Electroweak Effects in Antenna Parton Showers

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Chapter 1

Introduction

Particle physics is a field of study that is concerned with the nature of matter at the smallest length scales. The idea that matter can be broken down to elementary particles, the smallest constituents of matter with no further substructure, dates back to ancient Greek times [1]. However, the field of particle physics arguably truly commenced near the end of the 19th century with the discovery of the first elementary particle, the electron. On the theoretical front, the beginning of the 20th century saw the development of quantum mechanics, leading to a radically different understanding of the dynamics of subatomic particles. Later, it would be combined with special relativity to form the foundation of quantum field theory, which is to this day the primary theoretical framework used to describe our understanding of nature at the smallest length scales. Eventually, particle physics matured in the 1950s with the development of particle accelerators which could be used to collide subatomic particles at high energies. These high energies may be converted into new particles of which the remnants can be measured. Since that time, particle colliders have been constructed with the ability to collide subatomic particles at increasingly large energies. This has led to a plethora of discoveries of elementary particles and their properties and interactions, culminating in the confirmation of the existence of the Higgs boson when it was discovered by the ATLAS and CMS experiments at the LHC in 2012 [2, 3].

At the same time, elaborate theoretical models have been constructed to explain, describe and predict the results of the aforementioned collision experiments, and to gain insight in the fundamental laws of physics that govern the interactions between particles at the most fundamental level. The current understanding is compiled in the Standard Model, a theory that is able to describe all visible matter and its interactions due to three of the four fundamental forces. These interactions are the strong interaction, which is responsible for the confinement of quarks into hadrons such as
protons and neutrons, the weak interaction, which mediates radioactive decays and serves an essential role in nuclear fission and fusion processes, and electromagnetism, which is responsible for most macroscopic phenomena we encounter in daily life.

While the Standard Model has been extremely successful in its experimental predictions, it leaves some aspects of nature entirely unexplained. Most notably, it offers no fundamental description of the fourth fundamental interaction, gravity, at the microscopic level, and it offers no explanation for other physical phenomena such as dark matter, dark energy, baryon asymmetry and neutrino oscillations.

During the lifetime of the field of particle physics, the roles of the theoretical and experimental branches have shifted. While in the earlier stages the theory community faced the challenge of understanding the unexplained phenomena measured at experiments, after the development of the Standard Model one of the foremost purposes of new collider experiments became the confirmation of its predictions, culminating in the discovery of the Higgs boson as the final missing piece. Although the LHC was constructed to be a discovery machine and there were many reasons to suspect that new physics would be reachable at its operational energy, no signs of deviations from the Standard Model have yet been found. As such, it has become more likely that any signs of new physics will manifest themselves only as small discrepancies in the measurements of Standard Model processes. To reach any definite conclusions from such measurements, precise theoretical modelling of these processes must be emphasized.

Because the LHC collides hadrons, which are heavy composite particles, it is able to reach high energies and has a large discovery potential. This feature however comes at the price that the collision events are considerably more complicated than those of much cleaner lepton colliders. The main reason is that the constituents of hadrons are charged under the strong force, causing them to radiate large amounts of particles that lead to jets of hadrons. This phenomenon leads to experimental challenges because it means that large numbers of particles at high frequencies may be expected to enter the detectors, and the task of extracting interesting information is incredibly challenging. However, very different challenges appear on the theory side. It is extremely difficult to calculate and model these complex collision events with sufficient precision to be comparable with experiment. Theorists thus turn to computer simulations to aid in the required computational complexity. Such simulations are constructed to translate theoretical knowledge through a chain of mostly independent components to data that is directly comparable with measurements. One of those components are parton showers. These are Monte Carlo programs that are responsible for adding the large quantities of radiation to simulated collision events. Since
the strong interaction dominates the interactions at the LHC, most parton showers have been written to accurately simulate QCD radiation to the best possible theoretical accuracy. However, as more high-energy scattering events are measured, other types of radiation originating from the electroweak sector also become relevant. The inclusion of these types of radiation in a parton shower is the main topic of this work.

The outline of this thesis is as follows. In Chapter 2 the basics of the Standard Model are revisited, and Chapter 3 contains an overview of the procedures of event generation and simulation for the LHC. These chapters serve as the foundation upon which the new contributions of this thesis are built. Chapter 4 covers the fundamentals of Monte Carlo techniques with a particular focus on the so-called Sudakov veto algorithm. While this algorithm is a well-known and heavily used component of parton shower programs, an original calculational method is outlined in Chapter 4 that is used to analyze and extend it. Chapter 5 then presents an overview of the Vincia parton shower, a plugin to the widely used event generation program Pythia, which serves as the framework for the rest of the thesis. The Vincia parton shower has reached high levels of maturity after more than a decade of development, but several new contributions related to the inclusion of particle masses and different recoil strategies in the initial state kinematic mappings are highlighted in Chapter 5. In Chapter 6 the Vincia formalism is extended to incorporate QED effects, which comes with a unique set of problems as compared with the usual QCD machinery. Finally, Chapter 7 describes the implementation of a full-fledged Electroweak parton shower in Vincia. It discusses both its theoretical foundation as well as its practical implementation, and the large number of features and peculiarities that are unique to the electroweak theory. The content of these chapters is completely original. We conclude in Chapter 8 and defer some technical details of the shower implementations to four appendices.
Chapter 2

The Standard Model

This chapter contains a brief introduction to the Standard Model (SM) of particle physics. We review its particle content and the fundamental interactions between them. These interactions come about as a consequence of non-abelian gauge symmetry \( SU(3)_C \times SU(2)_W \times U(1)_Y \). The particle content of the Standard Model is shown in Figure 2.1. The interactions stemming from the

\[ SU(3)_C \] symmetry lead to the interactions described in the theory of Quantum Chromodynamics (QCD) [5, 6]. The fundamental charge of QCD is called colour, and it describes the interactions between the fermionic quarks and the gauge-bosonic gluons. QCD interactions dominate the production processes observed at the Large Hadron Collider (LHC). An adequate understanding of QCD is therefore required from the phenomenological viewpoint of translating theory to realistic predictions. The remaining \( SU(2)_W \times U(1)_Y \) symmetries describe the electroweak sector of the Standard Model.

![Particle Content of the Standard Model](Figure 2.1: The particle content of the Standard Model.)
CHAPTER 2. THE STANDARD MODEL

Model \[ \text{[7] [8]. However, this symmetry is broken by the Higgs mechanism, giving mass to the remaining weak gauge bosons } W^{\pm} \text{ and } Z. \text{ Only the photon remains as the massless gauge boson of the remaining } U(1)_{\text{em}} \text{ symmetry group associated with electromagnetic charge. The electroweak production cross sections are typically subdominant to the QCD ones due to its smaller coupling constant and the large masses of the massive electroweak gauge bosons. However, at the energy scales of the LHC and upcoming colliders, the masses are not as suppressing as they have been in the past, and the electroweak theory provides important contributions to hadronic scattering processes. We now briefly describe the theoretical foundations of QCD and the electroweak theory.}

2.1 Quantum Chromodynamics

Quantum Chromodynamics is the theory that describes the strong nuclear interaction. It is a non-abelian gauge theory with gauge group \( SU(N_c) \), where \( N_c \) is the number of so-called colour charges the theory couples to. It has been determined experimentally that \( N_c = 3 \), but we will later see that the limit of \( N_c \rightarrow \infty \) can be useful in phenomenological calculations. The QCD Lagrangian is

\[
\mathcal{L}_{QCD} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \bar{q}_i (i\slashed{D} - m) q_i. \tag{2.1}
\]

The first term contains the field strength

\[
F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_s f^{abc} A_\mu^b A_\nu^c, \tag{2.2}
\]

where \( A_\mu^a \) is the gluonic gauge field in the adjoint representation, \( g_s \) is the strong coupling constant and \( f^{abc} \) are the structure constants of the \( SU(3) \) gauge group. Because the gluons live in the adjoint representation of \( SU(3) \), there are \( N^2 - 1 = 8 \) of them, indicated by the index \( a \). The field strength term contains the dynamics of the gluon field, as well as its self-interactions. The second term of eq. \( \text{(2.1)} \) contains the quark fields \( q_i \). These fermionic fields live in the fundamental representation of \( SU(3) \), and thus there are \( N = 3 \) of them corresponding with each of the colour charges. The covariant derivative is given by

\[
D_\mu = \partial_\mu - i \frac{g_s}{2} A_\mu^a T^a \tag{2.3}
\]

where \( T^a \) are the generators of the \( SU(3) \) symmetry group in the fundamental representation. Note that the mass terms in eq. \( \text{(2.1)} \) are absent at high energies because they are incompatible with the weak symmetry group \( SU(2)_W \). However, due to the
Higgs mechanism and spontaneous symmetry breaking, fermion masses do appear and can be included in the Lagrangian of the QCD sector. From Figure 2.1 we can see that the index $i$ runs over six types of quarks divided into three families. From the perspective of QCD, the quarks in these three families are identical with the exception of their mass. The covariant derivative contains the dynamics of the quarks, as well as their interactions with gluons. QCD is usually quantized using the Fadeev-Popov procedure [9], leading to the introduction of an additional gauge fixing term in the Lagrangian.

When performing calculations in the QCD framework, ultraviolet divergences appear. These can be dealt with by the procedure of renormalization, which involves absorbing the divergent terms in the QCD coupling constant. The result is that the coupling becomes a scale dependent quantity that evolves through the renormalization group. The running of the coupling is usually indicated by the renormalization group equation

$$\mu^2 \frac{\partial}{\partial \mu^2} \alpha_s(\mu^2) = \beta(\alpha_s(\mu)).$$

(2.4)

where $\alpha_s = g_s^2/4\pi$. The right-hand side of eq. (2.4) is called the $\beta$-function of QCD. Its most important property is that it causes the coupling $\alpha_s(\mu^2)$ to become small for large scales $\mu$, but large for small energy scales. As a consequence, at large energies like those encountered at the LHC, scattering processes of coloured particles can be calculated using perturbation theory. The property of decreasing coupling at higher energy scales is known as asymptotic freedom. At a hadron collider, particles that are themselves colour-neutral but contain coloured particles are made to collide. At high enough energies, these collisions can be viewed as collisions of the coloured constituents of the hadron, which are collectively named partons. If the outgoing particles produced by such a collision are also coloured, they radiate other partons and evolve to lower scales of energy. At some point, this energy scale drops below some value beyond which QCD is no longer perturbative. That scale is called $\Lambda_{\text{QCD}}$, which can experimentally found to be approximately equal to 1 GeV. Below this scale, the strong interaction becomes sufficiently strong to bind all partons into hadrons. This property of QCD is called colour confinement and implies that coloured particles can not be observed directly. Instead, they appear as jets of a collimated bunch of hadrons that move in similar direction. Jets may be identified by their originating parton by using jet algorithms and their associated definitions of jet resolution [10]. These jet resolutions have to be infrared safe, meaning that they should not be sensitive to the addition of extra jet constituents which are collinear with other partons (move in exactly the same direction) or soft (have vanishing energy).
2.1.1 Factorization

Because of the large difference in behaviour of QCD between low-scale and high-scale physics, it is in principle difficult to acquire theoretical predictions covering both regimes. Fortunately, physics at low scales and high scales turns out to factorize. This principle is called the *factorization theorem*[^11][^12], and for hadronic cross sections it is usually written as

\[
\sigma_{AB \rightarrow X} = \sum_{a,b} \int dx_a dx_b f_{a/A}(x_a, \mu_F^2) f_{b/B}(x_b, \mu_F^2) \hat{\sigma}_{ab \rightarrow X}(x_a, x_b; \mu_F^2, \mu_R^2). \tag{2.5}
\]

The quantity \(\hat{\sigma}_{ab \rightarrow X}\) is the perturbative cross section for a branching of partons \(a\) and \(b\) to some final state \(X\). The sums run over all incoming partons that can scatter to that final state. The cross section is a function of two scales that both serve to regularize divergent behaviour in higher orders of perturbation theory. The renormalization scale \(\mu_R\) regularizes the ultraviolet divergences that emerge as part of the renormalization procedure. It only appears as the argument of the strong coupling, parameterizing its running as described above. The factorization scale \(\mu_F\) regulates infrared divergences associated with emissions from the initial state. It can be thought of as the scale separating the low-scale nonperturbative and high-scale perturbative physics. Partonic emissions of energy scales lower than \(\mu_F\) are absorbed into the functions \(f_{i/I}(x, \mu_F^2)\) which are called the *parton density functions* (PDFs). They represent the probability densities to find a parton \(i\) in hadron \(I\) carrying a momentum fraction \(x\) of the hadron momentum at scale \(\mu_F^2\).

The left-hand side of eq. (2.5) should be independent of the factorization and renormalization scales, and they can thus in principle be chosen arbitrarily. However, if the partonic cross section is computed perturbatively instead of to all orders in \(\alpha_s\), some spurious dependence on these scales remains. Variations in these scales are therefore often used as a way of estimating theoretical uncertainty due to higher orders of the perturbative expansion being left out.

The low-scale behaviour of QCD contained in the PDFs is not easily accessible through theoretical calculations, with the exception of numerical simulation on discrete space-time lattices. As such, the PDFs must be obtained from experiment. An example of PDFs as function of the momentum fraction \(x\) is shown in Figure 2.2. Note that at low scales the *valence* quarks that primarily constitute the proton dominate over the majority of the range of momentum fractions, while for higher scales it becomes more likely to find gluons and other types of quarks. The enhancement of these *sea* quarks and gluons is a consequence of perturbative physics. To capture this behaviour, an evolution equation can be constructed for the PDFs in analogy with a
Figure 2.2: The NNPDF3.1 NNLO proton PDF set, evaluated at $\mu_F^2 = 10$ GeV\(^2\) (left) and $\mu_F^2 = 10^4$ GeV\(^2\) (right) taken from [13].

This equation is known as the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) equation [14, 15, 16]. It describes the perturbative variation of the PDF as a function of the factorization scale. At leading order, the DGLAP equation can be interpreted to say that this variation with energy scale originates from all other available parton densities $j$ splitting into parton $i$. The functions $P_{ji}(z)$ are the regularized Altarelli-Parisi splitting functions. They can be interpreted as a measure of probability for a particle $j$ to split to a particle $i$ with fraction $z$ of the momentum of $j$. These functions are perturbatively calculable in the collinear limit. The splitting functions play a significant role in Chapters 5, 6 and 7, and will be described in further detail there.

2.1.2 The Leading Colour Limit

We conclude this section with a brief overview of the leading colour limit $N_c \rightarrow \infty$ mentioned earlier. This approximation is a very common phenomenological consid-
eration that leads to very significant simplifications \cite{17}. The starting point is to reorganize the colour structures that appear in the QCD Feynman rules in order to eliminate the structure constants $f^{abc}$ in favor of the fundamental representation generators $T^a$. Using the usual normalization $\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$, the relationship between the generators and the structure constants is

$$[T^a, T^b] = i f^{abc} T^c. \quad (2.7)$$

Using this expression, the structure constants that appear as part of purely gluonic vertices can be eliminated. The result may be reduced further by use of the $SU(N_c)$ Fierz identity \cite{17}

$$(T^a)^i_l (T^a)^j_k = \delta^i_j \delta^k_l - \frac{1}{N_c} \delta^i_l \delta^j_k. \quad (2.8)$$

The Fierz identity encodes the statement that the generators $T^a$ form a set of traceless Hermitian matrices. It can be understood physically to imply that a gluon consist of a component that behaves like an independent colour-anticolour pair and a component that is a $SU(3)$ singlet with a negative contribution. By application of the Fierz identity, it can be proven that purely gluonic QCD scattering amplitudes can be decomposed in the so-called *colour flow* decomposition \cite{18, 19, 20}

$$M = \sum_{\sigma \in S_n/Z_n} \delta^{i_{\sigma_1}}_{j_{\sigma_2}} \delta^{i_{\sigma_2}}_{j_{\sigma_3}} \ldots \delta^{i_{\sigma_n}}_{j_{\sigma_1}} A(\sigma_1, \ldots, \sigma_n) \quad (2.9)$$

where the sum runs over all cyclically-inequivalent permutations. The amplitudes $A(\sigma_1, \ldots, \sigma_n)$ are called partial amplitudes. They are colour-ordered, meaning that they are stripped of all colour factors and they only receive contributions from planar diagrams with a particular ordering of the gluons. Similar expressions can be derived for amplitudes that involve quarks. Next, after computing the squared amplitude and summing or averaging over external colours, it can be shown that the differential cross section takes the form

$$d\sigma \propto N_c^n \left( \sum_{\sigma \in S_n/Z_n} |A(\sigma_1, \ldots, \sigma_n)|^2 + O(1/N_c^2) \right). \quad (2.10)$$

This equation implies that the most important contributions come from a sum of positive terms that all have a definite colour ordering. All other contributions that contain interferences between colour orderings are suppressed with at least a factor of $1/N_c^2 = 1/9$. Dropping the suppressed terms corresponds with taking the limit of $N_c \to \infty$. It is extremely useful in the context of a Monte Carlo simulation, where the strictly positive contributions with a definite colour state are readily translated to a simulation procedure.
2.2 The Electroweak Theory

The electroweak theory is a unified description of electromagnetism and the weak interaction. The associated symmetries $U(1)_Y$ and $SU(2)_W$ are broken by the Higgs mechanism, giving mass to many of the Standard Model particles. The Lagrangian of the electroweak sector can be written as

$$\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{fermion}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}}. \quad (2.11)$$

The contribution from the gauge sector is similar to that of QCD. It is

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} W^a_{\mu\nu} W^{\mu\nu a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}, \quad (2.12)$$

where $W^a_{\mu\nu}$ is the field strength tensor for the $SU(2)_W$ gauge field $W^a_i$ and $B_{\mu\nu}$ is the field strength tensor for the $U(1)_Y$ gauge field $B_\mu$. They are given by

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g \epsilon^{abc} W^b_\mu W^c_\nu$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \quad (2.13)$$

The fermionic piece of the Lagrangian is

$$\mathcal{L}_{\text{fermion}} = i \bar{Q}_i \not\!D Q_i + i \bar{u}_i \not\!D u_i + i \bar{d}_i \not\!D d_i + i \bar{L}_i \not\!D L_i + i \bar{e}_i \not\!D e_i \quad (2.14)$$

where the index $i$ runs over the three fermion generations. The first three terms describe the dynamics of the quarks fields. The field $Q$ is the left-handed $SU(2)_W$ doublet containing the up- and down-type quark, while the fields $u$ and $d$ are the right-handed singlet up- and down-types which are not charged under the weak interaction. Similarly, the field $L$ contains the left-handed electron- and neutrino-type leptons, while the field $e$ is the right-handed counterpart of the electron-like lepton. The right-handed neutrino field is absent because it has never been observed. It is theoretically allowed to exist, and serves as one of the candidates for dark matter or several other unexplained phenomena [21].

The definition of the covariant derivative is also similar to that of QCD given in eq. (2.3). It is

$$D_\mu = \partial_\mu + ig R^a W^a_\mu + ig' Y B_\mu. \quad (2.15)$$
The matrices $\tau^a$ are normalized Pauli spin matrices, which are the generators of $SU(2)_W$. $Y$ is the hypercharge operator for $U(1)_Y$, and $g$ and $g'$ are the gauge coupling constants of $SU(2)_W$ and $U(1)_Y$ respectively.

The next contribution to the electroweak Lagrangian is the Higgs piece. It is

$$L_{\text{Higgs}} = (D^\mu \phi)^\dagger D_\mu \phi - \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2,$$  \hspace{1cm} (2.16)$$

where $\mu^2 = m_h^2/2$ is the Higgs mass and $\lambda$ is the Higgs self coupling. The Higgs field $\phi$ is a complex scalar doublet field of hypercharge one, and thus also comes with a covariant derivative. The last two components of eq. (2.16) are the Higgs potential, which cause spontaneous symmetry breaking. The covariant derivative contains the interactions of the Higgs with the electroweak gauge fields, while the term proportional to $\lambda$ leads to Higgs self interactions.

The last component of the Lagrangian of the electroweak theory is the Yukawa component

$$L_{\text{Yukawa}} = -y_{uij} \bar{Q}_i \tilde{\phi} u_j - y_{dij} \bar{Q}_i \phi d_j - y_{lij} \bar{L}_i \phi e_j$$  \hspace{1cm} (2.17)$$

where $\tilde{\phi} = i\sigma^2 \phi^\dagger$. These interactions will lead to mass terms for the fermions after spontaneous symmetry breaking, and their masses will be proportional to the coupling constants $y_{uij}$, $y_{dij}$ and $y_{lij}$.

### 2.2.1 Spontaneous Symmetry Breaking

Due to the chiral nature of the weak gauge symmetry $SU(2)_W$, mass terms cannot be added directly to the Lagrangians of QCD or the electroweak theory. Instead, particles get their mass from the Higgs mechanism \cite{22, 23, 24}. Consider the Higgs sector of the electroweak Lagrangian given in eq. (2.16). If the parameter $\mu^2$ is negative, the Higgs potential, consisting of the last two terms of that equation, has a minimum at

$$\phi = \sqrt{-\frac{\mu^2}{2\lambda}} = \frac{v}{\sqrt{2}},$$  \hspace{1cm} (2.18)$$

where $v$ is the vacuum expectation value. Using transformations of the symmetry groups, the minimum of the Higgs field can be taken to be

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}.$$  \hspace{1cm} (2.19)$$

Because this non-trivial ground state changes under the symmetry operations of the Standard Model, those symmetries are no longer respected. This type of symmetry breaking as a consequence of internal degrees of freedom is referred to as spontaneous symmetry breaking.
2.2. THE ELECTROWEAK THEORY

symmetry breaking. The field $\phi$ can now be expanded around this ground state $\langle \phi \rangle$. The constant term produces mass matrices for the electroweak vector bosons due to the covariant derivative term in eq. (2.16). After diagonalizing, we find the newly defined gauge bosons

$$
W_\mu^\pm = \frac{1}{\sqrt{2}} (W_\mu^1 \mp W_\mu^2) \\
Z_\mu = \frac{1}{g^2 + g'^2} \left( gW_\mu^3 - g'B_\mu \right) \\
A_\mu = \frac{1}{g^2 + g'^2} \left( g'W_\mu^3 + gB_\mu \right)
$$

with masses

$$
m_w = \frac{g}{2} v, \quad m_z = \frac{\sqrt{g^2 + g'^2}}{2} v = \frac{m_w}{c_w}, \quad \text{and} \quad m_a = 0.
$$

We have here defined $c_w = g/\sqrt{g^2 + g'^2}$ as the cosine of the Weinberg angle. After spontaneous symmetry breaking, the leftover symmetry is $U(1)_{em}$ with the photon field $A_\mu$ being its associated gauge boson. Its gauge operator is given by

$$
Q_{em} = I_3 + Y
$$

where $I_3$ is the third component of the $SU(2)_W$ isospin. The electromagnetic coupling constant can be found to be

$$
\epsilon = g' c_w.
$$

A very similar procedure can be carried out in the Yukawa sector. For instance, for the up-type quarks a mass matrix

$$
M_{uij} = y_{uij} \frac{v}{\sqrt{2}}
$$

is found. Diagonalization leads to the CKM matrix for quarks \cite{25, 26} and the PMNS matrix for leptons \cite{27}. These matrices encode the flavour-changing behaviour of the weak interaction and contains the only source of CP violation in the Standard Model.

The complex scalar $SU(2)_W$ doublet $\phi$ originally had four real components. After symmetry breaking three of these four components are absorbed by $W^\pm$ and $Z$ vector bosons, manifesting as their longitudinal polarizations. The fourth component remains as an individual scalar field which manifests as the Higgs boson. With the discovery of the Higgs boson \cite{2, 3} all components of the Standard Model have now been found. However, with open issues like the nature of dark matter which may well be weakly interacting \cite{28} as well as the nature of neutrino oscillations, it is not unlikely new physics may still appear at energies close to the electroweak scale.
Chapter 3

Event Generation

In this chapter we give an overview of the process of event generation for particle physics, and in particular for hadron colliders like the LHC. Hadron colliders have the advantage of being able to reach higher center-of-mass energies than lepton colliders, but the trade-off is that the collision events are much more complicated. A single scattering event at the LHC will for instance typically lead to the detection of $\mathcal{O}(100)$ particles. Because of the large multiplicity and the non-abelian nature of QCD leading to colour confinement, the only feasible option to produce phenomenological predictions is through the use of Monte Carlo event generators. Currently, three major general-purpose event generator programs are widely used: Pythia 8 [29, 30], Herwig 7 [31, 32] and Sherpa [33]. The event generation procedure is modular in the sense that it consists of several individual steps that are strung together after one another. A pictorial representation of a Monte Carlo event is shown in Figure 3.1.

Event generation starts with a hard scattering interaction, typically involving two or three final-state particles. These scatterings occur at sufficiently high energies such that QCD is weakly coupled and perturbation theory is applicable. The next step is to apply a parton shower. The parton shower is responsible for adding radiative corrections to the hard scattering creating the majority of the large multiplicity of hadron collision events. Parton showers are also based on perturbative physics, but they involve several approximations to allow for the simulation of a large number of radiative effects. Their analytical function is to sum up radiative corrections to all orders of perturbation theory in a process called resummation. The three major general-purpose event generators all have multiple parton showers available that differ significantly in a number of ways. In this chapter, we will describe the parton showering procedure in detail before specializing to one particular formalism in Chapter 5.

Since parton showers are also based on perturbative physics, matrix element cal-
Figure 3.1: A representation of a typical simulated LHC event. The incoming protons are indicated by the green ovals. Constituents of the protons collide with each other in the hard scattering indicated in blue. The energy scale associated with the hard scattering is $Q_{\text{hard}}^2$. Next, the parton shower adds radiation to the incoming and outgoing particles. The QCD shower continues until $\Lambda_{\text{QCD}}$, while electroweak radiation may be allowed to continue to lower scales $Q_{\text{cut}}^2$. The coloured partons are hadronized at $\Lambda_{\text{QCD}}$, and the outgoing hadrons are subsequently allowed to decay. Collisions of other constituents of the proton at energy scales smaller than the hard scattering scale $Q_{\text{hard}}^2$ are simulated by the multiple parton interactions model. These are indicated in red at the bottom.
3.1. THE HARD SCATTERING

culations may be used to improve the parton shower at least in the first couple of radiative corrections. In this respect, two different procedures are often distinguished: matching which refers to the matching of the parton shower to next-to-leading order (NLO) or higher matrix element calculations, and merging which refers to the inclusion of high-multiplicity leading order (LO) matrix element calculations.

Another important step in the event generation process involves the simulation of the fact that multiple partons in a single hadron-hadron collision may scatter. The simulation of this process is done with the multiple parton interaction (MPI) model. Most relevant nonperturbative physics is captured in the hadronization procedure where the colour-charged partons resulting from the parton shower are transformed into colour-neutral hadrons as the energy scale of the event becomes sufficiently low to cause the colour-charges to become strongly coupled. We review these steps involved in event generation briefly, putting some more emphasis on the parton shower as it is the topic of the rest of this thesis. Note that the discussion in this chapter is too short to be complete, and more elaborate reviews can be found in [34, 35].

3.1 The Hard Scattering

The generation of a Monte Carlo event starts with the computation of a hard scattering cross section to some order in perturbation theory. These calculations have now been automated at the NLO level in several matrix element generator programs like MadGraph5_aMC@NLO [36], WHIZARD [37] and Helac-Phegas [38]. Since then these generators have mostly replaced the traditional LO calculations. The basis for the calculation of scattering cross sections is the factorization theorem discussed in Chapter 2. It can be written as

\[
\sigma_{AB \to X} = \sum_{a,b} \int dx_a dx_b \int d\Phi_n f_{a/A}(x_a, \mu_F^2) f_{b/B}(x_b, \mu_F^2) \frac{1}{2s} |M_{ab \to X}(\Phi_n, \mu_F, \mu_R)|^2,
\]

(3.1)

where \( n \) is the multiplicity of the final state \( X \). Note that fragmentation functions may in principle be included to describe the hadronization of the final state. The partonic cross section \( \sigma_{ab \to X} \) has been replaced by several components. The squared matrix element \( |M_{ab \to X}(\Phi_n, \mu_F, \mu_R)|^2 \) provides the fully differential perturbative prediction from theory. It can be calculated up to some order in perturbation theory and depends on the factorization and renormalization scales respectively through the argument of the running coupling and the regulation of infrared divergences related to radiation from the initial state. One important property of the matrix element is that it consists
of several contributions

\[ |M_{ab\to X}(\Phi_n, \mu_F, \mu_R)|^2 = \sum_{h_i, c_j} |M_{ab\to X}^{(ij)}(\Phi_n, \{h_i\}, \{c_j\}, \mu_F, \mu_R)|^2 \]  

(3.2)

where the sums run over all helicity and colour configurations. Since all of these contributions are positive definite, it is possible to sample them individually and produce events that have a definite helicity and colour assignment. The factor \(1/(2\hat{s})\) is the partonic flux factor. \(\hat{s} = x_a x_b s\) is the partonic center-of-mass energy squared, while \(s\) is that of the hadrons. The integration element \(d\Phi_n\) represents the \(n\)-body phase space integral. It is given by

\[ d\Phi_n = (2\pi)^{4-3n} \delta^4(p_a + p_b - \sum_{i=1}^n p_i) \prod_{i=1}^n d^4 p_i \delta(p_i^2 - m_i^2). \]  

(3.3)

The phase space runs over all possible momentum configurations of the final-state particles given the constraints that they should respect momentum conservation and be on their mass shell. In many cases, one is not interested in the total cross section \(\sigma\), but rather in some differential distribution with respect to some observable \(O\). The distribution \(d\sigma/dO\) can then be computed by projecting out the observable from the phase space integral. Furthermore, kinematic cuts are often imposed for several reasons. For instance, they may reflect the detector geometry, they may be required to regulate singular behaviour or they may be imposed to separate some signal process from background.

When the cross section is calculated at higher accuracy than leading order, one not only has to include virtual corrections from loop diagrams to the lowest order Born process, but real radiative corrections should also be included. A cross section calculated at NLO thus consists of three parts, which can schematically be represented with

\[ d\sigma_{NLO} = d\Phi_n \left[ B(\tilde{\Phi}_n) + \alpha_s(\mu_R^2) V(\tilde{\Phi}_n) \right] + d\Phi_{n+1} \alpha_s(\mu_R^2) R(\tilde{\Phi}_{n+1}). \]  

(3.4)

where we have defined

\[ d\tilde{\Phi}_m = \frac{1}{s} \frac{dx_a}{x_a} \frac{dx_b}{x_b} d\Phi_m \]  

(3.5)

Similarly, the terms \(B(\tilde{\Phi}_n)\), \(V(\tilde{\Phi}_n)\) and \(R(\tilde{\Phi}_{n+1})\) include the PDFs. They are respectively the Born contribution and the virtual and real emission components.

The computation of eq. (3.4) is made difficult by the appearance of divergences. Ultraviolet divergences are readily dealt with by their absorption into the bare QCD parameters, but infrared divergences are more problematic. They appear in both the virtual and real corrections, and according to the Bloch-Nordsieck (BN)
and Kinoshita-Lee-Nauenberg (KLN) \cite{40,41} theorems, they should cancel between
the two for any sufficiently inclusive and infrared safe observables. However, real
corrections are integrated over a higher-dimensional phase space than the Born and
virtual corrections, making it difficult to incorporate this cancellation.

Several strategies are available to tackle this problem. The most common are
known as infrared subtraction methods, with the main implementations being the
Catani-Seymour (CS) dipole subtraction scheme \cite{43,44} or the antenna subtraction
scheme \cite{45,46,47}. Their basis is the fact that the infrared divergences in the real
emission corrections exhibit a universal structure which can be factorized from the
Born matrix element in the appropriate regions of phase space. A subtraction term
$S$ can be written down that contains all of the singular behaviour of the radiative
component $R$ such that $R - B \otimes S$ is finite over all of phase space. The convolutional
product indicates that a summation over internal quantum numbers may be
performed. This subtraction term is then integrated over the radiative phase space
$d\Phi_{n+1}/d\Phi_n$ and added back to the virtual term $V$. The total cross section can then
be written as
\begin{equation}
\sigma_{\text{NLO}} = \int d\Phi_n B(\Phi_n) + \alpha_s(\mu_R^2) \int d\Phi_n \left[ V(\Phi_n) + B \otimes \int \frac{d\Phi_{n+1}}{d\Phi_n} S(\Phi_{n+1}/\Phi_n) \right] + \alpha_s(\mu_R^2) \int d\Phi_{n+1} \left[ R(\Phi_{n+1}) - B(\Phi_n) \otimes S(\Phi_{n+1}/\Phi_n) \right].
\end{equation}

With the phase space factorization
\begin{equation}
d\Phi_{n+1} = \frac{d\Phi_{n+1}}{d\Phi_n} d\Phi_n
\end{equation}
comes a mapping of momenta from a phase space point in $d\Phi_{n+1}$ to a point in $d\Phi_n$.
The difference between the CS and antennae subtraction scheme is then the definition
of the subtraction terms, the phase space factorization eq. (3.7) and the momentum
mapping. A number of modern parton showers are also based on these components,
which will be discussed in more detail in the next subsection and the rest of this
thesis. We finally note that although the discussion in this section and the following
are based on QCD, the procedure for electroweak processes is very similar.

\section{Parton Showers}

In this section we describe the general formalism of a parton shower. We describe
the theoretical foundations and relevant approximations, but defer the details of the
\footnote{Counterexamples are known to exist, e.g. \cite{42}}
Monte Carlo implementation to Chapter 4.

The calculation of matrix elements quickly becomes very difficult as the number of external particles increases. The usual argument from perturbation theory indicates that higher-order corrections should decrease with corresponding powers of $\alpha_s$, and it should therefore be sufficient to restrict oneself to calculating the first few orders. However, in the regions of phase space close to the infrared divergences a higher-order matrix element may become enhanced. If these enhancements are such that the $\alpha_s$ suppression no longer guarantees that higher-order corrections are smaller than the lower order ones, the convergence of the perturbative series is spoiled.

### 3.2.1 Leading Logarithms

To investigate these enhancements, let us assume we consider an observable that is sensitive to infrared physics. Many very fundamental observables such as particle multiplicity or jet measures have this sensitivity. The infrared divergences in higher-order matrix elements are regulated by an energy scale close to $\Lambda_{\text{QCD}}$ that separates perturbative and non-perturbative physics. As this scale decreases, the contribution from the higher-order matrix elements grows, for instance causing the average number of particles to increase. At NLO, the infrared divergences leave their trace in contributions to infrared sensitive observables of the form

$$\alpha_s \log^2 \left( \frac{Q^2_{\text{hard}}}{Q^2_{\text{IR}}} \right) \quad \text{and} \quad \alpha_s \log \left( \frac{Q^2_{\text{hard}}}{Q^2_{\text{IR}}} \right),$$

where the scale $Q^2_{\text{hard}}$ is an energy scale associated with the hard scattering and $Q^2_{\text{IR}}$ is some resolution scale that regularizes the infrared divergences. Owing to the universal factorization properties of the infrared divergences that were also used in section 3.1, these contributions generalize to all orders in $\alpha_s$ as

$$\alpha_s^n \log^{2n} \left( \frac{Q^2_{\text{hard}}}{Q^2_{\text{IR}}} \right) \quad \text{and} \quad \alpha_s^n \log^{2n-1} \left( \frac{Q^2_{\text{hard}}}{Q^2_{\text{IR}}} \right).$$

These terms appear with $n$ powers of the coupling $\alpha_s$, and thus originate from $n$ emissions of additional partons associated with scales $Q^2_1, ..., Q^2_n$. A further requirement for these logarithms to appear is the strong ordering condition

$$Q^2_{\text{hard}} \gg Q^2_1 \gg ... \gg Q^2_n.$$  

The logarithms that originate from these ordered scales are called leading logarithms (LL). If not all scales are strongly ordered, further logarithms

$$\alpha_s^n \log^{2n-2} \left( \frac{Q^2_{\text{hard}}}{Q^2_{\text{IR}}} \right) \quad \text{and} \quad \alpha_s^n \log^{2n-3} \left( \frac{Q^2_{\text{hard}}}{Q^2_{\text{IR}}} \right),$$
and so on may still appear. These are referred to as next-to-leading log (NLL) or further. In short, if large differences in scales appear, it becomes necessary to sum up higher-order corrections to all orders in perturbation theory. This procedure is called \textit{resummation}, and it can be achieved through analytic methods (see [48] for a review). Another option is to cast the factorization of the infrared divergences in a probabilistic picture to be interpreted as a Monte Carlo algorithm. This is the starting point for the parton shower. We now describe its theoretical foundation due to [49, 50] in more detail.

### 3.2.2 Matrix Element Factorization

Parton showers are based on the factorization of infrared divergences from matrix elements. For simplicity, we will at first restrict ourselves to QCD radiation from the final state. There are two distinct phase space regions where these factorizations occur. The first is the \textit{quasi-collinear} limit [51, 52]. There, the factorization can be written as

\[
|M_{n+1}(\ldots, p_i, p_j, \ldots)|^2 = \frac{\alpha_s}{2\pi (p_i + p_j)^2} P_{I \rightarrow ij}(z) |M_n(\ldots, p_i + p_j, \ldots)|^2.
\]

The quasi-collinear limit is the phase space region where

\[
p_i \cdot p_j \approx m_i^2, m_j^2 \quad \text{and} \quad E_i^2, E_j^2 \gg p_i \cdot p_j.
\]

In other words, the partons $i$ and $j$ have energies much larger than their mass, but are moving in almost the same direction. The argument of the Altarelli-Parisi splitting function $P_{I \rightarrow ij}(z)$ is the momentum fraction $z$ carried by parton $i$. It may for instance be taken to be

\[
z = \frac{E_i}{E_i + E_j} \quad \text{or} \quad z = \frac{|\vec{p}_i|}{|\vec{p}_i| + |\vec{p}_j|}
\]

which are equivalent up to terms that are small in the quasi-collinear limit. For QCD, the Altarelli-Parisi splitting functions are given by

\[
P_{q \rightarrow gq}(z) = C_F \left[ 1 + z^2 \frac{m_q^2}{p_i \cdot p_j} \right]
\]

\[
P_{g \rightarrow gg}(z) = 2C_A \left[ \frac{z}{1 - z} + \frac{1 - z}{z} + z(1 - z) \right]
\]

\[
P_{g \rightarrow q\bar{q}}(z) = T_R \left[ z^2 + (1 - z)^2 + \frac{m_q^2}{p_i \cdot p_j + m_q^2} \right]
\]

where we have used the colour factors $T_R = 1/2$, $C_F = (N_C^2 - 1)/2N_C = 4/3$ and $C_A = N_C = 3$. The second divergent phase space region is the soft gluon limit, where
we consider the factorization of the matrix element in the leading colour limit

\[ |M_{n+1}(\ldots, p_i, p_j, p_k, \ldots)|^2 = \frac{\alpha_s}{4\pi} C \left[ 2 \frac{p_i \cdot p_k}{(p_i \cdot p_j)^2} - \frac{m_i^2}{(p_i \cdot p_j)^2} - \frac{m_k^2}{(p_j \cdot p_k)^2} \right] |M_n(\ldots, p_i, p_k, \ldots)|^2 + O(1/N_C^2). \]

The soft factorization in the leading-colour limit is illustrated in Figure 3.2. In this limit, the colour structure reduces to an assignment of colour ordering. This means that the only contributing diagrams to the emission of gluon \( j \) are those where the gluon is emitted from a parton adjacent to it in this ordering. The factorization in the soft (and quasi-collinear) limits may then be understood to appear because in both cases the propagator associated with the emission of \( j \) from an external line is almost on its mass shell. The diagram is therefore enhanced and dominates over other diagrams that are not enhanced.

### 3.2.3 The Parton Shower Formalism

Using the factorization properties of matrix elements, we begin constructing the parton shower starting from some hard process \( H \). For some observable \( \mathcal{O} \), the differential Born cross section is

\[
\frac{d\sigma_H}{d\mathcal{O}} = \int d\Phi_H |M_H^{(0)}|^2 \delta(\mathcal{O} - \bar{\mathcal{O}}(\{p\}_H)),
\]

where the \((0)\) refers to the number of loops. The delta function projects out the observable \( \mathcal{O} \) using its evaluation function \( \bar{\mathcal{O}}(\{p\}_H) \). We now insert an operator that is responsible for the inclusion of all higher-order corrections to the differential distribution eq. (3.17)

\[
\frac{d\sigma_H}{d\mathcal{O}} = \int d\Phi_H |M_H^{(0)}|^2 S(\{p\}_H, \mathcal{O}),
\]

where \( S(\{p\}_H, \mathcal{O}) \) is the showering function.
where the delta function is now incorporated in the operator $S$. We consider the first few terms stemming from the NLO corrections. From eq. (3.6), the operator at NLO precision should be

$$S^{(1)}(\{p\}_H, \mathcal{O}) = \left(1 + \frac{2\Re[M_H^{(0)}M_H^{(1)*}]}{|M_H^{(0)}|^2}\right) \delta(\mathcal{O} - \bar{\mathcal{O}}(\{p\}_H)) + \int \frac{d\Phi_{H+1}}{d\Phi_H} \frac{|M_{H+1}^{(0)}|^2}{|M_H^{(0)}|^2} \delta(\mathcal{O} - \bar{\mathcal{O}}(\{p\}_{H+1})).$$  \tag{3.19}

The second line evaluates the observable $\mathcal{O}$ from the final-state momenta $\{p\}_H+1$ with an extra emitted parton. The two higher-order contributions are separately divergent. In eq. (3.6) this was dealt with using the KLN theorem to define subtraction terms that remove the divergent contributions. Here, we will instead regulate the divergences by introducing the regulator $Q_{\text{Res}}^2$ that slices the phase space into a resolved and an unresolved piece. The definition of the scale $Q_{\text{Res}}^2$ should thus be able to classify all divergent phase space regions to be unresolved [53]. Any phase space point that is assigned a scale lower than $Q_{\text{Res}}^2$ is deemed invisible to the observable $\mathcal{O}$, and we can thus rewrite eq. (3.19) as

$$S^{(1)}(\{p\}_H, \mathcal{O}) = \left(1 + \frac{2\Re[M_H^{(0)}M_H^{(1)*}]}{|M_H^{(0)}|^2} + \int_{Q_{\text{Res}}^2} \frac{d\Phi_{H+1}}{d\Phi_H} \frac{|M_{H+1}^{(0)}|^2}{|M_H^{(0)}|^2} \right) \delta(\mathcal{O} - \bar{\mathcal{O}}(\{p\}_H))$$

$$+ \int_{Q_{\text{Res}}^2} \frac{d\Phi_{H+1}}{d\Phi_H} \frac{|M_{H+1}^{(0)}|^2}{|M_H^{(0)}|^2} \delta(\mathcal{O} - \bar{\mathcal{O}}(\{p\}_{H+1})).$$  \tag{3.20}

The divergences are now all contained in the first line of eq. (3.20). Next, we make use of the KLN theorem [40, 41] which can be written as

$$2\Re[M_H^{(0)}M_H^{(1)*}] + \int \frac{d\Phi_{H+1}}{d\Phi_H} \frac{|M_{H+1}^{(0)}|^2}{|M_H^{(0)}|^2} = K_H^{(1)},$$  \tag{3.21}

where $K_H^{(1)}$ contains no divergences. Using this expression, we rewrite eq. (3.20) to find

$$S^{(1)}(\{p\}_H, \mathcal{O}) = \left(1 + K_H^{(1)} - \int_{Q_{\text{Res}}^2} \frac{d\Phi_{H+1}}{d\Phi_H} \frac{|M_{H+1}^{(0)}|^2}{|M_H^{(0)}|^2} \right) \delta(\mathcal{O} - \bar{\mathcal{O}}(\{p\}_H))$$

$$+ \int_{Q_{\text{Res}}^2} \frac{d\Phi_{H+1}}{d\Phi_H} \frac{|M_{H+1}^{(0)}|^2}{|M_H^{(0)}|^2} \delta(\mathcal{O} - \bar{\mathcal{O}}(\{p\}_{H+1})).$$  \tag{3.22}

The NLO correction to the cross section is now entirely contained in the term $K_H^{(1)}$. Furthermore, the remaining ratio of matrix elements can be written as

$$\frac{d\Phi_{H+1}}{d\Phi_H} \frac{|M_{H+1}^{(0)}|^2}{|M_H^{(0)}|^2} = \frac{d\Phi_{H+1}}{d\Phi_H} K_H^{(0)} + \sum_r \frac{d\Phi_{H+1}}{d\Phi_H} S_r,$$  \tag{3.23}
where $K^{(0)}_{H+1}$ is process-dependent, but the additional terms $S_r$ are the universal singular contributions referred to as *branching kernels*. Their universal nature is due to the matrix element factorization properties of eq. (3.12) and eq. (3.16). The sum over $r$ runs over radiators whose definition depends on the parton shower model. The sum will include modes of gluon emission of different external legs, but it may also sum over other types of branchings like gluon splitting to $q\bar{q}$. We give an overview of the commonly used models at the end of this section.

Up to this point, eq. (3.22) is still exact. The major approximation of the parton shower is to drop the non-singular terms $K^{(1)}_H$ and $K^{(0)}_{H+1}$ which are deemed to be small with respect to the singular contributions as long as there is a large scale separation between $Q^2_{\text{Hard}}$ and $Q^2_{\text{Res}}$. What remains is

$$
S^{(1)}(\{p\}_H, \mathcal{O}) = \left(1 - \sum_r \int_{Q^2_{\text{Res}}} d\Phi_{H+1} S_r \right) \delta(\mathcal{O} - \bar{\mathcal{O}}(\{p\}_H)) + \sum_r \int_{Q^2_{\text{Res}}} \frac{d\Phi_{H+1}}{d\Phi_H} S_r \delta(\mathcal{O} - \bar{\mathcal{O}}((\{p\}_{H+1})).
$$

From this expression, it is clear that the parton shower is *unitary*. This means that it does not change the total cross section, which can be seen from the fact that the radiative corrections cancel from eq. (3.24) if the measurement delta functions were removed. Unitarity would be violated if the process-dependent terms $K^{(1)}_H$ and $K^{(0)}_{H+1}$ were not dropped and it is indeed to be expected that the full NLO correction alters the total cross section. In this regard, the parton shower approximation can be understood as expanding the universal structure of the infrared singularities to all of phase space. The parton shower has little accuracy outside the singular region, but it comes with the advantage that it is process-independent and is able to capture the largest enhancements of the matrix element.

Owing to the universal structure of the factorization properties eq. (3.12) and eq. (3.16), the parton shower approximation can be applied at all orders of perturbation theory. The resulting shower operator can be defined in terms of the *Sudakov form factor*

$$
\Delta(\{p\}, Q^2_1, Q^2_2) = \exp \left[- \sum_r \int_{Q^2_{\text{Res}}} \frac{d\Phi_{H+1}}{d\Phi_H} S_r \right],
$$

which exponentiates all radiative corrections. The shower operator becomes

$$
S(\{p\}_H, Q^2_{\text{Hard}}, Q^2_{\text{Res}}, \mathcal{O}) = \Delta(\{p\}, Q^2_{\text{Hard}}, Q^2_{\text{Res}}) + \sum_r \int_{Q^2_{\text{Res}}} \frac{d\Phi_{H+1}}{d\Phi_H} S_r \Delta(\{p\}, Q^2_{\text{Hard}}, Q^2_{H+1}) S(\{p\}_{H+1}, Q^2_{H+1}, Q^2_{\text{Res}}, \mathcal{O}).
$$

(3.26)
The above expression serves as a complete description of a parton shower. After expanding the form factors, the NLO result of eq. (3.24) is indeed recovered. Eq. (3.26) is cast in a recursive form, which can be expanded up to arbitrary perturbative order. The process is Markovian, meaning that the shower operator is defined as a sequence of iterative steps where every iteration only depends on the current state.

The first term on the right-hand side of eq. (3.26) is associated with exclusive $H$ production, meaning no additional branchings from the hard state $H$ occur. It comes with the form factor $\Delta(\{p\}, Q_{\text{Hard}}^2, Q_{\text{Res}}^2)$, which contains all virtual and unresolved real corrections. Since this term is always positive and smaller than one, a probabilistic interpretation can be assigned to it. We conclude that the Sudakov form factor $\Delta(\{p\}, Q_1^2, Q_2^2)$ represents the no-branching probability for a parton shower evolution between resolution scales $Q_1^2$ and $Q_2^2$. Similarly, the second term in equation eq. (3.26) is associated with inclusive $H + 1$ production, meaning that at least one additional branching has occurred at scale $Q_{H+1}^2$, but more may occur as a consequence of the iterative application of the shower operator. The term may be understood probabilistically as the no-branching probability between scales $Q_{\text{Hard}}^2$ and $Q_{H+1}^2$ given by the form factor, multiplied with the branching probability given by $S_r$. The total probability is guaranteed to evaluate to unity because of the unitary nature of the parton shower.

Note that the scales are ordered in the sense that $Q_{\text{Hard}}^2 > Q_{H+1}^2 > Q_{H+2}^2 > \ldots > Q_{\text{Res}}^2$, which is contrasted with the strong ordering condition eq. (3.10). The parton shower implementation of strong ordering is required for the exponentiation of the higher-order correction and a full coverage of phase space. If no ordering condition were in place, the parton shower would indeed double-count states with more than a single branching. The ordering of the parton shower is therefore chosen such that the leading logarithmic enhancements are incorporated.

We have so far only described parton showers for final-state branchings. For branchings from the initial state, the parton shower evolution is best captured by backward evolution which was first derived in [54]. As is the case with final-state evolution, the initial-state shower is started from the hard scattering scale $Q_{\text{Hard}}^2$. The parton shower evolution then tracks the initial state back in time, unfolding the DGLAP evolution of eq. (2.6) and increasing the initial-state momentum fractions as particles are emitted. Recalling the factorization theorem eq. (3.1), the no-branching probability should be modified to be

$$\Pi(\{p\}, x_a, x_b, Q_1^2, Q_2^2) = \frac{f_{a/A}(x_a, Q_{2}^2) f_{b/B}(x_b, Q_{2}^2)}{f_{a/A}(x_a, Q_1^2) f_{b/B}(x_b, Q_1^2)} \Delta(\{p\}, Q_1^2, Q_2^2).$$

(3.27)

The ratio of PDFs corrects the previous resolution scale $Q_1^2$, which starts from the
factorization scale at the hard scattering, to the new resolution scale $Q_2^2$. The phase space integral in the Sudakov form factor now also involves the momentum fractions $x_a$ and $x_b$. It can be written as

$$\Delta(\{p\}, Q_1^2, Q_2^2) = \exp \left[ -\sum_r \int_{Q_1^2}^{Q_2^2} \frac{d\Phi_{H+1}^r}{d\Phi_H} \frac{x_a x_b}{x_{\bar{a}} x_{\bar{b}}} S_r \right],$$

(3.28)

where $x_{\bar{a}}$ and $x_{\bar{b}}$ are the post-branching momentum fractions. Note that $\bar{a}$ and $\bar{b}$ may be different from $a$ and $b$ since the initial-state parton may change due to a branching. The sum over radiators now also contains the sum over all initial-state branchings. The ratios of momentum fractions $x_a/x_{\bar{a}}$ and $x_b/x_{\bar{b}}$ reduce to the initial-state splitting momentum fraction $z$ in the collinear limit. For collinear initial-state branchings, the branching kernels $S_r$ must thus reduce to

$$S_{r, \text{init}} = \frac{1}{z} P(z)$$

(3.29)

compensating the momentum fraction ratio. For final-state branchings, the momentum fractions remain unchanged and the ratio vanishes. The DGLAP equation can be used to rewrite eq. (3.27) to

$$\Pi(\{p\}, x_a, x_b, Q_1^2, Q_2^2) = \exp \left[ -\sum_r \int_{Q_1^2}^{Q_2^2} \frac{d\Phi_{H+1}^r}{d\Phi_H} S_r f_a(x_{\bar{a}}, Q_1^2) f_b(x_{\bar{b}}, Q_2^2) f_a(x_a, Q_1^2) f_b(x_b, Q_2^2) \right].$$

(3.30)

Using this new no-branching probability in the parton shower evolution operator eq. (3.26) results in an evolution that is consistent with the DGLAP equation [55], and thus adequately unfolds the PDF evolution as a function of the resolution scale.

### 3.2.4 Parton Shower Implementations

We conclude this section with an overview of currently available parton showers and briefly describe their major differences. The first parton showers used the Altarelli-Parisi splitting functions as their branching kernels. Examples are the Herwig angular-ordered shower [56, 57], the default Pythia shower [58] and the earlier Sherpa shower [59]. These showers automatically get the collinear factorization of eq. (3.12) right, but they do not include the soft factorization of eq. (3.16) naturally. The soft limit can be incorporated approximately by ensuring that subsequent emissions always decrease in their opening angle [56], which can be achieved by either defining an ordering variable that includes the opening angle, or by a veto on emissions that would violate the angular ordering requirement.

While the branchings of these types of parton showers are defined dynamically as $1 \rightarrow 2$ as encoded in the Altarelli-Parisi splitting functions, the phase space factorization and the associated kinematic mapping has to be either $2 \rightarrow 3$ or some
form of global recoil. This is required to ensure that all partons can remain on their mass shell, which is not possible for $1 \rightarrow 2$ branchings without violating momentum conservation.

Another type of parton showers are based on the subtraction terms used in eq. (3.6). They may either be based on the Catani-Seymour subtraction terms, as is the case in the Sherpa dipole shower [60] and the Herwig dipole shower [61], or on the antenna subtraction terms, as is the case in the ARIADNE shower [62] and the Vincia shower [50, 63]. The DIRE shower [55] fits in between the Altarelli-Parisi and dipole approaches. Another approach is used by the DEDUCTOR parton shower [64], which is based on quantum density matrices. These types of showers fit in the $2 \rightarrow 3$ picture from both the dynamic and kinematic point of view. Their major difference is that in the case of dipole showers, one of the two pre-branching momenta is assigned to be the ‘brancher’ while the other is the ‘spectator’. On the other hand, in the antenna picture no such distinction is made and both particles participate in a branching equally. Because of their $2 \rightarrow 3$ construction, these showers can be built to get the soft and collinear factorization properties right automatically.

### 3.3 Matching and Merging

While the parton shower approximation to the matrix element performs well in the soft and collinear parts of phase space, it may become very inaccurate outside of these regions. On the other hand, fixed-order calculations do describe these regions accurately, but only up to some order in perturbation theory. The goal of matching and merging algorithms is to combine the best of both approaches. The most pressing issue to be avoided is double-counting between the matrix element and the parton shower. The parton shower already includes the leading logarithmic contributions of every perturbative order which overlap with the higher-order corrections to matrix elements.

The term matching is typically used for methods that correct the first parton shower emission such that it is accurate to at least NLO precision. The two main methods are POWHEG [65] and MC@NLO [66]. The POWHEG method corrects the branching kernels of the parton shower using the NLO real emission matrix element. A normalization factor is then applied to correct for the total NLO cross section. The MC@NLO method does not correct the branching kernel, but adds a normalization that contains the LO contribution and the virtual contribution regularized with a subtraction term. The real corrections are then added on separately to recover the full NLO cross section.
The term merging instead refers to the inclusion of higher-order tree-level matrix elements to correct or replace the first few emission of the parton shower. Many approaches to merging are available. The most prominent one is CKKW(-L) [67, 68, 69, 70], although other approaches such as MLM [71], METS [72], and UMEPS [73, 74] are available. There are technical differences between the algorithms, and the more modern ones have considerable advantages over the older ones, but they all rely on the use of external event samples of events with higher multiplicities. These events must then be made exclusive by application of the appropriate Sudakov factors, and they may be corrected for scale variations in the strong coupling. Finally, the event samples are merged with the parton shower in a way to avoid double-counting.

Other methods for matrix element corrections to parton showers exist. Methods for NLO merging are also available [75, 76, 77], and the GKS matrix element correction scheme implemented in Vincia [49, 50, 78] combines matching and merging in a single formalism. Merging with matrix elements containing up to 9 jets has recently been achieved [79].

### 3.4 Multiple Parton Interactions

In hadronic collision events that contain highly energetic parton-parton interactions, the collision is likely to be very central. In those cases, it is very likely that other constituents of the hadrons interact, leaving an impact on many types of observables. This observation is known as the jet pedestal effect where hard outgoing jets are accompanied by a large amount of underlying activity.

In a Monte Carlo context, this effect is modelled by a multiple parton interaction (MPI) model [80]. These models are based on the observation that the vast majority of coloured scattering processes are $t$-channel exchanges between particles. The differential cross section of these processes behaves as

$$d\hat{\sigma}_{\text{QCD}} \propto \frac{dp_{\perp}^2}{p_{\perp}^4},$$  \hspace{1cm} (3.31)

where the transverse momentum now serves as a resolution scale. The average number of interactions will thus increase as the resolution scale $p_{\perp}$ decreases. In analogy with a parton shower, MPI can then be modelled by defining a no-scattering probability

$$P_{\text{MPI}}(b, p_{\perp}^2, p_{\perp}^2_{\text{max}}) = \exp\left(-A(b) \int_{p_{\perp}^2}^{p_{\perp}^2_{\text{max}}} \frac{d\hat{\sigma}_{\text{QCD}}}{dp_{\perp}^2}\right), \hspace{1cm} (3.32)$$

where $b$ is the hadronic impact parameter functioning as a measure of centrality of the collision. The parameter $A(b)$ encodes the fact that more interactions should occur for
more central collisions, as well as a normalization to the total hadronic cross section. This no-scattering probability can then be used to interleaves MPI modeling with the parton showers [58]. Every time an MPI scattering occurs, a subsequent shower may be initiated in the new MPI system starting from its associated scattering scale. The effects of these additional scattering systems does usually not lead to easily identifiable additional jets, but it provides a significant contribution to the total scattered energy and causes many more colour charges to appear in the event. The details of the treatment of MPI for the three general-purpose event generators can be found in [58, 81, 82].

3.5 Hadronization

As the parton shower and MPI evolution progress, the resolution scale approaches the nonperturbative regime of QCD. Once the scale reaches approximately $\Lambda_{\text{QCD}}$, QCD becomes strongly coupled and its colour confinement property must be modelled in some way. Not much is known about QCD in the nonperturbative regime, but some observations from lattice QCD have led to sufficient knowledge to build practical Monte Carlo models. The two main hadronization models are the string model [83] implemented in Pythia and the cluster model [84, 85] implemented in Herwig and Sherpa.

The string model is based on the observation that the QCD potential $V(r)$ between a quark-antiquark pair rises linearly with their distance $r$, as given by $V(r) = \kappa r$ where $\kappa \approx 2$ GeV/fm is the string tension. This fact is translated to a model where a string with an associated tension is spanned between the pair. As the quarks move apart, the potential energy contained in the string increases. At sufficiently high energies, the string may break and new quark-antiquark pairs are produced. Eventually, the energy density is not high enough to produce new pairs anymore, and quark-antiquark pairs that are strung together form mesons. Baryon production may also be included by allowing the production of diquark pairs. Gluons form kinks in the string, having an effect on the multiplicity and kinematic distribution of the produced hadrons. As a consequence, the model predicts that hadron production for final states with a quark-antiquark pair and a gluon is suppressed in the angular regions between the quark and the antiquark. This was confirmed in experiment [86].

The cluster model is based on the preconfinement property of QCD [87]. It implies that the parton shower prefers to create colour-singlet combinations with a universal invariant mass distribution. Models based on this fact enforce gluon splittings to quark-antiquark pairs at the end of the parton shower evolution. Colour-connected
quarks are then formed into clusters that display the preconfinement property. Low-mass clusters are transformed into hadrons, while high-mass clusters may first decay into more clusters before being turned into hadrons.
Chapter 4

Monte Carlo Techniques

In this chapter we discuss the topic of Monte Carlo techniques. For the purpose of translating theoretical particle physics models to quantitative predictions that may be compared to data from experiments, complicated, high-dimensional integrals often have to be calculated. These integrals are typically too cumbersome to compute analytically, so one often resorts to numerical methods. Monte Carlo methods are a subclass of these numerical methods which make use of random numbers to approximate integrals. They have some distinct technical advantages over other numerical integration strategies that will be discussed below, but one major upside is that integrals are approximated by drawing random samples from the differential space. In a physical context this space usually represents some physical concept. For example, in particle physics integrals over phase space are often performed. The samples generated by the Monte Carlo techniques then correspond to physical events, and these events can be used as a fully differential probe of the underlying physical distribution. In most of this thesis the sampling of events is in fact the primary purpose of Monte Carlo techniques, and they are used for all steps of the event generation procedure. In the following sections we discuss some generalities of Monte Carlo techniques and continue with an elaborate treatment of the so-called Sudakov veto algorithm which is heavily used in the rest of this thesis and in the construction of parton shower algorithms in general. The results presented in this section have partly been published in [88].

4.1 Monte Carlo Integration

Suppose we are interested in integrating some function $f(x)$ of an $n$-dimensional variable $x$. Many numerical techniques are available for this purpose as long as $n$ is not too large. Examples include Newton-Cotes formulas and Gaussian quadratures
These methods all suffer from the fact that they become increasingly inefficient as \( n \) increases to the point where they are no longer practically useful. In these cases Monte Carlo integration offers a more efficient solution. The integral we are interested in is

\[
I_f = \int_V dx \, f(x),
\]

where \( V \) is the volume of the integration space or phase space. Furthermore, let us assume that we are able to sample independent and identically distributed (iid) points \( x_i \) from a probability distribution \( g(x) \) over the integration region \( V \). In the simplest possible case, one may take the points \( x_i \) to be uniformly distributed according to \( g(x) = 1/\text{Vol}(V) \), but we will see that more involved choices may have an advantage. We then define the weight

\[
w(x) = \frac{f(x)}{g(x)}.
\]

The expectation values of powers of these weights are

\[
J_n = \langle w^n \rangle = \int_V dx \, g(x) w^n(x).
\]

From the fact that \( g(x) \) is a probability distribution, we find \( J_0 = 1 \). Similarly, \( I_f = J_1 \). Next, we define the Monte Carlo estimators

\[
E_1 = \frac{1}{N} \sum_{i=1}^{N} w(x_i)
\]

\[
E_2 = \frac{1}{N^2(N-1)} \left( N \sum_{i=1}^{N} w^2(x_i) - \left( \sum_{i=1}^{N} w(x_i) \right)^2 \right).
\]

Using eq. (4.3) and the assumption that the points \( x_i \) are independent, it can be proven that

\[
\langle E_1 \rangle = J_1
\]

\[
\langle E_2 \rangle = \frac{1}{N} \left( J_2 - J_1^2 \right) = \sigma^2(E_1),
\]

where \( \sigma^2(E_1) = \langle E_1^2 \rangle - \langle E_1 \rangle^2 \) is the standard deviation of \( E_1 \), which is itself a random variable. The expectation value of the Monte Carlo estimator \( E_1 \) is therefore equal to the integral we were interested in computing. The estimator \( E_2 \) provides us with an estimation of the standard deviation of \( E_1 \), and it can be interpreted as a measure of its error. Note that \( \sigma(E_1) \) decreases with \( 1/\sqrt{N} \), and this factor is independent of the dimension \( n \) of the points \( x_i \).

We pointed out earlier that there is a choice to be made for the probability distribution \( g(x) \). The estimator \( E_2 \) will be smaller whenever the weights \( w(x_i) \) fluctuate
less, which will lead to faster convergence to the desired integral. The choice of a suitable probability distribution for \( g(x) \) is known as importance sampling. One common pitfall is that the probability distribution \( g(x) \) may become zero in some region in \( V \). The variance is infinite in those regions, but this problem is hidden since no points will ever be sampled there. In practice, the procedure is typically to select a functional form for \( g(x) \) that approximates the function \( f(x) \) as closely as possible while still remaining simple enough to sample from it efficiently. An algorithm commonly used to achieve this is VEGAS [90, 91, 92], a form of adaptive stratified sampling. It constructs a stratified approximation of \( f(x) \) by filling \( V \) with a piecewise flat distribution. As the Monte Carlo integration progresses, the approximation is updated using the information gained from previous samples. Another common technique is multi-channeling [93], which can be used in case \( f(x) \) can be written as a sum over multiple contributions. The distribution \( g(x) \) is then also split up into multiple pieces and an optimal balance is found between them. In practical applications a combination of the two above methods is often used to sufficient efficiency.

In many applications such as event generation in particle physics, one is not only interested in the integral \( I_f \) but also (primarily) in the generated points \( x_i \). In those cases, it is often convenient to obtain a set of points (usually called events) which have unit weights. Looking back at eq. (4.2), this means that the points \( x_i \) should be distributed according to \( f(x) \) normalized over the integration region \( V \). This kind of sampling can be achieved using the rejection sampling algorithm that is explained in the next section.

### 4.2 The Unitary Algorithm Formalism

The unitary algorithm formalism is a method to analyze the kinds of probabilistic Monte Carlo algorithms that we will encounter in the rest of this chapter. It is useful as it provides a very direct way to translate algorithms in terms of pseudocode to mathematical statements that allow for the computation of the probability distribution generated by that algorithm. We first list a number of ingredients that the formalism makes use of, and then proceed to show how it functions on some standard Monte Carlo results.

Let \( f(x) \) be a probability density. The expression

\[
1 = \int dx f(x)
\]

(4.6)

corresponds to the algorithmic statement that \( x \) is a random number sampled according to the distribution \( f \). In pseudocode, we denote this as \( x \sim f(x) \). A simple
example is the generation of a random number $r$ between 0 and 1 that is part of an independent and identically distributed stream of numbers. In unitary language, this corresponds to

$$1 = \int dr \, \theta(0 < r < 1) = \int_0^1 dr.$$  \hspace{1cm} (4.7)

The probability distribution is in this case defined by the generalized Heaviside step function

$$\theta(S) = \begin{cases} 
1 & \text{if the statement } S \text{ is true}, \\
0 & \text{if the statement } S \text{ is false}.
\end{cases}$$  \hspace{1cm} (4.8)

A second ingredient is the assignment operation

$$1 = \int dy \, \delta(y - h(x)),$$  \hspace{1cm} (4.9)

which is equivalent to the pseudocode statement

$$y \leftarrow h(x).$$  \hspace{1cm} (4.10)

We shall of course use the standard result

$$\delta(y - h(x)) = \sum_j \frac{1}{|h'(x_j)|} \delta(x - x_j),$$  \hspace{1cm} (4.11)

where the sum runs over the roots $x_j$ of $h(x) = y$ (all assumed to be single). It should be noted here that the integral over $y$ runs over all real values, but if the range of $h$ is restricted to $h_0 \leq h(x) \leq h_1$, the corresponding bounds on $y$ automatically follow. Using these two simple ingredients, almost all Monte Carlo-type algorithms can be analyzed to find their resulting probability distribution. We show how this can be achieved by considering some standard results.

### 4.2.1 Inversion Sampling

If the primitive $F(x)$ of a one-dimensional probability distribution $f(x)$ is available, one can use the process of inversion sampling to generate points distributed according to $f(x)$. We use this algorithm as a very simple example of the formalism described above. The procedure is

<table>
<thead>
<tr>
<th>Algorithm 1 Inversion sampling</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r \sim \theta(0 &lt; r &lt; 1)$</td>
</tr>
<tr>
<td>$x \leftarrow F^{-1}(r)$</td>
</tr>
</tbody>
</table>
We translate this to an expression for the probability distribution $p(x)$ using the ingredients of eq. (4.6) and eq. (4.9). We find

$$p(x) = \int_0^1 dr \delta(x - F^{-1}(r)). \quad (4.12)$$

The domain of $F^{-1}$ is indeed the same as that of $r$ since $f$ is a probability distribution. Eq. (4.12) is just the subsequent application of eq. (4.6) and eq. (4.9) following along with the algorithm. The $x$-integral is absent since it is the output of the algorithm. The mathematical representations of the components of the algorithm are all equal to one, indicating the preservation of probability (unitarity) of the algorithm. In other words, if we were to add on the $x$-integral, we would find

$$\int p(x) = 1 \quad (4.13)$$

as is required. Performing the $x$-integral in eq. (4.12) indeed leads to the expected result $p(x) = f(x)$.

From here on, we will denote uniformly sampled random numbers between 0 and 1 with $r$, adding subscripts $r_i$ if an algorithm uses several.

### 4.2.2 Rejection Sampling

In many cases, the probability distribution $f(x)$ is not simple enough to directly sample using inversion sampling. In those cases, the next best option is to use rejection sampling. This algorithm comes with the upside that $f(x)$ is not required to be normalized. Instead, the algorithm will take care of that automatically. Let $g(x)$ be a probability density that we can generate, for instance with inversion sampling, and $c$ a number such that $c g(x) \geq f(x)$ over the support of $f(x)$. The rejection algorithm then reads

**Algorithm 2** The rejection algorithm

```
loop
    x \sim g
    if cr \leq f(x)/g(x) then
        return x
    end if
end loop
```

Algorithm 2 is recursive. This is conveniently incorporated by writing

$$p(x) = \int dy g(y) \int_0^1 dr \left[ \theta \left( r \leq \frac{f(y)}{c g(y)} \right) \delta(x - y) + \theta \left( r > \frac{f(y)}{c g(y)} \right) p(x) \right]. \quad (4.14)$$
This expression is again straightforwardly constructed by following along with the algorithm and inserting the unitary statements of eq. (4.6) and eq. (4.9), as well as step functions that encode the if-statement and add up to one. Working out the integrals leads to

\[ p(x) = \int dy \left[ \frac{f(y)}{c} \delta(x - y) + g(y) p(x) - \frac{f(y)}{c} p(x) \right] \]

\[ = \frac{1}{c} f(x) + p(x) - \frac{1}{c} \int dy \ f(y) p(x), \]

from which we see that \( p(x) \) is the normalized probability density

\[ p(x) = \frac{f(x)}{\int dy f(y)}. \]

In the next section, we will use the formalism described here to analyze the Sudakov veto algorithm, an algorithm that is heavily used in parton showers.

4.3 The Sudakov Veto Algorithm

We now turn our attention to the Sudakov veto algorithm, which is commonly and heavily used in parton shower programs. The algorithm may be viewed as a variation on the rejection sampling algorithm, and it can similarly be used to sample efficiently from complex probability distributions that may appear as part of parton showers simulations. To ease readability, we denote the squared ordering scale \( t = Q^2 \) in this section. This scale is typically evolved down from an initial scale \( u \), generating ordered branchings of partons. The scale of the next branching is selected according to a probability distribution of the form

\[ p(t; u) = f(t) \Delta_f(t, u) \theta(0 < t < u) \text{ where } \Delta_f(t, u) \equiv \exp \left( - \int_{t}^{u} dx f(x) \right), \]

where the function \( f(t) \) denotes the branching kernel and \( \Delta_f(t, u) \) denotes the Sudakov form factor. We first establish the normalization of eq. (4.17):

\[ p(t; u) = \frac{\partial}{\partial t} \Delta_f(t, u) \rightarrow \int_{0}^{u} dt \ p(t; u) = 1 - \exp \left( F(0) - F(u) \right). \]

Here, \( F(t) \) is the primitive function of \( f(t) \). The distribution \( p(t; u) \) is thus normalized if \( F(t) \rightarrow -\infty \) as \( t \rightarrow 0 \). To sample from eq. (4.17) using inversion sampling, the inverse of the primitive of the branching kernel is required. Unfortunately, \( f(t) \) is typically not simple enough for this inverse to be analytically calculable. The Sudakov veto algorithm is then available as a very efficient alternative.
4.3. THE SUDAKOV VETO ALGORITHM

The Sudakov veto algorithm relies on the existence of an overestimate function $g(t) \geq f(t)$ which does have an invertible primitive. The algorithm is given in pseudocode in Algorithm 3.

Algorithm 3 The Sudakov veto algorithm

\begin{algorithm}
\begin{algorithmic}
\State $t \leftarrow u$
\Loop
\State $t \leftarrow G^{-1}(\log(r_1) + G(t))$
\If {$r_2 < f(t)/g(t)$} \Return $t$ \EndIf
\EndLoop
\end{algorithmic}
\end{algorithm}

The first step in the loop is an application of the inversion sampling as shown in Algorithm 1. It is equivalent to solving the equation

\begin{equation}
    r = \Delta_g(t; u). \tag{4.19}
\end{equation}

The first step in the loop therefore generates scales $t$ distributed according to eq. (4.17), but with $f(t)$ replaced by $g(t)$. The rest of the algorithm then corrects for this, recovering eq. (4.17) fully. The Sudakov veto algorithm is very similar to the rejection sampling algorithm as shown in Algorithm 2. The difference is that every time a scale is vetoed, the next scale is guaranteed to be smaller than the previous scale. The veto algorithm thus describes a downward evolution of the ordering scale. The larger the overestimate $g(t)$, the higher the average sampled scale will be. These high scales are then rejected more often as $f(t)$ is smaller than $g(t)$, continuing the evolution downward. The rejection sampling algorithm on the other hand resets completely after rejecting a sample. To confirm that Algorithm 3 produces the probability distribution given by eq. (4.17), we use the unitary algorithm formalism to write

\begin{equation}
    p(t; u) = \int_0^u d\tau g(\tau) \Delta_g(u, \tau)
    \times \int_0^1 dr \left[ \theta \left( r < \frac{f(\tau)}{g(\tau)} \right) \delta(\tau - t) + \theta \left( r > \frac{f(\tau)}{g(\tau)} \right) p(t; \tau) \right]. \tag{4.20}
\end{equation}

After generating a trial scale $\tau$, the random number $r$ and the step functions guide the algorithm to either accept the generated scale, or to start over using $\tau$ as the new starting point. Next, the integral over $r$ is worked out, yielding

\begin{equation}
    e^{G(u)} p(t; u) = \int_0^u d\tau e^{G(\tau)} \left[ f(\tau) \delta(t - \tau) + (g(\tau) - f(\tau)) p(t; \tau) \right]. \tag{4.21}
\end{equation}
Taking the derivative with respect to \( u \), we find a differential equation:

\[
\frac{\partial}{\partial u} p(t; u) = f(u) \delta(t - u) - f(u) p(t; u). \tag{4.22}
\]

It is indeed solved by

\[
p(t; u) = f(t) \exp \left( - \int_{t}^{u} dx p(x) \right) \theta (0 < t < u), \tag{4.23}
\]

which is eq. (4.17). However, eq. (4.17) is not the most general solution to eq. (4.22). We will consider this issue more carefully in the next section.

### 4.4 Extending the Veto Algorithm

Next, we consider the Sudakov veto algorithm in a more practical setting. The algorithm needs to be extended in several ways to be applicable in a real parton shower. Below we discuss several extensions separately, noting that many of them can easily be combined to obtain the benefits from all of them. For the sake of readability, we keep the overlap to a minimum.

#### 4.4.1 Introducing Auxiliary Variables

The scale variable \( t \) is usually not enough to parameterize the entire branching phase space. At least one auxiliary variable \( z \) has to be introduced. In traditional parton showers, this parameter is the energy fraction carried by a newly created parton. However, in the more modern dipole-type showers, it is just a variable that parametrizes the factorized phase space. The boundaries of the branching phase space translate to scale-dependent boundaries on \( z \). Depending on the situation, more variables may be required. In a simple parton shower algorithm, the azimuthal angle \( \phi \) typically appears as a third variable. We will here assume \( \phi \)-independent overestimate functions such that the additional variable \( z \) is the only addition. We note however that the extension described here is easily expanded to multiple relevant additional variables.

The target distribution is now

\[
p(t, z; u) = f(t, z) \Delta(u, t) \theta (0 < t < u), \tag{4.24}
\]

where

\[
\Delta(u, t) = \exp \left( - \int_{t}^{u} d\tau \int_{z-(\tau)}^{z+(\tau)} d\zeta f(\tau, \zeta) \right). \tag{4.25}
\]
This distribution is normalized as

$$\int_0^u dt \int_{z_-(t)}^{z_+(t)} dz \, p(t, z; u) = 1. \quad (4.26)$$

We now need to produce pairs \((t, z)\) distributed according to \(p(t, z; u)\). A difficulty lies in the dependence of the range of \(z\) on the scale. In order to generate a value for \(t\), the \(\zeta\) integral in the Sudakov factor is required, which depends on \(t\). On the other hand, \(z\) cannot be generated first, since its boundaries depend on \(t\). To deal with this problem, an additional veto condition is introduced. We introduce a constant overestimate of the \(z\)-range as \(z_- \leq z_- (t)\) and \(z_+ \geq z_+ (t)\). Additionally, we require the overestimate function to be factorized as \(g(t, z) = h(t) s(z)\) where still \(g(t, z) \geq f(t, z)\).

Then, we define

$$\Sigma = \int_{z_-}^{z_+} d\zeta \, s(\zeta). \quad (4.27)$$

The algorithm is given in Algorithm 4.

**Algorithm 4** The Sudakov veto algorithm with an auxiliary variable

1. \(t \leftarrow u\)
2. loop
   1. \(t \leftarrow H^{-1}(\log(r_1)/\Sigma + H(t))\)
   2. \(z \leftarrow S^{-1}(r_2 \Sigma + S(z_-))\)
   3. if \(r_3 < f(t, z)/g(t, z)\) and \(z_- (t) < z < z_+ (t)\) then
      1. return \(t\)
   4. end if
3. end loop

The step of the algorithm that generates \(t\) is just the same as before, with the exception that the branching kernel is now \(\Sigma h(t)\). The step that generates \(z\) is just another instance of inversion sampling. It produces samples of \(z\) distributed according to the normalized distribution \(s(z)/\Sigma\). Introducing the notation

$$\theta^\tau(\zeta) \equiv \theta(z_-(\tau) < \zeta < z_+ (\tau)), \quad (4.28)$$

we now analyze Algorithm 4.

\[
p(t, z; u) = \int_0^u \Sigma h(\tau) \Delta_{\Sigma h}(u, \tau) \int_{z_-}^{z_+} d\zeta \frac{s(\zeta)}{\Sigma} \int_0^1 dr \]
\[
\times \left\{ (1 - \theta^\tau(\zeta)) p(t, z; \tau) + \theta^\tau(\zeta) \theta \left( r > \frac{f(\tau, \zeta)}{h(\tau)s(\zeta)} \right) p(t, z; \tau) \right. \]
\[
+ \left. \theta^\tau(\zeta) \theta \left( r < \frac{f(\tau, \zeta)}{h(\tau)s(\zeta)} \right) \delta(\tau - t) \delta(\zeta - z) \right\}. \quad (4.29)
\]
Evaluating the integrals and taking the derivative with respect to $u$ leads to:

$$\frac{\partial}{\partial u} p(t, z; u) = \int_{z_-(t)}^{z_+(t)} d\zeta \left[ f(u, \zeta) \delta(u - t) \delta(z - \zeta) - f(u, \zeta) p(t, z; u) \right],$$

(4.30)

which is solved by eq. (4).

### 4.4.2 Introducing an Infrared Cutoff

In a realistic parton shower, the values of the scale $t$ are not allowed to reach zero. In the case of QCD, a cutoff value $\mu$ is set to approximately $\Lambda_{\text{QCD}}^2 \approx 1 \text{ GeV}^2$, below which the perturbative parton shower approach is no longer viable. Eq. (4.17) now no longer represents a probability distribution, since its unit normalization depends on the scale running to 0. The following algorithm, also discussed in [94], allows for the introduction of a cutoff.

**Algorithm 5** The Sudakov veto algorithm in the presence of a cutoff $\mu$

1. $t \leftarrow u$
2. loop
3. if $r_1 < \Delta_g(t, \mu)$ then
4. return $\mu$
5. else
6. $t \leftarrow G^{-1}(\log(r_1) + G(t))$
7. if $r_2 < f(t)/g(t)$ then
8. return $t$
9. end if
10. end if
11. end loop

The difference with the standard veto algorithm is that any scales that would be generated below the cutoff $\mu$ terminate the algorithm immediately. The probability of this occurring from the current scale $t$ is $\Delta_g(t, \mu)$, which is the no-emission probability between $t$ and $\mu$. In unitary language, the algorithm translates to

$$p(t; u) = \int d\tau \left[ \Delta_g(u, \mu) \delta(\tau - \mu) + g(\tau) \Delta_g(u, \tau) \theta(\mu < \tau < u) \right]
\times \left\{ \theta(\tau = \mu) \delta(t - \mu) + \theta(\tau \neq \mu) \right\}
\times \int_0^1 dr \left[ \theta \left( r < \frac{f(\tau)}{g(\tau)} \right) \delta(t - \tau) + \theta \left( r > \frac{f(\tau)}{g(\tau)} \right) p(t; \tau) \right].$$

(4.31)
Working out the integrals, we find
\[ e^{G(u)p(t;u)} = e^{G(\mu)\delta (t - \mu)} + \int_\mu^u d\tau e^{G(\tau)}\left[f(\tau)\delta (t - \tau) + (g(\tau) - f(\tau)) p(t;\tau)\right]. \] (4.32)

After taking the derivative with respect to \( u \), the first term drops out and the \( \mu \)-dependence disappears from the second. Therefore, eq. (4.22) is recovered. However, eq. (4.17) is not the only solution to this differential equation. We can more formally solve eq. (4.22) by writing
\[ p(t,u) = \hat{p}(t.u)e^{-F(u)}. \] (4.33)

Substituting that into eq. (4.22) leads to
\[ \frac{\partial}{\partial u} \hat{p}(t;u) = f(t)e^{F(t)}\delta (t - u). \] (4.34)

Integrating that yields
\[ p(t;u) = f(t)\Delta_f(u,t)\theta (\sigma < t < u) + p_0(t,\sigma). \] (4.35)

The new parameter \( \sigma \) is some lower integration boundary that is not determined by the differential equation. The function \( p_0(t,\sigma) \) is the integration constant which can not depend on the upper bound \( u \). Both should be fixed by the boundary conditions. In this case, they can be fixed by requiring that \( p(t;u) \) reduces to a delta function when \( u \to \mu \). The full solution therefore is
\[ p(t;u) = f(t)\Delta_f(u,t)\theta (\mu < t < u) + \Delta_f(u,\mu)\delta (t - \mu). \] (4.36)

This expression is also easily checked to solve eq. (4.32).

### 4.4.3 Nonsingular Branching Kernels

Another issue that may appear is the fact that branching kernels do not always diverge for \( t \to 0 \). We will encounter such situations in Chapter 7. In the electroweak theory, the widths of resonances may prevent the branching kernels from diverging. The consequence is again that eq. (4.17) is not properly normalized, and in this case the distribution has to be modified to solve this issue.

Since the normalization is not unity anymore, we normalize the distribution explicitly. The distribution we wish to generate becomes
\[ p(t;u) = \frac{1}{1 - \Delta_f(u,0)} f(t)\Delta_f(u,t)\theta (0 < t < u), \] (4.37)
which is again properly normalized. Note that the normalization factor reduces to 1 in the usual case where $\Delta f(u, 0) \to 0$. An algorithm that generates this distribution is given in Algorithm 6.

**Algorithm 6** The Sudakov veto algorithm for nonsingular branching kernels

```
t ← u
loop
    if $r_1 < \Delta g(t, 0)$ then
        $t ← u$
    else
        $t ← G^{-1}(\log(r_1) + G(t))$
        if $r_2 < f(t)/g(t)$ then
            return $t$
        end if
    end if
end loop
```

Every time the loop is passed through, there is a probability of $\Delta g(t, 0)$ to reset the current value of the evolution scale all the way back to the starting scale $u$. This algorithm differs significantly from the ones we have seen before in that it is non-Markovian. In a Markovian stochastic process, all involved conditional probability distributions only depend on the current state of the system. Usually, with every pass of the loop, the actions of the veto algorithm only depend on the previous scale $t$. In this algorithm however, there is some finite probability to drop in scale several times, but then to reset back to $u$. The algorithm essentially has an internal memory of the highest scale it started at. Because of this property, the analysis of Algorithm 6 is not as straightforward. We split up the corresponding expression into two pieces and immediately solve the $r$-integrals

\[
p(t; u) = \Delta g(u, 0)p(t; u) + \int_0^u \! d\tau \Delta g(u, \tau) \left[ f(\tau)\delta(t - \tau) + (g(\tau) - f(\tau))q(t; \tau) \right]
\]

\[
q(t; \tau) = \Delta g(\tau, 0)p(t; u) + \int_0^\tau \! d\tau' \Delta g(\tau, \tau') \left[ f(\tau')\delta(t - \tau') + (g(\tau') - f(\tau'))q(t; \tau') \right].
\]

Equation (4.38)

Splitting the expression up in two pieces is required because the very first iteration of Algorithm 6, or the first iteration after a reset, is special. In that first iteration, the starting scale and the reset scale are both equal to $u$. In subsequent iterations, after
at least one scale has been vetoed, the starting scale \( \tau \) is no longer equal to \( u \) and the usual recursive behaviour is again present. The first two lines of eq. (4.38) thus represent the first iteration after a reset, while the last two lines represent all further iterations. In particular, it follows from eq. (4.38) that

\[
q(t; u) = p(t; u) \tag{4.39}
\]

and it will suffice to solve \( q(t; \tau) \). Using eq. (4.39) we can also rewrite the second part of eq. (4.38) to

\[
q(t; \tau) = \Delta g(\tau, 0)q(t; u) + \int_0^\tau d\tau' \Delta g(\tau, \tau') \left[ f(\tau')\delta(t - \tau') + (g(\tau') - f(\tau'))q(t; \tau') \right], \tag{4.40}
\]

which is now expressed in terms of the function \( q \) only. As usual, we take the derivative with respect to the upper bound, which is \( \tau \) in this case. This leads to

\[
\frac{\partial}{\partial \tau} q(t; \tau) = f(\tau)\delta(t - \tau) - f(\tau)q(t; \tau) \tag{4.41}
\]

which is again the same as eq. (4.22), and it can be solved in the same way using eq. (4.35). We do need a different integration constant \( p_0(t) \) this time. Since the solution should be eq. (4.37), the integration constant should have been

\[
p_0(t) \propto \frac{1}{1 - \Delta f(u, 0)} f(t)\Delta(u, t). \tag{4.42}
\]

The solution should also be normalized, leading to the proportionality factor. We find

\[
q(\tau, t) = f(t)\Delta f(\tau, t)\theta(t < \tau) + \frac{\Delta f(\tau, 0)}{1 - \Delta f(u, 0)} f(t)\Delta(u, t)
\]

\[
= \frac{1}{1 - \Delta f(u, 0)} f(t)\Delta f(\tau, t) \left( \theta(t < \tau) + \Delta f(u, 0)\theta(\tau < t < u) \right). \tag{4.43}
\]

The constant multiplying the integration constant in the first line of eq. (4.43) ensures the distribution is normalized. This expression can then be shown to solve eq. (4.40), verifying its validity. Finally, we indeed find that for \( \tau \to u \) eq. (4.43) reduces to eq. (4.37).

4.4.4 The Weighted Veto Algorithm

The Sudakov veto algorithm can be generalized to accommodate different veto probabilities. The usual veto probability \( f(t)/g(t) \) is replaced by some other probability
b(t). The effect of the difference can be compensated by weighting the resulting events. This version of the veto algorithm may be used for many different purposes. Examples are generating parton showers with branching kernels that are not positive-definite [95], enhancing certain branching probabilities [96, 97] and automatically accounting for parton shower variations [98, 99, 100]. The algorithm is

\textbf{Algorithm 7} The weighted Sudakov veto algorithm

\begin{verbatim}
    t ← u
    w → 1
    loop
        t ← G^{-1}(\log(r_1) + G(t))
        if r < b(t) then
            w ← \frac{f(t)}{g(t)} \frac{1}{b(t)} w
            return t, w
        else
            w ← \frac{1 - f(t)/g(t)}{1 - b(t)} w
        end if
    end loop
\end{verbatim}

The analysis of this algorithm includes the weights as multiplicative factors of the corresponding probabilities. It is

\[
p(t; u) = \int_0^u d\tau g(\tau) \Delta_g(u, \tau) \int_0^1 dr \left[ \theta (r < b(\tau)) \frac{f(\tau)}{g(\tau)} \frac{1}{b(\tau)} \delta (\tau - t) + \theta (r > \frac{f(\tau)}{g(\tau)}) \frac{1 - f(\tau)/g(\tau)}{1 - b(\tau)} p(t; \tau) \right].\tag{4.44}
\]

Performing the \(r\)-integral and taking the \(u\)-derivative again leads to eq. (4.22).

4.4.5 Radiation from Multiple Channels

The final extension we discuss is the inclusion of multiple branching channels. In the parton shower formalism, these channels may originate from either the presence of multiple radiating particles or dipoles, or from multiple modes of branching for a single radiator. Let us assume there are \(n\) branching channels, each characterized by a branching kernel \(f_i(t)\). The density \(p(t; u)\) now contains a Sudakov factor representing the no-branching probability for all channels, which is just the product of the individual Sudakov factors. Similarly, the branching kernel is now the sum of the
individual kernels. For the total branching kernel, we introduce the notation
\[
\tilde{f}(t) \equiv \sum_{i=1}^{n} f_i(t).
\] (4.45)

The probability distribution then is
\[
p(t; u) = \tilde{f}(t) \Delta f(t, u) \theta(0 < t < u).
\] (4.46)

This distribution can be attained by generating multiple scales and selecting the highest. This can be shown using the following result:
\[
p(t; u) = \left[ \prod_{i=1}^{n} \int_0^{u} d\tau_i f_i(\tau_i) \Delta f_i(u, \tau_i) \right] \sum_{j=1}^{n} \left[ \prod_{k \neq j} \theta(\tau_j > \tau_k) \right] \delta(t - \tau_j)
\]
\[
= \sum_{i=1}^{n} \left[ \prod_{j \neq i} \int_0^{\tau_i} d\tau_j f(\tau_j) \Delta f(u, \tau_j) \right] \int_0^{u} d\tau_i f_i(\tau_i) \Delta f_i(u, \tau_i) \delta(t - \tau_i)
\]
\[
= \sum_{i=1}^{n} f_i(t) \Delta f_i(u, t) \prod_{j \neq i} \Delta f_j(u, t) = \tilde{f}(t) \Delta f(u, t).
\] (4.47)

The first line can be understood as generating a scale \( \tau_i \) from every channel \( i \), and then setting \( t = \max(\tau_i) \). The veto algorithms that have been discussed in the previous sections can be used to produce the densities that appear in the first line of eq. (4.47). A second option is available. Instead of running a veto algorithm for every branching channel, one can also replace the branching kernels \( f_i(t) \) in eq. (4.47) by their overestimates \( g_i(t) \). The scales generated by this procedure are then distributed according to a Sudakov distribution with kernel \( \tilde{g}(t) \). The trial scale is then accepted using the acceptance probability of the selected channel. Both procedures result in eq. (4.46). In the next section, we will further detail these algorithms and compare them.

Algorithm 8 shows a very different algorithm that also produces the same density.

**Algorithm 8** A different competition Sudakov veto algorithm

\[
t \leftarrow u
\]

**loop**

\[
t \leftarrow \tilde{G}^{-1} \left( \log(r_1) + \tilde{G}(t) \right)
\]

Select \( i \) between 1 and \( n \) with probability \( g_i(t)/\tilde{g}(t) \)

**if** \( r_2 < f_i(t)/g_i(t) \) **then**

**return** \( t \)

**end if**

**end loop**
We analyze this algorithm to show that it also produces eq. (4.46):

\[
p(t; u) = \int_0^u d\tau \tilde{g}(\tau) \Delta_g(u, \tau) \int_0^1 dr_1 \sum_{i=1}^n \theta \left( \frac{\sum_{j=0}^{i-1} g_j(\tau)}{\tilde{g}(\tau)} < r_1 < \frac{\sum_{j=0}^i g_j(\tau)}{\tilde{g}(\tau)} \right)
\times \int_0^1 dr_2 \left[ \theta \left( r_2 < \frac{f_i(\tau)}{g_i(\tau)} \right) \delta(t - \tau) + \theta \left( r_2 > \frac{f_i(\tau)}{g_i(\tau)} \right) p(t; \tau) \right],
\]  

(4.48)

where \( g_0(t) \equiv 0 \). We go through the usual steps, noting that after doing the \( r_1 \) integral, the new sum over step functions yields terms \( g_i(\tau)/\tilde{g}(\tau) \) representing the probabilities to select the corresponding channels. The differential equation becomes

\[
\frac{\partial}{\partial u} p(t; u) = \tilde{f}(u) \delta(t - u) - \tilde{f}(u) p(t; u),
\]

(4.49)

which is solved by eq. (4.46).

Algorithm 8 requires the generation of trial scales using \( \tilde{g}(t) \) as overestimated branching kernel. In practice, this is often not much harder than generating trial scales for individual channels, since the kernels \( g_i(t) \) can usually be chosen to have the same \( t \)-dependence. In such a case, the channel selection step in Algorithm 8 does not even require the evaluation of the kernels at the trial scale anymore. We note that Algorithm 8 can still be used in more complicated situations by using the procedure outlined in eq. (4.47) to split \( \tilde{g}(t) \) up into groups of similar channels. In the next chapter, we incorporate the extensions discussed here into a full, practical veto algorithm. Since it was found there are multiple ways to handle competition, these algorithms are then tested for efficiency.

4.5 Testing the Competition Algorithms

We test the aforementioned competition algorithms by implementing them in a relatively simple antenna shower close to what is described in [50, 101]. This shower handles QCD radiation in the final state using an antenna scheme to include collinear and soft enhancements. It is very basic compared with the parton showers of the major event generators [29, 33, 32], including only the absolute necessities for a functional parton shower.

We first give a complete description of the algorithms that we compare. As discussed earlier, many of the extensions of the veto algorithm can readily be combined. In this case, we will need the auxiliary variable to generate realistic phase space points, and the cutoff to avoid the nonperturbative regime. All competition algorithms can
be shown to produce the probability distribution

\[
p(t, z; u) = \delta(t - \mu) \exp \left( -\sum_{i=1}^{n} \int_{\mu}^{u} d\tau \int_{z_{i-}(\tau)}^{z_{i+}(\tau)} d\zeta f_i(\tau, \zeta) \right) + \sum_{i=1}^{n} f(t, z) \theta_i(z) \theta(\mu < t < u) \exp \left( -\sum_{i=1}^{n} \int_{\mu}^{u} d\tau \int_{z_{i-}(\tau)}^{z_{i+}(\tau)} d\zeta f_i(\tau, \zeta) \right).
\] (4.50)

We now describe the four algorithms.

**Veto-Max**

This algorithm handles competition using eq. (4.47). That is, the veto algorithm is applied to every channel individually, after which the highest of the generated scales is selected. This is the most common way of handling competition. It is usually cited in the literature as the competition algorithm [96, 94], and is used in most parton showers. The full algorithm is detailed in Algorithm 9.

**Algorithm 9** The Veto-Max Sudakov veto algorithm

```plaintext
for all 1 \leq i \leq n do
    t_i \leftarrow u
    loop
        if \( r_{i,1} < \Delta_{\Sigma_i} h_i(u, \mu) \) then
            t_i \leftarrow \mu
            break
        else
            t_i \leftarrow H_i^{-1}(\log(r_{i,1})/\Sigma + H_i(t_i))
            z_i \leftarrow S_i^{-1}(r_{i,2} \Sigma_i + S_i(z_{i-}))
            if \( r_{i,3} < f_i(t_i, z_i)/g_i(t_i, z_i) \) and \( z_{i-}(t) < z_i < z_{i+}(t) \) then
                break
            end if
        end if
    end loop
end for
j \leftarrow \text{index(max}(t_i))
return \( t_j, z_j, j \)
```

**Max-Veto**

This algorithm also uses eq. (4.47), but with \( f_i(t, z) \rightarrow g_i(t, z) \). That is, trial pairs \((t, z)\) are generated according to the overestimate kernels. The highest of these scales
is selected, to which the veto step is applied using the branching kernel of the selected channel. This algorithm is for example employed in the default Pythia shower [29]. The algorithm is given in Algorithm 10.

Algorithm 10 The Max-Veto Sudakov veto algorithm

\[
\begin{align*}
    & \text{\texttt{Generate-Select}} \\
    & t \leftarrow u \\
    & \text{loop} \\
    & \quad \text{for all } 1 \leq i \leq n \text{ do} \\
    & \quad \quad \text{if } r_{i,1} < \Delta_{\Sigma_i} h_i(u, \mu) \text{ then} \\
    & \quad \quad \quad t_i \leftarrow \mu \\
    & \quad \quad \quad \text{else} \\
    & \quad \quad \quad \quad t_i \leftarrow H_i^{-1}(\log r_{i,1} + H_i(t)) \\
    & \quad \quad \quad \quad z_i \leftarrow S_i^{-1}(r_{i,2} \Sigma_i + S_i(z_i-)) \\
    & \quad \quad \text{end if} \\
    & \quad \text{end for} \\
    & j \leftarrow \text{index}(\max(t_i)) \\
    & \quad \text{if } t_j = \mu \text{ then} \\
    & \quad \quad \text{return } \mu, j \\
    & \quad \text{end if} \\
    & \quad \text{if } r_{i,3} < f_j(t_j, z_j)/\tilde{g}(t_j, z_j) \text{ and } z_{j-}(t) < z_j < z_{j+}(t) \text{ then} \\
    & \quad \quad \text{return } t_j, z_j, j \\
    & \quad \text{end if} \\
    & \text{end loop}
\end{align*}
\]

Generate-Select

This is the alternative algorithm described in the previous section. It generates trial scales \( \tau \) using the sum of the overestimate functions \( \tilde{g}(t, z) \). The overestimate functions are required to have the same \( z \)-dependence, that is

\[
g_i(t, z) = h_i(t) s(z). \tag{4.51}
\]

This way, \( \zeta \) can be generated using boundaries that are overestimates for all channels. Next, a channel \( i \) is selected with probability \( g_i(\tau)/\tilde{g}(\tau) \). Finally, the veto step is applied to this channel. The algorithm is detailed in Algorithm 11.

Select-Generate

Under certain circumstances, a variation of the Generate-Select algorithm is possible. If we require all overestimate functions \( g_i(t, z) \) to have the same scale dependence,
4.5. TESTING THE COMPETITION ALGORITHMS

Algorithm 11 The Generate-Select Sudakov veto algorithm

\[ t \leftarrow u \]

loop
if \( r_1 < \Delta_{\Sigma \tilde{h}}(u, \mu) \) then
  return \( \mu \)
else
  \[ t \leftarrow \tilde{H}^{-1} \left( \log(r_1) + \tilde{H}(t) \right) \]
  \[ z \leftarrow S^{-1}(r_2 \Sigma + S(z_{-})) \]
Select \( j \) between 1 and \( n \) with probability \( h_j(t) / \tilde{h}(t) \)
if \( r_3 < f_j(t, z) / g_j(t, z) \) and \( z_{-}(t) < z < z_{+}(t) \) then
  return \( t, z, j \)
end if
end if
end loop

this dependence drops out of the selection probabilities. That is, the overestimates are

\[ g_i(t, z) = h(t)s_i(z) \quad (4.52) \]

In that case, a channel can be selected before a scale is generated. As a consequence, the overestimate functions can have different dependence on \( z \), and universal overestimates of the boundaries are no longer required. The algorithm is given in Algorithm 12.

Algorithm 12 The Select-Generate Sudakov veto algorithm

\[ t \leftarrow u \]

loop
if \( r_1 < \Delta_{\Sigma \tilde{h}}(u, \mu) \) then
  return \( \mu \)
else
  Select \( j \) between 1 and \( n \) with probability \( \Sigma_j / \tilde{\Sigma} \)
  \[ t \leftarrow \tilde{H}^{-1} \left( \log(r_1) + \tilde{H}(t) \right) \]
  \[ z \leftarrow S_j^{-1}(r_2 \Sigma_j + S_j(z_{-})) \]
  if \( r_3 < f_j(t, z) / g_j(t, z) \) and \( z_{-}(t) < z < z_{+}(t) \) then
    return \( t, z, j \)
  end if
end if
end loop
CHAPTER 4. MONTE CARLO TECHNIQUES

We now briefly describe the parton shower model that is used to compare these algorithms. It is a very simplified version the Vincia parton shower, which will be described in detail in Chapter 5. The running of the strong coupling is taken into account by an overestimate

\[ \hat{\alpha}_s(t) = a \ln^{-1}(bt) \]  

where \( a \) and \( b \) are matched to the starting scale and the cutoff scale. This overestimate is corrected to the real one-loop running \( \alpha_s(t) \), which includes the proper flavor thresholds, by using \( \hat{\alpha}_s(t) \) for the overestimate kernels and \( \alpha_s(t) \) for the branching kernels.

The possible branchings for a QCD shower can be divided into two categories: emissions, where a quark or gluon sends out a new gluon, and splittings, where a gluon splits into a quark-antiquark pair. We use \( p_\perp \)-ordering \[50, 101\] for both for easier application of the Generate-Select and Select-Generate algorithms. The overestimates of the branching kernels are

\[
g_{\text{emit}}(t, z) = \frac{a C_A}{4\pi} \frac{1}{z(1-z)} \frac{1}{t \ln(bt)}
\]

\[
g_{\text{split}}(t, z) = \frac{a n_F T_R}{4\pi} \frac{1}{z(1-z)} \frac{1}{t \ln(bt)}.
\]

Note that a factor \( n_F \) is included in the overestimate of the splitting kernel. If a gluon splitting is selected through the Max-Veto algorithm, a quark flavor is chosen at random, as would be done by the Generate-Select algorithm. We use the antennae functions given in \[101\] for the splitting kernels. The code can be found at \[102\].

We compare the performance of the algorithms described above on this shower. For the Select-Generate algorithm, the bottleneck is the channel selection step. It is complicated by the fact that the \( z \)-integrals in the overestimates are different for every antenna. We use stochastic roulette-wheel selection \[103\] for the selection step, which achieves \( \mathcal{O}(1) \) complexity. Coincidentally, this is also a type of rejection algorithm and is easily provable using unitary language. The Generate-Select algorithm assigns the same boundaries to the \( z \)-integral for all channels. For \( n_F = 6 \) and the standard values \( C_A = 3 \) and \( T_R = 1/2 \), all overestimate functions are the same, and the channel selection step is trivial. In this sense, the difference between the Generate-Select and the Select-Generate algorithms is a trade-off between easier selection of a channel and lower veto rates. We produce 4 million events per algorithm. The initial scale is \( \sqrt{u} = 7 \) TeV and the cutoff scale is \( \sqrt{\mu} = 1 \) GeV. These settings produce events with multiplicities of \( \mathcal{O}(100) \), which are typical at the LHC. Figure 4.1 shows the average amount of CPU time the shower requires to produce an event. As expected, the Generate-Select and the Select-Generate algorithms are significantly more efficient.
Figure 4.1: The average CPU times required by the shower to produce events as a function of the multiplicities of those events.

than the Veto-Max and the Max-Veto algorithms. Although the shower used here is already fairly close to a real-world parton shower, all kinds of complications may occur that make the use of the Generate-Select and Select-Generate algorithms unpractical. One useful conclusion from the above discussion is that the algorithms can in such cases be combined, using the more efficient algorithms wherever possible.
Chapter 5

The Vincia Parton Shower

In this chapter, we give an overview of the Vincia parton shower and emphasize some recent developments. Vincia is a plugin to the Pythia event generator, replacing its default shower. As it is based on the antenna subtraction formalism, it differs significantly from many other parton showers. The Vincia formalism was set up in [49, 50] where final-state radiation from massless partons is described, as well as a unique technique for matrix element corrections to high orders. In [101], quark masses were included. A different sector-based treatment for gluon radiation was included in [104] to increase the efficiency of the matrix element corrections. The formalism was then extended to include radiation from the initial state in [63, 105]. Helicity-dependent shower evolution was described in [106] for final-state radiation and in [107] for initial-state radiation. This functionality will be important for the implementation of an electroweak shower that will be described in Chapter 7. Finally, the matrix element correction technique was extended to include NLO matrix elements in [78] and to a formalism that is able to merge matrix elements without introduction of a merging scale [108].

The main choices that have to be made in the construction of a parton shower are:

1. The branching kernels that capture the soft and collinear factorization properties of matrix elements. In traditional 1 → 2 parton showers, these are typically chosen to be the Altarelli-Parisi splitting functions, while in the more modern parton showers they are based on NLO subtraction kernels.

2. The phase space factorization that formalized the distinction between the post-branching phase space and the pre-branching phase space with an additional radiative piece. This factorization is often only exact in the collinear limit for 1 → 2 parton showers, but the more modern ones based on 2 → 3 branching
CHAPTER 5. THE VINCIA PARTON SHOWER

Figure 5.1: An example of a final-final antenna. The red line indicates the colour-ordered structure.

kernels make use of factorizations that are exact over all of phase space.

3. The *kinematics map* which specifies how the post-branching momenta depend on the pre-branching momenta and the additional phase space variables generated by the parton shower. It can be chosen to be locally momentum-conserving in $2 \rightarrow 3$ showers.

In this chapter, we discuss the above choices for the cases of final-state and initial-state radiation for QCD branchings in the leading colour limit. The colour structure of parton systems is then simplified to a colour ordering which can be evolved by the parton shower. A direct consequence is that only two external legs contribute to the emission of a new gluon and the radiation pattern is easily captured in a $2 \rightarrow 3$ branching kernel. We call a set of two partons that are able to branch an *antenna*, which differs from a *dipole* in the sense that no special role is assigned to either of the partons in the antenna. On the other hand, in a dipole-based shower one parton is assigned to be the brancher and the other to be the recoiler. The branching kernels of Vincia are called *antenna functions*. They have to capture the radiative collinear limits of both partons contained in the antenna, while the dipole-based branching kernels only capture the collinear behaviour of the brancher. Some subtleties appear because gluons are part of two antennae at the same time and the collinear limits have to be shared. This issue will be discussed below.

We give a brief overview of the Vincia shower in this chapter, emphasizing some new developments that include massive kinematics for branching involving radiation in the initial state, global recoil for initial-final antennae and treatment of radiation in resonance decays.
5.1 Final-Final Antennae

For antennae spanned between partons that are both in the final state, we denote the momenta involved in the $2 \to 3$ branching as

$$p_i + p_K \to p_i + p_j + p_k,$$

(5.1)

where $p_j$ may be considered the emitted parton. Local momentum conservation is imposed, such that $p_i + p_K = p_i + p_j + p_k$. The branching kernels and phase space factorization are defined in terms of Lorentz-invariant quantities like

$$s_{ij} \equiv 2p_i \cdot p_j \text{ and } m_{ij}^2 = s_{ij} + m_i^2 + m_j^2.$$

(5.2)

The parton shower approximation to a three-parton radiative matrix element is

$$|M_3(p_i, p_j, p_k)|^2 \approx 4\pi \alpha_s C a(s_{ij}, s_{jk}, s_{ik}) \times |M_2(p_I, p_K)|^2,$$

(5.3)

where $\alpha_s/4\pi$ is the strong coupling constant, $C$ is a colour factor that depends on the branching process and $a(s_{ij}, s_{jk}, s_{ik})$ is the antenna function. The colour factors are

$$\hat{C}_F = 2C_F = \frac{N_c^2 - 1}{N_c} = \frac{8}{3} \text{ for gluon emissions from quarks,}$$

$$C_A = N_c = 3 \text{ for gluon emissions from gluons,}$$

$$T_R = \frac{1}{2} \text{ for gluon splitting to a quark-antiquark pair.}$$

(5.4)

Note that for antennae for gluon emission from $qg$ or $g\bar{q}$, it is unclear which colour factor should be used. However, the difference between $\hat{C}_F$ and $C_A$ is explicitly subleading in colour and the discrepancy thus lies beyond the parton shower accuracy. Subleading colour corrections may be included for the first emission from $qg\bar{q}$ configurations by reweighting antenna functions with a factor including a negative antenna between the quark and antiquark [50]. Many other methods are available in other parton showers to improve upon the leading colour accuracy of parton showers [109, 110, 111, 112]. The dynamics of the branching process are captured in the antenna function, which may be calculated from ratios of matrix elements. For instance, the gluon emission antenna function for a quark-antiquark pair may be determined by calculating

$$a_{qg\bar{q}}(s_{ij}, s_{jk}, s_{ik}) = \frac{1}{4\pi \alpha_s \hat{C}_F} \frac{|M(X \to qg\bar{q})|^2}{|M(X \to q\bar{q})|^2}.$$

(5.5)

Depending on the choice of $X$, the antenna function may differ in terms that are nonsingular in the soft and collinear regions. In the process-independent formulation of the parton shower, these terms are in principle arbitrary and may be varied to assess
parton shower uncertainties. Alternatively, the antennae functions may be explicitly constructed to reproduce the soft and collinear limits. Excluding non-singular terms, the antenna functions for spin-summed branchings are

\[
a_{\text{emit}}(s_{ij}, s_{jk}, s_{ik}) = 2 \frac{s_{ik}}{s_{ij}s_{jk}} - 2 \frac{m_i^2}{s_{ij}^2} - 2 \frac{m_k^2}{s_{jk}^2} + \delta_{iq} \frac{s_{jk}}{s_{IK} s_{ij}} + \delta_{kq} \frac{s_{ij}}{s_{IK} s_{jk}} + \delta_{ig} \frac{s_{jk} - s_{ik}}{s_{IK} s_{jk}} + \delta_{kg} \frac{s_{ij} - s_{ik}}{s_{IK} s_{ij}} + \delta_{iq} \frac{m_i^2}{s_{IK} s_{jk}} + \delta_{kq} \frac{m_k^2}{s_{IK} s_{ij}} + \delta_{ig} \frac{m_i^2}{s_{IK} s_{ij}} + \delta_{kg} \frac{m_k^2}{s_{IK} s_{jk}}.
\]

(5.6)

where the Kronecker deltas are 0 or 1 if \(i\) and \(k\) are either (anti)quarks or gluons. The soft eikonal factor appears in all cases of gluon emissions. The other terms are additional collinear singularities that fix the collinear limits. The \(s_{ij}\)-collinear limit is given by \(s_{ij} \to 0, s_{ik} \to zs_{IK}\) and \(s_{jk} \to (1 - z)s_{IK}\). The antennae become

\[
a_{\text{emit}, q}^{\text{col}} = \frac{1}{s_{ij}} \left( \frac{2z}{1 - z} + (1 - z) - 2 \frac{m_i^2}{s_{ij}} \right) = \frac{1}{s_{ij}} \left( 1 + \frac{z^2}{1 - z} - 2 \frac{m_i^2}{s_{ij}} \right)
\]

\[
a_{\text{emit}, g}^{\text{col}} = \frac{1}{s_{ij}} \left( \frac{2z}{1 - z} + z(1 - z) \right)
\]

\[
a_{\text{split}}^{\text{col}} = \frac{1}{2m_{ij}^2} \left( z^2 + (1 - z)^2 + 2 \frac{m_q^2}{m_{ij}^2} \right).
\]

(5.7)

Since gluons contribute to two antennae due to the colour structure, there are two contributions to their collinear limit. The total contribution is then given by the sum of two antennae in their collinear limits, where the replacement \(z \leftrightarrow 1 - z\) is made in one of them. We find

\[
a_{\text{emit}, q} = \frac{1}{s_{ij}} P_{q\to qg}(z)
\]

\[
a_{\text{emit}, g} + a_{\text{emit}, q}[z \leftrightarrow 1 - z] = \frac{1}{s_{ij}} P_{g\to gg}(z)
\]

\[
a_{\text{split}} + a_{\text{split}}[z \leftrightarrow 1 - z] = \frac{1}{s_{ij}} P_{g\to q\bar{q}}(z).
\]

(5.8)

The antenna functions therefore capture all relevant soft and quasi-collinear limits. Next, we consider the phase space factorization. The factorization used by Vincia is exact, remaining valid outside the collinear region. It is given by

\[
d\Phi_{n+1} = d\Phi_{\text{ant}} d\Phi_n,
\]

(5.9)

where

\[
d\Phi_{\text{ant}} = \frac{1}{16\pi^2} \lambda^{-\frac{1}{2}}(m_{IK}^2, m_I^2, m_K^2) ds_{ij} ds_{jk} \frac{d\varphi}{2\pi} \theta(s_{ik} > 0) \theta(G_{ijk} > 0).
\]

(5.10)
5.1. FINAL-FINAL ANTENNAE

We have introduced the Källén function

\[ \lambda(m_{IK}^2, m_i^2, m_k^2) = m_{IK}^4 + m_i^4 + m_k^4 - 2m_{IK}^2 m_i^2 - 2m_{IK}^2 m_k^2 - 2m_i^2 m_k^2 \]

and the three-body Gram determinant

\[ G_{ijk} = s_{ij}s_{jk}s_{ik} - s_{jk}^2 m_i^2 - s_{ik}^2 m_j^2 - s_{ij}^2 m_k^2 + 4m_i^2 m_j^2 m_k^2. \]  

The step functions apply constraints to the Lorentz-invariant phase space quantities \( s_{ij}, s_{jk} \) and \( s_{ik} \). The proof of eq. (5.10) is given in Appendix A.

We have until now left the integration boundaries of the antenna phase space in eq. (5.10) unspecified. For the purpose of the construction of a parton shower as explained in Chapters 3 and 4, the integral should not run over the entire radiative phase space but instead over a region bounded by the values of the ordering variable \( Q_2 \). The integrated radiative component of eq. (5.3) can then be written as

\[ I(Q_1^2, Q_2^2) = \int_{Q_1^2}^{Q_2^2} \frac{\alpha_s}{4\pi} C a(s_{ij}, s_{jk}, s_{ik}) ds_{ij} ds_{jk} d\phi_2, \]

which appears in the Sudakov form factor as

\[ \Delta(Q_1^2, Q_2^2) = \exp \left( -I(Q_1^2, Q_2^2) \right), \]

where it represents the no-branching probability between ordering scales \( Q_2^2 \) and \( Q_1^2 \).

To sample from the associated probability distribution, it is convenient to rewrite eq. (5.12) to

\[ I(Q_1^2, Q_2^2) = \int_{Q_1^2}^{Q_2^2} |J| \frac{\alpha_s}{4\pi} C a(s_{ij}, s_{jk}, s_{ik}) |dQ^2 dz| d\phi_2, \]

where \( |J| \) is the Jacobian factor associated with the transformation \( s_{ij}, s_{jk} \to Q^2, z \).

The functional form of \( Q^2 \) is important \[53, 113\] and different parton showers make use of different definitions. The differences between choices are beyond the leading logarithmic accuracy of the parton shower as long as the soft and quasi-collinear singularities are relegated to the phase space region where \( Q^2 \to 0 \). The definitions used by Vincia are

\[ Q^2 = p_{iK}^2 = \frac{s_{ij}s_{jk}}{s_{IK}} \quad \text{and} \quad Q^2 = m_{ij}^2. \]

The first choice is the 'ARIADNE-\( p_\perp \)' \[62\] which represents the transverse momentum of the emitted parton in the antenna rest frame where the pre-branching momenta are aligned with the z-axis. It is essentially the inverse of the eikonal factor that appears in the gluon emission antennae of eq. (5.6), and thus straightforwardly regularizes its singularities. In the case of gluon splitting, the antenna is only singular in the
Chapter 5. The Vincia Parton Shower

quasi-collinear limit. The inverse of its singular factor \( m^2_{ij} \) is thus sufficient as ordering variable for gluon emissions. Vincia allows for the use of \( p^2_\perp \) or \( m^2_{ij} \) for gluon splittings, and the differences have been investigated in [101].

The definition of the auxiliary variable \( z \) has no impact on the probability distributions generated by the parton shower. It can be chosen to make for an easily generated probability distribution for the application of the Sudakov veto algorithm. As an example, the gluon emission antenna may be overestimated by the function

\[
\hat{a}(s_{ij}, s_{jk}, s_{ik}) = 2 \frac{s_{IK}}{s_{ij}s_{jk}}.
\]  

(5.16)

In this case, the choice

\[
z = \frac{s_{ij}}{s_{ij} + s_{jk}},
\]

(5.17)

is the most convenient. This particular choice also neatly separates the singular limits of the gluon emission antennae. The quasi-collinear limit \( s_{ij} \to 0 \) corresponds with \( z \to 0 \) while the other quasi-collinear limit \( s_{jk} \to 0 \) corresponds with \( z \to 1 \). The soft, noncollinear limit then occurs for \( z \approx 0.5 \). The phase space boundaries on \( z \) can be computed in the massless limit, where they are

\[
\frac{1}{2} \left( 1 - \sqrt{1 - \frac{Q^2}{m^2_{IK}}} \right) = z_\text{−} \leq z \leq z_\text{+} = \frac{1}{2} \left( 1 + \sqrt{1 - \frac{Q^2}{m^2_{IK}}} \right).
\]

(5.18)

The massive phase space boundaries are given by the step functions in eq. (5.10), but they are not easily translated to \( Q^2 \) and \( z \). The massive phase space is however always contained in the massless phase space, so it suffices to sample using the massless boundaries and veto any unphysical phase space points. Since the masses are typically small compared with the antenna invariant mass, this procedure does not cause any significant loss of efficiency. The Jacobian is \(|J| = 2s_{IK}/z(1 - z)\), and we can write

\[
\hat{T} = \int_{Q^2_1}^{Q^2_2} \frac{\alpha_s}{\pi} \frac{s_{IK}}{\lambda^4(m^2_{IK}, m^2_I, m^2_K)} \frac{dQ^2}{Q^2} \frac{dz}{z(1 - z)} \frac{d\varphi}{2\pi}
\]

(5.19)

for the overestimate of the radiative integral eq. (5.14). This distribution is easily sampled using the veto algorithm with auxiliary variables detailed in Section 4.4.1 First- and second-order running of the coupling constant can also be incorporated [50].

After generating \( Q^2 \), \( z \) and \( \varphi \), translating to \( s_{ij} \) and \( s_{jk} \) and correcting to the real antenna function using the Sudakov veto algorithm, the post-branching momenta must be constructed. To agree with the factorization properties discussed in Chapter 3 this mapping must obey the following rules:
1. Soft safety: For $p_j \to 0$, $p_i = p_I$ and $p_k = p_K$

2. Collinear safety: For $p_j \parallel p_i$, $p_k = p_K$ up to mass-suppressed corrections + equivalent for $i \leftrightarrow k$.

Furthermore the antenna picture requires $p_i$ and $p_k$ to be treated on an equal footing. Vincia uses the massive generalization \cite{kosower-map} of the Kosower map \cite{kosower-map}. The pre-branching and post-branching momenta are related by

\[
p_I = x_i p_i + r p_j + x_k p_k
\]

\[
p_K = (1 - x_i) p_i + (1 - r) p_j + (1 - x_k) p_k.
\]

(5.20)

The parameters $x_i$ and $x_k$ can be fixed by setting $p^2_I = m^2_I$ and $p^2_K = m^2_K$. The parameter $r$ remains as an ambiguity in the mapping. It can be understood to parameterize the global orientation of the post-branching system with respect to the pre-branching one. The choice of $r$ must therefore be made to ensure the orientations overlap in the quasi-collinear limit, but this does not fix it entirely. In the massless case, a straightforward choice is

\[
r = \frac{s_{jk}}{s_{ij} + s_{jk}},
\]

(5.21)

which can easily be seen to fit the collinear safety condition. The massive generalization is

\[
r = \frac{m^2_{IK} + m^2_I - m^2_K}{2m^2_{IK}} + \frac{\sqrt{s^2_{IK} - 4m^2_I m^2_K}}{2m^2_{IK}} \left( \frac{s_{jk} - s_{ij}}{s_{ij} + s_{jk}} - 2m_j m_k - 2m_i m_j \right).
\]

(5.22)

This choice respects quasi-collinear safety and it has the desired property of treating $I$ and $K$ on an equal footing, in the sense that under $I \leftrightarrow K$ and $i \leftrightarrow k$, $r \leftrightarrow (1 - r)$. The parameter $r$ can be translated to an angle between the pre-branching and post-branching systems by computing

\[
\cos \psi = \frac{E_I E_i - x_i m^2_i - r s_{ij} - x_k s_{ik}}{\left| \vec{p}_I \right| \left| \vec{p}_i \right|}.
\]

(5.23)

The construction of the post-branching momenta then proceeds in the center-of-mass frame of the antenna with the pre-branching system aligned with the $z$-axis and the $xz$-plane representing the branching plane. The post-branching momenta are

\[
p_i = (E_i, 0, 0, |p_i|),
\]

\[
p_j = (E_j, -|p_j| \sin \theta_{ij}, 0, |p_j| \cos \theta_{ij})
\]

\[
p_k = (E_k, |p_k| \sin \theta_{ik}, 0, |p_k| \cos \theta_{ik})
\]

(5.24)
where $|\vec{p}_x| = \sqrt{E_x^2 - m_x^2}$ and the energies and angles are all fixed by the invariants $s_{ij}$, $s_{jk}$ and $s_{ik}$. They are given by

$$E_i = \frac{s_{ij} + s_{ik} + 2m_i}{2m_{IK}}, \quad E_j = \frac{s_{ij} + s_{jk} + m_j}{2m_{IK}}, \quad E_k = \frac{s_{ik} + s_{jk} + 2m_k}{2m_{IK}}$$

$$\cos \theta_{ij} = \frac{2E_iE_j - s_{ij}}{2|\vec{p}_i||\vec{p}_j|}, \quad \cos \theta_{ik} = \frac{2E_iE_k - s_{ik}}{2|\vec{p}_i||\vec{p}_k|}.$$  \hspace{1cm} (5.25)

A rotation with angle $\varphi$ is then performed in the $xy$-plane. The angle $\psi$ is then used to rotate the branching plane with respect to the pre-branching system. Finally, a Lorentz boost is performed back to the lab frame.

### 5.2 Initial-Initial Antennae

For antennae spanned between two partons that are both in the initial state, we denote the momenta as

$$p_A + p_B \rightarrow p_a + p_j + p_b.$$ \hspace{1cm} (5.26)

The incoming momenta are parallel with the beam axis and are always taken to be massless. Initial-state radiation is simulated through backwards evolution. Branchings are represented as steps back in time starting from the hard scattering. The local momentum conservation property is thus

$$p_A + p_B = p_a + p_b - p_j,$$ \hspace{1cm} (5.27)

which indicates that the total initial-state momentum must increase since some momentum is deposited in the final state in the form of $p_j$. For the purpose of defining the phase space factorization, we collect all other final-state momenta that are not involved in the branching in the momentum $p_R$. As indicated in Figure 5.3, the emission of the momentum $p_j$ causes the incoming momenta to acquire some transverse momentum, pushing them off the beam axis. The system is subsequently realigned with the lab frame by a boost which transforms $p_R$ to $p_r$. 

![Figure 5.2: An example of an initial-initial antenna. The green line indicates the colour-ordered structure.](image)
The antenna functions are most easily acquired by employing crossing symmetry on the final-state antennae. The result is

\[ a_{\text{emit}}^{II}(s_{aj}, s_{bj}, s_{ab}) = 2 \frac{s_{ab}}{s_{aj} s_{bj}} + \delta_{aq} \frac{1}{s_{AB}} \frac{s_{bj}}{s_{aj}} + \delta_{bg} \frac{1}{s_{AB}} \frac{s_{aj}}{s_{bj}} + \delta_{aq} \frac{s_{bj}}{s_{aj}} \left( \frac{s_{ab}}{s_{AB}} + \frac{s_{AB}}{s_{ab} + s_{aj}} \right) + \delta_{bg} \left( \frac{s_{ab}}{s_{AB}} + \frac{s_{AB}}{s_{ab} + s_{bj}} \right) \]

\[ a_{\text{split}}^{II}(s_{aj}, s_{bj}, s_{ab}) = \frac{1}{s_{aj}} \left( \frac{s_{AB}}{s_{ab}} + \frac{(1 - s_{AB})^2}{s_{ab} s_{AB}} \right) \]

\[ a_{\text{conv}}^{II}(s_{aj}, s_{bj}, s_{ab}) = \frac{1}{2} \frac{s_{bj}}{s_{ab}} + \frac{s_{ab}}{s_{AB}}. \] (5.28)

Here, the second antenna labeled with 'split' refers to a quark or antiquark in the initial state backwards-evolving into a gluon and emitting an antiquark or quark into the final state. In the forward direction in time, this branching corresponds with a gluon splitting. The 'conv' antenna on the other hand describes an initial-state gluon backwards-evolving into a quark or antiquark, which corresponds with a forward gluon emission where the gluon ends up in the initial state. In the \( s_{aj} \) collinear limits, the momentum fraction is \( z = s_{AB}/s_{ab} \). The antennae become

\[ a_{\text{emit},q}^{II} \text{col.} = \frac{1}{s_{aj} z} \frac{1}{1 - z} + \frac{1 + z^2}{1 - z} = \frac{1}{s_{aj} z} P_{q \rightarrow gq}(z) \]

\[ a_{\text{emit},g}^{II} \text{col.} = \frac{1}{s_{aj} z} \left( \frac{z}{1 - z} + \frac{1 - z}{z} + z(1 - z) \right) \]

\[ a_{\text{split}}^{II} \text{col.} = \frac{1}{s_{aj} z} \left( z^2 + (1 - z)^2 \right) = \frac{1}{s_{aj} z} P_{g \rightarrow gg}(z) \]

\[ a_{\text{conv}}^{II} \text{col.} = \frac{1}{2 s_{aj} z} \frac{1}{z} + \frac{1}{s_{ab} z} \frac{1}{z} + \frac{1}{s_{aj} z} P_{q \rightarrow gq}(z). \] (5.29)

The last two splitting functions correspond to their forward-time branching picture. An extra factor of \( 1/z \) is included as explained in Section 3.2. For initial-state radiation, the gluon emission and gluon splitting kernels produce the full Altarelli-Parisi splitting kernels, while they were partitioned between two contributions in the case of final-final antennae. The reason is that Vincia has no antenna contribution for gluon emission into the initial state. The full collinear singularity is thus contained in the initial-state emission kernels.

The phase space factorization is

\[ \frac{d x_a}{x_a} \frac{d x_b}{x_b} d \Phi_2(p_a + p_b \rightarrow p_j + p_r) = \frac{d x_A}{x_A} \frac{d x_B}{x_B} d \Phi_1(p_A + p_B \rightarrow p_R) d \Phi^{II}_{\text{ant}} \] (5.30)

where

\[ d \Phi^{II}_{\text{ant}} = \frac{1}{16 \pi^2} \frac{s_{AB}}{s_{ab}^2} ds_{aj} ds_{bj} \frac{2 \theta(G_{abj} > 0)}{2 \pi}. \] (5.31)
Since only $m_j$ is massive, the Gram determinant is $G_{abj} = s_{aj}s_{bj}s_{ab} - m_j^2s_{ab}^2$. A proof of the phase space factorization is given in Appendix A. In similar fashion to the case of final-final branchings, the antenna phase space is transformed to variables $Q^2$ and $z$, where the possible choices are

$$Q^2 = p_\perp^2 = \frac{s_{aj}s_{bj}}{s_{ab}}$$ and $$Q^2 = s_{aj} \text{ or } s_{bj}. \tag{5.32}$$

In this case, the transverse momentum is normalized to $s_{ab}$ which is the largest invariant. The second choice is again only applicable for antennae without a soft singularity. Depending on the antenna overestimate, the choice of ordering variable and the Jacobian factor that appears in the antenna phase space, the variable $z$ can again be selected to yield a distribution that is easily sampled.

The kinematic map implicitly incorporates the two-step procedure outlined in Figure 5.3, keeping the initial-state partons aligned with the beam axis and boosting the rest of the system $p_R$. The map is not completely fixed by the requirement that $m_R^2 = m_r^2$, so rapidity conservation

$$y_R = \frac{p_0^R + p_R^3}{p_0^R - p_R^3} = \frac{p_0^r + p_r^3}{p_0^r - p_r^3} = y_r \tag{5.33}$$

is also imposed. The mapping can then be written as

$$p_a = \sqrt{\frac{s_{ab}s_{ab} - s_{aj}}{s_{AB}s_{ab} - s_{bj}}} p_A$$

$$p_b = \sqrt{\frac{s_{ab}s_{ab} - s_{bj}}{s_{AB}s_{ab} - s_{aj}}} p_B$$

$$p_j = \sqrt{\frac{s_{bj}s_{ab} - s_{aj}}{s_{AB}s_{ab} - s_{bj}}} p_A + \sqrt{\frac{s_{aj}s_{ab} - s_{bj}}{s_{AB}s_{ab} - s_{aj}}} p_B + \sqrt{\frac{s_{aj}s_{bj}}{s_{ab}} - m_j^2} p_\perp$$

$$p_r = p_a + p_b - p_j, \tag{5.34}$$
where the vector $p_\perp = (0, \sin \varphi, \cos \varphi, 0)$ in the rest frame of the antenna encodes the dependence of the map on the azimuthal angle $\varphi$. This vector is then boosted to the lab frame before constructing the post-branching momenta. The consistency requirements $x_a < 1$ and $x_b < 1$ lead to the upper limits on $s_{aj}$ and $s_{bj}$

$$s_{aj}^+ = \frac{1 - x_B^2}{x_B(x_A + x_B)} s_{AB} \quad s_{bj}^+ = \frac{1 - x_A^2}{x_A(x_A + x_B)} s_{AB}. \quad (5.35)$$

Together with the positivity requirement of the Gram determinant, all phase space boundaries are fixed. The shower evolution proceeds in a very similar fashion to the final-final antennae. The PDF ratio is incorporated by adding in an overestimate and an additional contribution to the veto probability.

### 5.3 Initial-Final Antennae

For antenna spanned between an initial-state parton and a final-state parton, we denote the momenta as

$$p_A + p_K \rightarrow p_a + p_j + p_k. \quad (5.36)$$

Initial-final antennae are a kinematic hybrid of backward initial-state evolution and forward final-state evolution. Local momentum conservation is given by

$$p_A - p_K = p_a - p_j - p_k. \quad (5.37)$$
The other initial-state momentum $p_B$ remains unchanged. The rest of the final state is indicated by the momentum $p_R$. The antenna functions are

$$a_{\text{emit}}^I(s_{aj}, s_{jk}, s_{ak}) = 2\frac{s_{ak}}{s_{aj}s_{jk}} + \delta_{aq} \frac{1}{s_{AK}s_{aj}} + \delta_{jq} \left( \frac{1}{s_{AK}s_{bj}} - 2\frac{m_j^2}{s_{jk}} \right)$$

$$a_{\text{split}, I}^I(s_{aj}, s_{jk}, s_{ak}) = \frac{1}{s_{aj}} \left( \frac{s_{AK}}{s_{aK}} + \frac{(1 + s_{AB})^2}{s_{ak}s_{AK}} \right)$$

$$a_{\text{split}, F}^I(s_{aj}, s_{jk}, s_{ak}) = \frac{1}{2m_{jk}^2} \left( \frac{\frac{1}{s_{aj}} + \frac{\frac{1}{s_{ak}}}{s_{2AK}}}{2} + 2\frac{m_j^2}{m_{jk}^2} \right)$$

$$a_{\text{conv}}^I(s_{aj}, s_{jk}, s_{ak}) = \frac{1}{2s_{aj}} \frac{\frac{1}{s_{jk}} + \frac{\frac{1}{s_{ak}}}{s_{2AK}}}{s_{2AK}}.$$  \hspace{1cm} (5.38)

In analogy with the previous sections, these functions can be shown to produce the correct collinear limits on the initial-state side and the final-state side.

The phase space factorization is most generally written as

$$\frac{dx_a}{x_a}d\Phi_3(p_a + p_B \rightarrow p_j + p_k + p_R) = \frac{dx_A}{x_A}d\Phi_2(p_A + p_B \rightarrow p_K + p_R)d\Phi_{\text{ant}}^I$$  \hspace{1cm} (5.39)

where

$$d\Phi_{\text{ant}}^I = \frac{1}{16\pi^2} \frac{1}{s_{AK}^2} \frac{x_A^2}{x_a^2} ds_{aj} ds_{jk} d\phi \theta(G_{ajk} > 0).$$  \hspace{1cm} (5.40)

In this case, $p_j$ and $p_k$ may be massive such that the Gram determinant is $G_{ajk} = s_{aj}s_{jk}s_{ak} - m_j^2s_{ak}^2 - m_k^2s_{aj}^2$. The ordering variables are

$$Q^2 = p_1^2 = \frac{s_{aj}s_{jk}}{s_{AK} + s_{jk}} \text{ and } Q^2 = s_{aj} \text{ or } s_{jk}.$$  \hspace{1cm} (5.41)

The proof of the phase space factorization is given in Appendix A. It is complicated by the highly nontrivial kinematic mapping involving initial-final antennae. One of the leading principles of the antenna-based parton shower formalism is that the branching particles should be treated on an equal footing. In the case of initial-final antennae, this means that the initial state and the final state should both receive a portion of the transverse momentum imparted as a consequence of the branching to $p_j$. A general mapping however comes with many problems, including too many degrees of freedom, issues with infrared safety and general intractability when particle masses are involved. We therefore take a different approach, where for every parton shower branching one of two maps is selected. These maps correspond with dipole-style maps.
5.3. INITIAL-FINAL ANTENNAE

Figure 5.5: Illustration of the local initial-final kinematic map. Only the momentum $p_k$ receives transverse momentum in the antenna rest frame as a result of the emission of $p_j$.

where one particle is designated to be the brancher while the other is the spectator. To restore the antenna equal-footing principle, one of these two alternative maps is chosen with a particular probability that is derived by comparison to matrix elements. Note that similar issues relating to global recoil in initial-final colour connections have been discussed in [97, 114, 55].

Local Map

The first map is illustrated in Figure 5.5. In this map, the transverse momentum is fully absorbed by the final-state momentum $p_K$. The initial-state $p_A$ thus remains aligned with the beam axis and no boost is required, remaining locally momentum conserving. Requiring only longitudinal recoil for $p_A$ fixes the map entirely. Defining $\sigma_l = s_{aj} + s_{ak} = s_{AK} + s_{jk} + m_j^2 + m_k^2 - m_K^2$, it is given by

\[
\begin{align*}
p_a &= \frac{\sigma_l}{s_{AK}} p_A \\
p_j &= \frac{1}{s_{AK}\sigma_l} \left(\sigma_l(s_{aj} + m_j^2 - m_k^2) + m_K^2(s_{ak} - s_{aj}) - s_{AK}s_{ak}\right) p_A \\
&\quad + \frac{s_{aj}}{\sigma_l} p_K + \frac{1}{\sigma_l} \sqrt{G_{ajk}} p_\perp \\
p_k &= \frac{1}{s_{AK}\sigma_l} \left(\sigma_l(s_{ak} - m_j^2 + m_k^2) + m_K^2(s_{aj} - s_{ak}) - s_{AK}s_{aj}\right) p_A \\
&\quad + \frac{s_{ak}}{\sigma_l} p_K - \frac{1}{\sigma_l} \sqrt{G_{ajk}} p_\perp.
\end{align*}
\] (5.42)

The phase space boundaries are given by the usual positivity requirement on the Gram determinant, as well as the condition $x_A < 1$. This leads to a maximum on $s_{jk}$ given by

\[
s_{jk}^+ = \left(\frac{1}{x_A} - 1\right) s_{AK} - m_j^2 - m_k^2 + m_K^2.
\] (5.43)
Figure 5.6: Illustration of the global initial-final kinematic map. Only the momentum $p_a$ receives transverse momentum in the antenna rest frame as a result of the emission of $p_j$. The system is then boosted to realign $p_a$ and $p_b$ with the beam axis.

**Global Map**

In the other map the transverse momentum is instead absorbed by the initial-state momentum $p_A$. A Lorentz boost that returns the post-branching momentum $p_a$ to the beam axis while leaving $p_B$ invariant is thus required. The map is sketched in Figure 5.6. The global map is a great deal more complex than the local one. We first consider the massless limit. Having defined $\sigma_g = s_{AK} - s_{aj}$, the kinematic mapping of the first step indicated in Figure 5.6 is given by

$$
p_a = \frac{s_{ak}}{\sigma_g} p_A + \frac{s_{aj}s_{jk}}{s_{AK}\sigma_g} p_K + \frac{\sqrt{s_{jk}s_{aj}s_{ak}}}{\sigma_g} p_\perp$$

$$
p_j = \frac{s_{jk}}{\sigma_g} p_A + \frac{s_{aj}s_{ak}}{s_{AK}\sigma_g} p_K + \frac{\sqrt{s_{jk}s_{aj}s_{ak}}}{\sigma_g} p_\perp$$

$$
p_k = \frac{\sigma_g}{s_{AK}} p_K.
$$

Next, a Lorentz transform is applied to return the post-branching momenta and $p_R$ back to the lab frame such that $p'_B = p_B$ and $p'_a = (x_a/x_A) p_A$. The transformation can be parameterized in terms of the momenta $p_A$, $p_a$ and $p_B$. It is given by

$$\Lambda^{\mu\nu} = g^{\mu\nu} + \frac{p'_B p'_a - p'_a p'_B}{p_a \cdot p_B} + \frac{p'_A p'_B - p'_B p'_A}{p_A \cdot p_B} + \frac{p_A \cdot p_a}{(p_A \cdot p_B)(p_a \cdot p_B)} p'_B p'_B.
$$

A derivation of this transform can be found in appendix A. The new momentum fraction $x_a$ after the boost can be computed through

$$
x_a \over x_A = \frac{s_{aB}}{s_{AB}}
$$

since the boost to the lab frame leaves $p_B$ untouched. The invariants can thus be evaluated before the boost. Substituting the functional form of $p_a$ and considering the collinear limit $s_{aj} \to 0$ leads to

$$
x_a \col = \frac{s_{AK} + s_{jk}}{s_{AK}} + O(\sqrt{s_{aj}}),
$$

(5.47)
which is identical to the momentum fraction of the local map as is required by collinear safety. In practice, the momentum fraction of the local map is used in the antenna phase space factorization in all cases. Due to the complexity of the momentum $p_a$ after the boost, it is not straightforward to express the momentum fraction resulting from the global map in terms of the shower variables, making it difficult to sample. As such, the form of the local map is correct at least in the collinear limit, which is formally the only phase space region where the PDFs are reliable. Note that an exact phase space factorization may be required for matrix element corrections [49], but in that case the local map can always be used because the kinematic mapping has no effect on matrix-element-corrected branchings.

The $x_a < 1$ requirement leads to

$$ s_{jk}^+ = \left( \frac{1}{x_A} \frac{s_{AB}}{s_{AB} - s_{BK}} - 1 \right) s_{AK}. \quad (5.48) $$

The shape of the antenna phase space is shown in Figure 5.7. The phase space points are weighted with the Jacobian factor $x_A^2/x_a^2$. The global map shows much lower phase space density for large values of the invariants $s_{aj}$ and $s_{jk}$. This is due to the fact that points with $x_a < x_A$ are left out. These points are considered unphysical, since in the backwards evolution picture of initial-state radiation the partonic momentum fraction should increase. This is guaranteed by the local map, but it is not generally true for the global one. In fact, $x_a$ may be negative for $\sigma_l < 0$ leading to entirely
unphysical events.

The generalization of the global map to massive momenta is not easily computed analytically. The increased complexity is caused by the fact that the momentum $p_K$ can no longer just be rescaled, since that would alter its mass. Instead, the Vincia implementation finds the post-branching momenta numerically by parameterizing them with

$p_a = \alpha_a p_A + \beta_a p_K + \gamma p_\perp$
$p_j = \alpha_j p_A + \beta_j p_K + \gamma p_\perp$
$p_k = (\alpha_a - \alpha_j - 1)p_A + (\beta_a - \beta_j + 1)p_K. \quad (5.49)$

The five parameters are then determined by the system of equations

\begin{align*}
0 &= \alpha_a \beta_a s_{AK} + \beta_a^2 m_K^2 - \gamma^2 \\
\gamma_2^2 &= \alpha_j \beta_j s_{AK} + \beta_j^2 m_K^2 - \gamma^2 \\
m_k^2 &= (\alpha_a - \alpha_j - 1)(\beta_a - \beta_j + 1)s_{AK} + (\beta_a - \beta_j + 1)^2 m_K^2 \\
s_{aj} &= (\alpha_a \beta_j + \beta_a \alpha_j)s_{AK} + 2\beta_a \beta_j m_K^2 - 2\gamma^2 \\
s_{jk} &= (\alpha_j (\beta_a - \beta_j + 1) + \beta_j (\alpha_a - \alpha_j - 1))s_{AK} + 2\beta_j (\beta_a - \beta_j + 1)m_K^2, \quad (5.50)
\end{align*}

which solve the on-shell conditions and fix the invariants $s_{aj}$ and $s_{jk}$. This system is solved using multi-dimensional Newton-Raphson. If the algorithm is started from the values of the parameters in the massless limit, numerically solving the massive parameters is fast and stable, producing the solution in sufficient numerical precision within a few iterative steps. Since the phase space shrinks due to the parton masses, the boundary given by eq. (5.43) remains valid.

**Selection Probability**

To maintain the antenna picture of treating both participating particles on an equal footing, the above maps can be combined probabilistically. That is, for every sampled shower branching, one of the above maps is selected with some phase space dependent probability. To find an appropriate functional form for that probability, we consider the first radiative correction to the simple process $q l \rightarrow q l$ in the massless limit, where the emission of the gluon is governed by an initial-final antenna. The contributing diagrams are shown in figure 5.8. The expressions for the matrix elements can be obtained from the crossing of the elementary QED process $l^+ l^- \rightarrow q \bar{q}$ and its radiative correction. They are

$$|M_2|^2 = 32\pi^2 \alpha^2 Q_l^2 \frac{s_{AK}^2 + s_{AR}^2}{s_{AB}^2}$$

(5.51)
for the Born level and
\[
|M_3|^2 = 128\pi^2 \alpha^2 \alpha_s Q_c^2 C_F s_{AB}^2 + s_{kR}^2 + s_{AR}^2 + s_{Bk}^2 \frac{s_{ak}}{s_{AK}(s_{AK} + s_{jk} - s_{aj}) s_{aj}s_{jk}}
\]
for the radiative correction. The parton shower approximation is
\[
|M_3|^2 \approx |M_2|^2 4\pi\alpha_s C_F a^{\text{IP}}_{\text{emit},qq}(s_{aj}, s_{jk}, s_{ak}),
\]
where the quark-quark emission antenna is most conveniently written as
\[
a^{\text{IP}}_{\text{emit},qq}(s_{aj}, s_{jk}, s_{ak}) = \frac{(s_{AK} - s_{aj})^2 + (s_{AK} + s_{jk})^2}{s_{AK}s_{aj}s_{jk}}.
\]

The choice of kinematic map influences $|M_2|^2$ in eq. (5.53). Figure 5.9 shows the performance of the maps discussed in the previous chapter for these matrix elements. The plotted colour corresponds to the average value of
\[
c = \left| \tanh \left( \log \left( \frac{|M_2|^2 a^{\text{IP}}_{\text{emit},qq}(s_{aj}, s_{jk}, s_{ak})}{|M_3|^2} \right) \right) \right|
\]
across phase space points weighted with the radiative matrix element $|M_3|^2$. The absolute value of the logarithm is large when the parton shower approximation differs significantly from the exact matrix element. The invariants are plotted logarithmically to roughly represent the parton shower distribution.

From Figure 5.9, it is clear that the local map performs much better than the global map. This is partly caused by the fact that the latter is unable to cover the entire phase space without resorting to unphysical momenta with negative energies. In particular, even in the singular limits, the denominator $\sigma_l = s_{AK} - s_{aj}$ may still be negative, leading to radiative momenta with negative energies and thus an incomplete coverage of phase space. The local map on the other hand is able to cover all of phase
Figure 5.9: Comparison of the parton shower approximation to the full matrix element eq. (5.52) of the process shown in figure 5.8 as a function of the antenna phase space invariants $s_{aj}$ and $s_{jk}$. The phase space points are weighted with the radiative matrix element $|M_3|^2$. The partonic invariant mass $x_A x_B s$ is set to 1 TeV$^2$, where $s$ is the hadronic invariant mass. The cutoff scale $Q_{\text{cut}}^2$ is set to $\Lambda^2_{\text{QCD}} = 1$ GeV$^2$. Lower values correspond to better agreement between the parton shower and the matrix element.

The worsening behaviour in the central region of the $s_{aj}$-$s_{jk}$-plane is caused by phase space points where $s_{AK}$ is of the same order of size as $s_{aj}$ and $s_{jk}$, which is where the parton shower approximation is expected to break down.

Ignoring the issue of unphysical momenta for a moment, it is possible to combine the kinematic maps using some probability $P_g$ to select the global map and $P_l = 1 - P_g$ to select the local map such that the exact matrix element eq. (5.53) is reproduced across all of phase space. Substituting these probabilities and the kinematic maps into eq. (5.53) yields

$$|M_3|^2 = 32\pi^2 \alpha^2 Q^2_q \left( P_g \frac{s_{KR}^2 + s_{KB}^2}{(s_{AK} - s_{aj})^2} + (1 - P_g) \frac{s_{aB}^2 + s_{aR}^2}{(s_{AK} + s_{jk})^2} \right) \times 4\pi\alpha_s C_F \frac{(s_{AK} - s_{aj})^2 + (s_{AK} + s_{jk})^2}{s_{AK}s_{aj}s_{jk}}$$

such that with the choice

$$P_g = \frac{(s_{AK} - s_{aj})^2}{(s_{AK} + s_{jk})^2 + (s_{AK} - s_{aj})^2}$$

the parton shower approximation exactly reproduces eq. (5.52). However, to avoid unphysical momenta, the probability is slightly modified to

$$P_g = \frac{(s_{AK} - s_{aj})^2}{(s_{AK} + s_{jk})^2 + (s_{AK} - s_{aj})^2} \theta(s_{AK} > s_{aj}).$$

(5.57)
5.3. INITIAL-FINAL ANTENNAE

Figure 5.10: Comparison of the parton shower approximation to the full matrix element eq. (5.52) of the process shown in figure 5.8 as a function of the antenna phase space invariants $s_{aj}$ and $s_{jk}$ using the combined kinematic map with probability given by eq. (5.58).

Figure 5.10 shows the performance of the maps combined with this probability. It is not straightforward to extend this argument to massive partons or to higher-order matrix elements. We thus choose to use the probability defined in eq. (5.58) in all cases, modifying the step function to only select the global map when it is physically viable. A similar choice of kinematic mapping was used for final-final antennae in previous antenna showers [62, 116].

Figure 5.11 shows the difference between the previous Vincia implementation only using the local map and the new strategy of making a probabilistic choice between the global and local maps. The process under consideration is Drell-Yan production at center-of-mass energy $\sqrt{s} = 7$ TeV. Vincia is plugged into Pythia 8.2 [29] using the default tune and the NNPDF2.3 PDF sets [117]. Shown are the transverse momentum spectrum of the $Z$ boson and the angular correlation of the lepton pair. The analysis was performed using RIVET [118] and follows ATLAS analyses presented in [119, 120]. The Drell-Yan process is particularly suited to investigating the difference in initial-state maps since it has no final-state colour charges. Furthermore, the spectra shown in Figure 5.11 are a direct consequence of the application of the parton shower. The first branching is always of initial-initial type, emitting a gluon into the final state most often. After that, the remaining antenna are of initial-final type. We neglect to show comparison with data to highlight the differences between the mapping strategies and because it is well known that the parton shower does not describe these spectra with sufficient accuracy without matrix element corrections.
Figure 5.11: Comparison between the local map and the probabilistic choice between local and global map for initial-final antennae on the transverse momentum and angular correlation spectra in the Drell-Yan process at $\sqrt{s} = 7$ TeV.

## 5.4 Resonance Antennae

A final type of antenna appears in processes with resonances, the prime example being a top quark \[121\]. In the Pythia formalism \[30\], resonances can appear as part of the hard scattering or in the parton shower. These resonances decay at an energy scale that corresponds with their width. Since all Standard Model resonances have decay widths close to, or lower than the QCD cutoff scale, in practice the decay occurs after the shower evolution. However, once a heavy resonance decays to a pair of light particles, some new phase space becomes available over which the parton shower should be run. In the case of a top quark decay to a bottom quark and a $W$ boson, the QCD radiation in the decay system is described by an initial-final antenna. Kinematically, these antennae differ from regular initial-final antenna as the initial state is not part of the beam and should not increase in energy. The mapping can be thought of as an initial-final mapping and the subsequent application of a Lorentz transform that returns the initial-state momentum to its original form. For that reason, the phase space factorization and kinematic mapping are very similar to the final-final case. The phase space factorization is

$$d\Phi_3(p_A \rightarrow p_j + p_k + p_r) = d\Phi_2(p_A \rightarrow p_K + p_R)d\Phi_{\text{ant}}^{RF}$$

(5.59)

where

$$d\Phi_{\text{ant}}^{RF} = \frac{1}{16\pi} \lambda^{-1}(m_A^2, m_K^2, m_R^2)ds_{aj}ds_{jk}d\varphi \frac{d\varphi}{2\pi}.$$  

(5.60)
The phase space factorization can be proven in very similar fashion to the final-final phase space factorization as shown in Appendix A. The kinematics are also constructed similarly, using the rest of the system as momentum $p_R$. In the next chapter a QED shower implementation is discussed, where resonance-final antennae may also appear in $W$ decays.
Chapter 6

QED Radiation in Vincia

While QCD interactions dominate the radiative corrections in collision events with coloured objects, other types of radiation also contribute. Radiation of gauge bosons from the electroweak sector is suppressed with respect to gluonic radiation predominantly due to the much smaller coupling constant, but it may still yield a significant contribution. To properly incorporate these radiative corrections in the parton shower picture, they should be interleaved with the dominant QCD contributions. In this chapter, we discuss the inclusion of QED radiation in the Vincia formalism, which has been lacking until recently.

Most event generators implement QED radiation as part of the parton shower \cite{97,30}. In many other cases, photon radiation is added by an external program such as PHOTOS \cite{122}. These implementations are all formally not coherent, meaning they do not incorporate all soft eikonal components of the factorized photon emission amplitude correctly. Such contributions may in fact be important in some cases. For instance, initial-state radiation and its interference with final-state radiation has already been shown to be relevant for precision measurements at the LHC \cite{123,124} and for future colliders \cite{125,126}. On the other hand, YFS exponentiation \cite{127} is used by some event generators \cite{128,129} to add soft photon radiation to particle decays, and by others to simulate process-specific photon radiation for precision physics \cite{130,131,132,133}. This type of radiation is fully coherent, but collinear logarithms can only be included order-by-order, and it cannot be interleaved with QCD radiation.

Since one of the main advantages of the antenna formalism employed in the Vincia parton shower is the fact that it is automatically coherent in the leading-colour limit, an extension of this property to the QED shower is desirable. In this chapter, we thus describe a new and full-fledged QED shower available in Vincia. It includes coherent photon emission from the initial and the final state, as well as photon splitting to
fermion pairs. After setting up the theoretical framework, we validate the implementation by comparing against theoretical results and consider its effect on several observables at the LHC. An earlier version of the work in this chapter was published in [134].

6.1 Photon Emission

All antenna and dipole showers operate in the leading-colour QCD approximation. In this context, it makes sense to partition the total gluon contribution into antennae or dipoles corresponding with distinguished colour-ordered states. Because of the leading-colour approximation, the number of contributing antennae or dipoles is limited and their interference structure is automatically disentangled.

![Figure 6.1: Factorization of soft or collinear gluon emission in leading-colour QCD.](image)

In contrast, for photon emissions, there is no colour structure or leading-colour approximation. This means that every pair of charged particles contributes equally, and there is no way to divide the kernel into several disconnected pieces. As a consequence, every charged particle in principle participates in the emission of a photon simultaneously.

![Figure 6.2: Factorization of soft or collinear photon emission in QED.](image)
The implementation of such a procedure in a parton shower formalism would be problematic, especially if it is to be interleaved with a regular QCD shower. For that reason, we will employ an approach similar to the sector shower detailed in [104] to cast photon emissions in an antenna shower-like procedure. Below, we first derive appropriate emission kernels for photon emission before proceeding to detail two different parton shower implementations available in Vincia.

6.1.1 Antenna Functions

Due to the absence of a colour structure and a leading-colour approximation, the soft matrix element factorization has a more involved form [135]. It is given by

\[ |M_{n+1}(\{p\}, p_j)|^2 = -8\pi\alpha \left[ \sum_{x,y} \sigma_x \sigma_y Q_x Q_y s_{xy} + \sum_x Q_x^2 m_x^2 s^2_{xj} \right] |M_n(\{p\})|^2, \]  

(6.1)

where \(\alpha\) is the fine-structure constant, \(p_j\) is the emitted photon and the sum runs over all charged particles with momenta in the set \(\{p\}\). The factors \(Q_i\) are the charges of particle \(i\), while \(\sigma_i = \pm 1\) is a sign factor that has \(\sigma_i = 1\) for final-state particles and \(\sigma_i = -1\) for initial-state particles. Charge conservation for the total event is then given by

\[ \sum_x Q_x \sigma_x = 0. \]  

(6.2)

The quasi-collinear limit with charged particle \(i\) is given by

\[ |M_{n+1}(p_1, \ldots, p_i, \ldots, p_n, p_j)|^2 = 4\pi\alpha Q_i^2 \frac{2}{s_{ij}} P_{f \rightarrow f\gamma}(z)|M_n(p_1, \ldots, p_i + p_j, \ldots, p_n)|^2, \]  

(6.3)

where the Altarelli-Parisi splitting functions are given by

\[ P_{f \rightarrow f\gamma}(z) = \frac{1 + z^2}{1 - z} - 2 \frac{m_i^2}{s_{ij}^2} \]

\[ P_{W^\pm \rightarrow W^\pm\gamma}(z) = 2 - \frac{z}{1 - z} - 2 \frac{m_i^2}{s_{ij}^2} + \frac{4}{3} \left( \frac{1 - z}{z} + z(1 - z) \right). \]  

(6.4)

We refer to Chapter [7] for the derivation of these splitting functions. While the fermionic splitting function is identical to its QCD counterpart, the \(W^\pm\) splitting function differs significantly. The first two terms of \(P_{W^\pm \rightarrow W^\pm\gamma}(z)\) contain the soft photon singularity encapsulated in the eikonal factor. This term appears for all polarizations of the \(W\) boson. On the other hand, the last two collinear terms only appear for transverse polarizations and thus receive a smaller numerical factor as a consequence of averaging over the pre-splitting polarizations. The collinear limit for initial-state particles is similar, except for an additional factor \(1/z\).
Before detailing the parton shower approximation to the radiative matrix element, we first define the relevant antenna functions. For final-final configurations we find

\[ a_{Emit}^{FF}(s_{ij}, s_{jk}, s_{ik}) = 4 \frac{s_{ik}}{s_{ij}s_{jk}} - 4 \frac{m_{j}^2}{s_{ik}^2} - 4 \frac{m_{k}^2}{s_{jk}^2} + \delta_{if} \frac{2}{s_{IK}s_{ij}} + \delta_{kW} \frac{2}{s_{IK}s_{jk}} \]

\[ + \delta_{W} \frac{8}{3} \frac{1}{s_{ij}} \left( \frac{s_{jk}}{s_{IK} - s_{jk}} + \frac{s_{ik}(s_{IK} - s_{jk})}{s_{IK}^2} \right) \]

\[ + \delta_{W} \frac{8}{3} \frac{1}{s_{jk}} \left( \frac{s_{ij}}{s_{IK} - s_{ij}} + \frac{s_{ij}(s_{IK} - s_{ij})}{s_{IK}^2} \right). \]  

(6.5)

The antennae for the initial-state configurations can then be found by employing crossing symmetry. The initial-final antennae are

\[ a_{Emit}^{IF}(s_{aj}, s_{jk}, s_{ak}) = 4 \frac{s_{ak}}{s_{aj}s_{jk}} - 4 \frac{m_{a}^2}{s_{aj}^2} - 4 \frac{m_{k}^2}{s_{jk}^2} + \delta_{af} \frac{2}{s_{AK}s_{aj}} + \delta_{kW} \frac{2}{s_{AK}s_{jk}} \]

\[ + \delta_{W} \frac{8}{3} \frac{1}{s_{aj}} \left( \frac{s_{jk}}{s_{AK} + s_{jk}} + \frac{s_{aj}(s_{AK} - s_{jk})}{s_{AK}^2} \right) \]

\[ + \delta_{W} \frac{8}{3} \frac{1}{s_{jk}} \left( \frac{s_{aj}}{s_{AK} + s_{jk}} + \frac{s_{aj}}{s_{AK} + s_{jk}} - \frac{s_{aj}^2}{(s_{AK} + s_{jk})^2} \right). \]  

(6.6)

Note that some factors of \( s_{jk} \) are added to the denominators in the last line of eq. (6.6). This ensures positivity of the antenna function over all of phase space, and prevents it from becoming enhanced in the collinear limit \( s_{ak} \rightarrow 0 \). An enhancement in that limit can only occur at very high values of the ordering scale and is not physical, but rather an artefact of a direct translation from the splitting functions to antennae. Thus, a factor of \( s_{jk} \) may be added for regularization of such unphysical poles since it vanishes in the real quasi-collinear limit. The initial-initial antenna is

\[ a_{Emit}^{II}(s_{aj}, s_{jk}, s_{ak}) = 4 \frac{s_{ab}}{s_{aj}s_{bj}} + \frac{2}{s_{AB}} \left( \frac{s_{aj}}{s_{bj}} + \frac{s_{bj}}{s_{aj}} \right). \]  

(6.7)

It is restricted to the fermionic contribution, since \( W \) bosons do not appear in hadronic initial states. They can however appear in the initial state of a resonance antennae, so the initial-final contributions remain relevant.

Due to the absence of a colour structure in the case of photon emission, the structure of the parton shower differs significantly. A fully coherent branching kernel has to reduce to the complete soft limit given in eq. (6.1) as well as all relevant collinear limits given in eq. (6.3). One way to capture these limits is to construct the branching kernel

\[ a_{Emit}(\{p\}, p_j) = - \sum_{\{x,y\}} \sigma_x Q_x \sigma_y Q_y a_{Emit}(s_{xj}, s_{yj}, s_{xy}), \]  

(6.8)
where the notation \{x, y\} now implies a sum over all pairs of charged particles x and y. In fact, the sum over x and y includes the initial-state particles, and as such the emission antenna on the right-hand side of eq. (6.8) can be of initial-initial, initial-final and final-final types. The parton shower approximation to the radiative matrix element is

$$|M_{n+1} (\{p\}, p_j)|^2 \approx 4\pi\alpha a_{\text{Emit}} (\{p\}, p_j) |M_n (\{\bar{p}\})|^2,$$

where \{\bar{p}\} are the pre-branching momenta of the charged particles. It is not immediately obvious that eq. (6.8) reproduces the soft limit eq. (6.1) and the collinear limit eq. (6.3). This can however be shown using charge conservation of the entire event given by

$$\sum_x \sigma_x Q_x = 0.$$  

Considering the \(s_{ij}\)-collinear limit of eq. (6.8) leads to

$$a_{\text{Emit}} (\{p\}, p_j) = -\sigma_i Q_i \sum_{x \neq i} \sigma_x Q_x a_{\text{Emit}} (s_{ij}, s_{xj}, s_{ix}) + \mathcal{O}(1)$$

$$= -\sigma_i Q_i \frac{2}{s_{ij}} P_{I \rightarrow ij}(z) \sum_{x \neq i} \sigma_x Q_x$$

$$= Q_i^2 \frac{2}{s_{ij}} P_{I \rightarrow ij}(z).$$

As such, all collinear limits are included in eq. (6.8). In a similar fashion, it may be shown that the soft mass terms in eq. (6.1) are also properly incorporated.

### 6.1.2 The Coherent Algorithm

Even though the branching kernel given by eq. (6.8) contains all of the required infrared behaviour, it describes an \(n \rightarrow n + 1\) branching process while the iterated steps of the Vincia shower are \(2 \rightarrow 3\) branchings. The implementation should distribute emissions according to eq. (6.8), but it should also be ordered in an ordering variable that regulates all singular limits simultaneously. Furthermore, the kinematic mappings should be infrared safe, meaning that in all collinear limits all other charged particles should remain unaffected and in the soft limit no particle momenta should be modified. Finally, since the QED shower is interleaved with the QCD shower, the ordering variable should be directly comparable with the QCD ordering variable.

The above requirements lead us to consider a construction where the phase space is sectorized. To that end, we modify the parton shower approximation of eq. (6.9) to

$$|M_{n+1} (\{p\}, p_j)|^2 \approx a_{\text{Emit}} (\{p\}, p_j) \sum_{\{x,y\} \neq \{x,y\}} \Theta (Q_{xy}^2) |M_n (\{\bar{p}\}_{xy})|^2,$$
where

\[ \Theta(Q_{xy}^2) = \begin{cases} 1 & \text{if } \forall \text{ pairs } \{v, w\} \; Q_{xy}^2 \leq Q_{vw}^2 \\ 0 & \text{otherwise.} \end{cases} \] (6.13)

That is, the emissive phase space is divided into *sectors*. The step function defined in eq. (6.13) is nonzero for a single term in every point in the radiative phase space. As such, eq. (6.12) can be interpreted as meaning that the photon emission is performed by the pair of charged particles that has the lowest ordering scale with the photon. Correspondingly, the argument of the non-radiative matrix element \( \{\tilde{p}\}_{xy} \) indicates that only the momenta \( p_x \) and \( p_y \) are modified. The ordering scale is identical to the transverse momentum for gluon emissions

\[ Q_{xy}^2 = \begin{cases} s_x s_j / s_{XY} & \text{for final-final} \\ s_x s_j / (s_{XY} + s_{yj}) & \text{for initial-final} \\ s_x s_j / s_{xy} & \text{for initial-initial.} \end{cases} \] (6.14)

Eq. (6.12) ensures that, as long as the 2 \( \rightarrow \) 3 kinematics are infrared safe, all singular behaviour in the branching kernel is regulated. In fact, the shower implementation of eq. (6.12) corresponds to an ordering variable

\[ Q^2 = \min (Q_{xy}^2), \] (6.15)

which has the required property of ensuring that all soft and collinear regions are contained in the limit \( Q^2 \rightarrow 0 \), while still allowing for the use of regular 2 \( \rightarrow \) 3 shower kinematics. The corresponding parton shower distribution is given by

\[
S_{\text{Emit}} (\{p\}, p_j; Q_{\text{Start}}^2) = 
4\pi \alpha \theta (Q^2 < Q_{\text{Start}}^2) \sum_{\{x,y\}} \Theta (Q_{xy}^2) a_{\text{Emit}} (\{p\}_{xy}, p_j) \frac{f_a(x_a, Q^2)}{f_a(x_A, Q^2)} \frac{f_b(x_b, Q^2)}{f_b(x_B, Q^2)} 
\times \exp \left(-4\pi \alpha \sum_{\{x', y'\}} \int_{Q^2}^{Q_{\text{Start}}^2} dQ_{x'y'} \Theta (Q_{x'y'}^2) a_{\text{Emit}} (\{p\}_{x'y'}, p_j) \frac{f_a(x'_a, Q^2)}{f_a(x_A, Q^2)} \frac{f_b(x'_b, Q^2)}{f_b(x_B, Q^2)} \right). 
\] (6.16)

The PDF ratios required for initial-state radiation are now included as well. The Sudakov factor includes a sum over antenna phase space elements that are sectorized by the step function defined in eq. (6.13). When one or both of the particles in the pairwise sum is an initial-state quark, the corresponding antenna phase space integrates over the momentum fractions that appear in the PDF ratios. On the other hand, when neither is an initial-state quark, the partonic momentum fractions remain
6.1. PHOTON EMISSION

unmodified and the PDF ratio drops out. Since the sectorized approach is correct in the initial-state-collinear limit, eq. (6.16) is consistent with the required DGLAP evolution.

The distribution given by eq. (6.16) is quite complex, but it can be generated in a relatively straightforward way by use of the Sudakov veto algorithm. The shower generates a photon emission in every sector and then selecting the largest of those scale as per the competition algorithms elaborated on in Chapter 4. As such, a sector overestimate \( g_{xy}(Q^2_{xy}, z_{xy}) \) of eq. (6.8) is required, where \( Q^2_{xy} \) and \( z_{xy} \) are the ordering scales and auxiliary variables for photon emissions off \( x \) and \( y \). We construct the sector overestimate by first defining overestimates of the individual antennae given in eqs. (6.5), (6.6) and (6.7). For all configurations, we denote this overestimate as \( \bar{g}_{xy}(Q^2_{xy}, z_{xy}) \). The details of the construction of these overestimates are listed in Appendix A.3.

The branching kernel given in eq. (6.8) is a sum over these antennae that all depend on the kinematics of the branching. That is, if a photon is emitted by particles \( x \) and \( y \), all terms in the sum have to be evaluated using the newly emitted photon. Even though this function is very complex, it is possible to find a general overestimate due to our choice of ordering scale and the sectorization of phase space. Due to the presence of the step function defined in eq. (6.13), a photon emission is kinematically always performed in the sector with the smallest value of the ordering scale. This sector can thus be expected to be associated with a very large contribution to the sum in eq. (6.8). It is thus practical to construct the overestimate of the branching kernel as

\[
g_{xy}(Q^2_{xy}, z_{xy}) = c \bar{g}_{xy}(Q^2_{xy}, z_{xy}),
\]

where

\[
c = \sum_{\{x,y\}} \max(0, -Q_x Q_y).
\]

This value is found by first discarding the same-sign terms in eq. (6.8) which contribute negatively. Next, we realise that the antenna function in sector \( xy \) is likely to be the largest antenna to contribute to eq. (6.8). In fact, in the case where all charged particles are fermions, the overestimate of the branching kernel behaves like \( 1/Q^2_{xy} \) and eq. (6.17) is guaranteed to overestimate eq. (6.8) because of the requirement eq. (6.15). When \( W \)-bosons are involved, momentum configurations can be constructed where eq. (6.17) does not overestimate eq. (6.8), but these cases are so rare that they never occur in practice.

Having found a functional overestimate, the Sudakov veto algorithm is given in Algorithm 13. We provide a brief rundown of the algorithm. The general idea is to
Algorithm 13 The Sudakov veto algorithm for coherent photon emission

\[ Q^2 \leftarrow 0 \]

\[ p \leftarrow \bar{p} \]


do

\[ Q^2_{xy} \leftarrow Q^2_{\text{Start}} \]

\[ \tilde{Q}^2_{xy} \leftarrow Q^2_{\text{Start}} \]

\while \ Q^2_{xy} > Q^2 \end{while}

\[ Q^2_{xy}, z_{xy} \sim g_{xy}(Q^2_{xy}, z_{xy}) \exp \left( -4\pi\alpha \int_{Q^2_{xy}}^{Q^2_{xy}} dQ_{xy}^2 d\tilde{z}_{xy} g_{xy}(Q^2_{xy}, \tilde{z}_{xy}) \right) \]

\if \ Q^2_{xy}, z_{xy} \lie \text{within the antenna phase space} \then

\[ \varphi_{xy} \sim \theta(0 \leq \varphi_{xy} \leq 2\pi)/2\pi \]

Construct post-branching momenta \( p_x, p_y, p_j \) from \( p_X, p_Y, Q^2_{xy}, z_{xy}, \varphi_{xy} \)

\if \ \forall \ pairs \ x', y' \ Q^2_{x' y'} \geq Q^2_{xy} \then

\[ a_{\text{Emit}}(\{p\}, p_j)/c g_{xy}(Q^2_{xy}, z_{xy}) < r \sim \theta(0 \leq r \leq 1) \then

\[ Q^2 \leftarrow Q^2_{xy} \]

\[ p \leftarrow \bar{p} \] replacing \( p_X \to p_x \) and \( p_Y \to p_y \)

break

\end if

\end if

\end if

\[ Q^2_{xy} \leftarrow Q^2_{xy} \]

\end while

\end for
6.1. PHOTON EMISSION

generate a trial emission in all sectors $xy$. After the trial scale $Q_{xy}^2$ and the auxiliary variable $z_{xy}$ have been sampled, the algorithm proceeds to the first veto step where the phase space limits are checked. If those requirements are passed, $Q_{xy}^2$ and $z_{xy}$ are used to construct the post-branching momenta of the trial emission. Next, using the newly created trial momenta, the ordering scale with respect to all other sectors is computed. The event is then vetoed if any ordering scale is smaller than $Q_{xy}^2$. If this veto step is passed, the final veto step is the usual branching kernel veto. If that step is passed as well, the algorithm finally checks if the scale $Q_{xy}^2$ is larger than any other trial emission scale that has been generated in another sector. If $Q_{xy}^2$ is indeed the largest scale so far, it is saved along with the post-branching momenta. This process is repeated until trials have been generated in all sectors. The veto step that checks if $Q_{xy}^2$ is the smallest ordering scale ensures that every post-branching phase space point is only populated by a single sector. This is reflected in the step function eq. (6.13) that appears in eq. (6.16).

Some details have been left out of Algorithm 13 to increase readability. For instance, the veto step also includes the PDF ratio veto as explained in Appendix A.3. Furthermore, the running coupling is accounted for by setting $\alpha = \alpha(Q_{\text{start}}^2)$ and including the ratio $\alpha(Q^2)/\alpha(Q_{\text{start}}^2)$ in the veto probability. A cutoff scale is used to regulate the shower according to the Sudakov veto algorithm in the presence of a cutoff described in Section 4.4.2. Using the techniques described in that chapter, Algorithm 13 can then be shown to produce the probability distribution given by eq. (6.16).

6.1.3 Pairing Algorithm

While the algorithm discussed in the previous section is fully coherent, it comes at a large computational cost. Trial emissions for the full branching kernel have to be generated in every sector, and these emissions are often vetoed due to the sector condition. At first glance, it may seem like the sector veto step only keeps 1 out of $n^2$ emissions on average. Fortunately, the algorithm is a little more efficient than that, since trial emissions in sector $xy$ at low values of the ordering scale have a much higher chance to pass the sector requirement. However, the fact that samples are drawn from the full branching kernel in every sector still leads the cost of the algorithm to scale as $\mathcal{O}(n^2)$, where $n$ is the number of charged particles. This number may increase as the shower runs and new charged particles are produced. For collisions at large invariant mass where many charged particles may be produced, this scaling behaviour may become prohibitive. Therefore, we implement the option to select a secondary algorithm that is constructed to more closely mimic its QCD counterpart.
Rather than constructing a single branching kernel that captures all singular limits associated with photon emission at once, we instead split up the collinear singularities into pairs that radiate independently. The parton shower approximation to the matrix element can then be written as

\[ |M_{n+1}(\{p\}, p_j)|^2 \approx 4\pi\alpha \sum_{[x,y]} Q_{[x,y]}^2 a_{\text{Emit}}(s_{xj}, s_{yk}, s_{xy}) |M_n(\{\bar{p}\}_{xy})|^2. \]  

The sum now runs over pairings \([x,y]\) that have identical but opposite charge \(Q_{[x,y]}\). That is, every charged particle is paired with another oppositely charged particle and only appears in the sum once. Eq. (6.19) trivially reduces to the correct collinear limits, but the requirement that the paired charges are equal but opposite is a requirement for that. It is in fact possible to encounter charge configurations where finding such a pairing is not possible. For instance, in the showering of a top decay to a bottom and a \(W\) boson all involved particles have different charges and no pairing can be found. In these instances, the Vincia implementation will automatically resort to the coherent approach.

Furthermore, eq. (6.19) only contains a subset of the eikonal factors incorporated in eq. (6.8), and most notably any negative contributions are absent. However, by choosing a suitable method to pair up the charges, the missing interference structure can be approximated. Figure 6.3 illustrates a configuration of charges where two pairings are available and one pairing performs much better than the other. In this configuration, we consider two boosted \(e^+e^-\) pairs moving in opposite directions. The antennae spanned between the charges are indicated in blue for the positive...
contributions and orange for the negative contributions. Since the components of the pairs move in roughly the same direction, the charges of the electrons and positrons should be shielded and the radiation of photons should be suppressed. In other words, the antennae spanned between the pairs should largely cancel. The only remaining antennae are then those inside the pair, where the radiative phase space is restricted by the antenna invariant mass. If the chosen pairing would combine antennae between the pairs, the large negative contribution would be completely neglected. We therefore opt to pair up charges to minimize the sum of invariant masses of the pairs.

The pairing is performed by use of the Hungarian algorithm (also referred to as the Kuhn-Munkres algorithm) \[136\] \[137\]. This algorithm solves the so-called assignment problem, which for us exactly corresponds to finding a pairing of positive and negative charges such that the sum of the pairwise invariant masses is minimized. The time complexity of the algorithm is naïvely $O(n^4)$, but in a more sophisticated implementation this can be reduced to $O(n^3)$ \[138\]. The Vincia implementation makes use of an open-source C++ implementation that can be found at \[139\]. After a suitable pairing has been found, the paired antennae are allowed to radiate independently in close analogy with their QCD counterparts.

### 6.1.4 Photon Emission Below the Hadronization Scale

A major difference between the QED shower and QCD shower is the difference in the cutoff scale. All coloured particles shower down to the QCD cutoff scale, which is typically set to be at the characteristic scale $\Lambda_{QCD} \approx 1 \text{ GeV}$. Due to the simulation of multiple particle interactions, multiple QCD systems may be showered simultaneously. These systems, which originate from different interactions in the same hadron-hadron subsystem, are considered to remain separate up to energy scales of the order of $\Lambda_{QCD}$ as it represents the characteristic size of the proton. Resonance decays are also showered separately down to approximately $\Lambda_{QCD}$.

On the other hand, the QED shower may continue well below $\Lambda_{QCD}$ for interactions between leptons and photons, up to scales where additional radiation is either too soft or too collinear to be measured. In the Pythia shower, the default value of this scale is $O(10^{-6}) \text{ GeV}$. After the QCD shower terminates, the partons are converted to hadrons and hadronic decays are simulated. Since both the incoming protons and the outgoing hadrons carry charge, the remaining leptonic system below $\Lambda_{QCD}$ is not guaranteed to conserve charge. Both photon emission algorithms described previously require charge conservation to produce the correct collinear limits. In case of the coherent algorithm, charge conservation is explicitly used in eq. (6.11). In absence of charge conservation, the pairing algorithm will not be able to pair up all
remaining leptons, meaning some leptons may not radiate at all. Note that this issue does not appear in either the traditional Altarelli-Parisi-based showers or the more modern dipole showers, since these kernels only contain a single collinear limit.

Our strategy for photon emissions off leptons below the hadronization scale is therefore to make use of the pairing algorithm and to supplement the pool of charges with the available colour-neutral strings that are entering the hadronization stage. These strings function as a recoiler for the lepton as a whole, but they should not radiate photons themselves. The usual antenna function is then replaced by a dipole function

$$a^{\text{Dipole}}_{\text{FF}}(s_{ij}, s_{jk}, s_{ik}) = 4 \frac{s_{ik}}{s_{ij}(s_{ij} + s_{jk})} - 4 \frac{m_i^2}{s_{ij}^2} + 2 \frac{s_{jk}}{s_{IK} s_{ij}}$$

(6.20)

that only includes the collinear limits for the lepton.

### 6.2 Photon Splitting

Due to the absence of a soft singularity and thus of any form of interference, photon splitting is much simpler than photon emission. The only significant difference with respect to the QCD counterpart of gluon splitting is again the absence of a colour structure, which in this case serves to guide the choice for a recoiler. Aside from photon splitting in the final state, the Vincia implementation also allows for photons in the initial state to backwards-evolve into quarks. This process is referred to as photon conversion, and it is relevant for photon-initiated hard processes that are weighted by photon PDFs.

#### 6.2.1 Antenna Functions

For final-state photon splitting, the recoiler is always chosen to be part of the final state. The antenna function in that case is

$$a^{\text{FF}}_{\text{Split}}(s_{ij}, s_{jk}, s_{ik}) = \frac{1}{m_{ij}^2} \left( 2 \frac{s_{ik}^2 + s_{jk}^2}{s_{IK}^2} + 4 \frac{m_f^2}{m_{ij}^2} \right),$$

(6.21)

where \(I\) is the photon, \(i\) and \(j\) are the newly created fermion-antifermion pair and \(K\) recoils to \(k\). In the quasi-collinear limit, eq. (6.21) reduces to

$$a^{\text{FF}}_{\text{Split}}(s_{ij}, s_{jk}, s_{ik}) \propto \frac{2}{m_{ij}^2} \left( z^2 + (1 - z)^2 + 2 \frac{m_f^2}{m_{ij}^2} \right) = \frac{2}{m_{ij}^2} P_{\gamma \rightarrow ff}(z).$$

(6.22)

For photon conversion, the recoiler is instead always taken to be the other initial state. The antenna function can be found by crossing eq. (6.21) into the initial state.
6.2. PHOTON SPLITTING

It is

\[ a_{\text{conv}}^{\mu}(s_{aj}, s_{bj}, s_{ab}) = \frac{2}{s_{aj} s_{AB}} s_{ab}^2 + s_{bj}^2. \]  

(6.23)

In this case, \( a \) refers to the parton that ends up in the initial state and \( j \) to the parton that ends up in the final state. The backwards evolution of the photon into a quark or an antiquark are separate processes, and they are both included with the appropriate PDF ratio weights. The collinear limit of eq. (6.23) is

\[ a_{\text{conv}}^{\mu}(s_{aj}, s_{bj}, s_{ab})\text{col.} = \frac{2}{s_{aj} s_{ab}} \left( \frac{1}{z} + (1 - z)^2 \right) = \frac{2}{s_{aj}} P_{f \to \gamma f}(z). \]  

(6.24)

The collinear limit produces the photon emission splitting function, as can be expected from the corresponding forward-evolution picture of the photon conversion branching where a quark emits a photon into the initial state. The splitting function thus has the corresponding soft pole, but there is no interference since no other particle is able to emit a photon into the initial state.

6.2.2 Spectator Selection for Photon Splitting

The only significant complication of the simulation of photon splitting is the question of how to select a spectator for the splitting process. Since there is no soft divergence that can interfere with other contributions, the antenna function essentially only has to capture the collinear behaviour and the choice of spectator is formally beyond the parton shower precision. However, the choice may still impact the behaviour of the shower outside the singular region, it has been shown in [101, 134] by comparing to matrix elements that a random selection may lead to an oversampling of photon splittings. We generalize a method that has been employed by Vincia [101] and previously ARIADNE [62] for gluon splitting. Consider the shower history depicted in Figure 6.4 where a gluon splitting follows a gluon emission. In this case, the gluon is
colour-connected with the initial quarks, and both are a valid choice for the recoiler of the subsequent gluon splitting. A probabilistic choice is made between the two available recoilers with probability

\[ P_{Ai}^i = \frac{1/m_{ij}^2}{1/m_{ij}^2 + m_{jk}^2} = \frac{m_{jk}^2}{m_{ij}^2 + m_{jk}^2} \]

\[ P_{Ai}^k = 1 - P_{Ai}^i = \frac{m_{ij}^2}{m_{ij}^2 + m_{jk}^2}. \]  (6.25)

That is, the probability to select a recoiler that was able to emit the gluon is proportional to the propagator associated with that emission. Consider for instance the case where \( i \) is selected. The gluon splitting leaves the invariant mass of the gluon-recoiler system \( m_{ij}^2 \) unchanged. This invariant mass appears in the propagator of the gluon emission, and was thus an important component of the emission distribution. While \( m_{ij}^2 \) remains invariant, the gluon splitting will change \( m_{jk}^2 \) and thus the propagator structure of the gluon emission. In particular, if \( m_{jk}^2 \) was small and is increased significantly, the shift in the emission distribution may be particularly large. The probabilities given by eq. (6.25) correct for this effect by assigning large probability to select recoilers with low gluon-recoiler invariant mass. Note that under the strong ordering condition the scale of the gluon splitting is much smaller than that of the gluon emission. Therefore, the kinematic impact on the emission propagator is formally negligible, but in practice an effect may remain.

We extend eq. (6.25) to the case of photon splitting. Rather than a choice of two potential recoilers, the photon splitting should be allowed to recoil against any charged particle in the event since they may all have contributed to the photon emission. The probability to select a charged recoiler \( i \) is given by

\[ P_{Ai}^i = \frac{1/m_{ij}^2}{\sum x 1/m_{xj}^2}. \]  (6.26)

The details of the implementation of photon splitting and photon conversion can be found in appendix A.3.

### 6.3 Validation

In this section we validate the QED shower implementation by comparing against theoretical results from numerical DGLAP resummation, YFS radiation spectra in resonance decays and single-emission events generated directly according to eq. (6.8).
6.3 VALIDATION

6.3.1 Comparison with DGLAP Resummation

To validate that the QED shower correctly captures the logarithms associated with collinear photon emission, we compare the coherent and pairing shower algorithms with the result of a simple numerical DGLAP resummation. In the same way that initial-state radiation of the parton shower captures the DGLAP evolution of PDFs, final-state radiation describes the evolution of partonic fragmentation functions. Since we only consider QED radiation, a similar concept can be introduced for leptons where the evolution is dominated by photon emission. The function \( L(x, Q^2) \) describes the distribution for a lepton to retain a fraction \( x \) of its original energy at scale \( Q^2 < Q^2_{\text{Start}} \).

The initial condition is given by
\[
L(x, Q^2_{\text{Start}}) = \delta(x - 1).
\]

(6.27)

The evolution of \( L(x, Q^2) \) is then described by the DGLAP equation
\[
Q^2 \frac{\partial}{\partial Q^2} L(x, Q^2) = \int_1^x \frac{\alpha(Q^2)}{2\pi} dz \, P_{ll}(z) L\left(\frac{x}{z}, Q^2\right),
\]

(6.28)

where \( P_{ll}(z) \) is the regularized Altarelli-Parisi splitting function
\[
P_{ll}(z) = \frac{2}{(1 - z)_+} + \frac{3}{2} \delta(1 - z) - (1 + z).
\]

(6.29)

Numerous methods are available to solve DGLAP equations. Among the most popular techniques are those based on Laguerre polynomials [140, 141], Mellin transforms [142] and the brute-force method [143]. We opt to use the brute-force method for its simplicity. We start from a RAMBO-generated \( e^+e^-\mu^+\mu^- \) system to highlight differences between the coherent and the pairing algorithms. The definition of \( x \) is
\[
x = \frac{E_{Q^2_{\text{Cut}}}^e}{E_{Q^2_{\text{Start}}^{\mu}}^e}.
\]

(6.30)

The starting scale \( Q^2_{\text{Start}} \) for the parton shower and the DGLAP evolution is set to the minimum of the invariant masses of all pairs of leptons. This ensures that all sectors in the coherent algorithm are able to radiate. The coupling \( \alpha \) is fixed to the default Pythia value at the electron mass \( \alpha(m_e^2) = 0.00729735 \). The comparison between the parton shower and the numerical solution of the DGLAP equation is shown in Figure 6.5 for two values of the cutoff scale that roughly correspond to the default Pythia QCD and QED parton shower cutoffs. While the parton shower results closely match the DGLAP evolution for \( Q^2_{\text{Cut}} = 1 \text{ GeV}^2 \), the Pythia and Vincia results differ significantly for \( Q^2_{\text{Cut}} = 10^{-12} \text{ GeV}^2 \). The primary cause of this effect is the different definition of the parton shower ordering scale between Pythia and
Vincia. The distributions shown in Figure 6.5 are sensitive to hard collinear radiation, which is regulated by the cutoff. The difference between ordering scales may thus allow for different amounts of collinear radiation at the same cutoff value. Note that in a realistic scenario, the leptons would be dressed by radiated photons that are sufficiently collinear as determined by some minimum radial separation $\Delta R$ between the lepton and the photon. After clustering of these collinear photons, the difference between showers will largely be washed out.

One further interesting feature of Figure 6.5 is the fact that events appear with $x > 1$. While not allowed in the strictly collinear limit, they may occur in the parton approximation due to the $2 \rightarrow 3$ kinematic mapping occasionally increasing the energy of one of the leptons. These types of branchings are strongly suppressed in the coherent algorithm because the sector requirement encoded in eq. (6.13) does not favor branchings with an increasing lepton energy.

### 6.3.2 Comparison with YFS Simulation

Next, we compare the Vincia implementation of showers in resonance decays with results from the YFS formalism implemented in [128, 129]. The YFS formalism is able to incorporate all soft contributions, but collinear corrections are included order-
6.3. VALIDATION

![Z boson](image1)

![Z boson (soft only)](image2)

Figure 6.6: Photonic radiation profile of $Z \to \tau^+\tau^-$ for varying values of $m_\tau$.

by-order in a procedure similar to matrix element corrections in parton showers. To confirm that the soft behaviour of the sector approach is consistent with the YFS method, we display photon radiation profiles for the decay processes $Z \to \tau^+\tau^-$ in Figure 6.6 and $W^- \to \tau^-\nu_\tau$ in Figure 6.7.

In these figures, the variable $E_\gamma$ represents the total photonically radiated energy. In the left-hand figures, the complete antenna function is used while only the eikonal factor is included on the right-hand side. Their difference reveals that the collinear terms mainly impact the hard part of the spectrum. This may be expected since the antenna is dominated by the eikonal factor in the soft limit. In both cases, the shower is run from the kinematic limit to the hadronic cutoff $\Lambda_{QCD}$. In reality, the parton shower continues to evolve the leptons down to much lower scales, but they then no longer remain isolated in the resonance system. All graphs drop off sharply at $E_\gamma = m_{Z/W}/2$ due to kinematic constraints since higher values of $E_\gamma$ can only be reached if more than a single photon is emitted.

Similar radiation profiles are shown in Figure 1 in both [128] and [129] and we observe good agreement. One difference is the drop in $E_\gamma$ for very low values in Figures 6.6 and 6.7 which is caused by the cutoff scale.

6.3.3 Phase Space Discontinuities

A reason for concern with eq. (6.12) and the sector approach to coherent photon emission is the presence of discontinuities in the radiative phase space on the bound-
The parton shower is run from the kinematic limit on a RAMBO-generated $e^+e^-$ system event with $E_{CM} = 10^4$ GeV and $Q^2_{Cut} = 1$ GeV$^2$. The shower is terminated after a single emission, and only the events with an emission are considered. To remove the Sudakov suppression, a CKKW-L-like [69, 70] procedure is used where events are rejected with a probability that is generated using trial emissions from the scale of the actual emission. A directly generated event sample was compared with the unweighted parton shower sample, both with $O(10^9)$ events, in the emission scale, the photon energy and the various leptonic invariant masses. The samples match up extremely well for all variables, giving no cause for concern for any visible effects.

6.4 Results

In this section, we apply the new QED shower algorithms implemented in Vincia to Drell-Yan and $W^+W^-$ production at the LHC and investigate their differences. The two photon emission algorithms implemented in Vincia are compared with results from the application of the default Pythia parton shower, which includes photon
emission in the strictly collinear approximation. The Pythia results are produced with Pythia 8.2 \cite{29} using the default tune and the NNPDF2.3 PDF sets \cite{117}. The Vincia results are produced using the Vincia plugin \cite{105} with Pythia 8.2, using the default tune and the same PDF set. In all cases, MPI has been disabled since its effect on the results are minimal.

6.4.1 Drell-Yan

We first consider dilepton invariant-mass observables for Drell-Yan at centre-of-mass energy $\sqrt{s} = 14$ TeV with the cuts

$$p_\perp > 25 \text{ GeV and } \eta < 3.5$$

applied to the final-state leptons. Invariant-mass distributions are good candidates to probe the effects of the QED shower because the QCD evolution of the initial state only affects the final state through recoil imparted by a Lorentz boost. As such, invariant mass observables are unaffected by initial-state QCD radiation and isolate the QED corrections.

Drell-Yan production serves as a suitable test-case to probe the differences between coherent and incoherent approaches to photon radiation. While the initial state and final state are both charged, the scope of interference between the two is suppressed by a factor of the order of the off-shellness of the $Z$ \cite{145, 146}. This is a result of the relatively long-lived nature of the $Z$-boson close to its mass peak, causing the production and decay to remain separated. Pythia approximates this separation by showering the initial state and final state of Drell-Yan production separately. This is equivalent to making the narrow-width approximation. However, a more physical treatment may be considered where the initial state and final state are showered incoherently up to scales of the order of the off-shellness of the $Z$, and coherently from there on.

To investigate the differences between these approaches, we generate Born-level events with Madgraph5 \cite{36} using the same PDF sets and shower them with Pythia and Vincia. The results are shown in Figure 6.8. On the left-hand side, the invariant-mass distribution of the lepton pair is shown. The leptons are dressed by clustering them with photons within a cone distance $\Delta R = 0.2$. The blue and red lines show the result of the application of the pairing and coherent algorithms for the complete shower evolution, while the orange line refers to a mix of the two, allowing for only incoherent radiation at scales above the off-shellness of the $Z$ and coherent radiation below it. To show the importance of QED radiation on this observable, the result without QED showering is also included.
We observe that the results from the Pythia shower and the pairing algorithm match closely while the coherent and mixed algorithms show deviations of a few percent in the lower end of the spectrum. These differences are a consequence of the forward-backward asymmetry due to the axial nature of the $Z$ boson.

This difference may be understood by considering the leading-order partonic matrix element, which can be written as

$$|M_{qq\rightarrow e^+e^-}|^2 \propto \left( |A_{++}(s)|^2 + |A_{--}(s)|^2 \right) (1 - \cos(\theta_{eq}))^2$$

$$+ \left( |A_{+-}(s)|^2 + |A_{-+}(s)|^2 \right) (1 + \cos(\theta_{eq}))^2.$$  \hfill (6.32)

In this expression, $\theta_{eq}$ is the angle between the electron and the antiquark and

$$A_{\alpha\beta}(s) = \frac{Q_e Q_f}{s} \left[ \frac{(v_e - \alpha a_e)(v_f - \beta a_f)}{s - m_z^2 + i m_z \Gamma_z} \right].$$  \hfill (6.33)

where the definitions of the couplings can be found in Appendix A.3. The matrix element parameterized by eq. (6.32) is enhanced for small angles $\theta_{eq}$ in case $s < m_z^2$ and for large angles in case $s > m_z^2$. Because of the PDF weights, the initial-state quarks are most likely to be of up-type, which also have larger electromagnetic charge than down-type quarks and thus yield a larger contribution to the photon emission kernel. Therefore, it is more likely for the electron of hard scattering events of invariant mass lower than $m_z^2$ to point in the direction of a negatively charged initial-state antiquark. In the soft limit of the factorized photon emission amplitude eq. (6.1)
this configuration leads to destructive interference, and as such the photon emission probability is suppressed. These interference effects are neglected by the pairing algorithm and the Pythia shower. The photon emission probability is thus larger for incoherent showers and the emission kinematics are more likely to push such events to lower values of the leptonic invariant mass. On the other hand, events with invariant masses larger than $m^2$ will on average lead to constructive interference in eq. (6.1). However, since the shower kinematics almost always decrease the leptonic invariant mass, the differences between the algorithms are most pronounced in the left tail of the distribution.

On the right-hand side of Figure 6.8, the spectrum of the invariant mass of the lepton pair with an additional isolated photon is shown. The isolated photon is required to have an energy of at least 1 GeV, and no other particles that have a combined energy of 7% of the photon energy may be present in a cone of $\Delta R = 0.2$ around the photon. In this observable, differences between the pairing algorithm and the Pythia shower appear, but these remain minor compared with the difference with the coherent and mixed algorithms.

6.4.2 $W^+W^-$ Production

We consider $W^+W^-$ production with the leptonic decays $W^+ \rightarrow e^+\nu_e$ and $W^- \rightarrow \mu^-\bar{\nu}_\mu$ at center-of-mass energy $\sqrt{s} = 14$ TeV. The cuts

\[ p_\perp > 25 \text{ GeV and } \eta < 3.5 \]  

are applied to the charged leptons, as well as the missing transverse energy cuts

\[ E_\perp > 20 \text{ GeV} \]  

are applied to the neutrinos. We again consider final-state invariant mass quantities to minimize influence of the initial-state QCD radiation. In this case, due to the more complex structure of the hard scattering matrix element and the application of the phase space cuts, we expect fewer coherence effects to appear. However, differences between the Vincia and Pythia showers may be expected due to major differences in the treatment of photon radiation from $W$ bosons. The Pythia shower radiates photons from $W$ bosons using the fermionic splitting function while the Vincia antenna functions include the full Yang-Mills coupling and the effects of the longitudinal $W$ boson polarization. Furthermore, the Vincia shower includes effects from initial-state radiation from the $W$ boson in resonance showers, which is absent in the Pythia shower.
Figure 6.9: Invariant mass spectra of the $W^+W^-$ pair (left) and the $W^+$ with an isolated photon (right) for $W^+W^-$ production at $\sqrt{s} = 14$ TeV. Figures were made using RIVET \cite{118}.

Figure 6.9 shows the spectra of the invariant mass of $W^+W^-$ pair and the $W^+$ with an additional isolated photon. The differences between the Vincia shower and the Pythia shower are significant, spanning the entire spectrum in the case of the $W^+\gamma$ invariant mass. The discrepancy between the integrated cross sections is caused by events passing the selection cuts at different rates due to the differences between QED showers.
In this chapter, we describe a formalism for the inclusion of electroweak effects in a parton shower. Signs of new physics have yet to appear and the Standard Model has so far survived all forms of scrutiny. It is therefore likely that the Standard Model continues to describe nature accurately up to very high energy scales. At these very high energies heavy particles like electroweak gauge bosons, Higgs bosons and top quarks can start to appear as constituents of jets [147, 148] or otherwise contribute to radiative corrections. These types of electroweak radiative corrections have been shown to become sizeable even at LHC energies [149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162]. To highlight a few results, corrections to transverse momentum at LHC energies can already reach about 10% for exclusive dijet production [158, 159], and about 20% for single vector boson production [160, 161, 162], and they can be expected to grow even larger at future collider energies [163, 164]. Recently, ATLAS has reported on measurements that are sensitive to the collinear enhancements associated with $W$ radiation in jets [165]. There, it is also pointed out that these types of effects will play a significant role for several measurements at high energy scales, which will become more abundant as the LHC gathers more data. It is therefore desirable to incorporate these electroweak effects in a systematic way in Monte Carlo simulations.

Electroweak effects have been incorporated in parton showers in the past. An implementation is available in Pythia [166, 167] that only includes the radiation of electroweak gauge bosons and does not retain any spin information. The radiation of electroweak gauge bosons was similarly included in Sherpa to study $W$ emissions in jets [168]. Another approach was used in [169] where fixed-order matrix element calculations are combined with analytic Sudakov factors to achieve results similar to those of an electroweak parton shower. A more recent work [170] has implemented an electroweak shower in the Pythia 1 → 2 transverse momentum ordered shower
formalism that retains spin information and includes all electroweak branchings.

In this chapter, we set out to construct an implementation of an electroweak parton shower in the Vincia parton shower. Vincia already has helicity-dependent antenna functions and allows for QCD evolution with partons of definite spin states. The shower described in this chapter will be responsible for the electroweak component of the shower evolution which is interleaved with the QCD shower. It also facilitates evolution of helicity states, which is especially important in the electroweak theory due to its chiral nature. The shower formalism described here is based on a very different approach than that described in [170] and employs many different solutions for the subtleties involved in the construction of an electroweak shower. In particular, it makes use of the spinor-helicity formalism to compute its branching kernels. The methods employed here are comparable with those used in [171, 172] to compute helicity-dependent QCD antenna functions. However, due to significant differences in the details of our procedure, we start with a brief overview of the spinor-helicity formalism and the conventions used in the calculation of the branching kernels.

7.1 The Spinor-Helicity Formalism

Due to the chiral nature of the electroweak theory, it is important to calculate electroweak branching kernels for individual spin states. We choose to perform these calculations using the spinor-helicity formalism with definitions similar to those described in [173]. This formalism allows for the calculation of branching processes at the amplitude level. They can then be used to construct branching kernels, which represent the usual soft and collinear factorization of particle branchings. Furthermore, it does not require us to commit to a particular representation of the Dirac algebra or an explicit form for the spinors. Amplitudes can instead be computed analytically, to which end the methods are outlined in this section.

7.1.1 Spinors

To fix the phase convention of fermionic spinors, we introduce basis vectors $k_0$ and $k_1$ which obey the identities

$$k_0^2 = 0, \quad k_1^2 = -1, \quad k_0 \cdot k_1 = 0. \quad (7.1)$$

Using the basis vectors, the basis helicity spinors

$$u_- \equiv u_-(k_0) \quad \text{and} \quad u_+ \equiv \bar{k}_1 u_- \quad (7.2)$$
can be defined. In general, we will refer to the helicity indices of spinors with a parameter $\lambda = 1, -1 = +, -$. These basis spinors are eigenspinors of $\gamma^5$ and have eigenvalue 1 under the projection operators

$$\omega_{\lambda} = \frac{1}{2} (1 + \lambda \gamma^5).$$  \hspace{1cm} (7.3)

Furthermore, the relationship

$$u_{\lambda} \bar{u}_{\lambda} = \omega_{\lambda} \bar{k}_0$$  \hspace{1cm} (7.4)

defines the momentum argument $k_0$ of the basis spinors. Note that $k_0$ is lightlike, which implies that the antifermion spinor $v_{\lambda}(k_0)$ equals the fermion spinor $u_{\lambda}(k_0)$.

**Massless fermions**

For massless particles, the definitions of chirality and helicity coincide and represent Lorentz-invariant quantities. The helicity spinor for a massless momentum $p$ can be defined uniquely as

$$u_{\lambda}(p) = \frac{1}{\sqrt{2p \cdot k}}(\gamma^5 \gamma^\lambda - \gamma^\lambda \gamma^5)u_{-\lambda}. \Rightarrow (7.5)$$

This definition is easily checked to yield

$$u_{\lambda}(p) \bar{u}_{\lambda}(p) = \omega_{\lambda} \phi. \Rightarrow (7.6)$$

Furthermore, the Dirac equation

$$\gamma^\mu \gamma^\nu \phi u_{\lambda}(p) = 0 \Rightarrow (7.7)$$

holds in a straightforward manner.

**Massive fermions**

For massive momenta, the definitions of helicity and chirality no longer coincide. Furthermore, helicity is no longer a Lorentz-invariant quantity, becoming dependent on the local Lorentz frame. For a massive momentum $p$, a helicity spinor can still be defined as

$$u_{\lambda}(p) = \frac{1}{\sqrt{2p \cdot k}}(\phi + m)u_{-\lambda}(k) \text{ and } v_{\lambda}(p) = \frac{1}{\sqrt{2p \cdot k}}(\phi - m)u_{\lambda}(k). \Rightarrow (7.8)$$

Here, the vector $k$ is a lightlike reference vector that determines the meaning of the helicity of a massive fermion. Because of the presence of a particle mass, the spinors no longer adhere to eq. (7.6). The chiral projection operators $\omega_{\lambda}$ project onto single helicity states only in the case of massless fermions. For massive particles helicity
and chirality no longer coincide, and mass-suppressed mixed terms appear. However, the spin-summed completeness relations

$$\sum_{\lambda=\pm} u_\lambda(p) \bar{u}_\lambda(p) = \not{p} + m$$ and $$\sum_{\lambda=\pm} v_\lambda(p) \bar{v}_\lambda(p) = \not{p} - m$$ (7.9)

still holds. The massive spinors have further orthogonality relations

$$\bar{u}_\lambda(p) u_{\lambda'}(p) = 2m \delta_{\lambda\lambda'}$$ and $$\bar{v}_\lambda(p) v_{\lambda'}(p) = -2m \delta_{\lambda\lambda'}$$ (7.10)

as well as the massive version of the Dirac equations

$$(\not{p} - m) u_\lambda(p) = 0$$ and $$(\not{p} + m) v_\lambda(p) = 0.$$ (7.11)

To decide on a choice for the reference vector $k$, we compute the spin vector of the massive helicity spinors. For a general spinor $u(p, s)$ with momentum $p$ and spin vector $s$, the spin vector $s$ can be extracted by calculating

$$s^\mu = -\frac{1}{2m} \bar{u}(p, s) \gamma^5 \gamma^\mu u(p, s).$$ (7.12)

For the massive helicity spinors and antispinors, this leads to

$$s^\mu = \frac{\lambda}{m} \left( p^\mu - \frac{m^2}{p \cdot k} k^\mu \right).$$ (7.13)

We therefore choose the reference vector

$$k = (1, -\vec{e}),$$ (7.14)

where $\vec{e}$ is a unit vector pointing in the direction of $\vec{p}$. With this choice, the massive helicity spinors retain the usual meaning of helicity as the projection of spin along the direction of motion. This choice also clarifies the dependence on the current Lorentz frame. If a boost is applied, the momenta $p$ and $k$ will generally no longer align in the new frame, indicating that the spin vector defined in eq. (7.12) no longer points in the direction of motion.

In hadron colliders, the center-of-mass frame does not coincide with the lab frame as a consequence of the composite nature of the incoming particles. In particular, the parton shower may increase the total scattering energy of the partonic subsystem through backwards evolution of the initial-state partons. In those cases, the initial state acquires some transverse momentum and a Lorentz boost is applied to the entire event to realign the initial state with the beam axis. As such, Lorentz boosts may be imposed on massive final states at any point of the parton showering process, and it may be important to account for the misalignment between the resulting momentum
7.1. THE SPINOR-HELICITY FORMALISM

and the reference vector. One way to do that is to use the completeness relation eq. (7.9) to write out the relationship between massive spinors with differing reference vectors \( k \) and \( k' \) as

\[
    u_{\lambda}(p, k) = \sum_{\lambda'= \pm} F_{\lambda\lambda'}(p, k, k') u_{\lambda'}(p, k'),
\]

(7.15)

where \( F_{\lambda\lambda'}(p, k, k') \) is the normalized overlap between spinors of different reference vectors. The squared overlaps can be expressed in terms of the spin vectors

\[
    F_{\lambda\lambda'}^2(p, k, k') = \frac{1}{4m^2} |\bar{u}_{\lambda}(p, k)u_{\lambda'}(p, k')|^2 = \frac{1}{2} (1 + s_\lambda \cdot s'_{\lambda'}).
\]

(7.16)

The procedure is identical for antispinors and leads to the same overlap functions. These overlap functions can then be computed for reference vectors in the old and the new frame, and be interpreted as probabilities for a spin state to either retain or flip its spin as a consequence of a boost.

7.1.2 Polarization Vectors

Vector boson polarizations are defined using the helicity spinors described in the previous section. The polarization vectors for a massive vector boson with momentum \( p \) can be defined as

\[
    \epsilon^\mu_{\pm}(p) = \pm \frac{1}{\sqrt{2}} \frac{1}{2p\cdot k} \bar{u}_{\pm}(k)\gamma^\mu u_{\pm}(k) \quad \text{and} \quad \epsilon^\mu_0(p) = \frac{1}{m} \left( p^\mu - \frac{2m^2 \cdot k}{2p\cdot k}k^\mu \right).
\]

(7.17)

Here, \( \epsilon^\mu_{\pm}(p) \) are the transverse polarizations and \( \epsilon^\mu_0(p) \) is the longitudinal polarization which only exists for massive vector bosons. These definitions can be shown to obey the Lorentz condition

\[
    \epsilon_{\lambda}(p) \cdot p = 0
\]

(7.18)

as well as the completeness relation

\[
    \sum_{\lambda=\pm,0} \epsilon^\mu_{\lambda}(p)\bar{\epsilon}^\nu_{\lambda}(p) = -g^{\mu\nu} + \frac{1}{m^2} p^\mu p^\nu.
\]

(7.19)

Furthermore, \( \epsilon_{\lambda}(p) \cdot \bar{\epsilon}_{\lambda'}(p) = -\delta_{\lambda\lambda'}(p) \) implies that they are orthogonal and normalized. In fact, the reference vector defined in eq. (7.14) can also be used here. The transverse
polarizations of eq. (7.17) are explicitly defined to have $\epsilon_\pm \cdot k = 0$, such that the choice of $k$ to be opposite the direction of $p$, together with eq. (7.18), guarantees that the polarization is truly transverse. On the other hand, the longitudinal polarization is easily seen to be completely longitudinal. To translate between polarizations of different reference vectors, we can again write

$$\epsilon_\lambda(p, k) = \sum_{\lambda' = \pm, 0} V_{\lambda\lambda'}(p, k, k') \epsilon_{\lambda'}(p, k').$$

(7.20)

The overlap functions can in this case be computed directly from eq. (7.17). They evaluate to

$$V_{\pm\pm}(p, k, k') = 1 - m^2 \frac{k \cdot k'}{p \cdot k \cdot k'} + \frac{m^4}{4} \frac{(k \cdot k')^2}{(p \cdot k)^2 (p \cdot k')^2}$$

$$V_{\pm\mp}(p, k, k') = \frac{m^4}{4} \frac{(k \cdot k')^2}{(p \cdot k)^2 (p \cdot k')^2}$$

$$V_{\pm 0}(p, k, k') = V_{0\pm}(p, k, k') = m^2 \frac{k \cdot k'}{p \cdot k \cdot k'} - \frac{m^4}{2} \frac{(k \cdot k')^2}{(p \cdot k)^2}$$

$$V_{00}(p, k, k') = 1 - 2m^2 \frac{k \cdot k'}{p \cdot k \cdot k'} + m^4 \frac{(k \cdot k')^2}{(p \cdot k)^2}.$$  

(7.21)

Note that the spin-changing contributions are all mass-suppressed as expected.

### 7.1.3 Amplitude Evaluation

Having expressed all massive spinors and polarization vectors in terms of massless spinors, amplitudes for particles with definite helicities can now be calculated very efficiently. We first define the spinor product

$$S_\lambda(k_a, k_b) \equiv \bar{u}_\lambda(k_a) u_{-\lambda}(k_b)$$

(7.22)

for lightlike (reference) vectors $k_a$ and $k_b$. These spinor products have

$$S_{-\lambda}(k_a, k_b) = -S_\lambda(k_a, k_b)^*$$

(7.23)

and

$$S_\lambda(k_a, k_b) S_{-\lambda}(k_a, k_b) = |S_\lambda(k_a, k_b)|^2 = 2k_a \cdot k_b.$$  

(7.24)

Using the representation of massless helicity spinors given in eq. (7.5), spinor products can be written as

$$S_\lambda(k_a, k_b) = \frac{1}{\sqrt{2k_a \cdot k_0}} \frac{1}{\sqrt{2k_b \cdot k_0}} \text{Tr} \left( \omega_{-\lambda} k_0^\dagger k_a^\dagger k_b k_1 \right).$$

(7.25)
We now fix the global reference vectors $k_0$ and $k_1$ to explicit and simple values, taking care to avoid the $z$-direction since it typically has a special role as the orientation of the beam axis:

$$k_0 = (1, 1, 0, 0) \quad \text{and} \quad k_1 = (0, 0, 1, 0). \quad (7.26)$$

This choice leads to the simple form

$$S_\lambda(k_a, k_b) = (\lambda k_a^2 + i k_a^3) \left[ \frac{k_0^2 - k_b^2}{k_0^2 - k_b^2} - (\lambda k_b^2 + i k_b^3) \right] \sqrt{k_0^a - k_0^b}, \quad (7.27)$$

which is easily evaluated. Using the spinors and polarization vectors of the previous section, all amplitudes can be expressed in terms of these spinor products. The structures that may appear look like

$$S_\lambda(k_a, p_i, p_j, ..., k_b) \equiv \bar{u}_\lambda(k_a) \psi_i \psi_j ... u_{\pm \lambda}(k_b), \quad (7.28)$$

where $p_i, p_j, ...$ may be massive. These structures may be expressed in terms of the spinor products eq. (7.22) by defining

$$\hat{p}_i = p_i - \frac{p_i^2}{2 p_i \cdot k_i} k_i, \quad (7.29)$$

which is explicitly massless. Using the Dirac equation, eq. (7.28) then evaluates to

$$S_\lambda(k_a, p_i, p_j, ..., k_b) = \bar{u}_\lambda(k_a) \omega_{- \lambda} \hat{p}_i \hat{p}_j ... u_{\pm \lambda}(k_b)$$

$$= \bar{u}_\lambda(k_a) u_{- \lambda}(\hat{p}_i) \times \bar{u}_{- \lambda}(\hat{p}_i) \hat{p}_j ... u_{\pm \lambda}(k_b)$$

$$= S_\lambda(k_a, \hat{p}_i) S_{- \lambda}(\hat{p}_i, p_j, ..., k_b). \quad (7.30)$$

This procedure can be repeated, the next time making $p_j$ massless by subtracting $(p_j^2/2 \hat{p}_i \cdot p_j) \hat{p}_i$, until the expression consists of two-momentum spinor products which can be directly evaluated using eq. (7.27).

### 7.2 Electroweak Branching Amplitudes

We are now ready to use the spinor-helicity formalism to compute branching amplitudes for the electroweak theory. We first recount the phase space regions where the radiative amplitude factorizes into a non-radiative amplitude and a radiative correction. The momentum and helicity assignment for the amplitudes is given by

$$M_1 = \bullet \ p_I, \lambda_I \quad \quad M_2 = \bullet \ p_i, \lambda_i \quad (7.31)$$
In the quasi-collinear limit, where
\[ p_i \cdot p_j \approx m_i^2, m_j^2 \text{ and } E_i^2, E_j^2 \gg p_i \cdot p_j, \]  
(7.32)
the energy sharing variable \( z \) can be defined by
\[ p_i = z p_I \text{ and } p_j = (1 - z) p_I. \]  
(7.33)
In this limit, the matrix element factorizes as
\[ |M_2|_\text{coll.}^2 = Q^2_2 Q^2_4 + m^2_I \Gamma^2_I P(\lambda_I, \lambda_i, \lambda_j, z) |M_1|^2, \]  
(7.34)
where \( Q^2_2 = (p_i + p_j)^2 - m^2_I, \) \( \Gamma_I \) is the decay width of \( I \) and \( P(\lambda_I, \lambda_i, \lambda_j, z) \) is the helicity-dependent Altarelli-Parisi splitting kernel. This definition of \( Q^2_2 \) foreshadows that it will later on assume the role of the ordering scale of the electroweak shower. On the other hand, in the soft limit, where
\[ E_j \approx m_j \text{ and } E_i \gg E_j \]  
(7.35)
the amplitude exhibits the eikonal factorization
\[ M_2 \equiv M_1 \times c \frac{2 p_i \cdot \epsilon_{j \lambda_j}}{Q^2 + i m_I \Gamma_I} \delta_{\lambda_I \lambda_i}, \]  
(7.36)
where \( c \) is some spin-dependent coupling.

The electroweak branching kernels should reduce to the above soft and collinear limits eq. (7.34) and eq. (7.36) in their respective phase space regions eq. (7.32) and eq. (7.36). To compute them, we use the following procedure:

1. Write down an operator that creates a particle with definite helicity
2. Show that these operators indeed lead to polarized states in the radiative and non-radiative amplitudes given by eq. (7.31)
3. Compute the branching kernel from the ratio between the radiative and non-radiative amplitudes
\[ B_{\lambda_I, \lambda_i, \lambda_j}(p_I, p_i, p_j) = \left| \begin{array}{c} \bullet \\ p_i, \lambda_i \\ p_j, \lambda_j \end{array} \right|^2 \left/ \left| \begin{array}{c} \bullet \\ p_I, \lambda_I \end{array} \right|^2. \]  
(7.37)
One major difference between the methods used in [171, 172] and our method are the definitions of the operators that create particles with states of definite helicity. These
are represented by the black dots in eq. (7.31). We take their Feynman rules to be

\[
p, \lambda \rightarrow \bigcirc = u_{-\lambda}(k) \\
p, \lambda \rightarrow \bigcirc = \bar{u}_{-\lambda}(k) \\
p, \lambda \rightarrow \bigcirc = \epsilon^\mu_\lambda(\hat{p}) \\
p, \lambda \rightarrow \bigcirc = 1.
\] (7.38)

where the grey blob represents the rest of the Feynman diagram and the vector

\[
\hat{p} = p - \frac{p^2 - m^2}{2p \cdot k} k
\] (7.39)

is again introduced such that the momentum associated with the polarization vector is on shell. We now proceed with the procedure associated above on a number of illustrative electroweak processes to compute the branching kernels.

### 7.2.1 Vector Boson Emission from a Fermion

The first process under consideration is the emission of a vector boson from a massive fermion. If the vector boson is a photon, the splitting functions will be very similar to their helicity-dependent analogue for gluon emission from a quark \[106, 107\]. On the other hand, the vector boson may be a $Z$-boson, in which case the chiral nature of its coupling will be reflected in the splitting functions, or it may be a $W$-boson, in which case the flavour of the fermion also changes.

We first compute the non-radiative amplitude in case the outgoing particle is a massive fermion. We observe that

\[
\bar{u}_\lambda p_I u_{-\lambda_j}(k_I) = \sqrt{2p_I \cdot k_I} \delta_{\lambda, \lambda_j}. \] (7.40)

The Feynman rule given in eq. (7.38) thus does indeed create a state of definite helicity in the non-radiative amplitude. We can therefore choose the helicity in the operators to be $\lambda_I$ in both the non-radiative and radiative amplitude. The non-radiative amplitude is then simply given by

\[
M^f_I = p_I, \lambda_I \\
= \bar{u}_{\lambda_I} p_I u_{-\lambda_I}(k_I) = \sqrt{2p_I \cdot k_I}.
\] (7.41)
Using the electroweak Feynman rules from appendix A.3, the radiative amplitude evaluates to

\[ M_{f \to f'}^{V} \equiv \begin{array}{c}
\bullet \ v_{i}, \lambda_{i} \\
\downarrow \ p_{i}, \lambda_{i} \\
p_{j}, \lambda_{j} \end{array} = \frac{1}{Q^{2} + i m_{I} \Gamma_{I}} \bar{u}_{\lambda_{i}}(p_{i})(v + a\gamma^{5})\lambda_{j}(p_{j})(\phi_{ij} + m_{I})u_{-\lambda_{i}}(k_{ij}), \]  

(7.42)

where \( p_{ij} = p_{i} + p_{j} \). Using the Dirac equation, we modify the momentum flowing in the fermion propagator to be on-shell. It is

\[ \hat{p}_{ij} \equiv p_{ij} - \frac{Q^{2}}{2p_{ij} \cdot k_{ij}}, \]  

(7.43)

which indeed has \( \hat{p}_{ij}^{2} = m_{I}^{2} \). Using the spinor completeness relation eq. (7.9), we continue with eq. (7.42) to find

\[ M_{f \to f'}^{V} = \frac{1}{Q^{2} + i m_{I} \Gamma_{I}} \bar{u}_{\lambda_{i}}(p_{i})(v + a\gamma^{5})\lambda_{j}(p_{j}) \left( \sum_{\lambda=\pm} u_{\lambda}(\hat{p}_{ij}) \bar{u}_{\lambda}(\hat{p}_{ij}) \right) u_{-\lambda_{i}}(k_{ij}) \]

\[ = \frac{1}{Q^{2} + i m_{I} \Gamma_{I}} \bar{u}_{\lambda_{i}}(p_{i})(v + a\gamma^{5})\lambda_{j}(p_{j})u_{\lambda_{i}}(\hat{p}_{ij}) \times \sqrt{2\hat{p}_{ij} \cdot k_{ij}}. \]  

(7.44)

Note that eq. (7.44) explicitly shows that the fermion Feynman rule from eq. (7.38) creates a fermionic state of definite helicity \( \lambda_{I} \) which also propagates to the vector boson vertex. In fact, the amplitude factorizes into a radiative piece and an additional factor \( \sqrt{2\hat{p}_{ij} \cdot k_{ij}} \), which is equal to the non-radiative amplitude up to corrections that vanish in the collinear limit.

We now evaluate the radiative piece of eq. (7.44) for all helicity configurations, which will from now on be referred to as the branching amplitude. For the transverse polarizations, the so-called Chisholm identity [173]

\[ f_{\pm}^{\mu}(p) = \pm \frac{1}{\sqrt{2}} \frac{1}{2p \cdot k} \bar{u}_{\pm}(k)\gamma^{\mu}u_{\pm}(k)\gamma_{\mu} \]

\[ = \pm \frac{\sqrt{2}}{2p \cdot k} \left( u_{\pm}(k)\bar{u}_{\pm}(k)\phi - \phi u_{\pm}(k)\bar{u}_{\pm}(k) \right) \]  

(7.45)

is used. Defining the prefactor

\[ A_{\perp} = \frac{1}{2\sqrt{2}} \frac{\lambda}{\sqrt{p_{i} \cdot k_{i}} \sqrt{p_{ij} \cdot k_{ij} p_{j} \cdot k_{j}}}, \]  

(7.46)

the branching amplitude for transverse vector boson polarizations is

\[ M_{f \to f'}^{V}(\lambda_{I}, \lambda_{i}, \lambda_{j}) = \frac{A_{\perp}}{Q^{2} + i m_{I} \Gamma_{I}} \bar{u}_{\lambda_{i}}(k_{i})(\phi_{i} - m_{i})(v + a\gamma^{5}) \]

\[ \times \left( u_{\lambda_{j}}(k_{j})\bar{u}_{-\lambda_{j}}(k_{j})\phi_{j} - \phi_{j} u_{\lambda_{j}}(k_{j})\bar{u}_{-\lambda_{j}}(k_{j}) \right) \]

\[ \times (\phi_{ij} - m_{I})u_{\lambda_{i}}(k_{ij}). \]  

(7.47)
Evaluating this expression for all possible helicity configurations leads to

\[
M^{f 	o f' V}(\lambda, \lambda, \lambda) = \frac{A_\perp}{Q^2 + i m_I \Gamma_I} \left[ (v - \lambda a) S_{-\lambda}(k_i, p_i, p_j, k_j) S_{-\lambda}(k_j, p_{ij}, k_{ij}) + (v + \lambda a) m_i m_I S_{-\lambda}(k_i, k_j) S_{-\lambda}(k_j, p_{ij}, k_{ij}) \right]
\]

\[
M^{f 	o f' V}(\lambda, \lambda, -\lambda) = \frac{A_\perp}{Q^2 + i m_I \Gamma_I} \left[ (v - \lambda a) S_{-\lambda}(k_i, p_i, k_j) S_{-\lambda}(k_j, p_{ij}, k_{ij}) + (v + \lambda a) m_i m_I S_{-\lambda}(k_i, k_j) S_{-\lambda}(k_j, p_{ij}, k_{ij}) \right]
\]

\[
M^{f 	o f' V}(-\lambda, \lambda, -\lambda) = \frac{A_\perp}{Q^2 + i m_I \Gamma_I} \left[ m_I (v + \lambda a) S_{\lambda}(k_i, p_i, k_j) S_{-\lambda}(k_j, p_{ij}, k_{ij}) - m_i (v - \lambda a) S_{\lambda}(k_i, k_j) S_{-\lambda}(k_j, p_{ij}, k_{ij}) \right]
\]

\[
M^{f 	o f' V}(-\lambda, -\lambda, -\lambda) = \frac{A_\perp}{Q^2 + i m_I \Gamma_I} \left[ m_I (v + \lambda a) S_{\lambda}(k_i, p_i, k_j) S_{\lambda}(k_j, k_{ij}) - m_i (v - \lambda a) S_{\lambda}(k_i, k_j) S_{\lambda}(k_j, p_{ij}, k_{ij}) \right].
\]

When the vector boson polarization is longitudinal, the prefactor is

\[
A_L = \frac{1}{2} \frac{1}{m_j \sqrt{p_i \cdot k_i} \sqrt{p_{ij} \cdot k_{ij}}}
\]

(7.48)

The amplitude becomes

\[
M^{f 	o f' V}(\lambda, \lambda, 0) = \frac{A_L}{Q^2 + i m_I \Gamma_I} \bar{u}_\lambda(k_i) (\phi_i - m_i) (v + a\gamma^5) \left( \phi_j - 2 m_j \frac{k_j}{w_j^2} \right) (\phi_{ij} - m_I) u_\lambda(k_{ij}).
\]

(7.49)

At this point, a common problem related to longitudinal polarization in the electroweak theory appears. Eq. (7.49) contains a contribution of the form

\[
\not{\phi}_i \not{\phi}_j \not{\phi}_{ij} = Q^2 \phi_i^2 + m_i^2 \phi_i^2 - m_i^2 \phi_{ij}^2.
\]

(7.50)

The term proportional to \( Q^2 \) will cancel against the propagator up to a remainder proportional to the width. The result is a contribution that is not singular in the quasi-collinear or soft limit. To make matters worse, this term leads to unitarity-violation for high energies and thus indicates a pathology in the method. These unitarity-violating terms appear because the calculation of branching amplitudes as illustrated in Figure 7.1 is not gauge-invariant. Vector boson emissions are not described by the single Feynman diagram we have considered so far. Other diagrams
where the vector boson is emitted from either an internal line, or some other external line also contribute. A similar situation formally appears for the calculation of QCD or QED branching kernels. In those cases, the left-over terms originating from the gauge choice, or equivalently those from other diagrams, are no cause for concern. They are not singular in the soft and quasi-collinear phase space regions and thus do not contribute to the leading-log precision of the parton shower. Instead, the nonsingular terms may be used as parameters for uncertainty estimation [50].

In the electroweak theory, this behaviour is spoiled by the presence of longitudinal polarizations. The definition in eq. (7.5) reveals that the longitudinal polarization has two contributions, one that scales like $O(E/m)$ and one that scales like $O(m/E)$. The first term is a scalar piece that is a remainder of the Goldstone boson from the unbroken theory, while the second term is a vector piece from the $SU(2)$ and $U(1)$ gauge bosons. Having chosen to use eq. (7.17) as polarization vectors and using the reference vector of eq. (7.14) we have fixed our gauge and ended up with a gauge-dependent branching amplitude.

We point out that this same issue is discussed in [170]. There, the issue is dealt with by choosing a particularly suitable gauge and invoking the Goldstone equivalence principle. This method ensures that the unitarity-violations are automatically isolated and no gauge artifacts remain. As we have already fixed our gauge, this avenue is not open to us. Fortunately, due to the analytic nature of the spinor-helicity formalism, the problematic terms are very easy to identify and remove manually. We are reassured in our method by the knowledge that the terms in question have no factors of $Q^2$ in the denominator, meaning they must cancel to prevent unitarity violation. The remaining terms all have a factor of $Q^2$ in the denominator, and can therefore not be cancelled against any other diagram.

After removing the unitarity-violating terms, the amplitudes for longitudinal vec-
tor boson emission become

\[
M_{f \to f'V}(\lambda, \lambda, 0) = 2 \frac{A_L}{Q^2 + im_I \Gamma_I} \times \left[ S_-\lambda(k_i, (v - \lambda a)(m_I^2 p_i - m_i^2 p_{ij} - im_I \Gamma_I) + (v + \lambda a)m_I m_{ij} k_{ij}) - 2 \frac{m_i^2}{w_j^2} \right.
\]

\[
M_{f \to f'V}(\lambda, -\lambda, 0) = 2 \frac{A_L}{Q^2 + im_I \Gamma_I} \left[ S_-\lambda(k_i, (v - \lambda a)(k_i, p_i, k_j, p_{ij}, k_{ij}) + (v + \lambda a)m_I m_{ij} S_-\lambda(k_i, k_j, k_{ij}) \right]
\]

\[
\left. + m_I (v + \lambda a) S_-\lambda(k_i, p_i, p_j - 2 \frac{m_i^2}{w_j^2} k_{ij}) \right].
\]

(7.51)

The calculation for the cases of vector boson emission from an antifermion and vector boson splitting to a fermion-antifermion pair proceed in very similar fashion. Furthermore, the electroweak shower also includes vector boson emission from the initial state. The calculation of the amplitudes for those branchings is also very similar to the above calculation. Alternatively, the initial-state emission amplitudes can be obtained by employing crossing symmetry from the final-state ones. In the next section, we consider the collinear limits of these and all other branching amplitudes.

### 7.2.2 Vector Boson Emission from a Vector Boson

Next, we consider the branching amplitude for a vector boson emission from a vector boson. We again first make the observation that

\[
\epsilon_\lambda(p_I) \cdot \epsilon_{\lambda'}(p_I) = -\delta_{\lambda,\lambda'},
\]

(7.52)

and as such, the Feynman rules given by eq. (7.38) do indeed create states of definite polarization in the non-radiative amplitude and we can again choose the helicity in the operators to be \( \lambda_I \) in the non-radiative and radiative amplitudes. The non-radiative amplitude is

\[
M_{I}^V = \epsilon_{\lambda_I}(p_I) \cdot \epsilon_{\lambda_I}(p_I) = -1.
\]

(7.53)
Note that the momentum argument of the vector boson coupling is on-shell, such that \( \hat{p}_I = p_I \). The radiative amplitude evaluates to

\[
M^{V\rightarrow V'V''}_{2} \equiv \epsilon_{\lambda_i} \epsilon_{\lambda_j} \epsilon_{\lambda_k} \epsilon_{\lambda_l} = \frac{g_{VVV}}{Q^2 + i m_I \Gamma_I} \epsilon_{\lambda_i}^{\mu} \epsilon_{\lambda_j}^{\nu} \left( -g_{\mu\nu} + \frac{p_{ij}^\mu p_{ij}^\nu}{m_I^2} \right) \times Y \left( -p_{ij}, \nu, p_i, \epsilon_{\lambda_i}(p_i), p_j, \epsilon_{\lambda_j}(p_j) \right),
\]

where the Yang-Mills vertex written this way is understood to be contracted with the external polarization vectors \( \epsilon_{\lambda_i}(p_i) \) and \( \epsilon_{\lambda_j}(p_j) \). We now follow a very similar line of reasoning as for the fermion radiative amplitude. Continuing with the second line of eq. (7.54) and inserting the completeness relation eq. (7.19) leads to

\[
\frac{g_{VVV}}{Q^2 + i m_I \Gamma_I} \epsilon_{\lambda_i}^{\mu} \epsilon_{\lambda_j}^{\nu} \left( \sum_{\lambda \pm 0} \epsilon_{\lambda_i}(\hat{p}_{ij}) \epsilon_{\lambda_j}(\hat{p}_{ij}) + \frac{p_{ij}^\mu p_{ij}^\nu}{m_I^2} \right)
\]

\[
= \frac{g_{VVV}}{Q^2 + i m_I \Gamma_I} \left[ - \epsilon_{\lambda_i}^{\nu}(\hat{p}_{ij}) + \epsilon_{\lambda_i}(\hat{p}_{ij}) \cdot p_{ij} \cdot \epsilon_{\lambda_j}(\hat{p}_{ij}) \right].
\]

Recalling the polarization vectors defined in eq. (7.17), we immediately see that the second term in the second line of eq. (7.55) vanishes for transverse polarizations. For longitudinal polarizations we instead find

\[
- \frac{1}{Q^2 + i m_I \Gamma_I} \left( \epsilon_{\lambda_i}^{\nu}(\hat{p}_{ij}) + \frac{i \Gamma_I}{2 m_I^2} p_{ij}^\nu \right) - \frac{1}{2 m_I^3} p_{ij}^\nu.
\]

The second term outside the brackets would again lead to unitarity-violating contributions to the branching kernels, and must therefore cancel against other diagrams. The term linear in \( \Gamma_I \) is subdominant for Standard Model vector bosons which have \( \Gamma_I \ll m_I \). The radiative amplitude of eq. (7.54) can thus be written as

\[
M^{V\rightarrow V'V''} = -\frac{g_{VVV}}{Q^2 + i m_I \Gamma_I} \epsilon_{\lambda_i}(\hat{p}_{ij}) Y \left( -(p_i + p_j, \nu, p_i, \epsilon_{\lambda_i}(p_i), p_j, \epsilon_{\lambda_j}(p_j)) \right)
\]

\[
= -\frac{2 g_{VVV}}{Q^2 + i m_I \Gamma_I} \left( p_j \cdot \epsilon_i \epsilon_j \tilde{\epsilon}_{ij} - p_i \cdot \epsilon_j \epsilon_i \tilde{\epsilon}_{ij} + p_i \cdot \tilde{\epsilon}_{ij} \epsilon_i \epsilon_j \right),
\]

where we have dropped the momentum argument from the polarization vectors for brevity, noting that \( \epsilon_{\lambda_i}(\hat{p}_{ij}) \equiv \epsilon_{ij} \). Recalling that the non-radiative amplitude \( M^V = -1 \), the factorization in the singular limits is again clear. To compute the branching amplitude for all helicity configurations, we write out all possible products of momenta.
and polarization vectors
\[
\epsilon_\lambda(p_a) \cdot \epsilon_\lambda(p_b) = -\frac{1}{4} \frac{1}{p_a \cdot k_b} S_{-\lambda}(k_a, p_a, p_b, k_b) S_\lambda(k_b, k_a)
\]
\[
\epsilon_\lambda(p_a) \cdot \epsilon_{-\lambda}(p_b) = -\frac{1}{4} \frac{1}{p_a \cdot k_b} S_\lambda(k_a, p_a, k_b) S_{-\lambda}(k_a, p_b, k_b)
\]
\[
\epsilon_\lambda(p_a) \cdot \epsilon_0(p_b) = \frac{\lambda}{2\sqrt{2}} \frac{1}{m_b p_a \cdot k_a} \left( S_{-\lambda}(k_a, p_a, p_b, k_a) - \frac{m_b}{p_b \cdot k_b} S_{-\lambda}(k_a, p_a, k_b) \right)
\]
\[
\epsilon_0(p_a) \cdot \epsilon_0(p_b) = \frac{1}{m_a m_b} \left( p_a \cdot p_b - \frac{m_a}{p_a \cdot k_a} k_a - \frac{m_b}{p_b \cdot k_b} k_b + \frac{m_a}{p_a \cdot k_a} \frac{m_b}{p_b \cdot k_b} k_a k_b \right)
\]
\[
\epsilon_\lambda(p_a) \cdot p_b = \frac{\lambda}{\sqrt{2}} \frac{1}{p_a \cdot k_a} S_{-\lambda}(k_a, p_a, p_b, k_a)
\]
\[
\epsilon_0(p_a) \cdot p_b = \frac{1}{m_a} \left( p_a \cdot p_b - \frac{m_a^2}{p_a \cdot k_a} \right),
\]
where \(a, b \in (i, j, ij)\). The unitarity-violating terms are then removed by the substitutions
\[
p_i \cdot p_j \to \frac{1}{2} \left( m_i^2 - m_j^2 - m_j - i m_j \Gamma_I \right)
\]
\[
p_{ij} \cdot p_i \to \frac{1}{2} \left( m_i^2 + m_j^2 - m_j - i m_j \Gamma_I \right)
\]
\[
p_{ij} \cdot p_j \to \frac{1}{2} \left( m_i^2 - m_i^2 + m_j^2 - i m_j \Gamma_I \right).
\]
\[
\textbf{(7.58)}
\]

The amplitudes for the branchings \(V \to Vh\) and \(h \to VV\) are simpler than the pure vector boson case, involving only a single product of polarization vectors. They can be computed using the expressions in eq. \textbf{(7.58)}.

### 7.2.3 Higgs Splitting to Fermions

The final case to consider is a branching initiated by a Higgs boson. Since the \(h \to VV\) amplitudes can be computed using the procedure in the previous subsection, we instead consider Higgs splitting to fermions. The non-radiative amplitude is
\[
M^h_{1} = \bullet \cdots p_i, \lambda_I = 1.
\]
\[
\textbf{(7.60)}
\]
Since the Higgs is a scalar, the factorization of the radiative amplitude is trivial. It is given by
\[
M^h_{2} \to ff = \bullet \cdots \left[ p_i, \lambda_i \right] _{p_i, \lambda_i} = \frac{e}{2 s \cdot w} \frac{m_f}{m_w} \bar{u}_{\lambda_i}(p_i) v_{\lambda_j}(p_j).
\]
\[
\textbf{(7.61)}
\]
The two helicity amplitudes are found to be
\[
M^{h \to f \bar{f}}(h, \lambda, \lambda) = \frac{e m_w^2}{4 s_w m_w \sqrt{p_i \cdot k_i}} \sqrt{p_j \cdot k_j} S_{-\lambda}(k_i, p_j - p_i, k_j)
\]
\[
M^{h \to f \bar{f}}(h, \lambda, -\lambda) = \frac{e m_w^2}{4 s_w m_w \sqrt{p_i \cdot k_i}} \sqrt{p_j \cdot k_j} \left( S_{-\lambda}(k_i, p_i, p_j, k_j) - m_f^2 S_{-\lambda}(k_i, k_j) \right)
\]  
(7.62)

This concludes the calculation of the branching amplitudes. Using the information in this section, all helicity amplitudes can be obtained. A full list of branching amplitudes, including initial-state vector boson emissions is not listed here for brevity, but will be available in [174]. We implement these branching amplitudes for all possible electroweak branchings in all helicity configurations in the parton shower.

### 7.3 Singular Limits of the Branchings Amplitudes

In this section we discuss the singular limits of the branching amplitudes given by eq. (7.32) and eq. (7.35). We first list all quasi-collinear limits of all amplitudes in all helicity configurations. Next, the soft limits are discussed, and the construction of branching kernels that incorporate soft interference effects is outlined.

#### 7.3.1 Quasi-collinear Limits

In the quasi-collinear limit, the reference vectors simplify to
\[
k_i^{\text{col.}} = k_j^{\text{col.}} = k_{ij} \equiv k.
\]  
(7.63)

The branching amplitudes can be expressed in terms of the energy sharing variable $z$ by replacing
\[
p_i \to z p_{ij} \text{ and } p_j \to (1 - z) p_{ij}.
\]  
(7.64)

The only two remaining spinor products in the branching amplitudes are related by
\[
S_{-\lambda}(k, p_j, p_i, k) = -S_{-\lambda}(k, p_i, p_j, k).
\]  
(7.65)

Up to a phase factor, they are
\[
S_{-\lambda}(k, p_i, p_j, k) \propto 2 \sqrt{p_i \cdot k_i} p_j \cdot k_j \sqrt{Q^2 + m_i^2 - m_i^2 p_j \cdot k_i} - m_j^2 p_{ij} \cdot k_{ij} + m_j^2 p_{ij} \cdot k_{ij} \sqrt{z(1 - z)} \sqrt{Q^2}
\]  
(7.66)
7.3. SINGULAR LIMITS OF THE BRANCHINGS AMPLITUDES

Table 7.1: Table of branching amplitudes for vector boson emission off a fermion for all helicity configurations. For the antifermion, the interchange \((v - \lambda a) \leftrightarrow (v + \lambda a)\) is applied.

\[
\begin{array}{cccc}
\lambda_f & \lambda_i & \lambda_j & f \rightarrow f'V \times \frac{1}{Q^2 + m_f^2} \\
\lambda & \lambda & \lambda & \sqrt{2} \lambda(v - \lambda a) \sqrt{Q^2 \frac{1}{1-z}} \\
\lambda & \lambda & -\lambda & \sqrt{2} \lambda(v - \lambda a) \sqrt{Q^2 \frac{1}{1-z}} \\
-\lambda & \lambda & \lambda & \sqrt{2} \lambda \left[ m_f(v - \lambda a) \sqrt{z} - m_i(v + \lambda a) \frac{1}{\sqrt{z}} \right] \\
-\lambda & -\lambda & \lambda & 0 \\
\lambda & \lambda & 0 & (v - \lambda a) \left[ m_i^2 \sqrt{z} - m_j^2 \frac{1}{\sqrt{z}} - 2 m_j \frac{\sqrt{z}}{1-z} \right] + (v + \lambda a) \frac{m_i m_f}{m_j} \frac{1-z}{\sqrt{z}} \\
\lambda & -\lambda & 0 & \sqrt{Q^2 \frac{1}{1-z} \left[ m_i(v - \lambda a) - m_j(v + \lambda a) \right]} \\
\end{array}
\]

where

\[
\tilde{Q}^2 = Q^2 + m_f^2 - \frac{m_i^2}{1-z} - \frac{m_j^2}{z}. \tag{7.67}
\]

Tables 7.1-7.4 contain the collinear limits of all electroweak branching amplitudes. These limits are related to the Altarelli-Parisi splitting kernels by

\[
|M|^2 = \frac{1}{Q^2 + m_f^2} P(z), \tag{7.68}
\]

where \(M\) is the branching amplitude. For the sake of notation, the collinear limits are given using eq. (7.66) and are only correct up to a phase factor. Note that this phase factor is irrelevant for the calculation of the splitting kernels. It is however relevant for the calculation of branching kernels with interferences, and they are correctly accounted for in the expressions in the previous section. The splitting functions found here agree with the results of [170].

We briefly discuss some similarities and differences between the electroweak splitting functions and those normally encountered in QCD and QED. Considering the amplitudes in table 7.1, the first two splitting kernels correspond to a fermion emitting a transversely polarized vector boson and maintaining its helicity. These kernels display the standard spin-summed behaviour

\[
P_{f \rightarrow f'V} \propto \frac{1 + z^2}{1 - z}, \tag{7.69}
\]

as would be expected from the unbroken phase of the standard model where the vector bosons are massless and do not have a longitudinal polarization. Up to the coupling constant, the splitting function is identical to those for photon and gluon
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<table>
<thead>
<tr>
<th>$\lambda_I$</th>
<th>$\lambda_i$</th>
<th>$\lambda_j$</th>
<th>$V \to f\bar{f} \times \frac{1}{Q^2 + im_f V_f}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>$\lambda$</td>
<td>$-\lambda$</td>
<td>$\sqrt{2}\lambda(v - \lambda a)\sqrt{Q^2 z}$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$-\lambda$</td>
<td>$\lambda$</td>
<td>$\sqrt{2}\lambda(v + \lambda a)\sqrt{Q^2 (1 - z)}$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$\lambda$</td>
<td>$\lambda$</td>
<td>$\sqrt{2}\lambda[m_i(v + \lambda a)\sqrt{\frac{1 - z}{z}} + m_j(v - \lambda a)\sqrt{\frac{z}{1 - z}}]$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$-\lambda$</td>
<td>$-\lambda$</td>
<td>$0$</td>
</tr>
<tr>
<td>$0$</td>
<td>$\lambda$</td>
<td>$\lambda$</td>
<td>$\sqrt{Q^2} \left[ \frac{m_i}{m_f}(v + \lambda a) + \frac{m_j}{m_f}(v - \lambda a) \right]$</td>
</tr>
<tr>
<td>$0$</td>
<td>$\lambda$</td>
<td>$-\lambda$</td>
<td>$(v - \lambda a) \left[ 2m_I\sqrt{z(1 - z)} - m_i^2 m_f \sqrt{\frac{1 - z}{z}} - m_j^2 m_f \sqrt{\frac{z}{1 - z}} \right] + (v + \lambda a) \frac{m_i m_j}{m_f} \frac{1}{\sqrt{z(1 - z)}}$</td>
</tr>
</tbody>
</table>

Table 7.2: Table of branching amplitudes for vector boson splitting to fermions for all helicity configurations.

<table>
<thead>
<tr>
<th>$\lambda_I$</th>
<th>$\lambda_i$</th>
<th>$\lambda_j$</th>
<th>$V \to V'V'' \times \frac{g_{VV'V''}}{Q^2 + im_f V_f}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>$\lambda$</td>
<td>$\lambda$</td>
<td>$\sqrt{2}\lambda \sqrt{\frac{Q^2}{Z(1 - z)}}$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$-\lambda$</td>
<td>$\lambda$</td>
<td>$\sqrt{2}\lambda \sqrt{Q^2 z \sqrt{\frac{z}{1 - z}}}$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$-\lambda$</td>
<td>$-\lambda$</td>
<td>$0$</td>
</tr>
<tr>
<td>$0$</td>
<td>$\lambda$</td>
<td>$\lambda$</td>
<td>$m_f(2z - 1) + \frac{m_i^2}{m_f} - \frac{m_i^2}{m_f}$</td>
</tr>
<tr>
<td>$0$</td>
<td>$\lambda$</td>
<td>$-\lambda$</td>
<td>$m_i \left( 1 + 2\frac{1 - z}{z} \right) + \frac{m_j^2}{m_i} - \frac{m_i^2}{m_j}$</td>
</tr>
<tr>
<td>$0$</td>
<td>$\lambda$</td>
<td>$0$</td>
<td>$0$</td>
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</tr>
<tr>
<td>$\lambda$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\frac{m_i^2 + m_j^2 - m_f^2}{m_i m_j} \sqrt{Q^2} \sqrt{z(1 - z)}$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\frac{m_i^2 + m_j^2 - m_f^2}{m_i m_j} \sqrt{Q^2} \sqrt{\frac{1 - z}{z}}$</td>
</tr>
<tr>
<td>$0$</td>
<td>$\lambda$</td>
<td>$0$</td>
<td>$\frac{m_i^2 + m_j^2 - m_f^2}{m_i m_j} \sqrt{Q^2} \frac{1}{\sqrt{z}}$</td>
</tr>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$\lambda$</td>
<td>$\frac{1}{2} \frac{m_i^2}{m_f} \left( 2z - 1 \right) - \frac{m_i^2}{m_f} \left( \frac{1}{2} + \frac{1 - z}{z} \right) + \frac{m_i^2}{m_f} \left( \frac{1}{2} + \frac{z}{1 - z} \right)$</td>
</tr>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\frac{m_i m_j}{m_f} \left( 1 - \frac{z}{1 - z} \right) + \frac{m_i m_j}{m_f} \left( 2 + \frac{z}{1 - z} \right)$</td>
</tr>
</tbody>
</table>

$\lambda_I \rightarrow \lambda_i \rightarrow \lambda_j$ | $V \rightarrow V'V'' \times \frac{g_{VV'V''}}{Q^2 + im_f V_f}$ |

<table>
<thead>
<tr>
<th>$\lambda_I$</th>
<th>$\lambda_i$</th>
<th>$\lambda_j$</th>
</tr>
</thead>
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<td>$\lambda$</td>
<td>$\lambda$</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$-\lambda$</td>
<td>$\lambda$</td>
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<tr>
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<td>$-\lambda$</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$-\lambda$</td>
<td>$-\lambda$</td>
</tr>
</tbody>
</table>

Table 7.3: Table of branching amplitudes for vector boson emission off a vector boson and all Higgs-fermion branchings for all helicity configurations. Since the Higgs is blind to fermion helicity, the amplitudes are identical for the fermion and the antifermion.
7.3. SINGULAR LIMITS OF THE BRANCHINGS AMPLITUDES

Table 7.4: Table of branching amplitudes for Higgs emission off vector bosons and Higgs splitting to vector bosons for all helicity configurations.

<table>
<thead>
<tr>
<th>$\lambda_i$</th>
<th>$\lambda$</th>
<th>$V \rightarrow V h \times \frac{g_{hVV}}{Q^2+m_i\Gamma_i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>$\lambda$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$-\lambda$</td>
<td>$0$</td>
</tr>
<tr>
<td>$0$</td>
<td>$\lambda$</td>
<td>$\frac{1}{m_i\sqrt{2}} \sqrt{\tilde{Q}^2 z(1-z)}$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$0$</td>
<td>$\frac{1}{m_i\sqrt{2}} \sqrt{\tilde{Q}^2 \frac{1-z}{z}}$</td>
</tr>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$\frac{1}{2m_i^2} + \frac{1-z}{z} + z$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\lambda_i$</th>
<th>$\lambda$</th>
<th>$h \rightarrow V V \times \frac{g_{hVV}}{Q^2+m_i\Gamma_i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>$\lambda$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$-\lambda$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$0$</td>
<td>$\lambda$</td>
<td>$\frac{1}{m_i\sqrt{2}} \sqrt{\tilde{Q}^2 \frac{1-z}{z}}$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$0$</td>
<td>$\frac{1}{m_i\sqrt{2}} \sqrt{\tilde{Q}^2 \frac{1-z}{z}}$</td>
</tr>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$\frac{1}{2m_i^2} - 1 - \frac{1-z}{z} - \frac{z}{1-z}$</td>
</tr>
</tbody>
</table>

emission from a fermion. The presence of general particle masses and widths induces a shift of $1/Q^2 \rightarrow \tilde{Q}^2/(Q^4+m_i^2\Gamma_i^2)$. The fermionic mass corrections in $\tilde{Q}^2$ also appear for photon and gluon emission and reproduce the mass contributions to the eikonal factor in the soft limit. For $W$ and $Z$ emission, a vector boson mass correction is also included which would be absent in QCD or QED.

The following two amplitudes describe transverse vector boson emission with a fermionic spin flip. The nonzero amplitude is mass-suppressed with respect to the previous amplitudes, meaning it only contributes significantly for values of $Q^2 \approx m_i^2, m_I^2$. These amplitudes also exist for photon and gluon emission off massive fermions, but here the flavour-changing property of $W$-emissions means that two separate mass terms of the pre-branching and post-branching fermions appear.

The very last amplitude describes longitudinal vector boson emission with a fermionic spin flip. In the unbroken phase of the Standard Model, the equivalent process is the emission of the corresponding Goldstone boson. The scalar splitting function

$$ P_{f\rightarrow f\varphi} \propto (1-z) \tag{7.70} $$

is indeed recovered and the proportionality factor corresponds to the coupling to the Goldstone boson. The amplitude for longitudinal vector boson emission without spin flip is again mass-suppressed with respect to the spin flip case, which is the ‘natural’ mode of scalar emission. In contrast to the spin flip case, where only the scalar part of the longitudinal polarization survives, here a contribution from the vector piece proportional to the vector boson mass appears as well.

7.3.2 Soft Limits and Interference

The soft limit, as given by eq. (7.35), is only relevant for amplitudes that involve vector bosons in the post-branching configuration. Branchings that do not involve vector
bosons are not enhanced in the limit of either external line becoming soft. The soft limits of the branching amplitudes have the universal form given by eq. (7.36). In fact, the soft behaviour already appears in the collinear limits of the relevant branching amplitudes in the previous subsection, leading to the double logarithms that dominate the leading-log parton shower approximation. Soft enhancements factorize at the amplitude level, and many diagrams simultaneously become enhanced and interferences between them may be important. Since the spinor-helicity formalism allows for the calculation of branching processes at the amplitude level, it is fairly straightforward to include these kinds of interference effects. To this end, we first point out that the Feynman rules defined in eq. (7.38) can be applied multiplicatively to create states of multiple particles with definite spin. An example is

\[
\begin{align*}
\langle p_a, \lambda_a | p_b, \lambda_b | p_c, \lambda_c \rangle &= u_{-\lambda_a}(k_a) \bar{u}_{-\lambda_b}(k_b) \bar{\epsilon}_{\lambda_c}(\hat{p}_c),
\end{align*}
\]

where the Dirac-indices of the spinors are different. Branching kernels that incorporate soft interferences are most easily written down diagrammatically as

\[
B_{\text{soft}} = \left| \frac{\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3}}{\text{Diagram 4}} \right|^2 .
\]

Since the amplitudes individually adhere to the correct soft limit, this branching kernel includes all soft interference effects. All collinear limits are also included correctly, since a single diagram becomes dominant in its respective collinear limit. Due to the collinear safety of the kinematic maps, the momenta of the external particles not involved in the collinear emission remain nearly unchanged. Since they are factored from the radiative matrix element by construction, they will cancel against the same terms in the denominator, recovering the collinear branching kernels of eq. (7.37).

One further advantage of the branching kernel of eq. (7.72) is that, if there are many particles with $SU(2)$ or $U(1)$ charges in the current showering system, the evaluation of the branching kernel involves only $O(n)$ evaluations, while those used in Chapter 6 require $O(n^2)$.

### 7.4 The Electroweak Shower Implementation

In this section we describe how the branching amplitudes computed in the previous section are used to construct a parton shower. The electroweak shower is interleaved
with the Vincia QCD shower, which in turn is a plug-in to the Pythia event generator. Except for the parton shower, all aspects of the Monte Carlo event generation chain are handled by the Pythia framework. The treatment of resonance decays in Pythia deserves some separate attention, since it directly overlaps with branchings that occur in the electroweak shower, such as $t \rightarrow Wb$ or the decay of heavy vector bosons.

### 7.4.1 Interference Effects

Parton showers based on QCD typically make several approximations that involve discarding particular types of interference effects \[175\]. In this subsection, we discuss our approach to several types of interference by comparing to their QCD shower counterparts.

One important type of interference is that of soft enhancements between vector boson emission diagrams. In QCD showers in the leading-colour approximation, these interference effects can be approximately incorporated by angular ordering \[176\], or they can be accounted for automatically when the branching kernels are dipoles \[61\] \[177\], antennae functions \[62\] \[50\] \[63\] , or otherwise include soft corrections in the purely collinear Altarelli-Parisi splitting kernels \[55\]. In Chapter \[6\] we constructed an algorithm that extends the inclusion of soft interference effects to photon emissions. For the electroweak shower, the branching kernel described in subsection \[7.3.2\] can be used instead of a pairwise sum over antennae. Considering the number of available branchings in the electroweak shower, the set where the soft interferences can be applied is rather limited. First of all, soft interference is only relevant for vector boson emissions. However, under the emission of a $W$ boson, the flavour of the emitting particle changes, preventing interferences with emissions from other legs. Processes like $h \rightarrow ZZ$ do not contribute to soft interference either as they similarly change the flavour of the ‘emitting’ leg. The only branchings left are $f \rightarrow fZ/\gamma$ and $W^{\pm} \rightarrow W^{\pm}Z/\gamma$. But even in those branchings, the interferences are left to amplitudes in which the emitting particle maintains its helicity, since emissions from other legs will never induce spin flips. The remaining instances of interference have been treated in Chapter \[6\] in the case of photon radiation. The current Vincia implementation of the electroweak shower, which replaces the QED shower when enabled, does not yet incorporate soft coherence effects in $Z$ and $\gamma$ emission.

A further source of interference effects may come from particle spin. Typically, parton showers make use of spin-averaged branching kernels, neglecting any type of interference effects between spin configurations of intermediate particles. The formalism described above allows for the calculation of branchings at the amplitude level and could therefore be used to construct an electroweak parton shower that fully in-
corporates spin interference effects. In the case of QCD, several parton showers have accounted for these types of effects. They result in azimuthal correlations and effects on the connection between the shower and the hard process or particle decays. For instance, the Herwig $^{[31]}$ angular-ordered shower includes an algorithm accounting for spin interference effects described in $^{[179]}$ and the references therein. The algorithm essentially corrects missing interference effects in previous branchings by modifying its splitting kernels. On the other hand, the approach taken in $^{[180,181]}$ is more closely tied to the density matrix formalism. Due to the chiral nature of the electroweak theory, the effects of particle spin there range beyond azimuthal correlations.

Vincia currently evolves states of definite helicity, but does account for spin-related interference effects. Correspondingly, the same is true for the electroweak shower.

One further unique type of interference effect appears in the electroweak sector, namely the overlap between photons and transversely polarized $Z$-bosons or Higgs bosons and longitudinally polarized $Z$-bosons $^{[182]}$. In $^{[170]}$, a procedure involving mixed branching kernels and evolution of a density matrix is described. While this procedure is physically accurate, it is not straightforward to combine with the definite intermediate states that appear as part of the shower evolution and may lead to a computationally expensive algorithm. In this work, we instead opt for a simple approach that attempts to incorporate the most important physical effect at little computational cost. Consider the electroweak shower evolving an $f$ final state and ending up with an $ff'\bar{f}'$ configuration. The two shower histories $f \rightarrow f\gamma \rightarrow ff'\bar{f}'$ and $f \rightarrow fZ_T \rightarrow ff'\bar{f}'$ contribute to this final state separately for some particular spin configuration. For different helicities, the intermediate particles may instead be $h$ and $Z_L$. In a shower with full spin-interference effects all of these shower histories would contribute.

Without any treatment of bosonic interferences, the shower approximation to such a $1 \rightarrow 3$ branching process can be described diagramatically as

$$\left| \begin{array}{c} \bullet \\ \rightarrow \end{array} \right|^2 = \left| \begin{array}{c} \bullet \\ \rightarrow \end{array} \gamma \right|^2 + \left| \begin{array}{c} \bullet \\ \rightarrow \end{array} Z_T \right|^2 + \left| \begin{array}{c} \bullet \\ \rightarrow \end{array} \gamma \right|^2$$

where the two separate contributions correspond to the probabilities for the shower to emit either a photon or a $Z$-boson, and its subsequent splitting. To adjust for the interference contributions, we add an event weight when a neutral vector boson
splitting occurs. This event weight may be expressed diagrammatically as

\[ w_{\text{BI}} = \sum_i \left( \frac{\times \begin{array}{l} i \end{array}}{\times \begin{array}{l} \gamma \end{array}} \right)^2 + \frac{\times \begin{array}{l} i \end{array}}{\times \begin{array}{l} Z_T \end{array}} \right)^2 \]

and similar for \( h/Z_L \) interference. The index \( i \) runs over all electroweak charges that could have emitted the neutral boson. Note that if the initial fermion is massive, multiple fermionic spin states can contribute to the photon and \( Z \) emission diagrams. For example, the fermion may start out with a positive helicity and emit a neutral vector boson without flipping its spin, or it may start out with negative helicity and emit a vector boson while flipping to positive helicity. This type of interference between spin states is not included elsewhere in the shower, and as such the contributions are summed over incoherently in eq. (7.74). For similar reasons, interference between transverse and longitudinal intermediate vector bosons is not accounted for.

The weight of eq. (7.74) corrects the branching kernels of the shower, but not the higher-order corrections included in the Sudakov factor. In that sense, this reweighting procedure is not as accurate as a treatment that involves evolution of density matrices, but it is computationally much simpler and does not lead to the presence of mixed states in the event record. An inclusion of a correction factor like eq. (7.74) during the shower evolution is not straightforward because the neutral boson emission rates have to be modified prior to the splitting taking place. We also point out that eq. (7.74) has an upper bound of 2, and as such there is little danger of wildly fluctuating weights leading to inefficiencies.

### 7.4.2 Spectator Selection

In Chapter 6 a strategy was already outlined for the selection of a spectator for photon splitting. The physical consequences for previous branching kernels of finite recoil for the spectator are mitigated by selecting spectators with an appropriate probability. Here we expand on this strategy to generalize it to the electroweak theory. For the branching of particle \( i \), the probability to select a spectator \( j \) from a pool of \( N \) available ones is

\[ P_j = \frac{|M_{x\rightarrow ij}|^2}{\sum_{j=1}^N |M_{x\rightarrow ij}|^2}. \]

That is, for all \( N \) available spectators we check if the pair of particles \( i, j \) could have been created by the electroweak shower as a branching of particle \( x \). All of
the contributions in the denominator of eq. (7.75) correspond with possible shower histories that contribute to the current state. The selection of a spectator is thus more likely if the shower history where the current brancher and that spectator were created by a previous branching. To clarify this further in terms of diagrams, an example is

\[
\begin{align*}
&\begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array}^2 = \begin{array}{c}
\begin{array}{c}
\bullet
\end{array}^2 + \begin{array}{c}
\bullet
\end{array}^2
\end{array}^2 + \begin{array}{c}
\begin{array}{c}
\bullet
\end{array}^2 + \begin{array}{c}
\bullet
\end{array}^2
\end{array}^2
\end{align*}
\]

(7.76)

The spectator for the splitting of the vector boson is chosen to be either of the other external legs based on the probabilities that the vector boson was emitted by either of those legs. When the vector boson splits, it is brought off its mass shell by transferring some momentum of the spectator particle to the vector boson. Because the vector boson momentum is modified, the emission kernel that it was produced with is no longer completely correct. In the strong-ordering phase space region where \( Q^2_{\text{emit}} \gg Q^2_{\text{split}} \) relevant for the leading log contribution, this type of mismodelling is formally absent. In a parton shower strong ordering is implemented as \( Q^2_{\text{emit}} > Q^2_{\text{split}} \) and some recoil effects may appear, especially when the involved particles are as heavy as the electroweak gauge bosons. The \( 2 \rightarrow 3 \) kinematic map used by the antenna shower conserves the invariant mass of the original two-particle system, so by using the probability in eq. (7.75) the invariant mass of the system of emitter + vector boson that was most important in the emission process is most often conserved. In [170] this effect is called ‘kinematic back-reactions’ and it is accounted for as a multiplicative factor of the branching kernels. We instead choose to implement it as a spectator selection probability.

7.4.3 Ordering and Resonance Showers

The electroweak shower includes a number of branchings that would normally be associated with the decay of resonances by Pythia [30]. In the context of the electroweak shower, branchings that have

\[
m_l^2 > (m_i + m_j)^2
\]

(7.77)

may be considered as resonance branchings. The Standard Model particles that have such decay-like branchings are \( Z, W^\pm, \) Higgs and top. In Pythia, the scale of the
resonance decay is associated with the width of the resonance. This means that, for instance in case of the top, it is allowed to shower down to $\Gamma_t \approx \Lambda_{\text{QCD}}$. The top is subsequently made to decay to a bottom quark and a $W$ boson, and the disparity between the top mass and masses of the decay products leads to some new phase space to open up. The resonance system, now consisting of the decay products, is showered from its kinematic limit down to $\Lambda_{\text{QCD}}$. Note that this double shower procedure does not lead to a double coverage of the phase space, because the evolution before the resonance decay assigns positive virtualities to the top, while the showering of the resonance products is associated with negative top virtuality. Afterwards, the $W$ may decay to quarks which are then again allowed to shower down to $\Lambda_{\text{QCD}}$.

With the inclusion of an electroweak shower, the decay modes of the resonances are now also all present as shower branchings. The parton shower enables highly-energetic resonances to branch and disappear at scales much higher than their width, where they should indeed be treated as any other non-resonant particle. At scales close to the resonance width, the Breit-Wigner character of the invariant mass of the resonance decay should instead dominate the distribution. The strategy employed in Pythia is to produce all resonances with a mass sampled from a Breit-Wigner distribution. In the context of the electroweak shower, the resonances should instead always be produced on their mass shell and acquire their mass distribution by branching.

To be able to properly incorporate resonance branchings in the parton shower formalism, we first define a suitable ordering scale. Branchings with large ordering scales should correspond to particles that are very far off-shell and are very short-lived. Resonance branchings come with the feature that regions of phase space with $Q^2 < 0$ appear. These kinds of branchings should be considered to be off-shell, and the ordering scale should be large for very negative values of $Q^2$. As such, we define the ordering variable of the entire electroweak part of the parton shower to be

$$|Q^2| = |(p_i + p_j)^2 - m_I^2|. \tag{7.78}$$

In the case of a resonance branching, this ordering scale approaches the Breit-Wigner peak from both sides. For all other types of branchings, $Q^2$ is strictly positive and this ordering scale corresponds to virtuality ordering. One subtlety worth mentioning is that the parton shower phase space is restricted by $Q^2 > (m_i + m_j)^2 - m_I^2$. In the case of multiple decay chains, as in the case of $t \rightarrow W^+ b \rightarrow \bar{l} \nu b$, the top decay phase space is restricted by the bottom and $W$ masses, while those disappear from the final configuration. The parton shower treatment is therefore approximate, but for a top decay the kinematic limit is only reached at $Q^2 = \mathcal{O}(-100m_t\Gamma_t)$ making the correction tiny. It could however be alleviated formally by the introduction of $1 \rightarrow 3$ branchings.
For most types of non-resonant electroweak branchings the phase space is naturally cut off due to the masses of the post-branching momenta. In all other cases like photon emission, the electroweak shower is cut off at the same scale as the QCD shower. QED radiation below the hadronization scale is still included using the QED shower described in Section 6. For resonance branchings, at first glance the most sensible shower implementation of resonance branchings would be to use branching kernels directly with the resonance widths included. That is, the branching kernels scale like

\[
B_{\Gamma}(Q^2) \propto \frac{1}{Q^4 + m^2 \Gamma^2},
\]  

(7.79)

where we have dropped the other parton shower variables as arguments of the branching kernel. These kernels do not diverge for \(|Q^2| \to 0\), but this can be dealt with by using the Sudakov veto algorithm for nonsingular branching kernels discussed in Section 4.4.3. The positive and negative values of \(Q^2\) can be accounted for by multiplying the overestimate of the branching kernel by a factor of two and selecting a sign at random. No cutoff scale is required, meaning that the shower is able to populate the full phase space. The algorithm then produces a normalized probability distribution which guarantees that a resonance will decay. The problem with this approach is illustrated in Figure 7.2.

The black line shows the distribution generated by the parton shower. Clearly, the above prescription leads to the undesirable effect of suppressing the low-\(|Q^2|\) region where the distribution should instead be peaked. To illustrate the cause of the suppression, the blue line shows the parton shower distribution reweighted to remove the Sudakov factor. Since the Sudakov factor is the no-branching probability for a shower evolution between the starting scale and the branching scale, it can be computed numerically. For a large event sample of resonance branchings that all start from the same pre-branching state, the Sudakov factor for every event is thus the fraction of other events with lower branching scale. Since the infrared-singular behaviour of the resonance branching kernels given in eq. (7.79) is regulated by the presence of the decay width, the distribution shown in Figure 7.2 is dominated by the Sudakov suppression at low scales. The cause of this effect is a double-counting of virtual corrections. The parton shower folds virtual corrections to the resonance branching into the Sudakov form factor, but these corrections are also included in the decay width. As such, a more precise description may be achieved by matching the parton shower that uses branching kernels without widths to a pure Breit-Wigner distribution without Sudakov factor. One straightforward way to achieve this is to define a matching scale \(|Q^2_{\text{match}}|\) where the parton shower is replaced by a Breit-Wigner.
Figure 7.2: The invariant mass distribution of the shower branching $t \rightarrow W^+b$ using width-dependent branching kernels and the Sudakov veto algorithm for nonsingular branching kernels. The black line is the parton shower distribution. The blue line indicates the distribution that results from reweighting events to remove the Sudakov factor contribution, which is itself included in red.
The resulting distribution is
\[
S(Q_{\text{Start}}^2, Q^2) = \theta (|Q_{\text{Start}}^2| > |Q^2| > |Q_{\text{Match}}^2|) \left[ B_0(Q^2) \Delta (|Q_{\text{Start}}^2|, |Q^2|) + \theta (|Q_{\text{Match}}^2| > |Q^2|) \Delta (|Q_{\text{Start}}^2|, |Q_{\text{Match}}^2|) \frac{B_1(Q^2)}{N(|Q_{\text{Match}}^2|)} \right],
\]  
(7.80)

where the Sudakov factor is also defined in terms of branching kernels without decay width. Note that its arguments are absolute values of ordering scales since they concern the evolution between scales, while the branching kernels may depend on the sign of \( Q^2 \). The Sudakov factor in the second term is the no-branching probability of the parton shower between the starting scale and the matching scale. It is a constant that, together with the Breit-Wigner normalization \( N(|Q_{\text{Match}}^2|) \) ensures that the total distribution is normalized.

The distribution given by eq. (7.80) is however not necessarily continuous or smooth. One may fix the continuity constraint by finding an adequate matching scale, but this may still not lead to a smooth distribution. A more general approach is to sample the matching scale from some probability distribution \( P(|Q_{\text{Match}}^2|) \) such that the branching distribution becomes
\[
\tilde{S}(Q_{\text{Start}}^2, Q^2) = B_0(Q^2) \Delta (|Q_{\text{Start}}^2|, |Q^2|) \int_0^{|Q_{\text{Match}}^2|} d|Q_{\text{Match}}^2| P(|Q_{\text{Match}}^2|) + \int_0^{|Q_{\text{Start}}^2|} d|Q_{\text{Match}}^2| P(|Q_{\text{Match}}^2|) \Delta (|Q_{\text{Start}}^2|, |Q_{\text{Match}}^2|) \frac{B_1(Q^2)}{N(|Q_{\text{Match}}^2|)}.
\]  
(7.81)

The probability distribution \( P(|Q_{\text{Match}}^2|) \) can now be selected to yield a suitable function in the parton shower term. We make the choice
\[
\int_0^{|Q_{\text{Match}}^2|} d|Q_{\text{Match}}^2| P(|Q_{\text{Match}}^2|) = \frac{Q^4}{Q^4 + m^2 \Gamma^2},
\]  
(7.82)

which leads to
\[
P(|Q_{\text{Match}}^2|) = \frac{1}{N_P(|Q_{\text{Match}}^2|)} \frac{m^2 \Gamma^2 Q^2}{(Q^4 + m^2 \Gamma^2)^2}.
\]  
(7.83)

The choice of eq. (7.83) ensures that the probability distribution of eq. (7.81) is dominated by the parton shower contribution at high scales, while the Breit-Wigner contribution dominates at low scales.

The implementation within the parton shower framework is relatively straightforward. When a shower of a resonance is initiated, a matching scale \( |Q_{\text{Match}}^2| \) is sampled from eq. (7.83), which serves as the cutoff scale. If during showering the branching scale drops below the matching scale, a new scale is instead drawn from the Breit-Wigner distribution through rejection sampling. This scale may still be allowed to
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compete against other shower branchings, and the usual parton shower kinematic maps are used to perform the decay kinematics.

7.4.4 Overestimate Determination

One technical problem with the implementation of the parton shower is related to finding suitable overestimates, which are required both for the parton shower evolution and the direct sampling for resonance branchings. The branching kernels associated with the electroweak shower are defined in terms of spinor products, while the antenna phase space is given in terms of inner products of the post-branching momenta. The task of determining overestimates for the branching kernels is consequently not straightforward. Furthermore, due to the sheer volume of available electroweak branchings, it is desirable to automate this procedure.

Final-state branchings are only allowed to recoil against other final-state particles. The antenna phase space is given by

\[ d\Phi_{\text{ant}}^{\text{FF}} = \frac{1}{16\pi^2} m_{IK}^2 \lambda^{-\frac{1}{2}} (m_{IK}^2, m_I^2, m_K^2) ds_{ij} ds_{jk} \frac{d\varphi}{2\pi}, \]  

(7.84)

where \( I \) is the brancher and \( K \) is the spectator. All final-state electroweak branching kernels are overestimated by a parameterized function

\[ O_{\text{FF}} = c_{1}^{\text{FF}} \frac{|Q^2|}{Q^4 + m_I^2 \Gamma_I^2} + c_{2}^{\text{FF}} \frac{|Q^2|}{Q^4 + m_I^2 \Gamma_I^2} \frac{E_{IK}(E_{IK} + |\vec{p}_{IK}|)}{s_{ij} + s_{ik} + m_i^2}, \]

\[ + c_{3}^{\text{FF}} \frac{|Q^2|}{Q^4 + m_I^2 \Gamma_I^2} \frac{E_{IK}(E_{IK} + |\vec{p}_{IK}|)}{s_{ij} + s_{jk} + m_j^2} + c_{4}^{\text{FF}} \frac{m^2_I}{Q^4 + m_I^2 \Gamma_I^2}. \]  

(7.85)

In terms of the antenna phase space variables, the ordering scale is

\[ |Q^2| = |s_{ij} + m_i^2 + m_j^2 - m_{IK}^2|. \]  

(7.86)

The term multiplying \( c_{1}^{\text{FF}} \) reflects the general \( 1/Q^2 \) behaviour of the collinear limits of many branching kernels. In the presence of a width, this factor is modified to \( |Q^2|/(Q^4 + m_I^2 \Gamma_I^2) \), which is easily expressed in terms of the ordering scale eq. (7.86).

The terms multiplying \( c_{2}^{\text{FF}} \) and \( c_{3}^{\text{FF}} \) also incorporate the soft behaviour mostly associated with vector boson emission. In the center-of-mass frame of the pair \( I \) and \( K \), the energies of the post-branching momenta are given by

\[ E_{CM}^i = \frac{s_{ij} + s_{ik} + m_i^2}{2m_{IK}} \quad \text{and} \quad E_{CM}^j = \frac{s_{ij} + s_{jk} + m_j^2}{2m_{IK}}. \]  

(7.87)

By Lorentz boosting, an underestimate for these energies in the lab frame can be found. They are

\[ E_{\text{lab}}^i = \frac{E_{CM}^i E_{IK}^\text{lab} + \vec{p}_{IK} \cdot \vec{p}_{\text{lab}}}{m_{IK}} \geq E_{CM}^i E_{IK}^\text{lab} - |\vec{p}_{IK}| = E_{CM}^i \frac{m_{IK}}{E_{IK}^\text{lab} + |\vec{p}_{IK}|}. \]  

(7.88)
The second and third term of eq. (7.85) contain the ratios $E_{IK}/E_i \sim 1/z$ and $E_{IK}/E_j \sim 1/(1-z)$ expressed in the invariants that appear in the phase space using the above underestimate. In practice, these terms can lead to problematic behaviour when $E_{IK}/m_{IK} \gg 1$, corresponding to a strongly boosted brancher-recoiler pair. We therefore restrict the spectator selection to never select pairs that have a very large boost. The term multiplying $c_{\text{FF}}^p$ represents the mass corrections that may be present for massive branchers. The dependence on the post-branching masses are typically negative, and therefore do not improve the overestimate much.

Initial-state branchings are only allowed to recoil against other initial states. In this case, the antenna phase space is

$$d\Phi^{\text{ant}} = \frac{1}{16\pi^2} \frac{x_A^2 x_B^2}{x_a x_b s_{AB}} ds_{aj} ds_{bj} \frac{d\varphi}{2\pi},$$

(7.89)

where $A$ branches to $a$ and $j$, and $B$ is the recoiler. The electroweak shower currently only implements vector boson emission from fermions in the initial state, which are treated as massless by Vincia. The ordering variable is crossed into the initial state to give

$$Q^2 = s_{aj} - m_j^2.$$ 

(7.90)

An absolute value qualification is not required here since resonance type branchings do not occur in the initial state. A sufficient overestimate is

$$O^{\text{II}} = c_{\text{II}}^1 \frac{1}{Q^2} \frac{s_{ab}}{s_{AB}} + c_{\text{II}}^2 \frac{1}{Q^2} \frac{x_A s_{ab}}{x_A s_{bj}(s_{ab} - s_{bj}) + x_B s_{aj}(s_{ab} - s_{aj})}. $$

(7.91)

The factor $s_{ab}/s_{AB}$ accounts for the additional factor of $1/z$ that shows up in the Altarelli-Parisi splitting kernels when crossed to the initial state. The second term represents the $1/(1-z)$ contribution that may appear for vector boson emissions. It is constructed by making use of the fact that the vector $E_a p_b + E_b p_a$ is at rest in the lab frame, and thus

$$(p_0^0 p_b + p_0^0 p_a) \cdot p_j \propto E_j.$$ 

(7.92)

Making use of the Vincia kinematic map described in Chapter 5, the form in eq. (7.91) can be found.

The parameters $c_{\text{FF}}^p - c_{\text{FF}}^p$ and $c_{\text{I}}^1$ and $c_{\text{I}}^2$ are automatically determined for all available branchings in the electroweak shower. To do that, brancher-recoiler pairs are generated from antennae with randomly chosen invariant masses. Branchings are then generated with a distribution $|Q^2|/(Q^4 + m_I^2 \Gamma_I^2)$ for the final state or $s_{ab}/s_{AB} 1/Q^2$ for the initial state to roughly model the branching kernel behaviour. For every event $i$, the value of the branching kernel $B_i$ as well as the terms $A_{ij}$ multiplying the
parameters $c_j$ are stored. The problem of finding suitable values for the overestimate parameters can then be formulated as

$$\text{Minimize } \sum_{i=1}^{n} (Ac)_i - B_i$$

subject to $(Ac)_i \geq B_i$

and $c \geq 0$. \hfill (7.93)

The minimization condition minimizes the average difference between the branching kernel and its overestimate. The constraints ensure the overestimate is larger than the branching kernel for all samples and the parameters are positive definite. The above problem is an instance of a mathematical optimization problem known as linear programming, for which many libraries are available. We make use of the Python \cite{183} package PuLP \cite{184}.

### 7.4.5 Overview of the Shower Algorithm

We now give a short description of the full shower algorithm. Branching kernels are constructed using the formalism described in section \cite{[section]} for all possible electroweak branchings and all helicity configurations. Overestimates are found using the optimization algorithm of subsection \cite{[subsection]} where the final-state helicities are summed over. This leaves a total of 277 types of final-state branchings, of which 74 are resonance decays, and 90 types of initial-state branchings.

As the shower initializes, a recoiler is selected for all final-state particles that have $SU(2)$ or $U(1)$ charges making use of the selection probability described in subsection \cite{[subsection]}. Initial-state branchers are always paired with the other initial-state particle.

While the shower runs, electroweak branchings compete against the QCD branchings generated by Vincia. The overestimates are used to generate trial branchings which are accepted or rejected by either the usual Sudakov veto algorithm. For resonance decay branchings, the procedure outlined in subsection \cite{[subsection]} is used to match the shower to a Breit-Wigner distribution. We make use of the same kinematic maps as Vincia, and first-order running of the electroweak coupling constant is incorporated as part of the veto procedure. A difference between the electroweak shower and the QCD shower, and a definite downside of the spinor-helicity formalism is that the electroweak branching kernels are not functions of the variables that appear in the antenna phase space factorizations eq. (7.84) and eq. (7.89). This means that the kinematic mapping always has to be performed before the veto probability can be computed. Wherever applicable, the bosonic interference factor described in subsection \cite{[subsection]} is included. After accepting a branching, a helicity state is selected with
 probability

\[ P_{\lambda_1,\lambda_i,\lambda_j} = \frac{B_{\lambda_1,\lambda_i,\lambda_j}(p_I, p_i, p_j)}{\sum_{\lambda_i,\lambda_j} B_{\lambda_1,\lambda_i,\lambda_j}(p_I, p_i, p_j)}. \] (7.94)

The QCD shower and the electroweak shower run interleaved until the QCD cutoff scale is reached. Resonance decays can in principle continue below the QCD cutoff, but we currently choose to make the modification \(|Q^2| \to |Q^2| + (p^2_{\perp})_{\text{cut}}\). This way, all resonances are guaranteed to have decayed before exiting the parton level of event generation, which is what the hadronization model of Pythia currently expects.

### 7.4.6 Future Improvements

We wrap up this section by listing some potential future improvements for the electroweak shower.

**Soft Interference Effects**

As already mentioned previously, the spinor-helicity formalism allows for the calculation of branching kernels that incorporate soft interference effects. The construction of the branching kernels is straightforward and the algorithm of Chapter 6 can be used. Its implementation is thus mostly a technical matter. The determination of overestimates is no longer as straightforward as for \(1 \to 2\) kernels, although it is possible to use triangle inequalities to overestimate the branching kernel by a sum over individual branchings. It was already pointed out in Chapter 6 that the algorithm may be very inefficient, which may be exacerbated by these choices of overestimates.

**Ordering Scale and Resonance Matching**

The electroweak shower is currently restricted to exclusive use the ordering scale defined in eq. (7.78). While this choice leads to a simple formulation of the shower evolution, it does not align with the transverse momentum ordering of the Vincia QCD shower. In a future implementation, electroweak emission-type branchings should similarly be ordered in transverse momentum. On the other hand, the Vincia shower already allows gluon splittings to be ordered in virtuality or transverse momentum [50, 63], and a similar choice may be suitable for the electroweak shower. It may however not be straightforward to use a different scale for resonance decay-type branchings because it would necessarily be asymmetric in its approach to the Breit-Wigner peak from the positive and negative sides in \(Q^2\).

Furthermore, the matching strategy to the Breit-Wigner distribution may be improved further. While the sampled matching strategy outlined in section 7.4.3 leads
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to a smooth transition to the Breit-Wigner shape, the choice of the sampling proba-

bility distribution has an impact on that shape. The investigation of this impact and

an improved matching to decay matrix elements instead of branching kernels will be

the subject of future research.

The CKM Matrix

The CKM quark-mixing matrix is an important part of the electroweak theory. In

the context of the parton shower, it would contribute by allowing branchings like
d \rightarrow cW^- or t \rightarrow sW^+. Because of the weak universality condition

\[ \sum_i |V_{ik}|^2 = \sum_k |V_{ik}|^2 = 1 \]  

the squared matrix elements |V_{ik}|^2 can be interpreted as probabilities multiplying the

W-emission branching kernels. The implementation is therefore again just a technical

issue. The impact of the CKM matrix is not expected to be large since the off-diagonal

terms of the third-generation row and column are close to zero. The other off-diagonal

terms are not as small, but they mix quarks that are treated as massless in Vincia

anyway.

Bloch-Nordsieck Violations

One topic worth discussing are the appearances of so-called Bloch-Nordsieck viola-
tions. As explained in Chapter 3, the parton shower formalism is fundamentally

based on the principle of unitarity and the cancellation of infrared divergences be-
tween real and virtual corrections. Since the electroweak vector bosons are massive,
divergences associated with their emission are mass-regulated. The flavour-changing

nature of W-boson turns out to spoil the exact cancellation of the infrared diver-
gences and some mass-regulated logarithms may be left-over. This phenomenon is
called Bloch-Nordsieck violation [185, 154], and it has was already pointed out in the
first electroweak shower implementation in Pythia in [166].

Bloch-Nordsieck violations appear as a result of W-radiation from the initial state.
One important requirement for the cancellation of the infrared divergences via the
KLN theorem [40, 41] is to sum over the associated gauge group indices. This sum-
mation is in fact required for QCD due to colour confinement, but in the case of the
broken electroweak theory the components of the SU(2) doublets are very distinct
particles. In particular, violations appear in radiation from the initial state because
radiation of a W changes the flavour of the incoming quark, while a W-loop does not.
Because the PDFs for up-type and down-type quarks are not the same, this leads
to a mismatch between real and virtual corrections which manifests itself in $W$-mass regulated logarithms.

There is no straightforward method to incorporate such corrections in a parton shower formalism. Since these violations explicitly break unitarity, the only avenue appears to be to apply these effects as event weights. Fortunately, Bloch-Nordsieck violations are not particularly significant at the LHC [154, 159]. They only appear for $W$-radiation from the initial state, and only when both initial-state particles are quarks with the correct $SU(2)$ charge to emit a $W$ boson.

**Bosons in the Initial State**

Hard processes initiated by vector bosons have been considered for a long time [186, 187]. PDF sets with QED corrections have been available for some time [188, 189, 190], and recent progress was made towards PDFs with complete electroweak corrections [191, 192, 193]. The current shower implementation only allows for the emission of vector bosons from the initial state. The calculation of the other required initial-state branching kernels is in principle as straightforward as the calculation of those available already, but an implementation in the Pythia framework is likely not simple.

**Spin Projection**

In section 7.1 it was pointed out that helicities and polarizations of massive particles are not Lorentz invariant, and that the evolution of the parton shower may induce Lorentz boosts that can shift helicities. These projections are not required at the nominal accuracy of the parton shower because no boost is induced for strictly soft and quasi-collinear emissions. Outside those limits, one may indeed view the initial-initial kinematic map as giving the initial state a transverse momentum, which then has to be corrected for by boosting back to the lab frame. On the other hand, the map may be viewed as a global recoil against the rest of the final state, skipping the boost altogether. From the perspective of the Lorentz-invariant phase space factorization, these maps are equivalent. In the global recoil case the momenta of the recoilers are still shifted, but it is now more clear that due to the strong-ordering condition the effects on the helicity configuration should be relatively small.
7.5 Results

This section presents the result of the application of the electroweak shower in a variety of situations. The electroweak shower is implemented in the Vincia parton shower.

7.5.1 Branching Spectra

To get a general sense for the branching rates predicted by the electroweak shower, we consider emission spectra for several highly energetic particles that have an electroweak charge. Figures 7.3, 7.4, 7.5 and 7.6 show the invariant mass distribution for the branching of a left-handed $\tau$ and top, a transverse $W^+$ boson and a Higgs as a consequence of their first branching. All particles are produced at an energy of 1 TeV together with a recoiler that is uncharged under electroweak interactions. For photon emissions, a cutoff around $\Lambda_{QCD}$ is imposed. All other branchings are automatically regulated by the particle masses. For all resonance decay branchings, the sampled matching procedure outlined in section 7.4.3 is used.

Figure 7.3 shows the branching spectrum of a negative-helicity $\tau$. The two dominant photon production channels are those where the $\tau$ helicity is conserved. The mass-suppressed spin-flip mode only contributes at very small invariant masses, as is to be expected from the branching kernel behaviour of $m_\tau^2/Q^4$. The other spin-flip mode is highly suppressed in the collinear limit as is indicated in Table 7.1. For the
emission of other vector bosons, the spin-flip contributions do not become sufficiently enhanced to show up before the kinematic limit is reached. The longitudinal vector boson emission channels have a characteristic form which looks very similar for the $W_0^-$ and the $Z_0$ channels, and which becomes comparable to the transverse channels at scales close to the kinematic limit.

Figure 7.4 shows the branching spectrum of a negative-helicity top. The left graph displays the resonance branchings as generated by the sampled matching procedure outlined in subsection 7.4.3. The right and bottom graphs show all other branchings that are not of the resonance decay type. Spin-flip modes now show up for $t \rightarrow bW^+$, $t \rightarrow tZ$ and $t \rightarrow t\gamma$ due to the large top mass, and they show the expected $m_t^2/Q^4$ scaling with the emission scale. The ‘natural’ mode of spin-flip Higgs emission is relatively flat compared with the fermion mass scaling mode of Higgs emission without spin flip.

Figure 7.5 shows the branching spectrum of a transverse $W^+$. Resonance peaks only appear for decays to negative-helicity states due to their small masses. The branchings $W^+_t \rightarrow tb$ with a spin-flipped top do occur on the other hand. The $W^+_t \rightarrow W^+Z$ and $W^+_t \rightarrow W^+\gamma$ channels are dominated by the all-positive helicity configuration because of its $1/z(1-z)$ scaling in the collinear limit as can be seen in Table 7.3. The modes to opposite transverse helicities are almost identical for the $W^+Z$ channels due to symmetry in the collinear limit and almost identical mass, but they are widely different for low scales in the $W^+\gamma$ channels. This is caused by the $z^3/(1-z)$ and $(1-z)^3/z$ scaling of the collinear limits, where the photon can attain a very small collinear momentum fraction while that of the $W^+$ is constrained by its mass. The single-longitudinal channels in $W^+Z$ are also almost identical for very similar reasons. The $W_0^+Z_0$ is a mode that is related to the Goldstone bosonic part of the $W^+$ and $Z$, and it can be seen to be very similar to the $W_0^+h$ channel. On the other hand, the $W_+^+h$ mode differs significantly from the $W_0^+Z_0$ channel because it is dominated by the vectorial part of the longitudinal polarization.

Figure 7.6 shows the branching spectrum of a Higgs. The only significant resonance decay channels are $b_+\bar{b}_+$ and $\tau_+\bar{\tau}_+$ as may be expected due to the coupling to the fermion mass and the Higgs spin zero nature. On the other hand, the mass-suppressed $t_+\bar{t}_\pm$ channel is comparable with the natural $t_+\bar{t}_\mp$ channel. All channels to $W^+W^-$ and $ZZ$ are almost identical since their branching kernels only differ in the gauge boson mass and a factor of $1/c_w$ in the coupling. Also included is the $h \rightarrow hh$ cubic Higgs coupling which is proportional to the Higgs mass $m_h$, or equivalently the Higgs self-coupling $\lambda$. This is the only branching where it makes an appearance, and it can be seen to provide a significant contribution to the total branching rate.
Figure 7.4: Branching spectra of a 1 TeV top quark to $bW^+$ (left), $tZ$ (right) and $t\gamma/h$ (bottom).
Figure 7.5: Branching spectra of a 1 TeV $W_+^+$ to fermions (left), $W^+Z$ (right) and $W^+\gamma/h$ (bottom).
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h → fermions

h → VV/hh

Figure 7.6: Branching spectra of a 1 TeV Higgs boson to fermions (left) and VV/hh (right).

7.5.2 Bosonic Interference

We now consider the effect of the application of the bosonic interference factor described in section 7.4.1. Figure 7.7 shows rates for the shower histories $e_\to e_\gamma/Z_T \to e_\gamma X$ and $e_\to e_\gamma/Z_T \to e_\gamma X$ using a similar setup as in the previous subsection, but starting from a 10 TeV source electron. Multiple interesting features appear when the bosonic interference weight eq. (7.74) is included. The most striking difference occurs for the $W^+W^-$ channel, where the bosonic interference causes an increase in case of the $e_\to e_\gamma$, but a major decrease in case of the $e_\to e_\gamma$. This may be understood by considering the structure of the interfering branching amplitudes. Factoring out coupling constants and other kinematic components, the interference is proportional to

$$
\frac{1}{M_{WW}^2} + \frac{c_w}{s_w} \frac{1}{4s_w c_w} (1 - 4s_w^2 - \lambda_e) \frac{1}{M_{WW}^2 - m_z^2 + i m_z \Gamma_z},
$$

(7.96)

where the factor $c_w/s_w$ comes from the ZWW-coupling and $\lambda_e$ is the electron helicity. The second term in brackets interferes destructively with the photon contribution for sufficiently large values of $M_{WW}$, and the remaining terms in the $Z$ contribution cancel for $\lambda = 1$.

Another interesting effect of the interferences is the modification of the charged lepton rates. Depending on the electron helicity, one of the leptonic decay modes $l_\to l_\gamma$ and $l_\to l_\gamma$ becomes suppressed at high values of the ordering scale. The rate then drops
Figure 7.7: Differential rates of the shower histories $e_- \rightarrow e_- \gamma/Z_T \rightarrow e_- X$ (top) and $e_+ \rightarrow e_+ \gamma/Z_T \rightarrow e_+ X$ (bottom) without (left) and with (right) the bosonic interference correction. The showers are initiated from a 10 TeV electron with a neutral recoiler.
even further down right before reaching the Z mass, while it is enhanced afterwards. On the other hand, the other helicity channel is enhanced at high scales, but dips right after the Z peak. Noting that \(v_e \propto (1 - 4 s_w^2) \approx 0\), the interference structure can in this case be written as

\[
\frac{1}{M_{ll}^2} - \frac{1}{4 s_w c_w} \frac{\lambda_Z \lambda_l}{M_{ll}^2 - m_z^2 + i m_z \Gamma_z}
= \left( 1 - \frac{\lambda_Z \lambda_l}{4 s_w c_w} \right) \frac{1}{M_{ll}^2} \frac{1}{M_{ll}^2 - m_z^2 + i m_z \Gamma_z}.
\]

where \(\lambda_Z\) is the transverse helicity of the Z boson and \(\lambda_l\) is the helicity of the lepton. In case the second term in the inner brackets is positive, the distribution dips at \(M_{ll}^2 < m_z^2\), while for negative values it dips at \(M_{ll}^2 > m_z^2\). The constant \(1/4 s_w c_w \approx 1.68 \approx 1\) causes the dip to occur quite close to the Z mass. We point out that the rates close to the Z peak may be significantly affected by the simplified and preliminary method of matching to resonance decays as described in section 7.4.3, and will be improved upon in future work.

### 7.5.3 Resonance Matching

We consider the consequences of the resonance matching strategy outlined in eq. (7.81) and compare it with matching at a fixed scale. A transversely polarized Z boson with an energy of 1 TeV is showered from its kinematic limit together with a recoiler that is uncharged under the electroweak theory. For the sake of comparing the resonance branchings only, the non-resonant shower branchings are disabled.

The left-hand side of Figure 7.8 shows the invariant mass spectrum of the Z boson decay products as a result of running only the electroweak shower with vanishing decay width in the branching kernel propagator and of sampling from the normalized branching kernel with nonvanishing decay width. The right-hand side shows the result of the smooth matching procedure following the distribution given in eq. (7.81) as compared with matching at scales \(n \Gamma_Z\) according to eq (7.80). Note that towards the tails of the resonance peak the smooth matching strategy matches up well with the pure parton shower and the fixed matching scale results, save for a difference in normalization. In the peak region the distributions smoothly transitions into a Breit-Wigner-like shape, while the fixed-scale matching distributions show the expected discontinuities at the transition point. We note that the value of \(n\) may be selected numerically to ensure the distributions match up precisely, but the determination of \(n\) must ultimately be dependent on the parton shower starting scale since it appears in the Sudakov factor. On the other hand, this dependence is folded into the smooth
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Figure 7.8: Invariant mass spectra of the resonance decay of a 1 TeV transversely polarized $Z$ boson with a neutral recoiler. The pure spectra (left) correspond to sampling from only the shower and only the branching kernel including a decay width. The matched spectra (right) correspond to the smooth matching strategy given by eq. (7.81) and the fixed matching strategy given by eq. (7.80) at several values of the matching scale.
Figure 7.9: Electroweak shower approximation of electroweak virtual corrections to exclusive dijet production and exclusive $W^+ + \text{jet}$ production at center-of-mass energy $\sqrt{s} = 14$ TeV as a function of the transverse momentum of the hard scattering process $p_{\text{hard}}^\perp$. The solid line corresponds to the parton shower prediction without the QCD shower, while the dashed line shows the effect of interleaving with QCD radiation.

matching strategy by virtue of the stochastic sampling procedure. While the smooth matching procedure achieves its goal, a better theoretical understanding of the interplay between Sudakov resummation and particles with finite decay width will be required for future improvement.

### 7.5.4 Electroweak Corrections to Proton Collision Processes

We finally consider the parton shower predictions of electroweak corrections to some common proton collision processes at LHC energies. Since the weak vector bosons produced by the electroweak shower at high energies are massive and thus observable, they may provide a rich environment for phenomenological studies including kinematic effects on the hard scattering, jet substructure due to vector boson decay inside jet cones and external high-energy jet and lepton production.

With the goal of examining the general size of the effects of the electroweak shower in common LHC processes, we generate dijet and $W^+ + \text{jet}$ events at $\sqrt{s} = 14$ TeV using the default tune of Pythia 8.2 and the NNPDF2.3 sets. Figure 7.9 shows the approximate electroweak virtual corrections as predicted by the Pythia electroweak shower, which only incorporates vector boson emission from fermions, and the Vincia
electroweak shower. The virtual corrections may be estimated by counting the events that contain at least one weak vector boson emission. The probability for the shower to produce no additional weak bosons is given by the Sudakov factor

$$\Delta_{\text{EW}} = 1 - \mathcal{O}(\alpha),$$

and thus the $\mathcal{O}(\alpha)$ corrections are given by the probability for at least one weak boson emission. Virtual corrections to these processes were calculated in for example [158][159] for exclusive dijet production and in [161][162] for vector boson production. For dijet production, the results of the showers are similar and are comparable in size to analytic results found in [158][159]. In the case of $W^+$ plus jet production the substantial difference between the showers is caused by the absence of the Yang-Mills vector boson coupling in the Pythia shower. However, the Vincia shower still underestimates the analytical results significantly. In [170] a similar calculation was performed using a purely final-state electroweak shower. The effects of initial-state radiation were approximated by generating $pp \rightarrow W^+q$ and squaring the Sudakov factor associated with the quark. We however find that the contribution to the weak boson emission rates of the initial-state quarks is much smaller than that of the final state. At large $x$ and high scales the PDFs are predominantly quark-like and the hard scattering is dominated by $qq' \rightarrow W^+g$. Furthermore, the initial-state phase space at large $x$ is small. These effects lead to the decrease in the estimated virtual correction at large $p_{\text{Hard}}^\perp$.

Also shown in Figure [7.9] are the results of interleaving the electroweak shower with the QCD showers of Pythia and Vincia. In the strongly ordered limit shower branchings are unaffected by subsequent branchings, but subleading effects due to the kinematics and the creation of weakly charged quarks still lead to minor differences.

Figure [7.10] shows the average number weak boson emissions of the showers. In $W^+$ plus jet production the first purely vector boson branching is always $W^+ \rightarrow W^+Z$ explaining the large increase in the $Z$ boson emission rates. Similarly, Pythia’s $W^+$ rate is small since the final-state quark is always down-type. The increase in the Vincia shower is thus caused entirely by secondary emissions from prior weak vector boson emissions.
Figure 7.10: Average number of weak boson emissions in exclusive dijet production and exclusive $W^+ +$ jet production at center-of-mass energy $\sqrt{s} = 14$ TeV as a function of the transverse momentum of the hard scattering process $p_{\text{Hard}}^\perp$. 
Chapter 8

Summary and Conclusions

While the LHC was constructed to be a discovery machine, definitive signs of new physics have yet to be detected. With the upcoming luminosity increase and the vast amount of data already available, the requirements on the theoretical modelling of Standard Model background processes are steadily increasing. Theoretical predictions are predominantly modelled by Monte Carlo programs, providing simulated collision events that may be compared with experimental results directly. An important component of these programs are parton showers, which are responsible for the modelling of radiative corrections that produce the majority of the particles in proton collision events. Among the increased theoretical demands is the realization that electroweak effects in parton showers, while significantly smaller than their QCD counterparts, may in some cases be important. This thesis is therefore dedicated to the systematic inclusion of electroweak effects in the Vincia parton shower.

After having summarized the theoretical background of the Standard Model in chapter 2 and the process of event generation in chapter 3, a mathematical formalism was set up that may be used to analyse the Sudakov veto algorithm in chapter 4. This method was then used to consider many different varieties of the algorithm that are commonly used in parton shower programs. In particular, it was shown that several strategies exist to handle the competition between radiation channels and that these algorithms may be combined. Furthermore, the algorithms were tested for their speed using a simplified parton shower model.

The foundations of the Vincia parton shower were detailed in chapter 5. Its main features are branching kernels based on antenna factorization and exact phase space factorization. We put particular emphasis on the development of the inclusion of global recoil effects in initial-final antennae, which was achieved by defining a probabilistic choice between two dipole-style maps, and was tested in Drell-Yan production.

Chapter 6 was dedicated to the implementation of a QED shower in Vincia. The
main novel development was a treatment of photon emissions that is fully coherent, capturing the entire soft coherent structure associated with emissions from multiple charged particles. An alternative, more efficient but less precise algorithm was also presented. The shower implements all forms of QED radiation, including photon splitting, conversion and photon emission from fermions and $W$ bosons. The impact of the coherent algorithm was tested by comparing different shower algorithms for Drell-Yan production close to the $Z$ peak where QED effects are known to be important, and the coherent algorithm was found to yield significant differences in certain regions of phase space. Since the treatment of interference effects of the coherent algorithm is quite universal, it may be expected to be useful in other situations such as the modelling of subleading colour effects in QCD showers where complicated interference structures appear.

In chapter 7 the implementation of a full-fledged electroweak shower in Vincia was described. The effects of weak boson radiation are known to become significant already at LHC energies, in particular with the upcoming luminosity upgrade, and will be even more relevant at future colliders. One of the major challenges of the construction of such a shower is the calculation of the relevant branching kernels, which was done using the spinor-helicity formalism. Compared with QCD, the electroweak theory involves many theoretical subtleties that have to be handled carefully. One major issue is the chiral nature of the electroweak theory, which forces the shower to be helicity-dependent and leads to a large number of possible types of branchings. Longitudinal polarizations appear for the massive weak gauge bosons, which lead to gauge-dependent unitarity violating terms in the calculation of branching kernels that have to be removed manually. The electroweak shower also includes many branchings that would usually be considered to be decays of resonances, in which case the distribution follows a Breit-Wigner peak. A strategy to match the parton shower to a resonance decay was proposed, but this may likely be improved upon by a better understanding of the interplay between the virtual corrections contained in the Sudakov factor and the decay width. Further electroweak effects added to the shower include a recoiler selection procedure that compensated for recoiler effects of previous branchings and treatment of bosonic interference effects.

Some of the features of the electroweak shower were tested by generating branching spectra of several high-energy weakly charged Standard Model particles. The results show that electroweak branching rates are significant and uncover the complex helicity-dependent structure of the electroweak theory. The impact of the inclusion of bosonic interference effects is revealed by significant changes in the rates of secondary branchings. Finally, the electroweak shower was applied to dijet and $W^+\text{ plus jet}$
production to estimate the size of electroweak virtual corrections, which are found to be significant, and to investigate the average emission rates.

While the electroweak shower is already comprehensive in its inclusion of the large number of electroweak branchings and other electroweak effects, many further improvements are still possible. The coherent emission algorithm discussed in chapter 6 can be applied in the electroweak shower for photon and $Z$ emissions. The CP violating nature of the electroweak theory encoded in the CKM matrix may be incorporated as additional flavor-changing weak boson emissions. Another significant theoretical issue that is likely difficult to account for are Bloch-Nordsieck violations caused by differing PDFs for quarks that are part of weak doublets. In general, future versions of the Vincia implementation of the electroweak shower should more closely follow the choices of ordering scales and branching kernels of the Vincia shower to facilitate interleaving with its QCD shower.
Appendix A

Antenna Phase Space Factorization Proofs

In this appendix, we give detailed proofs of the phase space factorizations used in Vincia.

A.1 Final-Final

Note first that the two-body phase space measure can be written as

$$d\Phi_2(P \to p_I, p_K) = (2\pi)^{-2} d^4p_I d^4p_K \delta(p_I^2 - m_I^2) \delta(p_K^2 - m_K^2) \delta^4(P - p_I - p_K)$$

$$= \frac{1}{32m_I^2 m_K^2 \lambda_{1/2}^2(m_{IK}^2, m_I^2, m_K^2)} d^4\Omega_2,$$  \hspace{1cm} (A.1)

where $\lambda_{1/2}^2(m_{IK}^2, m_I^2, m_K^2)$ is the Källén function. We start from the $(n+1)$-body phase space

$$d\Phi_{n+1} = (2\pi)^{1-3n} \prod_{a=1}^{n+1} d^4p_a \delta(p_a^2 - m_a^2) \delta^4(P - \sum_{a=1}^{n+1} p_a).$$  \hspace{1cm} (A.2)

We factorize $p_i, p_j$ and $p_k$, denoting $Q = P - \sum_{a \neq i,j,k} p_a$.

$$d\Phi_{n+1} = (2\pi)^{1-3n} \prod_{a \neq i,j,k} d^4p_a \delta(p_a^2 - m_a^2) d^4p_i d^4p_j d^4p_k$$

$$\times \delta(p_i^2 - m_i^2) \delta(p_j^2 - m_j^2) \delta(p_k^2 - m_k^2) \delta^4(Q - p_i - p_j - p_k).$$  \hspace{1cm} (A.3)

We now continue with the factorized three-body piece which can be rewritten as

$$d\Phi_3 = \frac{d^3p_i d^3p_k}{2E_i 2E_k} \delta((Q - p_i - p_k)^2 - m_j^2)$$

$$= \frac{1}{4} |\vec{p}_i||\vec{p}_k| dE_i dE_k d(\cos \theta) d\varphi d\Omega_2 \delta((Q - p_i - p_k)^2 - m_j^2),$$  \hspace{1cm} (A.4)
where $\theta$ is the angle between $\vec{p}_i$ and $\vec{p}_k$. This form is no longer Lorentz-invariant, so from here we specify to work in the center-of-mass frame of the factorized momenta. We now first introduce the invariant for $p_i$ and $p_k$.

\[
d\Phi_3 = \frac{1}{4} |\vec{p}_i||\vec{p}_k| dE_i dE_k d(\cos \theta) d\varphi d\Omega_2 \delta((Q - p_i - p_k)^2 - m_j^2) \\
\times ds_{ik} \delta (s_{ik} - 2E_i E_k + 2|\vec{p}_i||\vec{p}_k| \cos \theta) \\
= \frac{1}{8} dE_i dE_k ds_{ik} d\varphi d\Omega_2 \delta((Q - p_i - p_k)^2 - m_j^2) \theta \left(-1 < \frac{2E_i E_k - s_{ik}}{2|\vec{p}_i||\vec{p}_k|} < 1\right).
\]

(A.5)

We replace the condition $-1 < \cos(\theta) < 1$ by the equivalent statement $\cos^2(\theta) < 1$ and proceed by adding the other two invariants, yielding

\[
d\Phi_3 = \frac{1}{8} dE_i dE_k ds_{ik} d\varphi d\Omega_2 \delta((Q - p_i - p_k)^2 - m_j^2) \theta \left(\frac{(2E_i E_k - s_{ik})^2}{2|\vec{p}_i||\vec{p}_k|} < 1\right) \\
\times ds_{ij} \delta (s_{ij} - 2E_i m_{ijk} + 2m_i^2 + s_{ik}) \\
\times ds_{jk} \delta (s_{jk} - 2E_k m_{ijk} + 2m_k^2 + s_{ik}),
\]

(A.6)

where we have used $Q = (m_{ijk}, 0, 0, 0)$ to find expressions for the particle energies in terms of the invariants. Due to the positivity of $E_i$ and $E_k$, the invariants are all positive. Integrating out $E_i$ and $E_k$, the expression reduces to

\[
d\Phi_3 = \frac{1}{32m_{ijk}^2} ds_{ik} ds_{ij} ds_{jk} d\varphi d\Omega_2 \delta(m_{ijk}^2 - s_{ij} - s_{jk} - s_{ik} - m_i^2 - m_j^2 - m_k^2) \theta (G_{ijk} > 0),
\]

(A.7)

where $G_{ijk}$ is the three-body Gram determinant defined in eq. (5.11). We now recognize the two-particle phase space of eq. (A.1) and insert it, leading to

\[
d\Phi_3 = \frac{1}{4} \lambda^{-\frac{1}{2}}(m_i^2 K, m_i^2, m_K^2) d^4 p_i d^4 p_K \delta(p_i^2 - m_i^2) \delta(p_K^2 - m_K^2) \delta^4(Q - p_i - p_K) \\
\times ds_{ij} ds_{jk} ds_{ik} d\varphi \delta(m_{ijk}^2 - s_{ij} - s_{jk} - s_{ik} - m_i^2 - m_j^2 - m_k^2) \theta (G_{ijk} > 0).
\]

(A.8)

We can now substitute this back into eq. (A.2) to recover the antenna phase space factorization

\[
d\Phi_{n+1} = d\Phi_n \frac{1}{16\pi^2} \lambda^{-\frac{1}{2}}(m_i^2 K, m_i^2, m_K^2) ds_{ij} ds_{jk} \frac{d\varphi}{2\pi}.
\]

(A.9)
A.2 Initial-Initial

We start from the post-branching phase space

$$d\Phi = \frac{dx_a}{x_a} \frac{dx_b}{x_b} d\Phi_2(p_a + p_b \rightarrow p_r + p_j)$$

$$= \frac{1}{(2\pi)^2} \frac{dx_a}{x_a} \frac{dx_b}{x_b} d^4p_r d^4p_j \delta(p_r^2 - m_R^2) \delta(p_j^2 - m_j^2) \delta^4(x_a p_+ + x_b p_- - p_r - p_j)$$

$$= \frac{1}{4\pi} \frac{dx_a}{x_a} \frac{dx_b}{x_b} \left| \vec{p}_j \right| dE_j d(\cos \theta) \frac{d\phi}{2\pi} \delta \left( (x_a p_+ + x_b p_-)^2 - m_R^2 \right), \quad (A.10)$$

where $p_+ = (1, 0, 0, 1)$, $p_- = (1, 0, 0, -1)$ and $\theta$ is the angle between $p_j$ and the positive $z$-axis. Since the mass of the recoiler system should be left invariant by the kinematic map, we have $p_r^2 = m_R^2$. We first transform variables from $E_j$ and $\cos \theta$ to

$$s_{aj} = 2p_a \cdot p_j = x_a \sqrt{s(E_j - |\vec{p}_j| \cos \theta)} \quad \text{and} \quad s_{bj} = 2p_b \cdot p_j = x_b \sqrt{s(E_j + |\vec{p}_j| \cos \theta)} \quad (A.11)$$

which has

$$|J(E_j, \cos \theta \rightarrow s_{aj}, s_{bj})| = \frac{1}{2x_a x_b |\vec{p}_j|}. \quad (A.12)$$

This results in

$$d\Phi = \frac{1}{8\pi s} \frac{dx_a}{x_a^2} \frac{dx_b}{x_b^2} ds_{aj} ds_{bj} \frac{d\phi}{2\pi} \delta(x_a x_b s - s_{aj} - s_{bj} - m_r^2). \quad (A.13)$$

The pre-branching momentum fractions are given by the kinematic map of eq. (5.34). Written in terms of the variables that appear in eq. (A.13), they are

$$x_A = x_a \sqrt{1 - \frac{s_{aj} + s_{bj} - m_j^2}{x_a x_b s}} \left( \frac{x_a x_b s - s_{bj}}{x_a x_b s - s_{aj}} \right)$$

$$x_B = x_b \sqrt{1 - \frac{s_{aj} + s_{bj} - m_j^2}{x_a x_b s}} \left( \frac{x_a x_b s - s_{aj}}{x_a x_b s - s_{bj}} \right), \quad (A.14)$$

which has

$$|J(x_a, x_b \rightarrow x_A, x_B)| = 1. \quad (A.15)$$

Changing variables and adding the definition of $p_R$ leads to

$$d\Phi = \frac{1}{8\pi s} \frac{dx_A}{x_a^2} \frac{dx_B}{x_b^2} \int ds_{aj} ds_{bj} \frac{d\phi}{2\pi}$$

$$\times d^4p_R \delta^4(p_R - x_A p_+ - x_B p_-) \delta(p_R^2 - m_r^2). \quad (A.16)$$

Note that the square roots in eq. (A.14) are less than unity, so the integration over $x_A$ and $x_B$ is restricted by $x_a$ and $x_b$. We now define $s_{ab} = 2p_a p_b = x_a x_b s$ and $s_{AB} = 2p_A p_B = x_A x_B s$ and substitute the pre-branching phase space

$$d\Phi_1(p_A + p_B \rightarrow p_R) = 2\pi d^4p_R \delta(p_R^2 - m^2) \delta^4(p_A + p_B - p_R) \quad (A.17)$$
to find
\[ d\Phi = \frac{1}{16\pi^2} \frac{dx_A}{x_A} \frac{dx_B}{x_B} d\Phi_1 (p_A + p_B \rightarrow p_R) ds_{a} ds_{b} \frac{d\varphi}{2\pi} \frac{s_{AB}}{s_{ab}^2}. \]  

(A.18)

### A.3 Initial-Final

The derivation of the phase space factorization for initial-final antennae is much more involved than the previous two types. The derivation here is given for massless momenta, but the result does not change for massive ones [55]. We break up the derivation into multiple steps.

#### Lorentz Transform

We first derive the Lorentz transform required for the global map to realign the initial state with the beam axis. The transform is defined by its action on the set of vectors \( p_a, p_B, p_1 \) and \( p_2 \) which span the entire four-vector space. Here, \( p_1 \) and \( p_2 \) define the transverse directions to the beam axis. They are given by
\[
p_1 = (0, 1, 0, 0), \quad p_2 = (0, 0, 1, 0),
\]
which have \( p_1^2 = p_2^2 = -1 \). The action of the Lorentz transform on \( p_B \) and \( p_a \) should be
\[
p_B' = p_B, \quad p_a' = \frac{p_a \cdot p_B}{p_a \cdot p_B} p_a.
\]
We now derive the transformed transverse directions \( p'_1 \) and \( p'_2 \). First, define
\[
n' = (p_a \cdot p_1)p_2' - (p_a \cdot p_2)p_1',
\]
which has the property that \( n' = n \) since the Lorentz transform is restricted to the \( p_a - p_B \) plane and \( n \cdot p_a = n \cdot p_B = 0 \). The transformation of \( p_1 \) and \( p_2 \) can now be derived by parameterizing
\[
p_1' = c_1 p_1 + c_2 p_2 + c_a p_a + c_B p_B \quad \text{and} \quad p_2' = d_1 p_1 + d_2 p_2 + d_a p_a + d_B p_B
\]
and making use of the set of invariants
\[
p_1' \cdot p_B = p_2' \cdot p_B = p_1' \cdot p_2' = 0 \quad \text{and} \quad p_1'^2 = p_2'^2 = -1
\]
and the equation
\[
(p_a \cdot p_1)p_2' - (p_a \cdot p_2)p_1' = (p_a \cdot p_1)p_2' - (p_a \cdot p_2)p_1'.
\]

(A.23)
The only physically acceptable solution is
\[ p'_1^\mu = p_1^\mu + \frac{p_0^1}{p_0^B} p_B^\mu \quad \text{and} \quad p'_2^\mu = p_2^\mu + \frac{p_0^2 p_a}{p_0^B} p_B^\mu. \] (A.24)

With the effects of the Lorentz transform on a set of vectors spanning the full four-vector space known, the Lorentz transform itself can also be found by parameterizing. It is
\[ \Lambda^{\mu\nu} = g^{\mu\nu} + p^{\mu}_a p^{\nu}_B - p^{\mu}_B p^{\nu}_A + \frac{p_0^a p_0}{(p_0^0 p_0^B)(p_0^0 p_B)} p_B^{\mu} p_B^{\nu}. \] (A.25)

**Phase Space Factorization**

The pre-branching phase space is given by
\[ d\Phi = d\Phi_A d\Phi_2 (p_A + p_B \rightarrow p_K + p_R) d\Phi_{\text{shower}}, \] (A.26)

where
\[ d\Phi_A = \frac{dx_A}{x_A} \Theta(x_A \leq 1) \]
\[ d\Phi_2 = \frac{1}{(2\pi)^2} d^4 p_K \delta(p_K^2) d^4 p_R \delta(p_R^2 - m_R^2) \delta^4(p_A + p_B - p_K - p_R) \]
\[ d\Phi_{\text{shower}} = ds_{aj} ds_{jk} \frac{d\varphi}{2\pi}. \] (A.27)

**Invariant phase space measures**

Since the global map incorporates a Lorentz transform, it is convenient to rewrite all pieces of eq. (A.27) in terms of Lorentz-invariant phase space measures. The \( x_A \) integral may be written as
\[ d\Phi_A = d^4 p_A \delta(p_A^2) \delta(p_A^1 p_1) \delta(p_A^2 p_2) \theta(p_A^0 p_\perp \leq \sqrt{s}/2), \] (A.28)

where \( p_\perp = (1, 0, 0, -1) \). The shower phase space \( d\Phi_{\text{shower}} \) may be transformed to new variables \( u \) and \( v \) which are given by the prefactors of \( p_A \) and \( p_K \) in the expression for \( p_j \) in eq. (5.42) and eq. (5.44). They are given by
\[ u_l = \frac{s_j s_{aj}}{s_{AK} \sigma_l} \quad v_l = \frac{s_{aj}}{\sigma_l} \]
\[ u_g = \frac{s_j}{\sigma_g} \quad v_g = \frac{s_{aj} s_{ak}}{s_{AK} \sigma_g}, \] (A.29)

where \( \sigma_l = s_{AK} + s_{jk} \) and \( \sigma_g = s_{AK} - s_{aj} \). Furthermore, we can write
\[ d\varphi = d\varphi dw \delta(w - \sqrt{uv s_{AK}}) \]
\[ = d\alpha d\beta 2\delta(\alpha^2 + \beta^2 - uv s_{AK}), \] (A.30)
where \( w \) is the prefactor of \( p_\perp \) in equations eq. (5.42) and eq. (5.44). The variables \( \alpha = w \sin(\varphi) \) and \( \beta = w \cos(\varphi) \) parameterize the two transverse directions in terms of the shower variable \( \varphi \). The parton shower phase space can now be written as

\[
d\Phi_{\text{shower}} = \frac{1}{\pi} \frac{s_{AK} \sigma^2}{s_{ak}} du dv d\alpha d\beta \left( \alpha^2 + \beta^2 - uw s_{AK} \right).
\]

(A.31)

Now define independent transverse directions \( p_\perp \alpha \) and \( p_\perp \beta \) which have

\[
p_{\perp(\alpha/\beta)} \cdot p_A = p_{\perp(\alpha/\beta)} \cdot p_K = p_\alpha \cdot p_\beta = 0 \quad \text{and} \quad p_{(\alpha/\beta)}^2 = -1.
\]

(A.32)

The momentum \( p_j \) can now be constructed by writing

\[
p_j = u p_A + v p_K + \alpha p_\perp \alpha + \beta p_\perp \beta
\]

(A.33)

which has

\[
p_j^2 = uw s_{AK} - \alpha^2 - \beta^2.
\]

(A.34)

The Jacobian determinant is given by

\[
|J(u, v, \alpha, \beta \to p_j)| = \sqrt{|g|} = \frac{2}{s_{AK}},
\]

(A.35)

where \( g \) is the metric in the coordinate system with basis vectors \( p_A, p_K, p_\perp \alpha, p_\perp \beta \). The parton shower phase space now becomes

\[
d\Phi_{\text{shower}} = \frac{2}{\pi} \frac{\sigma^2}{s_{ak}} d^4 p_j \delta(p_j^2).
\]

(A.36)

Using the above, eq. (A.27) can now be rewritten to the entirely Lorentz-invariant expression

\[
d\Phi = \frac{1}{2\pi^3} \frac{\sigma^2}{s_{ak}} d^4 p_A \delta(p_A^2) d^4 p_K \delta(p_K^2) d^4 p_R \delta(p_R^2 - m^2) d^4 p_j \delta(p_j^2)
\]

\[
\times \delta(p_A \cdot p_1) \delta(p_A \cdot p_2) \theta(p_A \cdot p_- \leq \sqrt{s/2}) \delta^4(p_A + p_B - p_K - p_R).
\]

(A.37)

### Transfomring the antenna momenta

Next, the pre-branching antenna momenta \( p_A \) and \( p_K \) have to be transformed to the post-branching momenta \( p_a \) and \( p_k \). The procedures for the two kinematic maps are different but similar. We show here the procedure for the global map and comment on the differences with the local map at the end of this appendix. The transformation is realized by means of the factor

\[
1 = dy \delta \left( y - \frac{\sigma_F}{s_{AK}} \right) d^4 p_k \delta^4 (p_k - yp_K)
\]

\[
\times d^4 p_a \delta^4 (p_a - p_A - p_j + (1 - y)p_K).
\]

(A.38)
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Following the algorithm, the scaling factor \( y \) is chosen to ensure \( p_a \) is massless. The \( y \)-delta can thus be rewritten as

\[
\delta \left( y - \frac{\sigma_g}{s_{AK}} \right) = \frac{s_{AK} s_{ak}}{\sigma_g} \delta(p_a^2). \tag{A.39}
\]

Similarly, \( \delta(p_A^2) \) can be rewritten to

\[
\delta(p_A^2) = \frac{y}{s_{AK}} \delta \left( y - \frac{\sigma_g}{s_{AK}} \right). \tag{A.40}
\]

Substituting these expressions into the phase space integral and eliminating the pre-branching antenna momenta and the scaling factor \( y \) leads to

\[
d\Phi = \frac{1}{2\pi^3} s_{AK} d^4 p_a \delta(p_a^2) d^4 p_j \delta(p_j^2) d^4 p_k \delta(p_k^2) d^4 p_R \delta(p_R^2 - m^2) \\
\times \delta(p_A \cdot p_1) \delta(p_A \cdot p_2) \theta(p_A \cdot p_- \leq \sqrt{s/2}) \delta(p_a + p_B - p_j - p_k - p_R). \tag{A.41}
\]

Since the phase space measure is written in a form that is entirely Lorentz-invariant, the transform to the lab frame can be performed implicitly. Using the properties of the Lorentz transform we can write

\[
\delta(p_A \cdot p_{1/2}) = \delta \left( \frac{s_{AB}}{s_{aB}} p_a' p_{1/2} \right) = \frac{s_{AB}}{s_{aB}} \delta(p_a' p_{1/2}). \tag{A.42}
\]

We thus finally find

\[
d\Phi = 16\pi^2 \frac{dx_a}{x_a} \theta(x_a \leq 1) d\Phi_3(p_a + p_B \to p_j + p_k + p_R) \frac{s_{AK}}{s_{AB}^2} \frac{s_{aB}^2}{s_{AB}^2}. \tag{A.43}
\]

The procedure for the local kinematics is very similar, with the exception of the omission of the Lorentz transform. Due to the presence of the directional delta functions for \( p_A \), residual Jacobian factors survive after eliminating the pre-branching antenna momenta and the scaling factor \( y \). For the global map these appear as a consequence of the Lorentz transform instead.
Appendix B

Relevant Feynman Rules of the Electroweak Theory

This appendix lists the Feynman rules of the electroweak theory that are relevant for the calculation of branching amplitudes. We elect to make use of a practical notation for the electroweak Feynman rules. It makes for simpler results, but some of the underlying group structure is obfuscated. We work in the unitary gauge, which has propagators

\[ \begin{align*}
    &\quad = i \frac{p + m}{p^2 - m^2 + i m \Gamma} \\
    &\quad = i \frac{-g_{\mu\nu} + \frac{v^\mu v^\nu}{m^2}}{p^2 - m^2 + i m \Gamma} \\
    &\quad = i \frac{1}{p^2 - m^2 + i m \Gamma}.
\end{align*}\]  

(B.1)
The vertex interactions are

\[ V = i(\nu + a\gamma^5)\gamma^\mu \]

\[ = i e \frac{m_f}{2 s_w m_W} \]

\[ = ig_h g^{\mu\nu} \]

\[ = -i \frac{3}{2} \frac{m_h^2}{m_w s_w} \]

where

\[ Y(p_1, \mu, p_2, \nu, p_3, \alpha) = (p_1 - p_2)^\alpha g^{\mu\nu} + (p_2 - p_3)^\mu g^{\nu\alpha} + (p_3 - p_1)^\nu g^{\mu\alpha} \]

is the Yang-Mills vertex. As usual, the weak mixing angle is defined as

\[ c_w \equiv \cos \theta_w = \frac{m_W}{m_Z} \]
\[ s_w \equiv \sin \theta_w. \]

The coupling constants are defined in Table B.1.
Appendix C

Details of the QED Shower Implementation

In this appendix we collect the details of the QED shower implementation. These include the overestimates of the antenna functions as well as the definitions of the parton shower variables, their phase space limits and the radiative integrals that are used to sample them.

C.1 Photon Emissions

The overestimates of the antenna functions given in eqs. (6.5), (6.6) and (6.7) are

\[
\hat{a}_{E_{\text{Em}t}}^{\text{FF}}(s_{ij}, s_{jk}, s_{ik}) = 4 \frac{s_{IK}}{s_{ij} s_{jk}} + \delta_{iW} \frac{8}{3} \frac{1}{s_{ij}} \frac{s_{IK}}{s_{I} - s_{jk}} + \delta_{jW} \frac{8}{3} \frac{1}{s_{jk}} \frac{s_{IK}}{s_{I} - s_{ij}}
\]

\[
\hat{a}_{E_{\text{Em}t}}^{\text{IF}}(s_{aj}, s_{jk}, s_{ak}) = 4 \left( \frac{s_{AK} + s_{jk}}{s_{AI} s_{aj} s_{jk}} + \delta_{iW} \frac{8}{3} \frac{1}{s_{AI}} \left( 2 \frac{s_{jk}}{s_{AK}} + \frac{1}{2} \frac{s_{jk}^2}{s_{AK}} \right) + \delta_{kW} \frac{8}{3} \frac{1}{s_{jk}} \frac{s_{AK}}{s_{I} - s_{aj}} \right)
\]

\[
\hat{a}_{E_{\text{Em}t}}^{\text{II}}(s_{aj}, s_{bj}, s_{ab}) = 4 \frac{s_{ab}^2}{s_{AI} s_{aj} s_{bj}}
\]

We now compute the radiative integral as shown in Chapter 5 for all types of radiation.

Final - Final

The ordering scale is

\[
Q_{\text{FF}}^2 = \frac{s_{ij} s_{jk}}{s_{IK}}.
\]

We make use of the auxiliary variables

\[
z_{1}^{\text{FF}} = \frac{s_{ij}}{s_{ij} + s_{jk}}, \quad z_{2}^{\text{FF}} = \frac{s_{jk}}{s_{IK}}, \quad z_{3}^{\text{FF}} = \frac{s_{aj}}{s_{IK}}.
\]

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which have phase space boundaries

$$z_{1}^{\pm} = z_{2}^{\pm} = z_{3}^{\pm} = \frac{1}{2} \left(1 \pm \sqrt{1 - 4 \frac{Q_{FF}^{2}}{s_{IK}^{2}}} \right).$$  \hspace{1cm} (C.4)

The radiative integral is divided into three pieces corresponding with the eikonal piece and the two additional \(W\)-related pieces. They are given by

$$I_{ff}^{IF} = 4\pi \alpha \hat{a}_{ff}^{IF}(s_{ij}, s_{jk}, s_{ik}) d\Phi_{ant}^{IF}$$

$$= -\frac{Q_{i}Q_{k}}{2\pi} \frac{s_{IK}}{\sqrt{\lambda(m_{IK}^{2}, m_{i}^{2}, m_{k}^{2})}} dQ_{FF}^{2} \frac{dz_{1}^{FF}}{z_{1}^{FF}(1 - z_{1}^{FF})} d\varphi$$

$$I_{WF}^{IF} = 4\pi \alpha \hat{a}_{WF}^{IF}(s_{ij}, s_{jk}, s_{ik}) d\Phi_{ant}^{IF}$$

$$= -\frac{2}{3} \frac{Q_{i}Q_{k}}{\pi} \frac{s_{IK}}{\sqrt{\lambda(m_{IK}^{2}, m_{i}^{2}, m_{k}^{2})}} dQ_{FF}^{2} \frac{dz_{2}^{FF}}{1 - z_{2}^{FF}} d\varphi$$

$$I_{fW}^{IF} = 4\pi \alpha \hat{a}_{fW}^{IF}(s_{ij}, s_{jk}, s_{ik}) d\Phi_{ant}^{IF}$$

$$= -\frac{2}{3} \frac{Q_{i}Q_{k}}{\pi} \frac{s_{IK}}{\sqrt{\lambda(m_{IK}^{2}, m_{i}^{2}, m_{k}^{2})}} dQ_{FF}^{2} \frac{dz_{3}^{FF}}{1 - z_{3}^{FF}} d\varphi. \hspace{1cm} (C.5)$$

Initial-Final

The ordering scale is

$$Q_{IF}^{2} = \frac{s_{aj}s_{jk}}{s_{AK} + s_{jk}}. \hspace{1cm} (C.6)$$

We make use of the auxiliary variable

$$z_{IF}^{\pm} = \frac{s_{aj}}{s_{AK} + s_{jk}} \hspace{1cm} (C.7)$$

which has phase space boundaries

$$z_{IF}^{-} = \frac{Q_{IF}^{2}}{s_{jk}^{2}}, \hspace{1cm} z_{IF}^{+} = \frac{s_{jk}^{2}}{s_{jk}^{2} + m_{k}^{2}}. \hspace{1cm} (C.8)$$

The upper boundary is a consequence of the positivity requirement of the Gram determinant. The resulting expression is maximized for \(s_{jk} = s_{jk}^{+}\). The radiative integrals are

$$I_{ff}^{IF} = 4\pi \alpha \hat{a}_{ff}^{IF}(s_{aj}, s_{jk}, s_{ak}) \hat{R}_{PDF}^{IF} d\Phi_{ant}^{IF}$$

$$= -\alpha \frac{Q_{i}Q_{k}}{\pi} \frac{dQ_{IF}^{2}}{Q_{IF}^{2}} \frac{dz_{IF}^{\pm}}{z_{IF}^{\pm}} \hat{R}_{PDF}^{IF}$$

$$I_{fW}^{IF} = 4\pi \alpha \hat{a}_{fW}^{IF}(s_{aj}, s_{jk}, s_{ak}) \hat{R}_{PDF}^{IF} d\Phi_{ant}^{IF}$$

$$= -\frac{2}{3} \alpha \frac{Q_{i}Q_{k}}{\pi} \frac{dQ_{IF}^{2}}{Q_{IF}^{2}} \frac{dz_{IF}^{\pm}}{1 - z_{IF}^{\pm}} \hat{R}_{PDF}^{IF}. \hspace{1cm} (C.9)$$
where

\[ R_{PDF}^{IF} = \frac{f_a(x_a, Q_W^2)}{f_A(x_A, \hat{Q}_W^2)} \]  

\[ (C.10) \]

are the PDF ratios and \( \hat{R}_{PDF}^{IF} \) is a constant overestimate. We only consider the contribution to the antenna function with a \( W \) in the final state, since no \( W \) bosons appear in the initial state. Note that a factor \( s_{AK}/(s_{AK} + s_{jk}) \) remains in \( T_{FW}^{IF} \). Since this factor is always less than unity, it can be implemented as a probability in the Sudakov veto algorithm.

**Resonance-Final**

The ordering scale is again

\[ Q_W^2 = \frac{s_{aj}s_{jk}}{s_{AK} + s_{jk}}. \]  

\[ (C.11) \]

We make use of the auxiliary variables

\[ z_{1R}^F = \frac{s_{aj}}{s_{AK} + s_{jk}}, \quad z_{2R}^F = \frac{s_{jk}}{s_{AK}}. \]  

\[ (C.12) \]

The phase space boundaries are determined by the limits on the invariants, which are

\[ s_{aj}^+ = m_a^2 - (m_k + m_{AK})^2, \quad s_{jk}^+ = (m_a - m_{AK})^2 - m_k^2. \]  

\[ (C.13) \]

These boundaries are reached in the instances where respectively \( p_k \) and \( p_r \) are at rest. These translate to the boundaries

\[ z_{1R}^{F-} = \frac{Q_{RF}^2}{(m_a - m_{AK})^2 - m_k^2}, \quad z_{1R}^{F+} = \frac{(m_a - m_k)^2 - m_{AK}^2}{m_a^2 - m_k^2 - m_{AK}^2}, \]  

\[ (C.14) \]

\[ z_{2R}^{F-} = \frac{Q_{RF}^2}{m_a^2 - (m_k + m_{AK})^2 - Q_{RF}^2}, \quad z_{2R}^{F+} = \frac{(m_a - m_{AK})^2 - m_k^2}{m_a^2 + m_k^2 - m_{AK}^2}. \]  

\[ (C.15) \]

Note that the upper bound on \( z_1 \) overestimates the actual boundary, which is difficult to express analytically. The expression in eq. (C.14) is found by just substituting \( s_{aj}^+ \) and \( s_{jk}^- = 0 \). The upper boundary reduces to 1 in case \( m_k = 0 \), but the only contribution to the antenna function that scales like \( 1/(1 - z) \) is related to a final-
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state \(W\), which is massive. The radiative integrals are

\[
\mathcal{I}_{ff}^{RF} = 4\pi \alpha \hat{a}^{RF}_{ff}(s_{aj}, s_{jk}, s_{ak}) d\Phi_{ant}^{RF} \\
= -\alpha \frac{Q_a Q_k}{s_{ak}} \frac{s_{AK}}{\sqrt{\lambda(m_A^2, m_{AK}^2, m_K^2)}} dQ_{1f}^2 d\varphi \frac{z_{1f}^{RF}}{z_{1f}^{RF}} (s_{AK} + s_{jk})^2
\]

\[
\mathcal{I}_{Wf}^{RF} = 4\pi \alpha \hat{a}^{RF}_{Wf}(s_{aj}, s_{jk}, s_{ak}) d\Phi_{ant}^{RF} \\
= -\frac{2}{3} \alpha \frac{Q_a Q_k}{s_{ak}} \frac{s_{AK}}{\sqrt{\lambda(m_A^2, m_{AK}^2, m_K^2)}} dQ_{1f}^2 d\varphi \frac{z_{2f}^{RF}}{Q_{1f}^2} (z_{2f}^{RF} + (z_{2f}^{RF})^2) \frac{d\varphi}{2\pi}
\]

\[
\mathcal{I}_{fW}^{RF} = 4\pi \alpha \hat{a}^{RF}_{fW}(s_{aj}, s_{jk}, s_{ak}) d\Phi_{ant}^{RF} \\
= -\frac{2}{3} \alpha \frac{Q_a Q_k}{s_{ak}} \frac{s_{AK}}{\sqrt{\lambda(m_A^2, m_{AK}^2, m_K^2)}} dQ_{1f}^2 d\varphi \frac{s_{AK} + s_{jk}}{Q_{1f}^2} \frac{d\varphi}{2\pi}
\]

Note that factors that are larger than unity remain in the first and third integrals. These factors are accounted for by overestimating them by their identical expressions using \(s_{jk}^+\), and are then corrected by a probabilistic veto.

**Initial-Initial**

The ordering scale is

\[
Q_{1i}^2 = \frac{s_{aj} s_{bj}}{s_{ab}}.
\]

We make use of the auxiliary variable

\[
z_{ii} = \frac{s_{aj}}{s_{ab}},
\]

which has phase space boundaries

\[
z_{ii}^\pm = \frac{1}{2s} \left( s - s_{AB} \pm \sqrt{(s - s_{AB})^2 - 4Q_{1i}^2 s} \right),
\]

where \(s\) is the hadronic invariant mass. The radiative integral is

\[
\mathcal{I}_{ff}^{II} = 4\pi \alpha \hat{a}^{II}_{ff}(s_{aj}, s_{bj}, s_{ab}) \hat{R}^{II}_{PDF} d\Phi_{ant}^{II} \\
= -\alpha \frac{Q_a Q_b}{\pi} \frac{dQ_{1i}^2}{Q_{1i}^2} \frac{d\varphi}{\varphi_{1i}^2} \frac{z_{1i}^{II}}{(1 - z_{1i}^{II})} \frac{d\varphi}{2\pi} \hat{R}^{II}_{PDF},
\]

where

\[
R_{PDF}^{II} = \frac{f_a(x_a, Q_{1i}^2)}{f_A(x_a, Q_{1i}^2)} \frac{f_b(x_b, Q_{1i}^2)}{f_B(x_b, Q_{1i}^2)},
\]

are the PDF ratios and \(\hat{R}_{PDF}^{II}\) is a constant overestimate.
C.2 Photon Splitting

The antenna function is overestimated by

\[ \hat{a}_{\text{split}}^{\text{FF}}(s_{ij}, s_{jk}, s_{ik}) = 4 \frac{1}{m_{ij}^2}. \]  

(C.22)

The shower variables are

\[ Q^2 = m_{ij}^2, \quad z = \frac{s_{ik}}{m_{IK}^2}, \]  

(C.23)

with phase space boundaries given by

\[ z_- = 0, \quad z_+ = 1 - \frac{Q^2}{m_{IK}^2}. \]  

(C.24)

The evolution integral is

\[ I_{\text{split}}^{\text{FF}} = 4\pi \alpha Q^2 f_{\hat{a}_{\text{split}}^{\text{FF}}}(s_{ij}, s_{jk}, s_{ik}) d\Phi_{\text{ant}}^{\text{FF}} \]

\[ = \alpha \frac{Q^2}{\pi} \frac{1}{1 - m_{IK}^2/m_{IK}^2} \frac{dQ^2}{Q^2} dz \frac{d\varphi}{2\pi}. \]  

(C.25)

C.3 Photon Conversion

The antenna function is overestimated by

\[ \hat{a}_{\text{Conv}}^{\text{II}}(s_{aj}, s_{bj}, s_{ab}) = 4 \frac{1}{s_{aj} s_{ab}^2}. \]  

(C.26)

The evolution variables are

\[ Q^2 = s_{aj}, \quad z = \frac{s_{ab}}{s_{AB}}, \]  

(C.27)

with phase space boundaries given by

\[ z_- = 1 + \frac{Q^2}{s_{AB}}, \quad z_+ = \frac{s}{s_{AB}}. \]  

(C.28)

The evolution integral is

\[ I_{\text{Conv}}^{\text{II}} = 4\pi \alpha Q^2 \hat{a}_{\text{Conv}}^{\text{II}}(s_{aj}, s_{bj}, s_{ab}) R_{\text{PDF}}^{\text{II}} d\Phi_{\text{ant}}^{\text{II}} \]

\[ = \alpha \frac{Q^2}{\pi} \frac{dQ^2}{Q^2} dz \frac{d\varphi}{2\pi} x_a x_b \hat{R}_{\text{pdf}}^{\text{II}}, \]  

(C.29)

where

\[ R_{\text{PDF}}^{\text{II}} = f_q(x_a, Q^2) f_B(x_b, Q^2) \]  

\[ \frac{f_q(x_a, Q^2)}{f_B(x_b, Q^2)}. \]  

(C.30)

The factor \( x_a x_b/x_{AB} \) is folded in with the PDF veto to ease the required overestimate \( \hat{R}_{\text{pdf}}^{\text{II}} \) due to the large difference between photon and quark PDFs.
Appendix D

Details of the EW Shower Implementation

In this appendix we collect the details of the EW shower implementation. These include the overestimates of the branching kernels as well as the definitions of the parton shower variables, their phase space limits and the radiative integrals that are used to sample them.

D.1 Final-Final

The ordering scale in the final state is

\[ |Q^2_{\text{FF}}| = |s_{ij} + m_i^2 + m_j^2 - m_I^2|. \]  

(D.1)

Defining the variable

\[ R = E_{IK}(E_{IK} + |\vec{p}_{IK}|), \]  

(D.2)

the branching kernel overestimate is given by

\[
\mathcal{O}^{\text{FF}} = c_1^{\text{FF}} \frac{|Q^2_{\text{FF}}|}{Q_{\text{FF}}^4 + m_I^2 \Gamma_I^2} + c_2^{\text{FF}} \frac{|Q^2_{\text{FF}}|}{Q_{\text{FF}}^4 + m_I^2 \Gamma_I^2} \frac{R}{s_{ij} + s_{ik} + m_i^2} + c_3^{\text{FF}} \frac{|Q^2_{\text{FF}}|}{Q_{\text{FF}}^4 + m_I^2 \Gamma_I^2} \frac{R}{s_{ij} + s_{jk} + m_j^2} + c_4^{\text{FF}} \frac{|Q^2_{\text{FF}}|}{Q_{\text{FF}}^4 + m_I^2 \Gamma_I^2} \frac{R}{s_{ij} + s_{jk} + m_j^2}. 
\]  

(D.3)

The auxiliary variables are

\[ z_1^{\text{FF}} = \frac{s_{jk}}{m_I^2}, \quad z_2^{\text{FF}} = \frac{s_{ij} + s_{ik} + m_i^2}{R}, \quad z_3^{\text{FF}} = \frac{s_{ij} + s_{jk} + m_j^2}{R}. \]  

(D.4)
The evolution integrals are

\[ I_{1}^{\text{FF}} = 4\pi \alpha c_{1}^{\text{FF}} \frac{1}{Q_{1}^{2}} \frac{d\Phi_{\text{ant}}^{\text{FF}}}{d\phi} \]

\[ = \frac{\alpha}{4\pi} \frac{m_{I}^{2}}{\sqrt{\lambda(m_{I}^{2}, m_{I}^{2}, m_{I}^{2})}} \frac{dQ_{1}^{2}}{Q_{1}^{2}} \frac{dz_{1}^{\text{FF}}}{2\pi} \]

\[ I_{2}^{\text{FF}} = 4\pi \alpha c_{2}^{\text{FF}} \frac{1}{Q_{2}^{2}} \frac{R}{s_{ij} + s_{ik} + m_{i}^{2}} \frac{d\Phi_{\text{ant}}^{\text{FF}}}{d\phi} \]

\[ = \frac{\alpha}{4\pi} \frac{R}{\sqrt{\lambda(m_{I}^{2}, m_{I}^{2}, m_{I}^{2})}} \frac{dQ_{2}^{2}}{Q_{2}^{2}} \frac{dz_{2}^{\text{FF}}}{2\pi} \frac{dz_{3}^{\text{FF}}}{2\pi} \]

\[ I_{3}^{\text{FF}} = 4\pi \alpha c_{3}^{\text{FF}} \frac{1}{Q_{3}^{2}} \frac{R}{s_{ij} + s_{ik} + m_{i}^{2}} \frac{d\Phi_{\text{ant}}^{\text{FF}}}{d\phi} \]

\[ = \frac{\alpha}{4\pi} \frac{R}{\sqrt{\lambda(m_{I}^{2}, m_{I}^{2}, m_{I}^{2})}} \frac{dQ_{3}^{2}}{Q_{3}^{2}} \frac{dz_{3}^{\text{FF}}}{2\pi} \frac{dz_{4}^{\text{FF}}}{2\pi} \]

\[ I_{4}^{\text{FF}} = 4\pi \alpha c_{4}^{\text{FF}} \frac{m_{j}^{2}}{Q_{4}^{2}} \frac{d\Phi_{\text{ant}}^{\text{FF}}}{d\phi} \]

\[ = \frac{\alpha}{4\pi} \frac{m_{I}^{2}}{\sqrt{\lambda(m_{I}^{2}, m_{I}^{2}, m_{I}^{2})}} m_{I}^{2} \frac{dQ_{4}^{2}}{Q_{4}^{2}} \frac{dz_{4}^{\text{FF}}}{2\pi} \frac{dz_{4}^{\text{FF}}}{2\pi}. \]  

(D.5)

The phase space boundaries are different for the parton shower and the Breit-Wigner sampling. In the parton shower evolution, the branching kernels and overestimates do not include the width and \( Q_{2}^{2} \) is strictly positive. The phase space boundaries of the auxiliary variables are given by

\[ z_{1}^{\text{FF}} = \frac{2m_{j}m_{k}}{m_{I}^{2}}, \quad z_{1}^{\text{II}} = 1 - \frac{Q_{2}^{2} + m_{j}^{2} + 2m_{i}m_{k} + m_{k}^{2}}{m_{I}^{2}}, \]

\[ z_{2}^{\text{FF}} = \frac{Q_{2}^{2} - m_{j}^{2} + m_{I}^{2} + 2m_{i}m_{k}}{R}, \quad z_{2}^{\text{II}} = \frac{m_{I}^{2} - (m_{j} + m_{k})^{2}}{R}, \]

\[ z_{3}^{\text{FF}} = \frac{Q_{2}^{2} - m_{j}^{2} + m_{I}^{2} + 2m_{i}m_{k}}{R}, \quad z_{3}^{\text{II}} = \frac{m_{I}^{2} - (m_{i} + m_{k})^{2}}{R}. \]

No resonance branchings exist with soft-\( p_{j} \) peaks. As such, \( c_{2}^{\text{FF}} \) is always zero. The remaining phase space boundaries are

\[ z_{1}^{\text{Res}} = \frac{2m_{j}m_{k}}{m_{I}^{2}}, \quad z_{1}^{\text{FF}} = 1 - \frac{(m_{i} + m_{j})^{2} + (m_{i} + m_{k})^{2} - m_{i}^{2}}{m_{I}^{2}}, \]

\[ z_{3}^{\text{Res}} = \frac{m_{j} + 2m_{i} + 2m_{k}}{R}, \quad z_{3}^{\text{FF}} = \frac{m_{I}^{2} - (m_{i} + m_{k})^{2}}{R}. \]

\[ D.2 \quad \text{Initial-Initial} \]

The ordering scale in the initial state is

\[ Q_{\text{II}}^{2} = s_{aj} - m_{j}^{2}. \]  

(D.6)
The branching kernel overestimate is
\[
\mathcal{O}^\text{II} = c^\text{II}_1 \frac{1}{Q^2_{\text{II}} s_{AB}} s_{ab} + c^\text{II}_2 \frac{1}{Q^2_{\text{II}} x_{ASb}(s_{ab} - s_{bj}) + x_{BSa}(s_{ab} - s_{aj})}. \tag{D.7}
\]

The auxiliary variables are
\[
z^\text{II}_1 = \frac{s_{AB}}{s_{ab}}, \quad z^\text{II}_2 = \frac{s_{bj}}{s_{ab}}, \tag{D.8}
\]
which have phase space boundaries
\[
z^\text{II}_1 - = \frac{s_{AB}}{s}, \quad z^\text{II}_1 + = \frac{s_{AB}}{s_{AB} + Q^2_{\text{II}}}, \quad z^\text{II}_2 - = 0, \quad z^\text{II}_2 + = \frac{1 - x^2_{A}}{x_{A}(x_{A} + x_{B})}.
\]

The evolution integrals are
\[
\mathcal{T}^\text{II}_1 = 4\pi \alpha c^\text{II}_1 \frac{1}{Q^2_{\text{II}} s_{AB}} \hat{R}^\text{II}_{\text{PDF}} d\Phi^\text{II}_{\text{ant}}
= \alpha \frac{dQ^2_{\text{II}}}{4\pi} \frac{dz^\text{II}_1}{z^\text{II}_1} \frac{d\varphi}{2\pi} \hat{R}^\text{II}_{\text{pdf}},
\]
\[
\mathcal{T}^\text{II}_2 = 4\pi \alpha c^\text{II}_2 \frac{1}{Q^2_{\text{II}} x_{ASb}(s_{ab} - s_{bj}) + x_{BSa}(s_{ab} - s_{aj})} \hat{R}^\text{II}_{\text{PDF}} d\Phi^\text{II}_{\text{ant}}
= \alpha \frac{dQ^2_{\text{II}}}{4\pi} \frac{dz^\text{II}_2}{(z^\text{II}_2)^2 + z^\text{II}_2 + d} \frac{d\varphi}{2\pi} \hat{R}^\text{II}_{\text{pdf}}, \tag{D.9}
\]
where
\[
d = \frac{x_{B}(Q^2_{\text{II}} + m^2_{j})(Q^2_{\text{II}} + s_{AB})}{x_{AS}^2 s_{AB}}. \tag{D.10}
\]

The second evolution integral can be sampled by underestimating \(d\) with \(d_c\) which is found by substituting the cutoff scale and adding a multiplicative term
\[
\frac{(z^\text{II}_2)^2 + z^\text{II}_2 + d_c}{(z^\text{II}_2)^2 + z^\text{II}_2 + d}
\]
to the veto probability.
Bibliography


Samenvatting

Natuurkundigen houden zich bezig met bestuderen van het gedrag en de eigenschappen van materie en energie, met als doel om de natuur op alle schalen te begrijpen. Het gaat daarbij om de allergrootste structuren in het universum zoals melkwegstelsels en zwarte gaten, maar ook de allerkleinste bouwstenen waaruit alle materie is opgebouwd, en alles daartussenin. De natuur is extreem complex, en het vergaren van dit soort kennis is een uitdaging die de mensheid al millennia lang bezig houdt.

Sindsdien heeft het onbegrip in steeds grotere mate plaats gemaakt voor een steeds duidelijker beeld van de natuurwetten die ten grondslag liggen aan de ogenschijnlijke complexiteit. In de zoektocht naar de simpelere basis van de complexe realiteit heeft het in het bijzonder geholpen om op zoek te gaan naar de aard van de allerkleinste bouwstenen van de materie om ons heen. Het blijkt dat alle vormen van voor ons zichtbare materie zijn opgebouwd uit slechts enkele soorten deeltjes. Zodra we begrijpen wat deze deeltjes zijn en hoe ze op elkaar reageren, kunnen we dit gebruiken om de natuur op grotere schaal ook te begrijpen. Dit is het doel van de deeltjesfysica; het begrip van de natuur op het allerkleinste niveau.

Het Standaard Model

Helaas vormt het bestuderen van de allerkleinste natuurlijke processen een enorme uitdaging. Aan de ene kant bestaat er inmiddels een theoretische beschrijving van de deeltjes en hun interacties: het Standaard Model. Dit omvat ons volledige begrip van de meest elementaire deeltjes en hun interacties en is ontwikkeld als resultaat van decennia aan theoretisch en experimenteel onderzoek. Het Standaard Model is een van de meest succesvolle theorieën in de geschiedenis van de wetenschap vanwege zijn alomvattendheid en de nauwkeurigheid van zijn voorspellingen. Echter, er zijn nog verschillende theoretische problemen waar het Standaard Model geen directe oplossing voor biedt. Hieronder valt bijvoorbeeld de afwezigheid van een beschrijving van donkere materie en donkere energie, geheimzinnige fenomenen die indirect kunnen worden waargenomen, maar waarvoor enig begrip ontbreekt. Een ander probleem is
dat het Standaard Model geen beschrijving voor zwaartekracht bevat. Hoewel de zwaartekracht van weinig betekenis is voor de interacties tussen elementaire deeltjes, is het wel cruciaal voor ons begrip van de natuur op grotere schaal.

**Experimenteel onderzoek**

Aan de andere kant staat het experimenteel onderzoek. In de afgelopen decennia zijn er steeds omvangrijkere experimenten ontwikkeld met als doel om elementaire deeltjes met steeds meer detail te kunnen onderzoeken. Op het moment is het belangrijkste experiment de Large Hadron Collider (LHC), een deeltjesversneller gebouwd door de Europese onderzoeksorganisatie CERN. Dit experiment bestaat uit een ondergrondse ringvormige tunnel met een omtrek van 27 kilometer op de Frans-Zwitserse grens dichtbij Genève. In deze tunnel worden deeltjes versneld tot enorm hoge energie, waarna ze botsen in een van de tientallen meters grote ondergrondse detectoren. Bij deze botsingen komen grote hoeveelheden deeltjes vrij die kunnen worden gedetecteerd. Hoe groter de snelheid, en dus de energie, van de botsende deeltjes, hoe meer kans dat er tussen die deeltjes een teken van een nieuw natuurkundig fenomeen te vinden is. De Large Hadron Collider is dus op zoek naar afwijkingen van de kennis van het Standaard Model, in de hoop dat er nieuwe soorten deeltjes te ontdekken zijn die een oplossing kunnen bieden voor bovenstaande problemen.

**Fenomenologie**

Om deze afwijkingen van ons huidig begrip te kunnen vinden, is het in de eerste plaats nodig om te weten wat het Standaard Model voorspelt voor de deeltjesbotsingen die plaats vinden in de LHC. Dit blijkt echter een extreem complexe uitdaging. De LHC laat namelijk zogenaamde protonen op elkaar botsen die zelf weer bestaan uit meerdere kleinere deeltjes. Als twee van zulke protonen op elkaar botsen, zorgen de hoge energie en de samengestelde aard van de protonen ervoor dat een grote hoeveelheid deeltjes de detector raakt. Het doen van voorspellingen voor dit soort botsingsprocessen is zo ingewikkeld dat het een eigen vakgebied vormt, genaamd de fenomenologie. Hoewel het Standaard Model ons toestaat om voorspellingen uit te rekenen voor eenvoudige botsingsprocessen, blijkt dit een stuk lastiger voor de botsingen zoals die bij de Large Hadron Collider plaatsvinden. Fenomenologen maken daarom gebruik van computersimulaties waarbij de uitdaging is om zoveel mogelijk kennis van het Standaard Model te verwerken in computercodes waarmee uiteindelijk data worden geproduceerd die direct vergelijkbaar zijn met de experimentele metingen.
Parton showers

Vanwege de complexiteit van de botsingsprocessen worden dit soort computersimulaties opgesplitst in een aantal onderdelen. Eerst wordt de fundamentele botsing gesimuleerd, waarbij twee deeltjes op elkaar botsen en twee of drie nieuwe, hoogenergetische deeltjes worden geproduceerd. Deze stap kan tot op hoge nauwkeurigheid gemodelleerd worden met de kennis van het Standaard Model. De volgende stap is de zogenaamde Parton Shower, wat ook het onderwerp van dit proefschrift is. Dit onderdeel simuleert het uitstralen van de grote hoeveelheden deeltjes door de eerder genoemde hoogenergetische deeltjes die uiteindelijk in te detector terecht komen. De theoretische beschrijving van deze procedure is een stuk minder nauwkeurig dan die van de eerste stap, en is daarom ook een actief onderzoeksgebied.

Een nieuw soort straling

De inhoud van dit proefschrift draagt bij aan de verbetering van een van de Parton Shower codes, genaamd Vincia, die gebruikt kan worden met de meest gangbare simulatiesoftware, genaamd Pythia. In het bijzonder beschrijft Vincia de afstraling van een ander type elementaire deeltjes dan de deeltjes die normaal gesproken worden gesimuleerd. Hoewel Vincia de belangrijkste soorten straling al simuleert, was deze andere soort straling tot nu toe nog niet aanwezig. In dit proefschrift wordt de vertaalslag gemaakt van de theoretische beschrijving van de nieuwe soort straling, die op veel vlakken verschilt van de vorige soort, naar de implementatie in de code. Het blijkt dat deze toevoeging een significante aanvulling kan leveren aan het begrip van de voorspellingen van het Standaard Model, vooral met het oog op de hoge energieën en grote hoeveelheden data die worden bereikt bij de LHC en bij toekomstige experimenten.
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