Radboud University Nijmegen

Master's Thesis

Observers in Curved-Spacetime Quantum Field Theory

Author:
Berend Visser

Supervisor:
Prof. Dr. Wim Beenakker

A thesis submitted in fulfillment of the requirements for the degree of Master's in Physics and Astronomy

in the

High Energy Physics Group
Institute for Mathematics, Astrophysics and Particle Physics

May 3, 2019

RADBOUD UNIVERSITY NIJMEGEN

Abstract

Faculteit der Natuurwetenschappen, Wiskunde en Informatica
Institute for Mathematics, Astrophysics and Particle Physics

Master's in Physics and Astronomy

Observers in Curved-Spacetime Quantum Field Theory

by Berend Visser

The vacuum of a quantum field theory becomes ambiguous when a curved background is considered instead of a flat one. In order to lift this ambiguity we attempt to find distinguished states corresponding to an observer measuring properties of the quantum field. One approach we take is to demand that the vacuum should locally give a specific tensorial form to the stress-energy tensor. Because the calculation of the renormalized stress-energy is both ambiguous and very hard to perform we conclude that this criterion is unsuitable as a prescription for defining a vacuum-state. After this we try to define states using a local frequency-splitting prescription. We find that this definition is mathematically sound, but yields us a large class of vacua in stead of just one because the proper time of an observer can only be defined locally. We formulate some conjectures regarding the unitary equivalence of these states.

Acknowledgements

I would like to thank all the members of the HEP department, who created the stimulating work-environment in which I could write this thesis. I've had interesting discussions with many of you and I really learned a lot about the current standing of research in theoretical physics. There are too many of you to name individually, but I would like to highlight a few people who I feel gave a particularly important contribution to this work.

First off, I would like to thank Melissa, Chris, Marrit, Bob, David and Sandra (in no particular order) for the many interesting discussions had over coffee and the support they gave me when I was feeling lost. In particular, discussions with Chris during the first few months of my internship stuck by me and were an important part of forming some of the ideas for this thesis. Special thanks also to David for giving the impetus to this research project and allowing me to build further on his ideas.

There are some people from the mathematics department who helped out with this thesis, and I would like to thank them for allowing me to keep a broad scope of the problem. I had interesting discussions with Klaas Landsman, who provided me with many interesting references that I could follow in the later stages of this work. Daan Janssen was also of great help to me and I would like to thank him especially. His research has many similarities to my own and who was one of the few with whom I could share all the mathematical details.

And, lastly, I would of course like to thank my supervisor Wim Beenakker for sticking with me during this project while I pelted him with tiresome mathematics. Wim was always able to give me a fresh take when I was myself too lost in the details to see the bigger picture. He thought me that it is important to take a step back every now and then, and to always keep a clear view of what you are doing.

Contents

Chapter 1

Introduction

In front of you is my master's thesis which I wrote at the department of high energy physics of the Radboud University in Nijmegen during my time there in 2017-2019. My research here concerned the study of quantum field theory in curved spacetimes (QFTC), and in particular the concept of the vacuum. In classical physics or general relativity the vacuum is a state that is 'just empty', in that no matter is present. The situation is not so simple when considering a quantum-vacuum, because here the demand that no matter is present is an unclear criterion. The vacuum here rather takes the role of a state of the system where no particles are present, but the concept of particle turns out to be quite nebulous.

This theory of quantum fields is a semi-classical approach to quantum gravity, since it concerns the regime where matter is described by a quantum field but the gravitational field is still described by general relativity. This is the regime where many cosmologically interesting scenario's take place: Energies are large enough for particles to be relativistic but much lower than the Planck mass, such that we can safely disregard any quantum character of gravity. For this reason the theory gained a lot of traction in the 60's and 70's when people like Hawking, Unruh, Fulling, Kay and Wald (just to name a few) attempted to formulate an approximate quantum theory of the universe, valid at energy-scales that are natural at this cosmological era.

As an approximate theory of the universe, little definite success has been made since then in this framework. For instance, a semiclassical Einstein-equation can be posed, where one uses the expectation value of the stress-energy tensor of the quantum field theory in the Einstein-equations. Whilst this idea seems to be quite natural it turns out to be far too complicated to ever be practically solvable, on top of the fact that this equation is likely badly behaved from a mathematical point of view. There also seems to be quite a large mismatch between the scales that the quantum theory predicts and the scales that are provided by astronomical observations. Because of reasons like these the framework was largely abandoned as a practical theory of nature, at least on cosmological scales.

There are still reasons to be interested in the framework however. A suitable semiclassical theory of quantum fields could give us some pointers that a theory of quantum gravity could use as a limit to work towards. The concept of Hawking radiation and the laws of black hole thermodynamics are often used for this purpose. On top of this our current understanding of quantum field theory is not as mathematically sound as one might like, and the effort to generalize the theory to curved backgrounds might also give some important insights into the flat-spacetime QFT.

In this thesis I will investigate the viewpoint that there is no global notion of vacuum and particles, but rather that it should depend on the observer that is present to measure it. In the 1970's, Fulling, Kay and Unruh derived that in flat spacetime uniformly accelerated observers see thermally distributed quanta inside the stationary vacuum. This has been widely regarded as an indication that the concept of a particle is local, rather than universal, in the sense that it depends on the observer that measures them. The situation is complicated, however, when one leaves the safety of flat space and looks at a more general situation. The situation in the Unruh effect is very much an idealized situation. If the Unruh effect indicates that the vacuum is an observer-dependent concept, then one should be able to generalize such an interpretation to any observer. Put saliently the question I will try to answer is the following:

If QFTC prohibits a unique notion of 'a particle', can I at least give a satisfactory definition of what 'my particle' is?

I will try to tackle this problem with two different approaches. The first approach is to try and define the vacuum by demanding that it should not give direction to any observable quantities. In particular, we try to pick a state such that locally the stress-energy tensor is proportional to the metric tensor. We try to apply this idea to an FLRW-universe, but find that the renormalization of the stress-energy tensor is in practice quite hard and, more seriously, quite ambiguous. For these reasons we abandon this criterion as a valid theoretical prescription for finding a vacuum state.

The second approach we try is a frequency splitting prescription with respect to the proper time of the observer. Our approach yields us not one but many vacua that turn out to be indistinguishable from a practical point of view. Sadly the different vacua are likely not unitarily equivalent to each other and we therefore do not obtain a well-posed theory of nature. However, our investigation raises the question whether it is even physically justifiable to talk about global states of a quantum field when there is no practical way to ever tell them apart.

As a note on prior knowledge: This thesis was written with master's students in physics in mind. I have tried to keep much of the material self-contained but I do suppose the reader to have some prior knowledge. Firstly of course is quantum field theory at an introductory level. Since the material in this thesis deals exclusively with free fields (which proves challenging enough) this knowledge need not be extensive. Second is of course a grasp of general relativity, differential geometry and tensor calculus. Lastly is some basic notion of functional analysis and C*-algebras. Since this is a thesis in physics and not in mathematics I have tried to keep this to a minimum, but some of the results presuppose facts that transcend an introductory level. I will state these as fact whenever they are needed and add some references to guide the interested reader further.

Finally, a few notes on conventions. In this thesis we always set all natural constants to one unless noted otherwise, which means that $c = \hbar = G_N = k_B = 1$. As a consequence all physical results will be given in units of energy and length, which is the inverse of energy. For the metric signature we will use the convention which is standard in relativity, namely $(-+++)$. In this way spacelike lengths are positive, in agreement with everyday experience. The usual convention in particle physics is opposite to this, and we caution the reader to keep this in mind when comparing this

work to other sources. As to inner products, these are always defined to be anti-linear in the first entry and linear in the second, as is the standard physics convention.

Chapter 2

Quantum fields in Minkowski spacetime

2.1 Introduction

In order to properly describe the problem of quantizing a free field against a curved background it is enlightening to first describe the theory on a flat background. I will first review the problem of canonical quantization on a finite dimensional phase space. Field theory enlarges this construction to infinite dimensional phase-space which adds a lot of extra complications. We pose the theory for smaller models first since this makes clear what our goal should be in general.

First I will review the way that the theory is usually set up in standard QFT-texts and point out some problems that can be seen at this early stage. In flat space-time these have a clear resolution, but it can already be seen that they will prove problematic once we start looking at general spacetimes.

After this I will describe a mathematically more rigorous construction that does not employ mode-functions. This will allow us to give a mathematically precise definition of a free quantum field theory whilst highlighting where exactly the freedom lies when applying the construction to curved backgrounds.

We will follow the method described in (Wald, 1995).

2.2 Elements of quantization

In classical mechanics the state of a system is encoded in the phase space, which keeps track of the canonical position and momentum coordinates of the system. In general, the phase space of a system with m degrees of freedom is a 2m-dimensional manifold. Usually the phase space is a vector space and we can write it in terms of the familiar canonical variables

$$\mathcal{M} = \{(q_1, \dots, q_m, p_1, \dots, p_m) \in \mathbb{R}^{2m}\}.$$

The dynamics of the system are prescribed by the Hamiltonian $\mathcal{H}$ of the system. This is a function depending on the state of the system, ie. the point in phase space, and the time. Note that in this formalism time and space are **not** treated in a similar manner; Time is an absolute unchanging variable which has no dynamical equation. Space is encoded in the canonical q variables, which do have a dynamical equation. The system satisfies the Hamilton equations:

$$\frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q_i} \qquad \frac{dq_i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i}, \tag{2.1}$$

which uniquely determine the evolution of the system for each chosen point in phase space at some initial time t_0.

The main technical device that we will need is the symplectic form σ. In the case of a linear phase space with canonical coordinates we define:

$$\sigma_{\mu\nu} = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}, \tag{2.2}$$

where I_m is the m-dimensional unit matrix. If we write

$$x^\mu(t) = (q_1(t), \ldots, q_m(t), p_1(t), \ldots, p_m(t)), \tag{2.3}$$

we can rewrite Hamilton's equations compactly as

$$\frac{dx^\mu}{dt} = \sigma^{\mu\nu}\frac{\partial \mathcal{H}}{\partial x^\nu} \equiv \sigma^{\mu\nu}\partial_\nu \mathcal{H}, \tag{2.4}$$

where summation of repeated indices is implied (and will be from this point onwards).

We indicate by $\sigma^{\mu\rho}$ the inverse of the symplectic form. [1] We then define the Hamiltonian vector field by

$$h^\mu = \sigma^{\mu\nu}\partial_\nu \mathcal{H}.$$

Temporal evolution is then given by the flow of this vector field across the manifold, as can be seen by comparing with equation 2.4. This completely fixes the time-dependence of the system.

For us the main point of this construction is not that of the Hamiltonian. It will turn out to lose a lot of its physical meaning when considered on a general curved background because it is not a covariant entity. The symplectic form is the most important part when considering quantization, since it guides us in the way that observables should behave in the quantum theory. A classical observable is any function from phase space to the real numbers that is sufficiently smooth. In particular the canonical variables $q_1, \ldots, q_m, p_1, \ldots, p_m$ are observables. We can put a bilinear structure on the space of observables by defining the poison-bracket as:

$$\{f, g\} \equiv \sigma^{\mu\nu}(\partial_\mu f)(\partial_\nu g). \tag{2.5}$$

Note that this bracket is anti-symmetric, because σ is. It also satisfies the Jacobi-identity. The reader will probably find it more familiar when it is applied to the canonical observables p_μ and q_μ:

$$\{q_\mu, q_\nu\} = \{p_\mu, p_\nu\} = 0, \tag{2.6}$$

$$\{q_\mu, p_\nu\} = \delta_{\mu\nu}. \tag{2.7}$$

[1] Note that at this point we have no notion of a metric that can be used to raise and lower indices. The raised indices are just present so that we can intuitively use the Einstein summation convention for the contraction of indices.

The idea of canonical quantization, due to Dirac in the 1930's, is the following: Identify some Hilbert space H that will serve as the space of quantum states of the system. The classical observables are mapped to self-adjoint operators on this Hilbert space, which is usually indicated by writing a hat on the classical observable. The main guiding principle for this map is that the Poisson-bracket on the space of classical observables should map into the commutator-bracket of the quantum observables:

$$\widehat{\{f,g\}} = -i[\hat{f},\hat{g}]. \tag{2.8}$$

The procedure does however not work for all observables in the classical theory; The algebra of classical observables is commutative, but the algebra of quantum observables is not, hence ordering ambiguities will arise when considering observables that are of higher than quadratic order in the canonical observables. Nevertheless the quantization goes through without problem if we only consider these canonical observables.

As a final step of abstraction we'll write the canonical variables in a way that does not presuppose a fixed basis. We'll suppose that our phase space $\mathcal{M}$ is linear, which means that we can identify points in phase space with vectors on phase space. If y is a point in $\mathcal{M}$ we can then consider the map $\sigma(y,.) : \mathcal{M} \to \mathbb{R}$. We can recover the coordinate functions from this by considering $y = (0,\ldots,0,q_i = 1,0\ldots,0)$ and noting that for this choice of y

$$\sigma(y,(q_1,\ldots,q_m,p_1,\ldots,p_m)) = -p_i, \tag{2.9}$$

Similarly, if we choose $y = (0,\ldots,0,p_i = 1,0\ldots,0)$, we find the relation

$$\sigma(y,(q_1,\ldots,q_m,p_1,\ldots,p_m)) = q_i. \tag{2.10}$$

This shows that considering the maps $\sigma(y,.)$ for general $y \in \mathcal{M}$ is the same as considering the canonical position and momentum observables, but in a way that is manifestly coordinate invariant. In this form the commutation relations take the form

$$[\hat{\sigma}(y_1,.),\hat{\sigma}(y_2,.)] = -i\sigma(y_1,y_2)I. \tag{2.11}$$

The reason that the above discussion is omitted in standard treatments of quantum mechanics is that it doesn't really matter. The Stone-Von Neumann theorem states (up to some technicalities) that for a **finite dimensional** symplectic space there is only one irreducible representation up to unitary equivalence. Writing the phase space in terms of arbitrary coordinates is therefore allowed.

The theorem fails when an infinite dimensional system is regarded. Suddenly the possibility is opened up for multiple unitarily inequivalent representations of the canonical commutation relations. Picking out a preferred basis is then no longer an inconsequential choice and care must be taken.

2.3 Quantizing the free-scalar field

The theory that we will be considering is that of the free scalar field. This is widely regarded as a toy model for more advanced theories, but it contains all the necessary apparatus for our discussion. Extending the construction to free fermions or spin 1 bosons is straightforward; The only essential ingredient is the symplectic form on the

space of solutions as well as the fact that the equations of motion need a well posed initial value formulation. This means that we require the equations of motion to have unique solutions once initial conditions have been chosen.

We also do not regard interacting theories here, since it greatly complicates the discussion and a precise mathematical formulation of interacting field theories is not available. For a review of interacting QFT on a curved background we refer the reader to (Hollands and Wald, 2015).

The real Klein-Gordon field is regarded to have the following lagrangian density:

$$\mathcal{L} = -\frac{1}{2}[\partial^\mu \phi \partial_\mu \phi + m^2 \phi^2]. \tag{2.12}$$

The equation of motion then follows from this equation by varying the action with respect to the field and reads:

$$\partial^\mu \partial_\mu \phi - m^2 \phi = 0, \tag{2.13}$$

which is the well known wave-equation with mass m. Once we choose the initial data $[\phi(0,x), \dot{\phi}(0,x)]$ on the $t = 0$ hyperspace, this will have a well-defined and unique solution. We can therefore interchangeably use the space of solutions and the space of initial conditions at $t = 0$ to describe the system. Note here that both these spaces are linear, since equation 2.13 is, but do not have finite dimension.

In conventional texts on QFT (Peskin and Schroeder, 1995) the approach is to Fourier-transform equation 2.13 and to quantize the Fourier components into ladder operators, thereby regarding the system as an infinite collection of harmonic oscillators. We will describe the procedure below and point out what the problematic parts of the construction are.

The equation 2.13 has an obvious set of solutions, the plane waves associated with momentum vectors $\mathbf{k}$ (where we use bold-font symbols to indicate 3-vectors):

$$e_{\mathbf{k}}(t,\mathbf{x}) = \frac{e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}, \tag{2.14}$$

$$\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}, \tag{2.15}$$

together with their complex conjugates $e^*_{\mathbf{k}}(t,\mathbf{x})$. We call the modes $e_{\mathbf{k}}$ positive frequency solutions, because their time-dependence has the form $e^{-i\omega t}$ for ω positive. Consequently the complex conjugate mode functions are negative frequency solution, their time-dependence is of the form $e^{i\omega t}$ for positive ω. The normalization of these modes comes from the Klein-Gordon inner product on the space of solutions, which reads:

$$\langle f, g\rangle_{\mathrm{KG}} \equiv i\int d^3x[f^*(\partial_t g) - (\partial_t f^*)g] = -i\int d^3x f^* \overleftrightarrow{\partial_t} g, \tag{2.16}$$

evaluated at $t = 0$. Here we use the notation $\overleftrightarrow{\partial_t} = \overleftarrow{\partial_t} - \overrightarrow{\partial_t}$. It can be checked to be time-invariant from the fact that f and g satisfy the KG-equation and using Gauss' lemma. We will not show this explicitly here, since we will prove this in a more

general context in equation 3.9. It is now easily checked that the modes $e_{\mathbf{k}}$ satisfy:

$$\langle e_{\mathbf{k}}, e_{\mathbf{l}} \rangle_{\mathrm{KG}} = \delta^3(\mathbf{k} - \mathbf{l}), \tag{2.17}$$
$$\langle e^*_{\mathbf{k}}, e^*_{\mathbf{l}} \rangle_{\mathrm{KG}} = -\delta^3(\mathbf{k} - \mathbf{l}), \tag{2.18}$$
$$\langle e^*_{\mathbf{k}}, e_{\mathbf{l}} \rangle_{\mathrm{KG}} = 0. \tag{2.19}$$

Moreover, this set of solutions is complete within the space of solutions, meaning that every solution can be written as an integral of these functions with some suitable coefficients:

$$\phi(t, \mathbf{x}) = \int d^3\mathbf{k} \Big[\tilde{\phi}(\mathbf{k}) e_{\mathbf{k}}(t, \mathbf{x}) + \tilde{\phi}(\mathbf{k})^* e^*_{\mathbf{k}}(t, \mathbf{x}) \Big], \tag{2.20}$$

where $\tilde{\phi}$ denotes the Fourier transform of ϕ, which is determined by the initial conditions put upon the field. Reality of ϕ is ensured by the fact that the Fourier coefficients of the negative frequency modes are adjoint to those of the positive frequency modes.

Now we can calculate the canonical momentum corresponding to ϕ, this is

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}. \tag{2.21}$$

Expressed in terms of the Fourier coefficients this reads:

$$\tilde{\pi}(\mathbf{k}) = -i\omega_{\mathbf{k}} \tilde{\phi}(\mathbf{k}) \tag{2.22}$$

We now wish to quantize these two objects into operators on some suitable Hilbert space, such that they satisfy the equal-time canonical commutation relations:

$$[\hat{\phi}(\mathbf{x}), \hat{\phi}(\mathbf{y})] = 0, \tag{2.23}$$
$$[\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = 0, \tag{2.24}$$
$$[\hat{\phi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}). \tag{2.25}$$

We try to emulate the case of the harmonic oscillator, since we can regard the Klein-Gordon field as an infinite collection of these oscillators. We therefore link each mode $e_{\mathbf{k}}$ with an annihilation operator $a_{\mathbf{k}}$ and $e^*_{\mathbf{k}}$ with a creation operator $a^\dagger_{\mathbf{k}}$ acting on a Fock space of state vectors. We impose the following algebra on these operators:

$$[a_{\mathbf{k}}, a_{\mathbf{l}}] = 0, \tag{2.26}$$
$$[a^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{l}}] = 0, \tag{2.27}$$
$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{l}}] = \delta^{(3)}(\mathbf{k} - \mathbf{l}). \tag{2.28}$$

We also define the vacuum state $|0\rangle$ as a vector that is annihilated by all $a_{\mathbf{k}}$ operators, and define the Fock space to be all vectors that can be obtained by acting with the creation operators. Emulating the case of the classical field, we try to expand the field observable in terms of the basis $e_{\mathbf{k}}$, but replace the Fourier components $\tilde{\phi}(\mathbf{k})$ by the operators $a_{\mathbf{k}}$:

$$\hat{\phi}(t, \mathbf{x}) = \int d^3\mathbf{k} \Big[e_{\mathbf{k}}(t, \mathbf{x}) a_{\mathbf{k}} + e^*_{\mathbf{k}}(t, \mathbf{x}) a^\dagger_{\mathbf{k}} \Big]. \tag{2.29}$$

It can be easily checked that the so defined quantity satisfies the canonical commutation relations from the relations on the ladder operators and the completeness of the

plane-wave modes. There are however a few problems with this construction which make it hard to generalize to curved spacetimes. We will discuss these points below.

2.3.1 Precise interpretation of the field observable

Firstly the operator defined by equation 2.29 is not an operator on the Fock space defined above. To see why, consider the (formal) state $\hat{\phi}(x)|0\rangle$, for x some point of spacetime[2] if we try to find its norm we get into trouble, since

$$\langle 0|\hat{\phi}(x)^2|0\rangle = \int d^3\mathbf{l} \int d^3\mathbf{k} \langle 0|e_{\mathbf{k}}(x)e^*_{\mathbf{l}}(x)a_{\mathbf{k}}a^\dagger_{\mathbf{l}}|0\rangle = \int \frac{d^3\mathbf{l}}{2(2\pi)^3\omega_{\mathbf{l}}}, \tag{2.30}$$

which is infinite. In fact every expectation value of $\hat{\phi}(x)$ in a state with a finite number of particles is infinite.

The problem here is that $\hat{\phi}$ is too sharply peaked. In emulating the Fourier series of ϕ with operators instead of coefficients we have thrown out most of the dampening that makes the integral 2.20 converge. The only thing that is left to restrain the high k behaviour in our expressions are the $1/\sqrt{\omega_{\mathbf{k}}}$ terms inside the mode functions, but this is not enough to render quadratic expressions in $\hat{\phi}$ finite, when one considers the case of coinciding points. If one considers different points x and y, the expectation value

$$D(x-y) = \langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle \tag{2.31}$$

is finite. If x and y are separated purely timelike by an amount t, then one can show that (see chapter 2.4 of (Peskin and Schroeder, 1995)) $D(x-y) \sim e^{-imt}$ for large t. For x and y being separated purely spacelike by a distance r we have $D(x-y) \sim e^{-mr}$ for large r. However, D will be singular in the limit $x \to y$.

We should therefore think of $\hat{\phi}$ as distribution; It is only defined when integrated against some suitable class of test-functions. It is perfectly possible to use distributions in formal calculations (we have implicitly done so above) as long as one exercises some care. This is what is done in standard QFT texts, and is perfectly admissible for flat spacetime QFT. In section 3.3 we will formalize in which way we can interpret $\hat{\phi}(x)$ as an operator-valued distribution. However, the approach expanding $\hat{\phi}$ in a mode sum of ladder operators will not be pursued further for two reasons.

First, the above formalism obscures the fact what it is exactly that we are trying to quantize. In the finite dimensional case there was no ambiguity of what the 'space of solutions' was exactly, but in gaining an infinite number of degrees of freedom it is not so clear which solutions of equation 2.13 are sufficiently well behaved for us to perform calculations with.

Second is the fact that in general spacetimes mode decompositions become less practical, since explicitly finding them becomes very hard, if not impossible. They therefore lose their use as a theoretical device to build a theory on, because we have no guarantee that complete sets of mode functions will exist in general or what their explicit form would be.

[2]Note our convention here, a boldface $\mathbf{x}$ refers to a 3-dimensional vector, whilst a normal x is any point of spacetime, which we might denote as $(x^0, \mathbf{x})$.

2.3.2 The role of the plane-wave basis

Whilst the points above are more mathematical than physical in nature and can be overcome, the second point is the more important one. The choice we made above was to split our space of solutions into two parts, the positive frequency modes e_k whose time evolution was of the form

$$\partial_t e_{\mathbf{k}} = -i\omega_{\mathbf{k}} e_{\mathbf{k}}, \tag{2.32}$$

and their negative frequency counterparts $e^*_{\mathbf{k}}$. We linked these basis vectors to ladder-operators, and defined the vacuum to be a state that was annihilated by all the annihilation operators $a_{\mathbf{k}}$. Since a basis is usually just a mathematical device used to explicitly describe certain objects, we could try to pick another one and see whether the result remains the same.

Hence consider a second basis of solutions $\{f_k, f^*_k\}$ where k is now some arbitrary label which might be discrete or continuous. We'll denote the set of labels by K, for the plane-wave basis this was $\mathbb{R}^3$. Furthermore, we assume that there is some suitable generalized sum or integral over K. With respect to this summation we define a delta-distribution by the fact that, for a suitably nice function f on K, we have

$$f(k) = \sum_{l \in K} \delta(k,l) f(l). \tag{2.33}$$

As can be easily checked, the entire construction above goes through without problems as long as this new basis is complete and orthonormal. In particular, we can expand the field observable according to

$$\hat{\phi}(x) = \sum_k [f_k(x) b_k + f^*_k(x) b^\dagger_k], \tag{2.34}$$

where the b_k's are operators satisfying the equations 2.26-2.28 with respect to the delta-distribution of K. We define a new vacuum state by

$$b_k |\tilde{0}\rangle = 0. \tag{2.35}$$

As usual we define the Fock-space by acting on this vector with the operators $b^\dagger_k$. In terms of quantizing the canonical commutation relations this construction is perfectly viable and we can ask whether it gives the same result, ie. a unitarily equivalent Fock-space. We'll investigate whether the vacua are equivalent in that they contain no particles. Since the two sets of modes are both complete we can express one in terms of the other:

$$f_k = \sum_{l \in \mathbb{R}^3} \alpha_{kl} e_l + \beta_{kl} e^*_l, \tag{2.36}$$

$$\alpha_{kl} = \langle e_l, f_k \rangle_{\mathrm{KG}}, \tag{2.37}$$

$$\beta_{kl} = -\langle e^*_l, f_k \rangle_{\mathrm{KG}}. \tag{2.38}$$

We have here also taken the l labels abstractly in the index set $\mathbb{R}^3$ in order to obtain matching notation, but in this case this sum is just the integral over 3-dimensional l vectors. The second and third equalities follow by applying orthonormality of the $\{e_l, e^*_l\}$ basis in the Klein-Gordon inner product.

The coefficients α and β are called *Bogolyubov coefficients*. Since both of the expansions of the field observable should give the same object, we can equate the expressions 2.29 and 2.34, which gives us:

$$\begin{aligned} a_l &= \langle e_l, \phi \rangle_{\mathrm{KG}}, && (2.39)\\ &= \langle e_l, \sum_{k\in K} [f_k b_k + f_k^* b_k^\dagger] \rangle_{\mathrm{KG}}, && (2.40)\\ &= \sum_{k\in K} [\alpha_{kl} b_k + \beta_{kl}^* b_k^\dagger]. && (2.41) \end{aligned}$$

We now apply the number operator from our original theory to the vacuum of our second theory to see whether the new vacuum is empty. We find:

$$\begin{aligned} \langle \tilde{0}| \sum_{l\in\mathbb{R}^3} a_l^\dagger a_l \, |\tilde{0}\rangle &= \sum_{l\in\mathbb{R}^3} \langle \tilde{0}| \Big(\sum_{k\in K} \alpha_{kl}^* b_k^\dagger + \beta_{kl} b_k \Big) \Big(\sum_{r\in K} \alpha_{rl} b_r + \beta_{rl}^* b_r^\dagger \Big) |\tilde{0}\rangle, && (2.42)\\ &= \sum_{l,k,r} \beta_{kl} \beta_{rl}^* \langle \tilde{0} | b_k b_r^\dagger | \tilde{0} \rangle, && (2.43)\\ &= \sum_{l\in\mathbb{R}^3} \sum_{k\in K} |\beta_{kl}|^2. && (2.44)\\ &= \mathrm{Tr}(\beta^\dagger \beta). && (2.45) \end{aligned}$$

In the last step we regarded β as a matrix from $\mathbb{R}^3$ to L. This means that our new vacuum is empty with respect to the original particles if and only if the new modes are purely positive frequency with respect to the old modes. If this is the case, we have simply changed bases for the positive frequency solutions and our particle representation will be the same. If β is non-zero it might still be possible for the two Fock-spaces to be unitarily equivalent, just with different vacuum-states. In general however the sum 2.45 might diverge, which implies that the two constructions are unitarily inequivalent.

The previous derivation was often a bit messy since nowhere in the intermediate steps have we proven anything about convergence of the sums, or justified swapping the different sums in our expressions. A more rigorous treatment of the unitary equivalence of representations will be given in section 3.5. Here we find that in fact the convergence of the sum 2.45 implies the unitary equivalence of the two constructions in a more general setting.

From a physical viewpoint we seem to have gotten ourselves into trouble here: Both of the above constructions seem like a valid way to set up our QFT, but we would expect that we could unambiguously define 'empty' space, but it turns out that we can not. Obviously some physical criterion is needed to lift this ambiguity. Luckily we have one in Minkowski space: Poincaré invariance. We can dispense of the new basis set as unphysical since the modes are not invariant under transformations from the Poincaré group, whilst the original set $e_{\mathbf{k}}$ is.

To see this we first have to introduce a bit of notation. For two 4-vectors v and w we write their Lorentzian product as $\eta(v,w) = \eta_{\mu\nu} v^\mu w^\nu$. Also, we have extended the 3-vector $\mathbf{k}$ to a 4-vector by setting $k^0 = \omega_{\mathbf{k}}$, which yields a 4-vector with $\eta(k,k) = -m^2$.

Above we defined our modes in terms of 3-vectors $\mathbf{k}$, and manually set the coefficient of t to be $-i\omega_{\mathbf{k}}$ such that the KG-equation is satisfied. However, we can also consider modes labeled by 4-vectors k with $\eta(k,k) = -m^2$:

$$e_k(x) = \exp(i\eta(k,x)), \tag{2.46}$$

which solves the KG-equation by the fact that k is on-shell. For a given 3-vector $\mathbf{k}$ there are two options to extend to a 4-vector, namely by setting $k^0 = \pm\omega_{\mathbf{k}}$. The first option gives the positive frequency solution, whereas the second yields a negative frequency solution.

Now, for the invariance of the vacuum, recall that the Poincaré group is generated by (proper ortochronous) Lorentz transformations and constant shifts. Obviously we have for each constant 4-vector a:

$$e_{\mathbf{k}}(x+a) = e^{i\eta(k,a)}e_{\mathbf{k}}(x). \tag{2.47}$$

Obviously this transformation does not mix positive and negative frequency modes and therefore the vacuum is invariant under the subgroup of constant shifts. For a Lorentz transformation, remember that these are defined as the group of linear transformations that leave the Lorentzian product invariant. Because of this we have

$$e_k(\Lambda x) = \exp(i\eta(k,\Lambda x)) = \exp(i\eta(\Lambda^{-1}k, x)) = e_{\Lambda^{-1}k}(x). \tag{2.48}$$

We see that the label k is transformed by the Lorentz-transform. Because we are looking at proper-ortochronous Lorentz transforms [3] this transform respects the positive mass-shell and $e_{\Lambda^{-1}k}$ is a positive frequency mode. Hence we see that both sets of transformations have $\beta = 0$. Therefore the vacuum is invariant under the Poincaré group, which is a result of the specific form of modes that is chosen.

2.4 The definition of the Fock space

The previous section was mathematically quite shaky. In order to solve the first problem sketched above a more mathematical approach will be taken in the following sections. As a first step, the definition of the Fock space as the 'space of states generated by creation operators' is vague, and this section will serve to give a sound definition.

Let $\mathcal{H}$ denote any Hilbert spaces and let $\mathcal{H}^*$ be its dual, the space of bounded complex linear maps $\mathcal{H} \to \mathbb{C}$. By Riesz's lemma we can identify these two spaces through the formula

$$\psi \in \mathcal{H} \to \langle \psi, . \rangle \in \mathcal{H}^*, \tag{2.49}$$

which is an anti-unitary map of Hilbert spaces. We then define the Hilbert tensor product of $\mathcal{H}$ with itself as

$$\mathcal{H} \otimes \mathcal{H} = \{\alpha : \mathcal{H}^* \times \mathcal{H}^* \to \mathbb{C} \mid \sum_{i,j} |\alpha(\bar{e}_i, \bar{e}_j)|^2 < \infty\}, \tag{2.50}$$

[3] ie. no parity or time-reversal.

where$\{\bar{e}_i\}$ is any orthonormal basis of $\mathcal{H}^*$. The inner product is defined as one would expect:

$$\langle \alpha, \beta \rangle = \sum_{i,j} \alpha(\bar{e}_i, \bar{e}_j)^* \beta(\bar{e}_i, \bar{e}_j), \tag{2.51}$$

for some orthonormal basis e_i of $\mathcal{H}$. We have the natural bilinear tensor map $\otimes : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ defined by

$$\otimes\,(v, w)(x, y) = x(v)y(w), \tag{2.52}$$

or, more in line with standard Dirac notation:

$$(|v\rangle \otimes |w\rangle)(\langle x| \times \langle y|) = \langle x, v\rangle\langle y, w\rangle. \tag{2.53}$$

One can prove that the tensor product defined above is the Hilbert space closure of these elementary tensors.

We similarly define $\bigotimes^n \mathcal{H}$ as the n-fold tensor product of $\mathcal{H}$. The symmetrized tensor product $\bigotimes_s^n \mathcal{H}$ is defined as the subspace of $\bigotimes^n \mathcal{H}$ of tensors that are symmetric in all arguments. All these sets are Hilbert spaces in the inner product 2.51 when the sum is taken over the appropriate amount of arguments.

We can then define the symmetrized Fock space corresponding to the one-particle Hilbert space $\mathcal{H}$ as

$$\mathcal{F}_s(\mathcal{H}) = \bigoplus^n (\otimes_s^n \mathcal{H}), \tag{2.54}$$

where the direct sum starts at $n = 0$ and we take $\otimes^0 \mathcal{H} = \mathbb{C}$. The direct sum here is a Hilbert direct sum, which is the completion of the algebraic direct sum. Concretely, we define the Fock space as:

$$\mathcal{F}_s(\mathcal{H}) = \Big\{(\alpha_0, \alpha_1, \alpha_2, \dots) \mid \alpha_i \in \otimes^i(\mathcal{H}) \text{ and } \sum_i \|\alpha_i\|^2 < \infty\Big\}. \tag{2.55}$$

We now define creation and annihilation operators on this space. Let $\bar{\psi} \in \mathcal{H}^*$, we can define a map from the n-fold tensor product to the $(n-1)$-fold tensor product by $\bar{\psi} \cdot \alpha = \alpha(\bar{\psi}, \dots)$ ie. by taking up one of the arguments of α (since we symmetrized the Fock-space it doesn't matter which one). If $\psi \in \mathcal{H}$ denotes the dual to $\bar{\psi}$ we can also define a map to the $(n+1)$-fold tensor product through $\psi \cdot \alpha = \psi \otimes_s \alpha$. We then define the creation and annihilation operators by:

$$a(\bar{\psi})(\alpha_0, \alpha_1, \alpha_2, \dots) = (\bar{\psi} \cdot \alpha_1, \sqrt{2}\bar{\psi} \cdot \alpha_2, \sqrt{3}\bar{\psi} \cdot \alpha_3, \cdots), \tag{2.56}$$

$$a^\dagger(\psi)(\alpha_0, \alpha_1, \alpha_2, \dots) = (0, \psi \cdot \alpha_0, \sqrt{2}\psi \cdot \alpha_1, \sqrt{3}\psi \cdot \alpha_2, \cdots). \tag{2.57}$$

I'll leave it to the reader to check that these maps are indeed each others adjoints. Note that we took $\bar{\psi}$ as an argument for the annihilation operator rather than ψ itself in order for everything to come out linear.

The commutators of these operators can now be calculated. Since the Fock space was set up to be symmetric it can easily be seen that the annihilation operators commute amongst themselves, as do the creation operators.We can calculate the commutator between the two operators.

$$\begin{aligned}
\bar{\phi}\cdot(\psi\cdot\alpha_n) &= \frac{\langle\phi,\psi\rangle}{n+1}\alpha_n + \frac{n}{n+1}\psi\cdot(\bar{\phi}\cdot\alpha_n),\\
a(\bar{\phi})a^\dagger(\psi)(\dots,0,\alpha_n,0,\dots) &= a(\bar{\phi})(\dots,0,\sqrt{n+1}\psi\cdot\alpha_n,0,\dots),\\
&= (n+1)(\dots,0,\bar{\phi}\cdot(\psi\cdot\alpha_n),0,\dots),\\
a^\dagger(\psi)a(\bar{\phi})(\dots,0,\alpha_n,0,\dots) &= a^\dagger(\psi)(\dots,0,\sqrt{n}\bar{\phi}\cdot\alpha_n,0,\dots),\\
&= n(\dots,0,\psi\cdot(\bar{\phi}\cdot\alpha_n),0,\dots).
\end{aligned}$$

We can then calculate the commutator from the above expressions to be

$$[a(\bar{\phi}),a^\dagger(\psi)] = \langle\phi,\psi\rangle I, \tag{2.58}$$

which reproduces the well know form of the canonical commutation relations upon picking an orthonormal basis and setting $a_k = a(\bar{e_k})$, because in that case this equation reads

$$[a_k,a_l^\dagger] = \delta_{kl}, \tag{2.59}$$

with all other commutators zero. Note that it is not needed to pick a basis, the advantage of this definition is that it is manifestly basis-invariant.

2.5 Reformulation of the theory

We will need to rein in the space of solutions somewhat to functions that are 'sufficiently nice'. Since we want to be able to differentiate functions it makes sense to ask our functions to be smooth, ie. infinitely differentiable. We also want our functions to be small enough such that various integrals in the following analysis converge. In view of the coming generalization we will take a simple criterion and ask that our functions have compactly supported initial data. This also corresponds nicely with our intuition that a wave is a local excitation that causally expands after we have 'let go' of the vibrating medium. Hence, we take as our phase space:

$$V = C_c^\infty(\mathbb{R}^3)\bigoplus C_c^\infty(\mathbb{R}^3),$$

which consists of pairs (ϕ,π) of initial data at $t=0$. On this space we have the symplectic form given by

$$\sigma((\phi_1,\pi_1),(\phi_2,\pi_2)) = \int d^3x\Big(\pi_1\phi_2 - \phi_1\pi_2\Big). \tag{2.60}$$

Equivalently, given the corresponding solutions ψ_1 and ψ_2 to the Klein-Gordon equation:

$$\sigma(\psi_1,\psi_2) = \int d^3x\psi_1\overleftrightarrow{\partial_t}\psi_2, \tag{2.61}$$

Note the relation to the Klein-Gordon inner product defined by equation 2.16. The main difference is the added complex conjugate in the first argument, the factor i is present to make the inner product hermitian. The idea behind the following procedure is to alter the symplectic form into a hermitian inner product, and complete the complexified space of solutions in order to obtain a one-particle Hilbert space.

To do this, we first have to split any $\psi \in V$ into positive and negative frequency parts.

$$\psi = \psi^+ + \psi^-. \tag{2.62}$$

These are defined by considering the Fourier-transform of ψ and taking for $\psi^\pm$ the part only containing positive/negative frequency modes. As such we have $\overline{\psi^\pm} = \psi^\mp$ by reality of ψ. We can , as above, define the Klein-Gordon product on the complexified space of solutions by

$$\langle \psi_1, \psi_2 \rangle_{\mathrm{KG}} = -i\sigma(\bar{\psi_1}, \psi_2), \tag{2.63}$$

which is positive definite on the space of positive frequency solutions and negative definite on the space of negative frequency solutions. Mixed products of positive and negative frequencies vanish. We can use this form to complete the positive frequency space to a Hilbert space $\mathcal{H}$, since it's an inner product on the positive frequency space. We hence have an embedding $K : V \to \mathcal{H}$ sending an element of V to its positive frequency part embedded in our Hilbert space $\mathcal{H}$.

We can then define our quantization: Let the Hilbert space on which the $\hat{\sigma}(\phi, .)$ act be $\mathcal{F}_s(\mathcal{H})$ and define its action by

$$\hat{\sigma}(\phi, .) = i[a(\overline{K\phi}) - a^\dagger(K\phi)]. \tag{2.64}$$

We then check:

$$\begin{aligned}
[\hat{\sigma}(\phi, .), \hat{\sigma}(\psi, .)] &= [a(\overline{K\phi}), a^\dagger(K\psi)] - [a(\overline{K\psi}), a^\dagger(K\phi)] \\
&= \langle K\phi, K\psi \rangle_{\mathrm{KG}} - \langle K\psi, K\phi \rangle_{\mathrm{KG}} \\
&= -i\Big(\sigma(\phi^-, \psi^+) + \sigma(\phi^+, \psi^-)\Big) \\
&= -i\Big(\sigma(\phi^-, \psi^+) + \sigma(\phi^+, \psi^-) + \sigma(\phi^-, \psi^-) + \sigma(\phi^+, \psi^+)\Big) \\
&= -i\sigma(\phi, \psi),
\end{aligned}$$

which is exactly the quantization we set out to achieve. This solves the mathematical problems pointed out in section 2.3.1: We did not need to consider field operators on one point and did not have to choose any preferred basis. We did however not remedy the second problem posed above, in that the division in equation 2.62 is still arbitrary.

2.5.1 Characterization of the available freedom

The need for frequency splitting arises because we need a positive definite inner product to define the one-particle Hilbert space $\mathcal{H}$. We can elegantly describe the freedom available as a choice of a real inner-product on the space of solutions.

Note that the essential features of the quantization are all contained in the choice of the one-particle Hilbert space in which V is densely embedded by some map K. Density is needed (and sufficient) to ensure that the representation is irreducible, see lemma A.2 of (Kay and Wald, 1991). After this we can construct the Fock space and define our quantization map by equation 2.64. From the calculation above we see that this means that the inner product on $\mathcal{H}$ should satisfy the following relation:

$$\mathrm{Im}\langle K\psi, K\phi \rangle_{\mathcal{H}} = -\frac{1}{2}\sigma(\psi, \phi). \tag{2.65}$$

Suppose that we have some Hilbert space $\mathcal{H}$ and a map $K : V \to \mathcal{H}$ satisfying these conditions. The real part of the inner product is still open to choice and we'll denote it by μ. Hence we have the equality

$$\langle K\psi, K\phi\rangle_{\mathcal{H}} = \mu(\psi,\phi) - \frac{i}{2}\sigma(\psi,\phi). \tag{2.66}$$

It is easily checked that this defines a real inner product on V, and that the norm of ψ in V is the same as that of $K\psi$ in H. By the Cauchy-Schwarz inequality we can then estimate:

$$\mu(\phi,\phi)\mu(\psi,\psi) = \|K\phi\|^2 \|K\psi\|^2 \geq |\langle K\phi, K\psi\rangle|^2 \geq \frac{1}{4}\sigma(\phi,\psi)^2, \tag{2.67}$$

or rather

$$\mu(\phi,\phi) \geq \frac{1}{4}\frac{\sigma(\phi,\psi)^2}{\mu(\psi,\psi)}. \tag{2.68}$$

We can strengthen this inequality to be satisfied as a supremum over ψ using the density of the image of K. Fix some $\phi \in V$, for an arbitrary ψ we can then calculate:

$$\frac{1}{2}\left\| i\frac{K\phi}{\|K\phi\|} - \frac{K\psi}{\|K\psi\|}\right\|^2 = 1 - \frac{\mathrm{Im}\langle K\phi, K\psi\rangle}{\|K\phi\|\,\|K\psi\|} = 1 - \frac{1}{2}\frac{\sigma(\phi,\psi)}{\mu(\phi,\phi)^{\frac{1}{2}}\mu(\psi,\psi)^{\frac{1}{2}}}. \tag{2.69}$$

By the assumption that the image of K is dense in $\mathcal{H}$ we can choose ψ such that the left hand side becomes arbitrarily small. If we let $\epsilon > 0$ and take the left-hand side smaller than $\frac{\epsilon}{2}$ then we can rewrite the above equation as

$$\mu(\phi,\phi) < \frac{1}{(2-\epsilon)^2}\frac{\sigma(\phi,\psi)^2}{\mu(\psi,\psi)}. \tag{2.70}$$

Hence we arrive at our result that μ can be characterized as follows:

$$\mu(\phi,\phi) = \sup_{\psi\in V}\frac{1}{4}\frac{\sigma(\phi,\psi)^2}{\mu(\psi,\psi)}. \tag{2.71}$$

2.5.2 Sufficiency of the characterization

This previous result in fact fixes all available freedom. Given a positive, symmetric bilinear form μ on V that satisfies equation 2.71 we can find a unique (up to unitary isomorphism) one-particle Hilbert space $\mathcal{H}$ and an injective map $K : V \to \mathcal{H}$ such that

(i) KV is dense in $\mathcal{H}$.

(ii) The inner product on KV is given by

$$\langle K\psi, K\phi\rangle_{\mathcal{H}} = \mu(\psi,\phi) + \frac{i}{2}\sigma(\psi,\phi). \tag{2.72}$$

In order to construct this Hilbert space we need to complexify the real space V somehow, without increasing its size too much so that we do not spoil density of KV inside $\mathcal{H}$. What we need for this is a *complex structure* J. This is an operator that satisfies $J^2 = -I$ and can as such be used to emulate the action of the complex unit vector i. In this case a complex structure is given to us by the symplectic form.

To see this, we first need to complete V into a real Hilbert space such that we can use standard functional analytic results. In picking a positive linear form μ we have fixed a topology on V, but there is no reason to expect this to be complete. A sequence of elements in V may seem to converge to some pair of functions on $\mathbb{R}^3$, but this need not be compactly supported or smooth, such that the limit does not lie in V. To remedy this, we need to Cauchy-complete V in the norm 2μ to obtain a real Hilbert space, which we'll call V_μ.

In case this construction is unfamiliar to the reader, the important points are that V_μ is a real Hilbert space with inner-product 2μ, and V is densely embedded inside. This means that every element in V_μ can be approximated by elements in V. So while V_μ is in general strictly larger than V, it's still possible to describe many things in terms of V. [4]

We can then complexify this to a complex Hilbert space $V_\mu^{\mathbb{C}} = V_\mu \otimes \mathbb{C}$. We extend μ and σ to V_μ by continuity[5], and then to $V_\mu^{\mathbb{C}}$ by complex linearity. We define a complex inner product on $V_\mu^{\mathbb{C}}$ by

$$\langle \psi | \phi \rangle = 2\mu(\bar{\psi}, \phi), \tag{2.73}$$

which turns $V_\mu^{\mathbb{C}}$ into a complex Hilbert space. Complex conjugation is defined with respect to the subspace V_μ, ie. if some vector in $\psi \in V_\mu^{\mathbb{C}}$ can be written as $\psi = \phi_1 + i\phi_2$ for $\phi_i \in V_\mu$ then

$$\bar{\psi} = \phi_1 - i\phi_2. \tag{2.74}$$

Because σ is bounded by equation 2.71, we can use Riesz's Lemma in order to obtain a bounded operator $J : V_\mu \to V_\mu$ defined by

$$\sigma(\psi, \phi) = 2\mu(\psi, J\phi), \tag{2.75}$$

which we extend to $V_\mu^{\mathbb{C}}$ by complex linearity. Note that because J is defined as a map on the real Hilbert space it equals its own complex conjugate: $\overline{J} = J$.

Because of the skew-symmetry of σ we have $J^\dagger = -J$. We can also show that J is norm preserving:

$$\mu(\psi, \psi) = \sup_{\phi \in V} \frac{1}{4} \frac{\sigma(\psi, \phi)^2}{\mu(\phi, \phi)} = \sup_{\phi \in V} \frac{\mu(J\psi, \phi)^2}{\mu(\phi, \phi)} \geq \mu(J\psi, J\psi), \tag{2.76}$$

by choosing $\phi = J\psi$ for the last inequality. But, using Cauchy-Schwarz again, we also have

$$\frac{\mu(J\psi, \phi)^2}{\mu(\phi, \phi)} \leq \frac{\mu(J\psi, J\psi)\mu(\phi, \phi)}{\mu(\phi, \phi)} = \mu(J\psi, J\psi). \tag{2.77}$$

[4] For a few practical examples, the completion of the rational numbers $\mathbb{Q}$ in the standard norm yields us the real numbers $\mathbb{R}$. For a more involved example, look at compactly supported continuous functions on $\mathbb{R}$, with respect to the supremum-norm $\|f\|_\infty = \sup\{f(x) \mid x \in \mathbb{R}\}$. The Cauchy completion of this set is the set $C_0(\mathbb{R})$ of functions that go to zero at infinity, but these need not have compact support, they merely need to be small outside every compact set

[5] μ is continuous on V per definition of the topology, and σ is continuous because it is bounded by equation 2.71. They therefore uniquely extend to V_μ

And hence we find that the inequality in 2.76 is in fact an equality. This implies that J must also preserve inner products and therefore that $J^\dagger J = I$. Hence we obtain $J^2 = -I$.

Using the spectral theorem for self-adjoint operators (iJ is self adjoint) we can infer that $V_\mu^{\mathbb{C}}$ splits into two eigenspaces of J with eigenvalues $\pm i$. We will call the $+i$ eigenspace $\mathcal{H}$ and the $-i$ eigenspace $\overline{\mathcal{H}}$. Notice that these two subspaces are related by conjugation: If $\psi \in \mathcal{H}$ and hence has eigenvalue i then

$$J\overline{\psi} = \overline{J\psi} = \overline{i\psi} = -i\overline{\psi}, \tag{2.78}$$

which justifies the notation $\overline{\mathcal{H}}$.

Note that $\overline{\mathcal{H}}$ is naturally isomorphic to $\mathcal{H}^*$, by considering the map $\overline{\psi} \to \langle\psi|$, which is linear because inner-products are anti-linear in their first argument. We will in the future interchangeably use these two definitions to write annihilation operators with arguments in $\overline{\mathcal{H}}$.

This $\mathcal{H}$ is the space we are looking for, with K being the orthogonal projection of $V_\mu^{\mathbb{C}}$ onto $\mathcal{H}$. We can use the calculation in equation 2.69 in order to see that the embedding of V is dense. Concretely, the embedding is given as by $K = \frac{1}{2}(I - iJ)$.

We can now see that our second demand is also satisfied:

$$\begin{aligned}\langle K\psi, K\phi\rangle = 2\mu(\overline{K\psi}, K\phi) &= \frac{1}{2}\mu((I+iJ)\psi, (I-iJ)\phi) \\ &= \mu(\psi,\phi) - i\mu(\psi, J\phi) = \mu(\psi,\phi) - \frac{i}{2}\sigma(\psi,\phi),\end{aligned}$$

which is the equality we set out to achieve. As a final note, we show that this is the same inner product as defined earlier on the space of positive frequency solutions. Note that on $\mathcal{H}$ we have $I = -iJ$, and therefore

$$\langle K\psi, K\phi\rangle = 2\mu(\overline{K\psi}, K\phi) = -i2\mu(\overline{K\psi}, JK\phi) = -i\sigma(\overline{K\psi}, K\phi) \tag{2.79}$$

which is the same expression as 2.63 once we restrict to positive frequency parts, and write $K\psi = \psi^+$.

2.5.3 Physical perspectives

In the previous rather technical sections we showed that all the freedom left in the quantization of the Klein-Gordon field can be described by a positive bilinear form μ on the space of solutions that satisfies

$$\mu(\phi,\phi) = \sup_{\psi\in V} \frac{1}{4}\frac{\sigma(\phi,\psi)^2}{\mu(\psi,\psi)}. \tag{2.80}$$

Therefore such a choice is the same as a choice of vacuum, but in a mathematically rigorous way.

We now turn to the obvious question: Which vacuum should we pick? There are in general a lot of options available to us and it is unclear from a mathematical point of view which one is to be preferred. Luckily in flat spacetime we have a clear directive that singles out a preferred construction: we have a global time parameter

upon which observers can agree. The split of a solution into positive and negative frequency parts with respect to this global time is not ambiguous since the modes used in making the split are invariant under the Poincaré group.

There is a problem going forward, however, since in a general spacetime there are no symmetries. This means that the concept of a particle is bound to become ambiguous in curved spacetimes. This need not be a problem. After all, quantum field theory is a theory of fields and not one of particles. Intuitively however, we like to think about particles because quantum mechanical experiments indicate that nature behaves 'particle-like'.

The main purpose of this thesis is to investigate the possibility to keep some notion of the particle concept that is not global, but rather one that is linked to an observer. As measurements in physics are often dependent on **who** performs the measurement, it seems reasonable that the same might hold true for the concept of quantum particles. We will therefore try to construct a vacuum that is preferred from the point of view of the observer, but that a different observer will likely disagree with.

This can be compared to classical mechanics: Suppose we have two observers, one accelerating and one stationary with respect to the earth. If they both do some experiments juggling massive objects, the stationary observer will come to the conclusion that objects move in a straight line unless some force acts on them, and mathematically describes the relation as $F = ma$. The accelerating observer will not find the same: His trajectories will not be straight and hence he will not find the law that $F = ma$.

If they later meet up and compare their results, they will not agree. They can however make the results match if they introduce fictitious forces in each others frames. As long as this can be done in such a way that both observers can agree on the situation and produce the same predictions, the situation is salvaged.

We would like to have something similar in quantum field theory: If we introduce observers into the equation, can we define different QFT's in such a way that the process is unambiguous? In this way, while observers may not agree on what particles are, they might agree on what the other would regard as his or her particles.

Chapter 3

Quantum Fields in Curved Spacetime

3.1 Introduction

The problem of quantization in curved spacetimes is now clearly visible. In Minkowski spacetime there is a large group of symmetries that enforces a particular choice of vacuum by demanding the vacuum to be invariant. Such a criterion is absent for a general spacetime $(\mathcal{M}, g)$. We therefore do not know which state to choose as the vacuum.

One might hope that the different prescriptions might be unitarily equivalent such that it doesn't matter which state one takes to define the theory. Sadly this is not the case: The Stone-Von Neumann theorem is no longer valid for systems with an infinite amount of degrees of freedom. This means that unitarily inequivalent representations of the canonical commutation relations will arise, and it is not clear which equivalence class of representations is the physical one.

In the second section of this chapter we review the notions of Cauchy surfaces and global hyperbolicity. The general collection of spacetimes is too large for quantum field theory, since the notion of causality is important to the setup of the theory. The demand of global hyperbolicity is that spacetime is causally similar to flat space on a global scale. In the third section we briefly review the generalization of the classical phase-space to such a background.

The fourth section is devoted to defining the concepts of observers and reference frames. In considering what role observers might play in QFT it is important to have a mathematically rigorous notion of observer. We pose a construction of a local reference frame corresponding to a geodesic observer.

The fifth section is allocated to the question when two different choices of μ give rise to unitarily equivalent QFT's. Sufficient and necessary conditions that are needed to ensure that two theories are equivalent are presented and their proofs sketched. From this the existence of inequivalent representations can also be seen as these requirements are not satisfied in general.

The last two sections of this chapter are allocated to a short review of the Unruh-effect as an example of what can happen in QFT in general spacetimes (although it is set in flat spacetime). First we review the connection between modes with respect to inertial time and ones with respect to accelerated time. This leads to the result that the Minkowski vacuum is a thermal state with respect to the Rindler-quantization.

After this we investigate the reality of this thermal bath by coupling the system to a model particle detector, which sheds some light on the interpretation of QFT as a theory of particles.

3.2 Global hyperbolicity and space-time splits

Since in section 2.5 we already did a lot of the work involved in the quantization of the free scalar field the rest of the task is now fairly straightforward. As a first task we need to generalize the classical space of solutions to more general spaces. In order to do this we need to single out some spacetimes that are sufficiently nice for the wave-equation to have solutions.

Let $(\mathcal{M}, g)$ be some four-dimensional spacetime with metric signature $(-,+,+,+)$. Throughout this thesis we will assume spacetime to be time-oriented: a global choice for 'future-pointing' has been made. The metric tensor g is abstractly defined as a map sending two vector-fields to a smooth, real function on spacetime. In terms of components this is given by the contraction of indices:

$$g(\mathbf{X}, \mathbf{Y})(x) = g_{\mu\nu}(x)\mathbf{X}^{\mu}(x)\mathbf{Y}^{\nu}(x), \tag{3.1}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are vector fields and $x \in \mathcal{M}$.

For each spacelike subset $S \in \mathcal{M}$ we can define the timelike future of the set as

$$I^{+}(S) = \{x \in \mathcal{M} \mid \text{There is a future pointing timelike curve connecting } S \text{ to } x\}.$$

We likewise define the timelike past of S. By *causal* we will always mean: timelike or lightlike. Hence we also define the causal past/future $J^{\pm}(S)$ of S as the sets of all points causally connected to S in the past/future. These sets are usually interpreted as all events that can be influenced by events in S, since light-signals travel along lightlike paths.

Related to this is the definition of the domain of dependence of the set S. This is the set of events that is completely and uniquely influenced by S. We define it as:

$$D^{+}(S) = \{x \in \mathcal{M} \mid \text{every past-pointing causal curve without past endpoint through } x \text{ intersects } S\},$$

with $D^{-}(S)$ defined similarly and $D(S) = D^{+}(S) \cup D^{-}(S)$. We see that any information reaching a point in $D^{+}(S)$ must also register on S, and any information leaving a point in $D^{-}(S)$ also does. Concretely: If we know what happens on S, we can infer all that happens in $D(S)$. We exclude curves with a past endpoint since we want to prevent points from falling outside of $D(S)$ simply because we stopped the curves through x before hitting S.

The extra demand that we will put on our spacetimes is that some closed surface Σ exists that is large enough to capture all that happens in $\mathcal{M}$. Concretely, we call a smooth, closed, achronal set $\Sigma \in \mathcal{M}$ a Cauchy surface if $D(\Sigma) = \mathcal{M}$. It follows that every inextendible causal curve in $\mathcal{M}$ hits Σ **exactly once**. We call a spacetime globally hyperbolic if it admits a Cauchy surface.

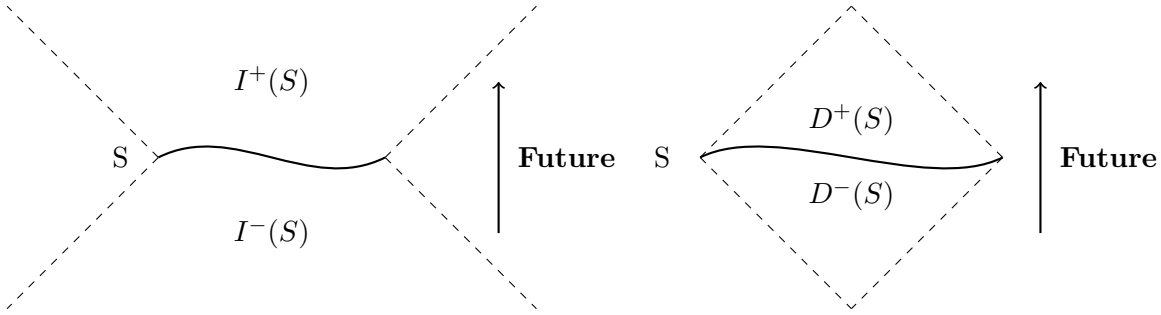

FIGURE 3.1: The left figure shows the timelike past/future of the closed set S. The causal past/future $J^{\pm}(S)$ is the union of the enclosed volume with the dashed boundaries. The right figure shows the past/future domain of dependence of S. Note that these sets are always contained within the causal past/future, but are in general a lot smaller.

A classic theorem by Geroch (Geroch, 1970) states that on a globally hyperbolic spacetime we can always find a global time-function, ie. a smooth function increasing on any future directed curve whose gradient is nowhere zero. Every surface of constant 'time' will then be a Cauchy surface. As such the topology of globally hyperbolic spacetimes is particularly simple, it is homeomorphic to $\mathbb{R} \times \Sigma$ for some 3-manifold Σ. A proof of this fact can be found in propostion 6.6.8 of (Hawking and Ellis, 1973).

This implies that a globally hyperbolic spacetime admits a foliation. A foliation is a global decomposition of spacetime into space and time. Concretely, it is a collection of smooth hypersurfaces Σ_t (which all have the same topology) labeled by a time-coordinate t such that the Σ_t's together cover the entire manifold, whilst no two different surfaces intersect. We can define a time function at a point by looking at which unique Σ_t the point is part of.

We can find coordinates adapted to the foliation as follows. If we choose spatial coordinates $\{x^i\}$ on (a patch of) Σ_t for $i = 1, 2, 3$ then $\{t, x^i\}$ forms a coordinate chart for M. This gives a basis for the tangent space: $\{\partial_t, \partial_i\}$. The vector field ∂_t connects the different slices of the foliation, but is in general not normal to the hypersurface Σ_t. This is because transport along this vector does not necessarily leave the spatial coordinates invariant. Hence we can decompose it into parts that are normal and tangential to Σ_t:

$$\partial_t = \alpha \mathbf{N} + \beta \tag{3.2}$$

Here $\mathbf{N}$ is the future-directed unit normal to Σ_t. We call α the **lapse** of the coordinate system, which is a scalar. The tangential part β is called the **shift** and is a spacelike vector. Loosely speaking the lapse indicates how far away a neighbouring hypersurface of constant time is, and the shift indicates how far one has to move the coordinates around in going to this surface. A straightforward calculation of the metric components in this coordinate chart gives:

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_i \beta^i & \beta_j \\ \beta_i & h_{ij} \end{pmatrix} \tag{3.3}$$

Where h_{ij} is the spatial metric induced on the tangent space of Σ_t by g in the coordinates $\{x_i\}$. The inverse metric can then be calculated:

$$g^{\mu\nu} = \begin{pmatrix} \frac{-1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & h^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix}. \tag{3.4}$$

The purpose of this formalism is to split spacetime into space and time separately. The covariant picture of space and time being the same is mathematically elegant but often not very practical in calculations. Choosing a Cauchy surface, a lapse and a shift effectively unravels the union of space and time. If coordinates are chosen such that β vanishes, the spacetime metric takes a form where space and time are not mixed at all.

It should be clear that these choices are not unique: We can slice spacetime in many different ways, and when we have done so there are many different choices for lapse and shift that are available. As such, no dependence of physical observables on these choices is allowed.

3.3 Generalization of the classical phase space

We now continue to the covariant generalization of the classical field system. Clearly we should swap out the partial derivatives for covariant derivatives in order to obtain a covariant equation. In fact, it is only because we posed the Klein-Gordon equation in Cartesian coordinates that we did not need to do so before, because in the flat case the Christoffel symbols vanish and covariant and partial derivatives are the same. From this point onward, we will always use the symbol ∇ to denote covariant derivatives, and the KG-equation becomes [1]:

$$(g^{\mu\nu}\nabla_\mu\nabla_\nu - m^2)\phi = (\Box - m^2)\phi = 0, \tag{3.5}$$

where the d'Alembertian operator is defined by $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. We have the following theorem, originally due to Leray:

Theorem. *Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime with Cauchy surface Σ and let N be the normal vector to Σ. If ϕ and π are two smooth functions on Σ supported within some compact subset K then there is a unique smooth solution ψ to the KG-equation such that $\psi|_\Sigma = \phi$ and $N^\mu\nabla_\mu\psi|_\Sigma = \pi$. This solution has compact support on any other Cauchy surface and is supported in $J^+(K) \cap J^-(K)$. Furthermore, if we vary the initial conditions outside of some closed subset S of Σ then the solution remains unchanged within $D(S)$.*

For a proof of this theorem we refer to section 7.4 of (Hawking and Ellis, 1973) or section 10.1 of (Wald, 1984). A self-contained treatment of wave equations in curved spacetimes can be found in (Bar, Ginoux, and Pfaffle, 2007).The theorem above states that in a globally hyperbolic spacetime solutions to the KG equation are uniquely characterized by their footprint on a Cauchy surface, and that this information is enough to reconstruct the solution. The propagation of information is causal

[1] The potential term admits a new term of the right physical dimensions, namely some constant times the Ricci scalar. This is often added with prefactor $\frac{1}{6}$, since this renders the equation conformally invariant. This allows many interesting examples to be explicitly calculated in conformally flat spacetimes. We will not add it here, for the reason that it adds little to the discussion and we see no reason to introduce some extra coupling to gravity on top of changing the background metric to a curved one.

in the sense that was discussed above.

In the previous chapter we described solutions to the KG-equation by the initial data they have at $t = 0$. Clearly the set $t = 0$ is a 3-surface in Minkowski space that hits every causal curve exactly once, and hence it is a Cauchy surface. The theorem above generalizes this: In a globally hyperbolic spacetime we pick some Cauchy surface Σ and set

$$V = C_c^\infty(\Sigma) \bigoplus C_c^\infty(\Sigma), \tag{3.6}$$

consisting of pairs (ϕ, π) of smooth functions of compact support on Σ. From the above theorem it follows that it is not important which Σ we take: If we take some compactly supported smooth initial conditions on one Cauchy surface, it uniquely infers smooth compactly supported data on any other. The symplectic form is generalized to

$$\sigma((\phi_1, \pi_1), (\phi_2, \pi_2)) = \int_\Sigma (\pi_1\phi_2 - \pi_2\phi_1)\sqrt{h}d^3x \tag{3.7}$$

By the theorem above, there is a one-to-one correspondence between V and solutions to the wave equation. If we make this identification between ψ and (ϕ, π) the symplectic form reads

$$\sigma(\psi_1, \psi_2) = \int_\Sigma (\psi_1 \overleftrightarrow{\nabla}_\mu \psi_2) d\Sigma^\mu. \tag{3.8}$$

Here $d\Sigma^\mu = N^\mu\sqrt{h}d^3x$ is the surface measure of Σ, where N^μ is the future pointing normal vector to Σ and h is the determinant of the spatial metric. We can use Gauss's theorem to see that this does not depend on the Cauchy-surface that is used: Suppose we have Σ_1, Σ_2 Cauchy surfaces and denote the volume enclosed by Ω, then Gauss's theorem gives us the equality:

$$\sigma_1(\psi_1, \psi_2) - \sigma_2(\psi_1, \psi_2) = \int_\Omega \nabla^\mu(\psi_1 \overleftrightarrow{\nabla}_\mu \psi_2) = \int_\Omega \psi_1(m^2 - m^2)\psi_2 = 0 \tag{3.9}$$

This gives us two equivalent ways of looking at the classical phase space. These constructions use 3 dimensional test-functions to smear the quantum field such that it is well-defined. An equivalent construction is to use 4-dimensional test functions, and we will sometimes also take this viewpoint.

Thus, look at the space of 4-dimensional test functions $C_0^\infty(M)$ of smooth functions of compact support on spacetime. This construction makes use of the fact that for $f \in C_0^\infty(M)$ we can solve the Klein-Gordon equation with source f. The retarded and advanced solutions are defined by

$$(\Box - m^2)Rf = f, (\Box - m^2)Af = f, \tag{3.10}$$

where $Rf = 0$ outside of the future of the support of f, and $Af = 0$ outside of the past of the support. Then $(A - R)f = Ef$ is a solution to the homogeneous wave equation, which registers compactly on any Cauchy-surface because the support of f is compact. Hence we find that E is a map of $C_0^\infty(M) \to V$. One can show (Wald, 1995) that this map is surjective, and that its kernel is exactly the image of $(\Box - m^2)$. Hence we find

$$V \cong C_c^\infty(M)/(\Box - m^2)C_c^\infty(M) \tag{3.11}$$

which is another equivalent way to portray the classical phase space. An important identity linking the symplectic forms is, for $f \in C_0^\infty(M)$ and ψ a solution to the wave equation:

$$\int_M f\psi dV = \sigma(Ef, \psi) \tag{3.12}$$

To see this, let $\Sigma^\pm$ be two Cauchy surfaces to the future resp. past of the support of f. We then calculate

$$\begin{aligned}
\int_M f\psi dV &= \int_M \Big((\Box - m^2)Af\Big)\psi dV \\
&= \int_M (Af)(\Box - m^2)\psi + \nabla^\mu((\nabla_\mu Af)\psi) - \nabla_\mu((Af)\nabla^\mu\psi)dV \\
&= \int_{\Sigma^-} Af \overleftrightarrow{\nabla}_\mu \psi d\Sigma^\mu - \int_{\Sigma^+} Af \overleftrightarrow{\nabla}_\mu \psi d\Sigma^\mu \\
&= \int_{\Sigma^-} Ef \overleftrightarrow{\nabla}_\mu \psi d\Sigma^\mu.
\end{aligned}$$

In the second step we used partial differentiation twice to move the derivatives to ψ. In the third step we used Gauss' lemma, as well as the fact that ψ solves the KG-equation. Note that Gauss' lemma instructs us to use the inward-facing normal, which is in this step minus the future pointing normal on Σ_+. In the last step we used that $Af = Ef$ on Σ^- and that $Af = 0$ on Σ^+.

We therefore see that on this realization of the symplectic space the symplectic form must be given by:

$$\sigma([f], [g]) = \sigma(Ef, Eg) = \int_M fEg dV. \tag{3.13}$$

We therefore define the four-smeared quantum field by:

$$\hat{\Phi}(f) = \int_M f(x)\hat{\Phi}(x)dV = \hat{\sigma}(Ef, .), \tag{3.14}$$

which gives rigorous meaning to $\hat{\Phi}(x)$ as an operator valued distribution.

3.4 Observers and reference frames

In this thesis we will be looking at the role observers play in quantum field theory, and hence a treatment of the concept of an observer is in order. We will loosely follow the treatment of the subject given in (Sachs and Wu, 1977). The account given here is relativistic rather than quantum mechanical: In order to measure some property of a quantum field we will first need to couple the field to a detector in order to be able to extract any information. The role of detectors will be discussed below in our treatment of the Unruh effect.

A realistic observer should move across a future directed worldline that describes the path he takes through spacetime. Hence we define an observer as any curve $\gamma : I \to M$, where I is some (possibly unbounded) interval in $\mathbb{R}$, whose tangent vector is everywhere future-directed timelike. Furthermore, we demand that the curve is parametrized in terms of its proper time, which means that we have at each point along γ

$$g(\dot{\gamma}, \dot{\gamma}) = -1. \tag{3.15}$$

Here $\dot{\gamma}$ denotes the derivative with respect to the proper time along the curve.

A special class of observers is that of the geodesic observers. These are observers that are freely falling, ie. not under any acceleration other than that coming from the background metric. They are defined by the demand that they should parallel-transport their own tangent vector:

$$\nabla_{\dot{\gamma}} \dot{\gamma} = \dot{\gamma}^\mu \nabla_\mu \dot{\gamma}^\nu = 0. \tag{3.16}$$

Written out in terms of coordinates this equation reads [2]:

$$\frac{d^2\gamma^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \frac{d\gamma^\rho}{d\tau} \frac{d\gamma^\sigma}{d\tau} = 0. \tag{3.17}$$

These equations express that there is no **covariant** acceleration along the worldline of the observer, and as such they are the generalization of a straight line in curved space. In flat spacetime, geodesic observers are inertial observers.

This definition of observer is very local in nature, since at each instance in proper time only one event of the spacetime is occupied by the observer. It is therefore unlikely that this observer would be able to make interesting measurements. It is the cooperation between different observers which allows meaning-full measurements to be made.

We therefore have to look at vector fields on $\mathcal{M}$, and not just at ones defined on the worldline of the observer. A vector field is a choice, at each point in $\mathcal{M}$, of a vector in the tangent space at that point. Moreover, this choice has to be made in such a way that the resulting collection of vectors varies smoothly across the manifold. Mathematically speaking, we define the vector fields $\mathcal{X}(\mathcal{M})$ of a manifold as the space of all derivations of smooth functions. These are linear operators $D : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ which satisfy the Leibniz-rule:

$$D(fg) = D(f)g + fD(g). \tag{3.18}$$

Other, equivalent definitions of vector fields are available, but we prefer this definition because it is mathematically clean and makes no reference to any specific coordinate chart. This definition is likely more familiar to the reader if we consider what a vector field looks like locally. Within some coordinate patch of $\mathcal{M}$ a vector field $\mathbf{X}$ has the following form:

$$\mathbf{X}(x) = \mathbf{X}(x)^\mu \nabla_\mu, \tag{3.19}$$

where the $\mathbf{X}^\mu$ are smooth functions of the coordinates and ∇ denotes the covariant derivative. This gives rigorous meaning to the intuition that a vector field is a smooth choice of a vector in each point.

Dual to the notion of a vector field is that of a 1-form. This is a smooth map sending any vector field to a real number, and we'll denote the set of 1-forms by $\Omega(\mathcal{M})$[3]. We

[2] Observe that because $\dot{\gamma}$ is a vector the covariant derivative does not simplify to a partial derivative. A Christoffel symbol therefore appears, and is contracted with the ν index.

[3] A superscript of $(0, 1)$ is usually added here to signify that this is a form as a function of 1 vectorfield. Since we will not need a notion of higher order tensors we'll omit this superscript.

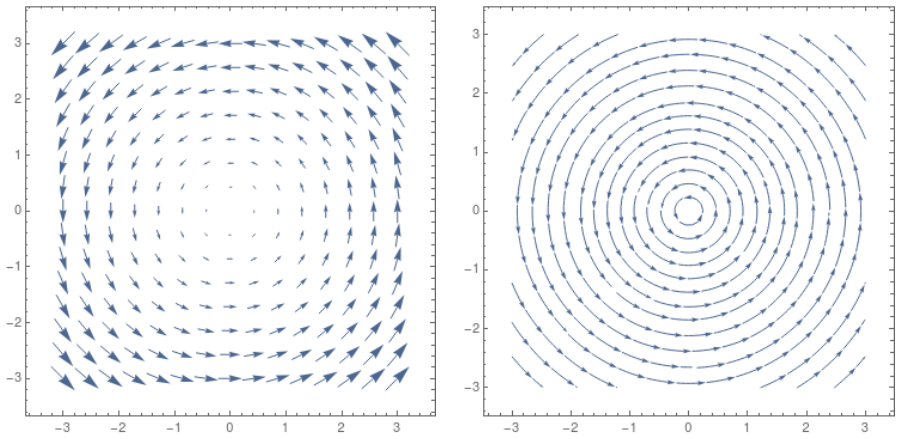

FIGURE 3.2: The vector field $\mathbf{X} = x\partial_y - y\partial_x$ and its integral curves.

have a natural class of 1-forms given by smooth functions f on $\mathcal{M}$. We define its derivative 1-form $df \in \Omega(\mathcal{M})$ as the map sending $\mathbf{X} \in \mathcal{X}(\mathcal{M})$ to

$$df(\mathbf{X}) = \mathbf{X}(f). \tag{3.20}$$

Because we have a non-degenerate metric on our spacetime there is a one-to-one link between vector fields and 1-forms. We define for $\mathbf{X}, \mathbf{Y} \in \mathcal{X}(\mathcal{M})$ and $\omega \in \Omega(\mathcal{M})$ the *musical isomorfisms* $\flat : \mathcal{X}(\mathcal{M}) \to \Omega(\mathcal{M})$ and $\sharp : \Omega(\mathcal{M}) \to \mathcal{X}(\mathcal{M})$ by the following relations:

$$\mathbf{X}^\flat(\mathbf{Y}) = g(\mathbf{X}, \mathbf{Y}), \tag{3.21}$$

$$\omega(\mathbf{Y}) = g(\omega^\sharp, \mathbf{Y}). \tag{3.22}$$

The gradient of a smooth function f is then defined as

$$\nabla f = df^\sharp, \tag{3.23}$$

in terms of components it is given by

$$\nabla^\mu f = g^{\mu\nu}\nabla_\nu f. \tag{3.24}$$

Intimately related to the concept of a vector field is that of its flow. The flow of a vector field is the collection of paths one obtains by 'following the arrows' of the field at each point. Concretely, we call a curve γ in $\mathcal{M}$ an integral curve of a vector-field $\mathbf{X}$ if it satisfies

$$\dot{\gamma}(t) = \mathbf{X}(\gamma(t)), \tag{3.25}$$

which means that the tangent vector to γ at some point should always be along $\mathbf{X}$. As an example, consider figures 3.2(a) and 3.2(b). Here we look at the vector field given by $\mathbf{X} = x\partial_y - y\partial_x$ in two dimensions. The left figure shows the vector field itself, while the right figure shows the path that you obtain by following the vector field. In this case these are circles of different radii.

With these preliminaries out of the way we can now to enlarge our concept of observer. We call a vector field $\mathbf{X}$ a *reference frame* if each of its integral curves is an observer. Concretely this means that X is future directed and that

$$g(\mathbf{X}, \mathbf{X}) = -1. \tag{3.26}$$

The interpretation of a reference frame is that it represents a collection of fictitious observers that can work together in order to establish non-local measurements. A reference frame is called *synchronizable* if functions t and h on M exist such that $\mathbf{X} = -h\nabla t$, and we call it *proper time synchronizable* if we can choose $h = 1$. These reference frames allow the different observers in the frame to synchronize their clocks [4] and define surfaces of constant time t. This effects a spacetime split: The observers agree that the surfaces of constant t form space and global time is equal to each observers proper time. This is the best case scenario, but we are not guaranteed the existence of a unique global synchronizable reference frame containing γ.

Clearly the concept of a reference frame is a global notion: Vector fields are defined on the whole of spacetime and are sensitive to the global properties of the manifold. As such it would be too optimistic to expect an observer to induce a unique reference frame that he is one of the observers of. In general there will be many different reference frames that extrapolate a single observer γ.

To restrict this choice, we can ask for some criterion to be satisfied which implements the idea that the reference frame 'behaves like the observer γ'. For example, since we will be looking at geodesic observers it would be natural to ask for a reference frame which is geodesic [5]. However, because gravity is attractive, geodesics are likely to cross and such a reference frame will not exist in any realistic model.

The problem is that, while there are many extensions of a single observer to a reference frame, there are in general no global extensions with nice properties. We can, however, **locally** define reference frames around the worldline of the observer. While this does not yield the full notion of a reference frame, we would argue that this notion is unphysical.

A realistic observer can not measure anything that has a large spatial separation from his own worldline. Cooperation between multiple observers can increase the range of measurements that can be performed, but this would always require different observers to communicate to compare their findings. This can only be done meaningfully if the different observers can synchronize their clocks, since then they can compare their measurements with an agreement to *when* they were made. There are schemes for synchronizing clocks between observers, such as the radar method. These methods are, however, not globally applicable, see for example (Alba and Lusanna, 2003).

We therefore take the viewpoint that a realistic reference frame should always be locally defined on some open set inside $\mathcal{M}$ which contains part of γ. This corresponds to the idea that the observer is able to operate some spatially extended apparatus to perform measurements away from the exact position of his worldline, and that he could communicate with other observers which are close. Global measurements are, however, not practically possible and we should therefore not try to relate γ to a

[4] Mathematically this is clear. Practically, the different observers within the frame which are close to each other can send each other messages by ways of light signals and use this information to synchronize their clocks. For a detailed description of how this would work we refer the reader to chapter 5 of (Sachs and Wu, 1977).

[5] In the sense that all its integral curves are geodesics.

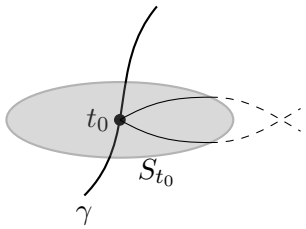

FIGURE 3.3: The local rest space S_{t_0} of the curve γ. We are guaranteed the existence of a neighbourhood such that no two different geodesics cross within. They will likely cross outside of this neighbourhood however, and at these points the defined set will probably not look like a 3-surface. Note that in this figure we depict S_{t_0} as 2-dimensional object, whilst really it is 3-dimensional. The situation is hard to visualize in 3 dimensions, but it should be clear that when geodesics start to cross the nature of S_{t_0} as a smooth 3-dimensional manifold can no longer be guaranteed.

global reference frame.

We describe two local reference frames below, which we will call the Fermi frame and the Gaussian frame. Both of these are synchronizable, as any useful local reference frame should be, and are defined for an open set around some point along the geodesic.

3.4.1 The Fermi Frame

The Fermi frame is based on the construction of Fermi normal coordinates, see (Manasse and Misner, 1963). This is an adapted form of the Riemann normal coordinates, which has a nice form on a geodesic, and not just in a single point.

One proceeds as follows: Along the geodesic γ we can define the *local rest space* at time t_0 by considering all the vectors normal to $\dot{\gamma}(t_0)$. This defines a spacelike linear subspace of the tangent space in this point. These vectors can be expanded geodesically [6] to yield a 3-surface S_{t_0} in $\mathcal{M}$ which we interpret as the local definition of 'space at time t_0' according to the observer γ. This surface is not global, because there is no guarantee that the collection of spacelike geodesics starting from $\gamma(t)$ continue to behave as a surface. Because of this we need to limit the curve-parameters of the geodesics so as to yield a smaller surface, which will in general not be Cauchy.

We can perform this procedure for each instant in proper time t. This gives us a collection of local rest spaces S_t along the geodesic. The rest spaces for different times may intersect, but we can always shrink them such that they do not [7]. We are in fact, as proven in section 4 of (Manasse and Misner, 1963), guaranteed an open neighbourhood of γ which is foliated by the S_t *for all times* t for which the geodesic is defined.

[6] We use the word geodesic here in the mathematical sense: It is a curve that parallel-transports its own tangent vector. Since these are *spacelike* geodesics we do not interpret these curves as paths a test-particle can take, as is often the meaning that is given to the word in physics literature.

[7] Look for example at figure 3.4(a). The different rest spaces S_{t_i} do not intersect here. However, if we were to extend S_{t_1} further, it might cross S_{t_0} or S_{t_2}. It is therefore important to choose the S_t suitably small.

The proper time function t then extends to this neighbourhood: The value of t at some point is determined by the unique S_t that point is part of. As such we have extended the proper time of the observer from the geodesic to a time function on the surrounding spacetime.

Now to define the reference frame: For each moment in time t the vector $\dot{\gamma}(t)$ is normal to S_t by construction. We can expand it by parallel transport along S_t, since for each point in S_t there is a unique radial geodesic connecting it to γ(t). Because parallel transport preserves inner products between vectors, the so defined vector field coincides with the normal vector field $\mathbf{n}$ to S_t.

Since the different S_t foliate spacetime around the geodesic γ this defines a vector-field $\mathbf{X}$, given by all the normal vectors to the rest spaces at different times. Smoothness of this vector field follows from the smoothness of the time-function t. To see this we note that we can write $\mathbf{X}$ as

$$\mathbf{X} = -N\nabla t, \qquad N = (-g(\nabla t, \nabla t))^{-1/2}, \tag{3.27}$$

where ∇t is the gradient vector field of t. Per definition of S_t as a surface of constant t, ∇t is orthogonal to S_t, and is therefore proportional to $\mathbf{X}$. Adding the factor N ensures equality because it makes the norms match.

By construction this vector-field is a reference frame: It is of unit length and is future directed. It is, however, not geodesic. To see this, we calculate the acceleration of $\mathbf{X}$:

$$\begin{aligned}
\mathbf{X}^\mu \nabla_\mu \mathbf{X}^\nu &= -\mathbf{X}^\mu \Big((\nabla_\mu N)(\nabla^\nu t) + N(\nabla^\nu \nabla_\mu t) \Big), && (3.28)\\
&= \mathbf{X}^\mu \Big((\nabla_\mu N)\frac{\mathbf{X}^\nu}{N} + N\nabla^\nu \Big(\frac{\mathbf{X}_\mu}{N} \Big) \Big), && (3.29)\\
&= \mathbf{X}^\mu \Big((\nabla_\mu N)\frac{\mathbf{X}^\nu}{N} - \frac{\nabla^\nu N}{N}\mathbf{X}_\mu + \nabla^\nu \mathbf{X}_\mu \Big), && (3.30)\\
&= (g^{\mu\nu} + \mathbf{X}^\mu \mathbf{X}^\nu)\nabla_\mu \log(N). && (3.31)
\end{aligned}$$

In the final step we used that $\mathbf{X}^\mu \mathbf{X}_\mu = -1$ and therefore that $\mathbf{X}^\mu \nabla^\nu \mathbf{X}_\mu = 0$.

We therefore see that the acceleration of our reference frame is equal to the spatial projection of the derivative of $\log(N)$. Because our foliation is not necessarily **proper time** synchronizable the function N is not constant along the S_t, and therefore this frame will in general not be geodesic.

The Fermi frame defined above has the interpretation of a rigid laboratory being dragged along by the observer γ. As such not all points of the frame will move freely when the force of gravity is not uniform. For our purposes this frame is the most natural: It describes the situation of a single observer whose worldline is extended in such a way that the other observers in the reference frame remain at a constant distance from him. For a more general treatment of accelerating laboratories in this context we refer also to section 2 of (Buchholz and Verch, 2016), where this frame is constructed for arbitrarily accelerated observers.

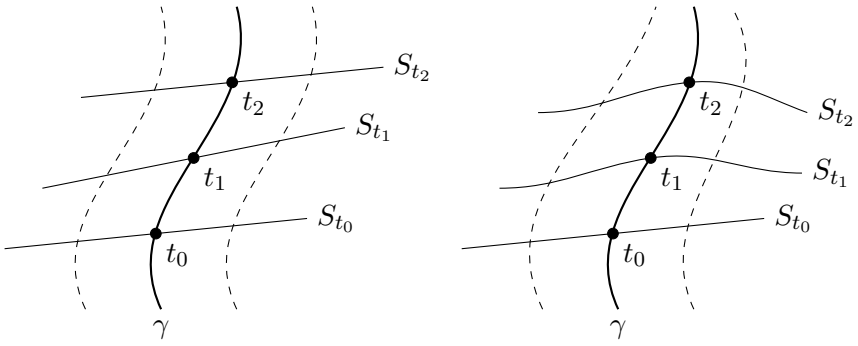

FIGURE 3.4: Our two local reference frames. The Fermi frame is defined by the local rest frames S_{t_0} and the normal vectors to these hypersurfaces. The Gauss frame is the same on t_0 but different afterwards, because we extend the vectors geodesically. The surfaces of simultaneity of this are therefore **not** the same as the S_t.

3.4.2 The Gaussian frame

A different approach is to use Gaussian normal coordinates. This procedure gives us a geodesic reference frame, but it is not defined for all times and depends heavily on the initial time chosen for the construction. The different observers in this frame are all treated equally and the central observer γ is of less significance. This reference frame has the interpretation of a freely falling dust which has no self-interaction. It is of less importance to the present investigation, but because it gives a nice illustration of the different constructions that are available, we will review it shortly.

First, we pick some initial moment of proper time t_0 and consider the local rest space S_{t_0}, as defined above. As with the Fermi frame, we geodesically transport $\dot{\gamma}(t_0)$ around this surface to obtain the normal vector field $\mathbf{n}$ to S_{t_0}. Now, at each point $x \in S_{t_0}$ we can look at the unique timelike geodesic $\gamma_x(s)$ defined by the initial conditions $\gamma_x(0) = x$ and $\dot{\gamma}_x(0) = \mathbf{n}(x)$. These geodesics will likely cross to the future of S_{t_0}, and therefore we have to restrict these curves in order for them to define a reference frame. Once we have done so, we define the Gaussian reference frame $\mathbf{X}$ by the tangent vectors to these geodesics. This gives us, by definition, a geodesic reference frame.

This construction also gives us a time function and a notion of spatial surfaces of simultaneity, but these will be different from the ones related to γ in the Fermi frame. For each point y within the neighbourhood where $\mathbf{X}$ is defined, there is a unique geodesic in the reference frame linking y to a point x in S_{t_0}, see figure 3.5. We define $t_G(y)$ as the geodesic distance along this curve to x. We can then define $\tilde{S}_t$ as the surface of constant $t_G = t$. By definition $\gamma(t)$ lies within $\tilde{S}_t$, but other than this point the $\tilde{S}_t$ will be different from the S_t defined above.

The two notions of simultaneity differ because of the order in which we evolve our geodesics. For the Fermi time, we first evolve along γ and then in the spatial directions orthogonal to γ . For the Gaussian frame, we first evolve in the spatial directions, and then along the geodesic γ_x. In the presence of curvature the path one takes becomes important, and therefore the two time functions do not match in general.

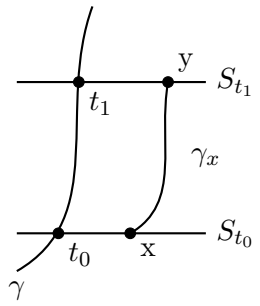

FIGURE 3.5: The difference between the Fermi time function and the Gaussian time function. The value of the Fermi time at y is equal to t_1. However, the value of the Gaussian time is given by the length of the geodesic path γ_x between S_{t_0} and S_{t_1}. In the presence of curvature the length of this path will not in general be the same as that of γ between the same two local rest spaces.

As a final word of warning, note that a reference frame is not the same as a coordinate system, although the concepts are very similar and sometimes interchangeable. Coordinate charts locally prescribe spacetime as a copy of $\mathbb{R}^4$ and are invaluable in performing practical calculations. They are the formal expression of the idea that a manifold is 'locally flat' for someone living on that manifold. They do not carry any physical significance by themselves however, and physical laws can not depend on the coordinates chosen to describe them in.

A reference frame is a covariant object and as such it does bear physical significance. It formalizes the idea of a family of observers that can measure properties of spacetime. It is not necessary that entities expressed in terms of different reference frames are the same: They are measurements related to different sets of observers and these may differ.

3.5 Unitary equivalence and particle production

Having reviewed the preliminaries from general relativity we now proceed to consider quantization of the system. In fact most of the work has now been done: In section 2 we generalized the classical field system that is under consideration. This is given by equations 3.6 and 3.8. As noted before, the Cauchy surface chosen in the definition is unimportant, except for the fact that one needs to exist. The only thing left to do is pick some real bilinear form on V that satisfies

$$\mu(\phi,\phi) = \sup_{\psi\in V} \frac{1}{4}\frac{\sigma(\phi,\psi)^2}{\mu(\psi,\psi)}. \tag{3.32}$$

After we have done so we can apply the construction of section 2.5 to obtain a Hilbert space with a representation of the canonical commutation relations. We no longer have the Poincare group in a general spacetime and as such we can not use it to pick out a preferred representation. As such the choice of the vacuum is no longer clear.

Systems in quantum mechanics are always defined up to unitary equivalence. Suppose we have some system $(\mathcal{H}_1, \mathcal{A}_1)$ of a Hilbert space together with an algebra of

observables. The physically measurable part of the theory are the expectation values of the observable, which are numbers of the form

$$\langle\psi|\mathcal{O}|\psi\rangle. \tag{3.33}$$

These numbers do not completely fix the Hilbert space and the algebra of observables. We might have some different quantum mechanical system $(\mathcal{H}_2, \mathcal{A}_2)$ that produces the same expectation values and hence describes the same physics. Suppose we have some bijective map $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U^*U = I$. This induces an action on the algebra of observables $U : \mathcal{A}_1 \to \mathcal{A}_2$ by

$$U(\mathcal{O}) = U\mathcal{O}U^*. \tag{3.34}$$

Then we have

$$\langle U\psi|U(\mathcal{O})|U\psi\rangle = \langle\psi|U^*U\mathcal{O}U^*U|\psi\rangle = \langle\psi|\mathcal{O}|\psi\rangle. \tag{3.35}$$

Hence if the action of U is also bijective on the algebra of observables then the two systems describe the same physics and we will call them unitarily equivalent. Hence some of the choices of μ that satisfy 3.32 might give rise to unitarily equivalent theories, and hence the amount of true freedom available in quantization might be smaller than it seems. In this section we will consider when two different choices of μ give rise to equivalent representations.

Suppose we have two bilinear forms μ_1 and μ_2 satisfying equation 3.32. From this we can construct two Fock-spaces $\mathcal{F}_1$ and $\mathcal{F}_2$ carrying two representations of the CCR $\hat{\sigma}_1$ and $\hat{\sigma}_2$ as was done in the previous chapter. From the fact that a unitary transform U between the two Fock spaces should intertwine the action of the observables we need, for any $\psi \in V$, that:

$$U\hat{\sigma}_1(\psi, .)U^* = \hat{\sigma}_2(\psi, .). \tag{3.36}$$

A first necessary condition that must be satisfied is that the completions of the vector space V to a real Hilbert space V_{μ_1} resp. V_{μ_2} must be the same. Loosely speaking, if one of the completions is larger than the other it will be impossible to map the two spaces bijectively into each other, since one will contain 'more' vectors than the other.[8]

Consequently, in order to ensure that the completions are the same we have to make sure that Cauchy sequences with respect to μ_1 are Cauchy sequences with respect to μ_2, and hence we need the existence of two positive constants C_1 and C_2 such that for all $\psi \in V$

$$C_1\mu_1(\psi, \psi) \leq \mu_2(\psi, \psi) \leq C_2\mu_1(\psi, \psi). \tag{3.37}$$

This ensures that a sequence $\{\psi_n\}$ is Cauchy with respect to one of the norms if and only if it is Cauchy with respect to the other. This effects that $V_{\mu_1} = V_{\mu_2}$, and that the two one-particle Hilbert spaces have the same size.

This first condition is not enough to ensure that the two constructions are equivalent, however. A second condition should be satisfied to ensure that the particle content of one of the vacua contains a finite amount of particles with respect to the particle concept of the other construction. I will sketch the procedure and the interpretation

[8] Both will be infinite dimensional Hilbert spaces so 'more vectors' is obviously a bit ambiguous. For a more rigorous treatment of the details, follow any of the references given below.

below. A more complete derivation can be found in (Wald, 1979), and we refer to (Araki and Yamagami, 1982) for a full treatment of all the mathematical details.

The necessary and sufficient second condition can be phrased as follows. Because μ_1 and μ_2 give rise to the same V_μ by the first condition, the one-particle Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are subspaces of the same larger Hilbert space $V_\mu^{\mathbb{C}}$. They differ in the complex structure that is used to split $V_\mu^{\mathbb{C}}$ into two conjugate spaces. We can define projections between the two different constructions by restricting the maps K_i. We write:

$$\begin{aligned} A &= K_1|_{\mathcal{H}_2} : \mathcal{H}_2 \to \mathcal{H}_1 \\ B &= \overline{K}_1|_{\mathcal{H}_2} : \mathcal{H}_2 \to \overline{\mathcal{H}}_1 \\ C &= K_2|_{\mathcal{H}_1} : \mathcal{H}_1 \to \mathcal{H}_2 \\ D &= \overline{K}_2|_{\mathcal{H}_1} : \mathcal{H}_1 \to \overline{\mathcal{H}}_2 \end{aligned}$$

Recall here that complex conjugation is defined with respect to the real Hilbert space V_μ inside $V_\mu^{\mathbb{C}}$. Earlier we defined the fundamental observables as

$$\hat{\sigma}(\phi, .) = i[a(\overline{K\phi}) - a^\dagger(K\phi)] = i[a(\overline{K}\phi) - a^\dagger(K\phi)]. \tag{3.38}$$

In the second step we used the identification between $\mathcal{H}_1^*$ and $\overline{\mathcal{H}}_1$ given by

$$\langle\phi| \to \overline{\phi}, \tag{3.39}$$

together with the fact that ϕ is real.

In the future we will also need the complex conjugate maps to A, B, C and D. These are defined on the conjugate domain and with conjugate image in the obvious way:

$$\begin{aligned} \overline{A} : \overline{\mathcal{H}_2} &\to \overline{\mathcal{H}_1}, \\ \overline{A}\overline{\psi} &= \overline{A\psi}, \end{aligned}$$

and similarly for the other operators.

Now suppose that the two constructions are related by some unitary U which satisfies equation 3.36. We can then look at the image of the vacuum of the first construction, which will be some nonzero vector in $\mathcal{F}_2$:

$$\Psi = U|0\rangle_1 \tag{3.40}$$

We write out equation 3.36 in terms of creation and annihilation operators:

$$U[a_1(\overline{K_1}\psi) - a_1^\dagger(K_1\psi)]U^* = a_2(\overline{K_2}\psi) - a_2^\dagger(K_2\psi). \tag{3.41}$$

Originally we defined the creation and annihilation operators only for real vectors ψ originating from V. However, the annihilation and creation operators are defined for any vector in H_i resp. $\overline{H}_i$. Because of complex linearity and continuity of $\hat{\sigma}_i$ [9]

[9] We implicitly assumed all representations to be strongly continuous. Some continuity assumption is reasonable, but since the annihilation and creation operators are not bounded operators we cannot

equation 3.41 holds for any $\psi \in V_\mu^{\mathbb{C}}$.

Now we choose some vector $\bar{\chi} \in \overline{\mathcal{H}_1}$ as the argument ψ. Then the creation operator drops out since $K_1\bar{\chi} = 0$. It can be shown that C is invertible and as such we can write $\chi = C^{-1}\xi$ for $\xi \in H_2$. We let this act on the vector Ψ created out of the vacuum by U. The left hand side is zero, since we act on the vacuum with an annihilation operator. Hence we find the following equation:

$$0 = \left[a_2(\bar{\xi}) - a_2^\dagger(\overline{DC^{-1}}\bar{\xi})\right]\Psi. \tag{3.42}$$

This equation expresses that, unless $DC^{-1} = 0$, the state Ψ is not the vacuum of $\mathcal{F}_2$, since this is defined by

$$0 = a_2(\bar{\xi})|0\rangle_2. \tag{3.43}$$

Hence, even if the two representations are unitarily equivalent, the vacuum is an ambiguous concept: The unitary transformation need not carry over the vacuum-vectors of the different theories in order to describe the same physics.

We still have to answer the question of whether a unitary map U exists. To do this we further examine equation 3.42. We can always write out Ψ in terms of n-particle vectors:

$$\Psi = (\psi_0, \psi_1, \psi_2, \dots), \tag{3.44}$$

since this is how the Fock-space is defined. Equation 3.42 must be satisfied in all entries, hence we get an infinite amount of equations that have to be satisfied. We write $E = \overline{DC^{-1}} : \overline{\mathcal{H}_2} \to \mathcal{H}_2$ and find:

$$\sqrt{i+1}\bar{\xi} \cdot \psi_{i+1} = \sqrt{i}(E\bar{\xi}) \cdot \psi_{i-1}. \tag{3.45}$$

It is understood that ψ_{-1} is zero in this equation. Since this equation has to be valid for any $\xi \in \mathcal{H}_2$ this means that $\psi_1 = 0$, and by induction it follows that ψ_i is zero for every odd i. Hence, in a free theory, particles are always produced in pairs.

Clearly then ψ_0 must be nonzero, otherwise the rest of the contributions would also vanish. We take it equal to 1 here and can later normalize the vector. We see then that all the contributions are fixed by ψ_2 and hence by E, since we find

$$E\bar{\xi} = \sqrt{2}\bar{\xi} \cdot \psi_2, \tag{3.46}$$

which links E to $\sqrt{2}\psi_2$ as maps from $\overline{\mathcal{H}_2}$ to $\mathcal{H}_2$. We can therefore identify E with $\epsilon = \sqrt{2}\psi_2$. This implies that the other ψ_i's are given by

$$\psi_i = \sqrt{\frac{i-1}{i}}\epsilon \otimes_s \psi_{i-2}, \tag{3.47}$$

which fixes all the ψ_i as symmetric tensor products of ϵ inductively. At last we come to the condition that must be satisfied by the two μ's. The vector $\epsilon \in \mathcal{H}_2 \otimes_s \mathcal{H}_2$ must be bounded in the inner product defined at the beginning of section 4 of the previous chapter. In order for E to give rise to a reasonable vector ϵ it should satisfy

$$\mathrm{Tr}(E^\dagger E) < \infty, \tag{3.48}$$

ask for norm-continuity. Because these operators are all defined on the same dense domain, namely the states with finite amounts of particles, we can reasonably ask for strong continuity, which is the next best thing.

which is not immediate for bounded operators on an infinite dimensional Hilbert space. This condition can be shown to be the same as the alternative condition that

$$\mathrm{Tr}(B^\dagger B) < \infty. \tag{3.49}$$

This is due to the fact that C is bounded, and $B^\dagger = -\overline{D}$. If this condition is satisfied the vector Ψ can be shown to be normalizable and the rest of the map U is defined straightforwardly, see the end of section 2 of (Wald, 1979) for details.

The condition on B allows a nice interpretation of this condition in terms of the number of particles inside Ψ. We write out the equation 3.36 for $\bar{\xi} \in \overline{\mathcal{H}}_2$ and shuffle it around to find

$$U^* a_2(\bar{\xi}) U = a_1(\overline{A\xi}) - a_1^\dagger(\overline{B\xi}). \tag{3.50}$$

We then apply this to the initial vacuum $|0\rangle_1$ and take its norm, this gives us the equation

$$\langle \Psi | a_2^\dagger(\xi) a_2(\bar{\xi}) | \Psi \rangle = \langle \Psi | \mathcal{N}_2(\xi) | \Psi \rangle = \langle \xi | B^\dagger B | \xi \rangle. \tag{3.51}$$

We then sum over some orthonormal basis for $\mathcal{H}_2$ to obtain the result:

$$\langle \mathcal{N}_2 \rangle_1 = \mathrm{Tr}(B^\dagger B). \tag{3.52}$$

We see that the two constructions are unitarily inequivalent if and only if the vacuum of the first construction contains finitely many particles with respect to the second particle notion. Note that this is the same result that was obtained in the previous chapter using the Bogolyubov-coefficients β, which is the less formal expression of these operators in terms of a set of modes.

The result above is nice from a physical perspective but can be rewritten into a more manageable form. Because μ_1 is continuous with respect to μ_2 we can use Riesz's lemma to find an operator S such that

$$\mu_2(\phi, \psi) - \mu_1(\phi, \psi) = \mu_2(\phi, S\psi). \tag{3.53}$$

The operator S describes the difference between the two norms. It can be found explicitly as

$$\mu_1(\phi, \psi) = \frac{1}{2}\sigma(\phi, -J_1\psi) = \mu_2(\phi, -J_2 J_1 \psi). \tag{3.54}$$

Therefore we have $S = I + J_2 J_1$. The above condition that B and D should be of Hilbert-Schmidt class can be reformulated as the condition that S should be of trace-class, ie $\mathrm{Tr}(S) < \infty$. The details of this equivalence are quite involved, and we refer the reader to (Shale, 1962) or (Araki and Yamagami, 1982) for a full treatment.

3.6 The Unruh effect

The results from the previous section imply that even between unitarily equivalent constructions, the notion of particle is ambiguous. It is only when a natural particle-interpretation exists, such as when a spacetime is Minkowskian in the (asymptotic) past or future, that an unambiguous interpretation can be given. During the 1970's a lot of effort was put into calculating particle-creation effects in expanding spacetimes or spacetimes containing a black hole.

At the same time a series of papers by Fulling, Davies and Unruh (Fulling, 1973; Davies, 1975; Unruh, 1976) established a related effect, which is often referred to as the Unruh effect. The result states that in flat spacetime uniformly accelerated observers with acceleration a do not regard the usual inertial vacuum as empty, but rather as a bath of particles at a temperature

$$T = \frac{a}{2\pi}. \tag{3.55}$$

It therefore seems that the notion of particle is dubious even before the step to curved spacetimes is undertaken. Below I will sketch the derivation of the effect, the reader is referred to the references given above for a complete treatment of the details.

The original motivation for the Unruh effect, first noted in (Fulling, 1973), is the construction of quantum field theory in static spacetimes. A spacetime is called stationary if it has some timelike Killing vector field. Hence there is some global time-direction in which the metric does not change. If in addition there is a Cauchy surface Σ such that the Killing field is orthogonal to Σ, then we call the spacetime static.

In a static spacetime we have a global time coordinate t and spatial slices such that $g_{\mu\nu}$ is constant in time and $g_{0i} = 0$ (which is not satisfied for stationary spacetimes). This means that the Klein-Gordon equation can straightforwardly be solved by separation of variables into solutions that oscillate like $\exp(\pm i\omega t)$. The quantization construction can then be unambiguously carried out in the same way as in Minkowski spacetime.

One can then observe that Minkowski spacetime has, in addition to the standard inertial Killing vector field, a second set of Killing-vector fields corresponding to the boosts. The boost Killing vector field in the x-direction reads

$$\mathbf{X} = a[x\partial_t + t\partial_x], \tag{3.56}$$

where a is some positive constant indicating the magnitude of the acceleration. This vector field is timelike in the wedge-shaped region defined by $|t| < x$. This region, called the right Rindler wedge, is a globally hyperbolic spacetime in its own right, and we will regard the Killing time defined by $\mathbf{X}$ as its global time.

The orbits of $\mathbf{X}$ are hyperbolas defined by $a^2(x^2 - t^2) = -\mathbf{X}^\mu\mathbf{X}_\mu = \text{constant}$, since the length of a Killing vector is always constant along its own orbit. We can calculate the proper acceleration along an orbit γ of $\mathbf{X}$:

$$A^\nu = \frac{d\gamma}{d\tau} = \frac{\mathbf{X}^\mu}{\sqrt{-\mathbf{X}^\rho\mathbf{X}_\rho}}\nabla_\mu\frac{\mathbf{X}^\nu}{\sqrt{-\mathbf{X}^\sigma\mathbf{X}_\sigma}} = \frac{\mathbf{X}^\mu\nabla_\mu\mathbf{X}^\nu}{\sqrt{-\mathbf{X}^\rho\mathbf{X}_\rho}} \tag{3.57}$$

We can then calculate it's magnitude as

$$A = \sqrt{A^\nu A_\nu} = \frac{1}{\sqrt{x^2 - t^2}} = \frac{a}{\sqrt{-\mathbf{X}^\mu\mathbf{X}_\mu}}. \tag{3.58}$$

We can therefore interpret a as the magnitude of the proper acceleration of the observer with $\mathbf{X}^\mu\mathbf{X}_\mu = -1$. Because of this fact $\mathbf{X}$ is regarded as 'the' reference frame belonging to an observer with constant acceleration a.

Similarly we define a left Rindler wedge for $|t| < -x$. Note that here we must take $-\mathbf{X}$ as the time-direction, since $\mathbf{X}$ is past directed in the left wedge. One can then, for both of these Rindler-wedges, carry out the exact same quantization-scheme as one does for Minkovksi space. This gives a different quantization and a different vacuum, the Rindler vacuum, which is not the same as the Minkowski vacuum. Note that we need both Rindler-wedges to tackle the problem, since otherwise we wouldn't be describing the same classical phase space.

To investigate particle creation one has to decompose the positive frequency solutions with respect to Rindler time into positive and negative frequency components with respect to inertial time. One way to do this would be to solve the Klein-Gordon equation in Rindler-coordinates to obtain a set of Rindler modes, and then explicitly calculate the Bogolyubov coefficients. This is what is done in (Fulling, 1973), but it is quite laborious and a better approach has been given in (Unruh, 1976), which we will follow below.

We first transform to null coordinates defined by

$$u = t - x, v = t + x \tag{3.59}$$

In these coordinates $\mathbf{X}$ reads

$$\mathbf{X} = a(v\partial_v - u\partial_u). \tag{3.60}$$

The trick we will employ is to look at the null surface defined by $v = 0$, which we will denote by Σ, and perform the Fourier-transforms there. This considerably simplifies matters since there is an easy correspondence between the accelerating time τ and u on Σ:

$$\partial_\tau = \mathbf{X} = -au\partial_u \tag{3.61}$$

and hence $\tau = -a^{-1}\ln(|u|)$. Now pick some solution $\phi_{\omega R}$ to the Klein-Gordon equation that has positive frequency ω with respect to τ and that is contained in the right wedge[10]. This solution infers initial data on the part of Σ where $u < 0$, since $\phi_{\omega R}$ is contained within the right wedge.

$$f_{\omega R} = \begin{cases} \phi(y,z)e^{-i\omega\tau} & u < 0 \\ 0 & u > 0. \end{cases} \tag{3.62}$$

Here ϕ is some complex function that describes the behaviour in the y and z directions. Note that a function f is positive frequency with respect to t if and only if it is positive frequency with respect to u and v, since

$$\exp(-i\omega_{\mathbf{k}}t + i\mathbf{k}\cdot\mathbf{x}) = \exp(-i\frac{\omega_{\mathbf{k}} + k^1}{2}u - i\frac{\omega_{\mathbf{k}} - k^1}{2}v + ik^2x^2 + ik^3x^3), \tag{3.63}$$

and always $\omega_{\mathbf{k}} \geq |k^1|$. Since we can always write f as an integral of plane waves, this implies the statement.

[10] We disregard the fact here that these solutions are not normalizable, since plane waves never are. This can be remedied by constructing wave packets of planes waves that are sharply peaked around ω. We refer the reader to chapter 5.1 of (Wald, 1995) for a more precise treatment of the mathematical details of the derivation.

Therefore, to split $\phi_{\omega R}$ into positive and negative frequency parts with respect to t, we can Fourier-transform $f_{\omega R}$ with respect to u on Σ. We get:

$$\hat{f}_{\omega R}(\zeta, y, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{+i\zeta u} f_{\omega R}(u, y, z) du, \tag{3.64}$$

$$= \frac{\phi(y, z)}{\sqrt{2\pi}} \int_{0}^{\infty} \exp(-i\zeta u + \frac{i\omega}{a} \log(u)) du. \tag{3.65}$$

To perform this integral, we complete the contour of the integration in the lower half plane, since here the $e^{-i\zeta u}$ term turns into a negative exponent for positive ζ. Since the integrand has no poles (putting the branch cut of the logarithm at the negative real axis), we get

$$\hat{f}_{\omega R}(\zeta, y, z) = -i \frac{\phi(y, z)}{\sqrt{2\pi}} \int_{0}^{\infty} \exp\Big(- \zeta u + \frac{i\omega}{a} \log(-iu)\Big) du. \tag{3.66}$$

Then using $\log(-iu) = \log(u) - \frac{i\pi}{2}$ we come to our final expression:

$$\hat{f}_{\omega R}(\zeta, y, z) = -i \exp\Big(\frac{\omega\pi}{2a}\Big) \frac{\phi(y, z)}{\sqrt{2\pi}} \int_{0}^{\infty} \exp\Big(- \zeta u + \frac{i\omega}{a} \log(u)\Big) du. \tag{3.67}$$

Now for negative frequencies we have to complete the contour in the upper half plane, but otherwise the calculation is similar. Using $\log(iu) = \log(u) + \frac{i\pi}{2}$ we find that:

$$\hat{f}_{\omega R}(-\zeta, y, z) = i \exp\Big(-\frac{\omega\pi}{2a}\Big) \frac{\phi(y, z)}{\sqrt{2\pi}} \int_{0}^{\infty} \exp\Big(- \zeta u + \frac{i\omega}{a} \log(u)\Big) du, \tag{3.68}$$

and we find the following relation

$$\hat{f}_{\omega R}(-\zeta, y, z) = - \exp\Big(- \frac{\omega\pi}{a}\Big) \hat{f}_{\omega R}(\zeta, y, z). \tag{3.69}$$

We now define the wedge-reflected solution by $\overline{\phi}_{\omega L}(t, x, y, z) = \phi_{\omega R}(-t, -x, y, z)$ which is a solution to the KG-equation that is completely contained in the *left* Rindler wedge. Furthermore, because of the reversal of time, this is a negative-frequency solution because we defined the positive τ direction by $-\mathbf{X}$ in the left Rindler wedge. Note that the bar over $\phi_{\omega L}$ does **not** denote complex conjugation of $\phi_{\omega R}$, but rather that this is a negative frequency solution with respect to Rindler-time in the left wedge.

In terms of initial data on Σ, the wedge-reflected solution gives:

$$\overline{f}_{\omega L} = \begin{cases} 0 & u < 0 \\ \phi(y, z) e^{i\omega\tau} & u > 0. \end{cases} \tag{3.70}$$

Because the initial data $\overline{f}_{\omega L}$ has time reflected behaviour, we can easily link its Fourier transform with respect to u to $\hat{f}_{\omega R}$:

$$\hat{\overline{f}}_{\omega L}(\zeta, y, z) = \hat{f}_{\omega R}(-\zeta, y, z) \tag{3.71}$$

The crucial result is then that the function defined by

$$F = (\phi_{\omega R} + e^{-\pi\omega/a} \overline{\phi}_{\omega L}) \tag{3.72}$$

is positive frequency with respect to inertial time. Indeed, if we look at the initial data of F on Σ and look at the Fourier-component of $-\zeta$ we get:

$$\hat{F}(-\zeta, y, z) = \hat{f}_{\omega R}(-\zeta, y, z) + e^{-\pi\omega/a}\hat{\bar{f}}_{\omega L}(-\zeta, y, z) \tag{3.73}$$

$$= -e^{-\pi\omega/a}\hat{f}_{\omega R}(\zeta, y, z) + e^{-\pi\omega/a}\hat{f}_{\omega R}(\zeta, y, z) = 0. \tag{3.74}$$

We can use this equality to express the inertial annihilation operators a in terms of right/left Rindler operators a_R and a_L. This gives us:

$$a(F) = a_R(\phi_{\omega R}) - e^{-\pi\omega/a}a_L^\dagger(\bar{\phi}_{\omega L}). \tag{3.75}$$

Similarly $G(u) = F(-u)^*$ is purely positive frequency and

$$a(G) = a_L(\phi_{\omega L}) - e^{-\pi\omega/a}a_R^\dagger(\bar{\phi}_{\omega R}). \tag{3.76}$$

From these equations we can calculate the number of Rindler particles with frequency ω that are present in the inertial vacuum. We apply relation 3.75 to the state $|0\rangle$ and take its norm. We find

$$\begin{aligned}\langle 0|a_R^\dagger(\bar{\phi}_{\omega R})a_R(\phi_{\omega R})|0\rangle &= e^{-2\pi\omega/a}\langle 0|a_L(\phi_{\omega L})a_L^\dagger(\bar{\phi}_{\omega L})|0\rangle \\ &= e^{-2\pi\omega/a}\Big(\langle 0|a_L^\dagger(\bar{\phi}_{\omega L})a_L(\phi_{\omega L})|0\rangle + 1\Big)\end{aligned}$$

A similar relation holds with right and left labels interchanged by considering equation 3.76. These equations are uniquely solved by

$$\langle \mathcal{N}_L(\phi_\omega)\rangle = \langle \mathcal{N}_R(\phi_\omega)\rangle = \frac{1}{e^{2\pi\omega/a} - 1}, \tag{3.77}$$

which are the occupation numbers for a thermal bath at temperature $a/2\pi$.

From the equations 3.75 and 3.76 a formal expression for the inertial vacuum can be written down in terms of Rindler states. This expression is only formal because the state that is defined is not normalizable. Nevertheless one can take this expression and trace out the Hilbert space corresponding to the left wedge. One then obtains a thermal density matrix for the right wedge. Heuristically: The inertial vacuum is a thermal state when restricted to one of the Rindler wedges.

The problem that arises above is that the two constructions considered are not unitarily equivalent. We tried to arrive at an expression for the vacuum in terms of only right-Rindler particles, but we first had to consider both the right and left wedges together in order to obtain a quantization of the entire symplectic vector space. A mathematically more rigorous formulation of the Unruh effect can be given in the framework of algebraic quantum field theory using the Bisognano-Wichmann theorem (Bisognano and Wichmann, 1976) as was first noted by Sewell (Sewell, 1982). A treatment of this more rigorous derivation of the Unruh effect would be well beyond this text, the interested reader is referred to (Kay, 1985) for a comprehensive review.

3.7 Particle Detectors

Of course the derivation above does not justify the interpretation that the particles in the Rindler quantization scheme are 'particles as observed by accelerating observers'. In order for such an interpretation to be valid one would have to look at the readings of a particle detector carried by the observer. If this detector responds as if it were immersed in a bath of thermally distributed particles, then the interpretation that the Minkowski vacuum is a thermal state according to an accelerating observer is credible.

We will consider a model originally coined by Unruh in (Unruh, 1976) and later adapted by DeWitt (DeWitt, 1980). The main result of the treatment by Unruh is that detectors respond to states that have positive frequency with respect to the detectors' proper time. We follow here the derivation that is given in (Unruh and Wald, 1984).

The original model under consideration was a box with a non-relativistic particle carried along the worldline of the observer. The particle is initially in its ground-state before the detector is switched on. After a finite amount of time the detector is switched off again and if it is found to be in a state other than the ground state it is said to have observed a particle.

For simplicity we assume the system to have two energy levels, disregarding the higher energy levels. Hence the system is two dimensional, spanned by two states $|-\rangle$ and $|+\rangle$. We have a raising operator $A^\dagger$ and a corresponding lowering operator A mapping these two states into each other. We model the detector system by the Hamiltonian

$$H_D = \Omega A^\dagger A, \tag{3.78}$$

where Ω is the energy of the $|+\rangle$ state, with the ground state energy taken to be zero. The interaction is given by the following Hamiltonian:

$$H_{\text{int}}(\tau) = \epsilon(\tau) \int_\Sigma \hat{\Phi}(\tau, \mathbf{x})[\psi(\mathbf{x})A + \psi^*(\mathbf{x})A^\dagger]\sqrt{-g}d^3\mathbf{x} \tag{3.79}$$

Here ϵ is some small switching function that switches the detector on for some finite time T. We assume that it is mostly constant during this time and zero outside the measurement interval. So as not to disturb the measuring process we also assume the switching of the detector to be slow with respect to the fluctuations of the quantum fields. The function ψ is some function that indicates the detector's cross section, which we will take to be compact such that we are considering a measuring device of finite extent.

Suppose then that before the measuring process the quantum field is in some normalized n-particle state in the mode ϕ:

$$|n_\phi\rangle = \frac{a^\dagger(K\phi)^n}{\sqrt{n!}}|0\rangle, \tag{3.80}$$

with the detector in the ground state $|-\rangle$. After switching the detector on and off during the finite interval of length T we obtain as the out-state of the total system

to first order in ϵ

$$\begin{aligned}|\Psi_{\text{out}}\rangle &= \Big(I - i\int_{-\infty}^{+\infty}(H_{\text{int}})_I d\tau\Big)|n_\phi;-\rangle,\\ &= |n_\phi;-\rangle - i\Big(\int e^{i\Omega s}\epsilon(s)\psi^*(\mathbf{x})\hat{\Phi}(s,\mathbf{x})\sqrt{-g}d^3\mathbf{x}ds\Big)|n_\phi;+\rangle.\end{aligned}$$

Because of the particular form of the interaction-term inside the integral we can recast this formula in terms of creation and annihilation operators. Denote by f the function

$$f(t,\mathbf{x}) = \epsilon(t)e^{i\Omega t}\psi^*(\mathbf{x}). \tag{3.81}$$

We use this function to smear the operator $\hat{\Phi}(t,\mathbf{x})$. In our original description of 4-smeared operator distributions we only admitted real test functions. Now we have a complex test-function f, and hence we need to extend the definition. We define for a complex test-function f:

$$\hat{\Phi}(f) = \hat{\Phi}(\text{Re}f) + i\hat{\Phi}(\text{Im}f) \tag{3.82}$$

We extend the map E by complex linearity and can therefore identify Ef and $E\overline{f}$ as elements of $V_\mu^{\mathbb{C}}$. A straightforward calculation shows that under these identifications, the expression in equation 3.14 generalizes to:

$$\hat{\Phi}(f) = \hat{\sigma}(Ef,.) = ia(\overline{KE\overline{f}}) - ia^\dagger(KEf). \tag{3.83}$$

As a consistency check, note that the extra bar on f makes sure that the right hand side is complex linear in f.

For the case at hand: Note that f is very nearly negative frequency since the switching of the detector is assumed to be slow with respect to the frequency of the detector Ω. The advanced and retarded solutions of a source that oscillates with respect to time, oscillates with the same frequency because the d'Alembertian splits into a spatial and a temporal part. Hence Ef is also nearly negative frequency, and therefore $KEf \approx 0$. As such we have:

$$\hat{\sigma}(Ef,.) \approx ia(\overline{KE\overline{f}}), \tag{3.84}$$

and the final state is seen to be

$$\begin{aligned}|\Psi_{\text{out}}\rangle &= |n_\phi;-\rangle - i\hat{\sigma}(Ef,.)|n_\phi;+\rangle\\ &\approx |n_\phi;-\rangle + \sqrt{n}\langle KE\overline{f},\phi\rangle|(n-1)_\phi;+\rangle.\end{aligned}$$

We can read off a particle interpretation from this result: After the measurement there is a chance that the system remains in its initial state, but there is also a chance that the detector undergoes a transition to the $|+\rangle$ state, thereby absorbing a ϕ quantum from the field. The transition amplitude is proportional to the integer n and therefore we think of the state $|n_\phi\rangle$ as 'the state with n particles with wave function ϕ'. It is clear from this result that when a detector follows an inertial trajectory in the Minkowski vacuum no particles are detected (up to a negligible contribution from the switching of the detector).

Note also that the transition amplitude is proportional to the overlap of the function $E\overline{f}$ and the mode ϕ. Since $E\overline{f}$ is spatially compactly supported (because $\psi(x)$ is)

the chance of detecting a particle with mode function spatially separated from the detector during the measuring process is zero. In this sense the detector described above is local, it only registers modes that overlap with the detector.

Now suppose we let this detector undergo a constant acceleration a. Again, the important thing to note is that the detector measures with respect to its own proper time and not with respect to global time. As such, a uniformly accelerated detector will measure 'particles' when they have positive frequency with respect to Killing-time. If the detector is contained within the right Rindler wedge it will therefore respond as if it were surrounded by a thermal bath at the Unruh temperature.

In this sense the Unruh effect is 'real': The splitting of functions into positive and negative frequency contributions is not just a trick to obtain quantization of the canonical commutation relations. When coupled to a detector model a natural interpretation in terms of particles arises, since the detector sees the quantum field as 'particle-like'. The notion of particles is, however, only as natural as the observers that measure them and in no way is it important for the construction of quantum field theory.

Chapter 4

The Stress-Energy Tensor

4.1 Introduction

In this chapter we try to define the vacuum by ways of the stress-energy tensor. This approach is a followup of earlier research by David Venhoek performed at this institute ("Quantum vacua in Curved Spacetime"). Rather than look at the vacuum as a state with no particles, we regard it as a state that gives $T_{\mu\nu}$ a specific form with regards to $g_{\mu\nu}$, namely:

$$\langle T_{\mu\nu}(x)\rangle_0 = \rho_{\text{vac}} g_{\mu\nu}(x), \tag{4.1}$$

There are a number of reasons to try such an approach. First is the idea that the vacuum is supposed to not give any 'direction' to physical quantities. In flat spacetime we had Poincaré invariance as a guiding principle, and we picked out the vacuum as the unique state that is invariant under the group action. We now no longer have a large group of symmetries, but we still have a metric and it would be natural for the stress-energy tensor to mimic its form as it's the most elementary tensor present in the problem.

On top of this, this definition of the vacuum is expressed in terms of covariant entities and as such it would be easy for observers to agree on it. If a constant value for ρ_{vac} can be found than we could tell each observer to pick his vacuum state such that equation 4.1 is satisfied.

The last motivation is that such a prescription would give us a notion of dark energy. Cosmological measurements have indicated that a cosmological constant is needed in order to explain the accelerated expansion of the universe. This yields an extra term of the form $\Lambda g_{\mu\nu}$ in the Einstein equations. If the stress-energy tensor satisfies equation 4.1 then we could interpret this cosmological constant as the energy present in the vacuum-fluctuations present in empty space.

The following sections are organized as follows. In section 2 we present the calculation of $\langle T_{\mu\nu}\rangle$ in Minkowski space using standard QFT techniques for the renormalization. In section 3 we try to extend the calculation to FLRW space, which is the simplest non-flat extension of Minkowski space. We discuss the results in section 4 and find that the problem of regularization becomes very complicated even in this highly symmetric model. We conclude that our regularization scheme is unsuited to the task at hand. In section 5 we discuss a different method of renormalization which is better suited to curved spacetime calculations and discuss how this relates to our definition of the vacuum.

4.2 The stress-energy tensor in Minkowski space

The calculation of $\langle T_{\mu\nu} \rangle$ is complicated because the naive expression one would use to perform the calculation involves divergent integrals. Divergences are common in QFT and a large amount of techniques is available in order to handle them. First one needs to regularize the divergent expression, which means to perform some mathematical trick that renders all expressions finite. Then one needs to subtract the divergent part of the expression or absorb it into one of the constants of the theory. After this the regularization can be undone and, if all steps are performed correctly, the final answer will be finite.

We will in this section present the calculation for Minkowski space using the technique of dimensional regularization, following the calculation in (Martin, 2012).

The stress energy tensor of the theory is obtained by varying the action with respect to the metric. The action for the free field is given by

$$S_{\text{matter}} = \int d^4x\sqrt{-g}\frac{1}{2}\Big(- g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - m^2\phi^2\Big) = \int d^4x\sqrt{-g}\mathcal{L}. \tag{4.2}$$

The stress-energy tensor is then given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla^\rho\phi\nabla_\rho\phi + m^2\phi^2). \tag{4.3}$$

Here we use that the variation of the metric determinant is given by

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}. \tag{4.4}$$

The approach to calculating the expectation value is to expand the field ϕ in a plane wave basis, and then obtain the stress-energy as a mode integral. We write

$$\hat{\phi}(\mathbf{x}) = \int d^3\mathbf{k}[e_{\mathbf{k}}(\mathbf{x})a_{\mathbf{k}} + e^*_{\mathbf{k}}(\mathbf{x})a^\dagger_{\mathbf{k}}]. \tag{4.5}$$

We can easily obtain an expression for $\langle T_{\mu\nu} \rangle$ using the commutation relations of the $a_{\mathbf{k}}$ and the fact that $a_{\mathbf{k}}|0\rangle = 0$. If we commute $a_{\mathbf{k}}$ operators to the right of the expression they act on the vacuum and hence yield zero. The commutator terms are then all that is left, which are delta functions of two momenta. We can then perform one of the integrations to obtain the final expression. We find the following expressions for the components of $T_{\mu\nu}$:

$$\langle T_{00} \rangle = \frac{1}{2(2\pi)^3}\int d^3\mathbf{k}\omega_{\mathbf{k}}, \tag{4.6}$$

$$\langle T_{ii} \rangle = \frac{1}{2(2\pi)^3}\int d^3\mathbf{k}\frac{k_i^2}{\omega_{\mathbf{k}}} = \frac{1}{6(2\pi)^3}\int d^3\mathbf{k}\frac{\mathbf{k}^2}{\omega_{\mathbf{k}}}, \tag{4.7}$$

$$\langle T_{0i} \rangle = -\frac{1}{2(2\pi)^3}\int d^3\mathbf{k}k_i = 0, \tag{4.8}$$

$$\langle T_{ij} \rangle = \frac{1}{2(2\pi)^3}\int d^3\mathbf{k}\frac{k_ik_j}{\omega_{\mathbf{k}}} = 0, \qquad (i \neq j). \tag{4.9}$$

We will follow the standard terminology of FLRW calculations and call the $\langle T_{00} \rangle$

component the density ρ, and the spatial components $\langle T_{ii}\rangle$ the pressure p.

Obviously these integrals diverge and one needs a way to regularize them if some sense is to be made of these expressions. We could impose a cut-off on the momentum $\mathbf{k}$ and justify this by saying that our theory is only valid up to this scale and we disregard any behaviour above it. This regularization method is ill-suited for the situation at hand, however, since this prescription is manifestly not Lorentz-invariant. More importantly, it does not reproduce a stress-energy tensor that is proportional to the metric because it breaks the Lorentz invariance of the theory.

A better method is that of dimensional regularization, where we calculate the integral in d dimensions rather than in 4, where d is understood to be 'close to 4'. Obviously a non-integer dimension is a strange concept, but we stress that we only use the dimension as a parameter in order to render the integrals finite. After this shift our integrals read

$$\langle T_{00}\rangle = \frac{\mu^{4-d}}{2(2\pi)^{d-1}} \int d^{d-1}\mathbf{k}\, \omega_{\mathbf{k}}, \tag{4.10}$$

$$\langle T_{ii}\rangle = \frac{\mu^{4-d}}{2(d-1)(2\pi)^{d-1}} \int d^{d-1}\mathbf{k} \frac{\mathbf{k}^2}{\omega_{\mathbf{k}}}. \tag{4.11}$$

Here the mass-scale μ is introduced in order to retain the right dimensions and is often called the renormalization scale. We can easily perform the angular integrals, yielding a factor

$$\int d^{d-2}\Omega = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}, \tag{4.12}$$

Where Γ is Euler's gamma function. It satisfies the defining equality

$$\Gamma(x+1) = x\Gamma(x). \tag{4.13}$$

The remaining integral over the radial part of $\mathbf{k}$ can now be performed. For later reference we establish a general formula for handling the divergent integrals we encounter, which is finite whenever d is non-integer. We denote $k = |\mathbf{k}|$ as the radial part of the momentum 3-vector. Let α and β be two complex numbers, then

$$\begin{aligned}\int d^{d-1}\mathbf{k}\, k^\alpha \omega_k^\beta &= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} \int dk k^{\alpha+d-2}(k^2+m^2)^{\frac{\beta}{2}} \\ &= \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} m^{\alpha+\beta+d-1} \int_0^1 dt (1-t)^{\frac{\alpha+d-3}{2}} t^{-\frac{\alpha+\beta+d+1}{2}} \\ &= \pi^{\frac{d-1}{2}} m^{\alpha+\beta+d-1} \frac{\Gamma(\frac{\alpha+d-1}{2})\Gamma(-\frac{\alpha+\beta+d-1}{2})}{\Gamma(\frac{d-1}{2})\Gamma(-\frac{\beta}{2})}.\end{aligned}$$

In the second equality we substituted

$$k = m\sqrt{\frac{1-t}{t}}.$$

The third equality is the integral representation of the beta-function, which is related to the gamma function by [1]

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{4.14}$$

We can now obtain the regularized components of the stress energy tensor by inserting the values $\alpha = 0, \beta = 1$ and $\alpha = 2, \beta = -1$, which gives us

$$\langle T_{00} \rangle = \frac{m^4}{2(4\pi)^{\frac{d-1}{2}}} \Big(\frac{m}{\mu}\Big)^{d-4} \frac{\Gamma(-\frac{d}{2})}{\Gamma(-\frac{1}{2})}, \tag{4.15}$$

$$\langle T_{ii} \rangle = \frac{m^4}{2(4\pi)^{\frac{d-1}{2}}} \Big(\frac{m}{\mu}\Big)^{d-4} \frac{1}{d-1} \frac{\Gamma(\frac{d+1}{2})\Gamma(-\frac{d}{2})}{\Gamma(\frac{d-1}{2})\Gamma(\frac{1}{2})}. \tag{4.16}$$

Finally, using equation 4.13 twice we find that

$$(d-1)\Gamma\Big(\frac{d-1}{2}\Big)\Gamma\Big(\frac{1}{2}\Big) = -\frac{d-1}{2}\Gamma\Big(\frac{d-1}{2}\Big)\Gamma\Big(-\frac{1}{2}\Big) = -\Gamma\Big(\frac{d+1}{2}\Big)\Gamma\Big(-\frac{1}{2}\Big). \tag{4.17}$$

From this we find that, as we set out to achieve, $\langle T_{00} \rangle = -\langle T_{ii} \rangle$ such that the stress-energy tensor is proportional to the metric. Of course the overall answer is still infinite since we need to take the limit of d to 4 and the gamma function is singular at negative integer values.

The technique that is used to discard the infinite terms inside these expressions is called renormalization. Now that we have calculated the vacuum energy in a regime where it is finite, ie. $d \neq 4$, we can isolate the divergent terms and absorb them into the constants of the theory, which are in this case Newton's constant and the cosmological constant. This subtraction is performed in equation 96 in (Martin, 2012) and hence we will not repeat it here. The result is here:

$$\rho_{\text{vac}} = \frac{m^4}{64\pi^2} \log\Big(\frac{m^2}{\mu^2}\Big). \tag{4.18}$$

This is a textbook calculation using an $\overline{\text{MS}}$-scheme, but in a certain sense it is a weakness. For the process of renormalization we need to match our theoretical prediction with an experimental value at some energy scale μ. After this our formula can predict what the value of the vacuum energy should be at a different energy scale, restoring predictive power to the theory.

However, it is unclear what we mean by the energy scale in this context: We only have one measurement for ρ, the one coming from cosmology, and it is unclear what energy scale we should associate with it. It is also unclear what we would mean with the vacuum energy at some different energy scale. We see that the terminology and techniques which are commonplace in particle physics do not carry over straightforwardly to the case at hand.

Clearly some more involved renormalization-scheme (or renormalization condition) is needed to make sense of the situation here. We will not go further into the details

[1] An additional cutoff is implicitly present here since the integral formula for the beta function is not valid for all values of α and β, and in particular the ones that we use our formula for. We thank Venhoek for pointing this point out to us.

of renormalization of the stress-energy here since it is an open problem which has no clear resolution as of yet. In what follows we will calculate regularized quantities, and leave the question of their renormalization open.

4.3 Stress-energy tensor in FLRW space

We now try to extend the above calculation to a simple cosmological model, that of a Friedmann–Lemaître–Robertson–Walker (FLRW) model. This model introduces an additional function $a(t)$ that describes the expansion of space. We consider here a spatially flat model, such that space is given by $\mathbb{R}^3$. The metric is given by

$$g_{\mu\nu} = a(\eta)^2 \eta_{\mu\nu}, \tag{4.19}$$

where we have switched already to conformal time η and $\eta_{\mu\nu}$ denotes the Minkowski metric.

We follow the adiabatic approach given in (Birrell and Davies, 1982) for solving the wave equations. Since the metric is diagonal we can separate the equation of motion into a spatial and a temporal part. We choose the spatial part of the modes to be given by the usual plane wave modes such that

$$u_{\mathbf{k}}(\eta, \mathbf{x}) = \chi_{\mathbf{k}}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{4.20}$$

where $\mathbf{k}$ indicates the spatial momentum of the mode, which ranges across $\mathbb{R}^3$. We can then write out the Klein-Gordon equation for these modes. Recall that we can write the d'Alembert operator in the following form:

$$\Box\phi = \frac{1}{\sqrt{-g}} \partial_\mu \Big(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi \Big). \tag{4.21}$$

If we insert our ansatz 4.20 into the Klein-Gordon equation we find

$$\ddot{\chi} + 2H\dot{\chi} + (\mathbf{k}^2 + a^2 m^2)\chi = 0, \tag{4.22}$$

where a dot denotes a derivative with respect to η and the Hubble-parameter H is given by $\frac{\dot{a}}{a}$. We see that this is the equation of a damped oscillator with time-dependent frequency. We can remove the damping from the equation by transforming to $\zeta = a\chi$. This removes the dampening term and leaves only a time-dependent frequency:

$$\ddot{\zeta} + \Big(\mathbf{k}^2 + m^2 a^2 - \frac{\ddot{a}}{a} \Big) \zeta = \ddot{\zeta} + \Big(\omega_{\mathbf{k}}(\eta)^2 - \frac{\ddot{a}}{a} \Big) \zeta = 0 \tag{4.23}$$

Now, expecting that the rate of expansion of the universe is much slower than the oscillations of the field we pose another ansatz, reflecting our expectation that ζ will be 'nearly harmonic'. We set:

$$\zeta(\eta) = \frac{\exp(-i \int^\eta W_{\mathbf{k}})}{\sqrt{2(2\pi)^3 W_{\mathbf{k}}(\eta)}} \tag{4.24}$$

Inserting this into equation 4.23 we find that $W_{\mathbf{k}}$ should satisfy the following equation:

$$W_{\mathbf{k}}^2 = \omega_{\mathbf{k}}^2 - \frac{\ddot{a}}{a} + \frac{3}{4}\frac{\dot{W}_{\mathbf{k}}^2}{W_{\mathbf{k}}^2} - \frac{1}{2}\frac{\ddot{W}_{\mathbf{k}}}{W_{\mathbf{k}}}. \tag{4.25}$$

Now, since the cosmic timescale is understood to be very large with respect to the frequency of the field , an adiabatic approach to solving this equation seems feasible. To zeroth adiabatic order, $W_{\mathbf{k}}$ is given by $W_{\mathbf{k}}^{(0)} = \omega_{\mathbf{k}}$. Because time derivatives are small, this is already a very good approximation, especially for the higher frequency modes. We can find the second order solution by inserting the zeroth order solution:

$$(W_{\mathbf{k}}^{(2)})^2 = \omega_{\mathbf{k}}^2 - \frac{\ddot{a}}{a} + \frac{3}{4}\frac{\dot{\omega}_{\mathbf{k}}^2}{\omega_{\mathbf{k}}^2} - \frac{1}{2}\frac{\ddot{\omega}_{\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{4.26}$$

and so on for higher adiabatic orders.

We are now in the position to start calculating the stress-energy tensor. Our modes now have the form

$$u_{\mathbf{k}}(\eta, x) = \frac{e^{i\mathbf{k}\cdot x}}{\sqrt{2(2\pi)^3}} \frac{\exp(-i\int^\eta W_{\mathbf{k}})}{\sqrt{W_{\mathbf{k}}(\eta)}a(\eta)} \tag{4.27}$$

and we can impose standard commutation relations for these modes. However, we will want these modes in d dimensions rather than 4 because we will use dimensional regularization. This has a few effects on the form of the modes. First, to makes sure that the normalization of the modes is correct, we have to replace the factor $(2\pi)^3$ by $(2\pi)^{d-1}$ and the factor a by $a^{\frac{d}{2}-1}$. Furthermore, the expression 4.25 for $W_{\mathbf{k}}$ picks up an extra term

$$-\frac{d-4}{2}\frac{\dot{a}^2}{a^4}, \tag{4.28}$$

but this will not prove of importance for the calculation we perform below (but will in higher order calculations). As such the modes in d dimensions read

$$u_{\mathbf{k}}(\eta, x) = \frac{e^{i\mathbf{k}\cdot x}}{\sqrt{2(2\pi)^{d-1}}} \frac{\exp(-i\int^\eta W_{\mathbf{k}})}{\sqrt{W_{\mathbf{k}}(\eta)}a(\eta)^{\frac{d}{2}-1}} \tag{4.29}$$

The only thing left to specify are the initial conditions for $W_{\mathbf{k}}$, which completely fixes the quantization. Since we are trying to obtain a specific form for the stress-energy tensor we defer making this choice until a later point. In FLRW space, the components take the following form, where we have already gone over to d dimensions:

$$\langle T_{00}\rangle = \frac{\mu^{4-d}}{2}\int d^{d-1}\mathbf{k}\Big(|\partial_\eta u_{\mathbf{k}}|^2 + (\mathbf{k}^2 + m^2a^2)|u_{\mathbf{k}}|^2\Big), \tag{4.30}$$

$$\langle T_{ii}\rangle = \frac{\mu^{4-d}}{2}\int d^{d-1}\mathbf{k}\Big(|\partial_\eta u_{\mathbf{k}}|^2 + \Big(\frac{2\mathbf{k}^2}{d-1} - (\mathbf{k}^2 + m^2a^2)\Big)|u_{\mathbf{k}}|^2\Big). \tag{4.31}$$

Again, this expression will diverge when $d = 4$ and we need to perform the calculation for d non-integer. Inserting our modes from equation 4.29 we obtain the following expression:

$$\langle T_{00}\rangle = \frac{\mu^{4-d}}{4(2\pi)^{d-1}a^{d-2}}\int \frac{d^{d-1}\mathbf{k}}{W_{\mathbf{k}}}\Big(W_{\mathbf{k}}^2 + \Big(\frac{1}{2}\frac{\dot{W}_{\mathbf{k}}}{W_{\mathbf{k}}} + H\Big)^2 + (\mathbf{k}^2 + m^2a^2)\Big), \tag{4.32}$$

$$\langle T_{ii}\rangle = \frac{\mu^{4-d}}{4(2\pi)^{d-1}a^{d-2}}\int \frac{d^{d-1}\mathbf{k}}{W_{\mathbf{k}}}\Big(W_{\mathbf{k}}^2 + \Big(\frac{1}{2}\frac{\dot{W}_{\mathbf{k}}}{W_{\mathbf{k}}} + H\Big)^2 - (\mathbf{k}^2 + m^2a^2) + \frac{2\mathbf{k}^2}{d-1}\Big). \tag{4.33}$$

In order for the stress-energy tensor to satisfy $\langle T_{00}\rangle = -\langle T_{ii}\rangle$ we pick initial conditions such that our integral has the same form as in the previous flat calculation. For this

to work we need to pick

$$W_{\mathbf{k}}(0) = \sqrt{\mathbf{k}^2 + m^2 a(0)^2} \qquad \dot{W}_{\mathbf{k}}(0) = -2HW_{\mathbf{k}}(0) \tag{4.34}$$

After these initial conditions are chosen the calculation of the integral is the same as it was in the flat case, just with mass equal to $ma(0)$. Hence, the stress-energy at $t = 0$ is given by

$$\langle T_{00} \rangle = \frac{m^4 a(0)^d}{2(4\pi)^{\frac{d-1}{2}} a(0)^{d-2}} \left(\frac{m}{\mu}\right)^{d-4} \frac{\Gamma(-\frac{d}{2})}{\Gamma(-\frac{1}{2})}, \tag{4.35}$$

$$\langle T_{00} \rangle = -\langle T_{ii} \rangle. \tag{4.36}$$

And we see that we exactly obtain

$$\langle T_{\mu\nu} \rangle = \rho_{\text{vac}} a^2 \eta_{\mu\nu} \tag{4.37}$$

with the same vacuum energy as before, given by equation 4.18. The effect of the curved metric is therefore, at $t = 0$, limited. This should not come as a surprise, since we engineered our initial conditions in such a way that we stayed as close to the flat case as possible.

Now note an unsatisfactory feature of our definition of the vacuum. We tried to get the stress-energy tensor to obey equation 4.1 and in order to do this we defined the vacuum by the initial conditions in equation 4.34. Since in physics measurements are always accompanied by an inherent uncertainty we can not put too much weight on what happens at $t = 0$ alone. It is not clear whether our definition of the vacuum is stable, in the sense that the form of the stress-energy tensor can deviate away from the surface where $t = 0$. We need to obtain the expectation value of $T_{\mu\nu}$ in some neighbourhood of our Cauchy surface that has nonzero temporal extent in order to say anything about the stability of the vacuum.

To investigate what happens away from $t = 0$ we try to perform the calculation for some small time ϵ later. We neglect all terms of order ϵ^2. Since these terms are bound to multiply some divergent integral it might seem weird to neglect them because their contribution could become very large. But, if we were to calculate the $O(\epsilon^2)$ terms and renormalize them in the same way that we do the $O(\epsilon)$ terms, the prefactors from the gamma functions would be comparable and the term would still be negligible.

Because we have chosen our initial conditions 4.34 with some care the calculations are still relatively easy. The term

$$\left(\frac{1}{2}\frac{\dot{W}_{\mathbf{k}}}{W_{\mathbf{k}}} + H\right)^2 \tag{4.38}$$

is of order ϵ^2 because we chose initial conditions that made it vanish at $t = 0$. We therefore do not have to include it in a calculation at this order, which would involve a lengthy expression involving equation 4.25 in order to obtain $\ddot{W}_{\mathbf{k}}(0)$.

We begin by Taylor-expanding the components inside the integrals in equations 4.32 and 4.33 up to order ϵ.

$$W_{\mathbf{k}}(\epsilon) = W_{\mathbf{k}}(0) + \dot{W}_{\mathbf{k}}(0)\epsilon = (1 - 2H\epsilon)\omega_{\mathbf{k}}, \tag{4.39}$$

$$W_{\mathbf{k}}^{-1}(\epsilon) = (1 + 2H\epsilon)\omega_{\mathbf{k}}^{-1}, \tag{4.40}$$

$$a(\epsilon) = a + \dot{a}\epsilon = (1 + H\epsilon)a, \tag{4.41}$$

$$a(\epsilon)^{2-d} = a^{d-2}(1 + (2-d)H\epsilon), \tag{4.42}$$

where we have introduced the shorthand notation $a = a(0)$. Inserting this into equation 4.32 we obtain

$$\langle T_{00}(\epsilon)\rangle = \frac{\mu^{4-d}}{4(2\pi)^{d-1}a^{d-2}}(1 + (2-d)H\epsilon)(1 + 2H\epsilon)\int \frac{d^{d-1}\mathbf{k}}{\omega_{\mathbf{k}}}\Big((2 - 4H\epsilon)\omega_{\mathbf{k}}^2 + 2m^2a^2H\epsilon\Big) \tag{4.43}$$

$$= (1 + (2-d)H\epsilon)\langle T_{00}(0)\rangle + \frac{\mu^{4-d}m^2H\epsilon}{2(2\pi)^{d-1}a^{d-4}}\int \frac{d^{d-1}\mathbf{k}}{\omega_{\mathbf{k}}} \tag{4.44}$$

$$\tag{4.45}$$

We can perform this integral using our general formula, to find that:

$$\int \frac{d^{d-1}\mathbf{k}}{\omega_{\mathbf{k}}} = \pi^{\frac{d-1}{2}}(ma)^{d-2}\frac{\Gamma(1 - \frac{d}{2})}{\Gamma(\frac{1}{2})} = d\pi^{\frac{d-1}{2}}(ma)^{d-2}\frac{\Gamma(-\frac{d}{2})}{\Gamma(-\frac{1}{2})}. \tag{4.46}$$

Inserting this result we find that

$$\langle T_{00}(\epsilon)\rangle = (1 + 2H\epsilon)\langle T_{00}(0)\rangle = \frac{a(\epsilon)^2}{a(0)^2}\rho_{\text{vac}}a(0)^2\eta_{00} = \rho_{\text{vac}}g_{00} \tag{4.47}$$

Now for the spatial components, we insert the order ϵ terms into the expression 4.33. This gives the expression

$$\langle T_{ii}(\epsilon)\rangle = \frac{\mu^{4-d}}{4(2\pi)^{d-1}a^{d-2}}(1+(4-d)H\epsilon)\int \frac{d^{d-1}\mathbf{k}}{\omega_{\mathbf{k}}}\Big(\frac{2\mathbf{k}^2}{d-1} - 2m^2a^2H\epsilon - 4H\epsilon\omega_{\mathbf{k}}^2\Big) \tag{4.48}$$

We note that the first term and the second terms together match with two terms in the expression for $\langle T_{00}(\epsilon)\rangle$ calculated above. The remaining terms are two integrals of the $-4H\epsilon\omega_{\mathbf{k}}^2$ terms. These have the same sign, and therefore do not cancel:

$$\langle T_{ii}(\epsilon)\rangle = -\langle T_{00}(\epsilon)\rangle + \frac{\mu^{4-d}}{4(2\pi)^{d-1}a^{d-2}}(1 + (4-d)H\epsilon)\int \frac{d^{d-1}\mathbf{k}}{\omega_{\mathbf{k}}}(-8H\epsilon\omega_{\mathbf{k}}^2) \tag{4.49}$$

$$= -\langle T_{00}(\epsilon)\rangle - 4\epsilon\dot{a}a\rho_{\text{vac}} \tag{4.50}$$

Which means that the pressure at time ϵ is given by

$$p(\epsilon) = -(1 + 4H\epsilon)\rho(\epsilon) \tag{4.51}$$

Hence we see that $T_{\mu\nu}$ does not satisfy equation 4.1 away from $t = 0$.

4.4 Discussion of results

At this point we can take a step back and interpret our results. We set out to define the vacuum by the demand that the expectation value of the stress energy should be proportional to the metric. We found that in FLRW space, we can get this demand to hold on some Cauchy surface of constant cosmological time. However, our result in equation 4.50 shows that it is only on the surface that this form is achieved, in any open neighborhood of the surface it starts to deviate. It is questionable how valuable such a prescription is from the point of view of a physical theory, since measurements are never exact.

The situation is not hopeless, however, at least in this case. We have one symmetry left in the system, namely the conservation of the stress energy tensor:

$$\nabla^\mu T_{\mu\nu} = 0, \tag{4.52}$$

which specializes in FLRW space to the equation

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{4.53}$$

We see that this dynamical equation is satisfied at $t = 0$ to first order in ϵ. This seems to imply that our technique is at least somewhat well-behaved.

We could thus take the viewpoint that the vacuum of the theory is an adiabatic quantity: At each moment in time we set our modes to have the initial conditions given by 4.34 (evaluated at time t rather than at 0) which makes sure that equation 4.1 is always satisfied. The modes will no longer be solutions to the Klein-Gordon equation in this case, but it will nonetheless be a good approximation to the dynamics of the theory if the spacetime geometry is slowly varying. For a good review of adiabatic states in a more general context we refer the reader to (Junker and Schrohe, 2002).

The author is not convinced by this approach, at least not in the form that it is posed in above. If we try to get equation 4.53 to hold at $t = \epsilon$ we get the prediction

$$\dot{\rho}(\epsilon) = 12H^2\epsilon\rho(\epsilon) \tag{4.54}$$

which seems to imply, disregarding any effects from higher orders in ϵ, that a positive energy density will tend to grow in a universe undergoing accelerated expansion, albeit very slowly. This could be a valuable prediction for cosmological models, but a zeroth order adiabatic approach would miss such a phenomenon. Setting the dynamics of the vacuum to be trivial by hand means the theory will lose its predictive power.

The prediction given in equation 4.54 seems to imply that in order to have any chance of keeping our stress-energy tensor in the form $T_{\mu\nu} = \rho g_{\mu\nu}$ we should devise a renormalization scheme which sets $\rho = 0$ in flat spacetime. This could be a demand on any renormalization scheme designed for tackling stress-energy calculations in curved spacetimes: When applied to flat spacetime the result should be zero, or otherwise the stress-energy tensor will depend heavily on the time chosen to perform the quantization. To make a more substantiated judgement on this, however, we would need a better calculation of $\langle T_{\mu\nu} \rangle$ at times later than $t = 0$.

We might try to resolve some of the questions posed above by calculating higher terms in ϵ to gain more feeling on the behaviour of $\langle T_{\mu\nu} \rangle$ around $t = 0$. However, the $O(\epsilon^2)$ calculation becomes very complicated very quickly because in this case we would have to use the evolution equation 4.25 for $W_{\mathbf{k}}$ in order to obtain $\ddot{W}_{\mathbf{k}}(0)$. One could approach this problem numerically, but we find it unlikely that such an expression would be very enlightening.

At this point one could note that our method of regularization is unsuited for the situation at hand: We used a technique that works in flat spacetime, and tried to expand it to a curved model. Because the wave equation is no longer analytically solvable for general expansion factors we are forced to resort to an adiabatic approach.

Because the error we make in approximating a true solution by an adiabatic one is summed over all values of $\mathbf{k}$ we might obtain a very large correction, or even an infinite one. It is in this case not so clear whether using an adiabatic expansion to calculate the stress-energy tensor is a justified approach, because it might not reflect the true value of the stress-energy tensor even if we could perform a calculation to all orders in ϵ. However, in absence of an analytic expression for our mode functions we do not really have any other choice.

As to some concluding remarks from this investigation. We see that our regularization scheme is unsuited for the problem at hand. The equations are too complicated to be handled analytically and it is unclear if our adiabatic approach is a justified approximation.

On top of this it is unclear how this scheme might be expanded to general globally hyperbolic spacetimes. In FLRW space we have $d-1$ spatial dimensions and it is not hard to generalize this away from $d = 4$ because all dimensions are equivalent. In a model were the symmetry between the different spatial directions is not present this can no longer be done.

Lastly, we have relied quite heavily on the fact that we have access to a spatially isotropic Cauchy surface on which we based our calculation. As we have discussed in the previous chapter, we find the assumption that one has access to such a global surface of simultaneity to be unphysical.

4.5 Different renormalization techniques

In this section we briefly touch on a few different available techniques which could be used in this context. The most universally accepted prescription is that of point-split renormalization. Put forward by Wald in (Wald, 1977), the aim of this scheme of renormalization is to pose four axioms that the expectation value of the stress-energy should satisfy and find a definition that satisfies them. The four axioms are, loosely phrased:

1. The difference in stress-energy between two states is a finite expression and the renormalized stress energy tensor should give the same results.

2. The value of $\langle T_{\mu\nu} \rangle_\omega$ should be 'suitably local' in the state ω, as discussed in chapter 4.6 of (Wald, 1995).

3. We have $\nabla^\mu \langle T_{\mu\nu} \rangle = 0$.

4. In Minkowski spacetime $\langle T_{\mu\nu}\rangle_0 = 0$ such that the semi-classical Einstein equation holds in this situation.

Note that axiom 4 matches with the conclusions from our previous investigation: We need to pose that ρ should be zero in flat spacetime in order to obtain a stable vacuum.

It turns out that these axioms uniquely fix the expectation value up to the addition of a local curvature tensor. A prescription that satisfies these axioms is to be preferred above the previous dimensional regularization setup because it aims at defining the stress-energy tensor in such a way that the semi-classical Einstein equations might, in principle, be satisfied:

$$G_{\mu\nu} = 8\pi G_N \langle T_{\mu\nu}\rangle \tag{4.55}$$

A prescription that satisfies these 4 axioms is the point-splitting renormalization method alluded to above. As described in the first two chapters of this work the field observable $\phi(x)$ is not truly an observable, but rather an operator-valued distribution. As such the product $\phi(x)\phi(x')$ of two field observables is a bi-distribution which is well defined as an object that should be smeared by two test functions. However, the coincidence limit of $x' \to x$ is singular, since the square of a distribution is ill-defined. Because terms of the form $\phi(x)^2$ feature in the stress-energy tensor it is no surprise that its expectation value diverges.

When viewed from this perspective the problem of divergences in QFT originates from the fact that products of field observables are evaluated in one point of spacetime. As such it is natural to separate the points in order to regularize the expression. We define the symmetrized two-point function in some state by

$$G(x,x') = \frac{1}{2}\langle \phi(x)\phi(x') + \phi(x')\phi(x)\rangle. \tag{4.56}$$

If one were to proceed naively, we would define the stress-energy tensor by differentiating on this two point function and take the coincidence limit of the resulting expression:

$$\langle T_{\mu\nu}(x)\rangle = \frac{1}{2}\lim_{x'\to x}\Big(\nabla_{\mu'}\nabla_\nu G + \nabla_\mu\nabla_{\nu'}G - g_{\mu\nu}\nabla_\alpha\nabla^{\alpha'}G - m^2 g_{\mu\nu}G\Big). \tag{4.57}$$

This will be a divergent expression, which is understandable since the two-point function G diverges on the diagonal. The idea of point-split renormalization is to subtract from the two point function the parts divergent in the coincidence limit of $x' \to x$. After this, we can apply the formula 4.57 to this new renormalized two-point function and obtain a finite result.

For this subtraction, one uses a locally constructed Hadamard solution to the wave equations. This is a function H of two variables, satisfying the wave equation in the first, with the following singularity structure:

$$H(x,x') = \frac{U(x,x')}{\sigma} + V(x,x')\log(\sigma) + \text{smooth part} \tag{4.58}$$

where σ is half the geodesic distance between x and x'. Assuming that H satisfies the wave equation fixes U and V completely by the local geometry. The smooth part is fixed by axiom 4 above in order to obtain the right subtraction in flat spacetime.

One now puts as a demand on the state used to calculate the two-point function G that it has the same singularity structure as H. A state that does (such as the vacuum in any static spacetime (Fulling, Narcowich, and Wald, 1981)) is called a **Hadamard state**. For a Hadamard state the renormalized two-point function may now be defined as

$$F = G - H \tag{4.59}$$

Which is now smooth when $x' \to x$. As such the renormalized stress-energy tensor can be defined by

$$\langle T_{\mu\nu}(x) \rangle = \frac{1}{2} \lim_{x' \to x} \Big(\nabla_{\mu'} \nabla_{\nu} F + \nabla_{\mu} \nabla_{\nu'} F - g_{\mu\nu} \nabla_{\alpha} \nabla^{\alpha'} F - m^2 g_{\mu\nu} F \Big), \tag{4.60}$$

which is finite because we are no longer regarding a function with singular parts.

The Hadamard condition is a demand on the short-range behaviour of the quantum field. It formalizes the demand that locally the high energy modes behave as they would in the Minkowski space. Since spacetime is locally flat, it is reasonable to assume that the high-energy/short-range modes that probe only the local geometry behave as such. Because of these facts the Hadamard condition is often regarded as the definition of a physically reasonable state (Fulling, Sweeny, and Wald, 1978).

The above treatment of point-split renormalization and the Hadamard condition is by no means meant to be exhaustive and is intentionally vague, since the details of the procedure are quite involved and not all known to the author. Here we cite a number of sources the interested reader might follow.

For a mathematically precise definition of the Hadamard condition we refer to (Kay and Wald, 1991), as well as to (Fulling, Sweeny, and Wald, 1978; Fulling, Narcowich, and Wald, 1981) for properties of these states. The axiomatic approach of Wald to stress-energy renormalization and the point-splitting procedure are described in (Wald, 1977; Wald, 1978). A more practical approach to point-splitting calculations is given in (Christensen, 1976) and (Adler and Lieberman, 1978), as well as in (DeWitt, 1975) by DeWitt who coined the procedure originally.

Now to return to the problem of renormalization in FLRW space. A technique called adiabatic renormalization, which can be shown to be equivalent to point-splitting renormalization (Birrell, 1978), can be practically employed to calculate $\langle T_{\mu\nu} \rangle$ in the adiabatic vacuum (which is comparable to our definition in the previous section). Proposed by Parker and Fulling in (Parker and Fulling, 1974), the idea here is to calculate the function W_k from equation 4.25 recursively. One can then calculate the density and pressure per mode and subtract the zeroth, second and fourth order terms **under the integral sign**. The resulting integral is then convergent and can be calculated numerically.

A sixth order calculation has been performed by Kaya and Tarman in (Kaya and Tarman, 2011). They find that, for an expansion factor of the form

$$a(\eta) = \left(\frac{\eta}{\eta_0} \right)^n \tag{4.61}$$

the ratio between pressure and density is given by

$$p = \frac{n-2}{n}\rho. \tag{4.62}$$

Hence proportionality of the stress-energy tensor to the metric only happens for $n = 1$, ie. in de Sitter space. The magnitude of the vacuum energy in this procedure is

$$\rho \approx \frac{H^6}{m^2}, \tag{4.63}$$

which is much smaller than the calculation based on dimensional regularization. This is due to the fact that in adiabatic regularization the terms lowest in adiabatic order are discarded **completely** to ensure conservation of the stress-energy tensor. Such discrepancies are inherent to the procedure of renormalization, since after regularization terms are subtracted and there is no clear prescription on what to subtract.

To lift this ambiguity we could look at experiment to see which prediction matches up the best. Sadly, there is a large difference between the observed value and theoretical predictions. From the accelerated expansion of the universe, the vacuum energy is measured to be of the order $\rho \approx 10^{-47}\text{GeV}^4$ (Martin, 2012). The mass of the Higgs boson is of the order 10^2 GeV whilst the Hubble constant H has a magnitude of 10^{-42} GeV. Clearly both theoretical predictions have a large mismatch in scale with regards to the measured value. Even worse, the scales of m and H can not be combined in any way that comes as close as 10 orders of magnitude to the measured value.

The above problem is called the cosmological constant problem: The scales present in QFT show a large mismatch with observation, no matter how one tries to calculate the vacuum energy. Since the techniques used in QFT have had remarkable success when confronted with particle physics experiments one would hope calculations using the same techniques to be valid in cosmological models. They sadly miss the right result by a large margin. This suggests that some new physics is needed to understand the measured value of the cosmological constant.

To conclude this chapter: We set out to define the vacuum by the expectation value of the stress-energy tensor, but we find that such a definition is impractical for several reasons.

Firstly, the renormalization of $\langle T_{\mu\nu}\rangle$ becomes a very hard problem in curved spacetimes. Even if it can be performed, it is problematic to backtrack to the definition of a state from this result because the renormalization step obscures much of the state-sensitive information.

Second, the subtraction of divergences leaves a large amount of ambiguity. The form of $\langle T_{\mu\nu}\rangle$ may very well depend quite drastically on the renormalization technique used. Because a preferred prescription is absent it is not even clear what $\langle T_{\mu\nu}\rangle$ is if we do decide on a vacuum state.

Lastly, we started considering equation 4.1 because we wanted to define the vacuum in terms of some local observable. However, a state in quantum field theory is not a local object, but a global one. A quantum field 'feels' the global properties of the spacetime it is in and this is also reflected in the expectation value of the stress-energy

tensor (compare for example with the Casimir energy). It is therefore hard to imagine how we can get equation 4.1 to hold since the right hand side is local, but the right hand side is not.

Chapter 5

Observer Vacua

5.1 Introduction

In this chapter we propose a construction for obtaining a collection of vacua according to a geodesic observer. If we follow Unruh's result that the notion of particles depends on the observer that measures them, we should also define the vacuum separately for each observer. To do this, we try to obtain a suitable generalization of the frequency-splitting procedure with respect to the observer's proper time.

We describe a procedure coined by Ashtekar and Magnon in (Ashtekar and Magnon, 1975) and later formalized by Kay and Chmielowski in (Kay, 1980; Chmielowski, 1994). This procedure is used to define a generalized frequency splitting vacuum for stationary spacetimes. With some slight modification this state can be carried over to more general situations. We use this procedure in the context of a single observer to define local frequency-splitting states.

Because the proper time of the observer can not be extended uniquely we find that we do not obtain a single vacuum , but rather a large class of vacua, which can not be distinguished by local measurements. We discuss unitary equivalence of these states, but find it unlikely that they are all equivalent for a general spacetime.

Of course in the presence of curvature we can not expect the vacuum to stay empty on cosmological timescales: Cosmological particle creation is a well known prediction of the theory. In order to claim that the states we defined are as close to vacuous as possible we would need to calculate the response of a particle detector for some small interval of time. Sadly this calculation becomes quite involved when not considering a static spacetime, and we have not been able to perform it. As such the precise physical interpretation of the states we have defined is still an open question.

The following sections are organized as follows. In section 2 we describe the AMK-vacuum of a stationary spacetime, and show that it matches the standard Minkowski vacuum when considering flat spacetime. Afterwards, in section 3, we specialize it to the case of a single observer, which yields us a local vacuum state. We describe a procedure for extending these states to the full spacetime. Finally, in section 4, we discuss unitary equivalence between the possible extensions and the question of the detector response in one of these states. A discussion of the physical significance of our construction is given in the outlook in the next chapter.

5.2 The AMK construction in stationary spacetimes

We first review the Ashtekar-Magnon-Kay (AMK) construction in stationary spacetimes. This means that there is some timelike Killing vector field ξ present that generates an isometry of the metric. As such, the Lie-derivative of the metric with respect to ξ is zero:

$$\mathcal{L}_\xi g = 0. \tag{5.1}$$

In local coordinates this can be rewritten as

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0. \tag{5.2}$$

This indicates that the metric obeys some symmetry: If we choose time in the direction of ξ, then the metric will not change with respect to time. We will view this time direction as the preferred time-direction of the model.

The presence of a Killing vector implies that there are conserved quantities in the theory. To be precise, we can define for a classical solution of the wave equation ϕ the stress-energy tensor by

$$T_{\mu\nu}(\phi) = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\rho \phi \nabla_\rho \phi - \frac{1}{2} m^2 g_{\mu\nu} \phi^2. \tag{5.3}$$

Clearly this tensor satisfies the conservation equation

$$\nabla^\mu T_{\mu\nu} = 0 \tag{5.4}$$

because ϕ satisfies the Klein-Gordon equation. We can then contract one of the indices with our Killing vector to obtain a conserved quantity:

$$\nabla^\mu (T_{\mu\nu} \xi^\nu) = (\nabla^\mu T_{\mu\nu}) \xi^\nu + T_{\mu\nu} (\nabla^\mu \xi^\nu) = 0, \tag{5.5}$$

where for the last equality we used the fact that $T_{\mu\nu}$ is symmetric in its indices and $\nabla^\mu \xi^\nu$ is anti-symmetric. This implies that we can define the classical-energy of the field ϕ by the equation

$$E(\phi) = \int_\Sigma T_{\mu\nu}(\phi) \xi^\nu d\Sigma^\mu \tag{5.6}$$

calculated on some Cauchy-surface Σ. Because of the conservation equation 5.5 Gauss' theorem implies that E does not depend on the Cauchy surface chosen for the calculation.

The idea proposed by Ashtekar and Magnon is to pick out a representation of the CCR by demanding that the classical energy of the field ϕ should coincide with its quantum analog $\langle \phi, H\phi \rangle$ where H is the Hamiltonian corresponding to time evolution by ξ. To effect this, we define the following energy-inner product on the space of classical solutions V:

$$\mu_\xi(\phi, \psi) := \int_\Sigma T_{\mu\nu}(\phi, \psi) \xi^\nu d\Sigma^\mu. \tag{5.7}$$

Here we have re-interpreted $T_{\mu\nu}$ as a bilinear form of two arguments by

$$T_{\mu\nu}(\phi, \psi) = \frac{1}{2} \Big(\nabla_{(\mu} \phi \nabla_{\nu)} \psi - g_{\mu\nu} \nabla^\rho \phi \nabla_\rho \psi - m^2 g_{\mu\nu} \phi \psi \Big). \tag{5.8}$$

A straightforward calculation shows that this tensor is also conserved, such that μ_ξ is independent of the Cauchy-surface Σ. There are a few things that need to be checked in order for this form to define a quantum field construction. Clearly μ_ξ is bilinear and symmetric by construction. We need to show that it's positive, ie. $\mu(\phi,\phi) \geq 0$ with equality if and only if $\phi = 0$. We also need to check that it satisfies the bound

$$\mu(\phi,\phi) = \sup_{\psi \in V} \frac{1}{4} \frac{\sigma(\phi,\psi)^2}{\mu(\psi,\psi)}. \tag{5.9}$$

For these things to hold we need to assume that there exists a Cauchy surface such that there exist $\epsilon > 0$ such that the following bounds are satisfied:

$$-\xi^\mu N_\mu > \epsilon \tag{5.10}$$

$$\frac{\xi^\mu \xi_\mu}{\xi^\nu N_\nu} > \epsilon^2 \tag{5.11}$$

Also, we assume that our mass is strictly positive to avoid infrared divergences. Since we have no massless scalar fields in nature this should not be a problem, and we may assume $m^2 > \epsilon$ [1]. In terms of the lapse and shift of ξ, the first demand states that the lapse α should not tend to zero. The second demand states that ξ should not approach a light-like vector. Written out in terms of the shift vector, this demand reads

$$-\alpha + \frac{\beta^i \beta_i}{\alpha} > \epsilon \tag{5.12}$$

Now, the show positivity of the form μ_ξ we write out equation 5.7 explicitly, following the calculation in the proof of theorem 4.1 of (Kay, 1980). The most troublesome term is the $\nabla^\rho \phi \nabla_\rho \phi$ term, for which we need the inverse metric to raise the ρ index. Recall that the inverse metric is given in terms of the lapse and shift by

$$g^{\mu\nu} = \begin{pmatrix} \frac{-1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & h^{ij} - \frac{\beta^i \beta^j}{\alpha^2} \end{pmatrix}. \tag{5.13}$$

We then calculate:

$$\begin{aligned} \nabla^\rho \phi \nabla_\rho \phi &= \frac{-1}{\alpha^2} \nabla_0 \phi \nabla_0 \phi + 2 \frac{\beta^i}{\alpha^2} \nabla_0 \phi \nabla_i \phi + \left(h^{ij} - \frac{\beta^i \beta^j}{\alpha^2} \right) \nabla_i \phi \nabla_j \phi \\ &= -\left(\frac{\nabla_0}{\alpha} \phi - \frac{\beta^i \nabla_i}{\alpha} \phi \right)^2 + h^{ij} \nabla_i \phi \nabla_j \phi \\ &= -(N^\mu \nabla_\mu \phi)^2 + h^{ij} \nabla_i \phi \nabla_j \phi \end{aligned}$$

Where we have used the fact that the normal vector satisfies $N^\mu = \frac{1}{\alpha}(\xi^\mu - \beta^\mu)$. Recall that we can always link the solution ϕ to its initial conditions on Σ in terms of $\phi|_\Sigma$ and $\pi = N^\mu \nabla_\mu \phi|_\Sigma$ and we will liberally use this identification in the following.

[1] This is vague due to the fact that m has a nonzero mass dimension. In the following we would like to perform calculations with dimensionless objects, and therefore we regard all our objects as being dimensionless functions and parameters. Equivalently, we could pick some arbitrary reference-scale m_{ref} and divide it out of all our objects, such that for example $\phi = m_{\text{ref}} \tilde{\phi}$ with $\tilde{\phi}$ dimensionless. In the following we will assume that such a reference scale has been chosen and all quantities have been rewritten in terms of their dimensionless analogues.

We then calculate the value of $\mu_\xi(\phi,\phi)$:

$$\begin{aligned}\mu_\xi(\phi,\phi) &= \int_\Sigma \Big((\nabla_\mu\phi\nabla_\nu\phi) - \frac{1}{2}g_{\mu\nu}\Big(\nabla^\rho\phi\nabla_\rho\phi + m^2\phi^2\Big)\Big)\xi^\nu N^\mu\sqrt{h}d^3x \\ &= \int_\Sigma \Big(\alpha\pi^2 + (\beta^i\nabla_i\phi)\pi + \frac{\alpha}{2}\Big(-\pi^2 + h^{ij}\nabla_i\phi\nabla_j\phi + m^2\phi\Big)\Big)\sqrt{h}d^3x \\ &= \int_\Sigma \Big(\frac{\alpha}{2}\Big(\pi^2 + h^{ij}\nabla_i\phi\nabla_j\phi + m^2\phi\Big) + (\beta^i\nabla_i\phi)\pi\Big)\sqrt{h}d^3x \\ &= \frac{1}{2}\int_\Sigma \Big((\alpha - \frac{\beta^i\beta_i}{\alpha})\pi^2 + \alpha h^{ij}(\nabla_i\phi + \frac{\beta_i}{\alpha}\pi)(\nabla_j\phi + \frac{\beta_j}{\alpha}\pi) + \alpha m^2\phi^2\Big)\sqrt{h}d^3x.\end{aligned}$$

This is an integral of quadratic terms, which is hence larger than 0 (recall that h^{ij} is a euclidean metric) if ϕ and π are nonzero[2].
ter op de avon
If all the above demands on α, β and m^2 hold then we can bound μ_ξ in terms of the $L^2 + L^2$ norm on the space of initial data. We throw out the middle term to obtain

$$\mu_\xi(\phi,\phi) \geq \frac{1}{2}\epsilon^2\int_\Sigma(\pi^2+\phi^2) = \frac{1}{2}\epsilon^2\left\|(\phi,\pi)\right\|^2_{L^2+L^2}. \tag{5.14}$$

We can write the symplectic form in terms of the $L^2 + L^2$ inner product by including an operator switching the two components of the initial-conditions vector:

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

such that

$$\sigma((\phi_1,\pi_1),(\phi_2,\pi_2)) = \int_\Sigma(\phi_2\pi_1 - \phi_1\pi_2)\sqrt{h}d^3x = \langle(\phi_1,\pi_1), M(\phi_2,\pi_2)\rangle_{L^2+L^2}.$$

Clearly M is an operator with norm 1, and therefore we can bound σ in the following way:

$$\begin{aligned}\sigma((\phi_1,\pi_1),(\phi_2,\pi_2))^2 &= \langle(\phi_1,\pi_1), M(\phi_2,\pi_2)\rangle^2_{L^2+L^2} \\ &\leq \left\|(\phi_1,\pi_1)\right\|^2_{L^2+L^2}\left\|(\phi_2,\pi_2)\right\|^2_{L^2+L^2} \\ &\leq \frac{1}{4\epsilon^4}\mu_\xi(\phi_1,\phi_1)\mu_\xi(\phi_2,\phi_2).\end{aligned}$$

This is not the bound we set out to achieve, since we want to have an inequality that can be satisfied and it is not clear that it can in this case. The factor ϵ^4 can be absorbed into a redefinition of ξ but we can not get equation 5.4 to hold.

Looking back at the procedure of defining the one-particle Hilbert space in section 2.5 we see that the use of the saturation of the bound is twofold. First, we need it to prove that the complex structure J has unit norm, such that the Hilbert space $V_\mu^{\mathbb{C}}$ splits into $\pm i$ eigenspaces of J. The second was in equation 2.69 were we used it to prove that KV is dense in the Hilbert space H.

[2]Note that, for positive definiteness, we do not need the bound $m^2 > \epsilon$. But, since we want to bound the symplectic form by μ_ξ we cannot rely on derivative terms such as $h^{ij}\nabla_i\phi\nabla_j\phi$ because they have no connection to the size of ϕ and π. This means that, sadly, the construction does not work for massless fields.

As shown in appendix A of (Kay and Wald, 1991) a one-particle Hilbert space can in this case still be uniquely constructed, but the embedding of V is not dense. This means that the representation of the canonical commutation relations is not irreducible, as shown in lemma A2 of the same source. We could accept this and move on with a non-pure state, but a better alternative has been given by Chmielowski in (Chmielowski, 1994).

The idea is to start with a scalar product μ on V which satisfies for all $\phi, \psi \in V$

$$\sigma(\phi, \psi) \leq 4\mu(\phi, \phi)\mu(\psi, \psi). \tag{5.15}$$

We can Cauchy-complete V in the norm μ to obtain V_μ. Because of the bound on σ we can use Riesz's lemma to obtain an operator A such that

$$\sigma(\phi, \psi) = 2\mu(\phi, A\psi) \tag{5.16}$$

This will not be complex structure, since if the bound on σ is not saturated the norm of A is smaller than 1. However, we can look at the polar decomposition of the operator A and write $A = J|A|$, here J is a partial isometry and $|A|$ is self-adjoint. The trick is to associate with the scalar product μ its purification $\tilde{\mu}$ by dividing out $|A|$. We define $\tilde{\mu}$ by

$$\tilde{\mu}(\phi, \psi) = \mu(\phi, |A|\psi). \tag{5.17}$$

The proof that the so defined $\tilde{\mu}$ satisfies equation 5.4 is straightforward but tedious and we refer the reader to (Chmielowski, 1994) or (Verch, 1997) for details.

The above procedure is very abstract and a practical example is in order. We will perform the procedure in Minkowksi spacetime and show that it produces the standard result, a complex structure that splits the positive and negative frequency modes. This also gives a justification for the statement that this procedure is a rigorous form of frequency splitting.

First we show that, because we have a Killing vector, the Lie derivative of any solution of the KG-equation is again a solution to the KG equation:

$$\Box(\xi^\nu \nabla_\nu \phi) = (\Box \xi^\nu)\nabla_\nu \phi + \xi^\nu(\Box \nabla_\nu \phi) + 2(\nabla_\mu \xi_\nu)\nabla^\mu \nabla^\nu \phi.$$

The last term is zero by anti-symmetry of $\nabla_\mu \xi_\nu$. For the first term we take a derivative of the Killing equation:

$$\begin{aligned} 0 &= \nabla^\mu(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) \\ &= \Box \xi_\nu + \nabla_\mu \nabla_\nu \xi^\mu \\ &= \Box \xi_\nu + [\nabla_\mu, \nabla_\nu]\xi^\mu \qquad (\nabla_\mu \xi^\mu = 0) \\ &= \Box \xi_\nu + R^\mu_{\lambda\mu\nu}\xi^\lambda \\ &= \Box \xi_\nu + R_{\mu\nu}\xi^\mu. \end{aligned}$$

Here we used the definition of the Riemann-tensor as the commutator of the covariant derivative. For the second term we calculate:

$$\begin{aligned}\Box\nabla_\nu\phi &= \nabla^\mu(\nabla_\nu\nabla_\mu\phi)\\ &= \nabla_\nu\Box\phi + [\nabla_\mu,\nabla_\nu]\nabla^\mu\phi\\ &= m^2\nabla_\nu\phi + R^\mu_{\lambda\mu\nu}\nabla^\lambda\phi\\ &= m^2\nabla_\nu\phi + R_{\mu\nu}\nabla^\mu\phi\end{aligned}$$

Putting all of this together we find

$$\Box(\xi^\nu\nabla_\nu\phi) = m^2\xi^\nu\nabla_\nu\phi + R_{\mu\nu}(-\xi^\mu\nabla^\nu + \xi^\nu\nabla^\mu)\phi = m^2\xi^\nu\nabla_\nu\phi,$$

by symmetry of $R_{\mu\nu}$. Hence $\mathcal{L}_\xi$ acts on the space of classical solutions V as an (unbounded) operator. It turns out that this operator inverts the operator A defined above by use of Riesz's lemma. To see this, we calculate the following identity:

$$\sigma(\phi,\mathcal{L}_\xi\psi) = 2\mu_\xi(\phi,\psi). \tag{5.18}$$

This shows that for all ϕ and ψ we have

$$\mu_\xi(\phi,\psi) = \mu_\xi(\phi, A\mathcal{L}_\xi\psi), \tag{5.19}$$

implying that $A = \mathcal{L}_\xi^{-1}$. Now, to prove equation 5.18 we write out all the terms, using Gauss's theorem to move covariant derivatives around. This gives us volume terms over the volume to the future or past of the Cauchy surface which we will calculate separately.

$$\begin{aligned}\sigma(\phi,\mathcal{L}_\xi\psi) &= \int \phi\overleftrightarrow{\nabla_\mu}(\xi^\nu\nabla_\nu\psi)d\Sigma^\mu\\ &= \int(\nabla_\mu\phi\xi^\nu\nabla_\nu\psi - \phi(\nabla_\mu\xi_\nu)\nabla^\nu\psi - \phi\xi^\nu\nabla_\mu\nabla_\nu\psi)d\Sigma^\mu\\ &= \int(\nabla_\mu\phi\xi^\nu\nabla_\nu\psi + \phi(\nabla_\nu\xi_\mu)\nabla^\nu\psi - \phi\xi^\nu\nabla_\nu\nabla_\mu\psi)d\Sigma^\mu\\ &= \int(\nabla_\mu\phi\xi^\nu\nabla_\nu\psi - \nabla_\nu\phi\nabla^\nu\psi\xi_\mu - m^2\phi\psi\xi_\mu + \nabla_\nu\phi\nabla_\mu\psi\xi^\nu)d\Sigma^\mu\\ &= 2\int T_{\mu\nu}(\phi,\psi)\xi^\nu d\Sigma^\mu = 2\mu_\xi(\phi,\psi).\end{aligned}$$

The neglected volume term is

$$\begin{aligned}\int\nabla^\nu(\phi\xi_\mu\nabla_\nu\psi - \phi\xi_\nu\nabla_\mu\psi)d\Sigma^\mu &= \int\nabla^\mu\nabla^\nu(\phi\xi_\mu\nabla_\nu\psi - \phi\xi_\nu\nabla_\mu\psi)dV\\ &= \frac{1}{2}\int[\nabla_\mu,\nabla_\nu](\phi\xi^{[\mu}\nabla^{\nu]}\psi)dV\\ &= \int R_{\rho\nu}(\phi\xi^\rho\nabla^\nu\psi) - R_{\rho\nu}(\phi\xi^\nu\nabla^\rho\psi)dV = 0.\end{aligned}$$

In the last step we again used the definition of the Riemann tensor as the commutator of two covariant derivatives and the fact that the Ricci tensor is symmetric in its indices. The volume over which we perform the integral can either be the past or the future of the Cauchy surface Σ, but since the integrand vanishes the result is zero either way. Hence we have proved the equality 5.18.

Now we can explicitly write down the complex structure, it is given by

$$J_\xi = -|\mathcal{L}_\xi|^{-1}\mathcal{L}_\xi \tag{5.20}$$

Where $|\mathcal{L}_\xi|^{-1} = (-\mathcal{L}_\xi\mathcal{L}_\xi)^{-\frac{1}{2}}$ is defined through the spectral theorem, which is applicable since $\mathcal{L}_\xi$ is skew-symmetric:

$$\mu_\xi(\phi, \mathcal{L}_\xi\psi) = \mu_\xi(A\mathcal{L}_\xi\phi, \mathcal{L}_\xi\psi) = \mu_\xi(\mathcal{L}_\xi\phi, A^\dagger\mathcal{L}_\xi\psi) = \mu_\xi(\mathcal{L}_\xi\phi, -A\mathcal{L}_\xi\psi) = \mu_\xi(-\mathcal{L}_\xi\phi, \psi) \tag{5.21}$$

Technically, we need $\mathcal{L}_\xi$ to be *essentially skew adjoint* on V_μ for this theorem to be applicable. The problem is that $\mathcal{L}_\xi$ is defined on all of V but is not bounded. Therefore it defines an unbounded, densely defined operator on V_μ. For these operators, being self-adjoint is not the same as being symmetric because there is also a condition on the domain of the operator. It is hard to show that $\mathcal{L}_\xi$ it is essentially skew-adjoint, but it is true nonetheless. We will here not bother with the mathematical details, but assure the reader that they can be performed correctly, as is done in (Kay, 1980).

To see that this is the complex structure linking σ and $\tilde{\mu}$ we calculate:

$$\sigma(\phi, -J_\xi\psi) = 2\mu_\xi(\phi, A|\mathcal{L}_\xi|^{-1}\mathcal{L}_\xi\psi) = \mu_\xi(\phi, |\mathcal{L}_\xi|^{-1}\psi) = \mu_\xi(\phi, |A|\psi) = \tilde{\mu}_\xi(\phi, \psi) \tag{5.22}$$

which is the desired result. In the second step we used the fact that $|\mathcal{L}_\xi|^{-1}$ commutes with $\mathcal{L}_\xi$ because it is defined by the spectral theorem.

Now suppose we act on a positive frequency mode by J_ξ. The operator $\mathcal{L}_\xi$ multiplies it by $-i\omega$, and the $|\mathcal{L}_\xi|^{-1}$ operator multiplies it by $|\omega|^{-1}$, the result is an eigenvalue $+i$. Similarly, a negative frequency mode has eigenvalue $-i$. We see therefore that we have recovered in Minkowksi space (or in fact, any static [3] spacetime) the standard complex structure that splits positive and negative frequency solutions. The AMK formalism can thus be viewed as a mathematically rigorous generalization of frequency splitting to stationary spacetimes.

5.3 AMK states corresponding to an observer

The previous construction can be extended to situations where a Killing vector is not present. In the previous section we used the Killing vector ξ and a Cauchy surface Σ in order to define a scalar product by

$$\mu_\xi(\phi, \psi) = \int_\Sigma T_{\mu\nu}(\phi, \psi)\xi^\nu d\Sigma^\mu. \tag{5.23}$$

In fact, for this to define a state we do not need ξ to be Killing, just that it satisfies the bounds in equations 5.10 en 5.11 on the Cauchy surface Σ. If ξ is no longer Killing the choice of Cauchy surface becomes important, since the integrand is no longer conserved. This should not come as a surprise: If the geometry of spacetime is no longer independent of time the quantum field will spontaneously create particles and what is considered a vacuum state at one time will not remain one at some later time.

[3] By which we mean a spacetime with Killing-vector that is hypersurface orthogonal, such that the Klein-Gordon equation can be separated into a temporal and a spatial part without mixed terms.

We therefore find that the construction of a QFT hinges on the choice of a Cauchy surface and a vector field on that Cauchy surface. In order to determine what a geodesic observer would most naturally indicate as the vacuum we would like to pick a vector field that coincides with the observers' worldline. We saw previously that this can be done, but only locally. At some time t_0 we define a local rest frame S_{t_0} by extending vectors normal to $\dot{\gamma}(t)$ geodesically, until they no longer form a surface. [4] With respect to these rest surfaces, the most reasonable choice of reference frame is formed by the normal vectors to the different S_t, which we will call the observers' local reference frame.

This is not enough information to set up our QFT, we need a full Cauchy surface and a full vector field to perform the procedure of the previous section. We only obtain these objects on a partial set of the spacetime, namely a cylindrical region of spacetime around the worldline of the observer. We can, however, restrict our spacetime to the diamond shaped region $D(S_{t_0})$ which is globally hyperbolic with Cauchy surface S_{t_0}. [5] We then have to restrict to the classical phase space

$$V_{t_0} = C_c^\infty(S_{t_0}) \bigoplus C_c^\infty(S_{t_0}), \tag{5.24}$$

which we regard as the local class of observables corresponding to $D(S_{t_0})$. Then we can define our bilinear product by

$$\mu_{S_{t_0}}(\phi, \psi) = \int_{S_{t_0}} T_{\mu\nu}(\phi, \psi) N^\nu dS^\mu_{t_0} \tag{5.25}$$

and define our quantum field theory by means of its purification $\tilde{\mu}_{t_0}$.

This construction gives us a mathematically rigorous way of frequency splitting with respect to an observer's proper time, but clearly the formalism leaves things to be desired. Most of all is that we had to restrict the symplectic vector space V to a rather small subset of local observables. That we had to do this is reasonable, because we cannot really expect an observer to measure observables that are far away. However, we want a full description of the quantum field, not just a local one, especially since the worldline of the observer will likely leave $D(S_{t_0})$ after a finite amount of time.

We can view the state above as a restriction of a state on the entire spacetime. We can extend the local rest frame by use of the procedure described in (Bernal and Sánchez, 2006). Their theorem 1.1 reads:

Theorem (Bernal & Sanchez)**.** *Let (M, g) be a globally hyperbolic spacetime, and let $H \subset M$ be a spacelike and acausal compact m–submanifold with boundary. Then, there exists a spacelike Cauchy hypersurface S such that $H \subset S$.*

[4] This definition is somewhat vague because of the condition 'until they no longer form a surface'. More precisely, we define:

$$S^\epsilon_{t_0} = \{x \in \mathcal{M} | \exists \text{Geodesic } \zeta \text{ s.t. } \zeta(0) = \gamma(t_0), \dot{\zeta}(0) \perp \dot{\gamma}(t_0), |\dot{\zeta}(0)| = 1 \text{ and } \exists s \leq \epsilon \text{ s.t. } x = \zeta(s)\}.$$

We know that for small ϵ the S^ϵ_t are partial Cauchy surfaces, and we can define δ as the supremum of the values for which they are. Then we can define S_{t_0} as $S^\delta_{t_0}$.

[5] It is here that our Hilbert-space approach comes back to haunt us, since it is awkward to define something like a 'local state'. If we would have adopted the framework of algebraic quantum field theory we could have spoken of a state on a restricted part of the algebra of observables, which is mathematically cleaner. We chose not to use this formalism in order to stay more in touch with standard treatments of QFT.

We need S_{t_0} to be a compact hypersurface with boundary for this theorem to be applicable. We tacitly assumed S_{t_0} to be small (ie. compact) but in some models, such as Minkowksi space and FLRW-space, the surface might be unbounded and might even be a Cauchy surface. In the case that it is a Cauchy surface we are happy because our procedure provides us with a global vacuum. In general the absence of a boundary on S_{t_0} does not mean that its domain of dependence fills the entire spacetime. In any case, in the following we restrict S_{t_0} by hand such that it is compact, which can always be done by restricting the curve parameter of the geodesics normal to the worldline of our observer.

Hence we obtain a smooth Cauchy-surface Σ that contains S_{t_0}. On this Cauchy surface, we define the bilinear product

$$\mu_\Sigma(\phi,\psi) = \int_\Sigma T_{\mu\nu}(\phi,\psi)N^\nu d\Sigma^\mu \tag{5.26}$$

which defines a state, since the bounds in equations 5.10 and 5.11 are clearly satisfied. Then we, again, define the vacuum by the purification procedure assigning $\tilde{\mu}_\Sigma$ to μ_Σ. The choice of the extension Σ is not unique, and hence each choice gives, in general, a different vacuum state. They do coincide for the local observables contained in $D(S_{t_0})$, ie. for classical solutions whose initial data is contained in V_{t_0}. Hence we see that in our search for a vacuum state we have not found one, but many different states.

The situation can be understood as follows. We started our construction with the desire to perform a frequency splitting procedure with respect to the observers proper time. However, the proper time of the observer can only be defined locally. Hence we can not use this criterion to pin down what positive frequency means for the low frequency modes, whose wavelength is comparable to or larger than the size of the local rest-frame.

In order to measure modes of this size a longer measuring time is needed and such an observation can not be performed within the time frame given by the domain of dependence of the local rest frame. As such measurements performed on timescales were the observer is within $D(S_{t_0})$ can not distinguish between these different states and they are therefore equally suited as vacuum state.

From a practical viewpoint the appearance of a large multitude of vacua is not very alarming. If we consider an observer orbiting the sun, such as the reader or the author, the size of S_{t_0} is quite large. We expect breakdown of the local rest frame by gravitational lensing at the sun, which means that the size of S_{t_0} is of the order 10^{11} meters. This means that the vacuum defined above is unique for around 500 seconds. For all intents and purposes in QFT this is an eternity. The long range modes which prevent us from defining the vacuum uniquely have an energy smaller than

$$E = \frac{1}{2}\hbar\omega \approx 4\times 10^{-18}\text{eV} \tag{5.27}$$

which is negligible. It is thus unlikely that a measurement that separates these different vacuum states could ever be carried out in practice. In a way, we have reconfirmed what we already knew: From the point of view of a terrestrial observer the vacuum is practically unique and Minkowksi space is an excellent approximation of the curved spacetime that we inhabit.

Of course we have not really justified our calling any of the above states a vacuum. For this to be justified we would have to calculate the response a model particle detector gives. The problem is complicated in the presence of a curved background. Recall that after the measuring process described in section 3.7 the combined system is in the following state, to first order in the switching function ϵ

$$|\Psi_{\text{out}}\rangle = |0;-\rangle - i\hat{\sigma}(Ef,.)|0;+\rangle, \tag{5.28}$$

where $f = \epsilon(t)e^{i\Omega t}\psi^*(\mathbf{x})$ is the profile of the detector. We assume that the spatial detector profile ψ has small support and the detector is only switched on for a short time such that the support of f is within $D(S_{t_0})$.

Since the state we defined is not a frequency splitting vacuum in the literal sense of the word we cannot simply dispense of the situation by claiming that Ef is positive frequency, and therefore only gives rise to an annihilation operator. In the presence of curvature the operator E will not be so simple as it was in Minkowksi spacetime and a proper calculation would become quite involved.

We can however guess the order of magnitude of the detector response. To do this, we first neglect the time dependence of the curvature. Concretely, we construct a static spacetime that is the same as our original spacetime $\mathcal{M}$ on S_{t_0}. We look at the set $D(S_{t_0})$ and set the metric equal to

$$g(t,x) = g(t_0,x) \tag{5.29}$$

By construction the observer vector field is a static Killing field in this modified spacetime. As such, the vacuum defined by equation 5.25 defines the static vacuum of this spacetime, which is directly related to frequency splitting because the complex structure J is in the static case given by equation 5.20.

Therefore the question of the detector response does in this case also boil down to the decomposition of Ef into positive and negative frequency parts. Just like in Minkowksi space, the map E preserves harmonic oscillation in a static spacetime. To see this, suppose we have a test function $g = e^{i\omega t}\psi(\vec{x})$. To find the advanced and retarded solutions with source g, we have to solve

$$(\Box - m^2)Ag = g \tag{5.30}$$

In the static case, the d'Alembertian splits like

$$\Box = -\partial_t\partial_t + g^{ij}\nabla_i\nabla_j, \tag{5.31}$$

as can be seen from the representation

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\Big(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\Big). \tag{5.32}$$

Thus the equation for Ag (or equivalently for Rg) factorizes, and we see that the behaviour in the time direction of these operators is $Ag \propto e^{i\omega t}$. Since $E = A - R$ it behaves in the same way.[6]

[6]This derivation is quite handwaving because the function g does not have compact support in the t-direction and is therefore not a test function. If we were to form some wave packet we could force

Therefore we find in this case, just like in Minkowksi spacetime:

$$\hat{\sigma}(Ef, .) \approx ia(\overline{KEf}). \tag{5.33}$$

Thus the out state after the measurement is $|0, -\rangle$, no particles are detected [7].

Now, to go back to the case at hand. If our construction is continuous in the parameters of the theory, namely in the background metric, the static case will be a good approximation to the present situation if the metric changes slowly with respect to time. We then expect the probability of detecting a particle will be proportional to the deviation from the static case. A good measure of this deviation would be the derivative of some metric quantities with respect to the observers' proper time. It is also reasonable that this chance would increase with a longer measuring time. We therefore predict the following formula to hold

$$P(\text{particle detection}) \propto T\partial_\tau g. \tag{5.34}$$

Where g is some component of the metric relevant to the situation at hand.

A prescription that satisfies this formula is the best that we could hope for. It would be unrealistic for us to pose the existence of a vacuum state that remains vacuous for all time, since a large amount of examples of the creation of particles by curvature effects has been calculated in the past. If equation 5.34 were to hold for our construction we could confidently pose that we had constructed a vacuum-like state for the observer γ at t_0.

5.4 Discussion

Asking for the theory to be continuous in its parameters is quite a big if, and we have not done anything to justify it. Whilst the assumption might seem harmless enough, but things tend to behave badly or diverge in quantum field theory. The construction we used is nice for stationary spacetimes, but there is no guarantee that it will continue to behave in the case we have considered.

For example, the frequency splitting vacuum of a static spacetime is a Hadamard state (Fulling, Narcowich, and Wald, 1981) but we have no guarantee that the states we defined above are. As a different example where things fail for a slight generalization of the metric we refer to (Fulling, 1979). Here instantaneous vacuum states in FLRW space are considered, defined by setting initial conditions for the temporal behaviour of our modes as we did in the previous chapter. It is shown that adding a small anisotropy into the spacetime introduces infinite particle-production between quantizations at different times: They are not unitarily equivalent even for small differences in time. Hence a slight generalization of a procedure that seems reasonable might in fact be very badly behaved.

Our claim to the validity of equation 5.34 follows from the fact that we have generalized in a mathematically sound way a prescription that we know to be valid for static

the test-function to take a compact support, but we won't because it encumbers the notation. The important part is that the d'Alembertian splits like $-\partial_t^2 +$ spatial part, such that the E preserves harmonic oscillation in the time-direction.

[7] Of course, up to the time-derivatives of the switching function $\epsilon(t)$ which we neglected with respect to the oscillatory factor $e^{i\omega t}$.

spacetimes. This is however, not a valid reason to assume that our generalization works in the way that we would like it to. Perhaps a different generalization is better suited, or perhaps a proper generalization of the frequency-splitting prescription just doesn't exist.

All of this doesn't mean that our construction is wrong, but clearly more work has to be done in order to solidify our interpretation of the observers' vacuum as a valid physical prescription. We would at least have to show the following things:

First of all, we would have to show that the two-point function is of the Hadamard form. If our construction fails to have the right singularity structure we would not be justified in claiming our construction is 'practically Minkowskian' for terrestrial observers. Also, the calculation of the stress-energy tensor, which is the most interesting observable a free theory has to offer, would not be possible by means of the point-splitting procedure.

Second, a proper derivation of the detector response is in order if we want to call our state vacuum-like according to our observer. The method for calculating the detector response in (Unruh and Wald, 1984) is mathematically quite elegant, but it is not completely clear to the author that it can be carried over straightforwardly to curved spacetimes. A different framework for calculating the response of an Unruh-deWitt detector is also available, see for instance (Louko and Satz, 2008) and references therein. This method involves integrating the two-point function of the theory along the worldline of the observer.

In order to solve both these problems we would need the two-point function of the theory, namely

$$W(x,x') = \langle 0|\hat{\phi}(x)\hat{\phi}(x')|0\rangle. \tag{5.35}$$

The value of the two-point function is directly related to our choice of μ, since

$$\langle 0|\hat{\phi}(x)\hat{\phi}(x')|0\rangle = \mu(E(x),E(x')) - \frac{i}{2}\sigma(E(x),E(x')). \tag{5.36}$$

Where $E(x)$ is the advanced minus retarded Green's function [8] The distribution $E(x)$ might be calculated for points lying close to x by a suitable expansion in orders of derivatives of the metric. For example, in Riemann local coordinates the Feynman propagator can be calculated to second order in metric derivatives as (Bunch and Parker, 1979):

$$G_F(x,x') = (-g(x'))^{-\frac{1}{4}} \int \frac{d^4k}{(2\pi)^4} e^{iky}\Big(1 - \frac{R}{6}\frac{\partial}{\partial m^2}\Big)\frac{1}{k^2+m^2}, \tag{5.37}$$

where y is the difference between x and x' in normal coordinates. An expression for the advanced and retarded Green's functions can be derived from this by a suitable prescription of how the poles in the complex plane should be shifted when performing the k^0 integral, namely all in the upper half plane or all in the lower half plane. A similar procedure can be performed in Fermi-coordinates, and has in fact been done in a different context in (Rinaldi, 2008).

[8] Here we interpret $\hat{\phi}(x)$ as the field observable corresponding to a Dirac-delta 'function'. As such $E(x)$ is defined as the advanced minus retarded solution of the equation $(\Box - m^2)f = \delta(x)$, ie. the advanced and retarded Green's functions.

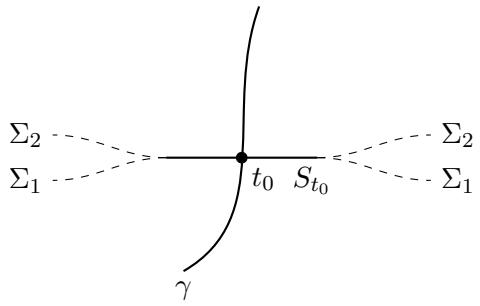

FIGURE 5.1: Two extensions of the same local rest frame S_{t_0}. Since the Cauchy surfaces Σ_1 and Σ_2 can float apart arbitrarily far at spatial infinity (within the constraint that both surfaces must be spacelike) it is unlikely that the two extensions $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are equivalent.

The real problem in calculating the two-point function is in the calculation of μ thereafter. We used Riesz's lemma and the spectral theorem in order to give our definition of the vacuum. Because these tools are quite abstract they allow us to define states in a general spacetime, but the drawback is that we have no solid handles for a practical calculation. Even if we were to go to the trouble to obtain a nice formula for $E(x)$, a practical calculation of $\tilde{\mu}(E(x), E(x'))$ would be far from trivial. The author is not familiar enough with the details of distribution theory and partial differential equations to estimate if such a calculation could be carried out in practice.

The third thing we would have to address is the question of unitary equivalence. As we have indicated before one of the main problems of QFT in curved spacetimes is the existence of unitary inequivalent constructions of the CCR. As an initial goal for considering this construction, we hoped that the set of vacua corresponding to geodesic observers would fall within one equivalence class of representations. Because these states have a clear distinction as 'states that can be measured' by reasonable observers we could, if this were the case, claim that this class of states is physically distinguished.

It is easily seen that for Minkowksi space it does work, the state for each geodesic observer is equal to the standard vacuum corresponding to inertial observers. In the general case however, we do not really have any reason to assume that even the different extensions of our vacuum state at some time t_0 are unitarily equivalent.

The problem is that the different extensions of the partial Cauchy surface S_{t_0} to a full Cauchy surface can be very far apart in the time-direction. Suppose we have two such extensions, Σ_1 and Σ_2. They are pinned together around the position of the observer, but far away from the worldline of the observer they can start to 'drift apart', one towards the future and one towards the past. This means that a temporally unbounded region is contained between the two Cauchy surfaces, which can contain some very wild curvature. It is hard to imagine that we could in that case obtain a bound between μ_1 and μ_2 that is uniformly valid for all spacetimes, not to mention the more difficult task of proving the operator B to be of Hilbert-Schmidt class.

What about our constructions at two different times for one observer? What clouds the discussion of unitary equivalence in this case is that it is really a question of comparing two **global** states, whereas we have taken an approach of defining a **local**

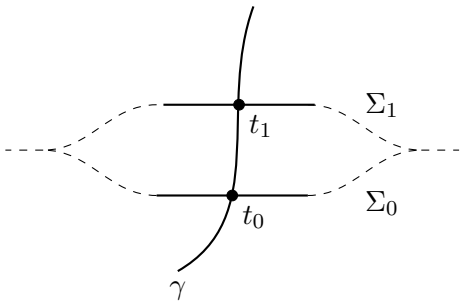

FIGURE 5.2: The extensions of S_{t_1} and S_{t_0} that might render the two constructions unitarily equivalent. The construction involves bending Σ_1 to the past and Σ_0 to the future such that they smoothly link up to form a Cauchy surface. In this case only a compact set is wedged in between the two surfaces.

state leaving the global extension open to choice. Because of this fact it is impossible to compare the local vacua defined at different times, because they are states of different quantum systems. It is only when we extend to the entire spacetime that we can compare these different states. We can thus really only ask the following question: Can we find an extension Σ_0 of $S_{t_0}$and an extension Σ_1 of S_{t_1} such that $\tilde{\mu}_{\Sigma_0}$ and $\tilde{\mu}_{\Sigma_1}$ give rise to unitarily equivalent representations?

There are some reasons to believe that we could, and we sketch here how this could be proven. Because of the finite difference between t_1 and t_0 we expect that it would be possible to find Cauchy surfaces Σ_0 and Σ_1 that coincide except for a compact part of spacetime, where one of the surfaces coincides with S_{t_0} and the other with S_{t_1}, see the figure 5.2 below.

The author has not been able to formulate a full proof of the existence of such a pair of Cauchy surfaces, but the construction would likely proceed as follows. Obviously a lot of mathematical detail is missing here and it is likely that not every statement is true in general. We find it likely that, for small differences between t_0 and t_1, it should be possible to perform the steps below in a mathematically rigorous fashion.

We can look at the future light-cone of S_{t_0} and the past light-cone of S_{t_1}. Then, the set $\mathcal{M}' = I^-(I^+(S_{t_0}) \cap I^+(I^-(S_{t_1})$ will be spacelike separated from both S_{t_0} and S_{t_1} and be globally hyperbolic. The set $\mathcal{M}'$ is the spacetime that one obtains from $\mathcal{M}$ by cutting out the future of S_{t_0} and the past of S_{t_1}.

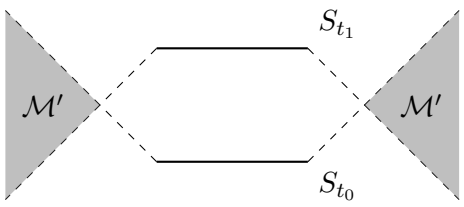

FIGURE 5.3: The spacetime $\mathcal{M}'$ obtained. From this figure it is clear that $\mathcal{M}'$ is the spacetime one obtains by removing from $\mathcal{M}$ the future of S_{t_0} and the past of S_{t_1}.

Global hyperbolicity of $\mathcal{M}'$ can be proved using the alternate, equivalent characterization of global hyperbolicity used in (Hawking and Ellis, 1973). It would go beyond

the scope of this text to discuss this alternate definition, but the interested reader can easily check that $\mathcal{M}'$ satisfies all demands because $\mathcal{M}$ does. Now, because $\mathcal{M}'$ is globally hyperbolic, it admits a Cauchy surface Σ, which is spacelike separated from both S_{t_0} and S_{t_1}. It should then be possible, by an adaptation of the smoothing procedure in (Bernal and Sánchez, 2006), to smoothly bridge the gap between Σ and the two different S_{t_i}. This would yield a pair of smooth Cauchy surfaces Σ_0 and Σ_1 which both contain Σ and S_{t_0} resp. S_{t_1}.

For the moment assuming such a pair of Cauchy-surfaces exists, now the difference between the two Cauchy-surfaces is compact, which makes the situation much more manageable. We can use the estimate on the energy inner-product in lemma 3.1 of (Verch, 1997) to find constants c_1, c_2 such that

$$c_1\mu_{\Sigma_0}(\phi,\phi) \leq \mu_{\Sigma_1}(\phi,\phi) \leq c_2\mu_{\Sigma_0}(\phi,\phi). \tag{5.38}$$

Because our Cauchy surfaces only differ up to a compact space, we can get this bound to hold uniformly for all $\phi \in V$. Therefore μ_{Σ_1} and μ_{Σ_2} define the same topology. What about $\tilde{\mu}_{\Sigma_1}$ and $\tilde{\mu}_{\Sigma_2}$? It is easily shown that the topology defined by the refinement of a product μ_i is weaker than that of the product itself, because J_i is bounded:

$$\tilde{\mu}_i(\phi,\phi) = \mu_i(\phi, |J_i|\phi) \leq \|J_i\|\, \mu_i(\phi,\phi). \tag{5.39}$$

And therefore we see that the topology defined by $\tilde{\mu}_i$ is weaker than that defined by μ_i, ie. $\tilde{\tau}_i \subseteq \tau_i$. However, this relation is likely strict, because in absence of the saturation property $|J_i|$ might not be invertible. While $|J_i|$ is injective on V because σ is non-degenerate, it might happen that there is a sequence of normalized vectors $\{\phi_n\}_{n\in\mathbb{N}}$ in V for which $|J_i|\phi_n \to 0$. In this case a reverse bound would not be possible, since it would imply

$$\mu_i(\phi_n,\phi_n) \leq C\mu_i(\phi_n, |J_i|\phi_n), \tag{5.40}$$

where the left hand side would be equal to 1, whereas the right hand side would shrink to 0. Hence we obtain

$$\tilde{\tau}_1 \subseteq \tau_1 = \tau_2 \supseteq \tilde{\tau}_2, \tag{5.41}$$

and clearly this is not good enough for our purposes. It might still happen that $\tilde{\tau}_1 = \tilde{\tau}_2$, but we can likely not prove this along these lines. [9]

The second demand is likely even harder to prove and we really only have circumstantial evidence to believe it can be done. In (Wald, 1979) Wald proved that for curvature of compact support between a flat in and out state $\mathrm{Tr}(B^\dagger B) < \infty$. The situation here is very similar, since we only have a region of compact support where our two states differ, and our states are generalizations of a static frequency-splitting prescription. We suspect that the proof by Wald can be modified to accommodate the situation we are in now, but at this time this is no more than a suspicion. More research is needed to give a definitive answer to this question.

[9] The author thought it likely that the relation5.38 would easily imply a similar relation would hold for the purifications. The problem is more involved than expected however, since the general case where 5.4 is not satisfied yields J's which are considerably less nice than the cases where it does satisfy this saturation property.

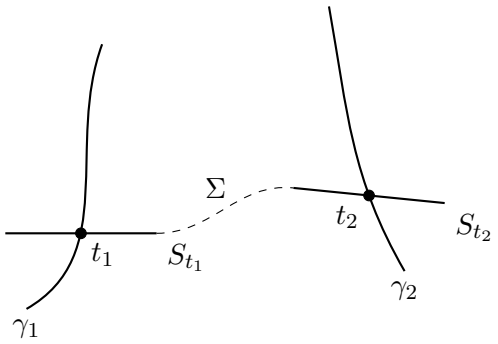

FIGURE 5.4: So long as the local rest frames of the two observers are spacelike seperated at their respective timestamps, an interpolating Cauchy-surface can be found that incorporates both. This means that both local states can be realized as part of the same global state.

As a final question of unitary equivalence, can the states defined by two **different** observers at some timestamps t_1 and t_2 be made to be unitarily equivalent? The answer is yes, if the two local rest frames are spacelike-separated, in which case they can even be chosen to be the same. If the two rest frames are spacelike separated their union forms a compact [10] spacelike hypersurface and hence we can find a Cauchy surface Σ that contains both local rest frames. We thus see that there is an extension Σ_0 of S_{t_0} (namely Σ) and an extension Σ_1 of S_{t_1} (also Σ) such that

$$\tilde{\mu}_{\Sigma_1} = \tilde{\mu}_{\Sigma_2}. \tag{5.42}$$

Hence we see that in this case our local states admit extensions that render them unitarily equivalent.

[10] Assuming again that S_{t_0} and S_{t_1} are compact themselves.

Chapter 6

Outlook

We saw in the discussion at the end of the last chapter that the question of unitary equivalence is a tricky one. Clearly a lot more work has to be done in order to resolve these questions, either in the positive or the negative. We now take a step back for a critical assessment of the previous chapter.

We started our investigation of observer-vacua from the desire to find a distinguished unitary equivalence class of representations of the CCR. We hoped that in imposing an extra condition, namely that a state should in principle be measurable by an observer, we would obtain a distinguished class of representations. However, we find that our observers are too local to define a state for the full quantum field.

The question of unitary equivalence therefore becomes hard to interpret, because it does not match up well with the local approach we took above. Unitary equivalence is a matter of global states, and we have not been able to prove that our class of observer-compatible vacua belongs to one unitary equivalence class. Even if we can choose extensions of local states such that they become equivalent it is not at all clear what the interpretation of these extended states should be. As explained above, the states we defined can not be separated by local measurements, and therefore they should all be regarded as physically equivalent. It is unreasonable to single out one extension just to ensure that we have internal mathematical consistency.

It is therefore unlikely that a further pursuit of this approach to QFT would be very fruitful. Giving some theoretical standing to the concept of observer does not help one gain a better understanding of the quantum field, because the field is so much larger than the observer. Our approach struggled because we have no definite control over the quantum field outside of the confines of the local rest frame. Because many interesting physical processes, such as the expansion of the universe, will play out over vastly larger regions of spacetime our set-up is unsuited as a theory of nature since it is unable to make definite predictions. There are however some lessons to be learned from our investigation.

Chiefly, we find that the idea that the particle content of a quantum field theory is observer-dependent is unfounded. Such an interpretation is usually justified by example of the Unruh effect. Our investigation shows that the particle interpretation is a consequence of the presence of the Killing fields and not of the presence of an observer. A choice of a reference frame is often interpreted as a choice of observers, but we find this interpretation unreasonable because a reference frame is much too global. The concept of a particle is one tied to isometries of the spacetime, not to an observer.

On top of this it is unclear what the exact relation is between the readout of a detector and the theoretical notion of a particle in a Fock space. In the case of a static spacetime, the calculation presented in section 5.3 shows that a clear one-on-one match can be made between the two notions of particles, but beyond this case this identification breaks down. For a general detector in curved spacetimes the calculation of the detector response is messy and a clear particle interpretation can likely not be given to the result. Therefore our approach of defining the vacuum of an observer as a state where he would measure no particles is dubious, because its justification hinges on a tenuous interpretation of what particles are.

In a recent set of articles by Bucholz and Verch, see (Buchholz and Verch, 2016) and references contained therein, further doubt is placed on the standing of the Unruh-deWitt detector as a particle detector. They show that according to an accelerated observer, the Minkowksi vacuum state does not look like a thermal state on a macroscopic scale because it fails to obey Tolman's law. This law states that for a gravitating gas in thermal equilibrium the following law must hold:

$$T(x)\sqrt{g_{00}(x)} = \text{constant} \tag{6.1}$$

which the Unruh-temperature does not satisfy, since it is uniform. They therefore pose that the Unruh-deWitt detector is not a good model for a particle detector, because it only registers microscopic properties of the state they are in, rather than macroscopic properties.

All of this poses the question of what the role of observables in quantum field theory should be, since it appears that there is no role to be played by observers. In our free theory we posed that our algebra of observables should be formed by the operators $\hat{\sigma}(f, .)$ where f is some initial data on a Cauchy surface. Such an f can have a very large support, such that it could never fall within the domain of one single observer, such that no single observer could ever measure this observable. It is thus at best dubious to give $\hat{\sigma}(f, .)$ the name of 'quantum observable'.

We can thus also look at the situation from a different perspective. Since we can only practically measure quantum field effects in a small area around us we could never hope to discriminate between the different vacuum states we defined. A scientist that switches on his measuring device for a finite time and measures no particles will in his description of nature say that the quantum field is in a vacuum state. He has no further knowledge that would allow him to give a qualified answer as to which extension is the one that is 'real'. However, our framework of quantum field theory forces him to pick one arbitrarily, because a global state is needed to set up the theory. Clearly there is a fundamental clash between theory and experiment here: The theory asks us to perform an experiment that can never be carried out.

There are two ways out of this. The first is, as is most often done, to just assume ourselves to be in Minkowski space and neglect any curvature effects. They are bound to be very small when full calculations are performed and we could therefore safely neglect them. If one wants to continue using the curved spacetime background then we come to the conclusion that a framework using global states is simply unsuited as a practical theory of nature. Any definition of quantum field theory should always be local.

We would like a framework that is set up to study local observables and states, since that is all that we can measure in practice. The necessity of choosing a global vacuum state should be avoided since it is not something that could ever be performed in practice.

One framework that does this is that of algebraic quantum field theory. Here an abstract algebra of observables is the central object of investigation, rather than a Hilbert space of states. We do not need a distinguished vacuum state in order to set up our theory. The definition of algebras of local observables is more natural in this context, meaning that we can dispense of our previous objections by only giving the local observables the interpretation of being measurable quantities.

Whilst it is true that this solves the problem of not having a vacuum state, at least for the construction of the theory, we feel that our investigation shows that perhaps the concept of observables has become ill-suited to be central to a quantum theory of nature. We would also like to be able to make predictions about situations where an observer could never be present, such as in the vicinity of a black hole. Problems with this interpretation are bound to increase when one tries to construct a theory of quantum gravity, since in these regimes the concept of an observer is completely absent in all usual senses of the word.

Our approach using canonical quantization dates back to Dirac in the 1930's, and has since seen little further refinement. We start with a classical mechanical system, which has some symplectic structure and a set of observables. Then we try to make a quantum analog using a Hilbert space of states, an algebra of operators corresponding to the observables and a commutator bracket that should mimic the action of the Poisson-bracket. This approach has had profound success when applied to finite dimensional systems, but we feel that this approach is unsuited for relativistic models.

Of course quantum field theory in flat spacetimes using the standard model has a remarkable degree of accuracy when confronted by experiment, but clearly this theory is not without severe mathematical problems. The concept of time is very complex in a general relativistic setting and is responsible for many of the problems we encountered. The author feels that a proper theory of relativistic quantum mechanics should not just try to apply the canonical quantization procedure to a relativistic system, but should also change something fundamental in the process of quantization.

Bibliography

Adler, Stephen L and Judy Lieberman (1978). "Trace anomaly of the stress-energy tensor for massless vector particles propagating in a general background metric". In: *Annals of Physics* 113.2, pp. 294–303. ISSN: 00034916. DOI: 10.1016/0003-4916(78)90206-3. URL: https://linkinghub.elsevier.com/retrieve/pii/0003491678902063.

Alba, D. and L. Lusanna (2003). "Simultaneity, radar 4-coordinates and the 3+1 point of view about accelerated observers in special relativity". In: p. 139. arXiv: 0311058 [gr-qc]. URL: http://arxiv.org/abs/gr-qc/0311058.

Araki, Huzihiro and Shigeru Yamagami (1982). "On Quasi-Equivalence of Quasifree States of the Canonical Commutation Relations". In: *Publications of the Research Institute for Mathematical Sciences* 18.2, pp. 703–758. ISSN: 00345318. DOI: 10.2977/prims/1195183576.

Ashtekar, A. and A. Magnon (1975). "Quantum Fields in Curved Space-Times". In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 346.1646, pp. 375–394. ISSN: 1364-5021. DOI: 10.1098/rspa.1975.0181. URL: http://rspa.royalsocietypublishing.org/cgi/doi/10.1098/rspa.1975.0181.

Bar, Christian, Nicolas Ginoux, and Frank Pfaffle (2007). *Wave equations on Lorenzian manifolds and quantization*.

Bernal, Antonio N. and Miguel Sánchez (2006). "Further results on the smoothability of cauchy hypersurfaces and cauchy time functions". In: *Letters in Mathematical Physics* 77.2, pp. 183–197. ISSN: 03779017. DOI: 10.1007/s11005-006-0091-5. arXiv: 0512095v1 [arXiv:gr-qc].

Birrell, N D (1978). "The application of adiabatic regularization to calculations of cosmological interest". In: *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* 361.1707, p. 513. URL: papers2://publication/uuid/3E51D417-453A-413A-879F-0ACB6627DC9A.

Birrell, N. D. and P. C. W. Davies (1982). *Quantum Fields in Curved Space*. Cambridge: Cambridge University Press. ISBN: 9780511622632. DOI: 10.1017/CBO9780511622632. URL: http://ebooks.cambridge.org/ref/id/CBO9780511622632.

Bisognano, Joseph J. and Eyvind H. Wichmann (1976). "On the duality condition for quantum fields". In: *Journal of Mathematical Physics* 17.3, pp. 303–321. DOI: 10.1063/1.522898. eprint: https://aip.scitation.org/doi/pdf/10.1063/1.522898. URL: https://aip.scitation.org/doi/abs/10.1063/1.522898.

Buchholz, Detlev and Rainer Verch (2016). "Unruh versus Tolman: on the heat of acceleration". In: *General Relativity and Gravitation* 48.3, p. 32. ISSN: 0001-7701. DOI: 10.1007/s10714-016-2029-2. arXiv: 1505.01686. URL: http://arxiv.org/abs/1505.01686.

Bunch, T. S. and Leonard Parker (1979). "Feynman propagator in curved spacetime: A momentum-space representation". In: *Physical Review D* 20.10, pp. 2499–2510. ISSN: 0556-2821. DOI: 10.1103/PhysRevD.20.2499. URL: http://link.springer.com/10.1007/BF01608497https://link.aps.org/doi/10.1103/PhysRevD.20.2499.

Chmielowski, Piotr (1994). "States of scalar field on spacetimes with two isometries with timelike orbits". In: *Classical and Quantum Gravity* 11.1, pp. 41–56. ISSN: 02649381. DOI: 10.1088/0264-9381/11/1/008.

Christensen, S. M. (1976). "Vacuum expectation value of the stress tensor in an arbitrary curved background: The covariant point-separation method". In: *Physical Review D* 14.10, pp. 2490–2501. ISSN: 0556-2821. DOI: 10.1103/PhysRevD.14.2490. URL: https://link.aps.org/doi/10.1103/PhysRevD.14.2490.

Davies, Paul (1975). "Scalar production in Schwarzschild and Rindler metrics". English (US). In: *Journal of Physics A: Mathematical and Theoretical* 8.4, pp. 609–616. ISSN: 1751-8113. DOI: 10.1088/0305-4470/8/4/022.

DeWitt, Bryce S. (1975). "Quantum Field Theory in Curved Space-Time". In: *Phys. Rept.* 19, pp. 295–357. DOI: 10.1016/0370-1573(75)90051-4.

— (1980). "QUANTUM GRAVITY: THE NEW SYNTHESIS". In: *General Relativity: An Einstein Centenary Survey*, pp. 680–745.

Fulling, S. A. (1979). "Remarks on positive frequency and hamiltonians in expanding universes". In: *General Relativity and Gravitation* 10.10, pp. 807–824. ISSN: 00017701. DOI: 10.1007/BF00756661.

Fulling, S.A, F.J Narcowich, and Robert M. Wald (1981). "Singularity structure of the two-point function in quantum field theory in curved spacetime, II". In: *Annals of Physics* 136.2, pp. 243–272. ISSN: 00034916. DOI: 10.1016/0003-4916(81)90098-1. URL: https://linkinghub.elsevier.com/retrieve/pii/0003491681900981.

Fulling, Stephen A. (1973). "Nonuniqueness of canonical field quantization in riemannian space-time". In: *Physical Review D* 7.10, pp. 2850–2862. ISSN: 05562821. DOI: 10.1103/PhysRevD.7.2850.

Fulling, Stephen A., Mark Sweeny, and Robert M. Wald (1978). "Singularity structure of the two-point function in quantum field theory in curved spacetime". In: *Communications in Mathematical Physics* 63.3, pp. 257–264. ISSN: 0010-3616. DOI: 10.1007/BF01196934. URL: http://link.springer.com/10.1007/BF01196934.

Geroch, Robert (1970). "Domain of dependence". In: *Journal of Mathematical Physics* 11.2, pp. 437–449. ISSN: 00222488. DOI: 10.1063/1.1665157.

Hawking, S. W. and G. F. R. Ellis (1973). *The Large Scale Structure of Space–Time*. Vol. 1. Cambridge: Cambridge University Press. ISBN: 9780511524646. DOI: 10.1017/CBO9780511524646.

Hollands, Stefan and Robert M. Wald (2015). "Quantum fields in curved spacetime". In: *General Relativity and Gravitation: A Centennial Perspective*, pp. 513–552. ISSN: 03701573. DOI: 10.1017/CBO9781139583961.015. arXiv: 1401.2026.

Junker, Wolfgang and Elmar Schrohe (2002). "Adiabatic vacuum states on general space-time manifolds: Definition, construction, and physical properties". In: *Annales Henri Poincare* 3, pp. 1113–1182. DOI: 10.1007/s000230200001. arXiv: math-ph/0109010 [math-ph].

Kay, Bernard S (1980). "Linear spin-zero quantum fields in external gravitational and scalar fields". In: *Communications in Mathematical Physics* 71.1, pp. 29–46. ISSN: 0010-3616. DOI: 10.1007/BF01230084. URL: http://link.springer.com/10.1007/BF01230084.

Kay, Bernard S. (1985). "The double-wedge algebra for quantum fields on Schwarzschild and Minkowski spacetimes". In: *Communications in Mathematical Physics* 100.1, pp. 57–81. ISSN: 00103616. DOI: 10.1007/BF01212687.

Kay, Bernard S. and Robert M. Wald (1991). "Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with

a bifurcate killing horizon". In: *Physics Reports* 207.2, pp. 49–136. ISSN: 03701573. DOI: 10.1016/0370-1573(91)90015-E.

Kaya, Ali and Merve Tarman (2011). "Stress-energy tensor of adiabatic vacuum in Friedmann-Robertson-Walker spacetimes". In: *Journal of Cosmology and Astroparticle Physics* 2011.4, pp. 1–15. ISSN: 14757516. DOI: 10.1088/1475-7516/2011/04/040. arXiv: 1104.5562.

Louko, Jorma and Alejandro Satz (2008). "Transition rate of the Unruh-DeWitt detector in curved spacetime". In: *Classical and Quantum Gravity* 25.5. ISSN: 02649381. DOI: 10.1088/0264-9381/25/5/055012. arXiv: 0710.5671.

Manasse, F. K. and C. W. Misner (1963). "Fermi Normal Coordinates and Some Basic Concepts in Differential Geometry". In: *J. Math. Phys.* 4, pp. 735–745. DOI: 10.1063/1.1724316.

Martin, Jér Ôme (2012). "Everything you always wanted to know about the cosmological constant problem (but were afraid to ask)". In: *Comptes Rendus Physique* 13.6-7, pp. 566–665. ISSN: 16310705. DOI: 10.1016/j.crhy.2012.04.008. arXiv: 1205.3365.

Parker, Leonard and S. A. Fulling (1974). "Adiabatic regularization of the energy-momentum tensor of a quantized field in homogeneous spaces". In: *Physical Review D* 9.2, pp. 341–354. ISSN: 0556-2821. DOI: 10.1103/PhysRevD.9.341. URL: https://link.aps.org/doi/10.1103/PhysRevD.9.341.

Peskin, Michael E. and Daniel V. Schroeder (1995). *An Introduction to quantum field theory*. Reading, USA: Addison-Wesley. ISBN: 9780201503975, 0201503972. URL: http://www.slac.stanford.edu/~mpeskin/QFT.html.

Rinaldi, Massimiliano (2008). "Momentum-space representation of Green's functions with modified dispersion relations on general backgrounds". In: *Physical Review D - Particles, Fields, Gravitation and Cosmology* 78.2, pp. 1–15. ISSN: 15507998. DOI: 10.1103/PhysRevD.78.024025. arXiv: /arxiv.org/abs/0803.3684 [http:].

Sachs, Rainer K. and Hung-Hsi Wu (1977). *General Relativity for Mathematicians*. Vol. 48. Graduate Texts in Mathematics. New York, NY: Springer New York. ISBN: 978-1-4612-9905-9. DOI: 10.1007/978-1-4612-9903-5. URL: http://link.springer.com/10.1007/978-1-4612-9903-5.

Sewell, Geoffrey L. (1982). "Quantum fields on manifolds: PCT and gravitationally induced thermal states". In: *Annals of Physics* 141.2, pp. 201–224. ISSN: 1096035X. DOI: 10.1016/0003-4916(82)90285-8.

Shale, David (1962). "Linear symmetries of free boson fields". In: *Transactions of the American Mathematical Society* 103.1, pp. 149–149. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-1962-0137504-6. URL: http://www.ams.org/jourcgi/jour-getitem?pii=S0002-9947-1962-0137504-6.

Unruh, W. G. (1976). "Notes on black-hole evaporation". In: *Physical Review D* 14.4, pp. 870–892. ISSN: 0556-2821. DOI: 10.1103/PhysRevD.14.870. URL: https://link.aps.org/doi/10.1103/PhysRevD.14.870.

Unruh, William G. and Robert M. Wald (1984). "What happens when an accelerating observer detects a Rindler particle". In: *Physical Review D* 29.6, pp. 1047–1056. ISSN: 05562821. DOI: 10.1103/PhysRevD.29.1047.

Venhoek, David. "Quantum vacua in Curved Spacetime". In: URL: https://www.ru.nl/publish/pages/913395/david{_}venhoek{_}masterscriptie.pdf.

Verch, Rainer (1997). "Continuity of Symplectically Adjoint Maps and the Algebraic Structure of Hadamard Vacuum Representations for Quantum Fields on Curved Spacetime". In: *Reviews in Mathematical Physics* 09.05, pp. 635–674. ISSN: 0129-055X. DOI: 10.1142/S0129055X97000233. arXiv: 9609004 [funct-an]. URL:

http://arxiv.org/abs/funct-an/9609004http://www.worldscientific.com/doi/abs/10.1142/S0129055X97000233.

Wald, Robert M. (1977). "The back reaction effect in particle creation in curved spacetime". In: *Communications in Mathematical Physics* 54.1, pp. 1–19. ISSN: 00103616. DOI: 10.1007/BF01609833.

— (1978). "Trace anomaly of a conformally invariant quantum field in curved spacetime". In: *Physical Review D* 17.6, pp. 1477–1484. ISSN: 0556-2821. DOI: 10.1103/PhysRevD.17.1477. URL: https://link.aps.org/doi/10.1103/PhysRevD.17.1477.

— (1979). "Existence of the S-matrix in quantum field theory in curved space-time". In: *Annals of Physics* 118.2, pp. 490–510. ISSN: 1096035X. DOI: 10.1016/0003-4916(79)90135-0.

— (1984). *General Relativity*. University of Chicago Press, pp. 1–116. ISBN: 9780226870335. DOI: 10.7208/chicago/9780226870373.001.0001. arXiv: QC173.6.W35. URL: http://www.bibliovault.org/BV.landing.epl?ISBN=9780226870335.

— (1995). *Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics*. Chicago Lectures in Physics. Chicago, IL: University of Chicago Press. ISBN: 9780226870274.