}$$

where λ is the spin-orbit interaction. As a result of using only the z-component of the curl any spin-orbit quantity is reduced to a two-dimensional quantity as it will only have components in the x- and y-directions.

To treat this extra term of the Hamiltonian perturbation theory can be used, since the spin-orbit parameter λ can be taken to be small compared to the energy term in the Green's functions. Interestingly however, the same can also apply to the exchange parameter Δ_{sd}, which leads to two different cases: one where Δ_{sd} is taken to still be a leading term over λ and therefore the spin-orbit term must be treated using perturbation theory, and the case where the exchange term must be treated using perturbation theory since we take λ to be a leading term instead. First of all, however, we will take a look at the cases without including any perturbation terms. If Δ_{sd} is the leading term, the Hamiltonian of the system is given by Eq.(84). However, if we take λ to be the leading term instead, the Hamiltonian becomes:

$$H = v_f(p\cos(\theta)\tau_1 + p\sin(\theta)\tau_2)\otimes\sigma_0 + \lambda(\sigma_1\otimes\tau_2 - \sigma_2\otimes\tau_1) \tag{90}$$

Since these Hamiltonians have no small terms perturbation theory does not need to be used yet. We can calculate $\tilde{F}$, and therefore by extension $\tilde{\sigma}$, $\tilde{R}$, and $\tilde{\Gamma}$ using Eq.(65) and Eq.(74), Eq.(75), and Eq.(76) respectively.

5.2 Presence of spin-orbit interaction

We can write out the entire Hamiltonian as:

$$H = H_p + H_\Delta + H_\lambda \tag{91}$$

where H_p, H_Δ, and H_λ are the Hamiltonian terms depending on the momentum vector $\mathbf{p}$, the exchange parameter Δ_{sd}, and the spin-orbit interaction λ respectively. The eigenvalues of both H_Δ and H_λ are small so that we can use perturbation theory. This gives two possibilities for the choice of unperturbed Hamiltonian H_0 and perturbation to the Hamiltonian δH:

- Case 1: $H = \underbrace{H_p + H_\Delta}_{H_0} + \underbrace{H_\lambda}_{\delta H}$
- Case 2: $H = \underbrace{H_p + H_\lambda}_{H_0} + \underbrace{H_\Delta}_{\delta H}$

First λ and Δ_{sd} are substituted by the dimensionless variables:

$$\overline{\lambda} = \frac{\lambda}{\epsilon} \tag{92}$$

$$\overline{\Delta} = \frac{\Delta_{sd}}{\epsilon} \tag{93}$$

since, for convenience, the energy ϵ can be set to 1. Furthermore, λ and Δ_{sd} are taken to be small compared to the energy so that perturbation theory can be used:

$$\begin{aligned}(\epsilon - H - \Sigma^{R(A)} - \delta H)^{-1} &= (\epsilon - H - \Sigma^{R(A)})^{-1} + (\epsilon - H - \Sigma^{R(A)})^{-1}\delta H(\epsilon - H - \Sigma^{R(A)})^{-1} \\ &= G^{R(A)} + G^{R(A)}\delta H G^{R(A)}\end{aligned} \tag{94}$$

In this case we only discuss the leading terms, so we take the perturbation only to first order. We can calculate the elements of the response matrix for this perturbed system using:

$$F'_{\alpha\beta} = \frac{\pi\alpha}{2}\mathrm{Tr}\int \frac{d\mathbf{p}}{(2\pi)^2}(G^R + G^R\delta H G^R)\omega_\alpha(G^A + G^A\delta H G^A)\omega_\beta \tag{95}$$

where $\omega_{\alpha,\beta}$ are basis elements of the basis Eq.(62). This expression can actually split up to calculate the perturbation terms of $F'_{\alpha\beta}$. The zeroth order term is given by Eq.(64). The first order perturbation term is simply:

$$F'_{\alpha\beta,1} = \frac{\pi\alpha}{2}\mathrm{Tr}\int \frac{d\mathbf{p}}{(2\pi)^2}[(G^R\delta H G^R)\omega_\alpha G^A\omega_\beta + G^R\omega_\alpha(G^A\delta H G^A)\omega_\beta] \tag{96}$$

and the second order perturbation term is:

$$\begin{aligned}F'_{\alpha\beta,2} = \frac{\pi\alpha}{2}\mathrm{Tr}\int \frac{d\mathbf{p}}{(2\pi)^2}[&(G^R\delta H G^R\delta H G^R)\omega_\alpha G^A\omega_\beta + G^R\omega_\alpha(G^A\delta H G^A\delta H G^A)\omega_\beta \\ &+ (G^R\delta H G^R)\omega_\alpha(G^A\delta H G^A)\omega_\beta]\end{aligned} \tag{97}$$

Analogously to Eq.(65) we can obtain the expression for the dressed matrix of the perturbed system:

$$\tilde{F}' = \frac{2}{\pi\alpha}\frac{F'_{\alpha\beta}}{1 - F'_{\alpha\beta}} \tag{98}$$

However it is possible that $F'_{\alpha\beta}$ has an eigenvalue that is equal to 1, meaning that $(1 - F'_{\alpha\beta})$ is not invertible. It is possible to project out the eigenvalue of 1 by finding a normalized eigenvector v of $(1 - F'_{\alpha\beta})$ that also has an eigenvalue of zero and constructing a projection operator:

$$P = 1 - v \otimes v \tag{99}$$

with it. Since the problematic eigenvalue in $(1 - F'_{\alpha\beta})$ is related to the self-energy in the Born approximation, the necessary eigenvector can be found by examining the self-energy in terms of the basis elements and constructing a vector from it. Then v becomes an eigenvector of the new matrix $(1 - P \cdot F'_{\alpha\beta})$. We use the notation

$$F = P \cdot F'_{\alpha\beta} \tag{100}$$

to denote the new term. Now that $(1-F)$ is invertible it is possible to apply perturbation theory to this expression:

$$(1-F-\delta F)^{-1} = (1-F)^{-1} + (1-F)^{-1}\delta F(1-F)^{-1} + \dots \tag{101}$$

This means the expression Eq.(98) becomes:

$$\tilde{F'} = \frac{2}{\pi\alpha}\frac{(F+\delta F)}{(1-F-\delta F)} \tag{102}$$

If we then sort the terms of F by their orders in either Δ_{sd} or λ depending on which term is the perturbation, we can determine how to calculate the matrices for only a certain order of these two variables. We define the matrix F_n as the matrix containing the n-th order terms of the pertubation parameter. For this the equation that needs to be written out is:

$$\frac{(F_0+F_1+F_2)}{(1-F_0-F_1-F_2)} = [(1-F_0)^{-1} + (1-F_0)^{-1}(F_1+F_2)(1-F_0)^{-1}](F_0+F_1+F_2) \tag{103}$$

where the perturbation is contained in the F_1 and F_2 matrices. For simplicity we only take terms up to second order perturbations, which are:

$$\begin{array}{ll} (1-F_0)^{-1}F_0 & \text{zeroth order term} \\ (1-F_0)^{-1}F_1(1-F_0)^{-1} & \text{first order term} \\ (1-F_0)^{-1}F_1(1-F_0)^{-1}F_1 + (1-F_0)^{-1}F_2(1-F_0)^{-1} & \text{second order term} \end{array} \tag{104}$$

The interesting results for these cases arise in the limits of these variables, notably the limits for very small (to zero) $\overline{\lambda}$ and $\overline{\Delta}$, and very large (to ∞) $\overline{\lambda}$ and $\overline{\Delta}$. There are two phenomena that we expect to see in these cases. The first effect is anisotropy, where the properties of the lattice which in this case are the conductivity, spin-orbit torque, and Gilbert damping, are different depending on the direction [9, 10]. The other effect is symmetry breaking, in which the tensors become asymmetric, implying that there is a stronger effect in some directions than others [10, 11]. Both these effects can be observed in the matrix form for the properties we look at. Symmetry breaking can be seen in the matrix if the matrix observed becomes asymmetric upon adding the perturbation term to it. Anisotropy is represented by one or more matrix elements on the diagonal being different from the other diagonal elements of the matrix.

6 Results

6.1 Cases without perturbation terms

6.1.1 Exchange interaction Hamiltonian

The Hamiltonian where the exchange parameter is the leading term is the previously mentioned low energy Hamiltonian Eq.(84):

$$H = v_f(\mathbf{p}\cdot\boldsymbol{\tau})\otimes\sigma_0 + \Delta_{sd}\tau_0\otimes(n_x\sigma_1 + n_z\sigma_3) \tag{105}$$

The bare Green's function without self energy for this Hamiltonian is:

$$G = \begin{pmatrix} \epsilon - n_z\Delta_{sd} & -n_x\Delta_{sd} & -p\cos\theta + ip\sin\theta & 0 \\ -n_x\Delta_{sd} & \epsilon + n_z\Delta_{sd} & 0 & -p\cos\theta + ip\sin\theta \\ -p\cos\theta - ip\sin\theta & 0 & \epsilon - n_z\Delta_{sd} & -n_x\Delta_{sd} \\ 0 & -p\cos\theta - ip\sin\theta & -n_x\Delta_{sd} & \epsilon + n_z\Delta_{sd} \end{pmatrix}^{-1} \tag{106}$$

where we for now ignore the $i0$-term in the expression for the Green's functions since it is effectively zero. Inserting this matrix into Eq.(32) gives the self-energy for this Hamiltonian:

$$\Sigma^R = -\frac{i\pi\alpha}{2}[\epsilon(\sigma_0\otimes\sigma_0) - n_x\Delta_{sd}(\sigma_0\otimes\sigma_1) - n_z\Delta_{sd}(\sigma_0\otimes\sigma_3)] \tag{107}$$

or is terms of basis elements:

$$\Sigma^R = -\frac{i\pi\alpha}{2}(\epsilon\omega_1 - n_x\Delta_{sd}\omega_2 - n_z\Delta_{sd}\omega_4) \tag{108}$$

Now this term can be added to the bare Green's functions to get the disorder-averaged Green's functions. Out of cautiousness it is better to also first construct the projection operator and apply it to the response matrix. This can be safely done since it would only project out the eigenvalue of 1 and not change anything else. The needed eigenvector is:

$$v = \left[\frac{\epsilon}{\sqrt{\epsilon^2+\Delta_{sd}^2}}, -\frac{n_x\Delta_{sd}}{\sqrt{\epsilon^2+\Delta_{sd}^2}}, 0, -\frac{n_z\Delta_{sd}}{\sqrt{\epsilon^2+\Delta_{sd}^2}}, 0,0,0,0,0,0,0,0,0,0,0,0\right] \tag{109}$$

where the denominator is needed to ensure that the vector is indeed normalized. We can then calculate the dressed response matrices using Eq.(65). This matrix contains a lot of expressions in terms of the disorder parameter α, but we can simplify this problem by treating the parameter as small and then expanding a Taylor series around the point $\alpha = 0$. Using this simplification the dressed conductivity tensor for this Hamiltonian according to Eq.(74) is:

$$\tilde{\sigma}_0 = \frac{e^2}{\pi}\frac{1}{\pi\alpha}\begin{pmatrix} -\dfrac{3\Delta_{sd}^2+\epsilon^2}{9\Delta_{sd}^2-\epsilon^2} & 0 \\ 0 & -\dfrac{3\Delta_{sd}^2+\epsilon^2}{9\Delta_{sd}^2-\epsilon^2} \end{pmatrix} \tag{110}$$

which has the following limits:

$$\tilde{\sigma}_0(\Delta_{sd}\to\infty) = \frac{e^2}{\pi}\frac{1}{\pi\alpha}\begin{pmatrix} -\dfrac{1}{3} & 0 \\ 0 & -\dfrac{1}{3} \end{pmatrix} \tag{111}$$

$$\tilde{\sigma}_0(\Delta_{sd}\to 0) = \frac{e^2}{\pi}\frac{1}{\pi\alpha}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{112}$$

However in integrating over residues we make the assumption that $\epsilon > \Delta_{sd}$. This means that even in the limit $\Delta_{sd} \to \infty$ we must still take ϵ to be the leading term, which means that the expression Eq.(111) is incorrect and must actually be:

$$\tilde{\sigma}_0(\Delta_{sd} \to \infty) = \frac{e^2}{\pi}\frac{1}{\pi\alpha}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{113}$$

since this would be the result if we take ϵ to be the leading term over Δ_{sd}. Calculating the dressed Gilbert damping matrix using Eq.(76) returns the following matrix:

$$\tilde{\Gamma}_0 = \Delta_{sd}^2\frac{2}{\pi\alpha}\begin{pmatrix} \frac{(-1+n_z^2)(3\Delta_{sd}^2\epsilon^2+\epsilon^4)}{2\Delta_{sd}^2(\Delta_{sd}^2+3\epsilon^2)} & 0 & -\frac{n_x n_z(3\Delta_{sd}^2\epsilon^2+\epsilon^4)}{2\Delta_{sd}^2(\Delta_{sd}^2+3\epsilon^2)} \\ 0 & 0 & 0 \\ -\frac{n_x n_z(3\Delta_{sd}^2\epsilon^2+\epsilon^4)}{2\Delta_{sd}^2(\Delta_{sd}^2+3\epsilon^2)} & 0 & -\frac{n_z^2(3\Delta_{sd}^2\epsilon^2+\epsilon^4)}{2\Delta_{sd}^2(\Delta_{sd}^2+3\epsilon^2)} \end{pmatrix} \tag{114}$$

which, in the limit $\Delta_{sd} \to \infty$ gives the matrix:

$$\tilde{\Gamma}_0(\Delta_{sd} \to \infty) = \Delta_{sd}^2\frac{2}{\pi\alpha}\begin{pmatrix} -\frac{3n_x^2\epsilon^2}{2\Delta_{sd}^2} & 0 & -\frac{3n_x n_z\epsilon^2}{2\Delta_{sd}^2} \\ 0 & 0 & 0 \\ -\frac{3n_x n_z\epsilon^2}{2\Delta_{sd}^2} & 0 & -\frac{3n_z^2\epsilon^2}{2\Delta_{sd}^2} \end{pmatrix} \tag{115}$$

and in the limit $\Delta_{sd} \to 0$ returns:

$$\tilde{\Gamma}_0(\Delta_{sd} \to 0) = \Delta_{sd}^2\frac{2}{\pi\alpha}\begin{pmatrix} -\frac{n_x^2\epsilon^2}{6\Delta_{sd}^2} & 0 & -\frac{n_x n_z\epsilon^2}{6\Delta_{sd}^2} \\ 0 & 0 & 0 \\ -\frac{n_x n_z\epsilon^2}{6\Delta_{sd}^2} & 0 & -\frac{n_z^2\epsilon^2}{6\Delta_{sd}^2} \end{pmatrix} \tag{116}$$

where the subscript zero denotes the order in the perturbation parameter, in similar fashion to the subscripts used in Eq.(103). We find in this case that the Gilbert damping is not equally strong in all directions, since we obtain terms that differ on their dependence on the direction vector components n_x and n_z.

It should also be noted that the dressed conductivity and Gilbert damping matrices imply a relation between the limits of the exchange interaction term Δ_{sd} and the energy ϵ. It turns out that the limit for small exchange interaction yields the same result as the limit for large energy and vice versa. This means we can take a look at the limits of quantities Eq.(92) and Eq.(93) since they turn out to be equivalent to the limits of λ and Δ_{sd}. This Hamiltonian does not have a spin-orbit torque, since it does not contain any spin-orbit term, therefore the spin-orbit torque matrix is simply:

$$\tilde{R}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{117}$$

6.1.2 Rashba spin-orbit

The Hamiltonian for this case is given by :

$$H = v_f(\mathbf{p}\cdot\boldsymbol{\tau})\otimes\sigma_0 + \lambda(\tau_1\otimes\sigma_2 - \tau_2\otimes\sigma_1) \tag{118}$$

where λ is the spin-orbit interaction. For this Hamiltonian the bare Green's function G is:

$$G = \begin{pmatrix} \epsilon & 0 & -p\cos\theta + ip\sin\theta & 0 \\ 0 & \epsilon & 2i\lambda & -p\cos\theta + ip\sin\theta \\ -p\cos\theta - ip\sin\theta & -2i\lambda & \epsilon & 0 \\ 0 & -p\cos\theta - ip\sin\theta & 0 & \epsilon \end{pmatrix}^{-1} \tag{119}$$

Inserting this matrix into the expression for the self-energy Eq.(32) gives:

$$\Sigma^R = -\frac{i\pi\alpha}{2}\epsilon(\sigma_0 \otimes \sigma_0) \tag{120}$$

which is

$$\Sigma^R = -\frac{i\pi\alpha}{2}\epsilon\omega_1 \tag{121}$$

in terms of basis elements. The conductivity tensor for this Hamiltonian is:

$$\tilde{\sigma}_0 = \frac{e^2}{\pi}\frac{1}{\pi\alpha}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{122}$$

The Gilbert damping matrix for this Hamiltonian is:

$$\tilde{\Gamma}_0 = \Delta_{sd}^2\frac{2}{\pi\alpha}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{123}$$

which is to be expected, as there is no z-direction term in the Hamiltonian and therefore there will be no z-direction term in the Gilbert damping matrix. Furthermore, both the conductivity and Gilbert damping matrices turn out to be independent of the parameter λ and will therefore be unchanged whether λ is very large or very small. This also implies the same as in the previous case, that we can use the substitutions Eq.(92) and Eq.(93) to rewrite the limits we look at, but it is less apparent since there are no λ-terms in the matrices. The spin-orbit torque on the other hand is:

$$\tilde{R}_0 = \Delta_{sd}\frac{2}{\pi\alpha}\begin{pmatrix}0 & -\frac{2\lambda}{\epsilon}\\ \frac{2\lambda}{\epsilon} & 0\\ 0 & 0\end{pmatrix} \tag{124}$$

and is linear in λ.

6.2 Perturbative cases

6.2.1 Case 1: $\lambda << \Delta_{sd}$

The Hamiltonian and its perturbation in this case are:

$$H = v_f(p\cos(\theta)\tau_1 + p\sin(\theta)\tau_2) \otimes \sigma_0 + \Delta_{sd}\tau_0 \otimes (n_x\sigma_1 + n_z\sigma_3) \tag{125}$$

$$\delta H = \lambda(\tau_1 \otimes \sigma_2 - \tau_2 \otimes \sigma_1) \tag{126}$$

This means we can define the eigenvector v needed to project out the eigenvalue 1 in Eq.(95) using expression Eq.(107) as:

$$v = (\frac{\epsilon}{\sqrt{\epsilon^2+\Delta_{sd}^2}}, -\frac{n_x\Delta_{sd}}{\sqrt{\epsilon^2+\Delta_{sd}^2}}, 0, -\frac{n_z\Delta_{sd}}{\sqrt{\epsilon^2+\Delta_{sd}^2}}, 0,0,0,0,0,0,0,0,0,0,0,0) \tag{127}$$

where the denominator term arises because v needs to be normalized. Now this eigenvector can be used to construct a projection operator to in turn construct the dressed response matrix. This can be done piecewise depending on the order of $\overline{\lambda}$ by calculating Eq.(96) and Eq.(97) and inserting those in Eq.(98) to obtain just the perturbation matrices. This gives the following matrices:

$$\tilde{\sigma}_1 = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \tag{128}$$

$$\tilde{\sigma}_2 = \frac{e^2}{\pi}\frac{1}{\pi\alpha}\begin{pmatrix} \frac{2(-\Delta_{sd}^4+10\Delta_{sd}^2\epsilon^2+7\epsilon^4+n_z^2(3\Delta_{sd}^2+\epsilon^2)^2)\lambda^2}{(\Delta_{sd}^2-\epsilon^2)(-9\Delta_{sd}^2+\epsilon^2)^2} & 0 \\ 0 & \frac{2((-9+n_z^2)\Delta_{sd}^4-2(3+5n_z^2)\Delta_{sd}^2\epsilon^2-(1+7n_z^2)\epsilon^4)\lambda^2}{(\Delta_{sd}^2-\epsilon^2)(-9\Delta_{sd}^2+\epsilon^2)^2} \end{pmatrix} \tag{129}$$

Since the first-order perturbation term is zero, we can simply ignore it and only look at the second-order perturbation term. The matrix $\tilde{\sigma}_2$ has the following limits:

$$\tilde{\sigma}_2(\overline{\Delta}\to\infty) = \frac{e^2}{\pi}\frac{1}{\pi\alpha}\begin{pmatrix} \frac{2(-1+9n_z^2)\overline{\lambda}^2}{81\overline{\Delta}^2} & 0 \\ 0 & -\frac{2(-9+n_z^2)\overline{\lambda}^2}{81\overline{\Delta}^2} \end{pmatrix} \tag{130}$$

and

$$\tilde{\sigma}_2(\overline{\Delta}\to 0) = \frac{e^2}{\pi}\frac{1}{\pi\alpha}\begin{pmatrix} -2(7+n_z^2)\overline{\lambda}^2 & 0 \\ 0 & -2(1+7n_z^2)\overline{\lambda}^2 \end{pmatrix} \tag{131}$$

where we have substituted λ and Δ_{sd} for the dimensionless variables $\overline{\lambda}$ and $\overline{\Delta}$ defined by Eq.(92) and Eq.(93). This means the total conductivity tensors in the observed limits are:

$$\tilde{\sigma}(\overline{\Delta}\to\infty) = \frac{e^2}{\pi}\frac{1}{\pi\alpha}\begin{pmatrix} 1+\frac{2(-1+9n_z^2)\overline{\lambda}^2}{81\overline{\Delta}^2} & 0 \\ 0 & 1-\frac{2(-9+n_z^2)\overline{\lambda}^2}{81\overline{\Delta}^2} \end{pmatrix} \tag{132}$$

and

$$\tilde{\sigma}(\overline{\Delta}\to 0) = \frac{e^2}{\pi}\frac{1}{\pi\alpha}\begin{pmatrix} 1-2(7+n_z^2)\overline{\lambda}^2 & 0 \\ 0 & 1-2(1+7n_z^2)\overline{\lambda}^2 \end{pmatrix} \tag{133}$$

We find that the conductivity becomes anisotropic when we introduce a perturbation to the system. However, if we assume the magnetization is perpendicular to the crystal surface, in other words $n_z = 1$, the conductivity becomes isotropic again. Since we chose $n_y = 0$ in our coordinate system this means that σ_{xx} is the conductivity for a current parallel to the magnetization, and σ_{yy} is the conductivity for a current perpendicular to the magnetization. For the spin-orbit torque, we find the perturbation terms to be:

$$\tilde{R}_1 = \Delta_{sd}\frac{2}{\pi\alpha}\begin{pmatrix} 0 & \frac{4n_z^2\epsilon\lambda}{-9\Delta_{sd}^2+\epsilon^2} \\ -\frac{4\epsilon\lambda}{-9\Delta_{sd}^2+\epsilon^2} & 0 \\ 0 & -\frac{4n_xn_z\epsilon\lambda}{-9\Delta_{sd}^2+\epsilon^2} \end{pmatrix} \tag{134}$$

and

$$\tilde{R}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{135}$$

The spin-orbit torque only has one non-zero matrix $\tilde{R}_1$ therefore taking the limits of only this matrix will result in the spin-orbit torque limits:

$$\tilde{R}(\overline{\Delta}\to\infty) = \Delta_{sd}\frac{2}{\pi\alpha}\begin{pmatrix} 0 & -\frac{4n_z^2\overline{\lambda}}{9\overline{\Delta}^2} \\ \frac{4\overline{\lambda}}{9\overline{\Delta}^2} & 0 \\ 0 & \frac{4n_xn_z\overline{\lambda}}{9\overline{\Delta}^2} \end{pmatrix} \tag{136}$$

and

$$\tilde{R}(\overline{\Delta}\to 0)=\Delta_{sd}\frac{2}{\pi\alpha}\begin{pmatrix}0 & 4n_z^2\overline{\lambda}\\ -4\overline{\lambda} & 0\\ 0 & -4n_xn_z\overline{\lambda}\end{pmatrix} \tag{137}$$

The perturbation terms of the Gilbert damping are:

$$\tilde{\Gamma}_1=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{138}$$

The second order perturbation matrix contains large polynomials up to the tenth order, so in this thesis we will only write down the leading terms in Δ_{sd}, λ, and ϵ since these are the only terms needed to determine the limits. The leading term matrix is:

$$\tilde{\Gamma}_2=\Delta_{sd}^2\frac{2}{\pi\alpha}\begin{pmatrix}-\frac{(6(-3+7n_z^4)\Delta_{sd}^{10}+(-1+n_z^4)\epsilon^{10})\lambda^2}{8\Delta_{sd}^6(9\Delta_{sd}^2-\epsilon^2)(\Delta_{sd}^2+3\epsilon^2)^2} & 0 & \frac{n_xn_z(6(6+7n_z^2)\Delta_{sd}^{10}+(1+n_z^2)\epsilon^{10})\lambda^2}{8\Delta_{sd}^6(9\Delta_{sd}^2-\epsilon^2)(\Delta_{sd}^2+3\epsilon^2)^2}\\ 0 & -\frac{(3\Delta_{sd}^2+\epsilon^2)\lambda^2}{9\Delta_{sd}^4-\Delta_{sd}^2\epsilon^2} & 0\\ \frac{n_xn_z(\epsilon^{10}+n_z^2(42\Delta_{sd}^{10}+\epsilon^{10}))\lambda^2}{8\Delta_{sd}^6(9\Delta_{sd}^2-\epsilon^2)(\Delta_{sd}^2+3\epsilon^2)^2} & 0 & \frac{n_z^2(6(-1+7n_z^2)\Delta_{sd}^{10}+(1+n_z^2)\epsilon^{10})\lambda^2}{8\Delta_{sd}^6(9\Delta_{sd}^2-\epsilon^2)(\Delta_{sd}^2+3\epsilon^2)^2}\end{pmatrix} \tag{139}$$

Taking the limits of this matrix gives the following:

$$\tilde{\Gamma}_2(\overline{\Delta}\to\infty)=\Delta_{sd}^2\frac{2}{\pi\alpha}\begin{pmatrix}\frac{(3-7n_z^4)\overline{\lambda}^2}{12\overline{\Delta}^2} & 0 & \frac{n_xn_z(6+7n_z^2)\overline{\lambda}^2}{12\overline{\Delta}^2}\\ 0 & -\frac{\overline{\lambda}^2}{3\overline{\Delta}^2} & 0\\ \frac{7n_xn_z^3\overline{\lambda}^2}{12\overline{\Delta}^2} & 0 & \frac{n_z^2(-1+7n_z^2)\overline{\lambda}^2}{12\overline{\Delta}^2}\end{pmatrix} \tag{140}$$

and

$$\tilde{\Gamma}_2(\overline{\Delta}\to 0)=\Delta_{sd}^2\frac{2}{\pi\alpha}\begin{pmatrix}\frac{(-1+n_z^4)\overline{\lambda}^2}{72\overline{\Delta}^6} & 0 & -\frac{n_xn_z(1+n_z^2)\overline{\lambda}^2}{72\overline{\Delta}^6}\\ 0 & \frac{\overline{\lambda}^2}{\overline{\Delta}^2} & 0\\ -\frac{n_xn_z(1+n_z^2)\overline{\lambda}^2}{72\overline{\Delta}^6} & 0 & -\frac{n_z^2(1+n_z^2)\overline{\lambda}^2}{72\overline{\Delta}^6}\end{pmatrix} \tag{141}$$

Combining these matrices with the matrices for the cases without perturbation terms yields the expression for the total Gilbert damping:

$$\tilde{\Gamma}(\overline{\Delta}\to\infty)=\Delta_{sd}^2\frac{2}{\pi\alpha}\begin{pmatrix}-\frac{3n_x^2}{2\overline{\Delta}^2}+\frac{(3-7n_z^4)\overline{\lambda}^2}{12\overline{\Delta}^2} & 0 & -\frac{3n_xn_z}{2\overline{\Delta}^2}+\frac{n_xn_z(6+7n_z^2)\overline{\lambda}^2}{12\overline{\Delta}^2}\\ 0 & -\frac{\overline{\lambda}^2}{3\overline{\Delta}^2} & 0\\ -\frac{3n_xn_z}{2\overline{\Delta}^2}+\frac{7n_xn_z^3\overline{\lambda}^2}{12\overline{\Delta}^2} & 0 & \frac{-3n_z^2}{2\overline{\Delta}^2}+\frac{n_z^2(-1+7n_z^2)\overline{\lambda}^2}{12\overline{\Delta}^2}\end{pmatrix} \tag{142}$$

$$\tilde{\Gamma}(\overline{\Delta}\to 0)=\Delta_{sd}^2\frac{2}{\pi\alpha}\begin{pmatrix}-\frac{n_x^2}{6\overline{\Delta}^2}+\frac{(-1+n_z^4)\overline{\lambda}^2}{72\overline{\Delta}^6} & 0 & -\frac{n_xn_z}{6\overline{\Delta}^2}-\frac{n_xn_z(1+n_z^2)\overline{\lambda}^2}{72\overline{\Delta}^6}\\ 0 & \frac{\overline{\lambda}^2}{\overline{\Delta}^2} & 0\\ -\frac{n_xn_z}{6\overline{\Delta}^2}-\frac{n_xn_z(1+n_z^2)\overline{\lambda}^2}{72\overline{\Delta}^6} & 0 & -\frac{n_z^2}{6\overline{\Delta}^2}-\frac{n_z^2(1+n_z^2)\overline{\lambda}^2}{72\overline{\Delta}^6}\end{pmatrix} \tag{143}$$

The Gilbert damping becomes anisotropic upon introducing a perturbation. We also observe symmetry breaking in the Gilbert damping upon introducing the perturbation term, if only in the limit for large exchange interactions.

6.2.2 Case 2: $\Delta_{sd} << \lambda$

$$H = v_f(p\cos(\theta)\tau_1 + p\sin(\theta)\tau_2) \otimes \sigma_0 + \lambda(\tau_1 \otimes \sigma_2 - \tau_2 \otimes \sigma_1) \tag{144}$$

$$\delta H = \Delta_{sd}\tau_0 \otimes (n_x\sigma_1 + n_z\sigma_3) \tag{145}$$

In this case the eigenvector v needed to project out the eigenvalue 1 is:

$$v = (1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) \tag{146}$$

according to Eq.(120). In similar fashion to the previous subsection we can obtain the perturbation terms for this perturbation by looking at the matrix terms per order of $\overline{\Delta}$. For the conductivity we find:

$$\tilde{\sigma}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{147}$$

and

$$\tilde{\sigma}_2 = \frac{e^2}{\pi}\frac{1}{\pi\alpha}\begin{pmatrix} -\dfrac{\Delta_{sd}^2((1+3n_z^2)\epsilon^2 - 4(1+2n_z^2)\lambda^2)}{4\epsilon^2(\epsilon^2-\lambda^2)} & 0 \\ 0 & \dfrac{\Delta_{sd}^2((1-5n_z^2)\epsilon^2 + 4(1+2n_z^2)\lambda^2)}{4\epsilon^2(\epsilon^2-\lambda^2)} \end{pmatrix} \tag{148}$$

which has the following limits:

$$\tilde{\sigma}_2(\overline{\lambda}\to\infty) = \frac{e^2}{\pi}\frac{1}{\pi\alpha}\begin{pmatrix} (-1-2n_z^2)\overline{\Delta}^2 & 0 \\ 0 & (-1-2n_z^2)\overline{\Delta}^2 \end{pmatrix} \tag{149}$$

and

$$\tilde{\sigma}_2(\overline{\lambda}\to 0) = \frac{e^2}{\pi}\frac{1}{\pi\alpha}\begin{pmatrix} \dfrac{(-1-3n_z^2)\overline{\Delta}^2}{4} & 0 \\ 0 & \dfrac{(1-5n_z^2)\overline{\Delta}^2}{4} \end{pmatrix} \tag{150}$$

Therefore the total conductivity tensors in the limits are:

$$\tilde{\sigma}(\overline{\lambda}\to\infty) = \frac{e^2}{\pi}\frac{1}{\pi\alpha}\begin{pmatrix} 1+(-1-2n_z^2)\overline{\Delta}^2 & 0 \\ 0 & 1+(-1-2n_z^2)\overline{\Delta}^2 \end{pmatrix} \tag{151}$$

and

$$\tilde{\sigma}(\overline{\lambda}\to 0) = \frac{e^2}{\pi}\frac{1}{\pi\alpha}\begin{pmatrix} 1+\dfrac{(-1-3n_z^2)\overline{\Delta}^2}{4} & 0 \\ 0 & 1+\dfrac{(1-5n_z^2)\overline{\Delta}^2}{4} \end{pmatrix} \tag{152}$$

The conductivity also becomes anisotropic for this perturbation, although the anisotropy actually vanishes in the limit of large spin-orbit parameters. While this limit is always isotropic, the limit of small spin-orbit parameters becomes isotropic if we again take $n_z = 1$. For the spin-orbit torque, the perturbation terms are:

$$\tilde{R}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{153}$$

and

$$\tilde{R}_2 = \Delta_{sd}\frac{2}{\pi\alpha}\begin{pmatrix} 0 & \frac{\Delta_{sd}^2(2(2+n_z^2)\epsilon^4-(13+5n_z^2)\epsilon^2\lambda^2+4(1+n_z^2)\lambda^4)}{4\epsilon^3\lambda(\epsilon^2-\lambda^2)} \\ -\frac{(1+n_z^2)\Delta_{sd}^2(3\epsilon^4-9\epsilon^2\lambda^2+4\lambda^4)}{4\epsilon^3\lambda(\epsilon^2-\lambda^2)} & 0 \\ 0 & \frac{n_x n_z\Delta_{sd}^2(\epsilon^2-2\lambda^2)}{\epsilon\lambda(\epsilon^2-\lambda^2)} \end{pmatrix} \tag{154}$$

This second-order term has the following limits:

$$\tilde{R}_2(\overline{\lambda}\to\infty) = \Delta_{sd}\frac{2}{\pi\alpha}\begin{pmatrix} 0 & -(1+n_z^2)\overline{\Delta}^2\overline{\lambda} \\ (1+n_z^2)\overline{\Delta}^2\overline{\lambda} & 0 \\ 0 & \frac{2n_x n_z\overline{\Delta}^2}{\overline{\lambda}} \end{pmatrix} \tag{155}$$

and

$$\tilde{R}_2(\overline{\lambda}\to 0) = \Delta_{sd}\frac{2}{\pi\alpha}\begin{pmatrix} 0 & \frac{(2+n_z^2)\overline{\Delta}^2}{2\overline{\lambda}} \\ -\frac{3(1+n_z^2)\overline{\Delta}^2}{4\overline{\lambda}} & 0 \\ 0 & \frac{n_x n_z\overline{\Delta}^2}{\overline{\lambda}} \end{pmatrix} \tag{156}$$

The total spin-orbit torque is then:

$$\tilde{R}(\overline{\lambda}\to\infty) = \Delta_{sd}\frac{2}{\pi\alpha}\begin{pmatrix} 0 & -(1+n_z^2)\overline{\Delta}^2\overline{\lambda}-2\overline{\lambda} \\ (1+n_z^2)\overline{\Delta}^2\overline{\lambda}+2\overline{\lambda} & 0 \\ 0 & \frac{2n_x n_z\overline{\Delta}^2}{\overline{\lambda}} \end{pmatrix} \tag{157}$$

and

$$\tilde{R}(\overline{\lambda}\to 0) = \Delta_{sd}\frac{2}{\pi\alpha}\begin{pmatrix} 0 & \frac{(2+n_z^2)\overline{\Delta}^2}{2\overline{\lambda}}-2\overline{\lambda} \\ -\frac{3(1+n_z^2)\overline{\Delta}^2}{4\overline{\lambda}}+2\overline{\lambda} & 0 \\ 0 & \frac{n_x n_z\overline{\Delta}^2}{\overline{\lambda}} \end{pmatrix} \tag{158}$$

For the Gilbert damping the perturbation matrices are:

$$\tilde{\Gamma}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{159}$$

where, for the second order matrix we again take leading terms:

$$\tilde{\Gamma}_2 = \Delta_{sd}^2\frac{2}{\pi\alpha}\begin{pmatrix} -\frac{\Delta_{sd}^2((-1+4n_z^2)\epsilon^8+32n_z^2\lambda^8)}{2\epsilon^4\lambda^2(\epsilon^4-5\epsilon^2\lambda^2+4\lambda^4)} & 0 & \frac{n_x n_z\Delta_{sd}^2(\epsilon^4-\epsilon^2\lambda^2-4\lambda^4)}{\epsilon^2\lambda^2(\epsilon^2-4\lambda^2)} \\ 0 & -\frac{\Delta_{sd}^2((-1+7n_z^2)\epsilon^6-16n_z\lambda^6)}{4\epsilon^4\lambda^2(\epsilon^2-\lambda^2)} & 0 \\ \frac{n_x n_z\Delta_{sd}^2(\epsilon^4-2\epsilon^2\lambda^2-2\lambda^4)}{2\lambda^2(\epsilon^4-5\epsilon^2\lambda^2+4\lambda^4)} & 0 & \frac{n_z^2\Delta_{sd}^2(2\epsilon^4-7\epsilon^2\lambda^2+4\lambda^4)}{\epsilon^2\lambda^2(\epsilon^2-4\lambda^2)} \end{pmatrix} \tag{160}$$

which has the limits:

$$\tilde{\Gamma}_2(\overline{\lambda}\to\infty) = \Delta_{sd}^2\frac{2}{\pi\alpha}\begin{pmatrix} -4n_z^2\overline{\Delta}^2\overline{\lambda}^2 & 0 & n_x n_z\overline{\Delta}^2 \\ 0 & -4n_z^2\overline{\Delta}^2\overline{\lambda}^2 & 0 \\ -n_x n_z\frac{\overline{\Delta}^2}{4\overline{\lambda}^2} & 0 & -n_z^2\overline{\Delta}^2 \end{pmatrix} \tag{161}$$

and

$$\tilde{\Gamma}_2(\overline{\lambda}\to 0)=\Delta_{sd}^2\frac{2}{\pi\alpha}\begin{pmatrix}-\frac{(-1+4n_z^2)\overline{\Delta}^2}{2\overline{\lambda}^2} & 0 & \frac{n_xn_z\overline{\Delta}^2}{\overline{\lambda}^2}\\ 0 & -\frac{(-1+7n_z^2)\overline{\Delta}^2}{4\overline{\lambda}^2} & 0\\ \frac{n_xn_z\overline{\Delta}^2}{2\overline{\lambda}^2} & 0 & \frac{2n_z^2\overline{\Delta}^2}{\overline{\lambda}^2}\end{pmatrix} \tag{162}$$

The total Gilbert damping becomes:

$$\tilde{\Gamma}(\overline{\lambda}\to\infty)=\Delta_{sd}^2\frac{2}{\pi\alpha}\begin{pmatrix}1-4n_z^2\overline{\Delta}^2\overline{\lambda}^2 & 0 & n_xn_z\overline{\Delta}^2\\ 0 & 1-4n_z^2\overline{\Delta}^2\overline{\lambda}^2 & 0\\ -n_xn_z\frac{\overline{\Delta}^2}{4\overline{\lambda}^2} & 0 & -n_z^2\overline{\Delta}^2\end{pmatrix} \tag{163}$$

and

$$\tilde{\Gamma}(\overline{\lambda}\to 0)=\Delta_{sd}^2\frac{2}{\pi\alpha}\begin{pmatrix}1-\frac{(-1+4n_z^2)\overline{\Delta}^2}{2\overline{\lambda}^2} & 0 & \frac{n_xn_z\overline{\Delta}^2}{\overline{\lambda}^2}\\ 0 & 1-\frac{(-1+7n_z^2)\overline{\Delta}^2}{4\overline{\lambda}^2} & 0\\ \frac{n_xn_z\overline{\Delta}^2}{2\overline{\lambda}^2} & 0 & \frac{2n_z^2\overline{\Delta}^2}{\overline{\lambda}^2}\end{pmatrix} \tag{164}$$

We also find anisotropy in the Gilbert damping, however it should be noted that the Gilbert damping matrix without perturbation terms $\tilde{\Gamma}_0$ was already anisotropic, as it had no component in the z-direction. In addition symmetry breaking occurs, although unlike in the previous case the asymmetry of the Gilbert damping matrix does not vanish in any of the limits.

6.3 Vector expressions

Ideally we would like to express both the spin-orbit torque and the Gilbert damping quantities in a vector form. In order to do so we must parameterize the Gilbert damping and spin-orbit torque matrices we obtained in the previous section. For the Gilbert damping matrices we use the following parameterization:

$$\tilde{\Gamma}=\Delta_{sd}^2\frac{2}{\pi\alpha}\begin{pmatrix}\zeta n_x^2+r_{11} & 0 & \zeta n_xn_z+r_{13}\\ 0 & r_{22} & 0\\ \zeta n_xn_z+r_{31} & 0 & \zeta n_z^2+r_{33}\end{pmatrix} \tag{165}$$

We can express the spin polarization $\delta\mathbf{s}$ in terms of the parameterized matrix:

$$\delta\mathbf{s}=\tilde{\Gamma}\frac{d\mathbf{n}}{dt} \tag{166}$$

which can be rewritten in terms of the parallel and perpendicular components of the direction of localized spins $\mathbf{n}$:

$$\delta\mathbf{s}_\Gamma=c_1\mathbf{n}_\parallel\times\left(\mathbf{n}_\parallel\times\frac{d\mathbf{n}_\parallel}{dt}\right)+c_2\frac{d\mathbf{n}_\parallel}{dt}+c_3\frac{d\mathbf{n}_\perp}{dt}+c_4\mathbf{n} \tag{167}$$

For the x-z plane these components are given by:

$$\mathbf{n}_\parallel=(n_x,0,0) \tag{168}$$

$$\mathbf{n}_\perp=(0,0,n_z) \tag{169}$$

with respect to the crystal surface which is in the x-y plane. The constants in Eq.(167) are given by:

$$\begin{aligned} c_1 &= \frac{r_{11} - r_{22} - r_{31}(1-n_z^2)/(n_x n_z)}{1-n_z^2} \\ c_2 &= r_{11} - r_{31}(1-n_z^2)/(n_x n_z) \\ c_3 &= r_{33} \\ c_4 &= \frac{r_{31}}{n_z}\frac{dn_z}{dt} + \zeta \frac{d\mathbf{n}}{dt}\cdot\mathbf{n} \end{aligned} \tag{170}$$

This parameterization also ensures that terms proportional to ζ do not contribute to the spin-polarization. This is because the inner product in constant c_4 is zero.

For the spin-orbit torque we use the matrix multiplication:

$$\delta\mathbf{s} = \Delta_{sd}\frac{2}{\pi\alpha}\begin{pmatrix} 0 & a \\ b & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} E_x \\ E_y \end{pmatrix} \tag{171}$$

to parameterize $\delta\mathbf{s}$ as:

$$\delta\mathbf{s}_R = A\hat{z}\times\mathbf{E} + B\mathbf{n}_\parallel \times (\mathbf{n}_\parallel \times (\hat{z}\times\mathbf{E})) + C\mathbf{n}_\parallel \times (\mathbf{n}_\perp \times (\hat{z}\times\mathbf{E})) \tag{172}$$

with constants:

$$A = a, \quad B = (a-b)/n_x^2, \quad C = -c/(n_x n_z) \tag{173}$$

$\mathbf{n}$ the direction of localized spins and $\mathbf{E}$ the electric field. The electric field is only present in the crystal plane, meaning it only has x- and y-components. We will now take a look at each case to construct a vector expression in the relevant limits.

6.3.1 Case 1: $\lambda << \Delta_{sd}$

If we take a look at the Gilbert damping matrix in the limit of infinite exchange energy Eq.(142) we find the following constants for our chosen parameterization Eq.(165):

$$\begin{aligned} \zeta &= -\frac{3}{2\overline{\Delta}^2} \\ r_{11} &= \frac{(3-7n_z^4)\overline{\lambda}^2}{12\overline{\Delta}^2} \\ r_{22} &= -\frac{\overline{\lambda}^2}{3\overline{\Delta}^2} \\ r_{31} &= \frac{7n_x n_z^3\overline{\lambda}^2}{12\overline{\Delta}^2} \\ r_{33} &= \frac{n_z^2(-1+7n_z^2)\overline{\lambda}^2}{12\overline{\Delta}^2} \end{aligned} \tag{174}$$

which, using the relations in Eq.(170) leads to the following parameterization constants:

$$\begin{aligned} c_1 &= \frac{7\overline{\lambda}^2}{12\overline{\Delta}^2} \\ c_2 &= \frac{(3-7n_z^2)\overline{\lambda}^2}{12\overline{\Delta}^2} \\ c_3 &= \frac{n_z^2(-1+7n_z^2)\overline{\lambda}^2}{12\overline{\Delta}^2} \\ c_4 &= \frac{7n_x n_z^2\overline{\lambda}^2}{12\overline{\Delta}^2}\frac{dn_z}{dt} \end{aligned} \tag{175}$$

Inserting these constants into Eq.(167) gives the following equation for the spin-polarization:

$$\begin{aligned} \delta\mathbf{s}_\Gamma(\overline{\Delta}\to\infty) = \Delta_{sd}^2\frac{2}{\pi\alpha}\Bigg[&\frac{7\overline{\lambda}^2}{12\overline{\Delta}^2}\mathbf{n}_\parallel\times\left(\mathbf{n}_\parallel\times\frac{d\mathbf{n}_\parallel}{dt}\right)+\frac{(3-7n_z^2)\overline{\lambda}^2}{12\overline{\Delta}^2}\frac{d\mathbf{n}_\parallel}{dt}+\frac{n_z^2(-1+7n_z^2)\overline{\lambda}^2}{12\overline{\Delta}^2}\frac{d\mathbf{n}_\perp}{dt} \\ &+\frac{7n_x n_z^2\overline{\lambda}^2}{12\overline{\Delta}^2}\frac{dn_z}{dt}\mathbf{n}\Bigg] \end{aligned} \tag{176}$$

where we add the subscript Γ to denote that we used the parameterization of the Gilbert damping. If we then look at the parameterization for the spin-orbit torque Eq.(171) in the limit of infinite exchange energy Eq.(136) we find that the constants are:

$$\begin{aligned} a &= -\frac{4n_z^2\overline{\lambda}}{9\overline{\Delta}^2} \\ b &= \frac{4\overline{\lambda}}{9\overline{\Delta}^2} \\ c &= \frac{4n_x n_z\overline{\lambda}}{9\overline{\Delta}^2} \end{aligned} \tag{177}$$

which, according to Eq.(173), leads to parameterization constants:

$$\begin{aligned} A &= -\frac{4n_z^2\overline{\lambda}}{9\overline{\Delta}^2} \\ B &= -\frac{4(n_z^2+1)\overline{\lambda}}{9n_x^2\overline{\Delta}^2} \\ C &= -\frac{4\overline{\lambda}}{9\overline{\Delta}^2} \end{aligned} \tag{178}$$

For these constants Eq.(172) will be:

$$\begin{aligned} \delta\mathbf{s}_R(\overline{\Delta}\to\infty) = -\Delta_{sd}\frac{2}{\pi\alpha}\Bigg[&\frac{4n_z^2\overline{\lambda}}{9\overline{\Delta}^2}\hat{z}\times\mathbf{E}+\frac{4(n_z^2+1)\overline{\lambda}}{9n_x^2\overline{\Delta}^2}\mathbf{n}_\parallel\times(\mathbf{n}_\parallel\times(\hat{z}\times\mathbf{E})) \\ &-\frac{4\overline{\lambda}}{9\overline{\Delta}^2}\mathbf{n}_\parallel\times(\mathbf{n}_\perp\times(\hat{z}\times\mathbf{E}))\Bigg] \end{aligned} \tag{179}$$

In the first term of Eq.(143) we can write:

$$-1+n_z^4 = -(1-n_z^2)(1+n_z^2) = -n_x^2(1+n_z^2) \tag{180}$$

which means we find the constants:

$$\begin{aligned}\zeta &= -\frac{1}{6\overline{\Delta}^2} - \frac{(1+n_z^2)\overline{\lambda}^2}{72\overline{\Delta}^6} \\ r_{11} &= 0 \\ r_{22} &= \frac{\overline{\lambda}^2}{\overline{\Delta}^2} \\ r_{31} &= 0 \\ r_{33} &= 0\end{aligned} \tag{181}$$

for in the limit of zero exchange energy. Using Eq.(170) gives the following parameterization constants:

$$\begin{aligned}c_1 &= -\frac{\overline{\lambda}^2}{(1-n_z^2)\overline{\Delta}^2} \\ c_2 &= 0 \\ c_3 &= 0 \\ c_4 &= 0\end{aligned} \tag{182}$$

According to Eq.(167) this results in the following spin-polarization:

$$\delta\mathbf{s}_\Gamma(\overline{\Delta}\to 0) = -\Delta_{sd}^2\frac{2}{\pi\alpha}\frac{\overline{\lambda}^2}{(1-n_z^2)\overline{\Delta}^2}\mathbf{n}_\parallel\times\left(\mathbf{n}_\parallel\times\frac{d\mathbf{n}_\parallel}{dt}\right) \tag{183}$$

For the spin-orbit torque in the zero exchange energy limit Eq.(137) we find that the constants needed in Eq.(171) are:

$$\begin{aligned}a &= 4n_z^2\overline{\lambda} \\ b &= -4\overline{\lambda} \\ c &= -4n_xn_z\overline{\lambda}\end{aligned} \tag{184}$$

Inserting these constants into Eq.(173) gives

$$\begin{aligned}A &= 4n_z^2\overline{\lambda} \\ B &= \frac{4(n_z^2+1)}{n_x^2}\overline{\lambda} \\ C &= 4\overline{\lambda}\end{aligned} \tag{185}$$

and the spin-polarization Eq.(172) becomes:

$$\begin{aligned}\delta\mathbf{s}_R(\overline{\Delta}\to 0) = \Delta_{sd}\frac{2}{\pi\alpha}\Big[&4n_z^2\overline{\lambda}\hat{z}\times\mathbf{E} + \frac{4(n_z^2+1)}{n_x^2}\overline{\lambda}\mathbf{n}_\parallel\times(\mathbf{n}_\parallel\times(\hat{z}\times\mathbf{E})) \\ &+ 4\overline{\lambda}\mathbf{n}_\parallel\times(\mathbf{n}_\perp\times(\hat{z}\times\mathbf{E}))\Big]\end{aligned} \tag{186}$$

6.3.2 Case 2: $\Delta_{sd} << \lambda$

Now we take a look at the results for the limits of the spin-orbit interaction λ. In the limit of infinite spin-orbit interaction Eq.(163) we find that the chosen parameterization Eq.(165) gives

the following constants:

$$\begin{aligned}\zeta &= 0\\ r_{11} &= 1-4n_z^2\overline{\lambda}^2\overline{\Delta}^2\\ r_{22} &= 1-4n_z^2\overline{\lambda}^2\overline{\Delta}^2\\ r_{31} &= -n_x n_z\frac{\overline{\Delta}^2}{4\overline{\lambda}^2}\\ r_{33} &= -n_z^2\overline{\Delta}^2\end{aligned} \tag{187}$$

Entering these constants into Eq.(170) gives:

$$\begin{aligned}c_1 &= \frac{\overline{\Delta}^2}{4\overline{\lambda}^2}\\ c_2 &= 1-4n_z^2\overline{\Delta}^2\overline{\lambda}^2+\frac{(1-n_z^2)\overline{\Delta}^2}{4\overline{\lambda}^2}\\ c_3 &= -n_z^2\overline{\Delta}^2\\ c_4 &= -\frac{n_x\overline{\Delta}^2}{4\overline{\lambda}^2}\frac{dn_z}{dt}\end{aligned} \tag{188}$$

The spin-polarization Eq.(167) becomes:

$$\begin{aligned}\delta\mathbf{s}_\Gamma(\overline{\lambda}\to\infty) = \Delta_{sd}^2\frac{2}{\pi\alpha}\Bigg[&\frac{\overline{\Delta}^2}{4\overline{\lambda}^2}\mathbf{n}_\parallel\times\left(\mathbf{n}_\parallel\times\frac{d\mathbf{n}_\parallel}{dt}\right)+\left(1-4n_z^2\overline{\Delta}^2\overline{\lambda}^2+\frac{(1-n_z^2)\overline{\Delta}^2}{4\overline{\lambda}^2}\right)\frac{d\mathbf{n}_\parallel}{dt}\\ &-n_z^2\overline{\Delta}^2\frac{d\mathbf{n}_\perp}{dt}-\frac{n_x\overline{\Delta}^2}{4\overline{\lambda}^2}\frac{dn_z}{dt}\mathbf{n}\Bigg]\end{aligned} \tag{189}$$

If we then take a look at the spin-orbit torque matrix in this limit Eq.(157) we find the constants:

$$\begin{aligned}a &= -(1+n_z^2)\overline{\Delta}^2\overline{\lambda}-2\overline{\lambda}\\ b &= (1+n_z^2)\overline{\Delta}^2\overline{\lambda}+2\overline{\lambda}\\ c &= \frac{2n_x n_z\overline{\Delta}^2}{\overline{\lambda}}\end{aligned} \tag{190}$$

Inserting these constants into Eq.(173) gives

$$\begin{aligned}A &= -(1+n_z^2)\overline{\Delta}^2\overline{\lambda}-2\overline{\lambda}\\ B &= -\frac{2(1+n_z^2)\overline{\Delta}^2\overline{\lambda}-4\overline{\lambda}}{n_x^2}\\ C &= -\frac{2\overline{\Delta}^2}{\overline{\lambda}}\end{aligned} \tag{191}$$

The spin-orbit polarization Eq.(172) becomes:

$$\begin{aligned}\delta\mathbf{s}_R(\overline{\lambda}\to\infty) = -\Delta_{sd}\frac{2}{\pi\alpha}\Bigg[&((1+n_z^2)\overline{\Delta}^2\overline{\lambda}+2\overline{\lambda})\hat{z}\times\mathbf{E}\\ &+\frac{2(1+n_z^2)\overline{\Delta}^2\overline{\lambda}-4\overline{\lambda}}{n_x^2}\mathbf{n}_\parallel\times(\mathbf{n}_\parallel\times(\hat{z}\times\mathbf{E}))\\ &+\frac{2\overline{\Delta}^2}{\overline{\lambda}}\mathbf{n}_\parallel\times(\mathbf{n}_\perp\times(\hat{z}\times\mathbf{E}))\Bigg]\end{aligned} \tag{192}$$

Next we take a look at the Gilbert damping in the limit of zero spin-orbit interaction Eq.(164). The constants in Eq.(165) in this case are:

$$\begin{aligned}
\zeta &= 0 \\
r_{11} &= 1 - \frac{(-1+4n_z^2)\overline{\Delta}^2}{2\overline{\lambda}^2} \\
r_{22} &= 1 - \frac{(-1+7n_z^2)\overline{\Delta}^2}{4\overline{\lambda}^2} \\
r_{31} &= \frac{n_x n_z \overline{\Delta}^2}{2\overline{\lambda}^2} \\
r_{33} &= \frac{2n_z^2\overline{\Delta}^2}{\overline{\lambda}^2}
\end{aligned} \tag{193}$$

This means the constants given by Eq.(170) are:

$$\begin{aligned}
c_1 &= -\frac{\overline{\Delta}^2}{4\overline{\lambda}^2} \\
c_2 &= 1 - \frac{3n_z^2\overline{\Delta}^2}{2\overline{\lambda}^2} \\
c_3 &= \frac{2n_z^2\overline{\Delta}^2}{\overline{\lambda}^2} \\
c_4 &= \frac{n_x\overline{\Delta}^2}{2\overline{\lambda}^2}\frac{dn_z}{dt}
\end{aligned} \tag{194}$$

This means the spin-polarization Eq.(167) becomes:

$$\begin{aligned}
\delta\mathbf{s}_\Gamma(\overline{\lambda}\to 0) =& \Delta_{sd}^2\frac{2}{\pi\alpha}\left[-\frac{\overline{\Delta}^2}{4\overline{\lambda}^2}\mathbf{n}_\parallel\times\left(\mathbf{n}_\parallel\times\frac{d\mathbf{n}_\parallel}{dt}\right)+\left(1-\frac{3n_z^2\overline{\Delta}^2}{2\overline{\lambda}^2}\right)\frac{d\mathbf{n}_\parallel}{dt}+\frac{2n_z^2\overline{\Delta}^2}{\overline{\lambda}^2}\frac{d\mathbf{n}_\perp}{dt}\right. \\
&\left.+\frac{n_x\overline{\Delta}^2}{2\overline{\lambda}^2}\frac{dn_z}{dt}\mathbf{n}\right]
\end{aligned} \tag{195}$$

For the spin-orbit torque in this limit Eq.(158) the constants in Eq.(171) are:

$$\begin{aligned}
a &= \frac{(2+n_z^2)\overline{\Delta}^2}{2\overline{\lambda}} - 2\overline{\lambda} \\
b &= -\frac{3(1+n_z^2)\overline{\Delta}^2}{4\overline{\lambda}} + 2\overline{\lambda} \\
c &= \frac{n_x n_z\overline{\Delta}^2}{\overline{\lambda}}
\end{aligned} \tag{196}$$

The constants given by Eq.(173) are:

$$\begin{aligned}
A &= \frac{(2+n_z^2)\overline{\Delta}^2}{2\overline{\lambda}} - 2\overline{\lambda} \\
B &= \frac{(7+5n_z^2)\overline{\Delta}^2}{4n_x^2\overline{\lambda}} - \frac{4\overline{\lambda}}{n_x^2} \\
C &= -\frac{\overline{\Delta}^2}{\overline{\lambda}}
\end{aligned} \tag{197}$$

The spin-polarization Eq.(172) becomes:

$$\begin{aligned}\delta\mathbf{s}_R(\overline{\lambda}\to 0) = \Delta_{sd}\frac{2}{\pi\alpha}\Bigg[&\left(\frac{(2+n_z^2)\overline{\Delta}^2}{2\overline{\lambda}} - 2\overline{\lambda}\right)\hat{z}\times\mathbf{E}\\ &+\left(\frac{(7+5n_z^2)\overline{\Delta}^2}{4n_x^2\overline{\lambda}} - \frac{4\overline{\lambda}}{n_x^2}\right)\mathbf{n}_\parallel\times(\mathbf{n}_\parallel\times(\hat{z}\times\mathbf{E}))\\ &-\frac{\overline{\Delta}^2}{\overline{\lambda}}\mathbf{n}_\parallel\times(\mathbf{n}_\perp\times(\hat{z}\times\mathbf{E}))\Bigg]\end{aligned} \tag{198}$$

6.4 Obtaining the rotation of magnetic moments

If we now look at Eq.(79) we can substitute the spin density $\mathbf{s}$ with the expression for spin-polarization. If we insert the expressions Eq.(167) and Eq.(172) we find:

$$\left(\frac{d\mathbf{n}}{dt}\right)_\Gamma = \mathbf{n}\times\Delta_{sd}^2\frac{2}{\pi\alpha}\left(c_1\mathbf{n}_\parallel\times\left(\mathbf{n}_\parallel\times\frac{d\mathbf{n}_\parallel}{dt}\right) + c_2\frac{d\mathbf{n}_\parallel}{dt} + c_3\frac{d\mathbf{n}_\perp}{dt} + c_4\mathbf{n}\right) \tag{199}$$

$$\left(\frac{d\mathbf{n}}{dt}\right)_R = \mathbf{n}\times\Delta_{sd}\frac{2}{\pi\alpha}(A\hat{z}\times\mathbf{E} + B\mathbf{n}_\parallel\times(\mathbf{n}_\parallel\times(\hat{z}\times\mathbf{E})) + C\mathbf{n}_\parallel\times(\mathbf{n}_\perp\times(\hat{z}\times\mathbf{E}))) \tag{200}$$

where the parallel and perpendicular components of $\mathbf{n}$ are given by Eq.(168) and Eq.(169) respectively. In Eq.(199) we find that the first and fourth terms vanish, since the outer product $\mathbf{n}_\parallel\times\frac{d\mathbf{n}_\parallel}{dt}$ is zero for Eq.(168) and $\mathbf{n}\times\mathbf{n}$ is always zero. Therefore Eq.(199) reduces to:

$$\left(\frac{d\mathbf{n}}{dt}\right)_\Gamma = \Delta_{sd}^2\frac{2}{\pi\alpha}\left(c_2 n_z\frac{dn_x}{dt} - c_3 n_x\frac{dn_z}{dt}\right)\hat{y} \tag{201}$$

Solving all the outer products in Eq.(200) gives:

$$\begin{aligned}\left(\frac{d\mathbf{n}}{dt}\right)_R = \Delta_{sd}\frac{2}{\pi\alpha}[&A(-n_zE_x\hat{x} - n_zE_y\hat{y} + n_xE_x\hat{z}) + B(n_x^2n_zE_x\hat{x} - n_x^3E_x\hat{z})\\ &+ Cn_x^2n_zE_y\hat{y}]\end{aligned} \tag{202}$$

Sorting this expression by directional components gives:

$$\begin{aligned}\left(\frac{d\mathbf{n}}{dt}\right)_R = \Delta_{sd}\frac{2}{\pi\alpha}[&(Bn_x^2n_zE_x - An_zE_x)\hat{x} + (Cn_x^2n_zE_y - An_zE_y)\hat{y}\\ &+ (An_xE_x - Bn_x^3E_x)\hat{z}]\end{aligned} \tag{203}$$

We take now take a look at the results of filling in these constants per case. In the case: $\lambda << \Delta_{sd}$ we obtain the expressions:

$$\left(\frac{d\mathbf{n}}{dt}\right)_\Gamma(\overline{\Delta}\to\infty) = \Delta_{sd}^2\frac{2}{\pi\alpha}\begin{pmatrix}\frac{n_z(3-7n_z^2)\overline{\lambda}^2}{12\overline{\Delta}^2}\frac{dn_x}{dt} - \frac{n_xn_z^2(-1+7n_z^2)\overline{\lambda}^2}{12\overline{\Delta}^2}\frac{dn_z}{dt}\\ 0\end{pmatrix} \tag{204}$$

$$\left(\frac{d\mathbf{n}}{dt}\right)_R(\overline{\Delta}\to\infty) = \Delta_{sd}\frac{2}{\pi\alpha}\begin{pmatrix}-\frac{4n_z\overline{\lambda}}{9\overline{\Delta}^2}E_x\\ -\frac{4n_z(1-2n_z^2)\overline{\lambda}}{9\overline{\Delta}^2}E_y\\ \frac{4n_x\overline{\lambda}}{9\overline{\Delta}^2}E_x\end{pmatrix} \tag{205}$$

$$\left(\frac{d\mathbf{n}}{dt}\right)_{\Gamma}(\overline{\Delta}\to 0)=\Delta_{sd}^2\frac{2}{\pi\alpha}\begin{pmatrix}0\\0\\0\end{pmatrix} \tag{206}$$

$$\left(\frac{d\mathbf{n}}{dt}\right)_{R}(\overline{\Delta}\to 0)=\Delta_{sd}\frac{2}{\pi\alpha}\begin{pmatrix}4n_z\overline{\lambda}E_x\\4n_z(1-2n_z^2)\overline{\lambda}E_y\\-4n_x\overline{\lambda}E_x\end{pmatrix} \tag{207}$$

In the case: $\Delta_{sd}<<\lambda$ we find:

$$\left(\frac{d\mathbf{n}}{dt}\right)_{\Gamma}(\overline{\lambda}\to\infty)=\Delta_{sd}^2\frac{2}{\pi\alpha}\begin{pmatrix}0\\n_z\left(1-4n_z^2\overline{\Delta}^2\overline{\lambda}^2+\frac{(1-n_z^2)\overline{\Delta}^2}{4\overline{\lambda}^2}\right)\frac{dn_x}{dt}+n_xn_z^2\overline{\Delta}^2\frac{dn_z}{dt}\\0\end{pmatrix} \tag{208}$$

$$\left(\frac{d\mathbf{n}}{dt}\right)_{R}(\overline{\lambda}\to\infty)=\Delta_{sd}\frac{2}{\pi\alpha}\begin{pmatrix}n_z\overline{\lambda}(6-(1+n_z^2)\overline{\Delta}^2)E_x\\\left(2n_z\overline{\lambda}+\frac{n_z(-2+\overline{\lambda}^2+(2+\overline{\lambda}^2)n_z^2)\overline{\Delta}^2}{\overline{\lambda}}\right)E_y\\-n_x\overline{\lambda}(6-(1+n_z^2)\overline{\Delta}^2)E_x\end{pmatrix} \tag{209}$$

$$\left(\frac{d\mathbf{n}}{dt}\right)_{\Gamma}(\overline{\lambda}\to 0)=\Delta_{sd}^2\frac{2}{\pi\alpha}\begin{pmatrix}0\\n_z\left(1-\frac{3n_z^2\overline{\Delta}^2}{2\overline{\lambda}^2}\right)\frac{dn_x}{dt}-\frac{2n_xn_z^2\overline{\Delta}^2}{\overline{\lambda}^2}\frac{dn_z}{dt}\\0\end{pmatrix} \tag{210}$$

$$\left(\frac{d\mathbf{n}}{dt}\right)_{R}(\overline{\lambda}\to 0)=\Delta_{sd}\frac{2}{\pi\alpha}\begin{pmatrix}-\frac{n_z(4+8\overline{\lambda}^2-(7+3n_z^2)\overline{\Delta}^2)}{4\overline{\lambda}}E_x\\-\frac{n_z(2-4\overline{\lambda}^2-(-2+n_z^2)\overline{\Delta}^2)}{2\overline{\lambda}}E_y\\\frac{n_x(4+8\overline{\lambda}^2-(7+3n_z^2)\overline{\Delta}^2)}{4\overline{\lambda}}E_x\end{pmatrix} \tag{211}$$

7 Conclusion

Regarding the effects of anisotropy and symmetry breaking, we find that they arise no matter the choice of Hamiltonian upon introducing a perturbation term. We observe that all unperturbed conductivity and Gilbert damping matrices are symmetric, but that while the conductivity matrices are also isotropic the Gilbert damping matrices are not. The Gilbert damping matrix of Eq.(87) is anisotropic since the elements depend on components of the direction of localized spins $\mathbf{n}$ and Gilbert damping matrix of Eq.(90) is anisotropic due to the lack of a z-component in the Hamiltonian. If we then introduce the perturbation terms and observe the limits we find that the conductivity matrices become anisotropic, with the exception of the large spin-orbit interaction limit. However if we take $n_z = 1$ the conductivity will always be isotropic. We also find that the conductivity matrices remain symmetrical, due to the lack of any off-diagonal elements. The Gilbert damping matrices become asymmetrical in all limits except for the limit of small exchange energy.

In the expressions for the spin-polarization we find terms both dependent on and independent of the vector components n_x and n_z. With our chosen parameterization, in the first case we find that the spin-polarization only has terms dependent on the second-order perturbation parameter $\overline{\lambda}^2$. This is because in that case the chosen perturbation allows us to absorb the unperturbed terms into ζ, which does not contribute. In the second case this cannot be done, and as such terms independent of the perturbation parameter $\overline{\Delta}$ appear in the expression for the spin-polarization.

In conclusion we find that the anisotropy of the conductivity can be used to determine the direction of localized spins $\mathbf{n}$ by sending a current through the crystal and observing the components of the conductivity parallel and perpendicular to the magnetization. We also find that the complex form of the Gilbert damping means we cannot use only one parameter to properly describe it. In the vector expressions from the Gilbert damping we see that the x- and z-components of are conserved, since the time-derivative of them is zero. Only the y-component changes over time. In the vector expressions from the spin-orbit torque this is not the case and all components change over time. Lastly, if we look at the constants in the Gilbert damping limits they depend on either $\frac{\lambda^2}{\Delta_{sd}^2}$ if $\lambda << \Delta_{sd}$ or $\frac{\Delta_{sd}^2}{\lambda^2}$ if $\Delta_{sd} << \lambda$. This means that despite λ and Δ_{sd} both being small terms compared to the energy the ratio between them is very important in determining the Gilbert damping.

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