Intuitionistic Probability Theory

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Introduction

In this thesis we deal with some aspects of an intuitionistic theory of probability. The first chapter provides a concise introduction to intuitionistic mathematics. We only briefly introduce the philosophy of intuitionism. Far better explanations of the rationale behind Brouwers reconception of mathematics can be found in Heyting’s introduction [He66] and Veldman’s lecture notes [Ve16]. Additionally Chapter 1 introduces the standard notation, terminology and tools of intuitionistic mathematics. The exposition is very incomplete, as we only cover the basics up until analysis and only state those results we will need in subsequent sections. We sometimes employ non-standard definitions, so the reader who is well-versed in constructive mathematics is invited to have a quick look as well. The second chapter is an adaptation of an article by Dijkman [Di65]. He considered basic, discrete probability spaces in an intuitionistic setting. The novelty of our approach stems mostly from Section 2.1. Here we provide rigorous definitions of objects Dijkman used in an intuitive way. We give several equivalent definitions of the probability of an event in Theorem 2.19. These allow the subsequent sections to develop Dijkman’s theory smoothly, providing clean proofs where the original article sometimes lacked an explanation. Another improvement upon Dijkman’s work is our definition of a stochastic variable. This is a direct analogue of that of measurable functions on the continuum and has many desirable properties. Dijkman’s space of stochastic variables on the other hand is not even closed under addition. We just discuss those results that are also present in Dijkman’s paper, but most theorems of classical, discrete probability theory, or suitable adaptations or refutations thereof, can be derived easily within this framework. The only proposition which appears troublesome is \( E(XY) = E(X)E(Y) \) for independent stochastic variables \( X \) and \( Y \). The final chapter deals with intuitionistic measure theory. Its centrepiece can be found in the last section. Sections 3.1 and 3.2 quickly build up the theory of the Brouwer-Lebesgue integral. We precisely discuss those results necessary to derive our main contribution in the form of Theorem 3.74. Here we prove that there exist distributions in intuitionistic measure theory: if \( f \) is a measurable function, then ‘most’ values of \( y \) yield a measurable set \( \{ x \in [0,1] \mid f(x) \leq y \} \). Such a result is conspicuously absent in the intuitionistic literature on measure theory, see for instance [vR54]. The constructive analyst Bishop did succeed in proving this theorem using his theory of ‘profiles’, which can be found in section 6.4 of [BB85]. Our adaptation appears to be more straightforward and elementary.
Chapter 1

Elementary intuitionistic mathematics

1.1 Philosophy and logic

*Intuitionism* is the name of a school of the philosophy of mathematics initiated by the Dutch mathematician L.E.J. Brouwer. The fundamental tenet of intuitionism is that mathematics is a construction of the mind. Thus mathematics itself is separated from *mathematical language*, which is an imperfect way of memorizing carried out constructions and communicating these between mathematicians. The origin of mathematics is the passing of time. At each time step the thinking mind elaborates upon a previously obtained construct. The subjective nature of the mathematics thus acquired is alleviated in the following way. When reading the word ‘mind’ in the previous sentences, we should not consider this to be the concrete thinking capacity of a specific mathematician. Rather we should keep in mind the thoughts of a *creating subject*. This is an idealized mathematician, whose constructions are not limited by the constraints of human physiology. The creating subject does not suffer from the limitations of time and space we experience, nor does she make false inferences or forget previously derived results. However, in a finite number of time steps she too can only carry out finitely many mental constructions.

A second basic standpoint of intuitionistic mathematics is the interpretation of the word ‘construction’. Contrary to other schools of thought, this does not mean an algorithmically carried out routine. At any stage in a mathematical process, the creating subject can invoke her *free will* in deciding the continuation of the evolving construction. When creating a numerical sequence for instance, in general this construction is not determined by some law given a priori. Instead, at any stage we append to the already obtained finite sequence a new number which we *choose*.

The mathematics obtained when following the rules of intuitionism profoundly differs from standard or *classical mathematics*. The first difference to note is that intuitionistic mathematics is *constructive*. The truth of a mathematical statement requires the actual completion of the construction of its proof. Among other things this implies that the ‘law of excluded middle’, for any sentence $A$ either $A$ is true or $A$ is not true, is not valid. For an arbitrary statement $B$ there is no reason to believe that we either have a proof of $B$, or a proof of the contradictoriness of $B$. In the same vein, an existential statement is true if and only if we can give a finite routine constructing the required object. Another crucial aspect is the intuitionist’s view on infinity. As we have seen, the creating subject has the possibility of continuing mathematical constructions indefinitely. At no time point will she have finished an actually infinite creation however. In intuitionism *potential* infinities exist, but actual infinity is not allowed. In this respect intuitionism does not differ from other schools of constructive mathematics. But the recognition of free choice as a valid means of construction, sets intuitionism apart from every other philosophy of mathematics. The resulting theory of the *continuum* greatly differs from both its classical counterpart and its equivalents in other branches of constructivism.
After Brouwer founded the intuitionistic school, his pupil A. Heyting formalized intuitionistic mathematics. It is generally believed that his interpretation of the logical symbols is in accordance with the intuitionistic notion of truth outlined on the previous page.

**Definition 1.1** (Brouwer-Heyting-Kolmogorov interpretation). Let $A, B$ be arbitrary statements and $S$ a class of objects. A statement $A$ is true in intuitionistic logic if we have a proof of $A$. The logical symbols and quantifiers are interpreted as follows:

(i) $\bot$ is a statement for which no proof can exist,
(ii) $A \lor B$ is true if we either have a proof of $A$ or a proof of $B$,
(iii) $A \land B$ is true if we have a proof of $A$ and a proof of $B$,
(iv) $A \rightarrow B$ is true if we have a routine which transforms any proof of $A$ into a proof of $B$,
(v) $\neg A$ is defined as $A \rightarrow \bot$,
(vi) $A \leftrightarrow B$ is defined as $(A \rightarrow B) \land (B \rightarrow A)$,
(vii) $\exists x \in S[A(x)]$ is true if we have a routine yielding an object $x$ from $S$ and a proof of $A(x)$,
(viii) $\forall x \in S[A(x)]$ is true if we have a routine yielding a proof of $A(x)$ for any object $x$ from $S$.

These interpretations as we state them are not formal definitions. They contain undefined notions such as proof, routine, object and class. Instead, we will simply use such symbols as ‘$\land$’ as shorthand notation to express statements as in Definition 1.1(iii). Adhering to the Brouwer-Heyting-Kolmogorov interpretation of the logical symbols, we lose some familiar rules of classical logic. Lemma 1.2 provides a list of inferences which are still valid in intuitionistic mathematics. On the other hand, Lemma 1.3 provides some examples of reasoning we can no longer employ when working constructively.

**Lemma 1.2.** Let $A, B$ be arbitrary statements and $S$ a class of objects. Then the following statements are intuitionistically true:

(i) $\neg (A \land \neg A)$,
(ii) $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$,
(iii) $(A \rightarrow B) \rightarrow (\neg \neg A \rightarrow \neg \neg B)$,
(iv) $A \rightarrow \neg \neg A$,
(v) $\neg A \rightarrow (A \rightarrow B)$
(vi) $\neg A \lor \neg B \rightarrow \neg (A \land B)$,
(vii) $\neg (A \lor B) \leftrightarrow (\neg A \land \neg B)$,
(viii) $\neg \neg \neg A \leftrightarrow \neg A$,
(ix) $(\exists x \in S[A(x)] \rightarrow B) \leftrightarrow (\forall x \in S[A(x) \rightarrow B])$,
(x) $(\neg \exists x \in S[A(x)]) \leftrightarrow (\forall x \in S[\neg A(x)])$,
(xi) $(\exists x \in S[\neg A(x)]) \rightarrow (\neg \forall x \in S[A(x)])$.

*Proof.* These statements can easily be derived by translating them into regular language using the Brouwer-Heyting-Kolmogorov interpretation.
Lemma 1.3. Let $A$ be an arbitrary statement and $S$ a class of objects. Then the following tautologies of classical logic are not derivable in intuitionistic logic:

(i) $A \lor \neg A$
(ii) $\neg\neg A \rightarrow A$
(iii) $\neg \forall x \in S[A(x)] \rightarrow \exists x \in S[\neg A(x)]$

Proof. From the Brouwer-Heyting-Kolmogorov interpretation of Definition 1.1 it is clear that (i), (ii) and (iii) are not necessarily true. Take (i) as an example and let $A$ be the statement ‘there exists a block of 99 consecutive 9’s in the decimal expansion of $\pi$’. Then as of yet there is no proof of $A$. But neither can we give method transforming any such proof into a proof of an unprovable statement. Thus we have no reason to believe (i) is true. Using Brouwer’s continuity principle, we can show that (i), (ii) and (iii) are truly contradictory in intuitionistic mathematics. This is done in Proposition 1.36.

Those statements $A$ for which we do have $A \lor \neg A$ deserve a special name. We can reason with such sentences as in classical mathematics, for instance by arguing by contradiction. We will do see freely and the reader should understand that those propositions in question are indeed decidable.

Definition 1.4. Let $A$ be an arbitrary statement, then we say $A$ is:

(i) decidable if $A \lor \neg A$ is true,
(ii) (as of yet) undecidable if there is as of yet no proof of $A \lor \neg A$.

Our introduction of intuitionistic logic was short and informal. In this thesis we will not be looking at intuitionistic logic as an interesting field of study in its own right. Instead, we believe the provability interpretation of logic of Definition 1.1 provides a quick and intuitive way of ascertaining whether certain mathematical inference is constructively valid. Switching from classical to intuitionistic reasoning can be confusing at first. By keeping in mind the Brouwer-Heyting-Kolmogorov interpretation, the reader can probably understand this thesis without constantly looking back to Lemma 1.2.
1.2 Species and spreads

In the previous section we have seen how mathematics originates from the move of time. This immediately yields the natural numbers $1, 2, ...$ as a potentially infinite construction $\mathbb{N}$. Moreover, there is the number 0. These objects can be manipulated in a familiar way by addition and multiplication. The principle of induction is available to us as well and so are the basic equality, inequality and order relations on the numbers. From these initial constructs we can build more intricate species.

Definition 1.5. A species is a property a mathematical object from a previously defined collection can possess. If $S$ is a species, then any such object $s$ with the property $S$ is called an element of $S$. We will sometimes denote this by $s \in S$ and say $S$ contains $s$. If it is contradictory that $s \in S$, we write $s \notin S$. The symbol $\emptyset$ denotes the species with no elements.

Definition 1.6. Let $P$ denote some mathematical property and let $S$ be a previously defined species. Then \{ \{ s \in S \mid P \} \} is the species of elements of $S$ which have property $P$. If $n$ is a natural number and $x_1, ..., x_n$ are mathematical objects, then \{ $x_1, ..., x_n$ \} is the species precisely containing these objects.

Definition 1.5 is not affected by problems as Russell’s paradox. As properties of previously defined objects, species are derived from already constructed collections. We simply ‘group together’ existing constructions according to a common property. At this point in our discussion for instance, the only mathematical objects we can use to define new species are the natural numbers. But there are of course several ways to create new species from existing ones.

Definition 1.7. Let $S$ and $T$ be species and let $S_n$ be a species for every $n \in \mathbb{N}$, we then define the following species:

(i) $S \cup T$ is the species of objects $s$ such that $s \in S \vee s \in T$, the union of $S$ and $T$,

(ii) $S \cap T$ is the species of objects $s$ such that $s \in S \wedge s \in T$, the intersection of $S$ and $T$,

(iii) $S \setminus T$ is the species of objects $s$ such that $s \in S \wedge s \notin T$,

(iv) $\bigcup_{n=1}^{\infty} S_n$ is the species of objects $s$ such that $\exists n[s \in S_n]$,

(v) $\bigcap_{n=1}^{\infty} S_n$ is the species of objects $s$ such that $\forall n[s \in S_n]$,

(vi) $S \times T$ is the species of pairs of objects $(s, t)$ such that $s \in S \wedge t \in T$.

Note that a species is not a definite object, but a property. Generally the collection of elements of a certain species is not a constructable set. Therefore we refer to a mathematical collection as a ‘species’ instead of ‘set’. The latter term conjures up an image of a finished and concrete totality. Referring to objects satisfying the condition of a species as ‘elements’ is but a syntactic trick. The following definitions contain examples of how species are more subtle objects than their classical counterparts. A good example is the concept of inhabitedness.

Definition 1.8. Let $S$ and $T$ be species, then we define the following properties of species:

(i) if $S \cap T = \emptyset$, we say that $S$ and $T$ are disjoint,

(ii) if $s \in S$ implies $s \in T$, we write $S \subseteq T$ and say that $S$ is a subspecies of $T$ or $T$ contains $S$,

(iii) if $S \subseteq T$ and $T \subseteq S$, we write $S = T$ and say that $S$ and $T$ are equal.

If $T$ is a subspecies of the species $S$ and $S$ is clear from the context, then we will sometimes use $T^c$ as a shorthand notation for the species $S \setminus T$ and call $T^c$ the complement of $T$. 
Definition 1.9. Suppose $S$ is a species and we can construct at least one object $s$ such that $s \in S$. Then $S$ is called an inhabited species.

Definition 1.10. Let $S$ be species and suppose $T$ is a subspecies of $S$. Then $T$ is a decidable or detachable subspecies of $S$, if for every $s \in S$ the statement $s \in T$ is decidable. If the species $S$ of which $T$ is a subspecies is clear from the context, we sometimes simply say that $T$ is decidable.

Definition 1.11. Let $S$ and $T$ be species, then a relation $R$ is a subspecies of $S \times T$. A relation $R \subseteq S \times T$ is decidable, if for every $s \in S$ and $t \in T$ the statement $(s,t) \in R$ is decidable.

Definition 1.12. Let $S$ be a species, then an equality relation $=$ on $S \times S$ is a relation such that for all $s,t,u \in S$:

(i) $s = s$,
(ii) $s = t$ implies $t = s$,
(iii) if both $s = t$ and $t = u$ hold, then $s = u$.

An example of an equality relation is the usual equality on natural numbers. Here the equality of two natural numbers $n,m$ means that $n$ and $m$ are identical. In Section 1.3 we will define an equality relation on the real numbers. In this case two equal numbers are not necessarily identical.

Definition 1.13. Let $S$ be a species with equality relation $=$, then an apartness relation $#$ on $S \times S$ is a relation such that for all $s,t,u \in S$:

(i) $\neg (s \# t)$ if and only if $s = t$,
(ii) if $s \# t$, then $t \# s$,
(iii) if $s \# t$, then $(s \# u) \lor (t \# u)$.

If $s \# t$, then we say that $s$ and $t$ are apart. If $A$ is a subspecies of $S$, then $s \# A$ denotes $\forall a \in A [s \# a]$.

Definition 1.14 below requires some justification and explanation. First note that $k_{99}$ is not a number and the '$<$' in $n < k_{99}$ is not an inequality. We are merely defining convenient shorthand notation. Statements as $\exists n [n = k_{99}]$ are useful to construct so-called weak Brouwerian counterexamples. This works as follows. From the assumption that a certain statement $Q$ is true, we derive a solution to an unsolved mathematical problem. The unsolved problem is the weak counterexample against $Q$. Thus our reasoning definitely does not prove that $Q$ is contradictory. Many of these weak counterexamples can be used to derive an actual contradiction however. This requires the use Brouwer’s continuity principle, see [Ve85] for instance. We will not concern ourselves with such strong Brouwerian counterexamples. A weak counterexample provides compelling evidence against the constructive validity of a statement. An example can be found in Example 3.49.

Definition 1.14. Let $n$ be a natural number, then we define the following abbreviations:

(i) $n < k_{99}$ stands for ‘there is no sequence of 99 9’s in the first $n$ decimals of $\pi$’,
(ii) $n \geq k_{99}$ stands for ‘there is a sequence of 99 9’s in the first $n$ decimals of $\pi$’,
(iii) $n = k_{99}$ stands for ‘$n$ is the least number such that $n \geq k_{99}$’,
(iv) $n \leq k_{99}$ stands for ‘$n < k_{99}$ or $n = k_{99}$’,
(v) $n > k_{99}$ stands for ‘$n \geq k_{99}$ but not $n = k_{99}$’.

Example 1.15. The statement $\exists n [n = k_{99}]$ is as of yet undecidable.
The next subjects we introduce are functions and the theory of equinumerosity. Here the subtle nature of intuitionistic mathematics is even more apparent than in our exposition of species. Simple, classical properties such as finiteness fall apart into several unequivalent notions. Moreover, the classical concept of a function necessarily involves extensionality and usually totality as well.

**Definition 1.16.** Let $A$ and $B$ be inhabited species, then a partial function $f$ from $A$ to $B$ is an effective routine that on input of certain elements $a$ of $A$ yields an output $f(a)$ in $B$. We sometimes denote this last statement with $f : A \to B$. If $f : A \to B$, then $A$ is the domain of definition of $f$ and $B$ is the range of definition.

**Definition 1.17.** Let $A$ and $B$ be inhabited species and suppose $f$ is a partial function from $A$ to $B$, then we define the following species:

1. The species $\mathcal{D}(f)$ is given by \{ $a \in A \mid \exists b \in B [f(a) = b]$ \} and is called the domain of $f$,
2. The species $\mathcal{R}(f)$ is given by \{ $b \in B \mid \exists a \in A [f(a) = b]$ \} and is called the range of $f$.

**Definition 1.18.** Let $A$ and $B$ be inhabited species and suppose $f$ is a partial function from $A$ to $B$, then $f$ is a total function if $A = \mathcal{D}(f)$.

**Definition 1.19.** Let $A$ and $B$ be inhabited species endowed with the equality relations $=_A$ and $=_B$ respectively. A partial function $f : A \to B$ is called extensional if $a =_A a'$ implies $f(a) =_B f(a')$ for all $a, a' \in A$.

**Definition 1.20.** Let $A$ and $B$ be inhabited species and let $f : A \to B$ be a function. Then $f$ is:

1. an injection if $f$ is total and $f(a) = f(a')$ implies $a = a'$ for every $a, a' \in A$,
2. a surjection if for every $b \in B$ there is an $a \in A$ such that $f(a) = b$,
3. a bijection if $f$ is an injection and a surjection.

**Definition 1.21.** Let $A$ and $B$ be species, then we say that $A$ and $B$ are equivalent or equinumerous if there exists an extensional bijection $f$ between $A$ and $B$.

**Definition 1.22.** Let $S$ be a species, then we say that $S$ is:

1. finite if there is an $n$ such that $S$ is equivalent to the species $\{1, 2, \ldots, n\}$,
2. finitely enumerable if there is an $n$ such that there is a surjection from $\{1, 2, \ldots, n\}$ to $S$,
3. countably infinite if $S$ is equivalent to $\mathbb{N}$,
4. infinite if $S$ contains a countably infinite subspecies,
5. enumerable if $S$ is equivalent to a decidable subspecies $A \subseteq \mathbb{N}$ with $n \in A \to \{1, \ldots, n\} \subseteq A$.

If $S$ is a finite species equivalent to $\{1, \ldots, n\}$ for some $n \in \mathbb{N}$, then we denote this by $|S| = n$.

The following examples contain the real number $\rho$, which is defined in Definition 1.50. The reader may wish to return to these constructions after finishing Section 1.3. Nonetheless the examples should be understandable, since the fundamental issue is just that $\rho = 0$ is not decidable.

**Example 1.23.** The species $\{0, \rho\}$ is finitely enumerable, but it is as of yet contradictory to say that it is finite. While $\mathbb{N} \cup \{\rho\}$ is obviously infinite, we can not yet state that it is countably infinite. Finally, the species $\{n \mid n \leq k_{99}\}$ is enumerable, but we can not decide whether it is finite or countably infinite.
Definition 1.24. Let $S$ be a species endowed with an apartness relation $\#$ and let $A$ be a subspecies of $S$. Then the species $A$ is co-finite if there is a finite species $B \subseteq S$ such that $s \# B$ implies $s \in A$ for all $s \in S$. The term co-enumerable is defined similarly.

As mathematics finds its origin in the passing of time, the notion of a sequence is part of our theory from its conception. This is a tricky subject nonetheless. As we have seen, sequences can be constructed over time by employing free choice. More information on these so-called choice-sequences can be found in [Tr69]. We restrict our attention to sequences of objects of a simple kind. The only sequences of a more complex nature we consider, will consist of real numbers. Here we shall employ an ad hoc argument to justify their existence. This approach stays clear of difficult philosophical issues, while at the same time avoiding an overly formal and logical notation.

Definition 1.25. Let $s$ be a mathematical object, then we say that $s$ is definite if the creating subject can finish the construction of this object in finitely many steps.

Definition 1.26. The symbol () denotes the empty sequence and $S^0 = \{()\}$ for all species $S$. Let $S$ be an inhabited species of definite elements and let $n$ be a natural number, then we define:

(i) $S^n$ is the species of finite sequences $(s_1, ..., s_n)$ of length $n$ with $\forall i \in \{1, ..., n\}[s_i \in S]$,

(ii) $S^*$ is defined as $\bigcup_{n=1}^{\infty} S^n \cup S^0$,

(iii) $S^n$ is the species of infinite sequences $(s_1, s_2, ...)$ with $\forall i[s_i \in S]$.

If $s \in S^*$ or $\alpha \in S^N$, then we will also call $s$ and $\alpha$ sequences of elements of $S$. We sometimes write $\alpha \in S^N$ more explicitly as $(\alpha_n)n$.

Definition 1.27. Let $S$ be an inhabited species of definite elements and let $k, m, n$ be natural numbers with $k \leq n$. Moreover let $s = (s_1, ..., s_n) \in S^n$ and $t = (t_1, ..., t_m) \in S^m$, then:

(i) $s \ast t$ is the sequence $(s_1, ..., s_n, t_1, ..., t_m) \in S^{n+m}$, we call $s \ast t$ an (finite) extension of $s$,

(ii) $\tilde{s}k$ is the sequence $(s_1, ..., s_k) \in S^k$, we call $\tilde{s}k$ an initial segment of $s$,

(iii) the function $l : S^* \rightarrow \mathbb{N}$ yields the unique natural number $l(s)$ such that $s \in S^l(s)$,

(iv) the function $\phi : S^* \rightarrow S$ is defined by $\phi(s) = \tilde{s}l(s)$, we call $\phi(s)$ the final entry of $s$.

If $1 \leq i \leq n$, then we call $s_i$ the $i$'th entry of $s$ and sometimes write $s(i)$ instead of $s_i$. If $u \in S$ and $s \in S^n$, then $s \ast (u)$ is an immediate extension of $s$.

Definition 1.28. Let $S$ be an inhabited species of definite elements and let $n \in \mathbb{N}$. Moreover suppose $s \in S^n$ and $\alpha \in S^N$, then:

(i) $s \ast \alpha$ is the sequence $(s_1, ..., s_n, \alpha_1, \alpha_2, ...) \in S^N$, we call $s \ast \alpha$ an infinite extension of $s$,

(ii) $\tilde{\alpha}n$ is the sequence $(\alpha_1, ..., \alpha_n) \in S^n$, we call $\tilde{\alpha}n$ an initial segment of $\alpha$.

If $1 \leq i \leq n$, then we call $\alpha_i$ the $i$'th entry of $\alpha$ and sometimes write $\alpha(i)$ instead of $\alpha_i$.

Definition 1.29. We define the following important species of sequences:

(i) the species $\mathbb{N}^N$ is called Baire space and is denoted by $\mathcal{B}$,

(ii) the species $\{0, 1\}^N$ is called Cantor space and is denoted by $\mathcal{C}$.

Note that we do not need a concept of actual infinity to work with infinite sequences. Every such sequence is an ever unfinished construction developing through time, exactly like $\mathbb{N}$. Moreover, it is important to emphasize again that a species as $\mathcal{N}$ does not contain only sequences a priori given by some algorithm. At any step in the construction of an $\alpha \in \mathcal{N}$, only a finite initial segment of $\alpha$ has been defined. Subsequently the natural number that is appended to the already obtained segment, is constructed through the free choice of the creating subject. In particular this implies that the species $\mathcal{N}$ is not enumerable. This is false if every sequence in $\mathcal{N}$ is algorithmic or lawlike.
So far our exposition has been restricted to species and sequences. A species consists of objects which have previously been constructed. We simply group together mathematical constructs with similar properties. Another way of creating new objects is through a construction evolving over time. This brings us to Brouwer’s fundamental notion of a spread.

**Definition 1.30.** Let $S$ be a finite or countably infinite species of definite elements. Then a total function $\sigma : S^* \rightarrow \{0, 1\}$ is called a spread-law if:

(i) $\sigma((())) = 1$,

(ii) $\sigma(s) = 1$ if and only if there is a $t \in S$ such that $\sigma(s \ast (t)) = 1$.

If $\sigma(s) = 1$, we say that the finite sequence $s$ is admitted by $\sigma$.

**Definition 1.31.** Let $\sigma$ be a spread-law. Then $F_\sigma$ is the species of all infinite sequences $\alpha \in S^\aleph$ such that every initial segment of $\alpha$ is admitted by $\sigma$. A species $X \subseteq S^\aleph$ is called a spread if and only if there is spread-law $\tau$ such that $X = F_\tau$.

**Definition 1.32.** Let $B$ be a subspecies of $S^*$ and let $X$ be a subspecies of $S^\aleph$. Then $B$ is a bar in $X$ if $\forall \alpha \in X \exists n[\bar{\alpha}n \in B]$. If $s \in B$, then we say that the finite sequence $s$ meets the bar $B$.

**Definition 1.33.** Let $\sigma$ be a spread-law such that for every admitted $s$ the spread-law $\sigma$ only admits finitely many immediate extensions. Then we call $\sigma$ a fan-law and the corresponding spread $F_\sigma$ a fan.

**Example 1.34.** Define the spread-law $\sigma : \aleph^* \rightarrow \{0, 1\}$ by $\sigma(s) = 1$ for all $s \in \aleph^*$. Then we immediately see $F_\sigma = N$. A more interesting example is obtained by defining $\tau : \aleph^* \rightarrow \{0, 1\}$ by $\tau(s) = 1$ if and only if $\phi(s) \in \{1, 2, 3\}$ and additionally $\bar{s}(l(s) - 1) \leq \bar{s}l(s)$ if $l(s) > 1$. This yields the non-decreasing sequences of natural numbers less than 4. Note that $\tau$ is a fan-law, but $\sigma$ is not.

The foremost distinguishing feature of intuitionism is Brouwer’s continuity principle. This principle contradicts classical mathematics and is also not accepted by other constructive schools. Since we already defined spreads, we can give a very general version of the continuity principle. The ‘proof’ should be seen as an explanation of the necessity of the continuity principle when dealing with choice sequences.

**Theorem 1.35** (Continuity principle). Let $F_\sigma$ be a spread and let $R$ be a relation on $F_\sigma \times \aleph$. Suppose that $\forall \alpha \in F_\sigma \exists n[(\alpha, n) \in R]$, then we have $\forall \alpha \in F_\sigma \exists n, \forall \beta \in F_\sigma[\bar{n}m = \beta m \rightarrow (\beta, n) \in R]$.

**Proof.** The statement $\forall \alpha \in F_\sigma \exists n[(\alpha, n) \in R]$ is very strong. For every $\alpha$ we can effectively calculate an $n$ with $(\alpha, n) \in R$. At any point in time we have only constructed a finite initial part of the sequence $\alpha$. To determine a suitable $n$ whatever $\alpha$ may be, we must therefore be able to compute $n$ using only some finite information on $\alpha$. Moreover, we can effectively calculate the amount of entries needed to fix an $n$, using some initial segment of $\alpha$. It also follows then that any sequence $\beta$ having the same entries as $\alpha$ up to the point where we could pinpoint $n$, must also satisfy $(\beta, n) \in R$. □

**Proposition 1.36.** Brouwer’s continuity principle contradicts the law of excluded third.

**Proof.** Suppose the law of excluded third is true. In particular this means $\exists n[\alpha(n) = 1]$ is decidable for all $\alpha \in \mathcal{C}$. Define the relation $R$ on $\mathcal{C} \times \{1, 2\}$ by $(\alpha, 2) \in R$ if and only if $\exists n[\alpha(n) = 1]$ and $(\alpha, 1) \in R$ if and only if $\forall n[\alpha(n) = 0]$. Apply Brouwer’s continuity principle to the sequence $\beta$ given by $\beta(n) = 0$ for all $n \in \aleph$. We find a $k \in \aleph$ satisfying $\exists n \forall \gamma \in \mathcal{C}[\beta k = \bar{\gamma}k \rightarrow (\gamma, m) \in R]$. But $m = 2$ implies $(\beta, 1) \in R$. Similarly, $m = 1$ is also contradictory as this yields $(\beta k \ast (1) \ast \beta, 0) \in R$. □
We are now ready to formulate Brouwer’s fan theorem. The ‘proof’ of this theorem is controversial, since its nature is more philosophical than mathematical. This is apparent from the proof given below, which is taken from [Ve16]. The fan theorem should perhaps be seen as an axiom of intuitionistic mathematics. Without this axiom a constructive treatment of the continuum appears to be very difficult. Many approaches have to rely on awkward definitions to alleviate all sorts of problems. Even then insurmountable difficulties often arise. See for instance the article by Waaldijk [Wa05]. When combined with the continuity principle, the fan theorem is a powerful tool allowing us to prove many useful results. Often we only require a specific consequence of these two principles, which is given in Proposition 1.38

**Theorem 1.37** (Fan theorem). Let \( \mathcal{F} \) be a fan and let \( B \) be a bar in \( \mathcal{F} \). Then there exists a finite subspecies of \( B \) that is also a bar in \( \mathcal{F} \).

**Proof.** Let \( \sigma \) be a fan-law such that \( \mathcal{F} = \mathcal{F}_\sigma \). By assumption we are able to construct a bar \( B \) in \( \mathcal{F} \). Even more explicitly, this means that we have constructed a subspecies of the natural numbers \( B \) and a proof of the fact that \( B \) is a bar in \( \mathcal{F}_\sigma \). Then we assume that there also exists a canonical proof of that statement. Such a proof would look as follows.

For every \( s \in \mathbb{N}^* \), define the species \( \mathcal{E}_s^\sigma \) as those \( \alpha \in \mathcal{F}_\sigma \) such that \( \text{âl}(s) = s \). These are the sequences in \( \mathcal{F}_\sigma \) that start with the finite sequence \( s \). We say that the finite sequence \( s \) is *safe*, if \( B \) is a bar in \( \mathcal{E}_s^\sigma \). This means that any sequence in \( \mathcal{F}_\sigma \) starting with \( s \) will eventually meet the bar \( B \). So \( () \) is safe if and only if \( B \) is a bar in \( \mathcal{F}_\sigma \). The canonical proof uses four types of propositions to prove the latter statement. These propositions are defined as follows:

(i) The initial reasoning steps: we know \( \sigma(s) = 1 \) for some finite sequence \( s \) and have verified that \( s \) meets \( B \). Therefore \( s \) is safe.

(ii) The forward reasoning steps: we have established for some admitted \( s \) that the only immediate extensions of this finite sequence admitted by \( \sigma \) are \( s * (n_0), s * (n_1), \ldots, s * (n_k) \) for some \( k \in \mathbb{N} \). We also know that all these extensions are safe. Therefore \( s \) is safe.

(iii) The backward reasoning steps: we know that a finite sequence \( s \) is safe and that \( s * (n) \in \mathcal{F}_\sigma \) for some \( n \). Therefore \( s * (n) \) is safe.

(iv) The conclusion: () is safe.

The canonical proof starts with several sentences of the form (i). Subsequently it repeatedly applies steps (ii) and (iii) to possibly different sequences \( s \). Eventually these sequences ‘shrink’ to the empty sequence: the canonical proof ends with proposition (iv). Next we define that \( s \) is *supersafe* if there is a finite subspecies of \( B \) that is also a bar in \( \mathcal{E}_s^\sigma \). Now replace every instance of the word ‘safe’ in the canonical proof by the word ‘supersafe’. It is easily verified that the reasoning steps (i), (ii) and (iii) all remain valid in this new proof. Therefore the adjusted conclusion, () is supersafe, must also be correct. But this exactly states that there is a finite subspecies of \( B \) that is a bar in \( \mathcal{F} \).

**Proposition 1.38.** Let \( \sigma \) be a fan-law on a species \( S \) and suppose that \( P(x,n) \) is a proposition for each \( x \in \mathcal{F}_\sigma \) and \( n \in \mathbb{N} \). Then \( \forall x \in \mathcal{F}_\sigma \exists n [P(x,n)] \) implies \( \exists N \forall x \in \mathcal{F}_\sigma \exists n \leq N [P(x,n)] \).

**Proof.** We obtain \( \forall x \in \mathcal{F}_\sigma \exists n, m \forall y \in \mathcal{F}_\sigma [\bar{x}m = \bar{y}m \rightarrow P(x,n)] \) by applying Brouwer’s continuity principle to \( \forall x \in \mathcal{F}_\sigma \exists n [P(x,n)] \). Hence \( B = \{ s \in \mathbb{S}^* \mid \exists n \forall y \in \mathcal{F}_\sigma [\bar{y}l(s) = s \rightarrow P(y,n)] \} \) is a bar in \( \mathcal{F}_\sigma \). Construct a finite subspecies \( B’ \) of \( B \) that is also a bar in \( \mathcal{F}_\sigma \) by using the fan theorem. For every element \( s \) of \( B’ \) there is a natural number \( k_s \) such that \( \forall y \in \mathcal{F}_\sigma [\bar{y}l(s) = s \rightarrow P(y,k_s)] \). Define \( N = \max \{ \{ k_s \mid s \in B’ \} \} \). If \( x \) is an arbitrary element of \( \mathcal{F}_\sigma \), then there is a \( p \) such that \( \bar{x}p \in B’ \). Evidently \( k_{\bar{x}p} \) satisfies \( P(y,k_{\bar{x}p}) \) for all infinite extensions \( y \) of \( \bar{x}p \) and also \( k_{\bar{x}p} \leq N \).
We finish this section with a discussion of Markov’s principle. This principle is a defining feature of the Russian school of constructivism. In the Russian school every sequence in \( C \) is given by an explicit algorithm. We interpret Definition 1.39 in an intuitionistic way, meaning that \( \alpha \) is given by a construction in time which may not be completely algorithmic. In our formulation the principle is rejected by the intuitionists on philosophical grounds, even when we restrict \( C \) to the algorithmic sequences. Finally, note that Markov’s principle is not named after the famous Russian probabilist A.A. Markov, but another A.A. Markov noted for his contributions to constructive mathematics. Their initials do not coincide by chance: the former is the father of the latter.

**Definition 1.39.** Markov’s principle is the following statement. For any \( \alpha \in C \), if \( \neg \forall n[\alpha(n) = 0] \) then \( \exists n[\alpha(n) = 1] \).

From the above formulation it is clear that accepting Markov’s principle means allowing for ‘search operations’ which are a priori not constructively bounded. We know that it is contradictory that \( \alpha(n) = 0 \) for all \( n \), but have not been given any further information on how to effectively determine an \( m \) such that \( \alpha(m) = 1 \). The best approach we can take is simply calculate subsequent values of \( \alpha \) and wait. Thus it is natural to ask whether certain mathematical statement under investigation is equivalent to Markov’s principle. In this case the statement essentially depends on allowing not effectively bounded search operations. Examples of statements equivalent to Markov’s principle can be found in Lemma 1.68 and Theorem 2.46. Effective search operations can be carried out with the \( \mu \)-operator.

**Definition 1.40.** For each \( n \in \mathbb{N} \), let \( P(n) \) be a decidable statement and suppose \( \exists n[P(n)] \). Then we define \( \mu(n)[P(n)] \) as the least number \( m \) such that \( P(m) \).
1.3 Real numbers

The construction of the familiar species \( \mathbb{Z} \) and \( \mathbb{Q} \) follows in a manner similar to the classical procedure. Note that we do not need to define \( \mathbb{Q} \) as a quotient space of some equivalence relation. By simply defining suitable relations and operations on \( \mathbb{N} \), we obtain a species which behaves exactly like the classical rational numbers. Moreover, the usual inequality and equality relations on \( \mathbb{N}, \mathbb{Z} \) and \( \mathbb{Q} \) are decidable. We implicitly assume all of these construction have been carried out and continue our exposition with rational segments. Subsequently we construct the real numbers as special sequences of these segments. The construction of the equality, apartness and order relations on the real numbers is straightforward as well.

**Definition 1.41.** The species of rational segments \( S \) consists of the pairs of rational numbers \((p, q)\) with \( p \leq q \). We denote the segment \((p, q)\) by \((p, q)\). If \( s \in S \), then we define functions \( \cdot : S \to \mathbb{Q} \) and \( / : S \to \mathbb{Q} \) in such a way that \( s = (s', s'') \). The diameter \( \delta(s) \) of the segment \( s \) is simply \( s'' - s' \).

**Definition 1.42.** Let \( s \) and \( t \) be rational segments, then we define:

(i) \( s < t \) if and only if \( s'' < t' \),

(ii) \( s \not< t \) if and only if \( s < t \lor t < s \), \( s \) and \( t \) are disjoint,

(iii) \( s \leq t \) if and only if \( s' \leq t'' \),

(iv) \( s \approx t \) if and only if \( s \leq t \land t \leq s \), \( s \) and \( t \) overlap or partially cover eachother,

(v) \( s \subseteq t \) if and only if \( t' \leq s' \leq s'' \leq t'' \), \( t \) covers \( s \) or \( s \) is contained in \( t \),

(vi) \( s \subset t \) if and only if \( t' < s' \leq s'' < t'' \), \( t \) properly covers \( s \).

**Definition 1.43.** Let \( s \) and \( t \) be rational segments, then we define the following rational segments:

(i) \( s + t = (s' + t', s'' + t'') \),

(ii) \( -s = (-s'', -s') \),

(iii) \( s \cdot t = (\min(s't', s''t', s't'', s''t'''), \max(s't', s''t', s't'', s''t''')) \),

(iv) \( 1/s = (\min(1/s', 1/s''), \max(1/s', 1/s'')) \) if \( s', s'' \neq 0 \) and undefined otherwise,

(v) \( |s| = (\min(|s'|, |s''|), \max(|s'|, |s''|)) \),

(vi) \( \max(s, t) = (\max(s', t'), \max(s'', t'')) \),

(vii) \( \min(s, t) = (\min(s', t'), \min(s'', t'')) \).

**Definition 1.44.** Let \( \alpha \) be an element of \( \mathbb{S}^n \) satisfying the following requirements:

(i) \( \forall n [\alpha(n + 1) \sqsubset \alpha(n)] \), \( \alpha \) is a shrinking sequence of rational segments,

(ii) \( \forall m \exists n [\delta(\alpha(n)) \leq 1/2^m] \), \( \alpha \) is a dwindling sequence of rational segments.

Then we call \( \alpha \) a real number and denote the species of such sequences by \( \mathbb{R} \).

**Definition 1.45.** If \( \alpha \) is a real number such that:

(i) \( \forall n [\alpha(n) \sqsubset \alpha(n - 1)] \), \( \alpha \) is a strictly shrinking sequence of rational segments,

(ii) \( \forall n [\delta(\alpha(n)) \leq 1/2^{n-1}] \), \( \alpha \) is a quickly dwindling sequence of rational segments.

Then we call \( \alpha \) a canonical real number and denote the species of such real numbers by \( \mathbb{R}_{\text{can}} \).
Lemma 1.46. The species of real numbers is not a spread.

Proof. Suppose there is a spread law \( \sigma \) such that \( \mathcal{F}_\sigma = \mathbb{R} \). For every \( n \) we can define the real number \( 0_n \) by \( 0_n(m) = (0,1) \) if \( m \leq n \) and \( 0_n(m) = (0,0) \) if \( m > n \). Thus we must have \( 0_n \in \mathcal{F}_\sigma \) for every \( n \). In particular this implies \( \sigma(0_n) = 1 \) irrespective of \( n \). But this means that the sequence \( \alpha \) given by \( \alpha(k) = (0,1) \) for every \( k \) is admitted by \( \sigma \). Clearly \( \alpha \) is not a real number. \( \qed \)

Definition 1.47. Let \( \alpha \) and \( \beta \) be real numbers, then we define the following relations on \( \mathbb{R} \):

(i) \( \alpha = \beta \) if and only if \( \forall n \left[ \alpha(n) \approx \beta(n) \right] \), \( \alpha \) and \( \beta \) are equal,

(ii) \( \alpha < \beta \) if and only if \( \exists n \left[ \alpha(n) < \beta(n) \right] \),

(iii) \( \alpha \leq \beta \) if and only if \( \forall n \left[ \alpha(n) \leq \beta(n) \right] \),

(iv) \( \alpha \# \beta \) if and only if \( \alpha < \beta \lor \alpha > \beta \), \( \alpha \) and \( \beta \) are apart,

(v) if \( \neg(\alpha = \beta) \), then we say that \( \alpha \) and \( \beta \) are unequal.

If \( 0 < \alpha \), we say that \( \alpha \) is positive. Similarly \( \alpha \) is negative is defined as \( \alpha < 0 \). If \( 0 \leq \alpha \), then \( \alpha \) is non-negative and if \( \alpha \leq 0 \) then we say \( \alpha \) is non-positive.

Definition 1.48. Let \( q \) be a rational number, then \( \alpha_q \) is the canonical real number defined by \( \alpha_q(n) = (q - 1/2^n, q + 1/2^n) \). Slightly abusing notation, we will always refer to \( \alpha_q \) as \( q \in \mathbb{R} \). If \( \alpha \) is a real number with \( \alpha \# \mathbb{Q} \), then \( \alpha \) is called positively irrational.

Lemma 1.49. If \( \alpha \) is an arbitrary real number, then there exists an \( \alpha_{can} \in \mathbb{R}_{can} \) such that \( \alpha = \alpha_{can} \).

Proof. Trivial. \( \qed \)

Next we define certain special real numbers, which have unintuitive properties with respect to the ordering on the real numbers. As such, they are very useful in constructing weak counterexamples against statements in analysis and other fields. Examples include the intermediate value theorem, for which we refer the reader to \([Ve16]\). Real numbers such as \( \rho \) also provide an explanation why a result as Proposition 3.71 involves so many intricate estimates and constructions.

Definition 1.50. Define the real number \( \rho \) as follows:

\[
\rho(n) = (-1/2^n, 1/2^n) \quad \text{if} \quad n < k_{99},
\]

\[
\rho(n) = (1/2^{k_{99}}, 1/2^{k_{99}}) \quad \text{if} \quad n \geq k_{99}.
\]

Similarly define \( \rho' \) by:

\[
\rho(n) = (-1/2^n, 1/2^n) \quad \text{if} \quad n < k_{99},
\]

\[
\rho(n) = (1/2^{k_{99}}, 1/2^{k_{99}}) \quad \text{if} \quad n \geq k_{99} \text{ and } \mu(m)[m \geq k_{99}] \text{ is even},
\]

\[
\rho(n) = (-1/2^{k_{99}}, -1/2^{k_{99}}) \quad \text{if} \quad n \geq k_{99} \text{ and } \mu(m)[m \geq k_{99}] \text{ is odd}.
\]

Lemma 1.51. The following statements are as of yet undecidable:

(i) \( \rho = 0 \lor \rho > 0 \),

(ii) \( \neg(\rho \leq 0) \),

(iii) \( \rho' < 0 \lor \rho' = 0 \lor \rho' > 0 \).

Proof. Trivial. \( \qed \)
We continue with our investigation of the order relations on the real numbers. The proofs of the statements below are very straightforward. Nonetheless, working with the constructive real numbers takes some getting used to. While Proposition 1.52 collects many familiar properties, a notably absent statement is $\alpha \leq \beta \rightarrow (\alpha = \beta \lor \alpha < \beta)$. This is explained by Lemma 1.51. Moreover $\neg(\alpha = \beta)$ is not equivalent to $\alpha < \beta \lor \alpha > \beta$. Lemma 1.68(i) shows that a proof of this statement requires Markov’s principle. To avoid confusion, we shall not abbreviate $\neg(\alpha = \beta)$ to $\alpha \neq \beta$. In publications in Bishop style constructivism, the symbol ‘\neq’ usually denotes the apartness relation ‘\#' between real numbers.

**Proposition 1.52.** Let $\alpha$, $\beta$ and $\gamma$ be elements of $\mathbb{R}$, then we have:

(i) if $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$,

(ii) if $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$,

(iii) if $\alpha < \beta$ and $\beta \leq \gamma$, then $\alpha < \gamma$,

(iv) if $\alpha \leq \beta$ and $\beta < \gamma$, then $\alpha < \gamma$,

(v) if $\alpha < \beta$, then either $\alpha < \gamma$ or $\gamma < \beta$,

(vi) if $\alpha \# \beta$, then either $\alpha \# \gamma$ or $\gamma \# \beta$,

(vii) if $\neg(\neg(\alpha \leq \beta))$, then $\alpha \leq \beta$,

(viii) if $\neg(\neg(\alpha = \beta))$, then $\alpha = \beta$,

(ix) $\alpha \leq \beta$ if and only if $-(\alpha > \beta)$,

(x) $\alpha = \beta$ if and only if $-(\alpha \# \beta)$,

(xi) $\alpha = \beta$ if and only if $\alpha \leq \beta \land \alpha \geq \beta$,

(xii) if $\alpha < \gamma$, then $\neg(\beta < \alpha \lor \alpha \leq \beta < \gamma \lor \gamma \leq \beta)$.

**Proof.** The proofs of all these propositions are trivial and can also be found in [He66]. As examples, we prove statement (v), (viii), (ix) and (xii). If $\alpha < \beta$, then there exist an $n$ such that $\alpha(n) < \beta(n)$. Calculate an $m$ such that $\delta(\gamma(m)) = \beta(n) - \alpha(n)^m$. If $\alpha(n)^m < \gamma(m)$, then clearly $\alpha < \gamma$ and we are done. So suppose $\gamma(m)^m \leq \alpha(n)^m$. Then $\gamma(m)^m \leq \alpha(n)^m + \delta(\gamma(m)) < \beta(n)^m$. This proves $\gamma < \beta$.

To prove (viii) note that $-(\neg(\alpha = \beta))$ means $\neg \forall n[\alpha(n) \approx \beta(n)]$. From this we have to deduce $\forall n[\alpha(n) \equiv \beta(n)]$. So pick an arbitrary $m \in \mathbb{N}$, it suffices to derive a contradiction from $\alpha(m) \neq \beta(m)$. Clearly this means $\forall n[\alpha(n) \approx \beta(n)]$ is contradictory. But this contradicts $\neg \forall n[\alpha(n) \approx \beta(n)]$ and we have derived $\alpha(m) \approx \beta(m)$.

Next we show (ix) so first assume $\alpha \leq \beta$. This means $\forall n[\alpha(n) \leq \beta(n)]$ and it is clearly contradictory then that $\exists n[\beta(n) < \alpha(n)]$. Next assume $\neg \forall n[\beta(n) < \alpha(n)]$ and let $m \in \mathbb{N}$ be arbitrary. We can now derive $\alpha(m) \leq \beta(m)$, since $-(\alpha(m) \leq \beta(m))$ exactly states $\beta(m) < \alpha(m)$ and we know that $\alpha(m) \leq \beta(m)$ is decidable.

Finally we prove (xii). From Lemma 1.2(vii) we see that we must derive a contradiction from $\beta \geq \alpha \land -(\alpha \leq \beta \land \neg(\gamma \leq \beta)$. By applying (v) we deduce that $\alpha < \beta$ or $\beta < \gamma$. In the first case, $\neg(\alpha \leq \beta < \gamma)$ implies $-(\beta < \gamma)$. This last statement clearly contradicts $-(\gamma \leq \beta)$. The case $\beta < \gamma$ can be dealt with similarly, so we are done.

The usual operations on the real numbers provide no special problems. As can be seen from Proposition 1.55, most familiar properties carry over from the classical theory. Moreover, we can define *intervals*. It will turn out to be beneficial to identify *rational, closed* intervals with rational segments and this is done in Definition 1.58. We also state a version of Cantor’s diagonal argument in the form of Proposition 1.63.
Definition 1.53. Let $\alpha$ and $\beta$ be real numbers. Define the following sequences in $\mathbb{S}^N$:

(i) $\alpha + \beta$ by $(\alpha + \beta)(n) = \alpha(n) + \beta(n)$ for all $n \in \mathbb{N}$,

(ii) $\alpha \cdot \beta$ by $(\alpha \cdot \beta)(n) = \alpha(n) \cdot \beta(n)$ for all $n \in \mathbb{N}$,

(iii) $-\alpha$ by $(-\alpha)(n) = -\alpha(n)$ for all $n \in \mathbb{N}$,

(iv) $\alpha/\beta$ by $(\alpha/\beta)(n) = \alpha(n) + m \cdot (1/\beta(n + m))$ for all $n \in \mathbb{N}$ with $m = \mu(k)[\beta(k)\prime, \beta(k)\prime\prime \neq 0]$ if $\beta \# 0$ and undefined otherwise,

(v) $|\alpha|$ by $|\alpha|(n) = |\alpha(n)|$.

Lemma 1.54. Let $\alpha$ and $\beta$ be real numbers. Then the sequences $\alpha + \beta, \alpha \cdot \beta, -\alpha$ and $|\alpha|, \max(\alpha, \beta)$ and $\min(\alpha, \beta)$ are all real numbers. The same holds for $\alpha/\beta$, under the condition that $\beta \# 0$.

Proof. Trivial.

Proposition 1.55. Let $\alpha$, $\beta$ and $\gamma$ be elements of $\mathbb{R}$ and $n$ a natural number, then:

(i) $|\alpha'(n) - \alpha| \leq \delta(\alpha(n))$ and $|\alpha''(n) - \alpha| \leq \delta(\alpha(n))$,

(ii) $\alpha \leq \max(\alpha, \beta)$ and $\max(\alpha, \beta) = \max(\beta, \alpha)$,

(iii) $\alpha \geq \min(\alpha, \beta)$ and $\min(\alpha, \beta) = \min(\beta, \alpha)$,

(iv) $\min(\alpha, \beta) \leq \max(\alpha, \beta)$,

(v) $\gamma > \max(\alpha, \beta)$ if and only if both $\gamma > \alpha$ and $\gamma > \beta$,

(vi) $\max(\alpha, \beta) + \min(\alpha, \beta) = \alpha + \beta$,

(vii) $|\alpha + \beta| = |\alpha| + |\beta|$,

(viii) $|\alpha||\beta| = |\alpha\beta|$,

(ix) $|\alpha| = |\alpha|$,

(x) $|1/\alpha| = 1/|\alpha|$ if $\alpha \# 0$.

Proof. All the proofs are trivial as the statements under investigation can easily be seen to hold when we consider their equivalents for rational segments. As an example, we prove (vii). So let $n \in \mathbb{N}$ be arbitrary, our task is to derive $|\alpha + \beta|(n) \leq (|\alpha| + |\beta|)(n)$. This is defined as $|\alpha + \beta(n)' \leq (|\alpha| + |\beta|)(n)'$. By definition we have $|\alpha + \beta(n)' = \min(|\alpha(n)' + \beta(n)'|, |\alpha(n)' + \beta(n)'|)$ and $(|\alpha| + |\beta|)(n)' = \max(|\alpha(n)'|, |\alpha(n)'|) + \max(|\beta(n)'|, |\beta(n)'|)$. Evidently this establishes $|\alpha + \beta(n)' \leq (|\alpha| + |\beta|)(n)'$, as $q_1 + q_2 \leq |q_1| + |q_2|$ holds for all rational numbers $q_1, q_2$.

From here we will drop the convention that real numbers are denoted by Greek letters. We no longer have to emphasize that real numbers are sequences created through time. This is a fundamentally different notion than an equivalence class of an undecidable equality, which we assume to be a definite object through an application of a choice axiom. Nonetheless, we shall often reason with real numbers in an intuitive way. Propositions such as 1.55 allow us to work as in classical mathematics most of the time. In many proofs the internal structure of real numbers as sequences of segments is not relevant.
Definition 1.56. Let \( a, b \) be real numbers satisfying \( a \leq b \), then we define the following intervals:

(i) \([a, b]\) is defined as the species \( \{x \in \mathbb{R} \mid a \leq x \leq b\} \), a closed interval,

(ii) \((a, b]\) is defined as the species \( \{x \in \mathbb{R} \mid a < x \leq b\} \), an open interval,

(iii) \([a, b)\) is defined as the species \( \{x \in \mathbb{R} \mid a \leq x < b\} \),

(iv) \((a, b)\) is defined as the species \( \{x \in \mathbb{R} \mid a < x < b\} \).

In all cases we refer to \( a \) and \( b \) as the endpoints of the interval. If the interval \( I \) has endpoints \( a \leq b \), then we define \( I^l = a \) and \( I^r = b \).

Definition 1.57. Let \( I \) be a closed interval, then \( \lambda(I) \) is the interval \( \lambda(I) = [I^l, (I^l + I^r)/2] \) and \( \rho(I) = \rho(I) = [(I^l + I^r)/2, I^r] \).

Definition 1.58. An interval with rational endpoints is a rational interval. The species of closed, rational intervals is denoted by \( \mathbb{I} \). We define the bijections \( \iota : \mathbb{S} \rightarrow \mathbb{I} \) and \( \zeta : \mathbb{I} \rightarrow \mathbb{S} \) by \( \iota(p, q) = [p, q] \) and \( \zeta([p, q]) = (p, q) \). We extend these bijections to bijections between \( \mathbb{S}^* \) and \( \mathbb{I}^* \) in the natural way.

Definition 1.59. Let \( a, b \in \mathbb{Q} \) be such that \( 0 < a < b < 1 \), then we define:

(i) the interior of \([0, 1]\) is the interval \([0, 1]\),

(ii) the interior of \([a, 1]\) is the interval \([a, 1]\),

(iii) the interior of \([0, b]\) is the interval \([0, b]\),

(iv) the interior of \([a, b]\) is the interval \([a, b]\).

Definition 1.60. A partition \( P \) of \([0, 1]\) is a finite sequence \((P_1, \ldots, P_n)\) of elements of \([0, 1]\) such that \( P_1 = 0, P_n = 1 \) and \( P_i < P_{i+1} \) for all \( i \in \{1, \ldots, n-1\} \). If \( P \) is a partition, then we also define:

(i) every interval \([P_i, P_{i+1}]\) for \( i \in \{1, \ldots, n-1\} \) is called a slice of \( P \),

(ii) the species of slices of a partition \( P \) is denoted by \( \mathcal{S}(P) \),

(iii) the upper distance of \( P \) is defined by \( d_u(P) = \max \{\{P_{i+1} - P_i \mid i = 1, \ldots, n-1\}\} \),

(iv) the lower distance of \( P \) is defined by \( d_l(P) = \min \{\{P_{i+1} - P_i \mid i = 1, \ldots, n-1\}\} \).

Definition 1.61. Let \( P \) and \( P' \) be partitions of \([0, 1]\). Then \( P' \) is called a refinement of \( P \) if for all \( i \in \{1, \ldots, l(P)\} \) there is a \( j \in \{1, \ldots, l(P')\} \) such that \( P_i = P'_j \).

Lemma 1.62. Let \( P \) be a partition with \( l(P) > 2 \) and let \( s \subseteq (0, 1) \) be a rational segment with \( \delta(s) < d_l(P) \). Then there is an \( i \in \{1, \ldots, l(P) - 2\} \) such that \( P_i < s' \leq s'' < P_{i+2} \).

Proof. Trivial.

Proposition 1.63. Let \((x_n)_{n \in \mathbb{N}}\) be a sequence of real numbers and let \( p, q \in \mathbb{Q} \) be such that \( p < q \). Then there exists a real number \( y \in [p, q] \) such that \( y \not\equiv x_n \) for all \( n \).

Proof. By scaling and translating the real numbers involved, we may without loss of generality assume \( p = 0 \) and \( q = 1 \). For every \( m \in \mathbb{N} \), we will construct the segments \( y(m) \) in such a way that \( y(m) \not\equiv x_m \). Calculate an \( n_1 \) such that \( \delta(x_1(n_1)) < 1 \), then evidently we can find a rational segment \( s \subseteq (0, 1) \) such that \( s \not\equiv x_1(n_1) \). Subsequently define \( y(1) = s \). For general \( m \), assume suitable segments \( y(1), \ldots, y(m-1) \) have already been defined. Construct an \( n_m \in \mathbb{N} \) such that \( \delta(x_m(n_m)) < 1/2 \cdot \delta(y(m-1)) \). Then it is easy to find a segment \( t \subseteq y(m-1) \) with \( \delta(t) < 1/2 \cdot \delta(y(m-1)) \) and \( t \not\equiv x_m(n_m) \). Defining \( y(t) = t \), the resulting sequence \( y \) clearly consists of shrinking and dwindling rational segments and \( y \not\equiv x_n \) for every \( n \).
Whenever we deal with species of real numbers, it is often necessary to work with species that ‘respect’ equality of real numbers. It would be hard to think of a theory of measure which considers species $A$ such that there are real numbers $x, y$ with $x = y$, $x \in A$ but at the same time $y \notin A$. To circumvent this issue, we introduce the notion of a completion. A completion is a good example of a species which truly deserves to be called a species. If $C$ is a completion of some $A \subseteq \mathbb{R}$ and $x$ a real number, then $x \in C$ is a meaningful statement. But the species $C$ is far from a finished collection.

**Definition 1.64.** Let $A$ be a subspecies of $\mathbb{R}$, then the species $\{x \in \mathbb{R} \mid \exists a \in A : x = a\}$ is called the completion of $A$ and is denoted by $C(A)$. If $A$ and $B$ are two subspecies of $\mathbb{R}$ such that $C(A) = C(B)$, then we say that $A$ and $B$ coincide. A subspecies $A \subseteq \mathbb{R}$ satisfying $C(A) = A$ is called a real species.

**Lemma 1.65.** Let $A$ and $B$ be real species and let $(A_n)_n$ be an enumerable collection of real species, then $A^e$, $A \cap B$, $A \cup B$, $\bigcap_{n=1}^{\infty} A_n$ and $\bigcup_{n=1}^{\infty} A_n$ are all real species.

**Proof.** Trivial.

**Example 1.66.** Define $x \in \mathbb{S}^N$ by $x(n) = (0, 0)$. Then $\{x\}$ is not a real species. But evidently $C(\{x\})$ is a real species and in fact $\{x\}$ coincides with $\{0\}$.

**Proposition 1.67.** Every closed interval coincides with a fan.

**Proof.** Without loss of generality suppose the interval under consideration is $[0, 1]$. We define the fan-law $\sigma : \mathbb{S}^* \rightarrow \{0, 1\}$ as follows. If $t = (t_1)$, then we define $\sigma(t) = 1$ if and only if $t_1 = (0, 1)$. For any given rational segment $s$, define the segments $s^\lambda$ and $s^\rho$ by $s^\lambda = (s', (1/3 \cdot s' + 2/3 \cdot s''))$ and $s^\rho = ((2/3 \cdot s' + 1/3 \cdot s''), t'')$. If $t$ is an admitted finite sequence of segments and $s$ is a rational segment, then we define $\sigma(t * (s)) = 1$ if and only if $s = \phi(t)^\lambda \lor s = \phi(t)^\rho$.

Now let $x$ be an element of $[0, 1]$, we have to find a $y \in \mathcal{F}_\sigma$ such that $x = y$. Without loss of generality we may assume $x(1) = (0, 1)$. Defining $y(1) = (0, 1)$, then clearly $x(1) \subseteq y(1)$. Let $n$ be an arbitrary natural number greater than 1. Suppose we have constructed $m \leq n$ initial segments of $y$ in such a way that $x(n) \subseteq y(m)$. We will now construct a further segment of $y(m)$ which totally covers $x(n + k)$ for some $k \geq 0$. Calculate a $k$ such that $\delta(x(n + k)) < 1/3 \cdot \delta(y(m))$. Then it is quite clear that at least one of the segments $y(m)^\lambda$ and $y(m)^\rho$ complements covers $x(n + k)$. We now simply define this to be our segment $y(m + 1)$. Obviously $y \in \mathcal{F}_\sigma$ and it is clear that $x(n) \approx y(n)$ for every $n$.

**Lemma 1.68.** The following statements are equivalent to Markov’s principle:

(i) $\forall x \in \mathbb{R}[\neg(x = 0) \to x \neq 0],$

(ii) $\forall x \in \mathbb{R}[\neg(x \leq 0) \to x > 0],$

(iii) $\forall x \in \mathbb{R}[\neg(x = 0) \to \exists y \in \mathbb{R}[xy = 1]].$

**Proof.** We only prove [i] the other equivalences can be shown in a similar way. First we derive Markov’s principle from statement [i] so let $\alpha$ be an element of $\mathcal{C}$ satisfying $\neg \forall n[\alpha(n) = 0]$. Define the real number $x$ as follows. If $\alpha(n) = 0$, define the segment $x(n)$ by $\langle -1/2^n, 1/2^n \rangle$ and if $\alpha(n) = 1$ define $x(n) = (1/2^n, 1/2^n)$. Recall that $0$ is the canonical real $y$ defined by $y(n) = (1/2^n, 1/2^n)$. Thus $\neg(x = 0)$ states $\neg(\forall n[x(n) \approx y(n)]$. Now clearly $\forall n[x(n) \approx y(n)]$ is contradictory, as this would mean $x(n) = (1/2^n, 1/2^n)$ for all $n$ and thus $\forall n[\alpha(n) = 0]$. From [i] we may therefore conclude $x \neq 0$, so we know $\exists n[x(n) < y(n) \land x(n) > y(n)]$. Evidently this can only be the case if there is an $n$ such that $x(n) = (1/2^n, 1/2^n)$, which establishes $\exists n[\alpha(n) = 1]$. The proof that Markov’s principle implies [i] runs along similar lines. Defining $\alpha \in \mathcal{C}$ by $\alpha(n) = 0$ if $x(n) \approx y(n)$ and $\alpha(n) = 1$ if $x(n) < y(n) \lor x(n) > y(n)$, we obtain the desired implication. □
1.4 Analysis

In analysis we will have to consider sequences of real numbers, something which Definition 1.26 does not allow. A sequence of rational segments is not a definite object. But by using some coding function, we can realize a sequence of real numbers as a single sequence in $\mathbb{S}^N$. Thus Definition 1.69 is correct. Having defined the species $\mathbb{R}^N$, the notions of limit, convergence and continuity pose no difficulties. Cauchy’s theorem is still true, but the proof requires more work than it does classically.

**Definition 1.69.** The species $\mathbb{R}^N$ consists of the infinite sequences $(\alpha_n)_n$ of real numbers. If $\alpha_n \in A$ for all $n \in \mathbb{N}$ and some species $A \subseteq \mathbb{R}$, then with a slight abuse of notation we sometimes write $(\alpha_n)_n \in A$. Moreover we will use obvious extensions of the notation of Definition 1.27 and 1.28 when dealing with elements of $\mathbb{R}^N$.

**Definition 1.70.** Let $(\alpha_n)_n \in \mathbb{R}^N$ and suppose there is an $N \in \mathbb{N}$ such that $|\alpha_n| < N$ irrespective of $n$. Then we call $(\alpha_n)_n$ a bounded sequence. If $\alpha_n \leq \alpha_{n+1}$ for every $n$, then $(\alpha_n)_n$ a non-decreasing sequence and non-increasing sequences are defined similarly.

**Definition 1.71.** Let $(\alpha_n)_n \in \mathbb{R}^N$ and let $\alpha$ be a real number, then we write:

(i) $\lim_{n \to \infty} \alpha_n = \alpha$ if $\exists m \forall p \geq k \left[|\alpha - \alpha_p| < 1/2^n\right]$, we say $(\alpha_n)_n$ converges to $\alpha$,

(ii) $\lim_{n \to \infty} \alpha_n = \infty$ if $\forall N \exists m \forall p \geq m [\alpha_p > N],$

(iii) $\lim_{n \to \infty} \alpha_n = -\infty$ if $\lim_{n \to \infty} -\alpha_n = \infty$.

The meaning of the expressions $\sum_{i=1}^{\infty} \alpha_i = \alpha$, $\sum_{i=1}^{\infty} \alpha_i = \infty$ and $\sum_{i=1}^{\infty} \alpha_i = -\infty$ should also be clear now. If $(\alpha_n)_n$ converges to $\alpha$, we also say that $\alpha$ is the limit of $(\alpha_n)_n$, write $\alpha_n \to \alpha$ and call $(\alpha_n)_n$ a convergent sequence.

**Definition 1.72.** Let $(\alpha_n)_n \in \mathbb{R}^N$ and let $\alpha$ be a real number, then we write:

(i) $\alpha_n \downarrow \alpha$ if and only if $\alpha_n \to \alpha$ and $\alpha_n \geq \alpha_{n+1}$ for every $n \in \mathbb{N},$

(ii) $\alpha_n \uparrow \alpha$ if and only if $\alpha_n \to \alpha$ and $\alpha_n \leq \alpha_{n+1}$ for every $n \in \mathbb{N}$.

**Example 1.73.** If $(\alpha_n)_n \in \mathbb{R}^N$ is bounded and non-decreasing, then $(\alpha_n)_n$ is not necessarily convergent. An example is the sequence $(\beta_n)_n$ given by $\beta_n = 0$ if $n < k_{99}$ and $\beta_n = 1$ if $n \geq k_{99}$. A proof that $\lim_{n \to \infty} \beta_n$ exists necessarily decides $\exists n [n = k_{99}]$.

**Lemma 1.74.** Let $(\alpha_n)_n$ be a convergent sequence with $0 \leq \alpha_n$ for every $n$. Then $0 \leq \lim_{n \to \infty} \alpha_n$.

**Proof.** Assume $\lim_{n \to \infty} \alpha_n < 0$, then we can find an $m$ such that $\lim_{n \to \infty} \alpha_n < \alpha_m < 0$. □

**Lemma 1.75.** Let $(\alpha_n)_n, (\beta_n)_n$ be convergent elements of $\mathbb{R}^N$ and let $\alpha, \beta \in \mathbb{R}$ be their respective limits. Then we know:

(i) $\lim_{n \to \infty} \alpha_n + \beta_n$ exists and equals $\alpha + \beta$,

(ii) $\lim_{n \to \infty} \alpha_n/\beta_n$ exists and equals $\alpha/\beta$,

(iii) $\lim_{n \to \infty} \alpha_n/\beta_n$ exists and equals $\alpha/\beta$, under the condition that $\beta \neq 0$.

**Proof.** Trivial. □

**Definition 1.76.** Let $A$ and $B$ be subspecies of $\mathbb{R}$. Then we say that the species $B$ is dense in $A$ if $\forall n \forall a \in A \exists b \in B [||a - b|| < 1/2^n]$. 
Definition 1.77. Suppose $$(\alpha_n)_n \in \mathbb{R}^\mathbb{N}$$ satisfies $$\forall m \exists \forall k, p > N \left[ |\alpha_k - \alpha_p| < 1/2^m \right]$$. Then $$(\alpha_n)_n$$ is called a Cauchy sequence.

Theorem 1.78. A sequence $$(\alpha_n)_n \in \mathbb{R}^\mathbb{N}$$ is convergent if and only if it is a Cauchy sequence.

Proof. If $$\lim_{n \to \infty} \alpha_n$$ exists, then it is easy to show that $$(\alpha_n)_n$$ is a Cauchy sequence. Assume that $$(\alpha_n)_n$$ is a Cauchy sequence. For every $$m$$, let $$N(m)$$ be a natural number such that $$N(m+1) > N(m)$$ and $$n, k \geq N(m)$$ implies $$|\alpha_n - \alpha_k| < 1/3 \cdot 1/2^m$$. Furthermore, for every $$m, p \in \mathbb{N}$$ let $$w(m, p)$$ be such that $$\delta(\alpha_p(w(m, p))) < 1/3 \cdot 1/2^m$$. Next write $$s_m = \alpha_{N(m)}(w(N(m), m))$$ and define $$\beta \in \mathbb{S}^\mathbb{N}$$ by $$\beta_m = s_m + (-1/2^{m-1}, 1/2^{m-1})$$. This proves $$\delta(\beta_{m+1}) < 1/2^m$$, so $$(\beta_m)_m$$ is a dwindling sequence of rational segments. Moreover we have:

$$
\beta_{m+1} - \beta_m = s_{m+1}' - s_m' + 2m - s_m - 1/2^{m-1} \\
\leq |s_{m+1}' - s_m'| - 1/2^m \\
\leq |s_{m+1}' - \alpha_{N(m+1)}| + |\alpha_{N(m+1)} - \alpha_{N(m)}| + |s_m' - \alpha_{N(m)}| - 1/2^m \\
< 1/3 \cdot 1/2^{m+1} + 1/3 \cdot 1/2^m + 1/3 \cdot 1/2^m - 1/2^m \\
< 0.
$$

Here (1.1) follows from Proposition 1.55(i). In a similar way we prove $$\beta_{m} - \beta_{m+1} < 0$$. This shows that $$(\beta_m)_m$$ is shrinking sequence of rational segments as well.

We will now prove $$\forall n \exists m \forall k \geq m \left[ |\beta - \alpha_k| < 1/2^n \right]$$. Fix $$n \in \mathbb{N}$$ and find an $$N$$ such that $$|\alpha_p - \alpha_m| < 1/2^n$$ for all $$m, p > N$$. Find a natural number $$l > n + 1$$ such that $$N(l) > N$$. If $$i > N(l)$$, then we deduce $$|\beta - \alpha_i| \leq |\beta - \alpha_{N(l)}| + |\alpha_{N(l)} - \alpha_i| \leq |\beta - \alpha_{N(l)}| + 1/2^{n+1}$$. As the segments $$\beta_i$$ and $$\alpha_{N(l)}(w(N(l), l))$$ are identical, we can bound $$|\beta - \alpha_{N(l)}|$$ by $$\delta(\beta_i) = 1/2^{l-1} \leq 1/2^{n+1}$$. This means we simply have to define $$m = N(l)$$ and we are done.

Definition 1.79. Let $$A$$ be a species and let $$f$$ be partial function $$A \to \mathbb{R}$$. Then we call $$f$$ a non-decreasing function if $$a \leq b$$ implies $$f(a) \leq f(b)$$ for all $$a, b \in \mathcal{D}(f)$$. Likewise, $$f$$ is strictly increasing if $$a < b$$ implies $$f(a) < f(b)$$ for all $$a, b \in \mathcal{D}(f)$$. The terms non-increasing and strictly decreasing are defined similarly. If $$f$$ is either non-decreasing or non-increasing, then we say that $$f$$ is a monotone function.

Definition 1.80. Let $$A$$ be a species and let $$f, g$$ be partial functions $$A \to \mathbb{R}$$. Then we call $$f$$ a positive function if $$\forall a \in \mathcal{D}(f) \ f(a) > 0$$ and denote this by $$f > 0$$. Similarly, we write $$f > g$$ if $$\mathcal{D}(f) = \mathcal{D}(g)$$ and $$f - g$$ is positive. We have similar definitions for negative, non-negative and non-positive functions and the expressions $$f < g$$, $$f \geq g$$ and $$f \leq g$$.

Definition 1.81. Let $$f : \mathbb{R} \to \mathbb{R}$$ be a partial function and suppose $$a \in \mathcal{D}(f)$$. Then $$f$$ is:

(i) continuous at $$a$$ if $$\forall n \exists m \forall x \in \mathcal{D}(f) \left[ |a - x| < 1/2^m \to |f(a) - f(x)| < 1/2^n \right]$$,

(ii) continuous if $$f$$ is continuous at $$x$$ for all $$x \in \mathcal{D}(f)$$,

(iii) sequentially continuous at $$a$$ if for all $$x_n \in \mathcal{D}(f)$$ converging to $$a$$ we have $$\lim_{n \to \infty} f(x_n) = a$$,

(iv) sequentially continuous if $$f$$ is sequentially continuous at $$x$$ for all $$x \in \mathcal{D}(f)$$,

(v) sequentially right continuous at $$a$$ if for all $$x_n \in \mathcal{D}(f)$$ with $$x_n \downarrow a$$ we have $$\lim_{n \to \infty} f(x_n) = a$$,

(vi) sequentially right continuous if $$f$$ is sequentially right continuous at $$x$$ for all $$x \in \mathcal{D}(f)$$,

(vii) uniformly continuous if $$\forall n \exists m \forall x, y \in \mathcal{D}(f) \left[ |x - y| < 1/2^m \to |f(x) - f(y)| < 1/2^n \right]$$.

Definition 1.82. Let $$f : \mathbb{R} \to \mathbb{R}$$ be a partial function and suppose $$b \in \mathbb{R}$$. Then $$b$$ is a continuity point of $$f$$ if $$\forall n \exists m \forall x, y \in \mathcal{D}(f) \left[ |x - b| < 1/2^m \wedge |y - b| < 1/2^m \to |f(x) - f(y)| < 1/2^n \right]$$. 

We are now ready to prove two of Brouwer’s most notorious results, through use of both the continuity principle and the fan theorem. The resulting theorems should not be seen as signs of the absurdity of intuitionistic mathematics. Rather, they show just how strict the notions of totality and extensionality are in a constructive context. We have to give up totality to obtain a useful notion of functions on the continuum. Measure theory provides us with a framework to analyze a whole range of such partial functions.

**Theorem 1.83.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a total and extensional function. Then \( f \) is continuous.

**Proof.** The fact that \( f \) is total implies \( \forall x \in \mathbb{R}_{can} \exists y \in \mathbb{R}_{can} [f(x) = y] \). Fix a natural number \( n \) and consider the function \( f_n : \mathbb{R}_{can} \to \mathbb{S} \) defined by \( f_n(x) = (f(x))(n) \). Every canonical real number is sent to the \( n \)’th segment of its image under \( f \). If \( e : \mathbb{N} \to \mathbb{S} \) is some arbitrary bijection between the natural numbers and rational segments, then we obtain \( \forall x \in \mathbb{R}_{can} \exists m [f_n(x) = e(m)] \). The continuity principle yields \( \forall x \in \mathbb{R}_{can} \exists m, k \forall y \in \mathbb{R}_{can} [(\bar{x}m = \bar{y}m) \to (f_n(y) = f_n(x) = e(k))] \).

Now pick arbitrary canonical real numbers \( x, y \in [0, 1] \), we will show that for every \( p \) there is an \( l \) such that \( |x - y| < 1/2^l \to |f(x) - f(y)| \leq 1/2^p \). By the previous exposition, find an \( m \) such that \( \bar{x}m = \bar{z}m \to f_p(x) = f_p(z) \) for every \( z \in \mathbb{R}_{can} \). Consider the segments \( x(m-1) \) and \( x(m) \). Since \( x \) is a canonical real, we know that \( x(m) \sqsubseteq x(m-1) \). Thus \( q = \min(x(m)\prime - x(m-1)\prime, x(m)\prime\prime - x(m-1)\prime\prime) \) is a rational number greater than 0. Calculate an \( l \) such that \( 1/2^l < q \). Then for any real number \( y \) with \( |x - y| < 1/2^l \) we know that \( x(m-1)\prime \leq y \leq x(m-1)\prime\prime \). Subsequently we can find a canonical real number \( y \) with \( y = z \) and moreover \( \bar{x}m = \bar{z}m \). Hence we obtain \( f_p(x) = f_p(z) \) and thus \( (f(x))(p) = (f(z))(p) \), which implies \( |f(x) - f(z)| \leq 1/2^p \). If \( x \) and \( y \) are not necessarily canonical real numbers, then the implication \( |x - y| < 1/2^l \to |f(x) - f(y)| \leq 1/2^p \) remains true because \( f \) is extensional. \( \square \)

**Theorem 1.84.** Let \( f : [0, 1] \to \mathbb{R} \) be a total and extensional function. Then \( f \) is uniformly continuous.

**Proof.** Let \( \mathcal{F}_\sigma \) be a fan coinciding with \( [0, 1] \) and let \( n \) be a natural number. From the previous theorem we obtain \( \forall x \in \mathcal{F}_\sigma \exists m \forall y \in \mathcal{F}_\sigma [(x - y) < 1/2^m \to |f(x) - f(y)| < 1/2^n] \). Applying Proposition \( \ref{prop:extensionality} \) we obtain \( \exists N \forall x \in \mathcal{F}_\sigma \exists m \leq N \forall y \in \mathcal{F}_\sigma [(x - y) < 1/2^m \to |f(x) - f(y)| < 1/2^n] \). Thus \( |x - y| < 1/2^N \) implies \( |f(x) - f(y)| < 1/2^n \) for all \( x, y \in \mathcal{F}_\sigma \). Moreover this implication remains true if \( x \) and \( y \) are elements of \( \mathcal{C}(\mathcal{F}_\sigma) = [0, 1] \) by the extensionality of \( f \). \( \square \)
Chapter 2

Discrete probability spaces

2.1 Preliminaries

In this section we pave the way for an efficient discussion of Dijkman’s article [Di65]. The main result is Theorem 2.19 where we give several equivalent definitions of the notion event. These allow us to develop Dijkman’s theory smoothly, as we can use whichever characterization is more practical in any given situation. Additionally, the original article lacked a definition of a sum as \( \sum_{i \in A} p_i \), where \( A \) is not necessarily a decidable subspecies of \( \mathbb{N} \). We start with the definition of a probability vector and derive some of its properties. Note that Lemma 2.2 is indeed trivial, but the inclusion of its proof has to do with Example 1.73.

Definition 2.1. Let \( p \) be an infinite sequence of real numbers. Then we call \( p \) a probability vector if \( 0 \leq p_i \leq 1 \) for all \( i \) and moreover \( \sum_{i=1}^{\infty} p_i = 1 \).

Lemma 2.2. Let \( p \) be a probability vector and suppose \( \Delta \) is a decidable subspecies of \( \mathbb{N} \). Then \( \sum_{i \in \Delta} p_i \) is a well defined real number.

Proof. We simply have to verify that the sequence \( (\sum_{i \in \Delta \cap \{1,\ldots,k\} } p_i)_k \) is a Cauchy sequence. This directly follows from the fact that \( p \) is a probability vector. \( \square \)

Lemma 2.3. Let \( p \) be a probability vector and let \( k \) be an arbitrary natural number. Then we can construct a finite, decidable species \( I \subseteq \mathbb{N} \) such that \( \sum_{i \in I} p_i \geq 1 - 1/2^k \) and \( p_i > 0 \) for every \( i \in I \).

Proof. Let \( N \) be such that \( \sum_{i=1}^{N} p_i \geq 1 - 1/2^{k+1} \). For each \( i = 1, \ldots, N \), calculate a number \( n_i \) such that \( \delta(p_i(n_i)) < 1/(2^{k+1}N) \). The decidable species of natural numbers \( S \) precisely consists of those \( i \in \{1,\ldots,N\} \) such that \( p_i(n_i)^t \leq 0 \leq p_i(n_i)^{\prime} \). By the definition of \( n_i \) and Proposition 1.55(i) we then know \( p_i \leq 1/(2^{k+1}N) \) for every \( i \in S \). This establishes \( \sum_{i \in S} p_i \leq 1/2^{k+1} \) and subsequently we define \( I = \{1, \ldots, N\} \setminus S \). \( \square \)

Corollary 2.4. Let \( p \) be a probability vector. Then there exists an \( i \in \mathbb{N} \) such that \( p_i > 0 \).

The first formulation we consider to define events is that of a summable species. This will turn out to be very useful when deriving results on the yet to be defined probability function. Especially theorems on the limit behaviour of probabilities such as Theorem 2.33 benefit from this approach. Dijkman’s original definition of an event can be found in Definition 2.9. Among other things it will be trivial to prove that the events in our discrete probability space form an algebra. Our final equivalent formulation is that of an almost everywhere decidable species. While its use in this chapter is limited, it is the direct analogue of the definition of a measurable subspecies of the continuum. In the remainder of this section, fix a probability vector \( p \).

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Definition 2.5. A species \( A \subseteq \mathbb{N} \) is **summable** if there is a real number \( c \) satisfying the following requirements:

(i) if \( \Delta \) is a decidable subspecies of \( A \), then \( \sum_{i \in \Delta} p_i \leq c \),

(ii) for every \( k \in \mathbb{N} \) there is a decidable subspecies \( \Gamma_k \) of \( A \) with \( \sum_{i \in \Gamma_k} p_i \geq c - 1/2^k \).

Lemma 2.6. Suppose the species \( A \) is summable. Then there is a unique real number \( c \) satisfying both requirements 2.5(i) and 2.5(ii).

Proof. Suppose \( c_1 \) and \( c_2 \) both satisfy the stated requirements, we are done if we can prove \( c_1 \leq c_2 \). Thus we have to derive a contradiction from \( c_1 > c_2 \). Find a number \( k \) such that \( 1/2^k < c_1 - c_2 \). By 2.5(ii) we can find a decidable species \( \Delta \) such that \( \sum_{i \in \Delta} p_i \geq c_1 - 1/2^k > c_2 \). This clearly contradicts 2.5(i) \( \square \).

Definition 2.7. If the species \( A \) is summable, then the unique real number \( c \) from Lemma 2.6 is denoted by \( f_{\sum_{i \in A}} p_i \).

Definition 2.8. Let \( p \) be a probability vector. If \( A \) is a subspecies of \( \mathbb{N} \), then we say that \( A \) is \((p-)\)almost everywhere decidable if for every \( k \in \mathbb{N} \) there is a decidable species \( \Gamma_k \) such that \( \sum_{i \in \Gamma_k} p_i \leq 1/2^k \) and \( i \notin \Gamma_k \rightarrow (i \in A \lor i \notin A) \) hold.

Definition 2.9. A species \( A \subseteq \mathbb{N} \) is called an event if \( p_i > 0 \) implies \( i \in A \lor i \notin A \) for all \( i \in \mathbb{N} \).

Lemma 2.10. Let \( k \) be a natural number, then:

(i) \( \delta_{k,i} = \mu[n] \left[ p_i(n) > 0 \lor p_i(n)'' < 1/2^{k+i} \right] \) is a well defined natural number,

(ii) the species \( \Delta_k = \{ i \mid p_i(\delta_{k,i}) > 0 \} \) is decidable.

Proof. We start with (i) and note that \( 0 < 1/2^{k+i} \) irrespective of \( i \). For every real number \( x \) it follows that \( x > 0 \) or \( x < 1/2^{k+i} \) by Proposition 1.52(v). This means the search operation we are conducting is bounded. The decidability of \( \Delta_k \) is then immediately clear. \( \square \).

Definition 2.11. For every \( k \in \mathbb{N} \) the species \( \Delta_k \) defined in the previous lemma is called the \( k \)'th order canonical decidable species.

Lemma 2.12. If \( n \leq m \), then \( \Delta_n \subseteq \Delta_m \).

Proof. Suppose \( i \in \Delta_n \), then we know \( p_i(\delta_{n,i})' > 0 \). Hence the least natural number \( k \) that is such that \( p_i(k)' > 0 \lor p_i(k)''' < 1/2^{n+i} \), satisfies \( p_i(k)' > 0 \). But then \( p_i(k)' > 0 \) also holds for the least \( k' \) satisfying \( p_i(k)' > 0 \lor p_i(k)''' < 1/2^{m+i} \). Obviously \( 1/2^{m+i} \leq 1/2^{n+i} \), so we in fact have \( k' = k \) and hence \( i \in \Delta_m \). \( \square \).

Lemma 2.13. If \( A \) is an event, then the species \( \Delta_k \cap A \) is decidable for every \( k \).

Proof. We can decide \( i \in \Delta_k \) for all \( i \in \mathbb{N} \). If \( i \) is indeed an element of \( \Delta_k \), then we can establish whether \( i \in A \) is also true. \( \square \).

Lemma 2.14. Let \( A \) be an event and let \( k \) be a natural number, then \( i \in A \setminus (\Delta_k \cap A) \) implies \( p_i < 1/2^{k+i} \) irrespective of \( k \) and \( i \).

Proof. We examine \( p_i(\delta_{k,i}) \). Suppose for now that \( p_i(\delta_{k,i})' > 0 \), which yields \( i \in A \). This would imply \( i \in \Delta_k \cap A \), hence \( p_i(\delta_{k,i})' > 0 \) is contradictory. As \( p_j \geq 0 \) is true for any \( j \), we now know \( p_i(\delta_{k,i})' \leq 0 \). By the definition of \( \delta_{k,i} \), the last statement implies \( p_i(\delta_{k,i})''' < 1/2^{k+i} \). Obviously this also entails \( p_i < 1/2^{k+i} \). \( \square \).
The next result shows that Definition 2.16 is justified. Here we define what will be the probability of an event in the next section. Also note the special role of Proposition 2.17 and 2.18. These verify that our notion of a sum over an event is indeed a consistent generalization of sums over decidable subspecies of \( \mathbb{N} \).

**Lemma 2.15.** If \( A \) is an event, then the limit \( \lim_{k \to \infty} \sum_{i \in \Delta k \cap A} p_i \) exists.

**Proof.** We verify that \( \left( \sum_{i \in \Delta k \cap A} p_i \right)_k \) is a Cauchy sequence. Let \( n \) be a natural number, then we have:

\[
\sum_{i \in \Delta k + n \cap A} p_i - \sum_{i \in \Delta k \cap A} p_i = \lim_{m \to \infty} \sum_{i=1}^{m} p_i - \lim_{m \to \infty} \sum_{i=1}^{m} p_i = \lim_{m \to \infty} \left( \sum_{i \in \Delta k + n \cap A} p_i - \sum_{i \in \Delta k \cap (A)} p_i \right) = \sum_{i \in (\Delta k + n \cap A) \setminus (\Delta k \cap A)} p_i \\
\leq \sum_{i \in A \setminus (\Delta k \cap A)} p_i \\
\leq \lim_{m \to \infty} \sum_{i=1}^{m} 1/2^{k+i} = 1/2^k.
\]

Equality (2.1) is a consequence of the fact that \( \Delta k \cap A \subseteq \Delta k + n \cap A \). The inequality of (2.2) follows from Lemma 2.14.

**Definition 2.16.** For every event \( A \) we denote the limit \( \lim_{k \to \infty} \sum_{i \in \Delta k \cap A} p_i \) by \( \oint_{\Delta k} p_i \).

**Proposition 2.17.** If \( \Delta \) is a decidable subspecies of \( \mathbb{N} \), then \( \oint_{\Delta} p_i = \sum_{i \in \Delta} p_i \).

**Proof.** The proof follows the exact same reasoning as the proof of Lemma 2.15 applied to the difference \( \sum_{i \in \Delta} p_i - \sum_{i \in \Delta \cap \Delta} p_i \) instead of \( \sum_{i \in \Delta k + n \cap A} p_i - \sum_{i \in \Delta k \cap A} p_i \).

**Proposition 2.18.** Let \( A \) be an event and let \( n \) be a natural number. Then \( \lim_{n \to \infty} \oint_{\Delta \cap \{1, \ldots, n\}} p_i \) exists and is equal to \( \oint_{\Delta \cap A} p_i \).

**Proof.** If we define \( A_n = A \cap \{1, \ldots, n\} \), then it is clear from Definition 2.16 that \( \oint_{\Delta \cap A} p_i \leq \oint_{\Delta \cap A_n} p_i \) irrespective of \( n \). We are done if we can show that the difference \( \oint_{\Delta \cap A} p_i - \oint_{\Delta \cap A_n} p_i \) gets arbitrarily small as \( n \to \infty \). For all \( k, n, m \) we have:

\[
\sum_{i \in \Delta k \cap A} p_i - \sum_{i \in \Delta k \cap A_n} p_i = \sum_{i \in (\Delta k \cap A) \setminus (\Delta k \cap A_n)} p_i = \sum_{i \in \Delta k \cap (A \setminus A_n)} p_i \\
\leq \sum_{i>n} p_i.
\]

Evidently this last expression can be made arbitrarily small by choosing \( n \) large enough.
Theorem 2.19. Let $A$ be a subspecies of $\mathbb{N}$, then the following statements are equivalent:

(i) $A$ is summable,

(ii) $A$ is an event,

(iii) $A$ is almost everywhere decidable.

If $A$ has one of the three equivalent properties above, then we also have $\mathcal{f}_{i \in A} p_i = \mathcal{f}'_{i \in A} p_i$.

Proof. We first prove (i) $\Rightarrow$ (ii) so assume $A$ is summable. To prove that $A$ is an event as well, let $i$ be such that $p_i > 0$. We need to show $i \in A \vee i \notin A$. Let $k \in \mathbb{N}$ be such that $p_i > 1/2^k$. Subsequently we use (2.5(ii)) to find a decidable subspecies $\Gamma_k$ of $A$ satisfying $\sum_{j \in \Gamma_k} p_j \geq c - 1/2^k$. If $i \in \Gamma_k$, we are clearly done. So suppose $i \notin \Gamma_k$, we wish to establish $i \notin A$. If we assume $i \in A$, then $\Gamma_k \cup \{i\}$ is a decidable subspecies of $A$. This yields $\sum_{j \in \Gamma_k \cup \{i\}} p_j = \sum_{j \in \Gamma_k} p_j + p_i \geq c - 1/2^k + p_i > c$, contradicting Definition 2.5(i).

Next we show the implication (ii) $\Rightarrow$ (i). Let $A$ be an event and define $c = \mathcal{f}'_{i \in A} p_i$. Suppose $\Delta$ is a decidable subspecies of $A$, then $\sum_{i \in \Delta} p_i \leq c$ is a direct consequence of Proposition 2.17. To prove that Definition 2.5(ii) is satisfied, fix some $k \in \mathbb{N}$ and recall that $\mathcal{f}'_{i \in A} p_i$ is defined as $\lim_{m \to \infty} \sum_{i \in \Delta_m \cap A} p_i$. So choose $m \in \mathbb{N}$ such that $\sum_{i \in \Delta_m \cap A} p_i \geq c - 1/2^k$. Obviously $\Delta_m \cap A$ is a decidable species, so we are done.

To derive (ii) $\Rightarrow$ (iii) suppose $A$ is an event. Let $k$ be an arbitrary natural number. Use Lemma 2.3 to construct a finite species $I$ such that $\sum_{i \in I} p_i \geq 1 - 1/2^k$ and $p_i > 0$ for every $i \in I$. Obviously the species $\Gamma_k = \mathbb{N} \setminus I$ is decidable and $\sum_{i \in \Gamma_k} p_i \leq 1/2^k$. Moreover $\mathbb{N} \setminus \Gamma_k = I$ by decidability of $I$. Because $A$ is an event, $i \in A$ is decidable for all $i \in I$. This establishes $i \notin \Gamma_k \rightarrow i \in A \vee i \notin A$, so $A$ is almost everywhere decidable.

We continue with a proof of (iii) $\Rightarrow$ (ii). Let $A$ be a summable species and suppose $i$ is such that $p_i > 0$. We wish to establish $i \in A \vee i \notin A$. Find a $k \in \mathbb{N}$ satisfying $p_i > 1/2^k$ and subsequently find a decidable $\Gamma_k \subseteq \mathbb{N}$ such that $\sum_{i \in \Gamma_k} p_i \leq 1/2^k$. Then $i \notin \Gamma_k$ is clearly contradictory. We can now directly conclude $i \in A \vee i \notin A$.

Finally we derive $\mathcal{f}_{i \in A} p_i = \mathcal{f}'_{i \in A} p_i$ for summable $A$. We will show that $\mathcal{f}'_{i \in A} p_i$ is the unique real number $c$ satisfying Definition 2.4(i) and 2.5(ii). For decidable $\Delta \subseteq A$ we have already seen $\sum_{i \in \Delta} p_i \leq \mathcal{f}'_{i \in A} p_i$. Moreover we can easily find a decidable $\Gamma_k$ making Definition 2.5(ii) true. Since $\mathcal{f}'_{i \in A} p_i = \lim_{k \to \infty} \sum_{i \in \Delta_k \cap A} p_i$, we simply define $\Gamma_k = \Delta_k \cap A$. \qed
2.2 Events and probabilities

The definition of a discrete probability space below is justified by Theorem 2.19. We will be using this theorem often throughout the remainder of this chapter. When we consider the species of events on such probability spaces, a notable difference with the classical theory arises. The events constitute an algebra, but not a $\sigma$-algebra. But the definition of independence carries over directly. Moreover, many familiar laws of probability theory still hold. It is only when we investigate infinite unions or intersections of events that our theory significantly diverges from its classical counterpart.

Definition 2.20. Define $\Omega = \mathbb{N}$ and let $p$ be a probability vector. Denote with $\mathcal{F}$ the species of almost everywhere decidable subspecies of $\Omega$. We then call the triplet $(\Omega, \mathcal{F}, p)$ a Dijkman discrete probability space (DDPS). We will refer to elements of $\mathcal{F}$ as events. If $A$ is an event, then we define $P : \Omega \to [0, 1]$ by $P(A) = \sum_{i \in A} p_i$. The real number $P(A)$ is called the probability of $A$.

Lemma 2.21. If $A$ and $B$ are events, then the following are events as well:

(i) $\Omega$,
(ii) $\Omega \setminus A$,
(iii) $A \cup B$,
(iv) $A \cap B$.

Proof. We use the fact that an event is an event. The first proposition is trivial. To prove (ii), let $p_i > 0$. We have to establish $i \in \Omega \setminus A$ or $i \notin \Omega \setminus A$. Since $A$ is an event, we can decide $i \in A \lor i \notin A$. In the first case, it is contradictory that $i \notin A$. Hence we obtain $\neg(i \in \Omega \setminus A)$. In the second case we immediately see that $i \in \Omega \setminus A$. Next we look at (iii). As we know $i \in A \lor i \notin A$ as well as $i \in B \lor i \notin B$ when $p_i > 0$, whether $i$ is an element of $A \cup B$ in this case is clear. Statement (iv) follows in the same way.

Example 2.22. Consider the DDPS $(\Omega, \mathcal{F}, p)$, where $p$ is such that $p_1 > 0$. Define the sequence of events $(A_n)_n$ as follows:

$A_n = \{n+1\}$ if $n < k_9$,
$A_n = \{1\}$ if $n = k_9$,
$A_n = \{n\}$ if $n > k_9$.

Then as of yet we cannot prove that $\bigcup_{n=1}^{\infty} A_n$ is an event. Deciding $1 \in \bigcup_{n=1}^{\infty} A_n \lor 1 \notin \bigcup_{n=1}^{\infty} A_n$ is equivalent to proving or disproving that in the decimal expansion of $\pi$ a sequence of 99 consecutive 9’s occurs.

Proposition 2.23. Let $A$ and $B$ be events, then:

(i) $P(\Omega) = 1$,
(ii) $0 \leq P(A) \leq 1$,
(iii) $P(A) \leq P(B)$ if $A \subseteq B$,
(iv) $P(A \cup B) \leq P(A) + P(B)$,
(v) $P(A \cup B) = P(A) + P(B)$ if $A$ and $B$ are disjoint.

Proof. Statement (i) follows directly from the fact that $\Omega$ is summable and Lemma 2.3. The other statements can be shown as follows. Fixing a $k$ and replacing every $A$ by $\Delta_k \cap A$ and $B$ by $\Delta_k \cap B$ the inequalities and equalities above are definitely valid. This is immediately clear from Proposition 2.18. Subsequently taking the limit as $k \to \infty$, we obtain the desired results.
Definition 2.24. Let $A$ be an event such that $\mathbb{P}(A) = 1$, then we say $A$ is almost full.

Lemma 2.25. An event $A$ is almost full if and only if $p_i > 0$ implies $i \in A$ for all $i \in \Omega$.

Proof. Suppose $A$ is almost full and let $i \in \Omega$ be such that $p_i > 0$. Because $A$ is an event, we know $i \in A \lor i \notin A$. To establish $i \in A$ it is therefore sufficient to derive a contradiction from $i \notin A$. Then Proposition 2.23(iii) would yield $\mathbb{P}(A) \leq \mathbb{P}(\{i\}^c)$. From Proposition 2.17 it follows that $\mathbb{P}(\{i\}^c) = 1 - p_i < 1$, which is contradictory. Next assume that $p_i > 0 \rightarrow i \in A$ for every $i \in \Omega$. Given some $k \in \mathbb{N}$, we can employ Lemma 2.3 to construct a subspecies $I$ of $A$ such that $\mathbb{P}(I) \geq 1 - 1/2^k$ and $i \in I \rightarrow p_i > 0$. Proposition 2.23(ii) and 2.23(iii) then imply that $\mathbb{P}(A) = 1$. \qed

Corollary 2.26. Let $(A_n)_n$ be a sequence of almost full events, then $\bigcap_{n=1}^{\infty} A_n$ is almost full.

Corollary 2.27. If $A$ is an almost full event, then $A$ is inhabited.

Proof. Combine Lemma 2.25 with Corollary 2.4 \qed

Proposition 2.28. Let $A$ and $B$ be events, then:

(i) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$,

(ii) $\mathbb{P}((A^c)^c) = \mathbb{P}(A)$,

(iii) $\mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) = \mathbb{P}(A)$.

Proof. The species $A \cup A^c$, $B \cup B^c$ and $(A^c)^c \cup A^c$ are almost full by Lemma 2.25. Combining this with Proposition 2.23(v) we easily obtain all statements. \qed

Definition 2.29. Let $A_1, ..., A_n$ be a sequence of events. Then we call $A_1, ..., A_n$ independent if for every decidable species $S \subseteq \{1, ..., n\}$:

$$\mathbb{P}(\bigcap_{i \in S} A_i) = \prod_{i \in S} \mathbb{P}(A_i).$$

An infinite sequence of events is independent if every finite subspecies of those events is independent.

Lemma 2.30. If the events $A_1, ..., A_n$ are independent, then so are $A_1, ..., A_{n-1}, A_n^c$.

Proof. Lemma 2.25 and Proposition 2.28 allow us to copy the classical proof of this statement. \qed

Definition 2.31. Let $A$ and $B$ be events and suppose that $p_i > 0 \land i \in A$ implies $i \in B$. Then we say that event $A$ implies $B$ and write $A \succ B$. If $A$ and $B$ both imply each other, then we call $A$ and $B$ equivalent events and denote this by $A \equiv B$.

Lemma 2.32. Let $A$ and $B$ be events, then:

(i) if $A \succ B$, then $\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A)$,

(ii) if $A \succ B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$,

(iii) if $A \equiv B$, then $\mathbb{P}(A) = \mathbb{P}(B)$.

Proof. It is quite clear that (iii) trivially follows from (ii) which easily follows from (i). So we only prove (i) It is evident that $\sum_{i \in \Delta_k \cap (B \setminus A)} p_i = \sum_{i \in (\Delta_k \cap B) \setminus (\Delta_k \cap A)} p_i$. Hence it remains to be shown that $\Delta_k \cap A \subseteq \Delta_k \cap B$ for all $k$. This immediately follows from $A \succ B$ and Definition 2.11. \qed
Theorem 2.33. Let \((A_n)_n\) be a sequence of events such that \(A_n \subseteq A_{n+1}\) for all \(n\) and \(\lim_{n \to \infty} \mathbb{P}(A_n)\) exists. Then:

(i) \(\bigcup_{n=1}^{\infty} A_n\) is an event,

(ii) \(\lim_{n \to \infty} \mathbb{P}(A_n) = \mathbb{P}(\bigcup_{n=1}^{\infty} A_n)\).

Proof. We first prove that \(\bigcup_{n=1}^{\infty} A_n\) is an event. Let \(i \in \mathbb{N}\) be such that \(p_i > 0\) and calculate an \(m\) such that \(\lim_{n \to \infty} \mathbb{P}(A_n) - \mathbb{P}(A_m) < p_i\). Determine whether \(i \in A_m\), if this is the case then obviously \(i \in \bigcup_{n=1}^{\infty} A_n\). But \(i \notin A_m\) implies that \(i \notin \bigcup_{n=1}^{\infty} A_n\) is also contradictory. This would mean \(\exists k > m[i \in A_k]\), which is impossible given our construction of \(m\).

Evidently we have \(\lim_{n \to \infty} \mathbb{P}(A_n) \leq \mathbb{P}(\bigcup_{n=1}^{\infty} A_n)\). Because \(\bigcup_{n=1}^{\infty} A_n\) is summable, for every \(k\) there is a decidable species \(\Delta \subseteq \bigcup_{n=1}^{\infty} A_n\) with \(\mathbb{P}(\bigcup_{n=1}^{\infty} A_n) - \mathbb{P}(\Delta) < 1/2^{k+1}\). Recalling Proposition 2.17 and 2.18, we can furthermore find a finite decidable subspecies \(F\) of \(\Delta\) with \(\mathbb{P}(\Delta) - \mathbb{P}(F) < 1/2^{k+1}\). Clearly we can construct an \(m\) such that \(F \subseteq A_m\), which means \(\mathbb{P}(\bigcup_{n=1}^{\infty} A_n) - \mathbb{P}(A_m) < 1/2^k\).

\(\Box\)

Corollary 2.34. Let \((A_n)_n\) be a sequence of disjoint events and moreover suppose \(\sum_{n=1}^{\infty} \mathbb{P}(A_n)\) converges. Then:

(i) \(\bigcup_{n=1}^{\infty} A_n\) is an event,

(ii) \(\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \mathbb{P}(\bigcup_{n=1}^{\infty} A_n)\).

Theorem 2.35. Let \((A_n)_n\) be a sequence of events such that \(A_{n+1} \subseteq A_n\) for every \(n\). Moreover suppose \(\bigcup_{n=1}^{\infty} A_n^c\) is an event. Then:

(i) \(\bigcap_{n=1}^{\infty} A_n\) is an event,

(ii) \(\lim_{n \to \infty} \mathbb{P}(A_n) = \mathbb{P}(\bigcap_{n=1}^{\infty} A_n)\).

Proof. We first prove the existence of the limit \(\lim_{n \to \infty} \mathbb{P}(A_n)\) by showing that \((\mathbb{P}(A_n))_n\) is a Cauchy sequence. Let \(k\) be a natural number. We need to find an \(N\) such that for each \(n > N\) and \(m \in \mathbb{N}\) we have \(\mathbb{P}(A_n) - \mathbb{P}(A_{n+m}) = \mathbb{P}(A_n - A_{n+m}) < 1/2^k\). Lemma 2.3 yields a decidable, finite species \(I\) of distinct indices such that \(i \in I\) implies \(p_i > 0\) and \(\sum_{i \in I} p_i > 1 - 1/2^k\). If we can construct \(N\) in such a way that \(I \cap (A_n \setminus A_{n+m}) = \emptyset\) if \(n > N\), so \(A_n \setminus A_{n+m} \sim I^c\), then we are clearly done. Pick an arbitrary \(i \in I\) and determine whether \(i \in \bigcup_{n=1}^{\infty} A_n^c\) is true. If \(i \notin \bigcup_{n=1}^{\infty} A_n^c\), then \(\exists n[i \in A_n^c]\) and hence \(\forall n[i \notin A_n^c]\). This means \(i \notin A_n \setminus A_{n+m}\) irrespective of \(n\) and \(m\). On the other hand, \(i \in \bigcup_{n=1}^{\infty} A_n^c\) yields \(\exists n[i \in A_n^c]\). So let \(n(i) \in \mathbb{N}\) be such that \(i \in A_n^c\). Since \((A_n)_n\) is a non-increasing sequence of species, it follows that \(i \notin A_n \setminus A_{n+m}\) for any \(n \geq n(i)\). Thus we simply define \(N = \max\{\{n(i) \mid i \in I\}\}\).

Next we show that \(\bigcap_{n=1}^{\infty} A_n\) is an event. So let \(i\) be such that \(p_i > 0\) and let \(k \in \mathbb{N}\) be such that \(1/2^k < p_i\). Since the limit \(\lim_{n \to \infty} \mathbb{P}(A_n)\) exists, we can find an \(N\) such that \(\mathbb{P}(A_n) - \mathbb{P}(A_{n+m}) = \mathbb{P}(A_n - A_{n+m}) < 1/2^k\) for every \(n > N\) and \(m\). Clearly \(i \notin A_n \setminus A_{n+m}\) and we now consider two cases. If \(i \in A_n\), we must conclude that \(i \in A_{n+m}\) irrespective of \(m\). Because \((A_n)_n\) is a non-increasing sequence of species, this yields \(i \notin \bigcap_{n=1}^{\infty} A_n\). So \(\bigcap_{n=1}^{\infty} A_n\) is indeed an event.
Finally we derive \( \mathbb{P}(\bigcap_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \mathbb{P}(A_n) \). To this end it is also sufficient to prove \( \mathbb{P}(\bigcap_{n=1}^{\infty} (A_n)^c) = \lim_{n \to \infty} \mathbb{P}(A_n^c) \) by Proposition 2.28. We know \( A_n^c \subseteq A_{n+1}^c \) from \( A_{n+1}^c \subseteq A_n \) for every \( n \). Defining \( A_0 = \Omega \), we see that \( \sum_{n=1}^{\infty} \mathbb{P}(A_n^c \setminus A_{n-1}^c) = \lim_{n \to \infty} \mathbb{P}(A_n^c) \) by Lemma 2.32(i). As the latter limit converges, we may use Corollary 2.34 to conclude that \( \bigcup_{n=1}^{\infty} (A_n \setminus A_{n-1}) \) is an event. Thus we are done if we can show that \( \mathbb{P}(\bigcap_{n=1}^{\infty} (A_n^c \setminus A_{n-1}^c)) = \mathbb{P}(\bigcup_{n=1}^{\infty} (A_n^c \setminus A_{n-1}^c)) \).

We first derive \( \mathbb{P}(\bigcap_{n=1}^{\infty} (A_n^c \setminus A_{n-1}^c)) \geq \mathbb{P}(\bigcup_{n=1}^{\infty} (A_n^c \setminus A_{n-1}^c)) \) from \( \sum_{n=1}^{\infty} \mathbb{P}(A_n^c \setminus A_{n-1}^c) \leq \lim_{n \to \infty} \mathbb{P}(A_n^c) \).

So suppose \( i \in \bigcup_{n=1}^{\infty} (A_n^c \setminus A_{n-1}^c) \), then there is an \( n \) such that \( i \in A_n^c \setminus A_{n-1}^c \). In particular this implies \( i \in A_n^c \) and so \( i \in \bigcap_{n=1}^{\infty} A_n \) is contradictory. Next we derive \( \mathbb{P}(\bigcap_{n=1}^{\infty} (A_n^c \setminus A_{n-1}^c)) \geq \mathbb{P}(\bigcup_{n=1}^{\infty} (A_n^c \setminus A_{n-1}^c)) \) by proving \( \mathbb{P}(\bigcap_{n=1}^{\infty} (A_n^c \setminus A_{n-1}^c)) \geq \mathbb{P}(\bigcup_{n=1}^{\infty} (A_n^c \setminus A_{n-1}^c)) \).

So let \( i \) be an element of \( \bigcap_{n=1}^{\infty} (A_n^c \setminus A_{n-1}^c) \) with \( p_i > 0 \), we shall derive a contradiction from \( i \in \bigcup_{n=1}^{\infty} (A_n^c \setminus A_{n-1}^c) \). The latter statement would imply \( i \in \bigcap_{n=1}^{\infty} (A_n^c \setminus A_{n-1}^c) \). Then in particular \( i \in A_n^c \) is contradictory for every \( n \). For if \( i \in A_n^c \) were true for certain \( m \), then by the fact that all the species \( A_1^c, \ldots, A_m^c \) are events we may assume that \( m \) is minimal with respect to this property. Thus the case \( m = 0 \) is immediately excluded and we obtain \( i \notin A_n^c \). This establishes \( i \in (A_m^c \setminus A_{m-1}^c) \), which contradicts \( i \in \bigcap_{n=1}^{\infty} (A_n^c \setminus A_{n-1}^c) \).


\[ \text{Corollary 2.36. Let } (A_n) \text{ be a sequence of events such that } A_{n+1} \subseteq A_n \text{ for every } n. \text{ If } \lim_{n \to \infty} \mathbb{P}(A_n) \text{ exists, then:} \]

(i) \( \bigcap_{n=1}^{\infty} A_n \) is an event,

(ii) \( \lim_{n \to \infty} \mathbb{P}(A_n) = \mathbb{P}(\bigcap_{n=1}^{\infty} A_n) \).

\[ \text{Proof. In the proof of Theorem 2.35, we only use the assumption that } \bigcap_{n=1}^{\infty} A_n^c \text{ is an event to show that } \lim_{n \to \infty} \mathbb{P}(A_n) \text{ exists.} \]

\[ \text{Corollary 2.37. Let } (A_n) \text{ be a sequence of almost full events, then } \bigcap_{n=1}^{\infty} A_n \text{ is almost full.} \]

\[ \text{Proof. Apply the previous corollary to the sequence of almost full events } (\bigcap_{m=1}^{n} A_m)_n. \]

In Theorem 2.35 the assumption that \( \bigcap_{n=1}^{\infty} A_n^c \) is an event, can not be weakened to the requirement that \( \bigcap_{n=1}^{\infty} A_n \) is an event. The theorem in this form is equivalent to Markov’s principle. Dijkman overlooked this fact in his article D65.

\[ \text{Proposition 2.38. Let } T \text{ denote the statement } T(\alpha) = \text{Let } (A_n) \text{ be a sequence of events satisfying } A_{n+1} \subseteq A_n \text{ for every } n \text{ and moreover suppose } \bigcap_{n=1}^{\infty} A_n \text{ is an event. Then } \lim_{n \to \infty} \mathbb{P}(A_n) \text{ exists and } \lim_{n \to \infty} \mathbb{P}(A_n) = \mathbb{P}(\bigcap_{n=1}^{\infty} A_n). \text{ Then } T \text{ is equivalent to Markov’s principle.} \]

\[ \text{Proof. We first show that Markov’s principle implies } T. \text{ Since } \bigcap_{n=1}^{\infty} A_n \text{ is an event, so is } \bigcap_{n=1}^{\infty} A_n^c = \{ i \in \Omega \mid \lnot(\forall n[i \in A_n]) \}. \text{ Pick an } i \in \Omega \text{ such that } p_i > 0, \text{ then we can decide whether } \lnot(\forall n[i \in A_n]) \text{ is true. Because of our choice of } i, \text{ the proposition } i \in A_n \text{ is decidable irrespective of } n. \text{ Thus } \lnot(\forall n[i \in A_n]) \text{ is equivalent to } \exists n[i \in A_n] \text{ by Markov’s principle. Hence the species } \bigcap_{n=1}^{\infty} A_n^c \text{ is also an event. We may now conclude } T \text{ from Theorem 2.35.} \]

Next assume that \( T \), we are to derive Markov’s principle. So let \( \alpha \) be an element of the Cantor space \( C \) satisfying \( \lnot(\forall n[\alpha(n) = 0]) \). Let \( (\Omega, \mathcal{F}, p) \) be a DDPS and define the non-increasing sequence of events \( (A_n) \) as follows. If \( \forall k \leq n[\alpha(k) = 0] \), define \( A_n = \Omega \) and if \( \exists k \leq n[\alpha(k) = 1] \), we define \( A_n = \emptyset \). We now show that \( \bigcap_{n=1}^{\infty} A_n \) is an event, by simply proving that \( \bigcap_{n=1}^{\infty} A_n = \emptyset \). So suppose \( i \in \bigcap_{n=1}^{\infty} A_n \), then \( i \in A_n \) for every \( n \). But this must mean that \( \forall n[\alpha(n) = 0] \), so \( \forall n[\alpha(n) = 0] \). This is a contradiction, hence \( \bigcap_{n=1}^{\infty} A_n = \emptyset \). We may now conclude from \( T \) that \( \lim_{n \to \infty} \mathbb{P}(A_n) \) exists. Quite clearly this limit can only have the values 0 or 1. If \( \lim_{n \to \infty} \mathbb{P}(A_n) = 1 \), then we must have \( A_n = \Omega \) for all \( n \). This again contradicts \( \lnot(\forall n[\alpha(n) = 0]) \). Hence we obtain \( \lim_{n \to \infty} \mathbb{P}(A_n) = 0 \), so we calculate an \( m \) such that \( p \geq m \) implies \( |\mathbb{P}(A_p)| < 1/2 \). But this must mean that \( \mathbb{P}(A_m) = 0 \), so \( A_m = \emptyset \) and therefore \( \exists k \leq m[\alpha(k) = 1] \). This establishes \( \exists n[\alpha(n) = 1] \).
2.3 Stochastic variables

Our theory of stochastic variables closely mirrors its classical equivalent. An exception is the fact that stochastic variables are partial functions. As we have already noted in the comments above Theorem 1.83, partial functions are much more natural than total functions in constructive mathematics. It is therefore unsurprising that the domains of our stochastic variables not necessarily encompass the whole probability space. Our theory significantly differs from Dijkman’s original approach. His stochastic variables had to adhere to very strict requirements. Because of these the stochastic variables did not form a vector space. But as is clear from the exposition below, we do not need his additional conditions to work with distributions of stochastic variables.

Definition 2.39. If \((\Omega, \mathcal{F}, p)\) is a DDPS, then a stochastic variable is a partial function \(X : \Omega \to \mathbb{R}\) whose domain \(\mathcal{D}(X)\) is almost full. We define \(X_i = X(i)\) for every \(i \in \mathcal{D}(X)\).

Lemma 2.40. Let \(X\) and \(Y\) be stochastic variables on \((\Omega, \mathcal{F}, p)\) and suppose \(g : \mathbb{R} \to \mathbb{R}\) is a total function. Then:

\[\begin{align*}
(i) \quad & X + Y \text{ is a stochastic variable}, \\
(ii) \quad & XY \text{ is a stochastic variable}, \\
(iii) \quad & g(X) \text{ is a stochastic variable}.
\end{align*}\]

Proof. Quite clearly \(\mathcal{D}(X) \cap \mathcal{D}(Y) \subseteq \mathcal{D}(X + Y)\), which establishes that \(\mathcal{D}(X + Y)\) is almost full. The proofs of the other cases are similar. \(\square\)

Lemma 2.41. Let \(A\) be an event and define the function \(I_A : \Omega \to \mathbb{R}\) as follows:

\[I_A(i) = 1 \text{ if } i \in A, \quad I_A(i) = 0 \text{ if } i \notin A.\]

Then \(I_A\) is a stochastic variable.

Proof. We know that \(i \in A\) is decidable if \(p_i > 0\). Therefore \(\mathcal{D}(I_A)\) is almost full. \(\square\)

Definition 2.42. Let \(A\) be an event, then the stochastic variable \(I_A\) defined in Lemma 2.41 is called the characteristic function of \(A\).

Definition 2.43. Let \(X\) be a stochastic variable and let \(x\) be a real number. Then the species \(\{X < x\}\) is defined as \(\{i \in \Omega \mid X_i < x\}\). If \(X < x\) is an event, we will often denote its probability \(\mathbb{P}(\{X < x\})\) by \(\mathbb{P}(X < x)\). We have similar definitions when the symbol ‘<’ is replaced by ‘\(\leq\)’, ‘\(>\)’ or ‘\(\geq\)’.

Definition 2.44. Let \(X\) be a stochastic variable, then the partial function \(F_X : \mathbb{R} \to [0, 1]\) defined by \(F_X(x) = \mathbb{P}(X \leq x)\) is called the distribution function of \(X\).

To avoid confusion, please note the following. Let \(X\) be a stochastic variable and suppose \(x, y \in \mathbb{R}\). In general we can not decide \(x \in \mathcal{D}(F_X)\). But what is clear is that if \(\{X \leq x\}\) is an event and \(x = y\), then \(\{X \leq y\} = \{X \leq x\}\) and hence \(F_X(x) = F_X(y)\). Thus it is evident that \(F_X\) is an extensional or well-defined partial function. However, from the definition of \(F_X\) it is not immediately obvious that \(\mathcal{D}(F_X)\) is even inhabited. Yet this is unrelated to the well-definedness of \(F_X\). Moreover, Lemma 2.45 in fact proves that the domain of \(F_X\) is much more than simply inhabited. Another difficulty involving distributions is the following. We can not state that distributions are right continuous on their whole domain. This would require an application of Markov’s principle. In some sense the right continuity of distribution functions can be salvaged however. Lemma 2.47 shows that the points where these functions are sequentially right continuous constitute a co-enumerable species. This is the same in the continuous case, but here the proof will cover the whole of Section 3.3.
Lemma 2.45. Let $X$ be a stochastic variable, then $\mathcal{D}(F_X)$ is co-enumerable.

Proof. Let $y \in \mathbb{R}$ be apart from the enumerable species \{X\_i | i ∈ Ω\}. Then $y < X_i$ or $y > X_i$ for every $i ∈ Ω$, so clearly \{i ∈ Ω | X_i ≤ y\} is an event. □

Theorem 2.46. Let $T$ denote the statement ‘Let $X$ be a stochastic variable and let $x$ be an element of $\mathcal{D}(F_X)$. Suppose $(x_n)_n$ is a sequence of real numbers in $\mathcal{D}(F_X)$ such that $x_n \downarrow x$, then $\lim_{n→∞} F_X(x_n) = F_X(x)$’. Then $T$ is equivalent to Markov’s principle.

Proof. Assume we may employ Markov’s principle, we shall derive $T$. We prove that $\lim_{n→∞} F_X(x_n)$ exists by showing that $(F_X(x_n))_n$ is a Cauchy sequence. Fix $k ∈ \mathbb{N}$ and construct a finite, decidable species $I$ such that $p_i > 0$ for each $i ∈ I$ and $\sum_{i∈I} p_i > 1 - 1/2^k$ by using Lemma 2.3. If we can find an $n$ such that $F_X(x_n) - F_X(x_{n+m}) < 1/2^k$ irrespective of $m$, we are done. Lemma 2.32 yields:

$$F_X(x_n) - F_X(x_{n+m}) = \mathbb{P}(\{X ≤ x_n\} - \{X ≤ x_{n+m}\})$$
$$= \mathbb{P}(\{i ∈ Ω | X_i ≤ x_n ∧ ¬(X_i ≤ x_{n+m})\})$$
$$≤ \mathbb{P}(\{i ∈ Ω | X_i ≤ x_n ∧ ¬(X_i ≤ x)\}). \tag{2.3}$$

Denoting the event \{i ∈ Ω | X(i) ≤ x_n ∧ ¬(X(i) ≤ x)\} with $A_n$, it suffices to prove $i \notin A_n$ for $i ∈ I$ if $n$ is large enough. This is possible by Proposition 2.28 and Lemma 2.32. We will construct a function $f : I → \mathbb{N}$ that will help us define such an $n$. Thereto fix a $j ∈ I$ and observe that, since $x ∈ \mathcal{D}(F_X)$, we may decide whether or not $X_j ≤ x$. If $X_j ≤ x$, define $f(j) = 1$. Otherwise, we deduce from $¬(X_j ≤ x)$ and Markov’s principle that $X_j > x$. Now define $f(j)$ by $f(j) < X_j$. We can find such a number since $x_n \downarrow x$. If we take $n > \max(\{f(j) | j ∈ I\})$, then we know $j \notin A_n$ for $j ∈ I$. This means $F_X(x_n) - F_X(x_{n+m}) < 1/2^k$ for every $m ∈ \mathbb{N}$ and the limit $\lim_{n→∞} F_X(x_n)$ exists. The previous exposition also showed that $\mathbb{P}(\{i ∈ Ω | X_i ≤ x_n ∧ ¬(X_i ≤ x)\}) = F(x_n) - F(x)$ goes to 0 as $n → ∞$. This establishes $\lim_{n→∞} F_X(x_n) = F_X(x)$.

Next we suppose that $T$ is true and derive Markov’s principle from this assumption. Let $α ∈ C$ be such that $¬(∀n[α(n) = 0])$. Define the real number $ρ_α$ by $ρ_α(n) = (−1/n, 1/n)$ if $∀k ≤ n[α(k) = 0]$ and $ρ_α(n) = (1/kα, 1/kα)$ if $∃k ≤ n[α(k) = 1]$. Finally consider the DDPS $(Ω, F, p)$ with $p_i = 1/2^i$. On this probability space we define the stochastic variable $X$ with $X(i) = ρ_α$ if $∀i ≤ n[α(i) = 0]$ and $X(i) = 0$ if $∃i ≤ n[α(i) = 1]$. We first establish $0 ∈ \mathcal{D}(F_X)$ by showing that $\{X ≤ 0\}$ is an event. For any $n ∈ \mathbb{N}$ it is decidable whether $∃k ≤ n[α(k) = 1]$, so we can effectively determine if $n ∈ \{X ≤ 0\}$. Similar reasoning yields that $1/n ∈ \mathcal{D}(F_X)$ for each $n ∈ \mathbb{N}$. By $T$ we know that $\lim_{n→∞} F_X(1/n)$ exists and equals $F(0)$. We shall use these facts to establish $∃n[α(n) = 1]$. If $α(1) = 1$ we are obviously done, so we may assume that $α(1) = 0$. Calculate, by the existence of the limit, an $m ∈ \mathbb{N}$ that is such that $F(1/m) - F(0) < 1/2$. Suppose for the moment that $∀m ≤ n[α(m) = 0]$, then obviously we now also have $ρ_α ≤ 1/m$. This establishes $F(1/m) = 1$ and therefore $F(0) > 1/2$. The latter fact means that we must have $1 ∈ \{X ≤ 0\}$, since $1 \notin \{X ≤ 0\}$ leads to $\{X ≤ 0\} \sim \{i | i > 1\}$ and hence $F(0) ≤ P(\{i | i > 1\}) = 1/2$. But clearly $X(1) = 0$ means $α(1) = 1$. This is a contradiction, so we have established $¬(∀k ≤ m[α(k) = 0])$. Obviously we now know $∃k ≤ m[α(k) = 1]$. □

Lemma 2.47. Let $X$ be a stochastic variable with distribution function $F_X$. If $x$ is a real number apart from $X_i$ for each $i ∈ \mathcal{D}(X)$, then $F_X$ is sequentially right continuous at $x$.

Proof. In the proof of Theorem 2.46 that Markov’s principle implies $T$, we only needed Markov’s principle to derive $¬(X_j ≤ x) → X_j > x$ for every element $j$ of a certain species $I ⊆ \mathcal{D}(X)$. But we can also conclude $X_j > x$ from $∀i ∈ \mathcal{D}(X)[x#X_i]$ and $¬(X_j ≤ x)$. □

Lemma 2.48. Let $X$ be a stochastic variable. If $x, y ∈ \mathcal{D}(F_X)$ satisfy $x ≤ y$, then $F_X(x) ≤ F_X(y)$.

Proof. This immediately follows from Proposition 2.23(iii). □
Lemma 2.49. Let $X$ be a stochastic variable and suppose $(x_n)_n$ is a sequence in $\mathcal{D}(F_X)$. If $\lim_{n \to \infty} x_n = \infty$, then $(F_X(x_n))_n$ converges to 1. Moreover if $\lim_{n \to \infty} x_n = -\infty$, then $(F_X(x_n))_n$ converges to 0.

Proof. We only prove the first claim, as the proof of the second statement is similar. Let $k \in \mathbb{N}$ be given and construct a finite, decidable species $I_k$ such that $\mathbb{P}(I_k) = 1 - 1/2^k$ and $p_i > 0$ for every $i \in I$. Next define $q \in \mathbb{Q}$ as $\max\{X_i(1)^n \mid i \in I_k\}$. Because $\lim_{n \to \infty} x_n = \infty$, we can furthermore find a number $N$ such that $n > N$ implies $q \leq x_n$. Then it is clear that the event $\{X \leq x_n\}$ implies the event $I_k$ and hence $\mathbb{P}(I_k) \leq F_X(x_n)$ for any $n > N$. This proves our claim.

The definition of independence of stochastic variables carries over directly from the classical case. Proving that a subspecies of independent variables also consists of independent stochastic variables is more difficult than it is in the classical case, but nonetheless easy. To relate the notion of independent stochastic variables to independent events however, we introduce a proof technique we have not employed before.

Definition 2.50. Let $X_1, \ldots, X_n$ be stochastic variables. Then we say that the variables $X_1, \ldots, X_n$ are independent if for all real numbers $x_1, \ldots, x_n$ such that the species $\{X_i \leq x_i\}$ are events for $i \in \{1, \ldots, n\}$ we have:

$$\mathbb{P}\left(\bigcap_{i=1}^{n}\{X_i \leq x_i\}\right) = \prod_{i=1}^{n}\mathbb{P}(X_i \leq x_i).$$

Proposition 2.51. If $X_1, \ldots, X_n$ are independent stochastic variables, then every subspecies of $\{X_1, \ldots, X_n\}$ also consists of independent stochastic variables.

Proof. Without loss of generality it suffices to prove that the variables $X_1, \ldots, X_{n-1}$ are independent. By using Lemma 2.45 and Lemma 2.48 we construct a sequence $(x_k)_k \in \mathcal{D}(F_{X_n})$ such that $\lim_{k \to \infty} F_{X_n}(x_k) = 1$. If $\{X_i \leq x_i\}$ is an event for $i = 1, \ldots, n-1$, this implies:

$$\mathbb{P}\left(\bigcap_{i=1}^{n-1}\{X_i \leq x_i\} \cap \{X_n \leq x_k\}\right) = \left(\prod_{i=1}^{n-1}\mathbb{P}(X_i \leq x_i)\right)\mathbb{P}(X_n \leq x_k).$$

Letting $k$ go to infinity on both sides of this equation, we find $\bigcup_{k=1}^{\infty}\left(\bigcap_{i=1}^{n-1}\{X_i \leq x_i\} \cap \{X_n \leq x_k\}\right)$ is an event with probability $\prod_{i=1}^{n-1}\mathbb{P}(X_i \leq x_i)$ by applying Theorem 2.33. Finally it is not hard to show that $\bigcap_{i=1}^{n-1}\{X_i \leq x_i\} \approx \bigcup_{i=1}^{n\infty}\left(\bigcap_{i=1}^{n-1}\{X_i \leq x_i\} \cap \{X_n \leq x_k\}\right)$.

Proposition 2.52. The events $A_1, \ldots, A_n$ are independent if and only if the indicator functions $I_{A_1}, \ldots, I_{A_n}$ are independent.

Proof. Suppose the stochastic variables $I_{A_1}, \ldots, I_{A_n}$ are independent. We deduce the independence of $A_1^c, \ldots, A_n^c$ from $\mathbb{P}\left(\bigcap_{i=1}^{n}\{I_{A_i} \leq 0\}\right) = \mathbb{P}\left(\bigcap_{i=1}^{n}\{I_{A_i} \leq 0\}\right)$. Lemma 2.30 then yields that the events $(A_1)^c, \ldots, (A_n)^c$ are independent as well. Since $\bigcap_{i=1}^{n}(A_i)^c \approx \bigcap_{i=1}^{n}A_i$ and $\mathbb{P}(A_i^c) = \mathbb{P}(A)$ for all events $A$, we see that $A_1, \ldots, A_n$ are independent.

Now assume that the events $A_1, \ldots, A_n$ are independent. Let the real numbers $x_1, \ldots, x_n$ be such that $B_i = \{I_{A_i} \leq x_i\}$ is an event for all $i \in \{1, \ldots, n\}$. We will show $\mathbb{P}\left(\bigcap_{i=1}^{n}B_i\right) = \prod_{i=1}^{n}\mathbb{P}(B_i)$ by proving $\neg(\neg(\bigcap_{i=1}^{n}B_i) = \prod_{i=1}^{n}\neg(\mathbb{P}(B_i))).$ This is sufficient by Proposition 1.52[viii]. So we are to derive a contradiction from $\neg(\neg(\bigcap_{i=1}^{n}B_i) = \prod_{i=1}^{n}\neg(\mathbb{P}(B_i)))$. For all $i = 1, \ldots, n$ define the proposition $P_i$ by $P_i = x_i < 0 \lor 0 \leq x_i \leq 1 < x_i$. Proposition 1.52[xii] states $\neg P_i$ for all $i \in \{1, \ldots, n\}$, which implies $\neg\forall i \in \{1, \ldots, n\} [P_i]$. Under the assumption $\forall i \in \{1, \ldots, n\} [P_i]$ it is easy to derive $\mathbb{P}\left(\bigcap_{i=1}^{n}B_i\right) \approx \prod_{i=1}^{n}\mathbb{P}(B_i)$. For instance, if $n = 2$, $0 \leq x_1 < 1$ and $0 \leq x_2 < 1$, then we obtain $\{A_1 \leq x_1\} \approx A_1^c$ and $\{A_2 \leq x_2\} \approx A_2^c$. This yields $\mathbb{P}\left(\{I_{A_1} \leq x_1\} \cap \{I_{A_2} \leq x_2\}\right) = \mathbb{P}(A_1^c \cap A_2^c)$. From the independence of the events $A_1$ and $A_2$ we subsequently see $\mathbb{P}(A_1^c \cap A_2^c) = \mathbb{P}(A_1^c)^2 = \mathbb{P}(A_1)^2$. Since $\mathbb{P}(\bigcap_{i=1}^{n}B_i) = \prod_{i=1}^{n}\mathbb{P}(B_i)$ is contradictory, we have obtained $\neg\forall i \in \{1, \ldots, n\} [P_i]$. But this is a contradiction and we are done.
2.4 Expected value

In this section we will consider the concept of expected value or expectation. To define this notion, we first need to prove a small lemma. Afterwards, the theory develops smoothly and we prove familiar properties of the function $\mathbb{E}$ without difficulties. Moreover, we derive the inequalities of Markov and Chebyshev. Only when we get to the strong law of large numbers do we require some ingenuity.

**Lemma 2.53.** Let $X$ be a stochastic variable and suppose that $\lim_{k \to \infty} \sum_{i \in \Delta_k} |X_i| p_i$ exists. Then the limit $\lim_{k \to \infty} \sum_{i \in \Delta_k} X_i p_i$ converges as well.

**Proof.** Notice that the sum $\sum_{i \in \Delta_k} |Y_i| p_i$ is well defined for any stochastic variable $Y$. Since $\mathcal{D}(Y)$ is almost full, we immediately see $\Delta_k \subseteq \mathcal{D}(Y)$ for all $k$. From Lemma 2.12 we deduce that $\sum_{i \in \Delta_k} X_i p_i - \sum_{i \in \Delta_k} |X_i| p_i = \sum_{i \in \Delta_k} |X_i| p_i$ for every fixed $n \in \mathbb{N}$. It is evident that $\sum_{i \in \Delta_k} X_i p_i \leq \sum_{i \in \Delta_k} |X_i| p_i$. The latter expression goes to 0 as as $k \to \infty$ because $\lim_{k \to \infty} \sum_{i \in \Delta_k} |X_i| p_i$ exists. This means $\lim_{k \to \infty} \sum_{i \in \Delta_k} X_i p_i$ converges by the Cauchy criterion. \qed

**Definition 2.54.** Let $X$ be a stochastic variable that is such that $\lim_{k \to \infty} \sum_{i \in \Delta_k} |X_i| p_i$ converges. We then define the expected value or expectation $\mathbb{E}(X)$ of $X$ as $\lim_{k \to \infty} \sum_{i \in \Delta_k} X_i p_i$.

**Corollary 2.55.** Let $X$ be a bounded stochastic variable, then $\mathbb{E}(X)$ exists.

**Lemma 2.56.** Let $X$ be a stochastic variable whose expectation exists and let $A$ be an almost full species. Then $\lim_{k \to \infty} \sum_{i \in \Delta_k \cap A} X_i p_i$ exists and equals $\mathbb{E}(X)$.

**Proof.** For every fixed $k \in \mathbb{N}$ we immediately see $\Delta_k \cap A = \Delta_k$ because $A$ is almost full. \qed

**Proposition 2.57.** Let $X$ and $Y$ be stochastic variables whose expectations exist and let $c \in \mathbb{R}$. Moreover suppose $A$ is an event, then:

(i) $\mathbb{E}(X + Y)$ exists and $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$,

(ii) $\mathbb{E}(cX)$ exists and $\mathbb{E}(cX) = c \mathbb{E}(X)$,

(iii) $\mathbb{E}(X) \leq \mathbb{E}(Y)$ if $\{X \leq Y\}$ is almost full,

(iv) $\mathbb{E}(I_A) = \mathbb{P}(A)$.

**Proof.** Fix $k \in \mathbb{N}$, then $\sum_{i \in \Delta_k} |X_i + Y_i| p_i \leq \sum_{i \in \Delta_k} (|X_i| + |Y_i|) p_i = \sum_{i \in \Delta_k} |X_i| p_i + \sum_{i \in \Delta_k} |Y_i| p_i$. Thus it is clear that $\mathbb{E}(X + Y)$ exists. In a similar way we can show $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$. In fact, both (ii) and (iii) follow by reasoning along similar lines. We conclude with a proof of (iv).

Existence of the expectation follows from Corollary 2.55. Fix a natural number $k$. Since $i \in \Delta_k \cap A$ implies $i \in A$, we obtain $\sum_{i \in \Delta_k \cap A} p_i = \sum_{i \in \Delta_k \cap A} I_A(i) p_i$. As $I_A(i)$ is equal to 0 if $i \notin A$, we can also write $\sum_{i \in \Delta_k \cap A} I_A(i) p_i = \sum_{i \in \Delta_k \cap (A \cup A^c)} I_A(i) p_i$. Because the species $A \cup A^c$ is almost full, letting $k$ go to infinity and applying Lemma 2.56 we are done. \qed

The following results pave the way for the laws of large numbers. Especially the proof of Lemma 2.61 seems complicated for such an innocent looking statement. But perhaps an approach with the method of Proposition 2.52 provides an easier route. The proof of the strong law of large numbers also contains unexpected intricacies. Its formulation is identical to the classical variant however.

**Lemma 2.58.** Let $F \subseteq \Omega$ be a finite species and $X$ a stochastic variable whose expectation exists. Then $\mathbb{E}(I_F X)$ exists as well and $\mathbb{E}(I_F X) = \sum_{i \in F} X_i p_i$.

**Proof.** It is clear that $\mathbb{E}(I_F X) \leq \sum_{i \in F} X_i p_i$. Fix an $n \in \mathbb{N}$ and define $M = \max \{|X_i| \mid i \in F\}$. Calculate a $k$ such that $1/2^k \cdot M < 1/2^n$. Then $i \in F \setminus (\Delta_k \cap F)$ implies $p_i < 1/2^k$, which proves that $\sum_{i \in F} X_i p_i - \sum_{i \in \Delta_k} I_F X_i p_i < 1/2^n$. Since $n$ was arbitrary the result has been shown. \qed
Proposition 2.59. Let \((X^n)_n\) be a sequence of stochastic variables whose expectations exist. Then \(X^n \downarrow 0\) implies \(\mathbb{E}(X^n) \downarrow 0\).

Proof. Because of Corollary 2.26 and Lemma 2.56 we may assume that the domain of all variables \(X^n\) is \(\mathcal{P}(X^1)\). Fix an \(m \in \mathbb{N}\) and find a \(k\) such that \(\mathbb{E}(X^1) - \sum_{i \in \Delta_k} X^1_i p_i < 1/2^{m+2}\). Furthermore construct a finite species \(F \subseteq \Delta_k\) such that \(\sum_{i \in \Delta_k} X^1_i p_i - \sum_{i \in F} X^1_i p_i < 1/2^{m+2}\). Finally let \(N \in \mathbb{N}\) be such that \(I_F X^N \leq 1/2^{m+1}\). This yields:

\[
\mathbb{E}(X^N) = \mathbb{E}(I_F X^N) + \mathbb{E}(I_{\overline{F}} X^N) \\
\leq 1/2^{m+1} + \mathbb{E}(I_{\overline{F}} X^1) \\
= 1/2^{m+1} + (\mathbb{E}(X^1) - \mathbb{E}(I_F X^1)) \\
= 1/2^{m+1} + (\mathbb{E}(X^1) - \sum_{i \in F} X^1_i p_i) \\
\leq 1/2^{m+1} + 1/2^{m+2} + 1/2^{m+2} = 1/2^m.
\]

Here we obtained (2.4) and (2.5) from Lemma 2.56. Equality (2.6) follows from the previous lemma. As the number \(m\) was arbitrary, the proof is completed.\(\square\)

Corollary 2.60. Let \((X^n)_n\) be a sequence of stochastic variables whose expectations exist. If \(X\) is a stochastic variable such that \(\mathbb{E}(X)\) exists and \(X_n \uparrow X\), then \(\mathbb{E}(X^n) \uparrow \mathbb{E}(X)\).

Lemma 2.61. Let \(X\) be a stochastic variable whose range only contains natural numbers and suppose \(X\) is bounded by \(N \in \mathbb{N}\). Then the expectation of \(X\) exists and \(\mathbb{E}(X) = \sum_{i=1}^N i \cdot \mathbb{P}(X = i)\).

Proof. Because \(X\) is bounded \(\mathbb{E}(X)\) exists and since \(X\) only takes on natural numbers as values the species \(\{X = i\}\) is an event for all \(i \in \{1, \ldots, N\}\). Define \(X^n\) by \(X^n = I_{\{1, \ldots, n\}} X\), then Corollary 2.60 yields \(\lim_{n \to \infty} \mathbb{E}(X^n) = \mathbb{E}(X)\). Thus we shall prove \(\sum_{i=1}^N i \cdot \mathbb{P}(X = i) = \lim_{n \to \infty} \mathbb{E}(X^n)\). Fix an \(n \in \mathbb{N}\), then we obtain \(\mathbb{E}(X^n) = \sum_{j=1}^n X_j p_j\) from Lemma 2.58. The statement \(X_j = i\) is decidable for all \(j \in \{1, \ldots, n\}\) and \(i \in \{1, \ldots, N\}\) and we only consider finitely many values of \(j\) in the sum \(\sum_{j=1}^n X_j p_j\). We may therefore rewrite \(\sum_{j=1}^n X_j p_j\) to \(\sum_{i=1}^N i \mathbb{P}(\{j \mid X_j = i \land j \leq n\})\). Defining \(A_n = \{j \mid X_j = i \land j \leq n\}\), we are done if we can prove \(\lim_{n \to \infty} \mathbb{P}(A_n) = \mathbb{P}(X = j)\). Note that the latter limit exists because the sequence \(\langle \mathbb{P}(A_n) \rangle_n\) is Cauchy. The difference between \(\mathbb{P}(X = j)\) and \(\mathbb{P}(A_n)\) can be made arbitrarily small by observing that \(\{X = i\} \setminus A_n \sim \{j \mid j > n\}\).\(\square\)

Lemma 2.62 (Markov’s inequality). Let \(X\) be a non-negative stochastic variable whose expectation exists. Moreover let \(a > 0\) be a real number such that \(\{X \geq a\}\) is an event, then \(\mathbb{P}(X \geq a) \leq \mathbb{E}(X)/a\).

Proof. Denoting the event \(\{X \geq a\}\) by \(A\) we obtain:

\[
a \cdot \mathbb{P}(X \geq a) = a \cdot \mathbb{E}(I_A) \\
= a \cdot \lim_{k \to \infty} \sum_{i \in \Delta_k} I_A(i) p_i \\
= \lim_{k \to \infty} \sum_{i \in \Delta_k} aI_A(i) p_i \\
\leq \lim_{k \to \infty} \sum_{i \in \Delta_k} X_i p_i \\
= \mathbb{E}(X).
\]

Here (2.7) is a consequence of the following facts. Both \(X\) and \(I_A\) are defined for all elements of \(\Delta_k\). Moreover, by the definition of \(A\) and non-negativity of \(X\), we obtain \(aI_A(i) \leq X(i)\) for all \(i \in \Delta_k\). Finally (2.8) is a direct application of Lemma 2.56.\(\square\)
Corollary 2.63 (Chebyshev’s inequality). Let $X$ be non-negative stochastic variable and suppose $\mathbb{E}(X^2)$ exists. Let $a > 0$ be such that $\{X \geq a\}$ is an event. Then we have $\mathbb{P}(X \geq a) \leq \mathbb{E}(X^2) / a^2$.

Proof. As $X$ is non-negative, clearly the event $\{X \geq a\}$ exactly equals $\{X^2 \geq a^2\}$. Now use the previous lemma.

Theorem 2.64 (Weak law of large numbers). Suppose $(A_n)_n$ is a sequence of independent events and such that $\mathbb{P}(A_n) = p$ for all $n \in \mathbb{N}$. Define the stochastic variable $S_n$ by $S_n = \sum_{i=1}^n I_{A_i}$. Let $a$ be a positive real number such that $\{|S_n/n - p| > a\}$ is an event for all $n$. Then we have $\lim_{n \to \infty} \mathbb{P}(|S_n/n - p| > a) = 0$.

Proof. Define $B_n = \{|S_n/n - p| > a\}$, then it is quite clear that $B_n$ can also be written as $\{(S_n - np)^2 > n^2a^2\}$. Then $\mathbb{P}(B_n) \leq \mathbb{E}((S_n - np)^2) / (n^2a^2)$ follows from Chebyshev’s inequality, if we can prove that $\mathbb{E}((S_n - np)^2)$ exists. To this end it suffices to prove the existence of $\mathbb{E}(S_n^2)$ and $\mathbb{E}(S_n)$, which follows from Corollary 2.55. Writing $\mathbb{E}((S_n - np)^2)$, we obtain several expectations that are easily calculated. The only exception is the term $\mathbb{E}(S_n^2)$. However, by first applying Lemma 2.61 this problem is reduced to writing out probabilities of the form $\mathbb{P}(\sum_{i=1}^n I_{A_i} = j)$ with $j \in \{0, \ldots, n\}$. Repeated use of Proposition 2.22 yields a sum of terms as $\mathbb{P}\{(i \in A_j \land i \notin A_k \mid j \in J, k \in K\}$, where $J$ and $K$ can range over the decided subspaces of $\{0, \ldots, n\}$ such that $|J| + |K| = n$. From the independency of the events $(A_n)_n$ we see $\mathbb{P}\{(i \in A_j \land i \notin A_k \mid j \in J, k \in K\} = p^{|J|}(1 - p)^{|K|}$ and we subsequently obtain $\mathbb{E}((S_n - np)^2) = np(1 - p)$. Thus we get $\mathbb{P}(B_n) \leq p(1 - p) / (na^2)$ and hence $\lim_{n \to \infty} \mathbb{P}(|S_n/n - p| > a) = 0$.

Theorem 2.65 (Strong law of large numbers). Let $(A_n)_n$ be a sequence of independent events and suppose $\mathbb{P}(A_n) = p$ for all $n \in \mathbb{N}$. Define the stochastic variable $S_n$ by $S_n = \sum_{i=1}^n I_{A_i}$. Then we have $\mathbb{P}\{(i \mid \lim_{n \to \infty} S_n(i)/n = p)\} = 1$.

Proof. Define the sequence of stochastic variables $(X_n)_n$ by $X_n = S_n/n - p$. Construct a sequence $(t_m)_m$ of positive real numbers such that $p \pm t_m \# q$ for every $q \in \mathbb{Q}$ and $\lim_{m \to \infty} t_m = 0$, by using Proposition 1.63. Then because of the properties of $t_m$ the species $A_{n,m} = \{|X_n| > t_m\}$ is an event for all $n, m$. Moreover $\lim_{m \to \infty} \mathbb{P}(A_{n,m}) = 0$ irrespective of $m$ by the weak law of large numbers.

Define the species $A$ by $A = \{i \mid \lim_{n \to \infty} X_n(i) = 0\}$. From the above we wish to conclude that $A$ is an event and $\mathbb{P}(A) = 1$. We can accomplish both these goals if we can show that for every $i \in \Omega$ with $p_i > 0$, we must also have $i \in A$. Since $\lim_{m \to \infty} \mathbb{P}(A_{n,m}) = 0$, we can find an $N_m$ such that $n > N_m$ implies $\mathbb{P}(A_{n,m}) < p_i$. This last statement means $i \notin A_{n,m}$, so $\neg(|X_n(i)| > t_m)$. From Proposition 1.52 we then conclude $|X_n(i)| < 2t_m$ by the positivity of $t_m$. Thus what we have shown is $\forall m \exists N \forall n > N\mid|X_n(i)| < 2t_m$. But this implies $\lim_{m \to \infty} X_n(i) = 0$.

Note how the weak law of large numbers is indeed weaker than the strong law of large numbers, just as in classical mathematics. Let $a \in \mathbb{R}$ be as in Theorem 2.64. Fixing $k \in \mathbb{N}$, it follows from $\mathbb{P}\{(i \mid \lim_{n \to \infty} S_n(i)/n = p)\} = 1$ that we can construct a finite species $F$ with $\mathbb{P}(F) > 1 - 1/2^k$ and $F \subseteq \{i \mid \lim_{n \to \infty} S_n(i)/n = p\}$. For every $i \in F$ there is an $n(i) \in \mathbb{N}$ such that $|S_n(i)/n(i) - p| < a$. Subsequently define $N = \max \{n(i) \mid i \in F\}$, then for every $n \geq N$ we get $\{|S_n/n - p| > a\} \sim F^c$. 

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Chapter 3

Measure theory

3.1 Preliminaries

In this chapter we develop the intuitionistic measure theory on $[0, 1]$ for bounded functions. The presentation is a combination of the expositions in [He66] and [Ve85]. Extensions to $[0, 1]^n$ and $\mathbb{R}^n$ can be made with some effort. How to extend our constructions to unbounded functions is less clear however. Many results of this section and even the definition of the integral itself depend on the boundedness of the functions under investigation. Throughout this chapter, the reader will find definitions and theorems analogous to those of Chapter 2. Examples include Definition 3.22, Proposition 3.45 and Theorem 3.74. We start with defining elementary domains and regions. These are real subspecies of $[0, 1]$ of a particular simple nature. Lemma 3.5 allows us to construct a pre-measure on our regions and elementary domains. This will later be extended to an integral and subsequently a measure on suitable subspecies of $[0, 1]$.

Definition 3.1. An elementary domain is a finite union of rational, closed intervals. The pre-measure $\mu^*(F)$ of an elementary domain $F$ is defined as its area in the ordinary sense, where parts of the continuum contained in $F$ several times are counted once. The interior of an elementary domain is the union of the interiors of its constituent intervals.

Lemma 3.2. Suppose $D$ and $F$ are elementary domains such that $D \subseteq F$, then there is an elementary domain $E$ with $F \setminus D \subseteq E$ and $\mu^*(E) = \mu^*(F) - \mu^*(D)$.

Proof. Trivial. □

Definition 3.3. A region is an infinite union of elementary domains $\bigcup_{n=1}^{\infty} F_n$ such that $F_n$ is a subspecies of the interior of $F_{n+1}$ for all $n$ and $\lim_{m \to \infty} \mu^*(\bigcup_{n=1}^{m} F_n)$ exists.

Example 3.4. Let $]a, b[ \subseteq [0, 1]$ be an inhabited open interval, then $]a, b[$ is a region.

Lemma 3.5. Let $R = \bigcup_{n=1}^{\infty} F_n$ be a region and suppose $F$ is an elementary domain such that $F \subseteq R$. Then there is an $m$ such that $F \subseteq \bigcup_{n=1}^{m} F_n$.

Proof. A straightforward extension of Proposition 1.67 yields that $F$ coincides with a fan $\mathcal{F}_\sigma$. We thus have $\forall x \in \mathcal{F}_\sigma \exists n [x \in F_n]$ and from Proposition 1.38 we deduce $\exists N \forall x \in \mathcal{F}_\sigma \exists n \leq N [x \in F_m]$. This establishes $F \subseteq \bigcup_{n=1}^{N} F_n$. □

Proposition 3.6. Let $\bigcup_{n=1}^{\infty} F_n$ and $\bigcup_{n=1}^{\infty} D_n$ be regions such that $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} D_n$. Then $\lim_{m \to \infty} \mu^*(\bigcup_{n=1}^{m} F_n) = \lim_{m \to \infty} \mu^*(\bigcup_{n=1}^{m} D_n)$.

Proof. We derive $\lim_{m \to \infty} \mu^*(\bigcup_{n=1}^{m} F_n) \leq \lim_{m \to \infty} \mu^*(\bigcup_{n=1}^{m} D_n)$ from $\bigcup_{n=1}^{\infty} F_n \subseteq \bigcup_{n=1}^{\infty} D_n$. Fix an $m \in \mathbb{N}$, then by the previous lemma there is a $k$ such that $\bigcup_{n=1}^{m} F_n \subseteq \bigcup_{n=1}^{k} D_n$. Evidently this means $\mu^*(\bigcup_{n=1}^{m} F_n) \leq \mu^*(\bigcup_{n=1}^{k} D_n) \leq \lim_{k \to \infty} \mu^*(\bigcup_{n=1}^{k} D_n)$.
It is a striking fact that the proof of a proposition as elementary and essential as the previous one depends on both the fan theorem and Brouwer’s continuity principle. Here it is essential that regions consist of closed intervals. By Proposition 3.6 we can extend the pre-measure \( \mu^* \) to regions.

**Definition 3.7.** The pre-measure \( \mu^*(R) \) of a region \( R = \bigcup_{n=1}^{\infty} R_n \) is \( \lim_{n \to \infty} \mu^*(\bigcup_{n=1}^{m} R_n) \).

**Lemma 3.8.** If \( R \) is a region and \( F \) an elementary domain such that \( F \subseteq R \), then \( \mu^*(F) \leq \mu^*(R) \).

*Proof.* This follows directly from Lemma 3.5.

**Proposition 3.9.** Let \( R \) and \( Q \) be regions, then:

(i) \( R \cup Q \) is a region and \( \mu^*(R \cup Q) \leq \mu^*(R) + \mu^*(Q) \),

(ii) \( R \cap Q \) is a region and \( \mu^*(R \cap Q) \leq \mu^*(R) \),

(iii) if \( R \subseteq Q \), then \( \mu^*(R) \leq \mu^*(Q) \).

*Proof.* Trivial.

**Lemma 3.10.** Let \( F \) be a finite domain with \( \mu^*(F) < 1 \) and let \( n \) be an arbitrary natural number. Then there exist regions \( R \) and \( Q \) such that:

(i) \( F \subseteq R \) and \( \mu^*(R) < \mu^*(F) + 1/2^n \),

(ii) \( Q \subseteq F \) and \( \mu^*(F) < \mu^*(Q) + 1/2^n \).

*Proof.* Write \( R = \bigcup_{n=1}^{\infty} F_n \), then from Lemma 3.5 we conclude that \( F \subseteq F_m \) for some \( m \). Subsequently apply Lemma 3.2 and the previous lemma to the sequence of species \( (F_{m+n} \setminus F_n) \).

**Theorem 3.12.** Let \( (R_n) \) be a sequence of regions such that \( \lim_{m \to \infty} \mu^*(\bigcup_{n=1}^{m} R_n) \) exists. Then \( \bigcup_{n=1}^{\infty} R_n \) is a region and \( \mu^*(\bigcup_{n=1}^{\infty} R_n) = \lim_{m \to \infty} \mu^*(\bigcup_{n=1}^{m} R_n) \).

*Proof.* Write \( R_n = \bigcup_{i=1}^{n} K_i^n \), where for all \( n, i \) the species \( K_i^n \) is an elementary domain and \( K_i^n \) is a subspecies of the interior of \( K_i^{n+1} \). Define \( A_n = \bigcup_{i=1}^{n} K_{n+1-i}^i \), then the sequence \( (A_n) \) consists of elementary domains and for all \( n \) the interior of \( A_{n+1} \) contains \( A_n \). Thus we are done if \( \lim_{n \to \infty} \mu^*(A_n) \) exists. It is evident that \( \mu^*(A_n) \leq \lim_{m \to \infty} \mu^*(\bigcup_{k=1}^{m} R_k) \). Fix an arbitrary \( p \in \mathbb{N} \), we shall prove that there is an \( N \) with \( \lim_{m \to \infty} \mu^*(\bigcup_{k=1}^{m} R_k) - \mu^*(A^N) < 1/2^p \). Firstly construct an \( I \in \mathbb{N} \) such that \( \lim_{m \to \infty} \mu^*(\bigcup_{k=1}^{m} R_k) - \mu^*(\bigcup_{k=1}^{I} K^k_k) < 1/2^{p+1} \). As a finite union of regions, the species \( \bigcup_{k=1}^{I} R_k \) is a region. So find \( h \in \mathbb{N} \) such that \( \mu^*(\bigcup_{k=1}^{I} R_k) - \mu^*(\bigcup_{k=1}^{h} K^k_k) < 1/2^{p+1} \). We then see that \( \bigcup_{i=1}^{h} R_i \bigcup_{k=1}^{I} K^k_k \subseteq A^h \), which proves \( \lim_{m \to \infty} \mu^*(\bigcup_{k=1}^{m} R_k) - \mu^*(A^h) < 1/2^p \).

**Corollary 3.13.** Let \( (R_n) \) be a sequence of regions such that \( \sum_{n=1}^{\infty} \mu^*(R_n) \) converges. Then \( \bigcup_{n=1}^{\infty} R_n \) is a region and \( \mu^*(\bigcup_{n=1}^{\infty} R_n) = \sum_{n=1}^{\infty} \mu^*(R_n) \).

**Lemma 3.14.** Let \((F_n)\) be a sequence of elementary domains such that \( \lim_{m \to \infty} \mu^*(\bigcup_{n=1}^{m} F_n) \) exists. Then for all \( k \) there is a region \( R \) such that \( \bigcup_{n=1}^{k} F_n \subseteq R \) and \( \mu^*(R) < \lim_{m \to \infty} \mu^*(\bigcup_{n=1}^{m} F_n) + 1/2^k \).

*Proof.* Fix a \( k \in \mathbb{N} \). For every \( m \in \mathbb{N} \) construct a region \( R_m \) such that \( \bigcup_{n=1}^{m} F_n \subseteq R_m \) and \( \mu^*(\bigcup_{n=1}^{m} F_n) \leq \mu^*(R_m) < \mu^*(\bigcup_{n=1}^{m} F_n) + 1/2^{k+m} \). This is possible by Lemma 3.10. Now apply Theorem 3.12 to \( R = \bigcup_{m=1}^{\infty} R_m \).
Lemma 3.15. Let \( R = \bigcup_{i=n}^{\infty} F_n \) be a region and suppose \( K \) is a rational, closed interval, then
\[
\mu^*\left(\bigcup_{n=1}^{m} F_n \cap K\right) \leq \mu^*\left(\bigcup_{n=1}^{k} F_n \cap K\right) + \mu^*(R) - \mu^*\left(\bigcup_{n=1}^{k} F_n\right) \quad \text{for all} \quad k, m \in \mathbb{N}.
\]

Proof. There is nothing to prove if \( m \leq k \). For every fixed natural number \( m > k \) a tedious calculation yields
\[
\mu^*\left(\bigcup_{n=1}^{m} F_n \cap K\right) \leq \mu^*\left(\bigcup_{n=1}^{k} F_n \cap K\right) + \mu^*(\bigcup_{n=1}^{k} F_n) - \mu^*\left(\bigcup_{n=1}^{k} F_n\right) \quad \text{for all} \quad l > k.
\]
Letting \( l \) go to infinity the result follows. \( \square \)

Next we derive Theorem 3.26 which states that complements of regions can be approximated by fans. The approach we follow is an adaptation of a proof by Veldman [Ve85], which improves upon Brouwer’s original work [Br19]. We first define when a rational, closed interval \( K \) is never-covered by a region \( R \). This term has an intuitive interpretation. If we divide the never-covered interval \( \lambda(K) \) by a region \( \mu(R) \), then at least one of these intervals is also never-covered by \( R \). We subsequently define a fan of sequences of nested, never-covered intervals whose lengths go to zero. Such a sequence \( x \) represents a real number whose segments are never-covered by \( R \), so \( x \notin R \).

Definition 3.16. Let \( R = \bigcup_{i=n}^{\infty} F_n \) be a region and \( K \) a rational, closed interval with \( \mu^*(K) > 0 \). Define \( N(K, R) \) by
\[
N(K, R) = \mu(m)\left[\mu^*(R) - \mu^*\left(\bigcup_{i=n}^{m} F_n \cap K\right)\right] < 1/2 \cdot \mu^*(K).
\]
If the elementary domain \( \bigcup_{i=n}^{N(K, R)} F_n \) covers at least half of \( K \), then we call \( K \) quickly-half-covered by \( R \).

Definition 3.17. Let \( R = \bigcup_{i=n}^{\infty} F_n \) be a region and \( K \) a rational, closed interval with \( \mu^*(K) > 0 \). Suppose there is a \( p \in \mathbb{N} \) such that
\[
\mu^*\left(\bigcup_{i=n}^{N(K, R)} F_n \cap K\right) < (1 - 1/2^p) \cdot \mu^*(K) \quad \text{for all} \quad m.
\]
Then we say \( K \) is never-covered by \( R \).

Lemma 3.18. Suppose \( R = \bigcup_{i=n}^{\infty} F_n \) is a region and \( K \) a rational, closed interval that is not quickly-half-covered by \( R \). Then \( K \) is never-covered by \( R \).

Proof. We can find a \( p \) such that
\[
\mu^*\left(\bigcup_{i=n}^{N(K, R)} F_n \cap K\right) < (1/2 - 1/2^p) \cdot \mu^*(K), \quad \text{because} \quad K \text{ is not quickly-half-covered}.
\]
An application of Lemma 3.15 completes the proof. \( \square \)

Proposition 3.19. Let \( R \) be a region and suppose the closed, rational interval \( K \) is never-covered by \( R \). Then at least one of the intervals \( \lambda(K) \) and \( \rho(K) \) is also never-covered by \( R \).

Proof. The fact that \( K \) is never-covered by \( R = \bigcup_{i=n}^{\infty} F_n \) implies that \( \mu^*(K) > 0 \). Suppose \( p \in \mathbb{N} \) is such that
\[
\mu^*\left(\bigcup_{i=n}^{m} F_n \cap K\right) < (1 - 1/2^p) \mu^*(K) \quad \text{for all} \quad m \in \mathbb{N}.
\]
Find a natural number \( k \) with
\[
\mu^*(R) - \mu^*\left(\bigcup_{i=n}^{k} F_n\right) < 1/2^{p+2} \mu^*(K).
\]
Consider which one of the intervals \( \lambda(K) \) and \( \rho(K) \) is the least covered by \( \bigcup_{i=n}^{k} F_n \). Without loss of generality assume it is \( L = \lambda(K) \). This yields for \( m \geq k \):
\[
\mu^*\left(\bigcup_{i=n}^{m} F_n \cap L\right) \leq \mu^*\left(\bigcup_{i=n}^{k} F_n \cap L\right) + \mu^*(R) - \mu^*\left(\bigcup_{i=n}^{k} F_n\right) \quad \text{(3.1)}
\]
\[
< \mu^*\left(\bigcup_{i=n}^{k} F_n \cap L\right) + 1/2^{p+2} \mu^*(K) \quad \text{(3.2)}
\]
\[
\leq 1/2 \cdot \mu^*\left(\bigcup_{i=n}^{k} F_n \cap K\right) + 1/2^{p+2} \mu^*(K) \quad \text{(3.3)}
\]
\[
\leq 1/2 \cdot \mu^*\left(\bigcup_{i=n}^{m} F_n \cap K\right) + 1/2^{p+2} \mu^*(K) \quad \text{(3.3)}
\]
\[
\leq 1/2 \cdot (1 - 1/2^p) \cdot \mu^*(K) + 1/2^{p+1} \mu^*(L) \quad \text{(3.4)}
\]
\[
= (1 - 1/2^{p+1}) \mu^*(L).
\]
Here (3.1) follows from Lemma 3.15 and our construction of \( k \) guarantees (3.2). Subsequently we derive (3.3) from the fact that \( \lambda(K) \) is the interval least covered by \( \bigcup_{i=n}^{k} F_n \) of \( \lambda(K_m) \) and \( \rho(K_m) \). Finally (3.4) is a consequence of the fact that \( K \) is never-covered by \( R \) and also \( L = \lambda(K) \). \( \square \)
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Proposition 3.20. Let $R$ be a region with $\mu^*(R) < 1$, then $R^c$ is inhabited.

Proof. Find a $p$ such that $\mu^*(R) < 1 - 1/2^p$. We see that the interval $[0, 1]$ is never-covered by $R$. Using Proposition 3.19 construct a sequence of rational, closed intervals $(K_n)_n$ satisfying the following requirements:

(i) $K_{n+1} \subseteq K_n$,
(ii) $\mu^*(K_n) = 1/2^{n-1}$,
(iii) for all $n$ there is a $p$ such that $\mu^*(R \cap K_n) \leq (1 - 1/2^p)\mu^*(K_n)$.

Define $x_n = \zeta(K_n)$, then $x = (x_n)_n$ is a real number. If we assume $x \in R = \bigcup_{n=1}^\infty D_n$, then there must be an $n \in \mathbb{N}$ such that $x \in D_m$. This means that $x$ is an element of the interior of $D_{m+1}$, so there is an open interval $I \subseteq R$ with $x \in I$. Thus we can find a $k$ satisfying $I^l < x(k)^l \leq x(k)^r < I^r$ and hence $K_k \subseteq R$. This contradicts (iii) so $x \in R^c$. □

Lemma 3.21. Let $F$ be an elementary domain and $R$ a region such that $\mu^*(F) > \mu^*(R)$. Then $F \setminus R$ is inhabited.

Proof. Suppose $F = \bigcup_{n=1}^m K_n$, where $K_n$ is a closed rational interval for each $n = 1, \ldots, m$. Without loss of generality assume the intersection of two intervals $K_i$ and $K_j$ with $i$ and $j$ unequal are all empty or consist of a single rational number. Moreover, let $R = \bigcup_{i=1}^\infty D_i$. Calculate a rational number $q_1$ satisfying $0 < q_1 \leq (\mu^*(F) - \mu^*(R \cap F))/m$ and let $q_2 \in \mathbb{Q}$ be such that $0 < q_2 < q_1$. For each $n \in \{1, \ldots, m\}$, there is a $k(n)$ such that $\mu^*(R \cap K_n) - \mu^*\left(\bigcup_{i=1}^{k(n)} D_i \cap K_n\right) < q_2$. Subsequently we define $k = \max\{\{k(n) \mid n = 1, \ldots, m\}\}$. Write $a_n = \mu^*(K_n) - \mu^*\left(\bigcup_{i=1}^{k} D_i \cap K_n\right)$ for every $n \in \{1, \ldots, m\}$, so $a_n \in \mathbb{Q}$ irrespective of $n$. Because of our assumption on the intervals $K_n$ we see $\sum_{n=1}^m a_n = \mu^*(F) - \mu^*\left(\bigcup_{i=1}^{k} D_i \cap F\right) \geq \mu^*(F) - \mu^*(R \cap F) > q_1$. This implies that there is a $j \in \{1, \ldots, m\}$ with $a_j \geq q_1$, so $\mu^*(K_j) - \mu^*\left(\bigcup_{i=1}^k D_i \cap K_j\right) \geq q_1$. We have reduced our problem to the case $m = 1$, because we now know $\mu^*(K_j) - \mu^*(R \cap K_j) \geq q_1 - q_2 > 0$. But inhabitedness of $K_j \setminus (R \cap K_j)$ can be derived from Proposition 3.20 after some shifting and multiplication. □

Definition 3.22. Let $A$ be a real subspecies of $[0, 1]$ such that there is a sequence of regions $(R_n)_n$ with $\lim_{n \to \infty} \mu^*(R_n) = 0$ and $R_n^c \subseteq A$ for all $n$. Then we call $A$ almost full or almost full by $(R_n)_n$.

Lemma 3.23. Suppose $A$ and $B$ are almost full species and let $C \subseteq [0, 1]$ be such that $A \subseteq C$. Then the following species are also almost full:

(i) $C$,
(ii) $A \cap B$.

Proof. The first two statement is trivial. If $A$ is almost full by $(R_n)_n$ and $B$ by $(S_n)_n$, then $A \cap B$ is almost full by $(T_n \cup S_n)_n$. □

Lemma 3.24. The positively irrational numbers constitute an almost full species.

Proof. Let $k$ be an arbitrary natural number and let $e : \mathbb{N} \to \mathbb{Q}$ be a bijection. Define the region $R^k_i$ by $R^k_i = (e(i) - 1/2^{k+i+1}, e(i) + 1/2^{k+i+1})$. Now $(R^k_i)_i$ is a sequence of regions such that $\sum_{i=1}^\infty \mu^*(R^k_i) = 1/2^k$. Hence $R_k = \bigcup_{i=1}^\infty R^k_i$ is a region and clearly $x \in R^k_k$ implies $x \not\in \mathbb{Q}$ for all $x \in [0, 1]$. Letting $k$ vary, we see that the positively irrational numbers are almost full by $(R_k)_k$. □

Definition 3.25. The $n$th order dyadic partition $\mathcal{D}_n$ is the partition $(0, 1/2^n, \ldots, (2^n - 1)/2^n, 1)$. We define the dyadic fan-law $\partial : \mathbb{N} \to [0, 1]$ by $\partial(t) = 1$ if and only if $t_1 = [0, 1]$ and $t_{n+1} = \lambda(t_n)$ or $t_{n+1} = \rho(t_n)$ for all $n = 1, \ldots, l(t) - 1$. 

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Theorem 3.26. Let $R$ be a region with $\mu^*(R) < 1/2^{p+3}$ for some $p \in \mathbb{N}$. Then there exists a region $U$ with $\mu^*(U) < 1/2^p$ and a fan-law $\sigma$ such that $U^c \subseteq C(F_\sigma) \subseteq R^c$.

Proof. Define the fan-law $\tau : \mathbb{I}^* \to \{0, 1\}$ as follows. A sequence of closed, rational intervals $J$ is admitted by $\sigma$ if and only if $\partial(J) = 1$ and $J_i$ is not quickly-half-covered by $R$ for all $i \in \{1, \ldots, |J|\}$. Since the interval $[0, 1]$ is never-covered by $R$, Proposition 3.19 guarantees that this indeed defines a fan-law.

Consider the species $O_n = \{K \in \mathcal{S}(D_n) \mid \exists J \in \mathbb{I}^{n-1}[\sigma(J) = 1 \land \sigma(J \ast (K)) = 0]\}$. From the definition of $\sigma$ it is clear that any closed, rational interval $K \in O_n$ is quickly-half-covered by $R$. Write $W_n = \bigcup_{m=1}^n O_n$ and define the elementary domain $W_n$ by $W_n = \bigcup_{K \in W_n} K$. We will show that $\lim_{n \to \infty} \mu^*(W_n)$ converges. Fix $k \in \mathbb{N}$, write $R = \bigcup_{n=1}^\infty F_n$ and construct a natural number $N$ such that $\mu^*(R) - \mu^*(\bigcup_{n=1}^N F_n) < 1/2^{k+3}$. Since $\lim_{n \to \infty} d_n(\mathcal{D}_n) = 0$, we can find an $M \in \mathbb{N}$ such that the species $Y_n = \{K \in \mathcal{S}(D_n) \mid K \subseteq \bigcup_{n=1}^N F_n\}$ satisfies $\mu^*(\bigcup_{n=1}^N F_n) - \mu^*(Y_n) < 1/2^{k+3}$ whenever $n \geq M$. Finally find a number $P$ such that $Y_M \subseteq W_P$. This can be done, because any interval in $Y_M$ is completely covered by $R$ and hence is also contained in a quickly-half-covered slice of $D_n$ for some $n$. Evidently this also yields $Y_M \subseteq R$ and $\mu^*(R) - \mu^*(Y_M) < 1/2^{k+2}$. Apply Lemma 3.11 to obtain a region $V$ such that $R \setminus Y_M \subseteq V$ and $\mu^*(V) < 1/2^{k+1}$. Fix an $l \in \mathbb{N}$ and consider the difference $\mu^*(W_{P+l}) - \mu^*(W_P)$. Any interval in $K \in O_{P+l}$ which does not satisfy $K \subseteq Y_M$, must be quickly-half-covered by the region $V$. This holds because our construction of $\sigma$ implies that $K$ and $Y_M$ are now disjoint and we already know that $K$ is quickly-half-covered by $R$. This means that $\mu^*(W_{P+l}) - \mu^*(W_P) \leq 2\mu^*(V) = 1/2^k$. We have shown that $((\mu^*(W_n))_n$ is a Cauchy sequence.

The species $W_n$ is a union of quickly-half-covered intervals irrespective of $n \in \mathbb{N}$, so $\lim_{n \to \infty} \mu^*(W_n) \leq 2\mu^*(R) = 1/2^{p+2}$ Next find a region $S$ such that $\bigcup_{n=1}^\infty W_n \subseteq S$ and $\mu^*(S) < 1/2^{p+1}$ by using Lemma 3.14. Furthermore construct the region $Q$ in such a way that $\mu^*(Q) < 1/2^{p+1}$ and $x \in Q^c \to x \# Q$. Such a region exists because the positively irrational numbers are almost full. This can be done because the positively irrational numbers are almost full. Define $U = S \cup Q$, then $U$ is a region with $\mu^*(U) < 1/2^p$. Finally define the fan-law $\sigma : S^* \to \{0, 1\}$ by $\sigma(s) = 1$ if and only if $\tau(\zeta(s)) = 1$.

We finally prove $U^c \subseteq C(F_\sigma) \subseteq R^c$. If $x \in [0, 1]$ is an element of $U^c$, then in particular $x \in Q^c$. Hence for every $n \in \mathbb{N}$ and $K \in \mathcal{S}(D_n)$, we can determine $x \in K \land x \notin K$. Additionally we know $x \notin S$. This means that for any $n$, the interval $\iota(x(n))$ is not quickly-half-covered by $R$. Thus $\iota(x(n))$ is contained in some never-covered interval $K \in \mathcal{S}(D_n)$ for some $m$. Evidently $x(n) \subseteq x(n+1)$ for all $n \in \mathbb{N}$ and this allows us to construct a sequence of rational segments $(y(n))_n$ with $y \in F_\sigma$ and $x = y$. This establishes $U^c \subseteq C(F_\sigma)$. Similarly, if $x \in C(F_\sigma)$ then it is contradictory that $x \in R$. This latter statement implies the existence of an $n$ such that $x \in F_n$. Thus $x$ is an element of the interior of $F_{n+1}$, so there is an $m$ with $\iota(x(m)) \subseteq F_{n+1}$. Then the interval $\iota(x(m))$ is completely covered by $R$, contradicting $x \in C(F_\sigma)$.

Corollary 3.27. Let $A$ be an almost full species, then for every $n \in \mathbb{N}$ there is a region $R$ and a fan-law $\sigma : S^* \to \{0, 1\}$ with $\mu^*(R) < 1/2^p$ and $R^c \subseteq C(F_\sigma) \subseteq A$. \qed
3.2 Measurable functions and species

The following collection of definitions are needed to define the Brouwer-Lebesgue integral. The reader may wish to return to Section 1.3 to refresh some definitions. These will be used throughout the remainder of this chapter.

**Definition 3.28.** A **tile** $t$ is a product of rational closed intervals $[a, b] \times [c, d]$ such that $0 \leq a < b \leq 1$ and $c < d$. We define the following functions on tiles:

(i) $\pi_x : \mathbb{I}^2 \to \mathbb{I}$ by $\pi_x([a, b] \times [c, d]) = [a, b]$, the **projected interval** of $t$,

(ii) $\pi_y : \mathbb{I}^2 \to \mathbb{I}$ by $\pi_y([a, b] \times [c, d]) = [c, d]$,

(iii) $w : \mathbb{I}^2 \to \mathbb{I}$ by $w(t) = \mu^x(\pi_x(t))$, the **width** of $t$,

(iv) $h : \mathbb{I}^2 \to \mathbb{I}$ by $h(t) = \mu^y(\pi_y(t))$, the **height** of $t$,

(v) $A : \mathbb{I}^2 \to \mathbb{Q}$ by $A(t) = w(t)h(t)$, the **area** of the tile $t$,

(vi) $A^u : \mathbb{I}^2 \to \mathbb{Q}$ by $A^u(t) = w(t)\pi_y(t)^r$, the **upper area** of the tile $t$,

(vii) $A^l : \mathbb{I}^2 \to \mathbb{Q}$ by $A^l(t) = w(t)\pi_y(t)^l$, the **lower area** of the tile $t$.

Two tiles $s$ and $t$ are **compatible** if $\pi_x(s) = \pi_x(t)$. If $s \cap t$ is inhabited, then $s$ and $t$ overlap.

**Definition 3.29.** Let $s = [a_1, a_2] \times [b_1, b_2]$ and $t = [a_1, a_2] \times [c_1, c_2]$ be compatible tiles and let $q$ be a positive rational number, then we define the following tiles:

(i) $q \cdot s = [a_1, a_2] \times [q \cdot b_1, q \cdot b_2]$,

(ii) $s + t = [a_1, a_2] \times [b_1 + c_1, b_2 + c_2]$.

**Definition 3.30.** Let $U$ be a finite species of tiles. Then the **projected domain** of $U$ is the unique elementary domain $\pi_x(U)$ such that $x \in \pi_x(U)$ if and only if there is a tile $[a, b] \times [c, d] \in U$ such that $x \in [a, b]$. The **projected length** $w(U)$ of $U$ is defined to be $\mu^x(\pi_x(U))$.

**Definition 3.31.** A **(horizontal) band** $B$ is a rational, closed interval. Let $B$ be a band and $t$ a tile, then we define:

(i) $t$ lies above $B$ if $\pi_y(t)^l > B^r$,

(ii) $t$ lies below $B$ if $\pi_y(t)^r < B^l$,

(iii) $t$ is decided for $B$ if $t$ lies above $B$ or below $B$,

(iv) $t$ is undecided for $B$ if $t$ neither lies above nor below $B$.

**Definition 3.32.** A **tiling** $T$ is a finite sequence $(T_1, ..., T_{l(T)})$ of tiles such that $\pi_x(T_1)^l = 0$, $\pi_x(T_{l(T)})^r = 1$ and $\pi_x(T_n)^r = \pi_x(T_{n+1})^l$ for every $n = 1, ..., l(T) - 1$. The species of all tilings is denoted by $T$. We define the following functions:

(i) $A : T \to \mathbb{R}$ by $A(T) = \sum_{i=1}^{l(T)} A(T_i)$, the **area** of $T$,

(ii) $A^u : T \to \mathbb{R}$ by $A^u(T) = \sum_{i=1}^{l(T)} A^u(T_i)$, the **upper area** of $T$,

(iii) $A^l : T \to \mathbb{R}$ by $A^l(T) = \sum_{i=1}^{l(T)} A^l(T_i)$, the **lower area** of $T$.

If $M \in \mathbb{N}$, then a tiling $T$ is $M$-**bounded** if every tile $t \in T$ satisfies $-M \leq \pi_y(t)^l \leq \pi_y(t)^r \leq M$. 
Definition 3.33. Let $T^1, T^2 \in \mathbb{T}$, then we say that $T^1$ and $T^2$ are compatible if $l(T^1) = l(T^2)$ and for every tile of $T^1$ there is a compatible tile in $T^2$.

Definition 3.34. Let $S, T \in \mathbb{T}$ be compatible tilings and let $q$ be a rational number. We then define the following tilings:

(i) $q \cdot S$ is defined by $(q \cdot S)_i = q \cdot S_i$ for $i = 1, \ldots, l(S)$,

(ii) $S + T$ is defined by $(S + T)_i = S_i + T_i$ for $i = 1, \ldots, l(S)$.

Lemma 3.35. Let $T$ be a tiling, then $A(T) = A^u(T) - A^l(T)$.

Proof. Trivial.

Definition 3.36. A point is a pair of real numbers. A point $(x, y)$ is captured or covered by a tile if $(x, y) \in t$. A tiling $T$ captures or covers a point $(x, y)$ if there is a tile in $T$ capturing $(x, y)$ and we write $(x, y) \in T$.

Definition 3.37. Let $f : [0, 1] \to \infty$ be a partial, extensional and bounded function and let $A$ be a subspecies of $\mathcal{D}(f)$. If $T$ is a tiling, then $T$ captures or covers $f$ on $A$ if for every $a \in A$ the point $(a, f(a))$ is captured by some tile in $T$.

This brings us to the construction of the Brouwer-Lebesgue integral. While the definition of an integrable function contains little surprises, proving that every integrable function has a unique integral takes some effort. But here the work we did in the previous section pays off and we obtain the desired result Proposition 3.40 fairly quickly.

Definition 3.38. Let $f$ be a partial, extensional function from $[0, 1]$ to $\mathbb{R}$ such that $\mathcal{D}(f)$ is almost full. Moreover assume $f$ is bounded, so $|f(x)| < M$ for some $M \in \mathbb{N}$. Then we call $f$ a Brouwer-Lebesgue integrable or measurable if:

(i) there exists a sequence of regions $(R_n)_n$ with $\lim_{n \to \infty} \mu^*(R_n) = 0$,

(ii) there exists a sequence of $M$-bounded tilings $(T_n)_n$ with $\lim_{n \to \infty} A(T_n) = 0$,

(iii) if $x \in R_n$, then $x \in \mathcal{D}(f)$ and the tiling $T_n$ captures the point $(x, f(x))$.

We will also call $f$ measurable by the sequences $(R_n)_n$ and $(T_n)_n$.

Suppose $f$ is Brouwer-Lebesgue integrable by the regions $(R_n)_k$ and tilings $(T_n)_n$. Fix a $k$ and consider $R_k$, $T_k$ and the tile $[a, b] \times [c, d] \in T_k$. It is important to realize that Definition 3.38 often implies that $R_k$ contains an interval $(p, q)$ with $p < a < q$. Since we can not decide $\alpha + \rho < \alpha$, $\alpha + \rho' = \alpha$ or $\alpha + \rho' > \alpha$, we can not be sure that $(\alpha + \rho', \alpha + \rho') \in T_k$. This problem can be alleviated by letting $R_k$ contain a sufficiently small, open, rational interval containing $\alpha$. As there are only finitely many endpoints of tiles of a given tiling, this imposes no difficult restrictions.

Proposition 3.39. Let $f$ be a Brouwer-Lebesgue integrable function that is measurable by the regions $(R_n)_n$ and tilings $(T_n)_n$. Then the limits $\lim_{n \to \infty} A^u(T_n)$ and $\lim_{n \to \infty} A^l(T_n)$ exist and $\lim_{n \to \infty} A^u(T_n) = \lim_{k \to \infty} A^l(T_n)$.

Proof. In view of Lemma 3.35 it suffices to prove that $\lim_{n \to \infty} A^u(T_n)$ exists. Suppose $|f(x)| \leq M$ for all $x \in \mathcal{D}(f)$. Let $p \in \mathbb{N}$ be given, we have to construct a $k$ such that $|A^u(T_k) - A^u(T_{k+m})| < 1/2^p$ for every $m$. Define the regions $S_{k,m}$ by $S_{k,m} = R_k \cup R_{k+m}$. As $\lim_{n \to \infty} \mu^*(R_n) = 0$, we can find an $l$ such that $\mu^*(S_{l,m}) < 1/(M \cdot 2^{l+2})$ irrespective of $m$. Because we also know $\lim_{n \to \infty} A(T_n) = 0$, we may additionally assume that $A(T^l) + A(T^{l+m}) < 1/2^{n+1}$ whatever $m$ may be. Fix such an $l$ and let $m \in \mathbb{N}$ be arbitrary.
Without loss of generality assume that \( T_l \) and \( T_{l+m} \) are compatible. Let \( U_1 \) be the species of tiles of \( T_l \) not overlapping with their compatible tiles of \( T_{l+m} \). We claim \( \mu^*(\pi_x(U_1)) \leq 1/(M \cdot 2^{p+2}) \). For suppose that \( \mu^*(\pi_x(U_1)) > 1/(M \cdot 2^{p+2}) \) were true. In this case \( \pi_x(U_1) \) \( \setminus \) \( S_{l,m} \) is inhabited by Lemma 3.21. Let \( x \) be an element of \( \pi_x(U_1) \) \( \setminus \) \( S_{p,m} \), then there are compatible tiles \( t_1 \in T_l \) and \( t_2 \in T_{l+m} \) such that \( x \in \pi_x(t_1) = \pi_x(t_2) \). From \( z \notin S_{l,m} \) we deduce that \( (x,f(x)) \in t_1 \) and \( (x,f(x)) \in t_2 \). But this clearly contradicts the fact that \( t_1 \) and \( t_2 \) do not overlap. Let \( U_2 \) be the species of tiles of \( T_{l+m} \) not overlapping with their compatible tiles of \( T_l \). We can then write:

\[
|A^u(T_l) - A^u(T_{l+m})| \leq \left| \sum_{t \in U_1^c} A^u(t) - \sum_{t \in U_2^c} A^u(t) \right| + \left| \sum_{t \in U_1} A^u(t) - \sum_{t \in U_2} A^u(t) \right| \\
\leq (A(T_l) + A(T_{l+m})) + \left| \sum_{t \in U_1} A^u(t) - \sum_{t \in U_2} A^u(t) \right| \\
\leq 1/2^{n+1} + 2M \cdot \pi_x(U_1) \tag{3.5}
\]

(3.5) follows because for every \( s_1 \in U_1^c \), there is a corresponding compatible \( s_2 \in U_2^c \) and vice versa. The construction of \( U_1 \) and \( U_2 \) ensures that \( s_1 \) and \( s_2 \) overlap and consequently we obtain \( |A^u(s_1) - A^u(s_2)| \leq A(s_1) + A(s_2) \). If however \( r_1 \in U_1 \) and \( r_2 \in U_2 \) are two compatible tiles, then we know that \( r_1 \) and \( r_2 \) do not overlap. Subsequently the best upper bound on \( |A^u(r_1) - A^u(r_2)| \) we can give is \( 2M \). This explains (3.6) and our bound on \( \pi_x(U_1) \) proves (3.7).

**Proposition 3.40.** Let \( f \) be a Brouwer-Lebesgue integrable function that is measurable via the sequence of regions \((R_n)_n\) and tilings \((T_n)_n\), but also by the region sequence \((Q_n)_n\) and tilings \((V_n)_n\). Then \( \lim_{n \to \infty} A^u(T_n) = \lim_{n \to \infty} A^u(V_n) \).

_Proof._ The proof is similar to the proof of Proposition 3.39.

**Definition 3.41.** Let \( f \) be a Brouwer-Lebesgue integrable function that is measurable by the regions \((R_n)_n\) and tilings \((T_n)_n\). Then the Brouwer-Lebesgue integral \( \int f \, d\mu \) or \( \int f(x) \mu(dx) \) of \( f \) is defined by \( \lim_{n \to \infty} A^u(T_n) \).

The notation \( \int f \, d\mu \) suggests the existence of a measure \( \mu \), but we defer the introduction of this object to Definition 3.54. At the end of this section the reader can find the consistency result linking \( \mu^* \) and \( \mu \) in the form of Corollary 3.59. First we must build up the theory of the Brouwer-Lebesgue integral however. As expected the integrable functions form a vector space and the integral is linear. We also derive the strong results Theorem 3.43 and Corollary 3.44 first proved in [vR54].

**Lemma 3.42.** Let \( f \) be a measurable function. If \( \{x \in [0,1] \mid |f(x)| \leq M\} \) is almost full for some \( M \in \mathbb{Q} \), then \( |\int f \, d\mu| \leq M \).

_Proof._ Let the species \( \{x \in [0,1] \mid |f(x)| \leq M\} \) be almost full by the regions \((R_n)_n\). Suppose \( f \) is measurable by the region sequence \((S_n)_n\) and tilings \((T_n)_n\). Then \( f \) is also measurable by \((R_n \cup S_n)_n\) and \((V_n)_n\), where \( V_n \) is obtained from \( T_n \) by replacing every tile \([a,b] \times [c,d] \in T_n\) by \([a,b] \times [\max(c,-M), \min(d,M)]\). This means \(-M \leq A^u(V_n) \leq M \) for every \( n \).

**Theorem 3.43.** Let \( f \) be a partial, extensional and bounded function from \([0,1]\) to \( \mathbb{R} \) and suppose that \( \mathcal{D}(f) \) is almost full. Then \( f \) is Brouwer-Lebesgue integrable.

_Proof._ Let the natural number \( n \) be given, we shall construct a region \( R \) with \( \mu^*(R) < 1/2^n \) and a tiling \( T \) with \( A(T) < 1/2^n \) capturing \( f \) on \( R^n \). We will use the fact that \( \mathcal{D}(f) \) is almost full. Employ Corollary 3.27 to construct a region \( S \) with \( \mu^*(S) < 1/2^n \) such that \( S^c \) is a subspecies of the fan \( \mathcal{F}_\sigma \) and \( \mathcal{C}(\mathcal{F}_\sigma) \subseteq \mathcal{D}(f) \). Thus we have \( \forall x \in \mathcal{F}_\sigma \exists y \in \mathbb{R} \, |f(x) = y| \). From the proof of Theorem 1.84 we see that \( f \) is uniformly continuous on \( \mathcal{C}(\mathcal{F}_\sigma) \). It is then evident that we can construct a tiling of arbitrarily small area capturing \( f \) on \( S^c \).
Proposition 3.44. Let $f_n$ be a sequence of measurable and uniformly bounded functions. If the species $\{x \in [0,1] \mid \lim_{n \to \infty} f_n(x) = 0\}$ is almost full, then $\lim_{n \to \infty} \int f_n(x) \mu(dx) = 0$.

Proof. Define $A = \{x \in [0,1] \mid \lim_{n \to \infty} f_n(x) = 0\}$. Fix $k \in \mathbb{N}$ and construct a region $R$ with $\mu^*(R) < 1/2^k$ and such that there is a fan-law $\sigma$ with $R^c \subseteq \mathcal{C}(\mathcal{F}_\sigma) \subseteq A$. Then we know that $\forall x \in \mathcal{F}_\sigma \exists N \forall n > N[|f_n(x)| < 1/2^k]$. As is to be expected we apply Proposition 1.38 to deduce $\exists M \forall x \in \mathcal{F}_\sigma \exists N \leq M \forall n > M[|f_n(x)| < 1/2^k]$, so $\exists M \forall x \in \mathcal{F}_\sigma \forall n > M[|f_n(x)| < 1/2^k]$. The sequence $(f_n)_n$ converges to 0 uniformly on $\mathcal{F}_\sigma$. Now use Lemma 3.42.

Proposition 3.45. Let $f$ and $g$ be Brouwer-Lebesgue integrable functions, then:

(i) $\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$.

(ii) $\int f \, d\mu \leq \int g \, d\mu$ if $\{x \in [0,1] \mid f(x) \leq g(x)\}$ is almost full,

(iii) $\int f \, d\mu = \int g \, d\mu$ if $\{x \in [0,1] \mid f(x) = g(x)\}$ is almost full.

Proof. Suppose $f$ is measurable by $(R_n)_n$ and $(T_n)_n$ and $g$ by $(S_n)_n$ and $(V_n)_n$. Then it is easy to see that $f + g$ is measurable by $(R_n \cup S_n)_n$ and $(T_n + V_n)_n$, proving (i). Next we consider statement (ii). Because of the previous result we can also show $\int f - g \, d\mu \leq 0$. But this latter fact follows from Lemma 3.42. Finally (iii) is a consequence of (ii) because both $\{x \in [0,1] \mid f(x) \leq g(x)\}$ and $\{x \in [0,1] \mid g(x) \leq f(x)\}$ are almost full in this case.

Corollary 3.46. Let $f_n$ be a sequence of measurable, uniformly bounded functions and suppose $f : [0,1] \to \mathbb{R}$ is a bounded, partial function. Suppose the species $\{x \in [0,1] \mid \lim_{n \to \infty} f_n(x) = f(x)\}$ is almost full. Then $f$ is measurable and $\lim_{n \to \infty} \int f_n(x) \mu(dx) = \int f(x) \mu(dx)$.

Proof. Combine Proposition 3.45 with Proposition 3.44 and Proposition 3.43.

Proposition 3.47. Let $f$ be a measurable function and let $a \in \mathbb{R}$, then $\int a f \, d\mu = a \int f \, d\mu$.

Proof. Let $f$ be measurable by the regions $(R_n)_n$ and tilings $(T_n)_n$. For every rational number $q$ it is clear that $\int q f \, d\mu = q \int f d\mu$, because $q f$ is measurable by $(R_n)_n$ and $(q \cdot T_n)_n$. For general $a \in \mathbb{R}$, construct a sequence $(q_n)_n \in \mathbb{Q}$ converging to $a$ and use Corollary 3.46.

This concludes our exposition of the Brouwer-Lebesgue integral. The next topic we discuss is the Brouwer-Lebesgue measure derived from this integral. Most of the theory is straightforward, but along the way will have to prove that every measurable species can be approximated by regions ‘from the outside’. We make this precise in Proposition 3.56. This result on the regularity of the Brouwer-Lebesgue measure is derived from the fundamental Corollary 3.27. In Theorem 3.57 we again find an example of a limit result which depends on an assumption which is superfluous in classical mathematics. By now the reader should not be surprised by this, nor by the delicate estimates in its constructive proof.

Definition 3.48. Let $A$ be a real subspecies of $[0,1]$ such that $A \cup A^c$ is almost full. Then we call $A$ an almost everywhere decidable or measurable species.

Example 3.49. Define $H = \{x \in [0,1] \mid \exists n[n \geq k_{99}]\}$, then as of yet we can not prove that $H$ is almost everywhere decidable. For suppose $H \cup H^c$ is almost full by the sequence of regions $(R_n)_n$. Without loss of generality assume $\mu^*(R_1) < 1/2$. Then $R_1^c$ is inhabited by Corollary 3.20 so let $x \in R_1^c$. This establishes $x \in H \cup H^c$, but $x \in H$ proves $\exists n[n \geq k_{99}]$ and $x \in H^c$ shows $\neg \exists n[n \geq k_{99}]$. 


Lemma 3.50. Let $A$ be an almost everywhere decidable species and define the partial function $I_A : [0, 1] \to \mathbb{R}$ by:

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Then $I_A$ is measurable.

Proof. Clearly $I_A$ is a bounded function and since $A \cup A^c = \mathcal{F}(I_A)$ its domain is almost full. Moreover $I_A$ is extensional because $A$ is a real species, if $x \in A$ and $x = y$ then $y \in A$. \qed

Definition 3.51. Let $A$ be an almost everywhere decidable species, then the function $I_A$ from the previous lemma is called the characteristic function of $A$. The measure $\mu(A)$ of a measurable species $A$ is defined by $\mu(A) \int I_A(x) \, d\mu(x)$.

Proposition 3.52. If $A$ and $B$ are almost everywhere decidable subspecies of $[0, 1]$, then the following are also almost everywhere decidable species:

(i) $[0, 1]$,
(ii) $[0, 1] \setminus A$,
(iii) $A \cup B$,
(iv) $A \cap B$.

Proof. The first claim is trivially true. If $A \cup A^c$ is almost full by $(R_n)_n$, then evidently so is $A^c \cup (A^c)^c$. Suppose additionally that $B \cup B^c$ is almost full by the sequence $(S_n)_n$. This means both $(A \cup B) \cup (A \cup B)^c$ and $(A \cap B) \cup (A \cap B)^c$ are almost full by $(R_n \cup S_n)_n$. \qed

Proposition 3.53. Let $A$ and $B$ be measurable subspecies of $[0, 1]$, then we have the following:

(i) $\mu([0, 1]) = 1$,
(ii) $0 \leq \mu(A) \leq 1$,
(iii) $\mu(B \setminus A) = \mu(B) - \mu(A)$ if $A \subseteq B$,
(iv) $\mu(A) \leq \mu(B)$ if $A \subseteq B$,
(v) $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$,
(vi) $\mu(A^c) = 1 - \mu(A)$,
(vii) $\mu((A^c)^c) = \mu(A)$.

Proof. The first claim follows from Lemma 3.54. The second statement can be derived from the third, which we prove as follows. As $A$ and $B$ are almost everywhere decidable, we know the that species $C = (A \cup A^c) \cap (B \cup B^c)$ is almost full. For every $x \in C$, it is easily verified that $I_{B \setminus A}(x) = I_B(x) - I_A(x)$. Applying Proposition 3.47(iii), we have shown (iii) and we get (iv) as an immediate corollary. The proof of (v) is similar. Since $A \cup A^c$ is almost full, we likewise derive (vi) from Proposition 3.47(iii) applied to $I_A$ and $1 - I_A$. Finally (vii) follows in a similar way. \qed

Lemma 3.54. Let $[a, b]$ be a rational closed interval, then $[a, b]$ is almost everywhere decidable and $\mu([a, b]) = \mu^*([a, b])$.

Proof. Trivial. \qed

Corollary 3.55. Let $F$ be an elementary domain, then $F$ is almost everywhere decidable and $\mu(F) = \mu^*(F)$.
Proposition 3.56. Let \( A \) be a measurable species. Then for every \( n \) there is region \( R \) such that \( R^c \subseteq A^c \) and \( \mu^*(R) \leq \mu(A) + 1/2^n \).

Proof. Fix an \( n \in \mathbb{N} \). By definition \( A \cup A^c \) is almost full and Corollary 3.27 yields a region \( S \) and a fan-law \( \sigma \) such that \( \mu^*(S) < 1/2^n \) and \( S^c \subseteq C(\mathcal{F}_\sigma) \subseteq A \cup A^c \). Thus we see \( \forall x \in \mathcal{F}_\sigma[x \in A \lor x \notin A] \). Applying the continuity principle we derive \( \forall x \in \mathcal{F}_\sigma \exists n \forall y \in \mathcal{F}_\sigma[\exists n = \exists n \rightarrow x, y \in A \lor x, y \notin A] \). This means that the species \( B = \{ s \in S \mid \sigma(s) = 1 \land \exists n \forall y, z \in S \rightarrow z = \exists n \rightarrow y, z \in A \lor z \notin A \} \) is a bar in \( \mathcal{F}_\sigma \). We can construct a finite subspaces \( B' \subseteq B \) that is also a bar in \( \mathcal{F}_\sigma \) using the fan theorem. Associated with each \( s \in B' \) there is a natural number \( k_s \) verifying that \( s \in B \). Define \( k = \max(\{k_s \mid s \in B'\}) \), then the species of rational segments \( F = \{ s(k_s) \mid s \in B' \} \) is finite. We may view the elements of \( F \) as rational, closed intervals. The properties of \( B' \) imply that if \( k \in F \), then either \( K \subseteq A \) or \( K \subseteq A^c \). Thus we may divide \( F \) into two elementary domains \( F_1 \) and \( F_2 \) with \( F_1 \cap F_2 = \emptyset, F_1 \cup F_2 = F \) and \( F_1 \subseteq A \) and \( F_2 \subseteq A^c \).

Construct a region \( Q \) such that \( F_1 \subseteq Q \) and \( \mu^*(Q) - \mu^*(F_1) < 1/2^{n+1} \). We claim that \( Q \cup S \) is a region having the desired properties. Suppose \( x \in (Q \cup S)^c \), then in particular \( x \in S^c \). Firstly this implies \( x \in C(\mathcal{F}_\sigma) \), which means \( x \in F_1 \cup F_2 \). Secondly we know \( x \in A \cup A^c \) as well. Assuming \( x \in A \), it is contradictory that \( x \notin F_2 \). Thus we deduce \( x \in F_1 \) and this is a contradiction with \( x \in (Q \cup S)^c \). So we have shown \( x \in A^c \) and Corollary 3.55 yields \( \mu^*(F_1) \leq \mu(A) \). We subsequently derive \( \mu^*(Q \cup S) \leq \mu^*(Q) + \mu^*(S) \leq \mu^*(F_1) + 1/2^{n+1} + 1/2^{n+1} \leq \mu^*(A) + 1/2^n \). □

Theorem 3.57. Let \( (A_n)_n \) be a sequence of almost everywhere decidable subspecies of \( [0,1] \) such that \( A_{n+1} \subseteq A_n \) for every \( n \). Moreover suppose \( \lim_{n \to \infty} \mu(A_n) \) converges. Then:

(i) \( \bigcap_{n=1}^\infty A_n \) is almost everywhere decidable,

(ii) \( \lim_{n \to \infty} \mu(A_n) = \mu\left(\bigcap_{n=1}^\infty A_n\right) \).

Proof. Fix a natural number \( k \) and define \( A = \bigcap_{n=1}^{\infty} A_n \). For every \( n \), there is a region \( Q_n \) such that \( \mu^*(Q_n) < 1/2^{k+n+2} \) and for every \( x \in [0,1] \) we have \( x \notin Q_n \rightarrow (x \in A_n \lor x \notin A_n) \). We may now construct the region \( Q = \bigcup_{n=1}^{\infty} R^n \) with \( \mu^*(Q) < 1/2^{k+1} \). Since \( \lim_{n \to \infty} \mu(A_n) \) converges, we can construct an increasing sequence \( (m(i))_i \) such that \( \mu(A_{m(i)} \setminus A_{m(i+1)}) < 1/2^{k+i+3} \). By using Proposition 3.56 we find regions \( S_i \) with \( \mu^*(S_i) < 1/2^{k+i+2} \) and \( S_i^c \subseteq (A_{m(i)} \setminus A_{m(i+1)})^c \). Subsequently define the region \( S \) by \( S = \bigcup_{i=1}^{\infty} S_i \). Then \( \mu^*(S) < 1/2^{k+1} \) and the region \( R_k = Q \cup S \) satisfies \( \mu^*(R_k) < 1/2^k \). Now suppose \( x \notin R_k \), then in particular \( x \notin Q_{m(i)} \). This means we know \( x \notin A_{m(i)} \lor x \notin A_{m(i)} \). If \( x \) is not an element of \( A_{m(i)} \), then obviously \( x \notin A \). So suppose \( x \in A_{m(i)} \). As \( x \notin R_k \), we know in particular that \( x \notin S_i \). This establishes \( x \in (A_{m(i)} \setminus A_{m(i)})^c \) and moreover from \( x \notin Q_{m(i)} \) we know that \( x \in A_{m(i+1)} \) is decidable. Now \( x \notin A_{m(2)} \) is contradictory, so we must have \( x \in A_{m(i)} \) for every \( i \), so \( x \in A \).

Letting \( k \) vary, the resulting sequence of regions \( (R_k)_k \) proves that \( A \) is almost everywhere decidable. Our construction of \( (R_k)_k \) also shows that \( \{ x \in [0,1] \mid \lim_{m \to \infty} I_{A_n}(x) = I_A(x) \} \) is almost full. An application of Corollary 3.46 concludes the proof. □

Theorem 3.58. Let \( (A_n)_n \) be a sequence of almost everywhere decidable subspecies of \( [0,1] \). Moreover suppose \( \lim_{m \to \infty} \mu(\bigcup_{n=1}^{m} A_n) \) converges. Then:

(i) \( \bigcup_{n=1}^{\infty} A_n \) is almost everywhere decidable,

(ii) \( \lim_{m \to \infty} \mu(\bigcup_{n=1}^{m} A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \).

Proof. The proof is very similar to that of the previous theorem and can also be found in Section 6.3 of [He66]. □

Corollary 3.59. Let \( R \) be a region, then \( R \) is almost everywhere decidable and \( \mu(R) = \mu^*(R) \).

Proof. Combine the previous theorem with Corollary 3.55 and the definition of a region. □
3.3 Distributions

In this section we will show that distributions of measurable functions are defined everywhere, up to an enumerable number of exceptional points. This result is obviously essential in a constructive development of a theory of measure. Nonetheless this fact was only established in 1972 by Bishop and Cheng, though Bishop had published another proof in 1967. An updated exposition of their theory can be found in Chapter 6 of [BB85]. While we restrict our attention to bounded functions, the following section presents a simpler and more direct approach to this problem. We first verify that a distribution \( F \) is densely defined. Subsequently the monotonicity of \( F \) coupled with the fact that \( \mathcal{U}(F) \) is dense, allows us to effectively isolate the potential 'jump-points' of \( F \). Any point \( x \) apart from all these jump-points is a continuity point of \( F \). Precisely this allows us to conclude that \( x \in \mathcal{U}(F) \) by Theorem 3.57. We start with defining distributions, the comments below Definition 2.44 should clarify why our definition is correct.

**Definition 3.60.** Let \( f : [0,1] \rightarrow \mathbb{R} \) be a Brouwer-Lebesgue integrable function. The partial function \( F : \mathbb{R} \rightarrow [0,1] \) defined by \( F(y) = \mu(\{x \in [0,1] \mid f(x) \leq y \}) \) is called the distribution of \( f \).

**Lemma 3.61.** Let \( f : [0,1] \rightarrow \mathbb{R} \) be a Brouwer-Lebesgue integrable function. Suppose \( \langle a, b \rangle \) is a rational segment and let \( k \in \mathbb{N} \) be given. Then we can construct a segment \( \langle c, d \rangle \) satisfying the following requirements:

(i) \( \langle c, d \rangle \subseteq \langle a, b \rangle \) and \( d - c < 1/2 \cdot (b - a) \),

(ii) there exists a region \( G \) with \( \mu^*(G) < 1/2^k \),

(iii) there exists a tiling \( T \) with \( A(T) < 1/2^{k+2} \),

(iv) on the species \( G^c \) the function \( f \) is captured by \( T \),

(v) every tile in \( T \) is either decided for \( [c, d] \), or its projected area is contained in \( G \).

**Proof.** By shifting and multiplying the function \( f \) and the other objects under investigation, we may assume without loss of generality that \( \langle a, b \rangle = (0,1) \). Define the bands \( B_i = [(i-1)/2^{k+2}, i/2^{k+2}] \) for \( i = 1,...,2^{k+2} \). Next construct a region \( S \) with \( \mu^*(S) < 1/2^{k+3} \) and tiling \( T \) with such that on the species \( S^c \) the function \( f \) is captured by \( T \) and \( A(T) < 1/2^{2(k+2)} \). This is possible because \( f \) is Brouwer-Lebesgue integrable. For each \( j = 1,...,2^{k+1} \), consider the band \( B_{2j} \) and the species \( U_j \) of tiles of \( T \) that are undecided for \( B_{2j} \). If \( m \) and \( n \) are unequal, it is clear that an arbitrary tile \( t \) can be both undecided for \( B_{2m} \) and \( B_{2n} \). But since we just consider the bands with even indices, this can only occur if \( h(t) \geq 1/2^{k+2} \). As the total area of \( T \) is \( 1/2^{3(k+2)} \), the total length of the projected intervals of the tiles in \( U_m \cap U_n \) does not exceed \( (1/2^{3(k+2)})/(1/2^{k+2}) = 1/2^{2(k+2)} \).

This means that the sum \( \sum_{1 \leq m \leq 2^{k+1}} w(U_m) \) is bounded by \( 1 + 1/2 \). Firstly any tile could occur in some species \( U_n \) for some \( n \). This explains the term \( 1 \) in our upper bound \( 1 + 1/2 \). Secondly, there may be tiles which are undecided for several bands. For any pair \( (m, n) \) with \( m < n \) the tiles in \( U_m \cap U_n \) have a projected length no larger than \( 1/2^{2(k+2)} \). As there are at most \( 2^{k+1} \cdot (2^{k+1} - 1)/2 \) such pairs, we can bound their total projected length by \( 1/2^{2(k+2)} \cdot 2^{k+1} \cdot (2^{k+1} - 1)/2 < 1/2 \). But the bound \( \sum_{1 \leq m \leq 2^{k+1}} w(U_m) \leq 1 + 1/2 \) must mean that there exists a \( p \) such that \( w(U_p) \leq (1 + 1/2)/2^{k+1} = 3/2^{k+2} \). Next consider the species \( F = \bigcup_{u \in U_p} \pi_x(u) \), which is an elementary domain. Evidently \( \mu^*(F) \leq 3/2^{k+2} \). Construct a region \( R \) such that \( F \subseteq R \) and \( \mu^*(R) - 1/2^{k+3} \). Define the region \( G \) by \( G = R \cup S \). Its pre-measure does not exceed \( 1/2^{k+3} + 3/2^{k+2} + 1/2^{k+3} = 1/2^k \). Finally set \( \langle c, d \rangle \) equal to \( \zeta(B_{2p}) \). It is quite clear that now (i), (ii), (iii) and (iv) are satisfied. Moreover we have included in \( G \) every projected area of tiles that are undecided for \( [c, d] \), so this establishes (v).
**Proposition 3.62.** Let \( f : [0, 1] \to \mathbb{R} \) be a Brouwer-Lebesgue integrable function. Then \( \mathcal{D}(F_f) \) is dense in \( \mathbb{R} \).

**Proof.** Let \([a, b] \subseteq [0, 1]\) be an arbitrary rational segment, if we can construct an \( y \in [a, b] \) such that \( y \in \mathcal{D}(F_f) \) we are done. To this end we successively apply the previous lemma. Thus we obviously obtain a shrinking and dwindling sequence of rational segments \( y \). Next we must establish that \( \{ x \in [0, 1] \mid f(x) \leq y \} \) is almost everywhere decidable. So let \( k \in \mathbb{N} \) be given. Our construction yields a rational segment \( y(k) = (c, d) \), a region \( G \) and tiling \( T \) satisfying Lemma 3.61. Let \( G \) meaning behind Definition 3.64 below should be clear. If \( \mathcal{D}(F_f) \) is monotone function and the interval \([a, b]\) is \((G, j, l, b)-bounded\), then the function \( G \) ‘jumps’ a minimum height of \( 1/j \) at most \( l \) times on \([a, b]\). Likewise, a real number \( x \in [0, 1] \) is \((G, j, l)-low\) if there is a segment \( x(k) \) of \( x \) such that the interval \( \mathcal{D}(x(k)) \) contains no ‘jump-points’ of \( G \) with a height of at least \( 1/j \). The definition of \((G, j, l)\)-low real number looks more complicated because of Lemma 1.62. If \( P \) is a partition, then in general we can not pinpoint a single slice of \( P \) such that \( x \) is an element of this slice.

**Lemma 3.63.** For any \( x \leq y \) with \( x, y \in \mathcal{D}(F) \) we have \( F(x) \leq F(y) \).

**Proof.** This directly follows from Proposition 3.53. \( \square \)

**Definition 3.64.** Let \( G : [0, 1] \to \mathbb{R} \) be a monotone function, let \([a, b] \subseteq [0, 1]\) be a closed interval and suppose \( j \) is a natural number and \( l \) a non-negative integer. Then the interval \([a, b]\) is \((G, j, l)-bounded\) if \( c, d \in \mathcal{D}(G) \) and there is an \( n \) such that \( |G(b) - G(a)|(n)^{\prime\prime} < (l + 1)/j \).

**Definition 3.65.** Let \( G : [0, 1] \to \mathbb{R} \) be a monotone function, let \( P \) be a partition and suppose \( j \) is a natural number. Then \( P \) is \((G, j, l)-bounded\) if there are natural numbers \( l_2, \ldots, l_n \) such that each slice \( [P_{i-1}, P_i] \) is \((G, j, l_i)-bounded\) and \( \sum_{i=2}^{n} l_i \leq l \).

**Lemma 3.66.** Fix a monotone function \( G : [0, 1] \to \mathbb{R} \) and natural number \( j \). Let \([a, b]\) be a \((G, j, l)-bounded\) interval. Suppose \( c \in \mathcal{D}(G) \) is such that \( a < c < b \). Then there exist non-negative integers \( l_1 \) and \( l_2 \) such that \([a, c]\) is \((G, j, l_1)-bounded\), \([c, b]\) is \((G, j, l_2)-bounded\) and \( l_1 + l_2 \leq l \).

**Proof.** Without loss of generality assume \( G \) is non-decreasing. Find a natural number \( n \) such that \( (G(b) - G(a))(n)^{\prime\prime} < (l + 1)/j \). Thus we see \((G(b))(n)^{\prime\prime} - (G(a))(n)^{\prime\prime} < (l + 1)/j \). Next construct an \( m > n \) with \( (G(c))(m)^{\prime\prime} - (G(c))(m)^{\prime} < 1/j \). This means the sum of \((G(b))(m)^{\prime\prime} - (G(c))(m)^{\prime}\) and \((G(c))(m)^{\prime} - (G(a))(n)^{\prime}\) is smaller than \( (l+2)/j \). Let \( l_1 \) be the least natural number such that \((G(b))(m)^{\prime\prime} - (G(c))(m)^{\prime} < (l_1 + 1)/j \) and define \( l_2 \) similarly. It is then obvious that \( l_1 + l_2 \leq l \). \( \square \)

**Corollary 3.67.** Fix a monotone function \( G : [0, 1] \to \mathbb{R} \), a natural number \( j \) and let \( P \) be a \((G, j, l)-bounded\) partition. If \( P' \) is a refinement of \( P \), then \( P' \) is \((G, j, l)-bounded\) as well.

**Lemma 3.68.** Let \( G : [0, 1] \to \mathbb{R} \) be a densely defined function and let \([a, b] \subseteq [0, 1]\) be a closed interval. Then there exists a real number \( c \in \mathcal{D}(G) \) such that \( |(a + b)/2 - c| < 1/6 \cdot (b - a) \).

**Proof.** Trivial. \( \square \)
Corollary 3.69. Let $G : [0,1] \to \mathbb{R}$ be a densely defined function with $0,1 \in \mathcal{D}(G)$. Then we can construct a sequence of partitions $(P^n)_n$ such that:

(i) $l(P^n) = 1 + 2^{n-1}$, 
(ii) $P^{n+1}$ is a refinement of $P^n$, 
(iii) $d_n(P^n) \leq (2/3)^{n-1}$, 
(iv) $P^n_i \in \mathcal{D}(G)$ for all $n$ and $i \in \{1, ..., l(P^n)\}$.

Proof. Define $P^1 = (0,1)$ and for $n > 1$ consider each slice $[P^n_i, P^n_{i+1}]$ of the partition $P^n$. Find a suitable $c_i \in [P^n_i, P^n_{i+1}]$ by using Lemma 3.68. Next define the partition $P^{n+1}$ with $l(P^{n+1}) = 1 + 2^n$ by $P^{n+1}_i = P^n_i$ if $i$ is odd and $P^{n+1}_i = c_i$ if $i$ is even. \hfill \Box

Definition 3.70. Let $G : [0,1] \to \mathbb{R}$ be a monotone function and let $j$ be a natural number. Suppose $x \in [0,1]$ satisfies one of the following conditions:

(i) there are $a, b, c \in \mathbb{R}$ with $a < b < c$, $a < x < c$ and both $[a, b]$ and $[b, c]$ are $(G, j, 0)$-bounded, 
(ii) there are $a, b \in \mathbb{R}$ with $a < x < b$ and $[a, b]$ is $(G, j, 0)$-bounded.

Then we call $x$ a $(G, j)$-low real number.

Proposition 3.71. Let $G : [0,1] \to \mathbb{R}$ be a monotone function with a dense domain and $0,1 \in \mathcal{D}(G)$. Let $j \in \mathbb{N}$ and suppose $[0,1]$ is $(G, j, l)$-bounded for some natural number $l$. Then there is a finitely enumerable species of real numbers $T \subseteq [0,1]$ such that all $x \# T$ are $(G, j)$-low.

Proof. Using Corollary 3.69, construct a sequence of partitions $(P^n)_n$, satisfying the requirements stated in this corollary. Carry out this construction in such a way that $P^1 = (0,1)$. Since the slice $[0,1]$ is $(G, j, l)$-bounded, we may also assume that all partitions $P^n$ are $(G, j, l)$-bounded by Corollary 3.67. Subsequently we can associate with each slice $[P^n_{i-1}, P^n_i]$ of $P^n$ a natural number $l^n_i$ such that this slice is $(G, j, l^n_i)$-bounded and $\sum_{i=2}^{l(P^n)} l^n_i \leq l$ for all $n$.

We shall construct $l$ sequences $K^1, ..., K^l$ of nested, closed intervals whose length goes to zero. Define $K^1 = \ldots = K^l = [0,1]$ and subsequently construct further entries of these sequences in a such way that the following requirements are met:

(i) $K^i_k$ is a slice of $P^k$ for all $i \in \{1, ..., l\}$ and $k \in \mathbb{N}$, 
(ii) $K^i_{k+1} \subseteq K^i_k$ for all $i \in \{1, ..., l\}$, 
(iii) for all $n$ and $i \in \{2, ..., l(P^n)\}$, there are at least $l^n_i$ copies of $[P^n_{i-1}, P^n_i]$ among $K^1_n, ..., K^l_n$.

This can be done due to Lemma 3.66 and Corollary 3.67. Since $\lim_{n \to \infty} d_n(P^n) = 0$, the resulting sequences $K^1, ..., K^l$ indeed consist of nested and vanishing intervals. Define the sequences of rational segments $t^1, ..., t^l$ by $t^n_i = \zeta(K^n_i)$ for all $i \in \{1, ..., l\}$ and $n \in \mathbb{N}$. Then clearly each such sequence is a real number. We define the species of real numbers $T$ by $T = \{t^1, ..., t^l\}$.

Let $x$ be a real number apart from $T$. Construct a $k \in \mathbb{N}$ such that $x(k) \not\approx t^n_k$ for every $i \in \{1, ..., l\}$. We can subsequently find a $p$ with $\delta(x(k+p)) < d_l(P^k)$. Lemma 1.62 then tells us that there is a $j \in \{1, ..., l(P^k) - 2\}$ with $P^k_j < x(k+p)' \leq x(k+p)'' < P^k_{j+2}$. Consider the slices $[P^k_j, P^k_{j+1}]$ and $[P^k_{j+1}, P^k_{j+2}]$. If for instance $[P^k_j, P^k_{j+1}]$ is not $(G, j, 0)$-bounded, then we must in fact have the stricter string of inequalities $P^k_{j+1} < x(k+p)' \leq x(k+p)'' < P^k_{j+2}$. A similar statement holds if $[P^k_{j+1}, P^k_{j+2}]$ is not $(G, j, 0)$-bounded. Moreover one of these two slices must be $(G, j, 0)$-bounded because of $x(k) \not\approx t^n_k$ for all $i \in \{1, ..., l\}$. The species $\{K^1_n, ..., K^l_n\}$ contains all slices of $P^k$ which are not $(G, j, 0)$-bounded. Whatever is the case, it is clear that $x$ is $(G, j)$-low. \hfill \Box
Corollary 3.72. Let $G : [0, 1] \to \mathbb{R}$ be a bounded, monotone function with a dense domain and $0, 1 \in \mathcal{D}(G)$. Then there is an enumerable species $T \subseteq [0, 1]$ such that all $x \# T$ are $(G, j)$-low for every $j \in \mathbb{N}$.

Proof. For every $j \in \mathbb{N}$, construct a species $T_j \subseteq [0, 1]$ such that every $x \# T_j$ is $(G, j)$-low using Proposition 3.71. Subsequently define $T = \bigcup_{j=1}^{\infty} T_j$. □

Proposition 3.73. Let $G : [0, 1] \to \mathbb{R}$ be a bounded, densely defined, monotone function with $0, 1 \in \mathcal{D}(G)$. Then the species of continuity points of $G$ is co-enumerable.

Proof. The previous corollary yields an enumerable species $T$ such that every $x \in [0, 1]$ apart from $T$ is $(G, j)$-low for all $j \in \mathbb{N}$. Let $w \# T$, we will prove that $w$ is a continuity point of $G$. Fix a $k \in \mathbb{N}$ and find $a, c \in \mathcal{D}(G)$ with $a < w < c$ and either $[a, c]$ is $(G, k)$-low, or there is a $b \in \mathcal{D}(G)$ such that both $[a, b]$ and $[b, c]$ are $(G, k)$-low. If $y, z \in \mathcal{D}(G)$, then in both cases we may conclude from $|y - w| < \min(c - y, y - a)$ and $|z - w| < \min(c - w, w - a)$ that $|G(z) - G(y)| < 2/k$. As $k$ was arbitrary, $w$ is a continuity point of $G$. □

We conclude this chapter with our main result in the form of Theorem 3.74. While the content is not new, the proof vastly differs from the one in [BB85]. We already stressed the importance of Theorem 3.74, but it is interesting to mention two examples illustrating this. Firstly many papers on constructive measure theory use the result as a basic assumption from which new theory is developed. Examples include [Sp03] and [Ch72]. Secondly there is the question of extending the Brouwer-Lebesgue measure to a Lebesgue-Stieltjes measure. This is the topic of Gibson’s PhD thesis [Gi67], which was finished in the same year as Bishop published Foundations of Constructive Analysis [Bi67]. Gibson rightly commented that an integrator $F$ in a constructive theory of the Lebesgue-Stieltjes integral can not be everywhere defined. Such an $F$ would be continuous and therefore the resulting measure $\int dF$ is not that interesting. Thus Gibson considers densely defined integrators, but in light of Bishop’s work this can probably be extended to distribution functions with co-enumerable domains.

Theorem 3.74. Let $f : [0, 1] \to \mathbb{R}$ be a Brouwer-Lebesgue integrable function. Then $\mathcal{D}(F_f)$ is co-enumerable.

Proof. Since $f$ is bounded, we may without loss of generality assume that $F_f(0) = 0$ and $F_f(1) = 1$. Proposition 3.73 tells us that the species of continuity points of $F_f$ is co-enumerable. Thus we are done if we can show that every continuity point $y$ of $F_f$ in fact belongs to the domain of $F_f$. Construct a sequence $(y_n)_n \in \mathcal{D}(F_f)$ such that $y_n \downarrow y$ using Proposition 3.62. Fix an $m \in \mathbb{N}$. Since $y$ is a continuity point, there is a $k$ such that $|F_f(x) - F_f(z)| < 1/2^k$ whenever $x, z \in \mathcal{D}(F_f)$, $|x - y| < 1/2^k$ and $|z - y| < 1/2^k$. Find an $N$ such that $n > N$ implies $y_n - y < 1/2^k$. This $N$ verifies that $(F_f(y_n))_n$ is a Cauchy sequence, so the limit $\lim_{n \to \infty} F_f(y_n)$ exists. The species $(\{x \in [0, 1] \mid f(x) \leq y_n\})_n$ constitute a decreasing sequence and Theorem 3.57 then yields that $\bigcap_{n=1}^{\infty} \{x \in [0, 1] \mid f(x) \leq y_n\}$ is measurable. It is easy to show that this latter species in fact equals $\{x \in [0, 1] \mid f(x) \leq y\}$, which proves $y \in \mathcal{D}(F_f)$. □
Bibliography


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