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The Abelian Higgs Model: Unitarity in the unitary Gauge

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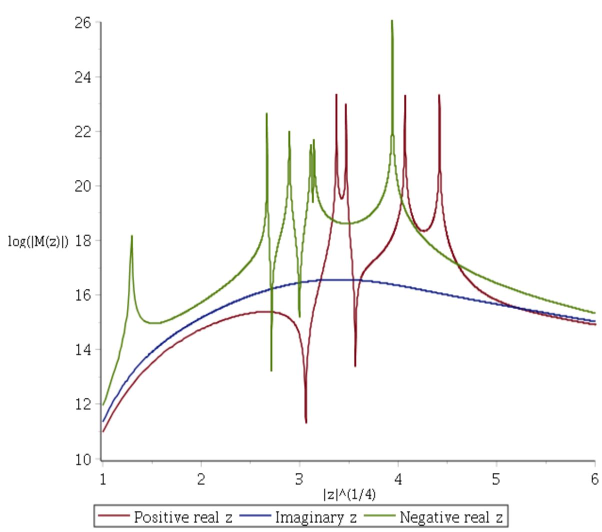


Abstract

The Standard Model of particles is one of the most successful theories in history explaining a grand variety of experimental measures. It uses Lagrangian densities to explain the dynamics of the quantum fields, the fundamental objects in the theory that correspond to particles in the physical reality. The Standard Model is also a gauge theory, which allows performing gauge transformations to the Lagrangian. These mathematical operations do not change the physics that the Lagrangian describe, but do change its form. This property is exploited in the theory to prove results using mathematical abstract concepts that do not correspond to any element in the physical reality. In concrete, proving the correct energy behavior of the matrix element using the equivalence theorem [1] uses non-physical fields in the Feynman-t'Hooft gauge - massless bosons, ghost particles, Faddeev-Popov ghosts - that allow the mathematical proof to be achieved. In this thesis we will focus on proving the correct energy behavior of a subgroup of the Standard Model - the Abelian Higgs model - working in the unitary gauge, in which every element has its counterpart in reality.

Contents

1 Introduction

This project tries to study the high energy behavior of a theory included in the Standard Model (SM) of Particles, the Abelian Higgs Model. This Model contains only two particles. These are two bosons, the Z and the Higgs Bosons. These two particles are massive and electrically neutral, they follow Bose-Einstein statistics - integer spin -, their interactions are limited and well-known, the Z boson carries a polarization while the Higgs does not, and at tree level there are no other particles involved.

In this context, the focus of the project is in proving that this theory is well-behaved at high-energies ($E_{scale} \gg M_Z, M_H$) in the physical gauge. The SM being a renormalizable theory, on which the divergence is solved by it, the energy behavior of the expressions for processes that can be experimentally measured - with a finite value - must have an upper limit if unitarity holds. All terms that have a higher energy dependence in the Matrix element have therefore to vanish. For the Standard Model this have been proven in the Feynman-t'Hooft gauge, using the equivalence theorem [1] to observe the highest energy dependence, and checking that it is under the limit energy behavior. In this gauge the particle spectrum allows Higgs ghosts and Faddeev-Popov ghosts to appear. The apparition of these particles is critical to see that the SM is well-behaved , but those are the mathematical residues of working on this gauge, since they do not exist in the physical world. Using the physical unitary gauge, in which the experimentally found particles interact, makes the expressions more complicated to simplify. This is why in this study, only the Abelian Higgs Model is taken into account.

1.1 Conventions

The conventions here used are the following:

- Natural units. $\hbar = c = 1$.
- Einstein summation notation of a repeated index.

$$\sum_{\mu} p^{\mu} q_{\mu} \equiv p^{\mu} q_{\mu}$$

- Partial derivatives are defined as: $\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$
- Incoming momentum for all particles in an interaction:

$$\sum_{i}^{n} p_i^{\mu} = 0$$

- External Z bosons always in the longitudinal polarization if not stated otherwise.

Later on the discussion some new conventions will be introduced to simplify the equations worked on.

1.2 Lagrangian formalism and gauge theories

In the history of physics, mathematics has been the basic tool that allow physicists to describe things in reality and perform operations on a system. This mathematical description also allows for predictions to be made. Developing new rigorous mathematical methods improves, simplifies and deepens the knowledge of a subject, at the cost of abstraction and an obscurer connection between the real objects and their mathematical representations.

In the 17th century, Leibniz and Newton developed calculus. This became an extremely powerful method by which many physical systems could be solved. Infinitesimals introduced continuous variables and their use in equations of motion allowed description and prediction of moving bodies. Classical mechanics was born.

Later in 1788, Joseph-Louis Lagrange reformulated classical mechanics. He introduced Lagrangian mechanics. The physics described were exactly the same as in classical mechanics, but the Lagrangian was a much more powerful and easy description to extract information from. A system could be described by kinetic and potential energies, and constrains, all this information contained in the Lagrangian (in non-relativistic mechanics $\mathbf{L}(q_j, \dot{q_j}, t) = T - V$, with q_j the coordinates of particle j, $\dot{q_j} = \frac{dq_j}{dt}$ the time derivative of the coordinate of particle j, and T and V the kinetic and potential energy respectively). The integral (over time) of the Lagrangian is the action ($\mathbf{S} = \int_{t_1}^{t_2} \mathbf{L} dt$). Applying then the principle of least action ($\delta \mathbf{S} = 0$), which looks for the path between the points t_1 and t_2 with a stationary action, led to the Euler-Lagrange equations of motions, equivalent to Newton's ones (full derivation can be found in [2]). In non-relativistic mechanics those are:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q_j}}) = \frac{\partial L}{\partial q_j} \quad (1)$$

This equations that describe point-like objects also can be extended to continuum fields. The coordinates become now fields (ψ_j) and their derivatives are now on a space-time point $\mathbf{s}$ on a Manifold $\mathcal{M}$. The action is now defined as the four dimensional integral over the Lagrangian density ($\mathscr{L}$) - usually just called Lagrangian. With the equivalent of the least action principle for continuum fields ($\frac{\delta \mathbf{S}}{\delta \psi_j} = 0$), we find the new equations of motion:

$$S[\psi_j] = \int \mathscr{L}(\psi_j, \frac{\partial \psi_j}{\partial s^\alpha}, s^\alpha) d^n s \quad (2)$$

$$\partial_\mu(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \psi_j)}) = \frac{\partial \mathscr{L}}{\partial \psi_j} \quad (3)$$

In the first quarter of the 20th century physicist were revolutionized with the new theories of general relativity and quantum mechanics. The latter showed that elementary particles could be described by point-like objects as well as by fields. Even though the Hamiltonian formulation was used first to describe quantum mechanics, Richard Feynman worked on a Lagrangian-based formulation, manifestly symmetric in time and space. The principle of least action was generalized to the path integral formulation. The idea behind it was that since particles are also waves, all possible paths between two points must be considered, each with a phase corresponding to the action, e^{iS}. This sum (functional integral) over all the paths is the calculation of the quantum amplitude, which squared is a probability density that allows to compute transition probabilities between states. Those paths further from the classic trajectory are suppressed, and in the classical limit the classic trajectory is recovered (more on the derivation at [3]).

Feynman also developed the Feynman diagrams, an easy and intuitive way to represent mathematical expressions for elementary-particle interactions in the form of pictures, or diagrams. Each element in the diagram (vertices, internal lines, external lines) has a mathematical equivalent that comes directly from the Lagrangian terms. The sum of all the possible Feynman diagrams for a process is the Matrix Element M, that when squared yields the quantum amplitude.

Once seen how the Lagrangian formalism arises, it is possible to look at its properties. Specifically at the continuous symmetries a Lagrangian has. A Lagrangian is symmetric under a transformation when it remains invariant under it, and the equations of motion prevail unchanged. Each symmetry is related to a conserved quantity by Noether's theorem. In the classical formulation, a symmetry under time transformation ($t \rightarrow t + \delta t$) implies that the total energy is conserved. Space symmetry ($q_k \rightarrow q_j + \delta q_j$) implies conservation of the momentum $\dot{q_j}$. When a theory has a Lagrangian invariant under a continuous group of transformations is called a gauge theory. For example, Maxwell's formulation of classical electrodynamics is a gauge theory, where the vector potential $\mathbf{A}^\mu = (V, \vec{A})$ can be transformed as $A^\mu \rightarrow A^\mu - \partial^\mu \Lambda$ leaving the electric and magnetic fields unaffected .This group of gauge transformations form a Lie group, which has a number of generators. Each generator will have an associated field that ensures Lagrangian invariance. When the theory is quantized, the quanta of the fields are the gauge bosons. A simple complex scalar theory can illustrate this. Start with the Lagrangian density for a complex scalar field Φ:

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \Phi^*)(\partial^\mu \Phi) - \frac{1}{2} m^2 \Phi^* \Phi \tag{4}$$

This is globally gauge invariant since a transformation $\Phi \rightarrow \Phi = e^{i\Lambda}\Phi'$, where Λ is real and does not depend on space nor time, leaves the Lagrangian invariant. But Local symmetry is also required. When the field Λ depends on space-time coordinates, the Lagrangian is not invariant anymore due to the derivatives acting on this field. To assure invariance under a local transformation the covariant derivative is defined: $D^\mu = \partial^\mu + igA^\mu$, where g is the coupling constant between the scalar and the gauge field A^μ. Now the Lagrangian is locally invariant under the transformation $\Phi \rightarrow \Phi = e^{i\Lambda(x^\mu)}\Phi'$ when A^μ transforms as $A^\mu \rightarrow A^\mu = A'^\mu - \frac{1}{g}\partial^\mu \Lambda(x^\mu)$. Some new interaction terms between Φ and A^μ also appear. It is then allowed to add a gauge fixing term to the Lagrangian which will give kinematics (and a propagator) to the gauge field. The A^μ vectorial gauge field has become a massless particle, a gauge boson, emerging from the symmetries of the Lagrangian.

1.3 Standard Model

The Standard Model of particle physics is a gauge quantum field theory in the group $SU(3) \times SU(2) \times U(1)$, which representations correspond with the elementary particles known, and accounts for their interactions via strong or electroweak forces. Its formulation has remain unchanged since the 1970s and has been consistent with experimental results and particles found (top quark (1995)[4], tau neutrino (2000)[5], Higgs boson (2012)[6][7]). It has an incredible predictive power to calculate scattering amplitudes and decays, but it leaves some phenomena unexplained[1]. Many other theories that try to solve this problems are extensions of this model with different groups or symmetries, because of how powerful the SM formulation has been. The Standard Model mathematical formalism uses a Lagrangian density [8] that respects the symmetries of the group and is renormalizable [9]. With renormalization, infinities arising in the calculations are reabsorbed into the parameters of the theory and it is then able to make predictions. The Standard Model is self-consistent as an effective field theory that arises from using perturbation theory, and is therefore only valid up to the Planck Energy scale ($\sim 10^{18} GeV$), were the effects of an underlying over-Planck-scale theory appear [10]. The Planck scale is, however, much larger than any parameter in the SM (The electroweak scale is $\sim$246 GeV and the Higgs mass is $m_H \simeq$125 GeV). This apparent discordance in the magnitudes is the so called hierarchy problem [11, Chapter 4.3]: some precise cancellations and fine-tunning would be required in order to find this experimental values. In this range of energies much higher than the SM parameters scale but before reaching the Planck scale, in which the theory still has predictive power, is where this project will focus on proving that the theory will indeed be still right.

2 Abelian Higgs Model

The Abelian Higgs Model is a model embedded in the SM. This model arises from ensuring unitarity, and therefore probability conservation, in the vector boson sector of the Standard Model. The vector bosons $W^{+/-}$, Z and their couplings, already fixed by the interaction of those bosons with fermions, have dangerous terms for all-longitudinal external vector boson scattering. In order to respect unitarity, a new particle H that interacts with these bosons is introduced. It must be a neutral, scalar particle so the amplitudes mediated by it have the same energy behavior. Since this particle is introduced for unitarity and to cancel some specific terms, relations between the coupling constants between this particle and the vector bosons are found. Reapplying the argument for different gedanken processes, the unitarity condition requires in the model

[1]No dark matter candidate, no dark energy explanation, no gravity quantization

a number of interactions, namely a ZZH and a ZZHH vertex for the process $ZZ \rightarrow HH$ to be well behaved, as well as self-interactions terms between H particles. Given these two particles, the interactions and their coupling constants are fixed just by the necessity of the fundamental and physical requirement of ensuring unitarity. Then it is easy to see that, at tree level (no internal loops in the process), the interactions between the particles Z and H can only lead to final states with the same particles.

Another approach to the model is considering the Lagrangian this system has. The Abelian Higgs Model is the simplest gauge theory in which spontaneous symmetry breaking is possible. The gauge invariant Lagrangian is composed of a massless vector field A^μ and a complex scalar field ϕ under a potential $V(\phi) = \lambda(|\phi|^2 - \psi)$ (Figure 1), ($\lambda < 0$ the coupling strenght) with $\psi = \frac{v^2}{2}$, the minimum of the potential.

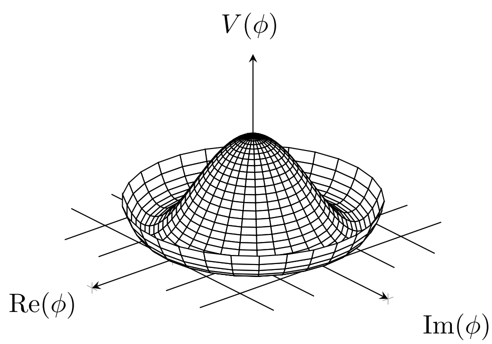


Figure 1: Mexican hat potential that the scalar particle ϕ is subject to. The minimum of the potential is a circle of points that has a fixed magnitude and a arbitrary phase that can be modified by gauge transformations.

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D^\mu\phi|^2 - \lambda(|\phi|^2 - \frac{v^2}{2})^2 \tag{5}$$

This is the Lagrangian for the Abelian Higgs model, where v is the scalar field vacuum expectation value (vev), $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ the A^μ field tensor and $D^\mu = \partial^\mu - ieA^\mu$ the gauge covariant derivative. Now we can break the U(1) symmetry by choosing one of the vev solutions for ϕ for the Mexican hat potential:

$$\phi = \frac{1}{\sqrt{2}}(v + h(x))e^{\frac{\omega(x)}{v}} \tag{6}$$

The fields $h(x)$ and $\omega(x)$ represent the modes of the solution around the vev v, distinguishing between perturbations in the magnitude and in the phase respectively.

Expanding the Lagrangian with the chosen solution for the scalar field:

$$\begin{aligned}\mathscr{L} = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + \frac{1}{2}e^2(v+h)^2|A_\mu - \frac{1}{ev}\partial_\mu \omega|^2 \\ & - \frac{\lambda}{4}h^4 - \lambda v^2 h^2 - \lambda v h^3 \end{aligned} \tag{7}$$

Now the gauge transformation $A_\mu \to A'_\mu = A_\mu - \frac{1}{ev}\partial_\mu \omega$ simplifies it to:

$$\begin{aligned}\mathscr{L} = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + \frac{1}{2}e^2(v+h)^2 A'_\mu A'^\mu \\ & - \frac{\lambda}{4}h^4 - \lambda v^2 h^2 - \lambda v h^3 \end{aligned} \tag{8}$$

In this Lagrangian it can be observed a vector boson A'^μ with mass $m_A = ev$, and a scalar particle h with mass $m_h = \sqrt{-2\lambda}v$ (m_h is real and positive due to $\lambda < 0$). With the Higgs mechanism, the massless Goldstone boson, ω field, disappears from the Lagrangian in order to give mass to the A boson. This gauge choice brings the Lagrangian to the unitary gauge. In the unitary gauge, the Goldstone modes are absorbed by the vector fields, giving them mass and a longitudinal polarization. In this case the fields can be identified with real particles from the SM. The massive vector boson A^μ is the Z boson, with it's three possible polarizations, and the massive scalar boson h is the Higgs particle.

In order to prove unitarity in the Abelian Higgs model, it will be approached firstly from the simplest perspective, the massless limit. In this limit, in which calculations are easier, recursive relations will be found and used to prove unitarity for the leading and next-to-leading order terms in energy. Finally, to extend the proof to all orders in energy, on-shell recursion relations (BCFW in chromodynamics) will be used. Partial fraction will be applied to the Matrix Element, expanding the phase space to more dimensions in order to create a deformation that allows us to rewrite amplitudes with lower-order, well-behaved diagrams, and infer the energy behavior for any possible process.

2.1 Interactions

The possible interactions in this model, as we can see from the Lagrangian terms are the following:

- ZZH: Three particles interaction, two Z and one Higgs bosons
- ZZHH: Four particles interaction, two Higgs and two Z bosons
- HHH: Three particle self-interaction of Higgs bosons
- HHHH: Four particle self-interaction of Higgs bosons

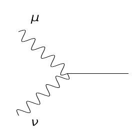
These four interactions generate all Feynman diagrams for any number of external legs. The coupling constant for these interactions are closely related and can be written in terms of the same coupling constants g as will be shown in the Feynman rules.

From this it concludes that the number of Z bosons will be even for any possible process, since the two interactions in which it participates have another Z boson created/annihilated.

2.2 Feynman rules

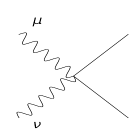
The set of starting and complete Feynman rules in the theory are (in natural units):

(i) ZZH three-point vertex:



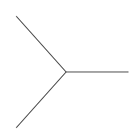
$$i\frac{em_z}{c_w s_w}g^{\mu\nu} = i\frac{em_z^2}{m_w s_w}g^{\mu\nu} = i2gm_z^2 g^{\mu\nu}$$

(ii) ZZHH four-point vertex:



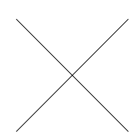
$$i\frac{e^2}{2s_w^2 c_w^2}g^{\mu\nu} = i2g^2 m_z^2 g^{\mu\nu}$$

(iii) HHH three-point vertex:



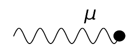
$$-i\frac{3em_h^2}{2m_w s_w} = -i3gm_h^2$$

(iv) HHHH four-point vertex:



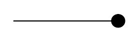
$$-i\frac{3e^2 m_h^2}{4m_w^2 s_w^2} = -i3g^2 m_h^2$$

(v) External Z boson (longitudinal):

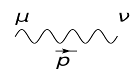


$$\epsilon^\mu = \frac{1}{m_z}(p^\mu - \frac{m_z^2}{(p \cdot t)} t^\mu)$$

(vi) External Higgs boson: 1

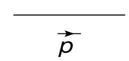


(vii) Internal Z propagator:



$$-i\frac{g^{\mu\nu} - \frac{p^\mu p^\nu}{m_z^2}}{p^2 - m_z^2} = -i\frac{T^{\mu\nu}}{p^2 - m_z^2} + i\frac{L^{\mu\nu}}{m_z^2}$$

(viii) Internal Higgs propagator:



$$\frac{i}{p^2 - m_h^2}$$

Here g is the coupling constant:

$$g = \frac{e}{2m_w s_w} \tag{9}$$

$$T^{\mu\nu} = g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \text{ and } L = \frac{p^\mu p^\nu}{p^2} \tag{10}$$

m_z and m_w are the Z and W boson masses respectively; s_w, c_w the sine/cosine of the weak mixing angle. The polarization has been written in the longitudinal case (the dot indicates that it is an external leg going into a diagram). The vector t^μ is a massless gauge vector so that the polarization conditions ($(\epsilon_i \cdot p_i) = 0$, $(\epsilon_i \cdot \epsilon_i) = -1$) hold.

This is the standard set of rules in which the Feynman diagrams can be computed. Later on this discussion, these rules will be modified for simplification of the calculations done and clarification of the notation.

2.3 Energy dependence limit of the matrix element

In order for the theory to be able to predict correct experimental results and to ensure unitarity, cross sections must be bounded. Using dimensional analysis we can establish a limit for the energy dependence of the matrix element. The expression to calculate the differential cross section for two particles scattering into n is:

$$d\sigma(2 \to m) = \Phi_\sigma \langle |M|^2 \rangle dV(p_a, p_b; p_1, .., p_m) F_{symm} \tag{11}$$

Where Φ_σ is the flux factor, M the matrix element, $dV(p_a, p_b; p_1, .., p_m)$ the phase space integration element that depends on all the particles momenta (subindex a and b denotes the two initial particles, 1,..,m the m resulting particles), and F_{symm} the symmetry factor. Knowing the energy dimensions of these elements, we can extract the matrix element dimensions:

$$[M_{m+2}] = [E^{2-m}] = [E^{4-n}] \tag{12}$$

Where $n = m+2$ is the total number of particles in the interaction. This energy dependence includes the energy behavior of all the parameters.

For a $2 \to 2$ process, the matrix element limit energy dependence is $[E^0]$. That means that any term in the matrix element with a dependence on energy higher than this has to vanish or the cross section would grow with the energy, predicting unphysical results.

2.4 Energy dependece

The energy dependence of the Feynman diagrams expressions comes from the number of external vector bosons (and their polarization) and the number of internal Higgs propagators the diagram has. The most extreme case, and therefore the one that this study focus most in, is when all the Z bosons are longitudinally polarized. This polarization is linear in energy at first order, and for each Z boson the energy dependence of the process increases by $[E]$. An external Higgs bosons do not change the energy dependence $[E^0]$, and a Higgs propagator diminishes it by two powers $[E^{-2}]$. A Z propagator also does not change the energy behavior (E^0), and neither do the vertices. Knowing the energy dependence of each element and with only two possible interactions (Higgs self-interactions always lower the energy dependence), the following relations between the number of ZZH vertices (n_3), the number of ZZHH vertices (n_4) and the number of internal and external Z/Higgs bosons ($i_{z/h}$, $n_{z/h}$) can be found:

$$\begin{aligned} n_3 + 2n_4 &= 2i_h + n_h \\ 2n_3 + 2n_4 &= 2i_z + n_z \\ n_3 + n_4 &= i_h + i_z + 1 \end{aligned} \tag{13}$$

This equations says that for each ZZH vertex there are two Z and one Higgs (internal or external), for each ZZHH vertex two Z and two Higgs, and this number of particles must be the same as the number of external and internal particles, knowing that an internal particle will end in two vertices (or that two vertices must be united by an internal line). Adding the second and third equation in (13), a relation between the number of internal Higgs and external Z can be found:

$$2i_h = n_z - 2$$

The matrix element energy dependence will be:

$$[M] \propto [E^{n_z}(E^{-2})^{i_h}] = [E^{n_z - 2i_h}] = [E^2] \tag{14}$$

Therefore, the highest energy order terms that can be found in any process in this model is $\mathcal{O}(E^2)$.

For each process with n external particles, with all the Z bosons in the longitudinal polarization, the Feynman diagrams terms that depend on energy as $\propto [E^2]$ down to $\propto [E^{4-n}]$ must vanish, if the theory is well-behaved. For a fixed Z boson polarization with maximum energy behavior $[E^n]$ only terms with dependece $[E^{n-2k}]$ are possible ($k \in \mathbb{N}_0$).

For example, a simple process like two Z bosons to two Higgs, $ZZ \to HH$, with the Z bosons in longitudinal polarization, must have the $[E^2]$ terms vanish, but the following order terms, $\mathcal{O}(E^0)$, do not need to vanish in order to have a well-behaved theory.

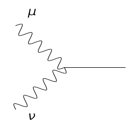
2.5 Hypothesis check

The hypothesis held here is that taking into account all the Feynman diagrams that contribute to a process with n external bosons, the mathematical terms that depend on the energy up to $\mathcal{O}(E^{4-n})$ will cancel between each other. First of all an explicit check of those cancellations is necessary. Which process is relevant enough to be calculated? It is important to notice that the number of possible Feynman diagrams for a process increases much faster than the number of external particles. For example, 4**Z** 1**H** has 21 diagrams, while 6**Z** 1**H** has 1770 diagrams. On the other hand, a trivial process like 4**Z**, with 3 diagrams, only needs the highest order in energy $\mathcal{O}(E^2)$ to cancel. The two processes chosen at first to see this cancellations are: 4**Z** 1**H**, with 21 diagrams, and 2**Z** 3**H**, with 25 diagrams. These two processes with 5 external particles need to cancel the terms with dependence $[E^2]$ and $[E^0]$ to be well behaved ($\mathcal{O}(E^{-1})$ terms are already safe) The calculations for these two process in all detail, for different polarizations of the Z bosons can be found in the Appendix A. After a notation simplification, the calculation of this processes will become much easier to perform due to the highly symmetric diagrams under label interchange and the use of combinatorics.

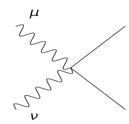
3 Massless approximation, E^2 terms

Once we have checked that these cancellations are happening, it is important to understand how they work. The $[E^2]$ terms, coming only from the $L^{\mu\nu}$ part of the Z boson propagator, must vanish by themselves. To take into account only those terms, set the particles in the model as massless. Higgs self-interactions lower the energy behavior by E^{-2}, and any Feynman diagram containing one can be neglected as $\mathcal{O}(E^0)$. In this limit, $m_z^2 = m_h^2 = 0$, the Feynman rules now become:

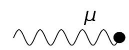
(i) ZZH three-point vertex: $i2g^{\mu\nu}$



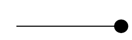
(ii) ZZHH four-point vertex: $i2g^{\mu\nu}$



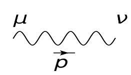
(iii) External Z boson (longitudinal): $\epsilon^\mu = p^\mu$



(iv) External Higgs boson: 1

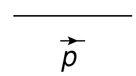


(v) Z propagator:



$$\simeq i\frac{p^\mu p^\nu}{p^2}$$

(vi) H propagator:



$$\simeq \frac{i}{p^2}$$

The factors m_z^{-2} from the propagators and m_z^{-1} from the longitudinal polarization cancels exactly to those m_z^2 from the vertices. With these propagators only the E^2 terms are taken into account, and they must vanish. This means they cancel completely or the result is $\propto q_i^2 = m_i^2 \simeq 0$, with q_i^μ the momentum of an external particle. In the latter case this will mean that they still play a role in the E^0 terms cancellation.

Being able to calculate the contribution from this terms will turn out to be an extremely powerful tool later. With the new Feynman rules and the following simplified notation the calculation for the first process 4 **Z** 1 **H** can easily be performed.

- Z bosons external momentum will be labeled by letters: $a^\mu, b^\mu, c^\mu, ...$
- Higgs bosons external momentum will be labeled by numbers: $1^\mu, 2^\mu, 3^\mu, ...$
- Four-vectors products depending on external momenta:

$$((a + b + c + ..) \cdot (a + 1 + 2 + ..)) \equiv (abc.. \cdot a12..)$$

- Four vectors squared: $(a{+}b{+}1{+}..)^2 \equiv (ab1..)^2 = 2(a{\cdot}b1..){+}2(b{\cdot}1..){+}2(1{\cdot}..)$

In this process we have three kinds of diagrams. First, there are 12 diagrams with only ZZH vertices, with the following form (figure 2):

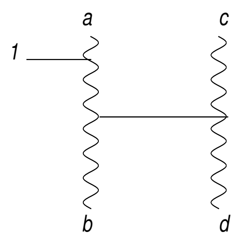


Figure 2: Feynman diagram for a process with 4 **Z**, 1 **H** with only ZZH vertex

$$M_1 = i8\frac{(a\cdot a1)(a1\cdot b)(c\cdot d)}{(a1)^2(cd^2)} = i2(a1\cdot b) \tag{15}$$

Where with massless particles $\frac{(a\cdot a1)}{(a1)^2} = \frac{(a\cdot 1)+a^2|_{m_z^2=0}}{2(a\cdot 1)} = \frac{1}{2}$.

The other diagrams are given by interchanges of indices. If we fix Higgs 1 to be emitted from particle a, interchanging b with c and d takes into account three diagrams, and the contributions will be:

$$i2(a1\cdot b) \xrightarrow[bcd]{\times 3} i2(a1\cdot bcd) = i2(a\cdot bcd) + i2(1\cdot bcd)$$

This includes three diagrams. The other 9 diagrams come from emitting Higgs 1 from the other Z bosons b, c, d. Interchanging then a with each of the others (4 possibilities) will take into account the 12 diagrams:

$$\begin{aligned} i2(a\cdot abcd) + i2(1\cdot bcd) \quad &\xrightarrow[abcd]{\times 4} \quad i2(abcd\cdot abcd) + i6(1\cdot abcd) \\ M_1 \quad &= \quad i4(1)^2 \end{aligned} \tag{16}$$

Introducing $(a\cdot a) = 0$ makes the first term more symmetric. For the second term, there are $\binom{4}{3}$ ways of choosing 3 out of the 4 momenta and each momentum will be in three of those terms. The term then is symmetrized and multiplied by 3. Following another reasoning: there are 3 terms of the form $2i(1\cdot A)$, which, after taking the symmetrization on 4 particles, will become 12 terms with $2i(1\cdot A)$, each momentum with the same number of terms. Since there are 4 possible momenta $12\cdot 2i(1\cdot A)\cdot\frac{1}{4} = 6i(1\cdot A)$ is the number of terms each momentum will have, and indeed corresponds with the previous result $6i(1\cdot abcd)$. Using momentum conservation $((abcd1) = 0)$, we can see that both terms are proportional to $\mathbf{1}^2 = 0$.

Next there are 6 diagrams with one ZZHH coupling. Following the Feynman rules, one of them looks like figure 3:

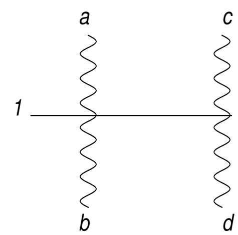


Figure 3: Feynman diagram for a process with 4 **Z**, 1 **H** with one ZZHH vertex

$$M_2 = -i4\frac{(a\cdot b)(c\cdot d)}{(cd)^2} = -i2(a\cdot b) \tag{17}$$

The six diagrams result from each of the ways of choosing two momentum out of the possible four.

$$-i2(a\cdot b) \xrightarrow[pairs]{\times 6} = -i2(a\cdot b) - i2(a\cdot c) - i2(a\cdot d) - i2(b\cdot c) - i2(b\cdot d) - i2(c\cdot d)$$

Also proportional to $1^2 = 0$. The other three diagrams that complete the set of 21 for this process have a three-point Higgs self-interaction vertex, which makes them $\mathcal{O}(E^0)$ at most.

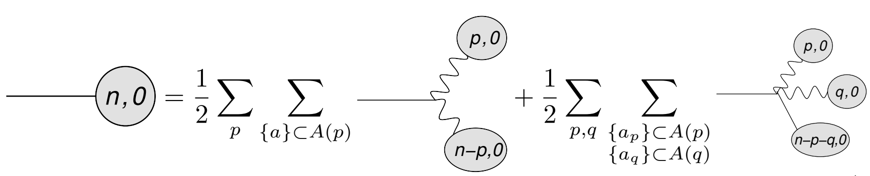
To extend this to any possible process we will use the Schwinger-Dyson equation for an internal Higgs or Z boson going to n Z and k Higgs.

3.1 $Z_{n,k}, H_{n,k}$

In the following it will be shown that, for an internal Z or Higgs going to n Z and k H, the sum of all possible diagrams in the massless case will be proportional to the momentum going through the internal particle. For consistency, the propagator factor is included in the calculations.

3.1.1 $Z_{n,0}, H_{n,0}$

First of all it is interesting to study the process when all the external particles are Z bosons. For $Z_{n,0}$, n must be odd (Z are always in pairs), and for $H_{n,0}$, n must be even. The Schwinger-Dyson equation for $H_{n,0}$ in diagrammatic and mathematical form respectively is:



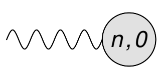
(18)

$$
\begin{aligned}
H_{n,0} &= \frac{i}{s^2}\Bigg[(i2)\frac{1}{2}\sum_p \sum_{\{a\}\subset A(p)} (a_1..a_p \cdot a_{p+1}..a_n) Z_{p,0} Z_{n-p,0} \\
&+ (i2)\frac{1}{2}\sum_{p,q} \sum_{\substack{\{a_p\}\subset A(p)\\ \{a_q\}\subset A(q)}} (a_1..a_p \cdot a_1..a_q) Z_{p,0} Z_{q,0} H_{n-p-q,0}\Bigg] = \\
&= -\Bigg[\sum_p \binom{n-2}{p-1} Z_{p,0} Z_{n-p,0} + \sum_{p,q} \binom{n-2}{p-1}\binom{n-p-1}{q-1} Z_{p,0} Z_{q,0} H_{n-p-q,0}\Bigg]
\end{aligned}
\tag{19}
$$

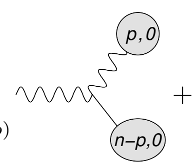
Where $\sum_p$ indicates a summation over the number of particles in the first blob, $\sum_{\{a\}\subset A(p)}$ is a summation over all possible partitions $\{a_1, .., a_p\}$ for p external bosons, a_i is the momentum of an external Z boson and s is the momentum going through the internal Higgs. The binomial coefficients arise when counting the $(a_i{\cdot}a_j)$ terms, symmetrizing them and using that $\sum_{i<j} 2(a_i \cdot a_j) = s^2$ to cancel against the denominator of the Higgs propagator.

For example, the first term generates products of the form $2(a_i \cdot a_j)$. a_i must be in the partition $\{a\}$ and a_j in the complementary partition $\{n-a\}$, and there is left $\binom{n-2}{p-1}$ ways of distributing the other $n-2$ momenta. This can be done for each pair of momenta and s^2 is recovered.

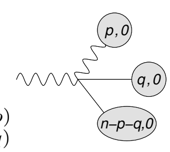
Similarly for an internal Z:



$$= \sum_p \sum_{\{a\}\subset A(p)}$$



$$+ \frac{1}{2}\sum_{p,q} \sum_{\substack{\{a_p\}\subset A(p)\\ \{a_q\}\subset A(q)}}$$



(20)

$$
\begin{aligned}
Z_{n,0} &= \frac{-2}{s^2}\Bigg[\sum_p \sum_{\{a\}\subset A(p)} (a_1..a_p \cdot s) Z_{p,0} H_{n-p,0} \\
&+ \frac{1}{2}\sum_{p,q} \sum_{\substack{\{a_p\}\subset A(p)\\ \{a_q\}\subset A(q)}} (a_1..a_p \cdot s) Z_{p,0} H_{q,0} H_{n-p-q,0}\Bigg] = \\
&= -2\Bigg[\sum_p \binom{n-1}{p-1} Z_{p,0} H_{n-p,0} + \frac{1}{2}\sum_{p,q} \binom{n-1}{p-1}\binom{n-p}{q} Z_{p,0} H_{q,0} H_{n-p-q,0}\Bigg]
\end{aligned}
\tag{21}
$$

The same procedure is followed to find the binomial coefficients, this time using conservation of momenta $\sum_i (a_i \cdot s) = s^2$.

With the starting values $H_{0,0} = 0$, $H_{1,0} = 0$, $Z_{0,0} = 0$, $Z_{1,0} = 1$, the recursive formula can be used to find other values. In table 1 is possible to observe that for a Z going to n Z, the coefficients follow a factorial product. The coefficients for a Higgs going to n Z follow a double factorial product:

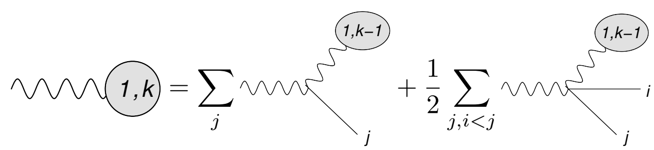
$$\begin{aligned} Z_{n,0} &= (n-1)! \\ H_{n,0} &= (-1)(n-3)!!(n-1)!! \\ n!! &= \prod_{k=0}^{\frac{n}{2}-1}(n-2k) \end{aligned} \tag{22}$$

$Z_{1,0}$	1	$H_{2,0}$	-1
$Z_{3,0}$	2	$H_{4,0}$	-3
$Z_{5,0}$	24	$H_{6,0}$	-45
$Z_{7,0}$	720	$H_{8,0}$	-1575
$Z_{9,0}$	40320	$H_{10,0}$	-99225

Table 1: Values for an internal Z/H going into n Z bosons

3.1.2 Finding $Z_{n,k}, H_{n,k}$

The behavior when emitting Z bosons is now known. The next step is to observe it when emitting Higgs. Since Higgs self-interactions are not taken into account, at least one or two Z will appear. The simples case is then $Z_{1,k}$:



(23)

$$\begin{aligned} Z_{1,k} &= \frac{-2}{s^2}\left[\sum_j (a1_1..1_{k-1}\cdot s)Z_{1,k-1} + \frac{1}{2}\sum_{j,i>j}(a1_1..1_{k-2}\cdot s)Z_{1,k-2}\right] \\ &= \frac{-2}{s^2}\left[\big[(k-1)s^2+\Delta\big]Z_{1,k-1} + \big[\frac{(k-1)(k-2)}{2}s^2+(k-1)\Delta\big]Z_{1,k-2}\right] \end{aligned} \tag{24}$$

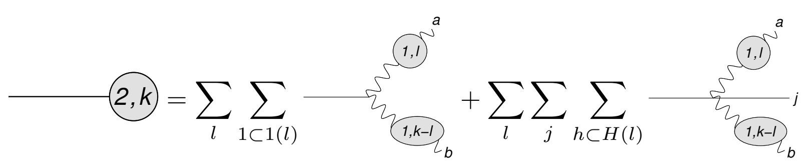
Where 1_i is the ith Higgs momentum, and $\sum_j$ is a summation over all possible Higgs momenta. Here there are two distinct cases: terms $(a\cdot s)\equiv\Delta$ and $(1_i\cdot s)$, where $\sum_i(1_i\cdot s) = s^2-(a\cdot s) = s^2-\Delta$. Here the binomials are included on the sum over j and i, $\binom{k-1}{1} = (k-1)$ and $\frac{1}{2}\binom{k-1}{1}\binom{k-2}{1} = \frac{(k-1)(k-2)}{2}$. Since the result has to be proportional to s^2, there are two equivalent recursive relations for $Z_{1,k}$:

$$-2Z_{1,k} - 2kZ_{1,k-1} = 0 \tag{25}$$

Expecting a $k!$ behavior, a generating function can be introduced:

$$\begin{aligned}
\phi(x) &= \sum_{k=0}^{\infty} \frac{x^k Z_{1,k}}{k!} = \sum_{k=1}^{\infty} \frac{x^k k Z_1, k}{k!} + 1(\Theta(k=0)) \\
\phi &= x \sum_{k=1}^{\infty} \frac{x^{k-1} Z_{1,k-1}}{(k-1)!} + 1 = x\phi(x) + 1 \qquad (26) \\
\phi &= \frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} Z_{1,k} \\
Z_{1,k} &= (-1)^k k! \qquad (27)
\end{aligned}$$

The second simplest case is $H_{2,k}$:



(28)

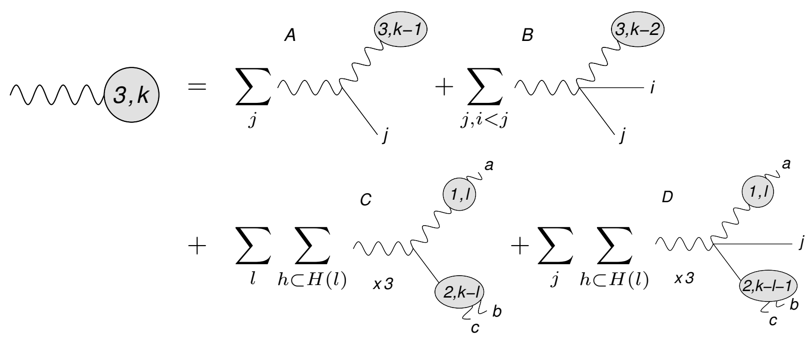
$$\begin{aligned}
H_{2,k} = \frac{-2}{s^2} \Bigg[& \sum_l \sum_{\{h\}\subset H(l)} (a1_1..1_l \cdot b1_{l+1}..1_k) Z_{1,l} Z_{1,k-l} \\
& + \sum_j \sum_l \sum_{\{h\}\subset H(l)}^{max} (a1_1..1_l \cdot b1_{l+1}..1_{k-l-1}) Z_{1,l} Z_{1,k-l-1} \Bigg] \qquad (29)
\end{aligned}$$

Where $\sum_l$ is a summation over the number of external Higgs on the first blob, $\sum_j$ and $\sum_{\{h\}\subset H(l)}$ are summations over all the subsets with 1 and l Higgs momenta respectively. The two Z momenta a and b are explicitly set in the diagram and has to be symmetrized. Having $Z_{1,k} = (-1)^k k!$ simplifies the binomial coefficients and the result is:

$$H_{2,k} = 2(-1)^{k+1}(k-2)! \sum_{l=1}^{k-1} l = (-1)^{k+1} k!$$

.

Now $Z_{3,k}$ can be calculated using the previous result for $H_{2,k}$:



(30)

$Z_{n,0} = (n-1)!$	$H_{n,0} = (-1)(n-3)!!(n-1)!!$
$Z_{1,k} = (-1)^k k!$	$H_{2,k} = (-1)^{k+1} k!$
$Z_{3,k} = (-1)^k (k+2)!$	$H_{4,k} = \frac{3}{2}(-1)^{k+1}(k+2)!$
$Z_{5,k} = (-1)^k (k+4)!$	

Table 2: Coefficients expressions for an internal Z/H to different external particles

The first two diagrams, A and B, can be included in C and D if a sum over the Z is performed. The $\times 3$ in the last two diagrams indicates that it has to be symmetrized between the three external Z boson momenta.

$$\begin{aligned}
A &= ((k-1)s^2 + \Delta) Z_{3,k-1} \\
B &= (\binom{k}{2} s^2 - (k-1)(s^2 - \Delta)) Z_{3,k-2} \\
C &= (-)^k (k+1)! (\frac{3}{2}s^2 - \frac{\Delta}{2}) \\
D &= (-)^{k-1} k! (\frac{3}{2}(s^2 - \Delta)(k-1) + k\Delta \\
Z_{3,k} &= \frac{-2}{s^2}\Big[A + B + C + D\Big]
\end{aligned} \quad (31)$$

As before, since the Δ terms must cancel, a recursive relation is found:

$$Z_{3,k} + kZ_{3,k-1} + (-1)^k 2(k+1)! = 0 \quad (32)$$

Using again a generating function:

$$\phi = \sum_{k=0}^{\infty} \frac{x^k}{k!} Z_{3,k}$$

The coefficients found are:

$$Z_{3,k} = (-1)^k (k+2)! \quad (33)$$

Following the same steps is possible to express the Schwinger-Dyson equation for any number of external Higgs and Z bosons. $H_{4,k}$ and $Z_{5,k}$ have also been calculated, finding the following coefficients:

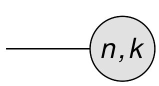
$$H_{4,k} = \frac{3}{2}(-1)^{k+1}(k+2)! \quad (34)$$

$$Z_{5,k} = (-1)^k (k+4)! \quad (35)$$

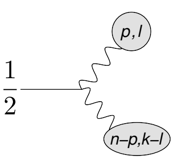
Table 2 gives a summary of the coefficients found. With this information is possible to conjecture a general formula for the coefficients for $Z_{n,k}$ and $H_{n,k}$:

$$
\begin{aligned}
Z_{n,k} &= \begin{cases} 0 & ,\text{for even n} \\ (-1)^k (n-1+k)! & ,\text{for odd n} \end{cases} \\
H_{n,k} &= \begin{cases} (-1)^{k-1}(n-2+k)!\beta_{(\frac{n}{2}-1)} & ,\text{for even n} \\ 1 & ,\text{for n=0 , k=1} \\ 0 & ,\text{for odd n and n=0 , k}\neq 1 \end{cases} \\
\beta_a &= \frac{(2a+1)!!(2a-1)!!}{(2a)!} = \begin{cases} \frac{(2a+1)!(2a-1)!}{2^{2a-1}a!(a-1)!(2a)!} & a \geq 1 \\ 1 & a = 0 \end{cases}
\end{aligned}
\tag{36}
$$

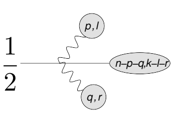
The Schwinger-Dyson diagrammatic equation for $H_{n,k}$ is:



$$= \sum_{\substack{p=1\\l=0}}^{\substack{p=n-1\\l=k}} \sum_{\substack{\{a_p\}\subset A(p)\\ \{h_l\}\subset H(l)}} \frac{1}{2}$$



$$+ \sum_{\substack{p=1\\l=0}}^{\substack{p=n-1\\l=k}} \sum_{\substack{q=1\\r=0}}^{\substack{q=n-p\\r=k-l}} \sum_{\substack{\{a_p\}\subset A(p)\\ \{a_q\}\subset A(q)}} \sum_{\substack{\{h_l\}\subset H(l)\\ \{h_r\}\subset H(r)}} \frac{1}{2}$$

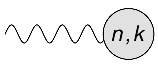


(37)

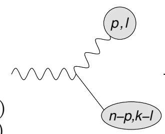
$$
\begin{aligned}
H_{n,k} = \frac{-2}{s^2} \Bigg[& \frac{1}{2}\sum_{p=1}^{p=n-1}\sum_{l=0}^{l=k}\Bigg[\binom{n-2}{p-1}\binom{k}{l}a^2 + \binom{n-1}{p-1}\binom{k-1}{l}(a\cdot h) \\
+ & \binom{n-1}{p}\binom{k-1}{l-1}(h\cdot a) + \binom{n}{p}\binom{k-2}{l-1}h^2\Bigg] Z_{p,l}Z_{n-p,k-l} \\
+ & \frac{1}{2}\sum_{p=1}^{n-1}\sum_{l=0}^{l=k}\sum_{q=1}^{n-p}\sum_{r=0}^{k-l}\Bigg[\binom{n-2}{p-1}\binom{n-p-1}{q-1}\binom{k}{l}\binom{k-l}{r}a^2 \\
+ & \binom{n-1}{p-1}\binom{n-p}{q}\binom{k-1}{l}\binom{k-l-1}{r-1}(a\cdot h) \\
+ & \binom{n-1}{p}\binom{n-p-1}{q-1}\binom{k-1}{l-1}\binom{k-l}{r}(h\cdot a) \\
+ & \binom{n}{p}\binom{n-p}{q}\binom{k-2}{l-1}\binom{k-l-1}{r-1}h^2\Bigg] Z_{p,l}Z_{q,r}H_{n-p-q,k-l-r}\Bigg]
\end{aligned}
\tag{38}
$$

Where a^2 is the sum of all Z bosons momenta squared, h^2 the sum of all Higgs momenta squared, and $(a\cdot h)$ the product between the sum of Z and Higgs momenta. $(a\cdot h)$ and $(h\cdot a)$ has been taken separately into account and they have different binomials but the contribution is the same. The combination $a^2+h^2+2(a\cdot h)=s^2$ is necessary to get rid of the propagator $\frac{1}{s^2}$, but there are some special cases: for $k=0$ there are no external Higgs and therefore $h^2=0$, $(a\cdot h)=0$, $a^2=s^2$ and for $k=1$, $h^2=m_h^2=0$.

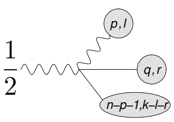
For $Z_{n,k}$:



$$= \sum_{\substack{p=1\\l=0}}^{\substack{p=n\\l=k}} \sum_{\substack{a_p\subset A(p)\\h_l\subset H(l)}}$$



$$+\sum_{\substack{p=1\\l=0}}^{\substack{p=n\\l=k}} \sum_{\substack{q=0\\r=0}}^{\substack{q=n-p\\r=k-l}} \sum_{\substack{a_p\subset A(p)\\a_q\subset A(q)}} \sum_{\substack{h_l\subset H(l)\\h_r\subset H(r)}} \frac{1}{2} \tag{39}$$



$$\begin{aligned} Z_{n,k} = \frac{-2}{s^2} \Bigg[& \sum_{p=1}^{p=n}\sum_{l=0}^{l=k}\left[\binom{n-1}{p-1}\binom{k}{l}(a\cdot s)+\binom{n}{p}\binom{k-1}{l-1}(h\cdot s)\right]Z_{p,l}H_{n-p,k-l} \\ + & \sum_{p=1}^{p=n}\sum_{l=0}^{l=k}\sum_{q=0}^{q=n-p}\sum_{r=0}^{r=k-l}\left[\binom{n-1}{p-1}\binom{n-1}{q}\binom{k}{l}\binom{k-l}{r}(a\cdot s)\right. \\ + & \left.\binom{n}{p}\binom{n-p}{q}\binom{k-1}{l-1}\binom{k-l}{r}(h\cdot s)\right]Z_{p,l}H_{q,r}H_{n-p-q,k-l-r}\Bigg] \end{aligned} \tag{40}$$

Where $(a\cdot s)$ is the product of the sum of all Z momenta with the momentum in the internal Z boson, $(h\cdot s)$ the product with the sum of all Higgs momenta. The combination to look for here is $(h\cdot s)+(a\cdot s)=s^2$, with a special case at $k=0$, where $(h\cdot s)=0$ and $(a\cdot s)=s^2$.

Solving these equations to find and expression for $H_{n,k}$ and $Z_{n,k}$ is hard due to the summation over indices in the binomials and the factorial coefficients in $Z_{p,l}$,$H_{q,r}$. Is possible, although, to check if the proposed solutions fulfill the equations. Using Maple it has been checked up to 100 Z and 100 Higgs external bosons. Although it works up to a large number of external particles, it is better to have a complete proof of these formulas. Using generating functions it can be shown that the solution is complete for any number of external particles. First of all is needed a generating function for $Z_{n,k}$ and $H_{n,k}$. Introducing the proposed solution into a generating function:

$$
\begin{aligned}
Z(x,y) &= \sum_{n\geq 1}^{\infty}\sum_{k=0}^{\infty} Z_{n,k}\frac{x^n y^k}{n!k!}\theta(\text{odd n}) \\
&= \sum_{n\geq 1}^{\infty}\sum_{k=0}^{\infty} \frac{x^n}{n!}\frac{(-y)^k(n+k-1)!}{k!}\frac{(n-1)!}{(n-1)!}\theta(\text{odd n}) \\
&= \sum_{n\geq 1}^{\infty} \frac{x^n}{n!}\frac{(n-1)!}{(1+y)^n}\theta(\text{odd n}) \\
&= \sum_{n\geq 1}^{\infty} \frac{1}{n}\left(\frac{x}{1+y}\right)^n \theta(\text{odd n}) \\
&= \frac{1}{2}\left(-\log\big(1-\frac{x}{1+y}\big) + \log\big(1+\frac{x}{1+y}\big)\right) \\
&= \frac{1}{2}\log\left(\frac{1+y+x}{1+y-x}\right) \qquad (41)
\end{aligned}
$$

$$
\begin{aligned}
H(x,y) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} H_{n,k}\frac{x^n y^k}{n!k!}\theta(\text{even n}) = \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} (-1)^{k+1}\frac{(n-2+k)!(n-1)!(n-3)!}{2^{n-3}(\frac{n}{2}-1)!(\frac{n}{2}-2)!(n-2)!}\frac{x^n y^k}{n!k!}\theta(\text{even n}) \\
&= -\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} (-1)^{k}\frac{(2m-2+k)!(2m-1)!(2m-3)!x^{2m}y^k}{2^{2m-3}(m-1)!(m-2)!(2m-2)!(2m)!k!} \\
&= -\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(2m-2+k)!x^{2m}(-y)^k}{2^{2m-1}(m-1)!m!k!} \\
&= -\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{(2n+k)!x^{2n+2}(-y)^k}{2^n n!(n+1)!} \\
&= y - \sum_{n=0}^{\infty} \frac{(2n)!}{2^n n!(n+1)!}\frac{x^{2n+2}}{2^{n+1}(1+y)^{2n+1}} \\
&= y - (1+y)\left(1 - \sqrt{(1-\frac{x^2}{(1+y)^2})}\right) \\
&= -1 + \sqrt{(1+y)^2 - x^2} \qquad (42)
\end{aligned}
$$

In the latter, we have replaced n by $2m$ in the third line, m by $n+1$ in the fifth line, and the following series expression has been used:

$$
1 - \sqrt{1-2x} = \sum_{n\geq 0}^{\infty} \frac{(2n)!}{2^n n!(n+1)!}x^{n+1}
$$

Expanding the generating functions in series of x and y give the correct coefficients for each power of the variables. If these generating functions are indeed the correct solution they must fulfill the Schwinger-Dyson equation. The summations over partitions and number of particles will be translated into derivatives and integrals on the generating functions.

Now we will rewrite the Schwinger-Dyson equation with the generating functions found.

For the $(a \cdot s)$ terms in Z_n, k, their generating function $A_1(x,y)$:

$$
\begin{aligned}
A_1(x,y) &= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{x^n y^k}{n!k!} A_{n,k} = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{p=1}^{p=n}\sum_{l=0}^{p=k} \binom{n-1}{p-1}\binom{k}{l} \frac{x^n y^k}{n!k!} Z_{p,l} H_{n-p,k-l} \\
&= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{p=1}^{p=n}\sum_{l=0}^{p=k} \frac{x^n y^k}{n(p-1)!l!(n-p)!(k-l)!} Z_{p,l} H_{n-p,k-l} \\
\frac{\partial A_1(x,y)}{\partial x} &= \sum \frac{x^{n-p} y^{k-l} H_{n-p,k-l}}{(n-p)!(k-l)!} \frac{x^{p-1} y^l Z_{p,l}}{(p-1)!l!} = \sum_{p,l} \frac{x^{p-1} y^l Z_{p,l}}{(p-1)!l!} H(x,y) \\
A_1(x,y) &= \int_0^x H(x',y) \partial_{x'}(Z(x',y)) dx' \qquad (43)
\end{aligned}
$$

The same procedure can be made for all the terms in $Z_{n,k}$. Defining the indefinite integral over x (or antiderivative) as $\int^x dx' f(x') = D_x^{-1}(f(x))$, the generating functions for each term are:

$$
\begin{aligned}
\text{ZZH } (a \cdot s) \to A_1(x,y) &= D_x^{-1}\big(H(x,y)\partial_x(Z(x,y))\big) \\
\text{ZZH } (h \cdot s) \to B_1(x,y) &= D_y^{-1}\big(H(x,y)\partial_y(Z(x,y))\big) \\
\text{ZZHH } (a \cdot s) \to A_2(x,y) &= \frac{1}{2} D_x^{-1}\big(H^2(x,y)\partial_x(Z(x,y))\big) \\
\text{ZZHH } (h \cdot s) \to B_2(x,y) &= \frac{1}{2} D_y^{-1}\big(H^2(x,y)\partial_y(Z(x,y))\big) \qquad (44)
\end{aligned}
$$

$$
\begin{aligned}
Z(x,y) = x + \frac{-2}{s^2} \Big[& \big(A_1(x,y) + A_2(x,y)\big)(a \cdot s) \\
+ & \big(B_1(x,y) + B_2(x,y)\big)(h \cdot s)\Big] \\
Z(x,y) = x + \frac{-2}{s^2} \Big[& D_x^{-1}\Big(\big(\frac{1}{2}(H(x,y)+1)^2 - \frac{1}{2}\big)\partial_x(Z(x,y))\Big)(a \cdot s) \\
+ & D_y^{-1}\Big(\big(\frac{1}{2}(H(x,y)+1)^2 - \frac{1}{2}\big)\partial_y(Z(x,y))\Big)(h \cdot s)\Big] \qquad (45)
\end{aligned}
$$

Where the x term is the source term for $Z_{1,0}$. With $D_x^{-1}\big(\partial_x(Z(x,y))\big) = Z(x,y)$

and the combination

$$\frac{-2}{s^2}\Big[-\frac{1}{2}Z(x,y)(a\cdot s)-\frac{1}{2}Z(x,y)(h\cdot s)\Big]=Z(x,y)$$

the functions must fulfill:

$$\begin{aligned}0 &= \Big[-x+D_x^{-1}\big((1+H(x,y))^2\partial_x(Z(x,y))\big)\Big](a\cdot s)\\ &+ \Big[-x+D_y^{-1}\big((1+H(x,y))^2\partial_y(Z(x,y))\big)\Big](h\cdot s)\end{aligned} \tag{46}$$

Each term $(a\cdot s)$ and $(h\cdot s)$ has to vanish by itself, except for the coefficients for x^1y^0 and x^1y^1. Renaming $(1+H(x,y))\equiv\omega(x,y)$ the partial derivatives of Z are: $\partial_x(Z(x,y))=\frac{(y+1)}{\omega^2}$ and $\partial_y(Z(x,y))=\frac{-x}{\omega^2}$:

$$\begin{aligned}D_x^{-1}\big(\frac{1+y}{\omega^2}\omega^2\big) &= x+yx\\ D_y^{-1}\big(\frac{-x}{\omega^2}\omega^2\big) &= -xy\end{aligned} \tag{47}$$

The only terms left represent the special cases mentioned. The term xy indicates that one Z and one Higgs boson are emitted. In that case $(a\cdot s)-(h\cdot s)=(a-h\cdot a+h)=a^2+h^2=m_z^2+m_h^2=0$. For the term x, only one Z is being emitted, therefore $(a\cdot s)=a^2=m_z^2=0$, $(h\cdot s)=0$.

The same procedure can be applied to the Schwinger-Dyson equation of $H(x,y)$. The generating functions for each of the terms are:

$$\begin{aligned}\text{ZZH } a^2\to AA_1(x,y) &= \frac{1}{2}D_x^{-2}\big((\partial_x Z(x,y))^2\big)\\ \text{ZZH } h^2\to BB_1(x,y) &= \frac{1}{2}D_y^{-2}\big((\partial_y Z(x,y))^2\big)\\ \text{ZZH } 2(a\cdot h)\to AB_1(x,y) &= \frac{1}{2}D_x^{-1}\big(D_y^{-1}\big(\partial_x(Z(x,y))\partial_y(Z(x,y))\big)\big)\\ \text{ZZHH } a^2\to AA_2(x,y) &= \frac{1}{2}D_x^{-2}\big((\partial_x Z(x,y))^2 H(x,y)\big)\\ \text{ZZHH } h^2\to BB_2(x,y) &= \frac{1}{2}D_y^{-2}\big((\partial_y Z(x,y))^2 H(x,y)\big)\\ \text{ZZHH } 2(a\cdot h)\to AB_2(x,y) &= \frac{1}{2}D_x^{-1}\big(D_y^{-1}\big(\partial_x(Z(x,y))\partial_y(Z(x,y))H(x,y)\big)\big)\end{aligned} \tag{48}$$

$$
\begin{aligned}
H(x,y) = y + \frac{-2}{s^2} \quad & \Bigg[\quad \Big(AA_1(x,y) + AA_2(x,y) \Big) a^2 \\
& + \quad \Big(BB_1(x,y) + BB_2(x,y) \Big) h^2 \\
& + \quad \Big(AB_1(x,y) + AB_2(x,y) \Big) 2(a \cdot h) \Bigg] \\
H(x,y) = y + \frac{-2}{s^2} \quad & \Bigg[\quad D_x^{-2}\big(\ (\partial_x Z(x,y))^2 (1 + H(x,y)) \big) \ a^2 \\
& + \quad D_y^{-2}\big(\ (\partial_y Z(x,y))^2 (1 + H(x,y)) \big) \ h^2 \\
& + \quad D_x^{-1}\big(D_y^{-1}\big(\ \partial_x (Z(x,y)) \partial_y (Z(x,y)) (1 + H(x,y)) \big) \big) \ 2(a \cdot h) \Bigg]
\end{aligned} \tag{49}
$$

Introducing $H(x,y)s^2 = H(x,y)[a^2 + h^2 + 2(a \cdot h)]$:

$$
\begin{aligned}
0 = \quad & \Big[\quad H(x,y) - y + D_x^{-2}\big(\ (\partial_x Z(x,y))^2 (1 + H(x,y)) \big) \Big] a^2 \\
+ \quad & \Big[\quad H(x,y) - y + D_y^{-2}\big(\ (\partial_y Z(x,y))^2 (1 + H(x,y)) \big) \Big] h^2 \\
+ \quad & \Big[\quad H(x,y) - y + D_x^{-1}\big(D_y^{-1}\big(\ \partial_x (Z(x,y)) \partial_y (Z(x,y)) \big) \big) \Big] \ 2(a \cdot h)
\end{aligned} \tag{50}
$$

With the result of the following integrals:

$$
\begin{aligned}
D_x^{-2}\big(\frac{(1+y)^2}{\omega^3} \big) \quad &= \quad -\omega \\
D_y^{-2}\big(\frac{x^2}{\omega^3} \big) \quad &= \quad -\omega \\
D_x^{-1}\big(D_y^{-1}\big(\frac{(1+y)x}{\omega} \big) \big) \quad &= \quad -\omega
\end{aligned} \tag{51}
$$

Each term vanishes except for $-y-1$ which correspond to the cases with only one Higgs being emitted, in which case $a^2 = 0$, $h^2 = m_h^2 = 0$, $(a \cdot h) = 0$, and no particles emitted, $a^2 = h^2 = (a \cdot h) = 0$.

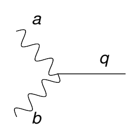
The proposed solutions, written in terms of their generating functions, fulfill the Schwinger-Dyson equation. Since the solution of these equations must be unique in this model with these interactions and boundary conditions, then the solution found is the only one possible.

Hitherto, it has been shown that all the terms in the matrix element that behave with E^2 will vanish for any process with any number of external Z and Higgs bosons. Furthermore, a general expression to find the coefficients for an internal Z or Higgs boson going to **n** Z and **k** H has been found in all generality, expressed in terms of a generating function. The next goal is to use these results to prove that, in the massive case now, the E^0 terms will also vanish.

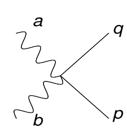
4 $\mathbf{E}^0$ terms

These are the second most dangerous terms that must vanish in order for the model to be well-behaved. To calculate these terms in the unitary gauge, the bosons must be considered as massive particles again, and the full propagator has to be taken into account. In a process with more than 4 external particles, these terms must vanish. They can be proportional to m_z^2 or m_h^2, and they do not mix together. Higgs self-interactions are now allowed (one at most), and they will contribute directly to the m_h^2 terms. It is important to understand and see where all the contributions to this terms come from and to be able to predict them. Each case will be taken separately into account. To easily see these contributions, a small change in the Feynman rules is made: all the momentum dependence is taken into account in the vertices directly. The new vertices now read:

- ZZH: $i(-a^2 - b^2 + q^2)$



- ZZHH: $i(-a^2 - b^2 + (q+p)^2)$

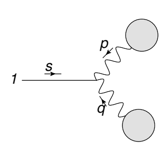


This rules do not depend on whether the particles are internal or external. External particles will have a m_i^2 in the first vertex, and propagators do not need to be expanded around $m_i^2 \simeq 0$.

4.1 m_h^2 terms

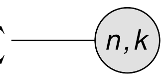
To be able to predict the cancellations of this terms proportionals to m_h^2, $[E^0]$, is important to see where all the contributions will come from. When there is a process with external Higgs, the $\propto E^2$ terms can have a contribution to m_h^2 since it might happen that they are proportional to $(1_i)^2 = m_h^2$. Also, as previously stated, any diagram with one Higgs self-interaction will also contribute to this terms. Finally, the first order expansion of the Higgs propagator when the internal momentum is much higher than its mass must be taken into account also.

With the new Feynman rules, an external Higgs will be a m_h^2 in the first vertex that is encountered.



$$= i2(m_h^2 - p^2 - q^2)$$

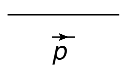
When considering the E^2 terms, it was shown that the sum of all the diagrams will be proportional to the momentum going through the propagator (a Higgs in this case). If this internal particle is instead an external one, the momentum squared will be m_h^2. Therefore the coefficient found for $H_{n,k}$ will be the contribution to m_h^2 for that particle. Each external Higgs will contribute to this. Since the coefficients include the imaginary unit from the propagator, a minus sign takes it into account. This are the first contributions to m_h^2:

$$\sum_k$$ 

$$= -ikm_h^2 H_{n,k}$$

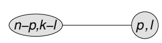
.

From the expansion of the propagator we can see that:



$$= \frac{i}{p^2 - m_h^2} \simeq \frac{i}{p^2}(1 + \frac{m_h^2}{p^2})$$

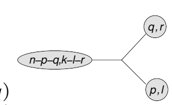
The first order terms in the expansion have a m_h^2. With the same procedure, the sum of all the possible diagrams to the right-hand side of the propagator will be proportional to p^2, as well as the ones in the left side. Then the product of those coefficients times the first order term in the propagator expansion will be proportional to m_h^2 and the product of the coefficients, with a minus sign to take out one imaginary unit. This must be summed over the number of particles in each side and over the possible partitions, and taking symmetry factors into account:

$$\sum_{p,l} \sum_{\substack{a_p \subset A(p) \\ h_l \subset H(l)}}$$ 

$$= \sum_{p,l} \sum_{\substack{a_p \subset A(p) \\ h_l \subset H(l)}} \left[-iH_{p,l} H_{n-p,k-l} \right]$$

Note that the previous m_h^2 contributions from an external Higgs could also be taken into account here.

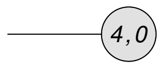
Finally, the last contribution is from the diagrams with Higgs self-interactions. Since this vertex reduces the energy order by 2, this diagrams will be just the E^2 coefficients multiplied by the vertex (with the sum over momentum in each blob and partitions, as before):

$$\sum_{p,l} \sum_{q,r} \sum_{\substack{a_p \subset A(p) \\ h_l \subset H(l)}} \sum_{\substack{a_q \subset A(q) \\ h_r \subset H(r)}}$$ 

$$=$$

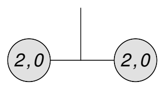
$$= \quad \sum_{p,l} \sum_{q,r} \sum_{\substack{a_p \subset A(p) \\ h_l \subset H(l)}} \sum_{\substack{a_q \subset A(q) \\ h_r \subset H(r)}} \left[-i3m_h^2 H_{p,l} H_{q,r} H_{n-p-q,k-r-l} \right]$$

Equivalently for the four Higgs self-interaction. Those are all the contributions to the $m_h^2[E^0]$ terms. Therefore their contribution must cancel. For the processes in which this has been seen, 4 **Z**, 1 **H**, applying now this rules:



$$= \quad -i(H_{4,0})m_h^2 = i3m_h^2$$

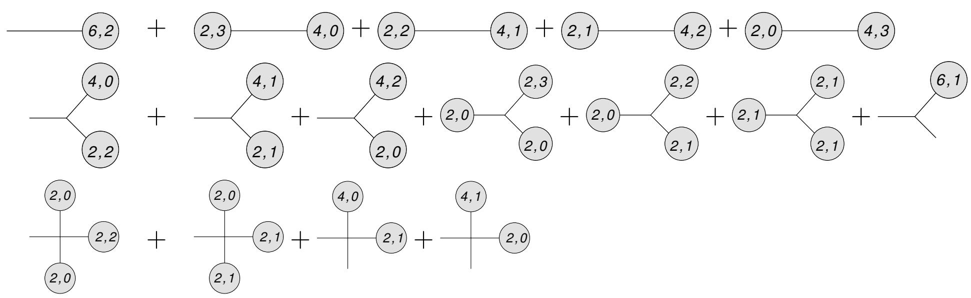
$$= \quad -im_h^2 \binom{4}{2} H_{2,0}H_{2,1} = i6m_h^2$$



$$= \quad -i3m_h^2 \frac{1}{2}\binom{4}{2} H_{2,0}H_{2,0} = -i9m_h^2 \tag{52}$$

The binomial coefficients express the number of ways particles can be chosen, the $\frac{1}{2}$ on the Higgs self-interaction is the symmetry factor for the two equal blobs. All the contributions cancel, meaning there will not be any left term with m_h^2.

For a more complicated process this still holds. For 6 **Z**, 3 **H**, for example:



This are all the possible diagrams that contribute to m_h^2. In numbers this becomes:

$$i9450m_h^2 - i12825m_h^2 + i3375m_h^2 = 0$$

The first term results from the contributions from the Higgs propagator (as well as external Higgs), the second from the diagrams with three point Higgs self-interaction, and the last from diagrams with four point Higgs self-interaction. The total contribution vanishes as expected.

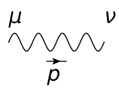
Being able to calculate all the contributions to m_h^2 allows us to observe explicitly and for any process that they cancel.

4.2 m_z^2 terms

The last terms with energy dependence $[E^0]$ are those proportional to m_z^2. At first glance, this kind of term has many sources: the contributions from the E^2, expansion on the $L^{\mu\nu}$ and the $T^{\mu\nu}$ part of the Z propagator and the t^μ vector from the longitudinal polarization.

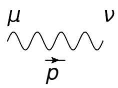
As in the m_h^2 case, it is known that the sum of all the possible diagrams on one side of a internal Z will be proportional to the momentum going through the propagator, in the massless limit. With this in mind, an expansion on the

Z propagator will be:



$$
\begin{aligned}
&= \frac{-i}{m_z^2}\left(\frac{m_z^2 g^{\mu\nu} - p^\mu p^\nu}{p^2 - m_z^2}\right) \simeq \frac{-i}{m_z^2}\left(\frac{m_z^2 g^{\mu\nu} - p^\mu p^\nu}{p^2}\right)(1 + \frac{m_z^2}{p^2}) \\
&\simeq \frac{-i}{m_z^2}\left(m_z^2 \frac{g^{\mu\nu}}{p^2} - \frac{p^\mu p^\nu}{p^2} - m_z^2 \frac{p^\mu p^\nu}{p^4} + \mathcal{O}(m_z^4)\right) \qquad (53)
\end{aligned}
$$

Contracting the propagator with something proportional to the momentum going through it from both sides:

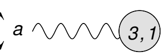


$$
= \frac{-i}{m_z^2}\left(-\frac{p^\mu p^\nu p_\mu p_\nu}{p^2} + m_z^2 \frac{g^{\mu\nu} p_\mu p_\nu}{p^2} - m_z^2 \frac{p^\mu p^\nu p_\mu p_\nu}{p^4}\right) = \frac{i}{m_z^2} p^2 \qquad (54)
$$

The contributions from the $g^{\mu\nu}$ part of the propagator and the expansion at first order of the $p^\mu p^\nu$ cancel exactly, and the propagator used to calculate the E^2 terms is recovered. Therefore the propagator can be simply taken as in the massless case.

So the sources of terms m_z^2 are reduced. Using now the last-introduced Feynman rule for the vertex, each external Z will have a m_z^2 at the first vertex. Therefore the contributions from the E^2 terms will be just those from an external Z going to the rest of particles, which is exactly the coefficients $Z_{n-1,k}$. For 4 **Z** 1 **H**, this is:

$$
\sum_a
$$



$$
= -im_z^2 4(-6) = -i24m_z^2 \qquad (55)
$$

The last contribution to this terms come from the t^μ vector in the longitudinal polarization. In the same way as before, for an internal Z, the sum of all the possible diagrams will be proportional to the momentum going through it. If instead of internal it is now an external Z in longitudinal polarization, the t^μ vector part: $-m_z^2 \frac{t^\mu}{(a \cdot t)} a_\mu Z_{p,l} = m_z^2 Z_{p,l}$. The contributions from the t^μ vector are exactly the opposite to those m_z^2 terms from the external Z. Since those are the only sources for this terms, all m_z^2 terms are automatically canceled.

5 On-shell recursion relations

The On-shell recursion relations ([12], [13]) are a method by which the amplitude of a process can be rewritten in terms of the amplitudes of smaller processes using partial fractioning. The goal in this thesis is not to use this method to calculate amplitudes, but to show that it is possible to apply this construction to the Abelian Higgs model and that the behavior of any process can be inferred from the behavior of fewer particles processes.

In the mentioned previous work ([12], [13]), this method is based on applying a complex deformation to the momenta such that, preserving the on-shell conditions of the external particles, an internal particle is modified to be on the mass shell, thereby splitting the diagram into two process with fewer external particles. This is performed in a Yang-Mills theory with massless particles, where the deformation is fulfilled by spinors.

For massive particles, as in this dissertation, the deformation is not so straightforward. Another solution is proposed here. Since there is no need to define any spinor, the phase space of the momenta can be extended to more dimensions, in which the deformation vectors exist[2]. The deformation of the momenta, adding four dimensions with metric signature $(+,-,-,-,-,-,-,+)$, will have the following form:

$$\begin{aligned} p_i^\mu &\rightarrow \widetilde{p}_i^\mu = p_i^\mu + z^{1/2}\eta_i^\mu \\ p_i^\mu &= (p_i^0, p_i^1, p_i^2, p_i^3) \\ &\rightarrow (p_i^0, p_i^1, p_i^2, p_i^3,\ 0,\ 0,\ 0,\ 0) \\ \eta_i^\mu &= (\ 0,\ 0,\ 0,\ 0, \eta_i^1, \eta_i^2, \eta_i^3, \eta_i^4) \end{aligned} \tag{56}$$

Where z can be a complex number and η_i^μ are massless with the four first components (corresponding to the usual 4-dimensions) set to zero, which means that any product between momentum (or polarization) and η_j^μ is zero ($(p_i \cdot \eta_j) = 0$). With the addition of new dimensions, new polarization vectors arise. Those are not a problem because there is still only one longitudinal polarization vector that depends on the energy. The rest are transversal, which leads to good energy behavior. The other properties of the deformation can be obtained preserving on-shell conditions for the external momenta:

$$\begin{aligned} \widetilde{p}_i^2 &= (p_i + z^{1/2}\eta_i)^2 = m_i^2 + z\eta_i^2 = m_i^2 \\ &\implies \eta_i^2 = 0 \end{aligned} \tag{57}$$

With the modified external momenta, the matrix element is now a function of z, $\mathcal{M}(z)$, and the original amplitude is recovered at z=0, $\mathcal{M}(z=0)$. It is possible then to write the original amplitude as the closed integral in the complex plane of $\frac{M(z)}{z}$ around the point $z \sim 0$ using Cauchy's residue theorem:

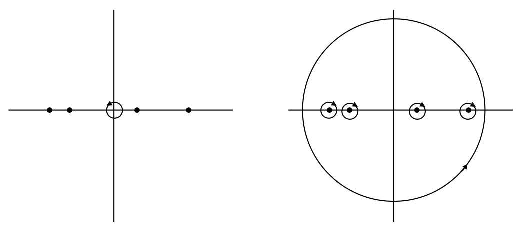
$$M(z=0) = \frac{1}{2\pi\imath} \oint_{z\sim 0} \frac{M(z)}{z} dz \tag{58}$$

Now deform the contour to surround all other poles in $M(z)$. For each propagator in the diagrams with a different combinations of external momenta going through, there will be a pole at some value of z. Defining Δ_s as the original propagator for the subset of external momenta s ($\Delta_s = p_s^2 - m_i^2$, $p_s = \sum_{i\in s} p_i$, m_i the mass of the propagator particle), the pole will be at z_s:

[2] At least 3 extra dimensions are needed, but any larger number is also acceptable.

$$
\begin{aligned}
\widetilde{\Delta}_s &= (p_s + z^{\frac{1}{2}}\eta_s)^2 - m_i^2 \\
&= p_s^2 + z\eta_s^2 - m_i^2 = \Delta_s + z\eta_s^2 \\
z_s &= -\frac{\Delta_s}{\eta_s^2}
\end{aligned}
\tag{59}
$$

Now the integral can be rewritten in terms of the residues at those poles and the contribution in the limit $z \to \infty$.



(a) Integral contour around $z = 0$.

(b) New integral contour around $M(z)$ poles at $z = z_s$.

Figure 4: Original contour around $z = 0$ and new deformed contour around all the other poles z_s and $z \to \infty$ contribution

$$
\begin{aligned}
M &= -\frac{1}{2\pi\imath}\oint_{z\sim z_s}\frac{dz}{z}\sum_{(s)}\frac{M_L(\widetilde{p}_s)M_R(\widetilde{p}_s)}{\Delta_s + z\eta^2} \\
M &= \sum_s M_L(z_s)\frac{i}{\Delta_s}M_R(z_s) + B
\end{aligned}
\tag{60}
$$

Where the sum is over all poles, and B is the residue of $\frac{M(z)}{z}$ at $z \to \infty$. M_L and M_R are now sub-amplitudes with fewer particles. For a process with **m** external particles, the maximum energy dependence to be well-behaved is $[E^{4-m}]$. Once the diagram has been split in two, with **p** particles in one side and **q** on the other ($p + q = m$), the energy dependence of each of the terms will be:

$$
[E^{4-m}] = [E^{4-(p+1)}][E^{-2}][E^{4-(q+1)}] \tag{61}
$$

Since the propagator is on-shell, each sub-amplitude has one more external particle, and the propagator diminishes by two the energy dependence. If this transformation is possible, then a process will be well behaved if the sub-amplitudes on which it depends are well behaved and the contour contribution (B) goes to zero in the $z \to \infty$ limit. The z dependence of the latter will be discussed below.

The η_i^μ vectors for the deformation can be created using a RAMBO-like algorithm [14], in which massless particles with momentum conservation can be generated, which are exactly the conditions η_i^μ vectors must fulfill. We use the following algorithm to create n deformation vectors η_i^μ (ρ_i is a random number):

$$
\begin{aligned}
&j{=}3..n;\\
&\eta_j^\mu = (\rho_1, \rho_2, \rho_3, \sqrt{\rho_1^2+\rho_2^2+\rho_3^2});\\
&q^\mu = -\textstyle\sum_i p_i^\mu;\\
&\vec{\eta}_1 = \tfrac{q^4+|\vec{q}|}{2|\vec{q}|}\vec{q}\,, \qquad \eta_1^4 = -\sqrt{\vec{\eta}_1^{\,2}}\ ;\\
&\vec{\eta}_2 = \tfrac{-q^4+|\vec{q}|}{2|\vec{q}|}\vec{q}\,, \quad\ \eta_2^4 = -\sqrt{\vec{\eta}_2^{\,2}};
\end{aligned}
$$

This vectors η_i are the 5th to 8th components of our 8-dimensional deformation vectors, and fulfill all the required conditions.

To see the behavior in the limit $z \to \infty$, a simple z power-counting can be performed. External particles do not have any z contribution (the external polarizations of the Z bosons are not deformed). All the z dependence must be in the internal propagators. Z propagators have a linear dependence on z both in the numerator and denominator, and therefore become constant as $z \to \infty$. Finally, Higgs propagator have a z^{-1} behavior. Any process with one internal Higgs will have a z^{-1} dependence, and will not contribute to the contour. More Higgs propagators will induce a better z-behavior to the process. The only processes left to check are those with no Higgs propagator, which also means that they only have two external Z bosons at most. These processes will have a product $(\epsilon_i \cdot \widetilde{p}_i + \widetilde{p}_j)$ at the first propagator encountered for each Z boson. With the product $z(\epsilon_i \cdot \eta_j) = 0$, one power of z is lost in these products, making these diagrams also behave as z^{-1}.

It has been seen that any possible process can be rewritten in lower-order processes after applying this deformation. Therefore, the energy behavior can be inferred from lower order, known processes using partial fractioning, and that the contour contribution at the limit $z \to \infty$ will be zero for any diagram.

Using Maple, a deformation for the process 4Z, 1H has been applied. The massless deformation vectors η_i^μ have been found for the 5 external particles, preserving on-shell conditions and momentum conservation. Then, it has been applied to the original Matrix Element in order to see the behavior $\lim\limits_{z\to\infty} M(z)$.

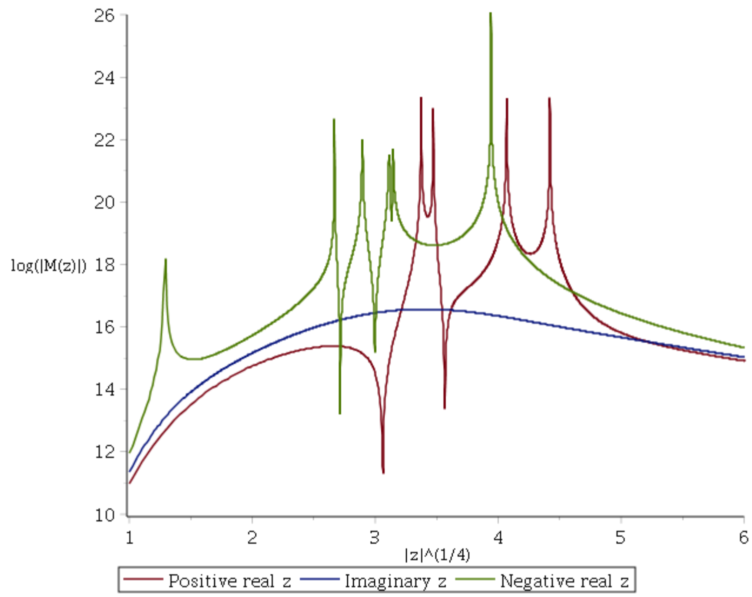


Figure 5: Logarithm of the absolute value of M(z) with $z^{1/4}$ values on the real axis

The matrix element has 10 poles which can be seen in figure 5. The 6 spiked maximums of the function M(z) in the green line mark the location of the poles on the negative part of the real axis, while the 4 red spikes are on the positive real axis. All the poles, as expected, are in the real axis of z. The behavior along the imaginary axis (blue line) shows that there are no poles there, and all three lines converge for large $|z|$.

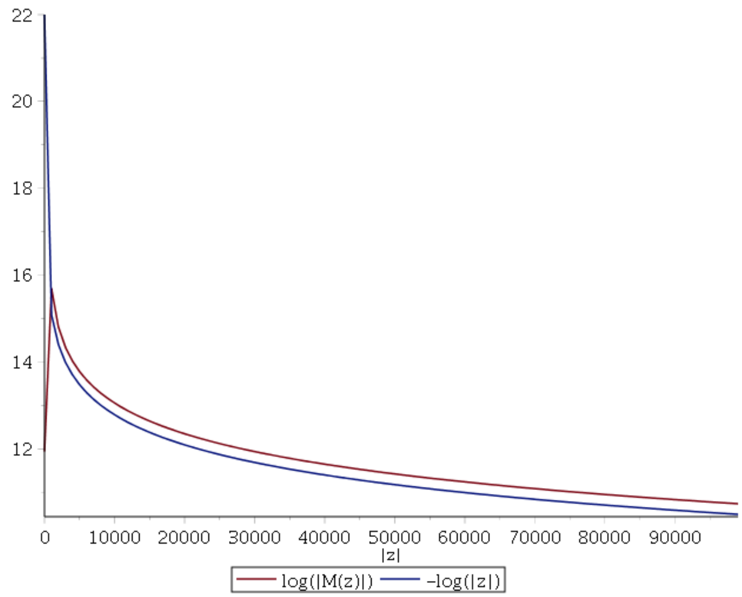


Figure 6: Logarithm of the absolute value of M(z) for large $|z|$, and logarithmic decay

Finally, in figure 6 we can see how $M(z)$ decreases as z^{-1} for large z values. The red line depicts $\log(|M(z)|)$ (it does not depend on the complex phase of z), while the blue line shows $-\log(z)$ with an arbitrary offset added.

6 Summary

With the initial objective of proving that the closed subgroup of the Standard Model including Z and Higgs bosons at tree level is well-behaved, an exhaustive study of the model has been done. The first realization is that this group corresponds to the representation of the Abelian Higgs Model after symmetry breaking, but it can also be inferred by ensuring unitarity in the Standard Model. Secondly, it has been shown that the matrix element for any Feynman diagram in this model has a maximum energy dependence $[E^2]$ and that for a process with **m** external particles, the remaining terms in the total matrix element can at most have an energy dependence of $[E^{4-m}]$.

Once the expected cancellations are known, a direct calculation for simple 5 particles processes (4**Z**, 1**H** and 2**Z**, 3**H**) was performed. The summation over all the possible diagrams that the interactions in the model allow has shown that cancellations between the diagrams are indeed occurring up to (but not further than) the expected energy behavior, with the vanishing of all elements with energy dependence $[E^2]$ and $[E^0]$, and all the left terms being at most

$\mathcal{O}(E^{-1})$[3].

Instead, and since all processes depend on energy as $[E^2]$ at most, a massless approximation in the unitary gauge is taken to work out these terms. The sum of all possible diagrams in this approximation shows that these terms indeed vanish if the particle going into the SDE is external, and also introduces a way of calculating the coefficients that this sums led to.

$$\begin{aligned} Z_{n,k} &= \begin{cases} 0 & \text{, for even n} \\ (-1)^k (n-1+k)! & \text{, for odd n} \end{cases} \\ H_{n,k} &= \begin{cases} (-1)^{k-1}(n-2+k)!\beta_{(\frac{n}{2}-1)} & \text{, for even n} \\ 1 & \text{, for n=0 , k=1} \\ 0 & \text{, for odd n and n=0 , k}\neq 1 \end{cases} \\ \beta_a &= \frac{(2a+1)!!(2a-1)!!}{(2a)!} = \begin{cases} \frac{(2a+1)!(2a-1)!}{2^{2a-1}a!(a-1)!(2a)!} & a \geq 1 \\ 1 & a = 0 \end{cases} \end{aligned} \tag{62}$$

Generating functions for the proposed solutions in equation (62) of the coefficients of a particle H or Z going into **n** Z, **k** H have been found. The recursivity of the Schwinger-Dyson equation only allows for one possible solution that fits given initial conditions, and therefore proved that these generating functions are the only possible solution for the model. These generating functions are:

$$\begin{aligned} Z(x,y) &= \frac{1}{2} log\left(\frac{1+y+x}{1+y-x}\right) \\ H(x,y) &= -1+\sqrt{(1+y)^2-x^2} \end{aligned} \tag{63}$$

These coefficients allow to calculate all the contributions to the second order terms in energy E^0 for any process in the massive case, identifying where this terms come from: m_z^2 from external Z particles, and m_h^2 from external and internal Higgs.

Finally, on-shell recursion relations have been used, showing that it is possible to move to a higher dimensional phase space and deform the momenta such that an internal particle is on-shell, and the amplitude can be rewritten as the product of sub-amplitudes of the process.

$$M = \sum_s M_L(z_s)\frac{i}{\Delta_s}M_R(z_s) + B \tag{64}$$

The energy dependence of each of the terms shows that if the sub-amplitudes are well-behaved and $\lim_{z\to\infty} M(z) = 0$, consequently, the total amplitude is also well behaved. Thus, having a starting set of well-behaved processes, as the

[3]Using handlebar diagrammatics, as in the Scalar QED model in [[15]], and applying it to the Schwinger-Dyson equation did not prove to be an effective method for massive particles in the unitary gauge.

ones calculated in Appendix A, any other process with more external particles involved can be rewritten as products of well-behaved, known processes, and that the contour contribution will be zero. It has been shown for a process that the result using this expression is well-behaved and the behavior of M(z) in the limit $z \to \infty$ do converge to zero as z^{-1} as expected.

Acknowledgement

I would like to offer my special thanks to my supervisor, Ronald Kleiss, for all his help and amazing insight into the physics discussed here. I would also like to express my appreciation to all the HEP department for the great work environment present.

A Appendix

In this appendix, a complete calculation of all the terms that must vanish for two processes with 5 external particles is performed explicitly.

A.1 4Z, 1H

For 4**Z**, 1**H**, there are 21 possible diagrams. In this case, the Z bosons labeled a and b will be taken as incoming particles, while the others are outgoing. The momentum conservation in this process $ZZ \to ZZH$ will then be $a^\mu + b^\mu = c^\mu + d^\mu + 1^\mu$. The 21 diagrams can be divided in three types: one with only ZZH vertices, one with a ZZHH vertex and finally one with Higgs self-interactions. The Mandelstam variables for this process are:

$$
\begin{aligned}
s &= (a+b)^2 = (d+c+1)^2 & s' &= (c+d)^2 = (a+b-1)^2 \\
u &= (a-c)^2 = (b-d-1)^2 & u' &= (b-d)^2 = (a-c-1)^2 \\
t &= (a-d)^2 = (b-c-1)^2 & t' &= (b-c)^2 = (a-d-1)^2
\end{aligned}
$$

The first 12 diagrams with only ZZH vertices have the following form:

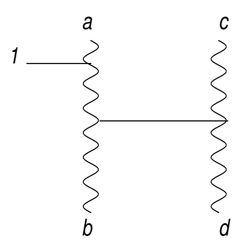


Figure 7: Diagram with only ZZH vertices

Using the Feynman rules, the matrix element corresponding to this diagram is:

$$M_1 = i8g^3 m_z^6 \frac{-(\epsilon_a \cdot \epsilon_b) + \frac{1}{m_z^2}(\epsilon_a \cdot (a-1))(\epsilon_b \cdot (a-1))}{(a-1)^2 - m_z^2} \frac{(\epsilon_c \cdot \epsilon_d)}{(s' - m_h^2)} \tag{65}$$

The other 11 diagrams with this form have the same formula with interchanged momentums.

$$\begin{aligned} M_2 =& M_1\big|_{a\leftrightarrow b} & M_8 =& M_4\big|_{b\leftrightarrow -c} \\ M_3 =& M_1\big|_{\substack{a\leftrightarrow -c\\ b\leftrightarrow -d}} & M_9 =& M_1\big|_{b\leftrightarrow -d} \\ M_4 =& M_1\big|_{\substack{a\leftrightarrow -d\\ b\leftrightarrow -c}} & M_{10} =& M_4\big|_{b\leftrightarrow -d} \\ M_5 =& M_1\big|_{b\leftrightarrow -c} & M_{11} =& M_2\big|_{a\leftrightarrow -c} \\ M_6 =& M_3\big|_{a\leftrightarrow -d} & M_{12} =& M_3\big|_{b\leftrightarrow -d} \\ M_7 =& M_2\big|_{a\leftrightarrow -d} & & \end{aligned}$$

Doing an expansion on the propagator at first order and setting the polarization as longitudinal, the matrix element M_1 becomes, taking into account only the relevant terms E^2 and E^0 and defining $\alpha_i \equiv (i\cdot 1)$ and $\beta_i \equiv (i\cdot t)$ (i is any external momentum):

$$\begin{aligned} M_1 &=& ig^3\left(1+\frac{m_h^2}{2\alpha_a}\right)\left(1+\frac{m_h^2}{s'}\right)\left(s-2\alpha_b\right) \\ &+& ig^3 m_z^2\left(\frac{s}{\alpha_a}-2\frac{s}{s'}-2+4\frac{\alpha_b}{s'}\right) \\ &+& ig^3 m_z^2\Bigg(-\frac{s\beta_1}{\alpha_a\beta_a}+2\frac{\alpha_b\beta_1}{\alpha_a\beta_a}-2\frac{\beta_a}{\beta_b}+2\frac{\beta_1}{\beta_b} \\ && \quad -2\frac{s}{s'}\left(\frac{\beta_c}{\beta_d}+\frac{\beta_d}{\beta_c}\right)+4\frac{\alpha_b}{s'}\left(\frac{\beta_c}{\beta_d}+\frac{\beta_d}{\beta_c}\right)\Bigg) \end{aligned} \qquad (66)$$

In M_1 is possible to see that the m_h^2 comes from the E^2 terms with the expansion on the Higgs propagator.

The next type of diagram are those with one ZZHH vertex. There are 6 diagrams, all related by label exchange. With the Feynman rules, the expression for this diagrams is:

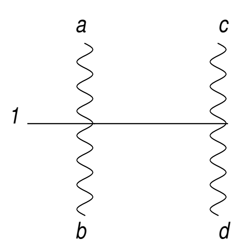


Figure 8: Diagram with one ZZHH vertex

$$M_{13} = -i4g^3 m_z^4\frac{(\epsilon_a\cdot\epsilon_b)(\epsilon_c\cdot\epsilon_d)}{s'-m_h^2} \qquad (67)$$

The other 5 diagrams come from the following transformations:

$$M_{14} = M_{13}\big|_{\substack{a\leftrightarrow -c\\ b\leftrightarrow -d}} \qquad M_{17} = M_{13}\big|_{b\leftrightarrow -d}$$
$$M_{15} = M_{13}\big|_{b\leftrightarrow -c} \qquad M_{18} = M_{13}\big|_{a\leftrightarrow -c}$$
$$M_{16} = M_{13}\big|_{a\leftrightarrow -d}$$

Performing the same operations as in M_1, the relevant terms in M_{13} are:

$$\begin{aligned} M_{13} &= ig^3\left(1+\frac{m_h^2}{s'}\right)(-s) \\ &+ ig^3 m_z^2\left(2+2\frac{s}{s'}\right) \\ &+ ig^3 m_z^2\left(2\frac{s}{s'}\left(\frac{\beta_c}{\beta_d}+\frac{\beta_d}{\beta_c}\right)+2\left(\frac{\beta_a}{\beta_b}+\frac{\beta_b}{\beta_a}\right)\right) \end{aligned} \quad (68)$$

Finally, the last three remaining diagrams that have a three point Higgs self-interaction, and therefore have an energy dependence E^0, have the following contribution:

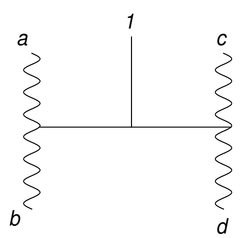


Figure 9: Diagram with one three point Higgs self-interaction

$$M_{19} = -i12g^3 m_z^4 m_h^2 \frac{(\epsilon_a\cdot\epsilon_b)}{s-m_h^2}\frac{(\epsilon_c\cdot\epsilon_d)}{s'-m_h^2} \quad (69)$$

$$M_{19} = -i3g^3 m_h^2 \quad (70)$$

$$M_{20} = M_{21} = M_{19}$$

Adding up all the relevant terms found, it is easy to use momentum conservation in the process to cancel the denominators and find just a coefficient for m_z^2 and m_h^2. These coefficients can be found in the table 3, where the terms from each matrix element with no Higgs self-interaction have been divided into those coming from the E^2 terms, those with m_h^2, $m_z^2 E^0$ and finally those with β_i.

E^2	$8ig^3m_z^2$
m_h^2	$i9g^3m_h^2$
$m_z^2E^0$	$-i16g^3m_z^2$
β_i	$i8g^3m_z^2$
$M_{19+20+21}$	$-i9g^2m_h^2$

Table 3: Coefficients taking into account all the Feynman diagrams for each kind of term present in the matrix element

The sum of all those contributions is indeed zero. The symmetry in the diagrams and momentum conservation work together so those terms that are dangerous disappear and the theory is well-behaved and protected at energies much higher than the masses of the particles involved.

In the case where not all Z bosons are in the longitudinal polarization, the energy dependence of the diagrams diminishes by one for each transversal Z. Only the 18 diagrams with the highest energy behavior need to be taken into account, since those that had an E^0 energy dependence will now be $\mathcal{O}(E^{-1})$.

With one Z in the transversal polarization, the matrix element are:

$$\begin{aligned} M_1 &= -i8g^3m_z\frac{(\epsilon_a\cdot 1)(b\cdot(a-1))}{(a-1)^2-m_z^2}\frac{(c\cdot d)}{s'-m_h^2} \\ M_{13} &= -i4g^3m_z\frac{(\epsilon_a\cdot b)(c\cdot d)}{s} \end{aligned} \tag{71}$$

The sum of all the diagrams is:

$$M_{transversalZ} = i2g^3m_z(\epsilon_a\cdot a) = 0 \tag{72}$$

The sum of all the diagrams when considering a Z in the transversal polarization is proportional to it's momentum. This realization is important for this study and is used on the derivation of a more general proof. With two transversal polarizations, the matrix element of order $[E^0]$ also vanishes; for each denominator (here particles a and b are in the transversal polarization):

$$i4\frac{(\epsilon_b\cdot d)}{(bd)^2}\Big[-(\epsilon_a\cdot 1)+(\epsilon_a\cdot(c+1))-(\epsilon_a\cdot c)\Big] = 0 \tag{73}$$

Having three Z bosons in the transversal polarization modifies the higher energy dependence terms to be $[E^{-1}]$, and therefore automatically well-behaved.

A.2 2Z, 3H

The second process for which the full calculation of the relevant terms has been done is $ZZ \to HHH$. As the one before, this process needs to cancel those terms with energy dependence E^0 or higher. There are 25 diagrams that contribute to the total matrix element, and from those, 12 have contributions to E^2, 10 have one Higgs self-interaction, and the other 3 left have an energy behavior of order $\mathcal{O}(E^{-2})$. The different matrix elements are:

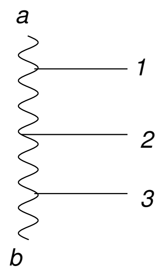


Figure 10: Diagram with no Higgs self-interaction and only ZZH vertices

$$\begin{aligned}
M_1 &= i8g^3m_z^6\frac{(\epsilon_a\cdot\epsilon_b)-\frac{1}{m_z^2}\Big((\epsilon_a\cdot(a-1))((a-1)\cdot\epsilon_b)+(\epsilon_a\cdot(b-3))((b-3)\cdot\epsilon_b)\Big)}{[(a-1)^2-m_z^2][(b-3)^2-m_z^2]}\\
&+ i8g^3m_z^2\frac{\Big((\epsilon_a\cdot(a-1))((a-1)\cdot(b-3))((b-3)\cdot\epsilon_b)\Big)}{[(a-1)^2-m_z^2][(b-3)^2-m_z^2]}\\
M_1 &= ig^3m_z^2\Big[4-2\frac{\beta_3}{\beta_b}\big(1+\frac{(a\cdot 1)}{(b\cdot 3)}\big)-2\frac{\beta_1}{\beta_a}\big(1+\frac{(b\cdot 3)}{(a\cdot 1)}\big)+2\frac{(a\cdot 1)}{(b\cdot 3)}+2\frac{(b\cdot 3)}{(a\cdot 1)}\Big] \qquad (74)
\end{aligned}$$

$$\begin{aligned}
M_2 &= M_1\big|_{2\leftrightarrow 3} & M_5 &= M_1\big|_{\substack{2\to 1\\ 1\to 3\\ 3\to 2}}\\
M_3 &= M_1\big|_{1\leftrightarrow 2} & &\\
M_4 &= M_1\big|_{\substack{2\to 3\\ 3\to 1\\ 1\to 2}} & M_6 &= M_1\big|_{1\leftrightarrow 3}
\end{aligned}$$

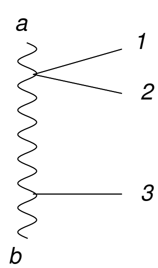


Figure 11: Diagram with no Higgs self-interaction and one ZZHH vertex:

$$M_7 = -i4g^3 m_z^4 \frac{-(\epsilon_a \cdot \epsilon_b) + \frac{1}{m_z^2}(\epsilon_a \cdot (b-3))((b-3)\cdot \epsilon_b)}{(b-3)^2 - m_z^2}$$

$$M_7 = i g^3 m_z^2 \Big[-2 + 2\frac{\beta_b}{\beta_a} - 2\frac{\beta_3}{\beta_a} + 2\frac{\beta_3}{\beta_b} + 2\frac{\beta_3}{\beta_b}\frac{(1\cdot 2)}{(b\cdot 3)} - 2\frac{(1\cdot 2)}{(b\cdot 3)} \Big] \quad (75)$$

$$M_8 = M_7\big|_{a\leftrightarrow b} \qquad M_{11} = M_7\big|_{2\leftrightarrow 3}$$
$$M_9 = M_7\big|_{1\leftrightarrow 3} \qquad M_{12} = M_7\big|_{\substack{a\leftrightarrow b\\ 2\leftrightarrow 3}}$$
$$M_{10} = M_7\big|_{\substack{a\leftrightarrow b\\ 1\leftrightarrow 3}}$$

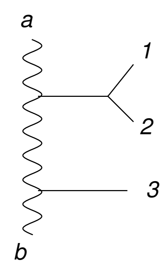


Figure 12: Diagram with a three point Higgs self-interaction and ZZH vertex

$$M_{13} = -i12g^3 m_z^4 m_h^2 \frac{-(\epsilon_a \cdot \epsilon_b) + \frac{1}{m_z^2}(\epsilon_a \cdot (b-3))((b-3)\cdot \epsilon_b)}{[(b-3)^2 - m_z^2][(1+2)^2 - m_h^2]}$$

$$M_{13} = -i3g^3 m_h^2 \frac{(a\cdot(b-3))}{(1\cdot 2)} \quad (76)$$

$$M_{14} = M_{13}\big|_{1\leftrightarrow 3} \qquad M_{17} = M_{14}\big|_{a\leftrightarrow b}$$
$$M_{15} = M_{13}\big|_{2\leftrightarrow 3} \qquad M_{18} = M_{15}\big|_{a\leftrightarrow b}$$
$$M_{16} = M_{13}\big|_{a\leftrightarrow b}$$

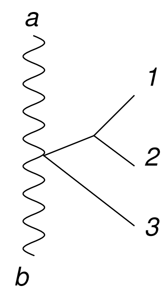


Figure 13: Diagram with a three point Higgs self-interaction and one ZZHH vertex

$$
\begin{aligned}
M_{19} &= i6g^3 m_z^2 m_h^2 \frac{(\epsilon_a \cdot \epsilon_b)}{(1+2)^2 - m_h^2} \\
M_{19} &= i3g^3 m_h^2 \frac{(a \cdot b)}{(1 \cdot 2)}
\end{aligned}
\tag{77}
$$

$$
M_{20} = \quad M_{19}\big|_{\substack{1\to 2\\ 2\to 3\\ 3\to 1}} \qquad M_{21} = \quad M_{19}\big|_{2\leftrightarrow 3}
$$

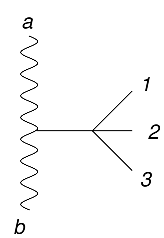


Figure 14: Diagram with one four point Higgs self-interaction

$$
\begin{aligned}
M_{22} &= i6g^3 m_z^2 m_h^2 \frac{(\epsilon_a \cdot \epsilon_b)}{(a+b)^2 - m_h^2} \\
M_{22} &= i3g^3 m_h^2
\end{aligned}
\tag{78}
$$

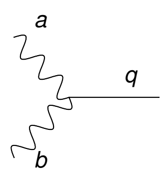
Adding all the relevant terms, in table 4 it can be seen what is the contribution of the different terms. All together, any term with energy dependence as E^0 or higher vanishes for this process, as expected.

β_i	$12ig^3m_z^2$
$m_z^2E^0$	$-12ig^3m_z^2$
m_h^2	$6ig^3m_h^2$
$M_{13..18} + M_{19..21} + M_{22}$	$-6ig^3m_h^2$

Table 4: Coefficients of the sum of all Feynman diagrams for the process $ZZ \to HHH$ from each kind of term

A.3 Three and Four particles processes

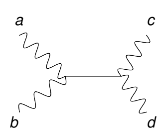
When using the on-shell recursion relations it is possible to split a diagram into a three or four particle process. This processes must be also well-behaved. A three-point process is well behaved if the energy behavior is $\mathcal{O}(E^1)$. For longitudinal Z bosons:



$$= i(m_h^2 - 2m_z^2) \tag{79}$$

Which has an energy dependence of E^0. With any Z boson in the transversal polarization, the process is also well behaved as it is $\mathcal{O}(E^1)$.

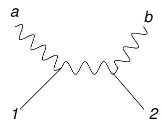
For four particles processes, and in the same way as for the three particle processes, only two cases are relevant: all Z in the longitudinal polarization or all except one Z, which is in the transversal polarization, all taken into account in the massless limit.

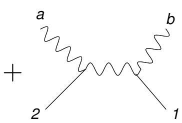


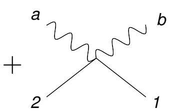
$$= -i4\frac{(\epsilon_a \cdot b)(c \cdot d)}{(cd)^2} \simeq -i2(\epsilon_a \cdot b) \tag{80}$$

Summing over the three possible channels, the result will be $-i2(\epsilon_a \cdot bcd) = 0$, for both longitudinal and transversal polarization. Choosing any other external Z to be in the transversal polarization has a symmetric result. This also can be inferred from our coefficients calculation, in which the sum of all the possible diagrams when looking at a Z going into the process is proportional to the momentum of the particle, which, by definition, will cancel when contracted against the polarization of the same particle.

The other four particles possible process:

$M =$ 

$+$ 

$+$ 

(81)

$$= -i2\left[2\frac{(\epsilon_a\dot{a}1)(a1\cdot b)}{(a1)^2} + 2\frac{(\epsilon_a\cdot a2)(a2\cdot b)}{(a2)^2} - (\epsilon_a\cdot b)\right]$$

$$= i2(\epsilon_a\cdot 12b) = -i2(\epsilon_a\cdot a) = 0 \qquad (82)$$

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