Radboud Universiteit

Quantum Synchronization and Entanglement calculations with the Lindblad equation

Sahel Katawazi S1036728

Supervisors: dr. Mikhail Titov, dr. Ivan Ado
Second reviewer: dr. Johan Mentink

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Contents

1 Introduction

A long time ago, in the 17th century, Christiaan Huygens discovered when two linked pendulums oscillate, they will eventually reach a state with the same frequency and perhaps a phase shift. His experiment was done by hanging two pendulum clocks on a wooden support and after some time he Huygens noted that the clocks had the same frequency but were swinging opposite ways[1]. This phenomenon of synchronization has also been noted in quantum mechanics, and subsequently gained much attention, for example with van der Pol oscillators[3]. It has already gained much attention analytically as well as experimentally [2]. Synchronization in the quantum case is similar to the classical case, but one needs to also consider how states are correlated to each other.

Synchronization could be consider a dissipation of a state to reach equilibrium or be tuned. As such, the Lindblad equation is a proper method of analyzing for quantum synchronization.

To understand the theory behind the Lindblad equation, we need to understand what we are working with, namely the fact that we are in the second quantization. Here, we work with operators that act on multiple particles at a time. Instead of looking at one single wave function, we need to work with ensembles of states, since we may not know which states become occupied. Since we have multiple states that may or may not be normalized, we require some other way to calculate probabilities of finding particles in a certain state. For this we introduce the density matrix $\rho = \sum_{i,j} c_{ij} \left|\psi_i\right\rangle \left\langle\psi_j\right|$, where $c_{ij} = \langle\psi_i|\psi_j\rangle$ are correlations between two states if $i \neq j$ and probabilities to find a state otherwise. One can easily show that with this definition, the density matrix obeys an equation for time evolution called the von Neumann equation (with $\hbar = 1$)[4]:

$$\frac{\partial \rho}{\partial t} = -i[\mathscr{H}, \rho], \tag{1}$$

where $\mathscr{H}$ is the system Hamiltonian. The Lindblad equation assumes that our system can be decomposed in two separate systems: our quantum system of interest and the heat bath. The latter is assumed to have infinitely more degrees of freedom than the quantum system. Where the full system governed by the two density matrices together is still described by the von Neumann equation as above, our quantum system of interest is of the form understood by the Lindblad equation, there are some important remarks to make with it:

$$\frac{\partial \rho}{\partial t} = -i[\mathscr{H}', \rho] + \sum_i \Gamma_i \Big(L_i \rho L_i^\dagger - \frac{1}{2}\Big\{L_i^\dagger L_i, \rho\Big\}\Big), \tag{2}$$

We immediately notice there is still a von Neumann part, however the Hamiltonian is shifted in energy. The Γ_i are terms that have knowledge about the coupling of our quantum system of interest to the bath, and in fact are responsible for thermalization of the density matrix. The L_i and its hermitian conjugate are operators of our system only. The $L\rho L^\dagger$ term is the quantum term of our equation and describes spontaneous jumps from one state to the other. The anticommutator is the more classical term and describes the slow dissipation[4, 5, 6, 7, 8]. Note also that we can generate a non-hermitian Hamiltonian by ignoring the quantum jump term, which is a possible approximation only at constantly applied external field.

There are many novel systems [10, 11] described by the Lindblad equation, some are simple two level systems, that can be solved analytically. An interesting property however may be to check entanglement in the Lindblad equation and subsequently, the synchronization between two quantum spins, driven by a time dependent magnetic field. In this thesis we will describe how a simple system of two quantum interacting spins coupled to a magnon bath may evolve in an external time dependent magnetic field. We derive the corresponding Lindblad equation and analyze quantum spin synchronization and the evolution of quantum spin entanglement.

1.1 Magnon bath

We begin with a simple square lattice on which we have our two spins and the spin waves[9]. Thus we begin with a Hamiltonian of the form:

$$\mathscr{H} = -\Delta(\sigma_z^{(1)} + \sigma_z^{(2)}) - I(\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}) \cdot \mathbf{S}_0 - \xi\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)} - \frac{J}{2}\sum_{\langle i,j\rangle} \mathbf{S}_i \cdot \mathbf{S}_j - H\sum_i \mathbf{S}_i, \tag{3}$$

where I is the coupling strength between the bath and the system, and ξ is the coupling between the two spins. Δ and H are defined by the external field, which is taken along the z-direction. and J is positive, meaning a ferromagnetic ground state. We further define operators acting on spin 1 as $A^{(1)} = A \otimes \mathbb{1}$ and for spin 2 it is $A^{(2)} = \mathbb{1} \otimes A$. Next we have the Pauli vectors for each spin as $\boldsymbol{\sigma}^{(i)} = (\sigma_x^{(i)}, \sigma_y^{(i)}, \sigma_z^{(i)})$, $i = 1,2$. The coupling between spin 1 and 2 is scalar in nature. The magnons we find by using the Holstein-Primakoff transformations:

$$S_i^+ = \sqrt{2S - a_i^\dagger a_i}\, a_i, \tag{4}$$

$$S_i^- = a_i^\dagger \sqrt{2S - a_i^\dagger a_i}, \tag{5}$$

$$S_i^z = S - a_i^\dagger a_i, \tag{6}$$

where $a_i^\dagger$ and a_i are bosonic creation and annihilation operators which follow the standard commutation relations $[a_i, a_j^\dagger] = \delta_{ij}$, $[a_i, a_j] = 0$ and $[a_i^\dagger, a_j^\dagger] = 0$. For spin waves we may consider only small deviations from the z-direction, thus the approximation $S_i^+ = \sqrt{2S}a_i$ and $S_i^- = a_i^\dagger\sqrt{2S}$ is sufficient. The Hamiltonian now takes the form:

$$\begin{aligned}\mathscr{H} = &-(\Delta + IS)(\sigma_z^{(1)} + \sigma_z^{(2)}) - \xi\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)} + I(\sigma_z^{(1)} + \sigma_z^{(2)})a_0^\dagger a_0 - \\ &I\sqrt{2S}((\sigma_+^{(1)} + \sigma_+^{(2)})a_0^\dagger + (\sigma_-^{(1)} + \sigma_-^{(2)})a_0) + \frac{JS}{2}\sum_{\langle i,j\rangle}(a_i^\dagger - a_j\dagger)(a_i - a_j) + H\sum_i a_i^\dagger a_i,\end{aligned} \tag{7}$$

where $\sigma_\pm^{(1,2)} = \frac{1}{2}(\sigma_x^{(1,2)} \pm i\sigma_y^{(1,2)})$. We can diagonalize the magnon bath by employing a Fourier transform to the creation and annihilation operators.

$$a_i = \frac{1}{\sqrt{N}}\sum_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}_i}\, a_{\mathbf{k}}, \tag{8}$$

now we may apply this to the magnon bath, and for a square lattice the result will be[15, 14]:

$$\frac{JS}{2}\sum_{\langle i,j\rangle}(a_i^\dagger - a_j\dagger)(a_i - a_j) + H\sum_i a_i^\dagger a_i = \sum_{\mathbf{k}} \omega_{\mathbf{k}}\, a_{\mathbf{k}}^\dagger a_{\mathbf{k}}, \tag{9}$$

where

$$\omega_{\mathbf{k}} = 2JS(2 - \cos(k_x b) - \cos\!\big(k_y b\big)) + H, \tag{10}$$

here b is the lattice constant. $\omega_{\mathbf{k}}$ is the magnon energy. Our Hamiltonian can then be decomposed into a form like:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I, \tag{11}$$

the first term describes the two separate systems and the second term is the interaction between the bath and the two spins.

$$\mathcal{H}_0 = -\frac{\Delta_0}{2}(\sigma_z^{(1)} + \sigma_z^{(2)}) - \xi \boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)} + \sum_{\mathbf{k}} \omega_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}}, \tag{12}$$

$$\mathcal{H}_I = -I\sqrt{\frac{2S}{N}} \sum_{\mathbf{k}} ((\sigma_+^{(1)} + \sigma_+^{(2)})\, a_{\mathbf{k}}^{\dagger} + (\sigma_-^{(1)} + \sigma_-^{(2)})\, a_{\mathbf{k}}) + \frac{I}{N} \sum_{\mathbf{k},\mathbf{k}'} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}'} (\sigma_z^{(1)} + \sigma_z^{(2)}), \tag{13}$$

where $\Delta_0 = 2(\Delta + IS)$. We have thus formulated a Hamiltonian describing the magnon bath, the two two quantum spins and their coupling, and how the quantum spins interact with the bath.

1.2 Entanglement

We are interested in how the entanglement of two spins evolve. The term describing how the two spins work together is the ξ term. Let us rewrite our $\mathcal{H}_0$ to better understand what it means, remember that the entanglement term is:

$$-\xi \boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)} = -\xi(\sigma_x^{(1)} \sigma_x^{(2)} + \sigma_y^{(1)} \sigma_y^{(2)} + \sigma_z^{(1)} \sigma_z^{(2)}), \tag{14}$$

using relations for $\sigma_{\pm}^{(i)}$ we find our entanglement term to be:

$$-\xi\, \sigma_z^{(1)} \sigma_z^{(2)} - 2\xi(\sigma_+^{(1)} \sigma_-^{(2)} + \sigma_-^{(1)} \sigma_+^{(2)}), \tag{15}$$

the spins are aligned to the z-direction and flipped oppositely. It is best to work in an entangled basis. Namely we choose:

$$|\psi_1\rangle = |\uparrow\uparrow\rangle, \tag{16}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \tag{17}$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\uparrow\downarrow\rangle), \tag{18}$$

$$|\psi_4\rangle = |\downarrow\downarrow\rangle, \tag{19}$$

or in vector notation:

$$|\psi_1\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad |\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}, \qquad |\psi_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \qquad |\psi_4\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \tag{20}$$

it is an orthonormal basis. We also need a way to measure the entanglement between the two spins. For this we will use the entropy of entanglement. It is defined as follows, we take $k_B = 1$:

$$\mathcal{S} = -\mathrm{Tr}(\rho \ln \rho), \tag{21}$$

the entanglement entropy describes how entangled two systems are, more specifically it can be shown that if the system is in a pure state (there is a one on the diagonal and zero everywhere else) the logarithm will vanish and entanglement entropy $\mathscr{S} = 0$. Otherwise, if $\mathscr{S} \neq 0$, it would mean our system can only be described by a mixed state. It is entangled.

Now we have the final form of our $\mathscr{H}_0$:

$$\mathscr{H}_0 = \frac{-\Delta_0}{2}(\sigma_z^{(1)} + \sigma_z^{(2)}) - \xi\,\sigma_z^{(1)}\sigma_z^{(2)} - 2\xi(\sigma_+^{(1)}\sigma_-^{(2)} + \sigma_-^{(1)}\sigma_+^{(2)}), \tag{22}$$

the Hamiltonian shows how the two quantum spins are interacting separately with the external field through Δ_0 and are coupled via ξ. The basis for this Hamiltonian is the entangled basis 20 with energies $-\Delta_0 - \xi$, 3ξ, $-\xi$ and $\Delta_0 - \xi$. with the ψ_2 state being the singlet state with eigenvalue 3ξ [12, 13].

1.3 The Lindblad equation

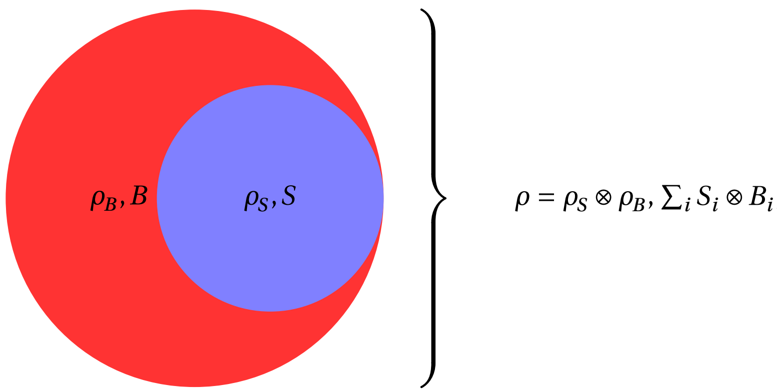

Figure 1: A system that can be decomposed in two by tensor products in the density matrix and the Hamiltonian. ρ_S describes the system of interest with operators S and ρ_B describes the bath with operators B, the interaction between the two is then given by $\sum_i S_i \otimes B_i$.

again, the Lindblad equation assumes a separable quantum system of interest and a big heat bath, as well as a separable interaction Hamiltonian of a form like in the above figure, with S and B being particle operators acting on the quantum system and the bath respectively. For our theory we consider a simple system, two spins coupled to a larger magnon bath. We assume that our system can be decomposed like that of 1. Because of that, we could simply trace the bath terms out of it. To do that, we first switch to the interaction picture, in Schrödinger picture time dependence is placed on wave functions and the operators are usually independent of time, in Heisenberg picture it is the opposite, in the interaction (or Dirac) picture, time dependence is both on operators and on wave functions. Wave functions in this picture look as follows:

$$|\psi_I\rangle = e^{i\mathscr{H}_0 t}|\psi\rangle, \tag{23}$$

where we take the unperturbed Hamiltonian. The operators in this picture become:

$$A_I = e^{i\mathscr{H}_0 t} A e^{-i\mathscr{H}_0 t}, \tag{24}$$

we work in second quantization, meaning we need an equation that describe the time evolution of the density matrix, of course for this we will use the famous von Neumann equation, it is defined as:

$$\frac{\partial \rho}{\partial t} = -i[\mathscr{H}, \rho], \tag{25}$$

where we take $\hbar = 1$. In the interaction picture, clearly the commutator will contain the interaction Hamiltonian as $\bar{\mathscr{H}}_I = e^{i\mathscr{H}_0 t}\mathscr{H}_I e^{-i\mathscr{H}_0 t}$. For the Lindblad equation, we need to iterate the equation. The result will be in integral form.

$$\frac{\partial(\rho_I^S \otimes \rho_I^B)}{\partial t} = -i\int_0^t dt' [\bar{\mathscr{H}}_I(t), [\bar{\mathscr{H}}_I(t'), (\rho_I^S \otimes \rho_I^B)(t')]], \tag{26}$$

first let us assume our bath is constant in time, more specifically:

$$\rho^B = Z^{-1}\prod_{\mathbf{k}} e^{-\beta(\omega_{\mathbf{k}}-\mu)\, a_{\mathbf{k}}^\dagger a_{\mathbf{k}}} \left|\Phi_{\mathbf{k}}\right\rangle \left\langle\Phi_{\mathbf{k}}\right|, \tag{27}$$

where:

$$Z = \prod_{\mathbf{k}} \left\langle\Phi_{\mathbf{k}}\right| e^{-\beta(\omega_{\mathbf{k}}-\mu)\, a_{\mathbf{k}}^\dagger a_{\mathbf{k}}} \left|\Phi_{\mathbf{k}}\right\rangle \tag{28}$$

$$= \prod_{\mathbf{k}} \frac{1}{1-e^{-\beta(\omega_{\mathbf{k}}-\mu)}}, \tag{29}$$

the bath density matrix remains unchanged in the interaction picture. Next we trace out the bath terms from equation 26. Afterwards we make a Markov approximation, which states that the interaction between the system and the bath starts at $t = 0$, it has no memory of the past. This is done easily by taking $\rho_I^S(t') = \rho_I^S(t)$. Before the approximation, the integral will look as follows:

$$\frac{\partial \rho_I^S}{\partial t} = -\int_0^t dt' \,\mathrm{Tr}_B [\bar{\mathscr{H}}_I(t), [\bar{\mathscr{H}}_I(t'), \rho_I^S(t) \otimes \rho_I^B]], \tag{30}$$

this integral can be solved quite easily, since most terms in the integral will disappear under the trace. By changing to the interaction picture and assuming a time-independent bath, we have an integral equation to formulate the dissipators in the Lindblad equation.

2 Results

For the switch to interaction picture, we need to take the exponent of $\mathscr{H}_0$, for this we first note that the magnon terms will commute with the rest of the Hamiltonian, so we will calculate those terms first. It is easy to show that the following holds:

$$e^{i\omega_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} t}\, a_{\mathbf{k}}^\dagger\, e^{-i\omega_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} t} = e^{i\omega_{\mathbf{k}} t}\, a_{\mathbf{k}}^{\dagger}, \qquad e^{i\omega_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} t}\, a_{\mathbf{k}}\, e^{-i\omega_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} t} = e^{-i\omega_{\mathbf{k}} t}\, a_{\mathbf{k}}, \tag{31}$$

next we check the rest of the Hamiltonian, the easiest way to do this is by taking the matrix form of the Hamiltonian, which in the basis of equation 20 is:

$$\mathscr{H}_0 = \begin{pmatrix} -\Delta_0 - \xi & 0 & 0 & 0 \\ 0 & \xi & -2\xi & 0 \\ 0 & -2\xi & \xi & 0 \\ 0 & 0 & 0 & \Delta_0 - \xi \end{pmatrix}, \tag{32}$$

the exponential of which will be:

$$e^{i\mathscr{H}_0 t} = \begin{pmatrix} e^{-it(\Delta 0+\xi)} & 0 & 0 & 0 \\ 0 & \frac{1}{2}e^{-i\xi t} + \frac{1}{2}e^{3i\xi t} & \frac{1}{2}e^{-i\xi t} - \frac{1}{2}e^{3i\xi t} & 0 \\ 0 & \frac{1}{2}e^{-i\xi t} - \frac{1}{2}e^{3i\xi t} & \frac{1}{2}e^{-i\xi t} + \frac{1}{2}e^{3i\xi t} & 0 \\ 0 & 0 & 0 & e^{it(\Delta 0-\xi)} \end{pmatrix}, \tag{33}$$

by simple matrix multiplication we get the following form for the interaction Hamiltonian in interaction picture:

$$\bar{\mathscr{H}}_I = -I\sqrt{\frac{2S}{N}}\sum_{\mathbf{k}}\left(e^{i(\omega_{\mathbf{k}}-\Delta_0)t}\,a^\dagger_{\mathbf{k}}(\sigma_+^{(1)}+\sigma_+^{(2)}) + e^{-i(\omega_{\mathbf{k}}-\Delta_0)t}\,a_{\mathbf{k}}(\sigma_-^{(2)}+\sigma_-^{(1)})\right) + \frac{I}{N}\sum_{\mathbf{k},\mathbf{k}'} e^{i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}\,a^\dagger_{\mathbf{k}}a_{\mathbf{k}'}(\sigma_z^{(1)}+\sigma_z^{(2)}), \tag{34}$$

an important observation here is that this Hamiltonian is independent of ξ, meaning this Hamiltonian knows nothing of the coupling between the two spins, it only knows that the two spins are coupled to the bath. This will have some important consequences for the Lindblad equation. Next we will look at the double commutator and trace out the bath terms. We note there are only three commutators that are relevant, since some combinations of the creation and annihilation operators under the trace will be zero. The following commutators we consider:

$$\left[e^{i(\omega_{\mathbf{k}}-\Delta_0)t}a^\dagger_{\mathbf{k}}\sigma_+^{(i)},\left[e^{-i(\omega_{\mathbf{k}}-\Delta_0)t'}a_{\mathbf{k}'}\sigma_-^{(j)},\rho_I^S(t')\otimes\rho^B\right]\right] = e^{i(\omega_{\mathbf{k}}-\Delta_0)(t-t')}\left[a^\dagger_{\mathbf{k}}\sigma_+^{(i)},\left[a_{\mathbf{k}'}\sigma_-^{(j)},\rho_I^S(t')\otimes\rho^B\right]\right], \tag{35}$$

$$\left[e^{-i(\omega_{\mathbf{k}}-\Delta_0)t}a_{\mathbf{k}}\sigma_-^{(i)},\left[e^{i(\omega_{\mathbf{k}}-\Delta_0)t'}a^\dagger_{\mathbf{k}'}\sigma_+^{(j)},\rho_I^S(t')\otimes\rho^B\right]\right] = e^{-i(\omega_{\mathbf{k}}-\Delta_0)(t-t')}\left[a_{\mathbf{k}}\sigma_-^{(i)},\left[a^\dagger_{\mathbf{k}'}\sigma_+^{(j)},\rho_I^S(t')\otimes\rho^B\right]\right], \tag{36}$$

$$\left[e^{i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}a^\dagger_{\mathbf{k}}a_{\mathbf{k}'}\sigma_z^{(i)},\left[e^{i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t'}a^\dagger_{\mathbf{p}}a_{\mathbf{p}'}\sigma_z^{(j)},\rho_I^S(t')\otimes\rho^B\right]\right] = e^{i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})(t-t')}\left[a^\dagger_{\mathbf{k}}a_{\mathbf{k}'}\sigma_z^{(i)},\left[a^\dagger_{\mathbf{p}}a_{\mathbf{p}'}\sigma_z^{(j)},\rho_I^S(t')\otimes\rho^B\right]\right], \tag{37}$$

where we need to change some indices since we will be summing over these commutators. However for the first two we may immediately take $\mathbf{k}=\mathbf{k}'$, since the terms where they are not equal will be zero under the trace. Taking $\mathrm{Tr}\left(a^\dagger_{\mathbf{k}}a_{\mathbf{k}}\right)=n_{\mathbf{k}}$ and $\mathrm{Tr}\left(a_{\mathbf{k}}a^\dagger_{\mathbf{k}}\right)=1+n_{\mathbf{k}}$, where $n_{\mathbf{k}}=n(\omega_{\mathbf{k}})=(e^{\beta(\omega_{\mathbf{k}}-\mu)}-1)^{-1}$, the Bose-Einstein distribution. for the third term we need to carefully consider combinatorics. Afterwards we make the Markov approximation, in mathematical terms this means $\int_0^t \to \int_0^\infty$ and $\rho_I^S(t')\to\rho_I^S(t)$. Our integral will look as follows:

$$\begin{aligned}\frac{\partial\rho_I^S}{\partial t} =& \frac{2SI^2}{N}\sum_{i,j}\sum_{\mathbf{k}}\int_0^\infty dt'\,e^{i(\omega_{\mathbf{k}}-\Delta_0)(t-t')}\left[n_{\mathbf{k}}\sigma_+^{(i)}\sigma_-^{(j)}\rho_I^S(t)-(1+n_{\mathbf{k}})\sigma_+^{(i)}\rho_I^S(t)\sigma_-^{(j)}-n_{\mathbf{k}}\sigma_-^{(j)}\rho_I^S(t)\sigma_+^{(i)}+\rho_I^S(t)(1+n_{\mathbf{k}})\sigma_-^{(j)}\sigma_+^{(i)}\right]\\ &+\frac{2SI^2}{N}\sum_{i,j}\sum_{\mathbf{k}}\int_0^\infty dt'\,e^{-i(\omega_{\mathbf{k}}-\Delta_0)(t-t')}\left[(1+n_{\mathbf{k}})\sigma_-^{(i)}\sigma_+^{(j)}\rho_I^S(t)-n_{\mathbf{k}}\sigma_-^{(i)}\rho_I^S(t)\sigma_+^{(j)}-(1+n_{\mathbf{k}})\sigma_+^{(j)}\rho_I^S(t)\sigma_-^{(i)}+\rho_I^S(t)n_{\mathbf{k}}\sigma_+^{(j)}\sigma_-^{(i)}\right]\\ &+\frac{I^2}{N^2}\sum_{\mathbf{k},\mathbf{p}}\int_0^\infty dt'\,e^{i(\omega_{\mathbf{k}}-\omega_{\mathbf{p}})(t-t')}\left[n_{\mathbf{k}}(1+n_{\mathbf{p}})+n_{\mathbf{p}}(1+n_{\mathbf{k}})\right]\left((\sigma_z^{(1)}+\sigma_z^{(2)})\rho_I^S(t)(\sigma_z^{(1)}+\sigma_z^{(2)})+\left\{\sigma_z^{(1)}+\sigma_z^{(2)},\rho_I^S(t)\right\}\right)\\ &+\frac{2I^2}{N^2}\sum_{\mathbf{k},\mathbf{p}}\int_0^\infty dt'\,n_{\mathbf{k}}n_{\mathbf{p}}\left((\sigma_z^{(1)}+\sigma_z^{(2)})\rho_I^S(t)(\sigma_z^{(1)}+\sigma_z^{(2)})+\left\{\sigma_z^{(1)}+\sigma_z^{(2)},\rho_I^S(t)\right\}\right),\end{aligned} \tag{38}$$

with $\{\cdot,\cdot\}$ being the anticommutator. This integral can be solved quite easily, note however that the final integral will lead to divergencies. To solve this, we may redefine Δ_0 to include the diverging term:

$$\Delta_0 = 2\left(\Delta + I\left(S - \frac{1}{N}\sum_{\mathbf{k}} n_{\mathbf{k}}\right)\right), \tag{39}$$

doing this means we made a mean-field approximation that describes thermalized magnons depolarizing our system. Finally, our Lindbladian will look as follows:

$$
\begin{aligned}
\frac{\partial \rho_I^S}{\partial t} &= 2\pi I^2 (\nu_0 S (n_0 \mathscr{D}_{-+}[\rho_I^S] + (1+n_0)\mathscr{D}_{+-}[\rho_I^S]) + \Gamma \mathscr{D}_{zz}[\rho_I^S]) \\
&= \mathscr{L}(\rho_I^S),
\end{aligned} \tag{40}
$$

where $n_0 = n(\Delta_0)$ and $\Gamma = \int d\omega \nu(\omega) n(\omega)(1+n(\omega))$ and $\nu_0 = \nu(\Delta_0)$ is the magnon density of states at the gap:

$$
\nu(\omega) = \frac{1}{N} \sum_{\mathbf{k}} \delta(\omega - \omega_{\mathbf{k}}), \tag{41}
$$

the terms

$$
\mathscr{D}_{-+}[\rho] = (\sigma_-^{(1)} + \sigma_-^{(2)})\rho(\sigma_+^{(1)} + \sigma_+^{(2)}) - \frac{1}{2}\Big\{(\sigma_+^{(1)} + \sigma_+^{(2)})(\sigma_-^{(1)} + \sigma_-^{(2)}), \rho\Big\}, \tag{42}
$$

$$
\mathscr{D}_{+-}[\rho] = (\sigma_+^{(1)} + \sigma_+^{(2)})\rho(\sigma_-^{(1)} + \sigma_-^{(2)}) - \frac{1}{2}\Big\{(\sigma_-^{(1)} + \sigma_-^{(2)})(\sigma_+^{(1)} + \sigma_+^{(2)}), \rho\Big\}, \tag{43}
$$

$$
\mathscr{D}_{zz}[\rho] = (\sigma_z^{(1)} + \sigma_z^{(2)})\rho(\sigma_z^{(1)} + \sigma_z^{(2)}) - \frac{1}{2}\Big\{(\sigma_z^{(1)} + \sigma_z^{(2)})(\sigma_z^{(1)} + \sigma_z^{(2)}), \rho\Big\}, \tag{44}
$$

are our Lindblad dissipators. These operators are clearly traceless, and thus conserve the trace of the density matrix. It is clear that if we switch back to Schrödinger picture, we get the commutator term in the equation back and we can rewrite the entire equation to be of the form 2. We also know that given enough time the two spins should equilibrate, or thermalize to the temperature of the bath. More specifically:

$$
\lim_{t\to\infty} \rho^S(t) = \rho_{therm}, \tag{45}
$$

where $\rho^S(t)$ is the solution to the Lindblad equation and ρ_{therm} is the thermal density matrix:

$$
\rho_{therm} = \frac{e^{-\beta\mathscr{H}}}{\mathrm{Tr}\big(e^{-\beta\mathscr{H}}\big)}, \tag{46}
$$

we may implicitly check if our equation is valid if this is indeed the case. The simplest way to do this is by simply plugging the thermal density matrix in, and checking if it is a stationary solution. There should in fact only be one stationary solution to the Lindblad equation, namely the thermal density matrix. One can check that indeed this is the case and our Lindblad equation is correct:

$$
\frac{\partial \rho_{therm}}{\partial t} = \mathscr{L}(\rho_{therm}) = 0 \tag{47}
$$

This is however an implicit proof, a more explicit proof can be found by solving the Lindblad equation. There is an analytical solution, which looks like this:

$$
\rho^S_{therm} \begin{pmatrix} \frac{(1-c)e^{2\beta\Delta 0}}{e^{\beta\Delta 0}+e^{2\beta\Delta 0}+1} & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & \frac{(1-c)e^{\beta\Delta 0}}{e^{\beta\Delta 0}+e^{2\beta\Delta 0}+1} & 0 \\ 0 & 0 & 0 & \frac{1-c}{e^{\beta\Delta 0}+e^{2\beta\Delta 0}+1} \end{pmatrix} \tag{48}
$$

where we have already diagonalized the density matrix. Because the Hamiltonian is off-diagonal originally, the density matrix is too, meaning the structure is correct. This diagonalized density matrix is clearly not the same as the thermalized density matrix. For one it is missing the coupling constant ξ but more importantly the population of the ψ_2 state is some arbitrary number! Whatever is in ψ_2 at $t = 0$ stays there for all time. What does this mean for our system? One may check for mathematical errors, however there are none. In fact if we look at the thermal density matrix:

$$\rho_{therm} = \begin{pmatrix} \frac{e^{\beta(\Delta0+\xi)}}{e^{\beta(\xi-\Delta0)}+e^{\beta(\Delta0+\xi)}+e^{-3\beta\xi}+e^{\beta\xi}} & 0 & 0 & 0 \\ 0 & \frac{e^{\beta\Delta0}}{e^{2\beta(\Delta0+2\xi)}+e^{\beta(\Delta0+4\xi)}+e^{\beta\Delta0}+e^{4\beta\xi}} & 0 & 0 \\ 0 & 0 & \frac{e^{\beta(\Delta0+4\xi)}}{e^{2\beta(\Delta0+2\xi)}+e^{\beta(\Delta0+4\xi)}+e^{\beta\Delta0}+e^{4\beta\xi}} & 0 \\ 0 & 0 & 0 & \frac{e^{4\beta\xi}}{e^{2\beta(\Delta0+2\xi)}+e^{\beta(\Delta0+4\xi)}+e^{\beta\Delta0}+e^{4\beta\xi}} \end{pmatrix}, \tag{49}$$

we find not equation 45, instead what we find is that one of our states is completely decoupled from our bath, namely the singlet state, $|\psi_2\rangle$ Our Lindblad density matrix doesn't thermalize to the regular thermal density matrix, but a very specific case. By virtue of equation 47 we know that the Lindblad equation should solve for all values of ξ. But this is not the case. What we see here is that our singlet state does not thermalize whatsoever, in fact however high the population in the state is at t=0, it will not change.

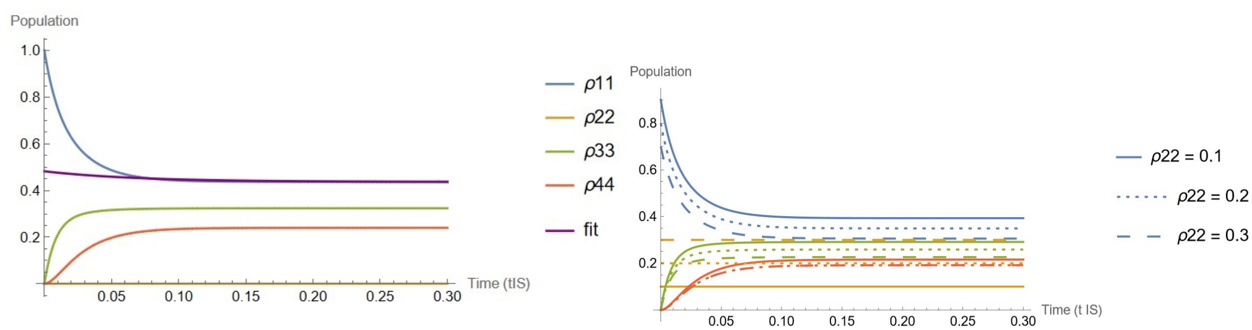

Figure 2: This graph shows the evolution of the Lindblad density matrix, the left graph shows the case $\xi = 0$, the right graph shows how for different initial values of ρ_{22} the other states behave . This is for the diagonalized case, and the purple line is a fit to determine the time at which the elements thermalize.The used quantities are $\Delta_0 = 0.3, \beta = 1, v_0 = 1, \Gamma = 0.1, I = 1, S = 1, \xi = 0.01$, although the value of ξ doesn't matter, since it disappears.

From the above figures we see that if we haven't put anything in our singlet state, it will remain empty forever, and if we do start of with some nonzero singlet state, the population will remain constant through all time, unless we somehow pump it, for example with a perturbation. Now we have solved the integral from equation 30 and formulated the dissipators and the Lindblad equation, we have also discovered that the thermalization procedure is different from what we expected, namely one state is decoupled from the bath.

2.0.1 Linearly polarized perturbation

We may also check different types of perturbations. This can be done easily by first transforming back to Schrödinger picture and simply perturbing the Hamiltonian:

$$\frac{\partial \rho_S}{\partial t} = -i[\mathscr{H}_0 + V_1(t), \rho_S] + \mathscr{L}[\rho_S], \tag{50}$$

with the perturbation being defined as follows:

$$\begin{aligned} V_1(t) &= \cos(\Omega t)\sigma_x^{(1)} \\ &= \cos(\Omega t)(\sigma_+^{(1)} + \sigma_-^{(1)}), \end{aligned} \tag{51}$$

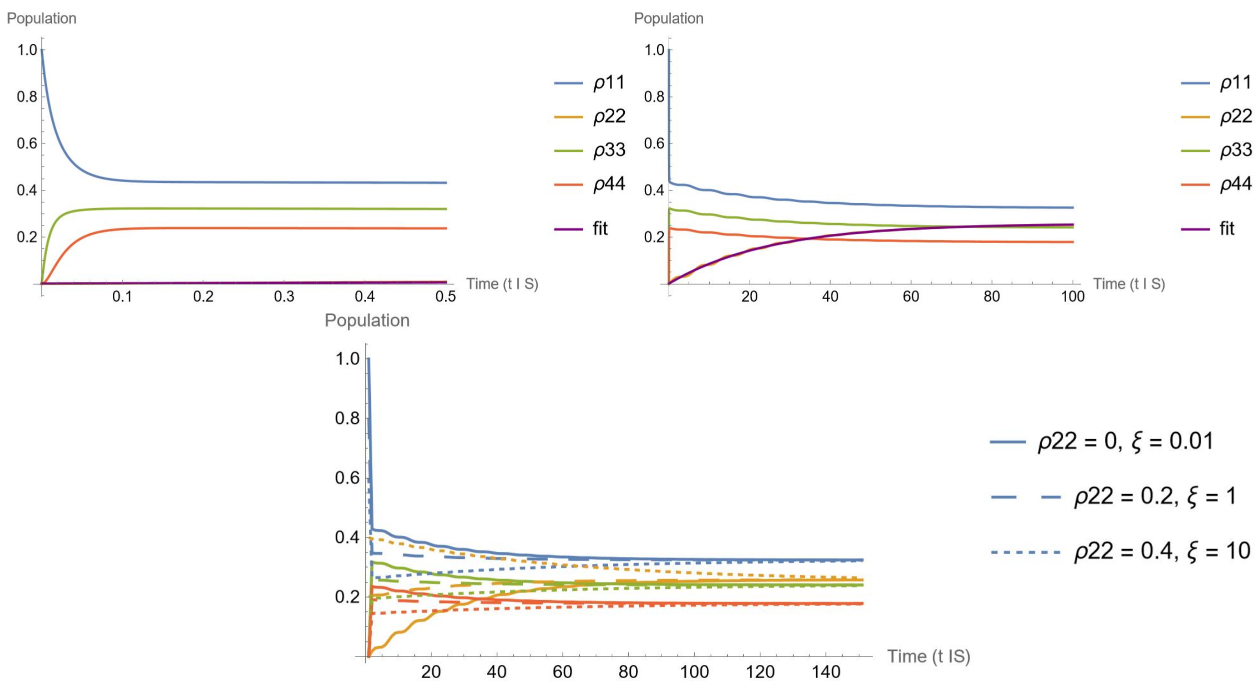

Figure 3: This figure shows how our system behaves under a linearly polarized perturbation,the left figure shows what happens on short timescales and the right on a long timescale the values used in the upper graphs are $\Delta_0 = 0.3, \beta = 1, \nu_0 = 1, \Gamma = 0.1, I = 1, S = 1, \xi = 0.01, \Omega = 0.5$. *The* ξ *was changed for the lower graph*

with any perturbation we expect the system to no longer thermalize, basically one of our spins is periodically flipped to the x direction and back to the z direction, thus we expect some oscillations to occur in the populations as well as the entanglement entropy. Indeed, we see from figure 3 that our system has oscillatory behaviour, and although looks to be thermalized, it is actually not. The off diagonal elements are still oscillating, even in large times, meaning it does not thermalize. Another important thing to note is that in this case, the perturbation directly influences the diagonal elements and thus the states and although one might think the singlet is coupled to the bath again, in reality the perturbation is simply pumping the singlet state. To keep the normalization condition of the density matrix, this means the other states will be altered, the lower graph in the figurer shows that thermalization is independent of ξ, which is in line with what we expect.

2.0.2 Circularly polarized perturbation

Another type of perturbation we can use is circularly polarized, namely our perturbation is nothing other than:

$$V_2(t) = \cos(\Omega t)\sigma_y^{(1)} \tag{52}$$

$$= -i\cos(\Omega t)(\sigma_+^{(1)} - \sigma_-^{(1)}), \tag{53}$$

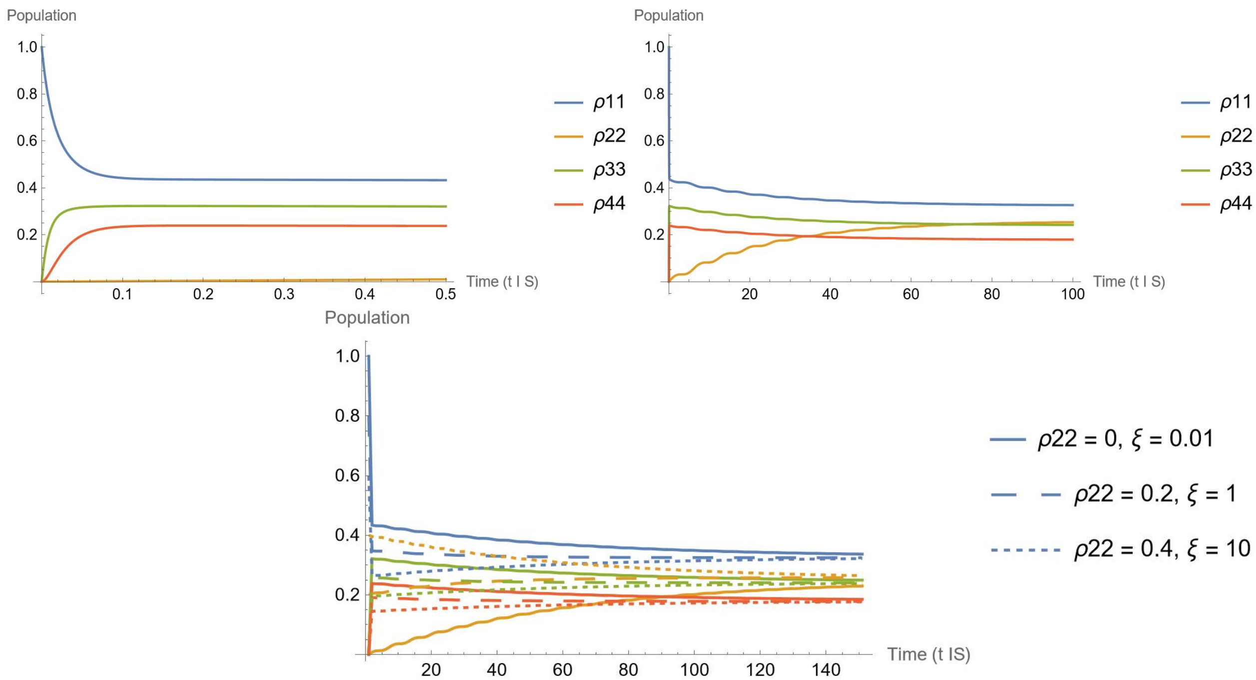

Figure 4: This figure shows how our system evolves under a circularly polarized perturbation, it is clear that even in large times the diagonal elements will remain constant, but the off diagonal elements oscillate and although are small, are not 0 meaning this will never thermalize. The used values are again: $\Delta_0 = 0.3, \beta = 1, \nu_0 = 1, \Gamma = 0.1, I = 1, S = 1, \xi = 0.01, \Omega = 0.5$. *Except for the lower graph, where* ξ *was changed.*

once again, we expect some oscillatory behaviour and that our density matrix never thermalizes, however as can be seen from figure 4, the oscillatory part is found in the off-diagonal components as well as the diagonal ones, this looks very similar to the population evolution of the linearly polarized perturbation. Again this means that our system indeed never thermalizes, since for thermalization it was required to have zero off-diagonal components, but our singlet remains decoupled from the bath, even though once again there are states pumped into the singlet. The lower graph shows again how the diagonal elements are independent of ξ and still approach the same value, even for different ξ's.

2.0.3 Time Scale of Thermalization

It is easy to see how there is a time scale for the thermalization in the unperturbed case (and the circularly polarized case), which is relatively short and that there is another time scale on the linearly polarized perturbation which is longer, we can check how there is a dependence on the coupling constant ξ for both cases. To this end, we fit a function of the form $a + be^{-ct}$ and extract the decay parameter for both time scales and see what the dependence looks like.

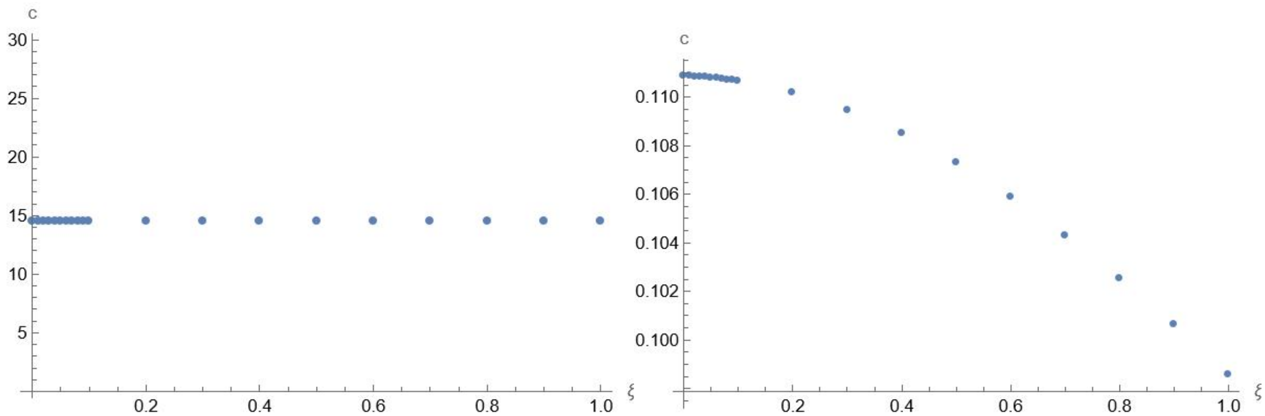

Figure 5: This figure shows how the decay parameter evolves with increasing ξ, first was the range from 0.01 to 0.1 plotted and afterwards from 0.1 to 1. Left shows the short timescale and right the long timescale.

From the above figure we see the short timescale has no dependence on ξ, as expected, since that is the original thermalization timescale whose equation did not contain any ξ terms, the long timescale does show some dependence, namely some decay, but no real conclusions can be made except for the fact that the dependence is very weak, this is the case for both the linear and circularly polarized perturbation.

2.1 Solution to the decoupling

To see why the ψ_2 state decouples, we have to go back to our Hamiltonian and find the eigenvalues of our system, from here we may infer what is behind the decoupling. The procedure is trivial, we find the following few eigenenergies related to our eigenvectors from 20:

$$\begin{aligned} E_1 &= -\Delta_0 - \xi, \\ E_2 &= 3\xi, \\ E_3 &= -\xi, \\ E_4 &= \Delta_0 - \xi, \end{aligned} \tag{54}$$

we have a singlet state, namely $|\psi_2\rangle$ with eigenenergie E_2 and a triplet of $|\psi_1\rangle, |\psi_3\rangle, |\psi_4\rangle$ with eigenergies E_1, E_3, E_4. We can already solve one mystery now, ξ disapears because the Lindblad equation describes how our spins switch from one state to the other, meaning if it goes from state one to state four, ξ would indeed disappear, the same goes for any combination of the triplet states. This however does not explain why the singlet state has become decoupled, clearly a transition from any state to state 2 will have a remainder of at least 2ξ or -4ξ. However if we take a closer look at $|\psi_2\rangle\langle\psi_2|$ we find the following operator:

$$|\psi_2\rangle\langle\psi_2| = \frac{1}{4}(\sigma_0^{(1)}\sigma_0^{(2)} - \boldsymbol{\sigma}^{\mathbf{(1)}} \cdot \boldsymbol{\sigma}^{\mathbf{(2)}}) = \chi, \tag{55}$$

this operator actually commutes with the original Hamiltonian $\mathscr{H}_0$, this is the case before any approximations are made. This means we have a conserved quantity in our system of which the expectation value that cannot be thermalized, because the root of this quantity lies in state 2, and as such, $|\psi_2\rangle$ will never thermalize under these circumstances. Unless we alter our original Hamiltonian such that there is not one external field acting on the two spins, but two separate external fields. Our equation thermalizes. Important to note here is, that we can use some perturbations like mentioned above to pump the singlet state, but that will not be thermalization.

2.2 Entanglement entropy

One of our goals was to check what happens to the entanglement between our spins in our system. A simple method to check is by using the entanglement entropy, defined as follows:

$$\mathscr{S} = -\mathrm{Tr}(\rho \log(\rho)), \tag{56}$$

it is not the actual entropy but gives a quantitative measure of how entangled our system is. We expect the entanglement entropy to grow, meaning our entanglement gets enhanced, up to a certain point, namely when it thermalizes. Because then it should saturate and remain constant,however we must also take into account the coupling term ξ, as it disappears from our Lindblad density matrix. Furthermore, because one of the matrix elements was zero, we need to do this numerically. The result is visible in figure 6, for a certain set of values.

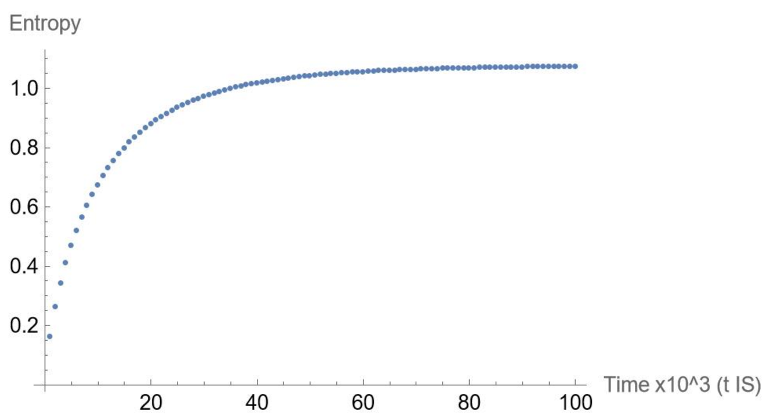

Figure 6: The real part of the entanglement entropy vs time (in arbitrary units), the time was from 0.001 to 0.1 in steps of 0.001.
The used quantities are $\Delta_0 = 0.3, \beta = 1, \nu_0 = 1, \Gamma = 0.1, I = 1, S = 1, \xi = 0.01$, although the value of ξ doesn't matter, since it disappears.

Clearly the figure fulfills our expectations of the growing entanglement and the saturation afterwards.

2.2.1 Entropy of linearly polarized perturbation

Next we can check how the entanglement entropy is influenced by our perturbation, we first check the linearly polarized perturbation.

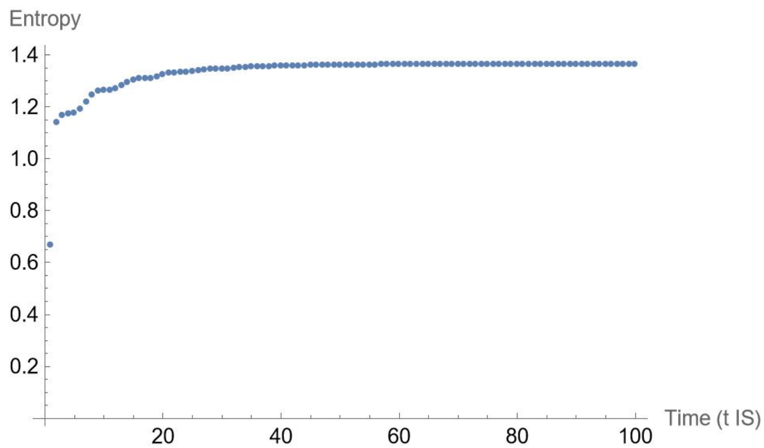

Figure 7: This graph the real part of the entanglement entropy vs time for a linearly polarized perturbation. The time is from 0.01 to 100 in steps of 0.1, The used quantities are $\Delta_0 = 0.3, \beta = 1, \nu_0 = 1, \Gamma = 0.1, I = 1, S = 1, \xi = 10^{-6}, \Omega = 0.5$

We can see from figure 7 that our entropy has some oscillatory behaviour, most likely because one spin is periodically flipped in the x direction, which causes the entropy to drop and then flipped back to z direction allowing the entropy to grow again. Thus we see again that because there is pumping to the singlet state, the entanglement entropy slightly changes.

2.2.2 Entropy of circularly polarized perturbation

We may also check how the circularly polarized perturbation acts on the entanglement entropy. Of course, we have already seen how the population in the states evolves in time for both perturbation, based on the similarities there we may assume that the entanglement entropy is also very similar.

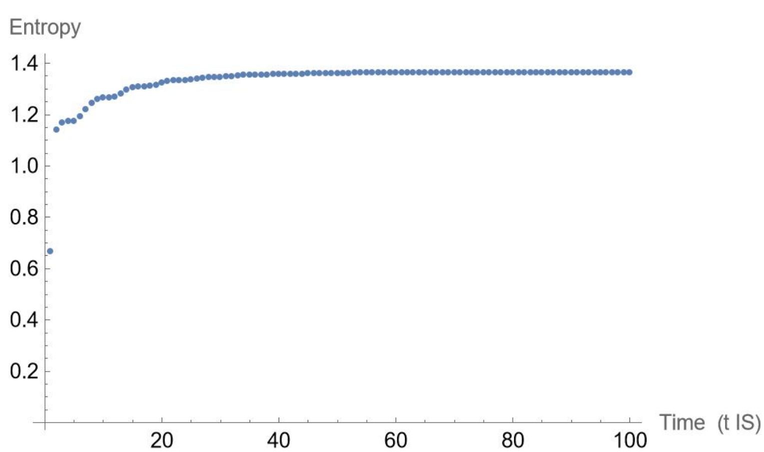

Figure 8: This graph the real part of the entanglement entropy vs time for a circularly polarized perturbation. The time is from 0.01 to 100 in steps of 0.1, The used quantities are $\Delta_0 = 0.3, \beta = 1, \nu_0 = 1, \Gamma = 0.1, I = 1, S = 1, \xi = 10^{-6}, \Omega = 0.5$

And indeed, the entanglement entropy is very similar to the case of linearly polarized perturbation, since once again states are pumped into the singlet and we have oscillations in the entanglement entropy.

2.3 Synchronization

One of our goals was to check for synchronization. We may do this by simply checking the expectation values of one of the spin directions. We choose $\langle\sigma_x\rangle = \mathrm{Tr}(\rho\sigma_x)$, because we know our perturbations have effect on σ_x. This is done for three ranges of ξ/IS and for the different perturbations. From there, the phase difference can be extracted with the help of Fourier analysis.

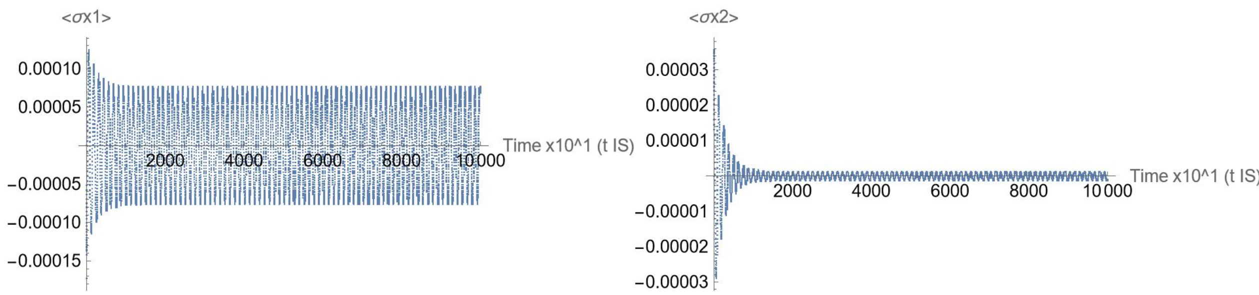

Figure 9: This figure shows the expectation value of the Pauli operators acting on spin 1 and 2. Time was from 0 to 1000 in steps of 0.1, this is in the case of the linearly polarized perturbation with $\xi/IS < 1$.

From the figure above we see that there are oscillations in both spins. From fitting we know that the frequency is equal for both, but the phases differ. We can check how the phase difference behaves for $\xi/IS < 1, \xi/IS = 1$ and $\xi/IS > 1$ for both perturbations.

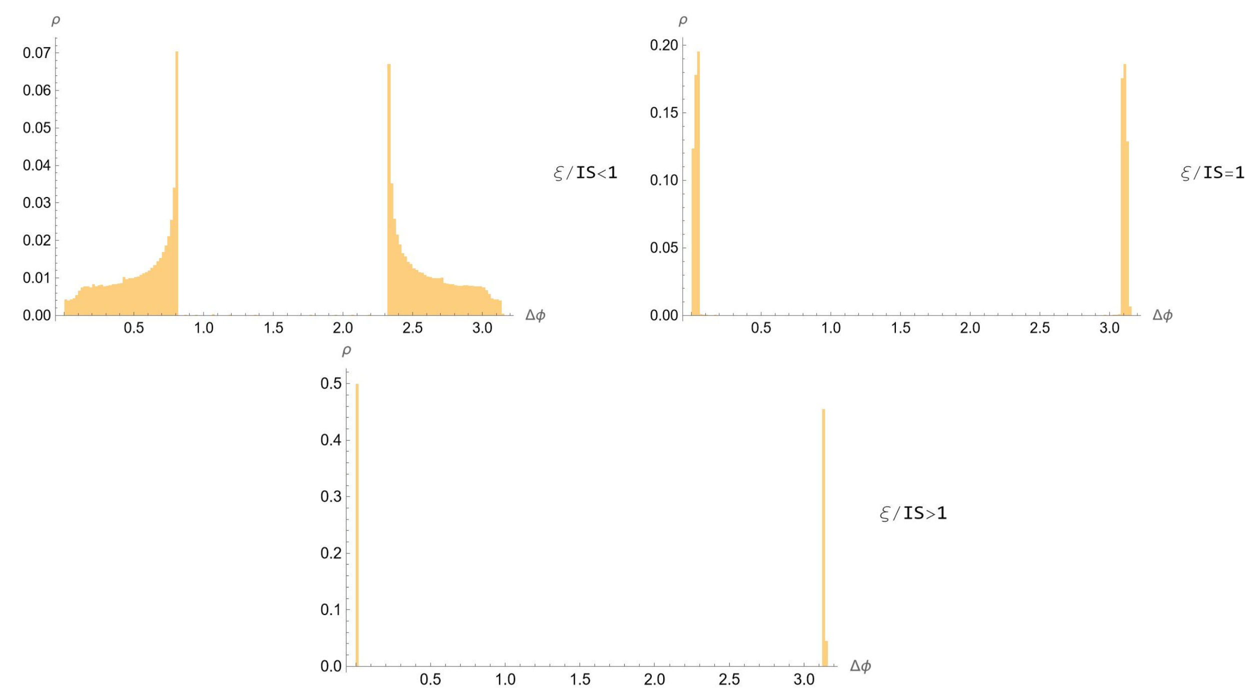

Figure 10: These histogram show the probability to find certain phase differences for different values of the fraction ξ/IS in the case of a linearly polarized perturbation.

From figure 10 we clearly note that when the bath term becomes IS becomes smaller than the coupling between the two spins, the spins are either aligned or anti aligned. Meaning that there is indeed synchronization of the spins.

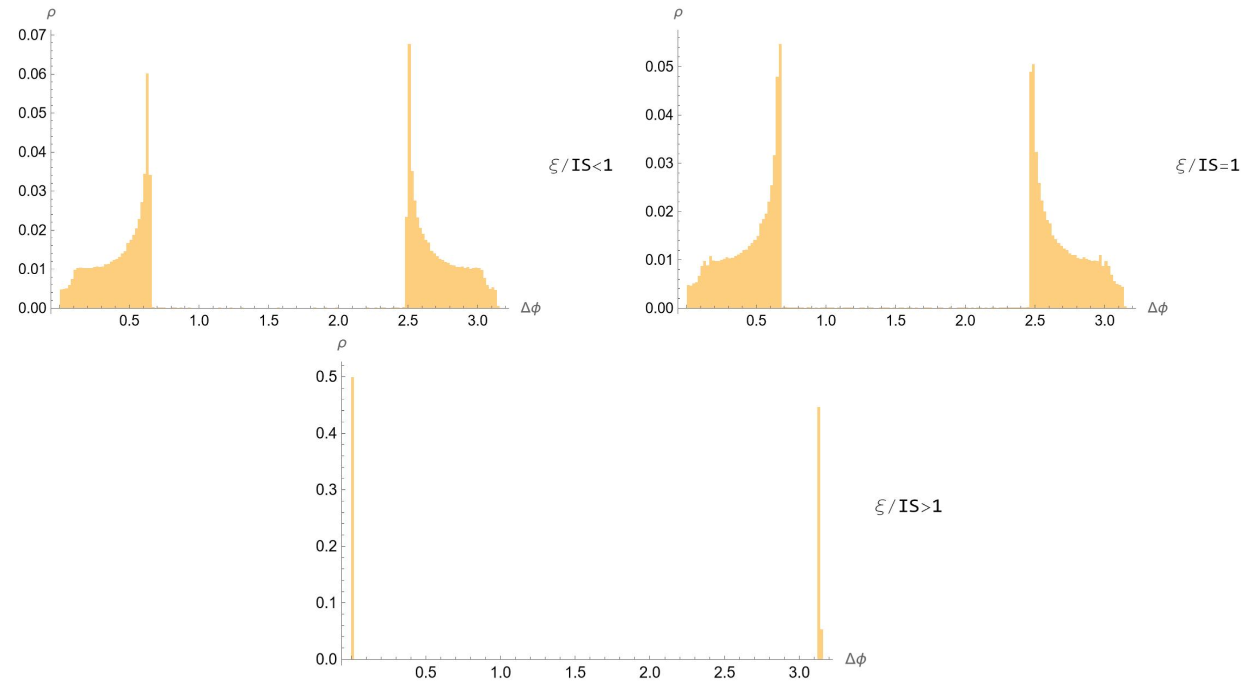

Figure 11: These histogram show the probability to find certain phase differences for different values of the fraction ξ/IS in the case of a circularly polarized perturbation.

Figure 11 shows that there are similarities in the phase difference probabilities for both perturbations, and although it happens slower, there is still synchronization between the two spins. Something important to discuss is that there is a large gap in the middle of the histogram, the origin of which is not yet known, there might be some quantum phenomena showing.

3 Conclusion and Discussion

3.1 Conclusion

The goal of this thesis was to check whether quantum synchronization occurs in an open system, namely two coupled spins in a magnon bath, furthermore we checked how the entanglement between the two spins evolves in time. We found that our system has a triplet state and a singlet state. The singlet state being completely decoupled from the bath and with help from perturbations can be pumped. The entanglement entropy, which we chose as a measure of the entanglement shows that the spins become more entangled up to the point of thermalization, and in the case of perturbation, through pumping of the singlet state there are oscillations visible in the entanglement entropy. Finally, we noted that quantum synchronization does occur in our system and when the coupling to the bath is smaller than the coupling between the spins, they either align or anti align.

3.2 Discussion

The methods used were purely theoretical simulations, meaning there might be difficulties for practical applications. A more intensive research might be needed to make proper conclusions, for example about the lifetime of the pumping, figures 5. Or the gap in the histograms, which has unknown origins.

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