Master thesis

Radboud university Nijmegen

$\mathcal{N} = 4$ supersymmetric Yang-Mills theory and its deformations

a functional renormalisation approach

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Abstract

We use $\mathcal{N} = 1$ superspace formalism and functional renormalisation group techniques, especially the Wetterich equation [1], to derive the β-function for $\mathcal{N} = 1$ supersymmetric Yang-Mills theory up to a sign discrepancy. We derive the super Yang-Mills action in superspace and apply Wetterichs method to calculate the flow of the effective average action for $\mathcal{N} = 1, 2, 4$ super Yang-Mills theories. The provided calculation follows [2, 3] closely and leads to the conclusion that $\mathcal{N} = 4$ super Yang-Mills has a vanishing beta function. Together with super Poincaré invariance this makes $\mathcal{N} = 4$ super Yang-Mills a super conformal field theory. Motivated by the AdS/CFT correspondence [4] we investigate the possibility of breaking supersymmetry to $\mathcal{N} = 1$ while maintaining super conformal invariance. It has been shown by Leigh and Strassler [5] that such a theory is described by a marginally deformed $\mathcal{N} = 4$ super Yang-Mills theory and the corresponding supergravity dual was first obtained by Lunin and Maldacena [6]. We believe that mentioned renormalisation techniques can be used to achieve a better understanding of the deformed supergravity theory as well.

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1 Personal foreword and acknowledgements

It was around march 2018 when I was taking the course "String theory" in Utrecht. The course had a great impact on my view on string theory and I was convinced that I wanted to write my master thesis on this subject. For technical reasons I was unable to find a position in this direct field of research but I call myself extremely lucky to have found such a great person as my supervisor Dr. Frank Saueressig at my University who wanted to give it a go in the end. He is therefore the person who made it possible for me to develop a better understanding of string theory and is definitely the one person I would like to thank the most. He was not only a great supervisor in always knowing some kind of way out, he also was just a very kind person. Together we came up with a project regarding AdS/CFT which initially was aimed at finding a relationship between the seven-dimensional AdS space from 11-dimensional M-theory, the associated six-dimensional SCFT and fixed points of this theory. This may have been a high aim and in the end turned out to be a dead end because it was far too complicated. We quickly went in to the direction of the SYM theory of which it seemed possible to calculate the β-function using superspace formalism. I have learned a lot during this time and will hopefully be able to continue in this direction.

Next to my supervisor Frank there are a couple more people that I would like to thank because they were there every day for me to listen to things they also did not understand. My good friends Nesta and Salem are two mates that I will never forget, thank you for being part of the whole journey. Furthermore I would like to thank Lando, Stach, Jochem, Joep, Lieke, and Remco. We were the master students of our time and had some discussions at the coffee table that made it a great time. I show great gratitude towards Benjamin, Chris, Joren and Markus who were there whenever I had some questions that required some discussion rather than answers. Finally, I would like to thank my parents and my brother who were there for me on a family level and never lost hope in me even though they have zero clue of what I was doing.

Before you now lays my master thesis as a result of roughly 10 months of work at the theoretical high energy department of the Radboud University Nijmegen. I hope you enjoy reading it as much as I had writing it.

Contents

2 Introduction, motivation and set-up

The human being, a roughly 1.75 meter tall, 75 kilograms heavy, becomes roughly 80 years of age and is an arguably smart creature that has been walking on earth for the past 200,000 years. We live in a universe that appears to be roughly 13.8 billion years old, therefore 8.8×10^{23} kilometers in diameter and has an estimated mass content of 4.5×10^{51} kilograms. Even though both of these scales appear to be extreme opposites of each other one can in fact consider an even more extreme that goes in to the small scale, or microscopic scale. Just consider the air that you breathe in daily to stay alive. The air that we breathe consist of roughly 78% nitrogen, 21% oxygen and other elements for $< 1\%$. The human eye can not observe these elements yet we know that they are there. But how? How do we dare make predictions about objects that we can not even observe around us? From 1700 onwards scientist have tried to make sense out of the question what "stuff" is made of. It took several hundreds of years of observations paired with theoretical predictions to conclude that everything is made out of tiny little particles called atoms. It did not stop here, several more discoveries were made in the 20th century and lead to discoveries of protons, neutrons and electrons as being constituents of said atoms. For a comparison, the scale of atoms hints at length scales of about 10^{-10} meters and less. Even more modern insights have shown that electrons are in fact elementary particles by which we mean that they are truly indivisible. On the other hand, protons are not. It has been shown much more recent that protons and neutrons are built out of sub-atomic, and elementary particles that are called quarks. Knowing this, it was then straightforward to assume that one can build other particles made out of these quarks. This opened the field of sub-atomic particle physics. In particle physics one identifies small length scales with high energy scales. This is a direct consequence of the de Broglie hypothesis in which one identifies a wavelength λ to a particle with energy E. The de Broglie wavelength is then given by $\lambda = \frac{hc}{E}$. This corresponds to the Compton wavelength which represents the relevant length scale for a quantum system with energy E. The small world of sub-atomic particles is what we refer to as the quantum regime whereas the large cosmological scale of the universe is called the gravitational regime because gravity seems to be the dominant force on this scale. It is of a physicists intrinsic interest to develop a better understanding of both regimes. The theories describing these regimes are **quantum field theory** for the quantum regime and **general relativity** for

the gravitational regime. By regime we mean to explain what the relevant interactions are. The quantum regime is dominated by three out of four fundamental forces: the electromagnetic, weak and strong force. Both theories, quantum field theory and general relativity seem very incompatible with each other because they are described by completely different formalisms.
One of the major problems in modern theoretical physics is to combine these two theories in one theory of **quantum gravity**. This is were our research starts. Since the 1970s one of today's best investigated attempts to a theory of quantum gravity is **string theory**. String theory is a theory that aims to describe gravity together with the other three fundamental forces of nature in one framework by replacing point particles by one-dimensional strings. It can be shown that string theory contains a massless spin-2 mode which one refers to as the graviton and is therefore regarded as a theory of quantum gravity which can be shown to contain general relativity. Together with string theory comes the concept of **supersymmetry**. Originally formulated as a purely bosonic theory it quickly became apparent that one needs to add fermions to the theory by hand to achieve self-consistency. The resulting theory is supersymmetric which means it is invariant under a transformation that relates bosons and fermions to each other. Supersymmetry is a spacetime symmetry which extends the Poincaré symmetry of conventional spacetime to a super group. The group contains extra fermionic generators that generate translations in an extended space called superspace. On superspace we can define scalar fields called superfields. Theories formulated using these concepts are manifest supersymmetric. In this thesis we will consider supersymmetric gauge theories known as Yang-Mills theory. A Yang-Mills theory is a gauge theory where one has a gauge field one-form A as a new dynamical field. It takes values in the gauge group G to maintain local gauge invariance. The dynamics of the gauge field is encoded in its associated two-form field strength F. The square of the field strength integrated over spacetime is then regarded as the Yang-Mills action and forms the starting point of our research. We will couple the Yang-Mills gauge field, which will be promoted to a superfield, to so called chiral superfields. These fields represent the matter content including chiral fermions and scalar fields. We are interested in the particular theory of $\mathcal{N} = 4$ supersymmetric Yang-Mills because it is widely known that this theory exhibits a scale invariant coupling constant g. In this thesis we will present a manifest supersymmetric and non-perturbative way to calculate the energy or momentum scale dependence of this coupling. Let $t = \log \frac{k}{k_0}$ be a dimensionless momentum scale for the momentum k with k_0

a reference scale. The object that we will calculate is the so called β-function given by:

$$\beta(g_k^2) = \partial_t g_k^2 \tag{1}$$

It is this point where we refer to the first part of this introduction where we asked ourselves; how do we dare make predictions about objects that we can not even observe? To answer this it is worthwhile mentioning that we can zoom in and out by considering small and large values of momentum. The smallest scales, which are of interest in the quantum gravity regime are described by high momentum values. This part is called the ultra-violet (UV), because it corresponds to short wavelengths. The low-energy regime is therefore a large scale configuration and is called the infra-red (IR). We will be using a technique called **functional renormalisation** to find the β-function. This technique starts from a reference scale k_0 and subsequently integrates out all quantum fluctuations with momenta $p^2 < k_0^2$ shell-by-shell in momentum space to arrive at an effective description of physics at scale k. This is done by introducing a so called regulator action to the original action that will serve as a cutoff. The regulator is a smooth function of the momentum scale k and will pick out the relevant modes $k = k_0$. The regulators are chosen such that in the IR one recovers the full effective action Γ and in the UV, which is the microscopic theory we have the bare action S. This then induces a **flow** for all coupling constants in the theory along the momentum scale. Furthermore, the notion of the **AdS/CFT** correspondence will be introduced to describe Yang-Mills theory using a certain type of string theory and we use this correspondence to describe a new class of conformal field theories with less supersymmetry.

In 1997 Juan Maldacena [4] has provided a new way of looking at strongly coupled quantum field theories. Building on work by 't Hooft [7] that discuss "Large N limits" of gauge theories, it was shown that one can find a duality between a gravitational theory as the weak energy limit of a stringy theory[1] and a conformal field theory. This duality goes by name AdS/CFT correspondence where AdS stands for anti-de Sitter spacetime geometry and CFT for conformal field theory. The AdS/CFT correspondence provides more than just a duality. It also serves as a map between strongly and weakly coupled theories, i.e. a strongly coupled conformal field theory corresponds to a weakly coupled gravitational theory. More details will be discussed in

[1]Including full string theory, supergravity and M-theory.

the rest of the thesis. Having this correspondence in the back of our heads we already see that this is in conflict with perturbative approaches taken in conventional quantum field theory. We therefore opt for a non-perturbative way of quantum field theory. Before we can go into this we need to understand the concept of renormalisation. Renormalisation is in fact a reinterpretation of the infinities that arise when calculating loop diagrams using perturbative methods. The aim is to put these infinities in non-observable quantities like the bare parameters of the theory. The only quantities that one has access to at this point are the renormalised coupling constants that are the perturbative expansion parameters. The reinterpretation lays in the technique where one makes the coupling constant itself infinite but keeps the observable finite. There is one especially nice way of doing this and it goes under the name of functional renormalisation explained earlier. This way of renormalising is handy because it is insensitive to large or small couplings. This is exactly what we want in the AdS/CFT correspondence and motivates studying AdS/CFT using these functional renormalisation techniques. We are especially interested in the best known example of the AdS/CFT correspondence [4] and will thus use functional renormalisation group techniques for the case of $\mathcal{N} = 4$ supersymmetric Yang-Millls theory.

In order to achieve this we will arrange the thesis in four parts. Part 1 contains an introduction to functional renormalisation and Yang-Mills theory. Part 2 is devoted to supersymmetry, superspace formalism and supersymmetric Yang-Mills theory. Part 3 will be about conformal field theory, string theory and the AdS/CFT correspondence. We finish the thesis with part 4 which contains an analysis of Leigh-Strassler deformations of the $\mathcal{N} = 4$ super Yang-Mills theory and the supergravity dual solution given by Lunin and Maldacena.

To finish this part we would like to mention that this master thesis is aimed at other master students in the field of theoretical physics with a background in quantum field theory and general relativity.

Part I
Functional renormalisation and Yang-Mills theory

3 The functional renormalisation group

Let us start by introducing the notion of renormalisation in general and then show the method used in this thesis, functional renormalisation. The fundamental theory underlying the world of sub-atomic physics, quantum field theory, is, similar to quantum mechanics, a probabilistic theory. The canonical way of doing quantum field theory calculations goes by the name of perturbative quantum field theory. There it is assumed that a theory is described by a Lagrangian $\mathcal{L}$ that depends on some field content $\{\phi_i\}$ connected by coupling constants g_j. In non-perturbative quantum field theory however observables can be expressed through the **path integral** $Z[\phi_i]$ given by the integral expression:

$$Z_k[J] = \int \mathcal{D}\phi \; e^{-S[\phi]+J_\phi \phi} \tag{2}$$

with $\mathcal{D}\phi = \prod_i d\phi_i$ and J_ϕ representing source fields for ϕ_i. It is also common to work in Euclidean spacetimes with positive definite metric rather than a Lorentzian metric. Considering the path integral as a probability density an observable O acquires an expectation through:

$$\langle O \rangle = \int \mathcal{D}\phi \; O(\phi) e^{-S[\phi]} \tag{3}$$

Generally speaking, a quantum field ϕ is regarded as an operator such that $\phi(x)$ can be found to have an expectation value, or one-point function, $\langle \phi(x) \rangle$ given by the functional variation of the path integral $Z[\phi]$ with respect to its associated source J_ϕ evaluated at zero source. The generating functional for all n-point functions is defined by:

$$W[J_\phi] \equiv \log(Z[J_\phi]) \tag{4}$$

The n-point function of a field ϕ_i arise when we take multiple derivatives of the path integral:

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = \left. \frac{\delta Z[J_{\phi_1}, \dots, J_{\phi_n}]}{\delta J_{\phi_1} \dots \delta J_{\phi_n}} \right|_{J=0} \tag{5}$$

To the expectation value $\langle \phi(x) \rangle$ we associate an **effective action** $\Gamma[\chi]$.

$$\chi \equiv \langle \phi(x) \rangle. \tag{6}$$

The effective action is given by the Legendre transform:

$$\Gamma[\chi] \equiv \sup_{J_\phi}(J_\phi \chi - W[J_\phi]) \tag{7}$$

where the product $J_\phi \chi$ is a functional innerproduct on D-dimensional Euclidean spacetime. The functional $W[J_\phi]$ generates all n-point functions that are allowed within the considered theory. These n-point functions can then be used to construct Feynman diagrams for all possible interactions. In order to compute full scattering amplitudes one would have to include all possible Feynman diagrams that are allowed. These are usually infinitely many. In turn, this yields the first divergence in this quantum field theoretic approach. Another divergence arises when considering all diagrams is the fact that higher powers of g_i appear. The only way we can tame this divergence is by assuming $g_i \ll 1$. It is then possible to take only most relevant diagrams in to account, most often tree diagrams[2]. If one however still needs to compute loop diagrams these are associated with a free loop momentum running through the loop. Feynman rules then tell us to integrate out all these momenta and the resulting integrals diverge. When we compute loop corrections to the bare propagator of a theory we observe that we still get infinities. The idea of **renormalisation** is then to put the divergences in to the self-coupling constants by promoting them to momentum, or energy, depended quantities.

Without going in to further detail on conventional renormalisation of self-energy diagrams let us start by what we mean with **functional renormalisation**. All functionals so far are still divergent in the original sense. We will now promote the path integral $Z[J]$ to a scale dependent object $Z_k[J]$ by adding a mass-like momentum-dependent to the path integral (2).

$$Z_k[J] = \int \mathcal{D}\phi \; e^{-S[\phi]+J_\phi \phi - \Delta_k S[\phi]} \tag{8}$$

This additional term in the action will function as an IR cutoff. From this it is then possible to construct what we call the **effective average action** $\Gamma_k[\phi]$. We follow the same procedure as above, now the n-point generating functional W is also promoted to a k dependent object. Then we define the effective average action to be:

$$\Gamma_k[\chi] = \tilde{\Gamma}_k[\chi] - \Delta_k S[\chi] \tag{9}$$

[2]Which means that the diagram does not contain any closed loops.

with $\tilde{\Gamma}_k[\chi] = \sup_{J_\phi}(J_{k,\phi}\chi - W_k[J_\phi])$ and χ as in (6).
For all practical purposes one chooses $\Delta_k S[\chi]$ to be quadratic in the quantum fields so that it takes the form of a mass term:

$$\Delta_k S[\phi] = \frac{1}{2}\int d^D x\ \phi(x) R_k(\Box)\phi(x) \tag{10}$$

A canonical choice for the operator $\Box$ is the d'Alembertian $-\partial^2 = p^2$ which organises the fluctuation field in terms of their momentum p^2. As we will see later different choices can be made. The function R_k is a function called the regulator of the theory and has certain properties that make the theory well behaved in the IR and UV. It should work in such a way that:

1. For $p^2 \gg k^2$ we require that R_k approaches zero so that these modes are unaffected.
2. For $p^2 \leq k^2$ we require R_k to be proportional to k^2 so that modes around the cutoff scale are suppressed.

It is then clear that if our cutoff scale k is infinite all modes will be suppressed. The limits of the effective average action if $k \to 0$ and $k \to \infty$ are given by $\Gamma_{k\to 0} = \Gamma$ and $\Gamma_{k\to\infty} = S$. Here S is the bare action describing UV physics and Γ is the full effective action (7). The regulator that we will use in this thesis is the exponential regulator[3]:

$$R_k(p^2) = k^2 R^{(0)}(p^2/k^2) = k^2 \cdot \frac{p^2/k^2}{\exp(p^2/k^2) - 1} \tag{11}$$

$R^{(0)}$ itself should satisfy following limits:

$$\begin{aligned} \lim_{k^2\to 0} R^{(0)} &= 0 \\ \lim_{k^2\to\infty} R^{(0)} &= 1 \end{aligned} \tag{12}$$

Having set up a scale-dependent system it is possible to construct an equation that describes its flow when changing k. Without deriving it we present the functional renormalisation group equation, or Wetterich equation [1]:

$$\partial_t \Gamma_k = \frac{1}{2}\mathrm{STr}\Big(\big(\Gamma_k^{(2)} + R_k)\big)^{-1}\partial_t R_k\Big) \tag{13}$$

[3] There are different choices for a regulator such as the Litim regulator: $R_k(p^2) = (k^2 - p^2)\Theta(k^2 - p^2)$ with $\Theta(x)$ the Heaviside step function.

with $t = \log(k/k_0)$ and k_0 a reference scale. Observe also how $\partial_t = k\partial_k$. This equation is an exact equation that can be derived by taking the k-derivative with respect to the effective average action Γ_k. STr stands for the supertrace of the operator between brackets [8]. This includes traces over gauge group indices and an integral over momentum space. We can decompose a matrix into bosonic and fermionic degrees of freedom. The supertrace collects all bosonic degrees of freedom with a plus while all fermionic degrees of freedom come with a minus sign. $\Gamma_k^{(2)}$ denotes the second functional variation with respect to each field that appears in the action. Because of the appearance of $\Gamma_k^{(2)}$ in this equation it is sufficient to write down the effective average action up to second order in the fields. So given an action $S[\phi]$ we can in principle construct the effective average action by introducing the mean field $\chi = \langle\phi\rangle$ and promoting the couplings to be k-dependent. The terms that we write down must be invariant under relevant symmetries of the theory under consideration. In this thesis we will be considering gauge transformations under Lie groups like $SU(N)$. The Wetterich equation gives then a set of relations between coupling constants and their derivatives. These coupling constants serve as coordinates in a space called **theory space** which in princinple corresponds to a space in which each point is a different action functional. Scale derivatives of coupling constants are called **β-functions** and are of main interest in this thesis. These are expressed as:

$$\beta(g_k) = \partial_t g_k \tag{14}$$

What we will be mostly interested in conformal field theories. These are theories that are not sensible to changes in momentum scale. It is clear from above definition that these theories should satisfy $\beta(g_k) = 0$. This means that g_k is a constant. The points in theory space where the β-functions vanish are called fixed points. The existence of fixed points is important in the sense that the whole theory flows to a fixed point for all values of k including $k = \infty$. The fixed points can then control UV-behaviour of such a theory. A theory for which $g_k = g_\star = 0$ is called Gaussian and the fixed point is called Gaussian fixed point. This corresponds to the free theory. For a theory with just one coupling constant λ the procedure becomes a lot easier because the fixed point requirement reduces to one equation. This will be the case for the rest of the thesis.

Let us now take a look at how one would actually solve the Wetterich equation. Considering the theory of scalar field with effective average action:

$$\Gamma_k[\phi] = \int d^D x \; \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V_k(\phi) \tag{15}$$

with $\partial_\mu\partial^\mu = \eta_{\mu\nu}\partial^\mu\partial^\nu$ and $\eta_{\mu\nu} = \mathrm{diag}(1, -1, \cdots, -1)$ in $D = d+1$ spacetime dimensions. $V_k(\phi)$ is some potential term. The second variation $\Gamma_k^{(2)} = \frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} = \Box + V_k''(\phi)$ with $''$ denoting the second derivative with respect to ϕ. For this example we choose the Litim regulator [9] $R_k(p^2) = (k^2 - p^2)\Theta(k^2 - p^2)$ with $\Theta(x)$ the Heaviside step function. Identifying $\Box = p^2$ the Wetterich equation (13) becomes:

$$\partial_t\Gamma_k = \frac{1}{2}\mathrm{STr}\Big(\big(p^2 + V_k''(\phi) + (k^2 - p^2)\Theta(k^2 - p^2)\big)^{-1} \times 2k^2\Big) \tag{16}$$

it is easy to see that we need the trace of a 1×1 matrix and expand this in terms of ϕ. This technique will be used when we start our computation. One computes the left-hand-side, and then tries to project out the relevant terms on the right hand side. The only trace that needs to be taken is a momentum integral over all values of p^2 and the internal indices are trivially contracted because there is just one field and no gauge indices. Since ϕ is bosonic we also get a plus sign. We get:

$$\partial_t\Gamma_k = \frac{1}{2}\int_{p^2\leq k^2} \frac{d^D p}{(2\pi)^D}\Big(2k^2\big(V_k''(\phi) + k^2\big)^{-1}\Big) \tag{17}$$

The integral yields the volume of a D-sphere which is given by the expression:

$$V_D(p) = \frac{\pi^{D/2}}{\Gamma(D/2+1)}p^D \tag{18}$$

with $\Gamma(x)$ the Gamma-function. Putting everything together yields the flow equation:

$$\partial_t\Gamma_k = \frac{k^{D+2}}{2^D\pi^{D/2}\Gamma(D/2+1)(k^2 + V_k''(\phi))} \tag{19}$$

To continue the calculation one may use the so called **local potential approximation** (LPA) in which we expand around $\phi = \phi_0 =$ constant. Let us pick massless ϕ^4 theory with $V_k(\phi) = \frac{g_k}{4!}\phi^4$ leading to $V_k''(\phi) = \frac{g_k}{2}\phi^2$. Using an expansion around $\phi = 0$ we can invert the second derivative term and find $\partial_t g_k$ to be:

$$\partial_t g_k = \beta(g_k) = \frac{6k^{D-4}}{2^D\pi^{D/2}\Gamma(D/2+1)}g_k^2 \tag{20}$$

In $D = 4$ this reduces to the well-known result for the ϕ^4 β-function:

$$\beta(g_k)_{\phi^4} = \frac{3}{16\pi^2} g_k^2 \tag{21}$$

where we have projected on the ϕ^4 term and we see that we have a fixed point solution $\beta(g_k) = 0$ for $g_k = 0$. This indeed corresponds to the theory of a free scalar field. Similar techniques will be used in this thesis.
Our theory of choice will be the supersymmetric version of Yang-Mills theory. We will generalise this formalism to supersymmetric field theories and use the superspace formulation in order to maintain supersymmetry. Even by adding the mass-like regulator the theory remains supersymmetric. To maintain gauge invariance of a gauge theory it is very convenient to use the background field method. We can then solve the Wetterich equation by splitting the quantum fields in a background and fluctuation part. The final equation will then be evaluated at zero fluctuation. We will briefly touch on this in the next section where we start with a short review on conventional Yang-Mills theory and gauge fixing.

4 Yang-Mills theory

In this section we will introduce Yang-Mills theory as being interesting for a couple of reasons. First, The standard model of particle physics is described by Yang-Mills theory with gauge group $SU(3) \times SU(2) \times U(1)$. Secondly, Yang-Mills theory has very nice properties, especially when we look at its supersymmetric version. It is a gauge theory that posesses non-abelian gauge groups that are most likely compact Lie groups like $SU(N)$. The general ansatz for a Yang-Mills theory is always based on the gauge symmetry that is obeyed by a theory. Suppose one has a Lagrangian that depends on some field content $\{\Phi\}$ and is invariant under a transformation $\zeta : \{\Phi\} \to \{\Phi\}, \Phi \to g\Phi$ with $g \in G$ some symmetry group of the Lagrangian. One then generally speaks of a gauge symmetry of the theory. As long as G is an Abelian group and the group parameters do not explicitly depend on spacetime there is not much of a problem. However, localising this symmetry, meaning the group parameters become spacetime dependent one in general has a problem with kinetic terms since these involve spacetime derivatives. There is then need for the introduction of a new derivative, called the gauge covariant derivative. This derivative, by construction, takes in to account the extra derivative of group parameters by adding a new dynamical field called the gauge field.

4.1 The Yang-Mills action

As explained in the previous section one adds a gauge field action to the full Lagrangian in order to make the gauge field dynamical. If we consider pure Yang-Mills theory this is constructed as follows. Let there be a system that is invariant under a transformation that is element of some symmetry group G. Let us call it a gauge transformation because the Lagrangian, that covers its physics, is apparently invariant under transforming the fields in a certain way. The point becomes apparent when we see that if the transformation applied does in fact lead to an almost-invariant theory. At this point we introduce the gauge field A. The gauge field A allows us to obtain again a full gauge invariant theory by introducing a covariant derivative. This new derivative must then transform in such a way such that the theory becomes invariant. An example is the the theory of quantum electro dynamics.
In the case for pure Yang-Mills we choose a gauge group G with gauge algebra $\mathfrak{g}$ spanned by Hermitian generators T^a in such a way that:

$$[T^a, T^b] = if^{abc}T^c \tag{22}$$

$$\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab} \tag{23}$$

with a, b, c enumerating the generators of the group and $[A, B] = AB - BA$ the commutator of two matrices. Let the gauge field A be a one-form field in the Lie algebra $\mathfrak{g}$. In order to make this field dynamical we introduce the so called field strength two-form $F \equiv dA + A \wedge A$ where d stands for the exterior derivative and $\wedge$ is the wedge product [10]. The term $A \wedge A$ does not vanish since A is Lie algebra valued. The covariant derivative is given by $D = d - A$. A gauge transformation is then a transformation of the gauge field according to:

$$A \to A' = A + d\alpha \tag{24}$$

The exterior derivative is a nilpotent operator, i.e. $d^2 = 0$. Therefore we see that F itself is not gauge invariant. The only special case where this happens is if the Lie algebra happens to be Abelian. In a more general context we can however create a gauge invariant quantity known as the Yang-Mills action which is given by [10]:

$$S_{YM} = -\frac{1}{4g^2}\int \mathrm{Tr}(F \wedge \star F) \tag{25}$$

where $\star$ is the Hodge star operator given for p-forms by:

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle d\text{Vol} \tag{26}$$

with dVol the volume form on the considered manifold and $\langle \cdot, \cdot \rangle$ the inner-product on p-forms. To write this expression in coordinate form we use that $F = dA + A \wedge A = d \wedge A + A \wedge A \Rightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$

$$S_{YM} = -\frac{1}{4g^2} \int d^D x \ \text{Tr}(F_{\mu\nu} F^{\mu\nu}) \tag{27}$$

Note that $F_{\mu\nu} = F^a_{\mu\nu} T^a$ is Lie algebra valued such that we also have: $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$. F transforms as a rank two tensor and the trace makes the whole combination gauge invariant. From now on we refer actions of type (27) as a Yang-Mills action.

4.2 Gauge fixing and ghosts

When quantising Yang-Mills theories one faces the conceptual difficulty that the path integral (2) integrates over an infinite amount of fields that are related by a gauge transformation. We need to divide out all gauge states that are related by gauge transformations. This is done by the path integral formalism and is known as under the name of gauge fixing. This means that we have to fix-a-gauge, or, pick a constraint that does change physics in overal but reduces the degrees of freedom. We then work only with states that satisfy this constraint known as the gauge condition. The most common gauge that one uses is the orthonormality condition called the Lorentz gauge [11] where $\partial_\mu A^\mu = 0$. In momentum space this corresponds to the transversality condition $p \cdot A = 0$. Let us call a gauge condition $G[A] = 0$ with in this case $G[A^a] = \partial \cdot A^a$ with a a group index. We thus want to implement this gauge on the whole system and therefore reduce the number of configurations. At the path integral level this is implemented as follows. The non gauge-fixed Lorentzian path integral reads:

$$Z_{YM}[A] = \int \mathcal{D}A \exp(iS_{YM}[A]) \tag{28}$$

The problem lies in the measure $\mathcal{D}A = \prod_i dA_i$. This measure is infinite due to gauge transformations between different A fields. What we would like to achieve is, dividing out all gauge equivalent fields. This is done by replacing the infinite measure by the finite regulated one:

$$\mathcal{D}A \rightarrow \frac{\mathcal{D}A}{\text{Vol(G)}} \tag{29}$$

where we denote the volume of a gauge group G by Vol(G)[4]. The next step is the so called Faddeev-Popov trick [11]. This encodes the way the gauge fixing is implemented in the path integral. Using the one-dimensional property:

$$1 = \int dx |g'(x)| \delta(g(x)) \tag{30}$$

of the Dirac δ -function can be expanded to a multi-dimensional version:

$$1 = \int \mathcal{D}\alpha \ \det(\frac{\partial G[A]}{\partial \alpha}) \delta(G[A]) \tag{31}$$

α denotes a gauge parameter and therefore we have that $\int \mathcal{D}\alpha = \text{Vol(G)}$. Inserting above in to equation (28) we find:

$$Z_{YM}[A] = \int \mathcal{D}A \exp(iS_{YM}[A]) \ \det(\frac{\partial G[A]}{\partial \alpha}) \delta(G[A]) \tag{32}$$

The factor $\det(\frac{\partial G[A]}{\partial \alpha})$ is known as the "Faddeev-Popov" determinant. Let us now set up everything we have. The gauge fixing function is $G[A] = \partial \cdot A - w(x)$. For non-abelian fields A we also use a covariant derivative in the gauge transformation of A given by: $D_\mu \alpha^a = \partial_\mu \alpha^a + g f^{abc} A^b_\mu \alpha^c$ and finally the gauge transformation of A given by $A \rightarrow A^a_\mu = A^a_\mu + g^{-1} D_\mu \alpha^a$. We can then easily see that after a gauge transformation $\det(\frac{\partial G[A]}{\partial \alpha}) = g^{-1} \partial_\mu D^\mu$. Furthermore, we have that the measure and the action are gauge invariant [11].

$$Z_{YM}[A] = \int \mathcal{D}A \ \ \exp(iS_{YM}[A]) \ \det(g^{-1} \partial_\mu D^\mu) \delta(\partial \cdot A - w(x)) \tag{33}$$

[4]In the language of the gauge parameter space Vol(G) is the integral along one gauge orbit.

We now recast this expression in a more managable form. In order to fix the gauge with the δ-function, use the mutli-dimensional version of $\int dx\ f(x)\delta(x-a) = f(a)$. We multiply both sides by a functional Gaussian $\int \mathcal{D}w\ \exp(-i\int dx\ w^2(x)/2\xi)$. The path integral Z_{YM} gets just multiplied by a number that depends on ξ but for the δ-function picks out the gauge fixing condition.

$$Z_{YM}[A] = \int \mathcal{D}A\ \exp(iS_{YM}[A])\ \det(g^{-1}\partial_\mu D^\mu)\exp(-i\int d^Dx\ (\partial\cdot A)^2/2\xi) \tag{34}$$

For the determinant we again use a trick invented by Faddeev and Popov. We introduce the notion of **Grassmann numbers**. These are, rather than commuting, anticommuting complex numbers c_i. For this kind of numbers the Gaussian integral involving and $n\times n$ matrix A

$$\int_{-\infty}^{\infty} d^nx\ \exp(-\frac{1}{2}x^TAx) = \sqrt{\frac{(2\pi)^n}{\det(A)}} \tag{35}$$

gets replaced by [11]:

$$\int dcd\bar{c}\ \exp(-\bar{c}Ac) = \det(A) \tag{36}$$

With A an $n\times n$ symmetric matrix. It is therefore costumary to take anti-commuting fields c_i as the fields that belong to the determinant factor in the Yang-Mills path integral. We write:

$$\det(g^{-1}\partial_\mu D^\mu) = \int \mathcal{D}c\mathcal{D}\bar{c}\ \exp(-g^{-1}\int d^Dx\ \bar{c}\partial_\mu D^\mu c) \tag{37}$$

Then the full path integral becomes:

$$\begin{aligned} Z_{YM}[A] = &\int \mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}\ \exp(iS_{YM}[A]) \\ &\times \exp(-i\int d^Dx\ (\partial\cdot A)^2/2\xi)\exp(-g^{-1}\int d^Dx\ \bar{c}\partial_\mu D^\mu c) \end{aligned} \tag{38}$$

Defining the gauge-fixed action:

$$S_{\text{GF}}[A,c,\bar{c}] = -\frac{1}{4g^2}\int d^Dx\ \text{Tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{2\xi}\int d^Dx\ (\partial\cdot A)^2 - \frac{1}{g}\int d^Dx\ \bar{c}\partial_\mu D^\mu c \tag{39}$$

we arrive at the full path integral expression:

$$Z_{YM}[A] = \int \mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}\ \exp(iS_{\text{GF}}[A, c, \bar{c}]) \tag{40}$$

The new dynamical fields c and $\bar{c}$ are called **ghost fields**[5]. The parameter ξ is an arbitrary number and can be chosen such that the final answer simplifies as much as possible, this is a matter of convention.
This is a prerequisite for our study and returns when we investigate supersymmetric Yang-Mills theory. In principle one can derive all Feynman-rules from this action and use it for a diagrammatic analysis of the theory. In the next section we will briefly explain the renormalisation of g by using the Abelian $U(1)$ action as a guide line.

4.3 Comments on the Yang-Mills β function

Yang-Mills theory is a gauge theory with one coupling constant g. The coupling constant gets renormalised. In order to find its momentum dependence we will consider what we have called the β-function. The renormalisability of a theory is captured in the mass dimension of its interaction coupling constant. We do this by power counting the mass dimension of relevant fields and couplings. Let us consider Yang-Mills theory described above. We start with the mass dimension of the Lagrangian $[\mathcal{L}] = D$. The covariant derivative has the same mass dimension as the partial derivative and is therefore $[D] = 1$. Hence, the gauge field A has $[A] = [g^{-1}] + 1$ or, $[A] = \frac{1}{2}[g^{-2}] + 1$.

$$\begin{aligned}
D &= [g^{-2}] + 2[F] \\
&= [g^{-2}] + 2[dA] \\
&= [g^{-2}] + 2(1 + [A]) \\
&= [g^{-2}] + 2(2 + \frac{1}{2}[g^{-2}]) \\
&\Rightarrow [g^{-2}] = \frac{D-4}{2} \\
&\Rightarrow [g] = 4 - D
\end{aligned} \tag{41}$$

From general quantum field theory we know that if the mass dimension is positive a theory is super-renormalisable, if it is zero the theory is renormalisable and if it is negative the theory is non-renormalisable. We see that

[5]This name is probably due to their off-statistics. They are fermionic but have bosonic equations of motion.

Yang-Mills is non-renormalisable for $D > 4$ and renormalisable in $D = 4$ Let us assume we are in $D = 4$ for now and also assume we have a $U(1)$ gauge group. The field strength is just $F = dA$ because all structure constants of $U(1)$ are zero and we only have a spacetime integral in the kinetic term for the gauge fields because the group trace is trivial. Now recall the Wetterich equation (13). It does not matter which regulator we pick for the equation to hold, as long it satisfies the properties for a regulator. We promote the coupling constant g to be momentum k dependent, i.e. g_k. Then the effective average action reads:

$$\Gamma_k[A, c] = \int d^D x \left(-\frac{1}{4g_k^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{2\xi}(\partial \cdot A)^2 - \frac{1}{g}\bar{c}\partial_\mu\partial^\mu c\right) \tag{42}$$

a quick look reveals that the left hand side of the Wetterich equation yields a term $\sim -\frac{\dot{g}_k}{g^3}\mathcal{O}(A^2) - \frac{\dot{g}_k}{g^2}$(ghost fields). On the righthand side however we find an expression that is a cosmological constant, a function of the cutoff momentum k. This is due to the fact that there is no interaction between ghosts and gauge fields i.e. the ghost determinant is not gauge field dependent. This means that:

$$\partial_t g_k = 0 \tag{43}$$

or equivalently:

$$\beta(g)_{U(1)} = 0 \tag{44}$$

If the gauge group is non-abelian we have to be a lot more careful with the computation. In this case the right-hand-side does depend on the gauge field giving a non-trivial β-function. We cite the result [12] for 1-loop pure Yang-Mills:

$$\beta(g)_{YM} = -\frac{11}{16\pi^2}g^3 + \mathcal{O}(g^5) \tag{45}$$

This β-function admits only 1 fixed point, the Gaussian fixed point where $g = 0$. This corresponds to the free theory where ghosts and gauge fields do not interact. The β-function is also negative, which means that the theory is asymptotically free. In the section on super Yang-Mills theory we will demonstrate the working of the Wetterich equation in more detail. We will now slowly build on our knowledge of supersymmetry to write down a supersymmetric version of Yang-Mills theory which finds applications in a rather different direction in physics as we will see later.

Part II

Supersymmetry and super Yang-Mills theory

5 Supersymmetry

In order to derive the renormalisation group flows of supersymmetric Yang-Mills theory we now introduce the concept of supersymmetry. We first state the Coleman-Mandula theorem: "Spacetime with the Poincaré group as its symmetry group, and internal symmetries denoted as G-symmetries can not be combined in any but a trivial way"[8]. However by extending the notion of a Lie group to a **graded** Lie group one could find a symmetry that includes the Poincaré group and the G-symmetry [13, 8, 14]. It was of great importance to physics to combine all symmetry groups in to one.

5.1 Symmetries

The original standard model of particle physics is defined by the particles that are representations of the Poincaré group $SO(1,d) \ltimes \mathbb{R}^{1,d} = ISO(1,d)$. This group represents the isometries of Minkowski space $\mathcal{M}^{1,d}$. The Poincaré group is generated by generators P_μ and $M_{\mu\nu}$ that generate translations and boosts together with spatial rotations respectively. they satisfy the following algebra [8]:

$$[P_\mu, P_\nu] = 0 \tag{46}$$

$$[M_{\mu\nu}, P_\rho] = i(P_\mu \eta_{\nu\rho} - P_\nu \eta_{\mu\rho}) \tag{47}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(M_{\nu\sigma}\eta_{\mu\rho} + M_{\mu\rho}\eta_{\nu\sigma} - M_{\mu\sigma}\eta_{\nu\rho} - M_{\nu\rho}\eta_{\mu\sigma}) \tag{48}$$

In $D = 4$ the algebra can be reorganised such that $\mathfrak{so}(1,3) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ becomes manifest. This allows to organise fields according to their transformation behaviour under Poincaré transformations. The group $SU(2)$ is labeled by an half-integer s called **spin**. These representations are $2s+1$ dimensional. From here we can deduce the following representations for

$0 \leq s \leq 2$ [8]:

$$\begin{aligned}
&\text{Scalar}(s=0) = (0,0)\\
&\text{Spinor}(s=\frac{1}{2}) = (\frac{1}{2},0) \oplus (0,\frac{1}{2})\\
&\text{Vector}(s=1) = (\frac{1}{2},\frac{1}{2}) = (\frac{1}{2},0) \otimes (0,\frac{1}{2})\\
&\text{Rarita-Schwinger}(s=\frac{3}{2}) = (\frac{1}{2},1) \oplus (1,\frac{1}{2})\\
&\text{2-Tensor}(s=2) = (\frac{1}{2},\frac{1}{2}) \otimes (\frac{1}{2},\frac{1}{2}) = (1,0) \oplus (0,1) \oplus (1,1) \oplus (0,0)
\end{aligned}$$

with $(\cdot,\cdot)$ denoting chirality. In order to form the standard model it is necessary to include internal symmetries that label extra quantum numbers carried by different particles. It is known that the group $SU(3)_c \times SU(2)_L \times U(1)_Y$ carries the gauge symmetry of the standard model. We therefore wish to include general symmetries, known as G-symmetries, to any global symmetry like the Poincaré group.

5.2 The supersymmetry algebra

From the Coleman-Mandula theorem we would conclude that it is impossible to combine spacetime symmetry and G-symmetries non-trivially. However, by extending the notion of a Lie group we can work around this. A Lie group G is a group that is also a differentiable manifold such that the maps $G \times G \to G : (g,h) \to gh$ and $G \to G; g \to g^{-1}$ with $g,h \in G$ are smooth. Furthermore, a Lie group has an associated Lie algebra which is a vector space $\mathfrak{g}$ together with a non associative, alternating bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (X,Y) \to [X,Y]$ with $[\cdot,\cdot]$ called the Lie bracket. The elements of $\mathfrak{g}$ are called the generators of the Lie group G and are the vectors of the tangent space of G at the unit element. The Lie group is generated by exponentiating its Lie algebra elements. For most physics applications the Lie group is taken to be a matrix group and so the Lie algebra consists of matrices. Then the Lie bracket is taken to be the commutator of two matrices defined by $[A,B] = AB - BA$. The generators $X^a \in \mathfrak{g}$ then satisfy the following commutation relation:

$$[X^a, X^b] = if^{abc}X^c \tag{49}$$

where f^{abc} are called structure constants of the Lie algebra and summation over c is implied. The next step is to expand the notion of such an algebra to what we shall call a **super algebra**. As far as we have seen all generators are **bosonic**. The notion of a superalgebra comes if we include generators that act **fermionic**. Let us introduce a so called graded Lie algebra with generators O_a where a are group indices. We then define a graded commutator $[\cdot,\cdot\}$ by $[O_a, O_b\} = O_a O_b - (-1)^{g_a g_b} O_b O_a$

Where g_a, g_b are the gradings of the operators and defined as:

$$g_a = \begin{cases} 0 & \text{if } O_a \text{ is bosonic} \\ 1 & \text{if } O_a \text{ is fermionic} \end{cases}$$

Denoting B =bosonic and F =fermionic, note:

$$[B, F\} = [B, F] \tag{50}$$
$$[B, B\} = [B, B] \tag{51}$$
$$[F, F\} = \{F, F\} \tag{52}$$

with $\{A, B\} = AB + BA$ the anticommutator. An algebra satisfying these rules is said to be $\mathbb{Z}_2$ graded. The earlier defined property that we get an element in the Lie group G by exponentiating the generators will be continued later when we indeed try to formulate the super Lie group of the super Lie algebra.

The only problem at this stage is that we do not have fermionic generators in our Poincaré algebra. In order to extend the Poincaré algebra to the super Poincaré algebra we introduce new fermionic generators Q^A_α and $\bar{Q}^A_{\dot{\alpha}}$ with the convention:

1. A numerates the fermionic generators
2. α indices: are spinor indices, no lorentz/spacetime indices. α indices represent the lefthanded part and $\dot{\alpha}$ indices represent the righthanded part of a spinor

We take $A = 1 \ldots \mathcal{N}$. The Q^A_α's and $\bar{Q}^A_{\dot{\alpha}}$ are spinorial objects and thus transform in the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of the Lorentz group.

We get a new closed algebra that we shall call the **supersymmetry algebra**. The supersymmetry algebra is given by [8]:

$$[Q^A_\alpha, M^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha^\beta Q^A_\beta \tag{53}$$
$$[Q^A_\alpha, P^\mu] = 0 \tag{54}$$
$$\{Q^A_\alpha, \bar{Q}^B_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \delta^{AB} \tag{55}$$
$$\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} Z^{AB} \tag{56}$$

here Z^{AB} is antisymmetric and referred to as the central charge[6] of the specific supersymmetry and the Q's generate the supersymmetry transformations. In particular for $\mathcal{N} = 1$ we have $Z^{AB} = 0$. There might also be an internal symmetry that transforms the "Q's in to Q's". This symmetry is called R-symmetry. In the simple case of $\mathcal{N} = 1$ we can transform $Q \to \exp(ia)Q$ with $a \in \mathbb{R}$ a constant and observe that the supersymmetry algebra is invariant. We therefore have a global $U(1)$ R-symmetry group. This R-symmetry can be a larger symmetry group than just $U(1)$. In general, a $\mathcal{N} > 1$ supersymmetric theory enjoys an $U(\mathcal{N})$ R-symmetry denoted usually by a subscript R. In the special case of $\mathcal{N} = 4$ this is reduced to $SU(4)_R$ [15, 14]. From the supersymmetry algebra it is possible to create the full Fock space representation of the theory for massless and massive states respectively.

5.3 Supermultiplets

In this section we will discuss the matter content of supersymmetry theories. This content will depend on $\mathcal{N}$, the number of supersymmetry generators. The collection of all particles that are allowed within the given representation is called a supermultiplet. Since the Q's are spinorial objects any quantum state with spin s will become a state with spin $s \pm 1/2$ under the action of Q. This means that $Q|B\rangle \sim |F\rangle$ and $Q|F\rangle \sim |B\rangle$. The operators Q do not change anything about the states except for their spin, this means the so called **superpartner** will have the same mass. We can further conclude that:

1. Any supersymmetric theory always has a positive groundstate energy E_0:
   To see this consider:

$$
\begin{aligned}
0 \leq |Q^A_\alpha|\Psi\rangle|^2 + |(Q^A_\alpha)^\dagger|\Psi\rangle|^2 &= \langle\Psi|(Q^A_\alpha)^\dagger Q^A_\alpha + (Q^A_\alpha)^\dagger Q^A_\alpha|\Psi\rangle \\
&= \langle\Psi|\{Q^A_\alpha, \bar{Q}^A_{\dot{\alpha}}\}|\Psi\rangle = 2(\sigma^\mu)_{\alpha\dot{\alpha}}\langle\Psi|P_\mu|\Psi\rangle
\end{aligned}
\tag{57}
$$

   Performing the sum, noting the $\alpha\dot{\alpha}$ actually represents the trace we find:

$$
\langle\Psi|P_0|\Psi\rangle \geq 0 \tag{58}
$$

   with $P_0 = E_0$.

[6] Central charges are operators that have the property that they commute with every other operator of the theory

2. Any supermultiplet contains an equal number of bosonic and fermionic degrees of freedom: Let us construct a matrix A that is diagonal with entries ± 1. This matrix can be represented as $A = (-1)^{N_F} I$. Here N_F refers to as the fermion number that is 0 for a bosonic and 1 for a fermionic state and I is the identity. In the following we leave out the I. We wish to show that $\mathrm{Tr}((-1)^{N_F}) = 0$ since bosonic and fermionic degrees of freedom would cancel each other exactly. Since this number operator $(-1)^{N_F}$ anticommutes with Q^A_α we have

$$\begin{aligned}\mathrm{Tr}\Big((-1)^{N_F}\{Q^A_\alpha, \bar{Q}^B_{\dot\beta}\}\Big) &= \mathrm{Tr}\Big((-1)^{N_F} Q^A_\alpha \bar{Q}^B_{\dot\beta} + (-1)^{N_F} \bar{Q}^B_{\dot\beta} Q^A_\alpha\Big)\\ &= \mathrm{Tr}\Big(-Q^A_\alpha (-1)^{N_F} \bar{Q}^B_{\dot\beta} + Q^A_\alpha (-1)^{N_F} \bar{Q}^B_{\dot\beta}\Big)\\ &= 0\end{aligned}$$

where in the second line we used the anticommutator and cyclicity of the trace. Comparing this with:

$$\mathrm{Tr}\Big((-1)^{N_F}\{Q^A_\alpha, \bar{Q}^B_{\dot\beta}\}\Big) = \mathrm{Tr}\Big((-1)^{N_F} 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu\Big) = 2(\sigma^\mu)_{\alpha\dot\beta} p_\mu \mathrm{Tr}\Big((-1)^{N_F}\Big)$$

where we replaced $P_\mu \to p_\mu$ which are the eigenvalues for each state. We conclude that the only consistent possibility is:

$$\mathrm{Tr}\Big((-1)^{N_F}\Big) = 0 \tag{59}$$

And since the trace is taken over the different states being either bosonic or fermionic we conclude that $n_F = n_B$.

Let us now discuss the $\mathcal{N} = 1$ case extensively and then summarise the result for $\mathcal{N} > 1$.

5.3.1 $\mathcal{N} = 1$ supersymmetry representations

To describe different representations we refer to the supersymmetry algebra (53) and look at the various actions of the supersymmetry generators. Here we discuss the massless representations of unextended supersymmetry. For a massless representation we have that $P^2 = 0$ and we can thus write $p_\mu = E(1, 0, 0, 1)$. Then we have:

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\beta} \tag{60}$$

thus

$$\{Q_1, \bar{Q}_{\dot{1}}\} = 4E \tag{61}$$
$$\{Q_2, \bar{Q}_{\dot{2}}\} = 0 \tag{62}$$

Observe that we have halved the amount of supersymmetry generators from $2\mathcal{N}$ to $\mathcal{N}$. We can cast this in a more recognizable way by defining $a = \frac{Q_1}{\sqrt{4E}}$ and $a^\dagger = \frac{\bar{Q}_{\dot{1}}}{\sqrt{4E}}$. These operators then satisfy the usual fermionic algebra, i.e. $\{a, a^\dagger\} = 1$. This means we can use the supersymmetry generators as ladder operators to build up the Fock space of states labeled by momentum p and helicity λ. The classical way to do this is by starting with a lowest weight state that is annihilated by a. Let us denote this state by $|\Omega\rangle = |P, \lambda\rangle$. We also define the helicity operator $J^3 = M^{12}$ like usual. We then find that $[Q_1, J^3] = \frac{1}{2}Q_1$ [8]. By using the definition of the creation $a^\dagger$ and annihilation a operators we can also write $[a, J^3] = \frac{1}{2}a$. Then for some non-minimal state $|P, \lambda\rangle$ we have that:

$$J^3 a|P, \lambda\rangle = (aJ^3 - [a, J^3])|P, \lambda\rangle = (\lambda - \frac{1}{2})a|P, \lambda\rangle \tag{63}$$

Here we have used the commutator and the fact that $J^3|\lambda\rangle = \lambda|\lambda\rangle$. If we had used the creation operator $a^\dagger$ we had found an eigenvalue of $\lambda + \frac{1}{2}$. It is now straightforward to find the full supermultiplet. Recall the anticommutator for the creation operator, it follows that $(a^\dagger)^2 = 0$[7]. This then means that there is no state that is proportional to $a^\dagger a^\dagger|\Omega\rangle$ since it gets annihilated just like $a|\Omega\rangle$. To write down a charge-parity-time (CPT) invariant[8] supermultiplet we have to add CPT conjugate states by hand. We therefore have the full $\mathcal{N} = 1$ massless supermultiplet:

$$|\Omega\rangle \text{ and } |\tilde{\Omega}\rangle = |p, \pm\lambda\rangle$$
$$|p, \pm(\lambda + \frac{1}{2})\rangle$$

We can now take the lowest value of λ and look at the matter content. Examples are:

[7]This is a manifestation of the Pauli exclusion principle for identical fermions.

[8]We strongly believe in CPT conservation in non-supersymmetric quantum field theories.

1. $\lambda = 0$ The chiral supermultiplet: This supermultiplet consists of a spin 0 and a spin $\frac{1}{2}$ state plus their CPT conjugates. This is sometimes abbreviated to $(0, \frac{1}{2}) + (-\frac{1}{2}, 0)$.

2. $\lambda = \frac{1}{2}$ The vector supermultiplet: This supermultiplet consists of a spin $\frac{1}{2}$ and a spin 1 state plus their CPT conjugates.

3. $\lambda = 1$ The gravitino supermultiplet: This supermultiplet contains a spin $\frac{3}{2}$ and a spin 1 state plus their CPT conjugates.

4. $\lambda = \frac{3}{2}$ The graviton supermultiplet: This supermultiplet contains a spin $\frac{3}{2}$ and a spin 2 state plus their CPT conjugates.

It is strongly believed that there is no fundamental particle with $\lambda > 2$. Thus supermultiplets that contain these states are typically not of interest. Futhermore, the gravitino, which would be the superpartner of the graviton[9] is to be considered only within a theory where we do have gravitons at all. Therefore it is impossible for the gravitino multiplet to exist on its own within the frame of $\mathcal{N} = 1$ supersymmetric theories. Quantum field theories that are of interest to us do not contain spin-2 fields. The relevant supermultiplet for Yang-Mills theory therefore consists of the chiral- and vector supermutliplet.

Let us now discuss massive particles. The starting point is the momentum p_μ which now can take the form $p_\mu = (m, 0, 0, 0)$ in the particle rest frame. From here we can write down the full supersymmetry algebra. Remember however that the anticommutator between Q_α and Q_β does vanish for unextended supersymmetric theories since there are no central charges. In the case for $\mathcal{N} = 1$ the supersymmetry algebra therefore doubles for massive representations, i.e.:

$$\{Q_1, \bar{Q}_{\dot{1}}\} = 2m \tag{64}$$

$$\{Q_2, \bar{Q}_{\dot{2}}\} = 2m \tag{65}$$

Thus we have the possibility of creating the state $(a_{\dot{1}})^\dagger (a_{\dot{2}})^\dagger |\Omega\rangle$ etcetera. This string of creation operators has twice the length of the massless case. If we allow for, lets say, $\mathcal{N} = 2$ we would have four times as much. We see a growth

[9] The graviton is the spin-2 particle that mediate the gravitational force.

that goes as a square compared to the massless case. Later we will make this more precise. The interesting physics start to happen when we look at the extended supersymmetry algebra with non zero central charges.

5.3.2 Extended supersymmetry representations

Considering the case $\mathcal{N} > 1$ for the massless representations with $Z^{AB} = 0$ will change a few things. The same arguments leading to (60) still applies, so:

$$\{Q_1^A, \bar{Q}_{\dot{1}}^B\} = 4E\delta^{AB}$$
$$\{Q_2, \bar{Q}_{\dot{2}}\} = 0$$

Thus only half of the supersymmetry generators remain. We then define creation and annihilation operators again, this time labeled by the specific supersymmetry generator it belongs to. So $(a_1^A) = \frac{Q_1^A}{\sqrt{4E}}$ and $(a_{\dot{1}}^B)^\dagger = \frac{\bar{Q}_{\dot{1}}^B}{\sqrt{4E}}$. These again satisfy the usual fermionic algebra. This time we do not have the criterium that we have to stop after two creation operators acting on the lowest spin state, instead, we can act with all different creation operators. This means that for some $k \leq \mathcal{N}$ we can build $(a^1)^\dagger \cdots (a^k)^\dagger|, \lambda\rangle = |p, \lambda + \frac{k}{2}\rangle$. Where we have suppressed the spinor index. the maximum value of $k = \mathcal{N}$. The number of states within the multiplet with k of these creation operators is just $\binom{\mathcal{N}}{k}$ where the lowest spin state is included. The full massless supermultiplet then has a total number of states given by this summed over all values of k:

$$n_{\text{massless}} = \sum_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k} = 2^{\mathcal{N}} \tag{66}$$

From here it is easy to find the supermultiplets for different values of $\mathcal{N}$. We still have to add the CPT conjugates by hand, these would double the number of states, so one could write $n_{\text{massless}} = 2^{\mathcal{N}+1}$. We can now first pick a value for $\mathcal{N}$ and then start writing down the supermultiplets we have encountered already, chiral, vector and graviton. However this time the language differs a little bit. We will list the different supermultiplets for $\mathcal{N} = 2, 4, 8$ below [8].

1. The $\mathcal{N} = 2$ hyper multiplet, $\lambda = -\frac{1}{2}$: This supermultiplet contains a fermion, each respective creation operator acting on it, and the two

distinct creation operators acting on it. This produces four possibilities, two spin 1/2 and two spin 0 states plus their CPT conjugates, which makes a total of 8 states.

2. The $\mathcal{N} = 2$ vector multiplet, $\lambda = 0$: This supermultiplet contains of one spin 1, two spin 1/2 and one spin 0 states plus their CPT conjugates.

   This exhausts supermultiplets for $\mathcal{N} = 2$ possible that have spin ≤ 1, and thus do not include gravitons.

3. The $\mathcal{N} = 4$ vector multiplet, $\lambda = -1$: This supermultiplet is constructed in the same way as above. There is one spin 1, four spin 1/2 and three spin 0 particles plus their CPT conjugates. This is the building block of the famous $\mathcal{N} = 4$ supersymmetric Yang-Mills theory.

   Finally, if we go above $\mathcal{N} = 4$ we necessarily have to include the spin 3/2, and therefore, the spin 2 particle e.g. gravity. There is one supermultiplet that is evenly distributed around spin 0.

4. The $\mathcal{N} = 8$ gravity supermultiplet, $\lambda = -2$: This multiplet uniquely consists of only increasingly longer strings of creation operators. It has a content of one spin 2, eight spin 3/2, 28 spin 1, 56 spin 1/2 and 70 spin 0 particles.

These are all massless representations. Going beyond $\mathcal{N} = 8$ would include higher spin particles which are typically not considered. Finally we briefly touch on massive representations in an extended supersymmetry algebra. We go in to the same Lorentz frame as earlier for the $\mathcal{N} = 1$ case, but this time we consider non-zero central charges. The supersymmetry algebra reads:

$$\{Q_1^A, \bar{Q}_{\dot{1}}^B\} = 2m\delta^{AB} \tag{67}$$

$$\{Q_2^A, \bar{Q}_{\dot{2}}^B\} = 2m\delta^{AB} \tag{68}$$

$$\{Q_1^A, Q_2^B\} = Z^{AB} \tag{69}$$

Here we have used that $\epsilon_{12} = 1$ is the only non-vanishing combination. The last anticommutator is straightforward for the $\bar{Q}$'s. Recall that $Z^{AB} =$

$-Z^{BA}$. Let us for simplicity only consider the the case $\mathcal{N} = 2$, the generalisation is then straightforward. Thus the general form for Z^{AB} is:

$$Z^{AB} = \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix} \tag{70}$$

We have

$$\{Q_1^1, Q_2^2\} = q \tag{71}$$

with the trivial counterpart, $\{Q_1^2, Q_2^1\} = -q$. We proceed as earlier, defining creation and annihilation operators that we can use to build the Fock space. It is however impossible to write this algebra in terms of a single set of creation or annihilation operators $a, a^\dagger$. It is necessary to include a second set $b, b^\dagger$. The corresponding annihilation operators are denoted by a_α^A and b_α^A with $A \in \{1 \cdots \mathcal{N}\}$. We define the following two operators in $\mathcal{N} = 2$ [8]:

$$a_\alpha = \frac{1}{\sqrt{2m}}(Q_\alpha^1 + \epsilon_{\alpha\dot{\beta}}\bar{Q}^{2,\dot{\beta}}) \tag{72}$$

$$b_\alpha = \frac{1}{\sqrt{2m}}(Q_\alpha^1 - \epsilon_{\alpha\dot{\beta}}\bar{Q}^{2,\dot{\beta}}) \tag{73}$$

$$a_{\dot{\alpha}}^\dagger = \frac{1}{\sqrt{2m}}(\bar{Q}_{\dot{\alpha}}^1 + \epsilon_{\dot{\alpha}\beta}Q^{2,\beta}) \tag{74}$$

$$b_{\dot{\alpha}}^\dagger = \frac{1}{\sqrt{2m}}(\bar{Q}_{\dot{\alpha}}^1 - \epsilon_{\dot{\alpha}\beta}Q^{2,\beta}) \tag{75}$$

These operators again satisfy the fermionic algebra with themselves. However, the anticommutator $\{a, a^\dagger\}$ changes due to cross terms. These can be expressed nicely in the anticommutators of the algebra. We compute:

$$\{a_\alpha, a_{\dot{\beta}}^\dagger\} = \frac{1}{2m}(2m + 2m + q + q)\delta_{\alpha\dot{\beta}} = (2 + \frac{q}{m})\delta_{\alpha\dot{\beta}} \tag{76}$$

$$\{b_\alpha, b_{\dot{\beta}}^\dagger\} = \frac{1}{2m}(2m + 2m - q - q)\delta_{\alpha\dot{\beta}} = (2 - \frac{q}{m})\delta_{\alpha\dot{\beta}} \tag{77}$$

We observe one interesting property of this algebra. Unitarity, or the absence of ghosts[10] requires the fermionic algebra be strictly positive by means of the eigenvalues of the operators. This is only satisfied if $2 - \frac{q}{m} \geq 0$. This means that we have to require:

$$2m \geq q \tag{78}$$

[10] States with a negative norm.

This bound is called the Bogomol'nyi-Prasad-Sommerfeld (BPS) bound and tells us that we have a positive mass that is larger than the central charge of the theory. States that fullfil the minimum requirement with $m = q/2$ are called BPS states and strictly require that only $a, a^\dagger$ are relevant operators. Multiplets consisting of these states are therefore called **short** supermultiplets. For the case that $m > q/2$ we now have a total of $2\mathcal{N}$ creation operators. This leads to the fact that $n_{\text{massive}} = 2^{2\mathcal{N}}$. This is the square of the massless case.

Using above insights the next step is to create the Fock space from the lowest spin state by acting with all different creation operators. The only difference to the case with $Z = 0$ is the fact we have the BPS boundary. This concludes our introduction to supersymmetry and its representations.

6 Superspace formalism

The following section will be devoted to develop the necessary tools in order to describe supersymmetric field theory in a compact way. To do so we aim for a way to keep supersymmetry manifest at every step in the calculation. This can be done if we expand our notion of usual spacetime to **superspace**. Fields living on superspace are called **superfields**. Their components will represent our conventional fields depending on normal spacetime. This whole section we will closely follow a set of lecture notes [13, 16, 17, 18, 8].

Before going to superfields we will introduce the notion of superspace related to Minkowski space. As it turns out if one wishes to write down actions for a supersymmetric theory it suffices to write them down in terms of superfields[11].

Without going into mathematical subtleties it is possible to write down smooth manifolds in terms of their corresponding Lie groups, since a Lie group is a smooth manifold by definition. Consider Minkowski space in four spacetime dimensions. The spacetime coordinates satisfy $[x^\mu, x^\nu] = 0$. The isometry group of $\mathcal{M}^4$ is the Poincaré group $ISO(1,3)$. This means that under the group action[12] $ISO(1,3) \curvearrowright \mathcal{M}^4 = \mathcal{M}^4$. Thus any point in $\mathcal{M}^4$ can be reached by an appropriate Poincaré transformation. Using the mathematical result: $Orb_G(x) \simeq \frac{G}{G_x}$ where $Orb_G(x)$ is the orbit of an element x under G defined as the set $\{gx | g \in G, x \in X\}$ with $X \subset G$. G_x is the

[11]Note that not everything will be supersymmetric invariant, only certain types of terms.

[12]The notation $G \curvearrowright M$ means that the Lie group G acts on the manifold M.

stabiliser of G on X and defined as the set $\{gx = x | g \in G, x \in X\}$. It is easy to show that for $G = ISO(1,3)$ the stabiliser is the Lorentz group $SO(1,3)$ since it leaves inner products on $\mathcal{M}^4$ invariant. Using the definition of the stabiliser it follows that the orbit of the Lorentz group is $\mathcal{M}^4$. We therefore conclude that:

$$\mathcal{M}^4 \cong \frac{ISO(1,3)}{SO(1,3)} \tag{79}$$

This provides a coset representation of Minkowski space. If one can write a manifold as a coset representation it is possible to write a point in that manifold as being generated by the generators of the Lie algebra. This means that some point in $\mathcal{M}^4$, say x^μ, is generated by $\exp(x^\mu P_\mu) \in G$ where P_μ is the generator of translations in $\mathcal{M}^4$.

In a similar fashion it is possible to define superspace as a coset by promoting the Poincaré group to the super Poincaré group which is the Poincaré algebra (46) extended with the supersymmetry algebra from (53). A point in superspace will still have the Lorentz group $SO(1,3)$ as stabiliser [13] so we can write down:

$$\mathcal{M}^{4|1} = \frac{\text{Super Poincaré}}{SO(1,3)} \tag{80}$$

with $\mathcal{M}^{4|1}$ denoting four-dimensional Minkowski space supplemented with $\mathcal{N} = 1$ supersymmetry. It is now possible to give a concrete meaning to this by introducing superspace coordinates. These coordinates consist of bosonic but also of fermionic coordinates. We recall the supersymmetry algebra (53). This algebra is written in anticommutator language, however in order to define a Lie algebra we have to use commutators. To turn this graded algebra of spinorial objects in to commutators we introduce new Grassmann valued spinorial objects θ and $\bar{\theta}$ that satisfy [16]:

$$\{\theta^\alpha, \theta^\beta\} = 0 \tag{81}$$

$$\{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} = 0 \tag{82}$$

$$\{\theta^\alpha, \bar{\theta}_{\dot{\beta}}\} = 0 \tag{83}$$

Starting from the supersymmetry algebra it is easily shown that in fact we can turn it to a commutator algebra by calculating:

$$[\theta Q, \bar{\theta}\bar{Q}] = \theta\bar{\theta}\{Q, \bar{Q}\} = 2\theta\sigma^\mu\bar{\theta}P_\mu \tag{84}$$

$$[\theta Q, \theta\bar{Q}] = 0 \tag{85}$$

In the second step we used the spinor properties of θ and $\theta Q = \theta^\alpha Q_\alpha$. Let us now choose an excplicit representation of superspace by introducing coordinates $z^m = \{x^\mu, \theta, \bar{\theta}\}$. From this it is clear that we now must consider translations in the fermionic sector as well. A general group element in the super Poincaré group looks like:

$$g(z) = \exp(ix \cdot P + i\theta \cdot Q + i\bar{\theta} \cdot \bar{Q} + \frac{1}{2} i\omega \cdot M) \tag{86}$$

where the $\cdot$ means inner product with respect to the index structure. If we restrict $g(z)$ to translations, i.e. $\omega^{\mu\nu} = 0$ we can read of the generators of translations in superspace.

6.1 $\mathcal{N} = 1$ superspace

The above discussion is not the most general ansatz for superspace. So far we introduced only a total of four new coordinates, these were labeled θ^α and $\bar{\theta}_{\dot{\beta}}$, with $\alpha, \dot{\beta} \in \{1, 2\}$. Just like we have labeled supersymmetry generators by an integer number $A \in \{1, \dots, \mathcal{N}\}$ we can also label our new fermionic coordinates by such a number. This would create $\mathcal{N} > 1$, or extended, superspace. This will not be needed in this work however and we stick to the $\mathcal{N} = 1$ superspace formulation. We now make the translations in superspace explicit. To do so we need differentiation operators that can act as translation generators on superspace. We define differentiation operators with respect to Grassmann parameters in the following way [17]:

$$\frac{\partial}{\partial \theta^\alpha} \theta^\beta = \partial_\alpha \theta^\beta = \delta_\alpha^\beta \tag{87}$$

$$\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} \theta^\alpha = \dot{\partial}_{\dot{\alpha}} \theta^\alpha = 0 \tag{88}$$

Integration for Grassmann numbers is defined as:

$$\int d\theta = 0 \tag{89}$$

$$\int d\theta \ \theta = 1 \tag{90}$$

and analogous for $\bar{\theta}$. Note the fact that differentiation and integration for Grassmann numbers are the same operation. Considering that we do not have just one single Grassmann number but a total of four we write down integrals over full superspace. Therefore we also define:

$$\begin{aligned} d^2\theta &= -\frac{1}{4}d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta} \\ d^2\bar{\theta} &= \frac{1}{4}d\theta^{\dot{\alpha}} d\bar{\theta}^{\dot{\beta}} \epsilon_{\dot{\alpha}\dot{\beta}} \\ d^6 z &= d^4 x d^2\theta \\ d^6 \bar{z} &= d^4 x d^2\bar{\theta} \\ d^8 z &= d^4 x d^2\theta d^2\bar{\theta} \end{aligned} \tag{91}$$

We have now layed out the playground in order to describe physics in superspace. Yet we still have to find the correct translation generators in superspace, to do so we need superfields.

6.2 Superfields

We will now introduce the notion of a superfield and clarify their importance. First let us review the computation of the translation operator for a scalar field $\phi(x)$ in Minkowski space. Under a translation in Minkowski space, $x^\mu \to x^\mu + a^\mu$. A scalar field transforms as $\phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x) + \cdots$. We see that ∂_μ is the generator of the translation. From equation (86) we also know that the field transforms as $\phi(x) \to \exp(ia \cdot P)\phi(x)$. Expanding the last expression gives the result: $P_\mu = -i\partial_\mu$.
Extending above argument to superspace requires the notion of a "field" in superspace. To keep it similar to the above scalar field we write for a generic superfield $S(z)$ the following transformation law:

$$S(z) \to S(z') = S(x, \theta, \bar{\theta}) = S(x + \delta x, \theta + \delta\theta, \bar{\theta} + \delta\bar{\theta}) \tag{92}$$

If we consider two subsequent supersymmetry transformations it is clear from the supersymmetry algebra that this genarates a spacetime translation. Therefore by just considering supersymmetry transformations spacetime transformations are built in. This makes sense because the supersymmetry algebra is an extension to the conventional Poincaré algebra. We define a superspace translation by:

$$
\begin{aligned}
&S(x+\delta x, \theta+\delta\theta, \bar{\theta}+\delta\bar{\theta}) \\
&= \exp(-i(\epsilon Q+\bar{\epsilon}\bar{Q}))S(x,\theta,\bar{\theta})\exp(i(\epsilon Q+\bar{\epsilon}\bar{Q}) \\
&= \exp(-i(\epsilon Q+\bar{\epsilon}\bar{Q}))\exp(-i(xP+\theta Q+\bar{\theta}\bar{Q})) \\
&\times S(0,0,0)\exp(i(xP+\theta Q+\bar{\theta}\bar{Q}))\exp(i(\epsilon Q+\bar{\epsilon}\bar{Q})
\end{aligned} \tag{93}
$$

with infinitesimal spinor parameters ϵ and $\bar{\epsilon}$. The above Q generators act as differential operators of which we still have to find the explicit form. Since we are dealing with non-commuting operators in the exponential we make use of the Baker-Campbell-Hausdorff formula [16]:

$$
\mathrm{e}^A\mathrm{e}^B = \exp\Big(\sum_{n=1}^{\infty}\frac{1}{n!}C_n(A,B)\Big) \tag{94}
$$

The first three non-trivial coefficients $C_n(A,B)$ are given by:

$$
\begin{aligned}
C_1 &= A+B \\
C_2 &= [A,B] \\
C_3 &= \frac{1}{2}[A,[A,B]] - \frac{1}{2}[B,[B,A]]
\end{aligned} \tag{95}
$$

Because of the already implemented factors of θ we indeed get commutators rather than anticommutators. From the supersymmetry algebra we also have that $Q^2 = 0$. So that the expansion of the exponential terminates at some order. We find the following transformations:

$$
\delta x^\mu = i(\theta\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\theta}) \tag{96}
$$

$$
\delta\theta = \epsilon \tag{97}
$$

$$
\delta\bar{\theta} = \bar{\epsilon} \tag{98}
$$

We can now write a more precise way of how a superfield should transform under an infinitesimal supersymmetry transformation $\delta_{\epsilon,\bar{\epsilon}}$:

$$
\delta_{\epsilon,\bar{\epsilon}}S(z) \equiv S(z') - S(z) \tag{99}
$$

such that

$$
\delta_{\epsilon,\bar{\epsilon}}S(x,\theta,\bar{\theta}) = (\epsilon^\alpha\partial_\alpha + \bar{\epsilon}^{\dot{\alpha}}\bar{\partial}_{\dot{\alpha}} + i(\theta\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\theta})\partial_\mu + \ldots)S(x,\theta,\bar{\theta}) \tag{100}
$$

Following the example of the scalar field we expand equation (93). We arrive at the transformation law for a generic superfield $S(z)$:

$$\delta_{\epsilon,\bar{\epsilon}}S(x,\theta,\bar{\theta}) = i(\epsilon Q + \bar{\epsilon}\bar{Q})S(x,\theta,\bar{\theta}) \tag{101}$$

It is common to define fields by their transformation behaviour. We therefore say that a superfield is a field that transforms under supersymmetry transformations according to (101). Furthermore we find an expression for the supersymmetry generators as differential operators on superspace by comparing (101) to (100).

$$\begin{aligned} Q_\alpha &= -i\partial_\alpha - \sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \\ \bar{Q}_{\dot{\alpha}} &= i\bar{\partial}_{\dot{\alpha}} + \theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \end{aligned} \tag{102}$$

It is straightforward to show that these representations actually satisfy the $\mathcal{N} = 1$ supersymmetry algebra (53). We will end this section by writing down an ansatz for a superfield. The idea is that we can write down expansion of a field that will depend on x, θ and $\bar{\theta}$. Fields on conventional Minkowksi space only depend on the spacetime coordinates x^μ. We therefore introduce fields that have spacetime dependend components but can be expanded in fermionic coordinates. Considering for the moment just one fermionic coordinate we write a superfield as:

$$f(\theta) = \sum_{n=0} a_n(x)\theta^n \tag{103}$$

Since θ satisfies the anticommutator algebra (81), we have $\theta^2 = 0$ and hence:

$$f(\theta) = a_0(x) + a_1(x)\theta \tag{104}$$

has a terminating Taylor series. In the general setting we expand in left and right chiral spinorial coordinates θ and $\bar{\theta}$:

$$\begin{aligned} S(x,\theta,\bar{\theta}) = A(x) + \theta B(x) + \bar{\theta}C(x) + \theta^2 D(x) + \bar{\theta}^2 E(x) + \\ \theta\sigma^\mu\bar{\theta}F_\mu(x) + \theta^2\bar{\theta}G(x) + \bar{\theta}^2\theta H(x) + \theta^2\bar{\theta}^2 I(x) \end{aligned} \tag{105}$$

All higher order terms in θ and $\bar{\theta}$ will vanish. We have also defined the square here $\theta^2 = \theta^\alpha\theta_\alpha \neq (\theta^\alpha)^2 = 0$. The superfield (105) is our starting point for every kind of superfield. In the following sections we will encouter different kinds of superfields that each have their special property. The general superfield has too many degrees of freedom to correspond to an irreducible representation of the supersymmetry algebra. We therefore need to put constraints on it in order to describe a set of particles which are the irreducible representations of the symmetry group. Furthermore, observe that the set $\{A(x), \ldots, I(x)\}$ denotes what we have earlier called a supermultiplet. In general the component fields of some superfield will be called the field content or the supermultiplet of the theory. Let us now go to the first special kind of superfield, the chiral superfield.

6.2.1 The chiral superfield

The superfield (105) is highly reducible [13, 17, 16]. We are looking for a way of writing down a theory in terms of superfields and a good way of finding an analogy to usual field theory is to start by defining differential operators on superspace. In fact we have only seen Grassmannian and spacetime derivatives so far. To define derivatives that will be invariant under supersymmetry transformations we need to introduce covariant derivatives acting on superspace. The first kind of covariant derivative will also be a Grassmannian derivatives but be extended by spacetime derivatives. These need to be introduced because the usual derivatives ∂_α do not transform covariantly under supersymmetry transformations. This can be seen from the fact that $[Q_\alpha, \partial_\mu] = 0$ but in general the Q's will act on the Grassmannian derivatives. We consider the transformation laws of x and θ from (96). From these we can construct a covariant derivative since θ and x mixes under supersymmetry transformations.

$$\frac{\partial}{\partial\theta^\alpha} \to \frac{\partial\theta^\beta}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta} + \frac{\partial x^\mu}{\partial\theta^\alpha}\frac{\partial}{\partial x^\mu} \tag{106}$$

While the first term on the right hand side gives a δ_α^β the second can be calculated easily from the transformation laws, we get:

$$\frac{\partial}{\partial\theta^\alpha} \to \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot\beta}\bar\epsilon^{\dot\beta}\partial_\mu \tag{107}$$

From this we define the super covariant derivative as:

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu \tag{108}$$

$$\bar D_{\dot\alpha} = \bar\partial_{\dot\alpha} + i\theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu \tag{109}$$

These two super covariant derivatives satisfy[13, 17]:

$$\{D_\alpha, \bar{D}_{\dot{\beta}}\} = -2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \tag{110}$$

$$\{D_\alpha, D_\beta\} = \{D_\alpha, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot{\beta}}\} = 0 \tag{111}$$

This corresponds to the supersymmetry algebra (53) up to a sign. We shall use super covariant derivatives to construct kinetic terms in supersymmetric actions, generating kinetic terms like $\mathcal{L} \sim \partial_\mu \phi \partial^\mu \phi$.
However, our first priority was to reduce the degrees of freedom of the general superfield. The above anticommutation relations of the super covariant derivatives commute with supersymmetry transformations $\delta_{\epsilon,\bar{\epsilon}}$. Hence we conclude that $\delta_{\epsilon,\bar{\epsilon}}(D_\alpha S) = D_\alpha(\delta_{\epsilon,\bar{\epsilon}} S)$. This means that given a superfield S, $D_\alpha S$ is also a superfield. We can use this property as a constraint on a superfield. Setting $D_\alpha S = 0$ would be a good supersymmetry invariant constraint that will reduce the degrees of freedom. This can be seen by the fact that every term in the superfield will be differentiated with respect to a Grassmann variable and we are thus led to set constant terms to zero. The same argument may be used with the super covariant derivative $\bar{D}_{\dot{\alpha}}$. Note however that $\partial_\mu S$ is a superfield because of the vanishing commutator $[Q, P]$. We now define a chiral superfield.
A chiral superfield is a superfield $\Phi(z)$ that satisfies $\bar{D}_{\dot{\alpha}}\Phi(z) = 0$ and an anti-chiral superfield is a superfield that satisfies $D_\alpha \bar{\Phi}(z) = 0$ [16]. Note that a product of chiral superfields $\Phi_1 \cdot \Phi_2$ is again a chiral superfield because the linear differential operators D_α obey the Leibniz rule: $\bar{D}_{\dot{\alpha}}(\Phi_1 \cdot \Phi_2) = (\bar{D}_{\dot{\alpha}}\Phi_1)\Phi_2 + \Phi_1(\bar{D}_{\dot{\alpha}}\Phi_2) = 0$. This can also be seen from the fact that the chiral superfield itself is bosonic and therefore the Grassmann derivatives themselves do obey the Leibniz rule. Let us introduce the supercoordinate $y^\mu \equiv x^\mu + i\theta^\alpha \sigma^\mu_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}}$, using the spinor identities from appendix A it can be shown that y^μ is covariantly constant, i.e. $\bar{D}_{\dot{\alpha}} y^\mu = 0$. It is then clear that the chiral superfield $\Phi(x, \theta, \bar{\theta}) = \Phi(y, \theta)$, which is easier to expand. For later convenience we will put a factor of $\sqrt{2}$ in to the following expression. Taylor expanding the above chiral superfield in the fermionic variable θ and using $\theta^3 = 0$ one gets:

$$\Phi(y, \theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y) \tag{112}$$

Rewriting in terms of x we find:

$$\Phi(x,\theta,\bar{\theta}) = \phi(x) + \sqrt{2}\theta\psi(x) + \theta^2 F(x) \\ + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) + \frac{i}{\sqrt{2}}\theta^2\sigma^\mu\bar{\theta}\partial_\mu\psi(x) \\ - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2\phi(x) \tag{113}$$

The first thing we observe is that chiral supermultiplet consists of: two bosons and one fermion that has two degrees of freedom. We conclude that this corresponds to the chiral supermultiplet. Using appendix B we see that the components of the chiral superfield transform as:

$$\delta_{\epsilon,\bar{\epsilon}}\phi(x) = \sqrt{2}\epsilon\psi \tag{114}$$

$$\delta_{\epsilon,\bar{\epsilon}}\psi(x) = \sqrt{2}i\bar{\epsilon}\sigma^\mu\partial_\mu\phi(x) - \sqrt{2}\epsilon F(x) \tag{115}$$

$$\delta_{\epsilon,\bar{\epsilon}}F(x) = \sqrt{2}i\bar{\epsilon}\sigma^\mu\partial_\mu\psi(x) \tag{116}$$

Indeed bosons are mapped to fermions and vice versa like intended by supersymmetry. The only thing we need to do here in order to evaluate $\delta_{\epsilon,\bar{\epsilon}}\Phi(y,\theta)$ is to rewrite the original supersymmetry generators Q in terms of the supercoordinates y. These are found to be [16, 17, 8]

$$Q'_\alpha = -i\partial_\alpha \\ \bar{Q}'_{\dot{\alpha}} = i\bar{\partial}_{\dot{\alpha}} + 2\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial y^\mu} \tag{117}$$

Also note the fact that the field $F(x)$, traditionally called "F-term", transforms as a total derivative. This term, if present in a Lagrangian, will by Stokes' theorem show up as a surface term and thus leave the total action invariant. This special feature will become more important later.

6.2.2 The vector superfield

On our way towards a superspace formulation of Yang-Mills theory it is of great importance to include fields that we know as vector fields. Together with them the full framework of gauge theory comes along. We start by writing down the definition of a vector superfield. By definition a vector superfield is a real superfield $V^\dagger(z) = V(z)$ [16, 17]. As an example observe that if Φ and $\bar{\Phi}$ are two chiral superfields that $\Phi(z)+\Phi^\dagger(z) \in \mathbb{R}$. We conclude

$\Phi(z) + \Phi^\dagger(z)$ is a real superfield. By definition of the vector superfield the number of degrees of freedom with respect to the general superfield given earlier has halved. The most general vector superfield takes the form:

$$V(x,\theta,\bar{\theta}) = A(x) + \theta B(x) + \bar{\theta}\bar{B}(x) + \theta^2 C(x) + \bar{\theta}^2 \bar{C}(x) \tag{118}$$
$$+ \theta\sigma^\mu\bar{\theta}V_\mu(x) + \theta^2\bar{\theta}\bar{\lambda}(x) \tag{119}$$
$$+ \bar{\theta}^2\theta\lambda(x) + \frac{1}{2}\theta^2\bar{\theta}^2 D(x) \tag{120}$$

Taking in to account reaility properties of the vector superfield we can Count degrees of freedom. There are in total 8 bosonic degrees of freedom from four scalars and four degrees of freedom from the vector. In addition we also find four spinors with each two propagating degrees of freedom. So that we again have an equal number of bosonic and fermionic propagating degrees of freedom. Observe how this superfield contains a vector field V_μ. It is therefore regarded as the vector supermultiplet. In the next step we want to understand the generalisation of a gauge transformation (24) to the vector superfield. For this consider the real superfield $\Phi(z) + \Phi^\dagger$. In component form:

$$\Phi(z) + \Phi^\dagger(z) = \phi(x) + \phi^\dagger(x) + \sqrt{2}\theta\psi(x) + \sqrt{2}\bar{\theta}\bar{\psi(x)} \tag{121}$$
$$+\theta^2 F(x) + \bar{\theta}^2 F^\dagger(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu(\phi(x) - \phi^\dagger(x)) + \frac{i}{\sqrt{2}}\bar{\theta}^2\bar{\sigma}^\mu\theta\partial_\mu\bar{\psi}(x)$$
$$+ \frac{i}{\sqrt{2}}\theta^2\sigma^\mu\bar{\theta}\partial_\mu\psi(x) - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2(\phi(x) + \phi^\dagger(x))$$

We then consider the following transformation:

$$V \to V + \Phi + \Phi^\dagger \tag{122}$$

This induces transformations on the components of the vector superfield V. We are especially interested in the transformation of the vector component. This component transforms as:

$$V_\mu \to V_\mu + i\partial_\mu(\phi(x) - \phi^\dagger(x)) \tag{123}$$

Comparing to (24) we define supersymmetric gauge transformation by (122). So for V a vector superfield the transformation (122) takes the role of a gauge transformation because the vector component of the vector superfield

transforms like an ordinary gauge field.
We therefore talk about $\delta_g V = \Phi + \Phi^\dagger$ as a gauge transformation with gauge parameter the chiral superfield Φ and gauge field V. Like for usual gauge theories, it is possible to choose a possible gauge due to the freedom we have. Looking at the transformations of the component fields under a super gauge transformation, it is possible to set certain fields to zero. This includes what we have called the fields $A(x)$, $B(x)$ and $C(x)$. This choice is called the Wess-Zumino gauge and the vector superfield in Wess-Zumino (WZ) gauge takes the form:

$$V_{WZ}(x,\theta,\bar{\theta}) = \theta\sigma^\mu\bar{\theta}V_\mu(x) + \theta^2\bar{\theta}\lambda(x) + \bar{\theta}^2\theta\bar{\lambda}(x) + \frac{1}{2}\theta^2\bar{\theta}^2 D(x) \tag{124}$$

We note that since the Wess-Zumino vector superfield V_{WZ} has a minimum θ dependence of $\theta\bar{\theta}$ in the vector component that V_{WZ}^2 is the highest power of V_{WZ} that is non-zero, i.e. $V_{WZ}^n = 0$ for $n \geq 3$. This concludes our brief introduction to vector superfields. In the next sections we will start talking more about super gauge invariance and how we can build supersymmetric invariant actions from that. This will ultimately lead to a good description of supersymmetric models in the superspace formalism.

6.2.3 Supersymmetric actions: abelian gauge theory

We now use the superspace formalism to construct manifestly supersymmetric action functionals. These give equations of motion for the component fields. In order to construct supersymmetric actions, invariant under supersymmetry transformations $\delta_{\epsilon,\bar{\epsilon}}$, it is important to recall that any Lagrangian that changes as $\mathcal{L} \to \mathcal{L} + \delta\mathcal{L}$ leaves the action $S = \int dx\, \mathcal{L}$ invariant provided $\delta\mathcal{L} = \partial_\mu(\text{something})^\mu$. This statement follows directly from Stokes' theorem [10] $\int_{\partial\Omega}\omega = \int_\Omega d\omega$ where ω is a differential form on some manifold Ω and $\partial\Omega$ denotes its boundary, provided $\omega|_{\partial\Omega} = 0$. This will be starting point for our first supersymmetric Lagrangian. First we recall that the F-term in the chiral superfield had the transformation (114). For a chiral superfield we will denote its F-term by: $\Phi|_F$. this means that a Lagrangian that is supersymmetric looks like $\mathcal{L} = \Phi|_F$ with corresponding action $S = \int d^4x\, \Phi|_F$. In general the F-terms can be singled out by integrating $\mathcal{L}$ with respect to spacetime and Grassmann variables θ: $S = \int d^4x d^2\theta\, \Phi$. However, if we can take the F-term of a chiral superfield this means that actually any holomorphic[13] function of a chiral superfield $W(\Phi) = a\Phi + b\Phi^2 + c\Phi^3 + \dots$ is a good candidate for a supersymmetric action[17, 13]. This means the action could contain a general term that looks like $S = \int d^4x d^2\theta \, W(\Phi)$. $W(\Phi)$ looks like a ordinary potential term hence the name **superpotential**. The superpotential takes into account interactions between different chiral superfields. We have shown that a product of chiral superfields is also chiral, therefore the superpotential is also a chiral superfield:

$$\bar{D}_{\dot{\alpha}} W(\Phi) = \frac{\partial W(\Phi)}{\partial \Phi} \bar{D}_{\dot{\alpha}} \Phi = 0 \tag{125}$$

Looking at a general superfield and taking the transformations of the components under supersymmetry transformations from appendix B we observe that for a general superfield its $\theta^2\bar{\theta}^2$-term, called the D-term, also transforms as a full derivative. Thus for a general superfield $S(z)$ the term $S|_D$ also leads to a supersymmetric action. The D-term can be obtained by integrating over both Grassmann variables in superspace. In the most general setting any **non**-holomorphic real function $K(\Phi, \Phi^\dagger)$ is a good candidate for generating a D-term. This means the action:

$$S = \int d^4x \, \mathcal{L} = \int d^4x d^2\theta d^2\bar{\theta} \, K(\Phi, \Phi^\dagger) \tag{126}$$

is supersymmetric. The term $K(\Phi, \Phi^\dagger)$ is called the Kähler potential.
In order to include vector fields we can just pick a vector superfield V and take its D-term. This will of course also lead to a supersymmetric action. Therefore we propose our first supersymmetric action in superspace formalism:

$$S = \int d^4x d^2\theta d^2\bar{\theta} \, (K(\Phi_i, \Phi_j^\dagger) + \xi V) + \int d^4x d^2\theta \, (W(\Phi_i) + \text{h.c.}) \tag{127}$$

The second D-term has the name Fayet-Iliopoulos term, or FI-term for short and ξ is a constant. We have also generalised to the case where we have a set of chiral superfields Φ_i. With the eye on a super gauge invariant action it is important to note that after introducing the FI-term this action is not super gauge invariant. We have also not included a kinetic term for the vector superfield V.

[13] i.e. $\frac{\partial}{\partial z} f(z, \bar{z}) = 0 \to f = f(\bar{z})$ for a holomorphic function of variables z and $\bar{z}$.

6.2.4 Supersymmetric gauge theory

We now look for a gauge invariant kinetic term for a vector superfield generalising (27). In order to talk about super gauge transformation recall that we defined a super gauge transformation by considering $V \to V + \Phi + \Phi^\dagger$ with V a vector superfield and Φ a chiral superfield. Starting with the Kähler potential $K(\Phi, \Phi^\dagger) = \Phi^\dagger\Phi$. This is however not super gauge invariant. If we let $\Phi \to \exp(-iq\Lambda)\Phi$ transform under a local $U(1)$ transformation where Λ is a chiral superfield, it is clear that

$$\begin{aligned}\Phi^\dagger\Phi &\to \Phi^\dagger \exp(iq\Lambda^\dagger)\exp(-iq\Lambda)\Phi \\ &= \Phi^\dagger \exp(-iq(\Lambda - \Lambda^\dagger))\Phi\end{aligned} \tag{128}$$

Where we used the fact that Λ is a chiral superfield and is not Lie algebra valued. This would then be the gauge transformation of the Kähler potential, which is clearly not invariant. Note however that the exponential contains exactly the gauge transformation of a gauge field. We therefore introduce the gauge field V and propose the following:

$$\Phi^\dagger\Phi \to \Phi^\dagger \exp(qV)\Phi \tag{129}$$

This is gauge invariant when also $V \to V + i(\Lambda - \Lambda^\dagger)$. We now generalise this observation to non-abelian gauge groups where the chiral superfield Λ takes values in the adjoint representation, i.e. $\Lambda = \Lambda^a T^a$ with T^a generator of the gauge group. Then:

$$\Phi \to \exp(-iq\Lambda^a T^a)\Phi \tag{130}$$

In this case the general super gauge transformation takes the form [16]:

$$V \to V + i(\Lambda - \Lambda^\dagger) - \frac{i}{2}[V, \Lambda + \Lambda^\dagger] + \dots \tag{131}$$

where we have suppressed the gauge indices. This expression, by the Campbell-Baker-Hausdorf formula, corresponds to:

$$\exp(V) \to \exp(-i\Lambda^\dagger)\exp(V)\exp(i\Lambda) \tag{132}$$

It is constructed such that the following expression is super gauge invariant under non-abelian super gauge transformations:

$$\begin{aligned}\Phi^\dagger \exp(V)\Phi &\to \Phi^\dagger \exp(i\Lambda^\dagger)\exp(-i\Lambda^\dagger)\exp(V)\exp(i\Lambda)\exp(-i\Lambda)\Phi \\ &= \Phi^\dagger \exp(V)\Phi\end{aligned} \tag{133}$$

We can therefore write a super gauge invariant Kähler term:

$$K(\Phi, \Phi^\dagger, V) = \Phi^\dagger \exp(qV)\Phi \tag{134}$$

Which is invariant under the combined transformations (130) and (132). In order to make the action from the previous section super gauge invariant the next term to consider is the superpotential. Since $W(\Phi)$ is a holomorphic function it is, by construction, not gauge invariant. Thus gauge invariance requires $W(\Phi) = 0$. Since the exponentials are matrix valued having a set of chiral superfields Φ^i we can construct the transformations $\Phi_i \to \big(\exp(iq\Lambda)\big)^i_j \Phi^j$. It is then attainable to construct a non vanishing superpotential[17, 13, 16].

The final term we encoutered in (127) was the FI-term. This term is only super gauge invariant under abelian transformations because it breaks any non-abelian super gauge invariance. This can be seen from the fact that the D-term of of a chiral superfield is not necessarily zero. In the next part we will omit the gauge charges in the expressions to keep them smaller.

Finally we need to discuss the generalisation of the gauge field strength $F_{\mu\nu}$ in supersymmetric gauge theories. This term is of course necessary as its square provides us with a kinetic term for gauge fields. In the non-supersymmetric case the kinetic term for the gauge field A_μ is given by the Yang-Mills action (27). This term is gauge invariant if $F_{\mu\nu}$ transforms in the adjoint representation of the gauge group G. The extension to the superspace formalism exploits the fact that the field strength is defined as the exterior derivative of the gauge field. For us the gauge field is the vector superfield V. The first thing one could try would be a field strength $D_\alpha V$. However, one quickly realises that these terms do not transform properly under super gauge transformations.

The super gauge field strength G_α is a chiral superfield defined by[14]:

$$G_\alpha = -\frac{1}{8}\bar{D}^2(\exp(-V)D_\alpha \exp(V)) \tag{135}$$

where D_α the covariant derivative from(108). This field strength is also gauge group valued, $G_\alpha = G^a_\alpha T^a$. We will not provide a proof of the fact that G_α is a chiral superfield, but we will check that it transforms like a chiral superfield.

[14]For an alternative definition in terms of (anti)commutators we refer to appendix C.

First we verify:

$$\exp(V)\exp(-V) \to [\exp(-i\Lambda^\dagger)\exp(V)\exp(i\Lambda)] \times [\exp(-i\Lambda)\exp(-V)\exp(i\Lambda^\dagger)] = 1 \tag{136}$$

Then:

$$\begin{aligned} G_\alpha \to & -\frac{1}{8}\bar{D}^2\Big(\exp(-i\Lambda)\exp(-V)\exp(i\Lambda^\dagger)D_\alpha\exp(-i\Lambda^\dagger)\exp(V)\exp(i\Lambda)\Big) \\ &= -\frac{1}{8}\bar{D}^2\Big(\exp(-i\Lambda)\exp(-V)D_\alpha\exp(V)\exp(i\Lambda)\Big) \\ &= -\frac{1}{8}\bar{D}^2\Big(\exp(-i\Lambda)\exp(-V)\big((D_\alpha\exp(V))\exp(i\Lambda)+\exp(V)D_\alpha\exp(i\Lambda)\big)\Big) \\ &= \exp(-i\Lambda)G_\alpha\exp(i\Lambda) - \frac{1}{8}\bar{D}^2\Big(\exp(-i\Lambda)(D_\alpha\exp(i\Lambda))\Big) \end{aligned} \tag{137}$$

where we used that Λ is a chiral superfield so that $\bar{D}_{\dot\beta}\exp(i\Lambda)=0$.

$$\begin{aligned} &\frac{1}{8}\bar{D}^2\Big(\exp(-i\Lambda)(D_\alpha\exp(i\Lambda))\Big) = -\frac{1}{4}\exp(-i\Lambda)\bar{D}_{\dot\beta}\bar{D}^{\dot\beta}(D_\alpha\exp(i\Lambda)) \\ &= -\frac{1}{8}\exp(-i\Lambda)\bar{D}_{\dot\beta}\Big(\bar{D}^{\dot\beta}D_\alpha + D_\alpha\bar{D}^{\dot\beta}\Big)\exp(i\Lambda) \\ &= -\frac{1}{8}\exp(-i\Lambda)\bar{D}_{\dot\beta}\{\bar{D}^{\dot\beta}, D_\alpha\}\exp(i\Lambda) \\ &= 0 \end{aligned} \tag{138}$$

The last line uses that the anticommutator is proportional to the partial derivative and it commutes with $\bar{D}_{\dot\beta}$ We conclude that

$$G_\alpha \to \exp(-i\Lambda)G_\alpha\exp(i\Lambda) \tag{139}$$

This indeed the correct transformation of a chiral superfield in the adjoint representation of the gauge group. This motivates the fact that G_α is indeed a chiral superfield. It is then convenient to also introduce a super gauge covariant derivative ∇_α [16].

$$\nabla_\alpha\Phi = \exp(-V)D_\alpha(\exp(V)\Phi) \tag{140}$$

This is needed since the covariant derivative D_α is not super gauge invariant:

$$D_\alpha\Phi \to D_\alpha(\exp(-i\Lambda)\Phi) = D_\alpha(\exp(-i\Lambda)\Phi + \exp(-i\Lambda)D_\alpha\Phi \tag{141}$$

In general the first term does not vanish and therefore the derivative does not transform covariantly under super gauge transformations. In contrast, the transformation of (140) is.

This derivative will transform like a chiral superfield under super gauge transformations:

$$\begin{aligned}\nabla_\alpha\Phi &\to e^{-i\Lambda}e^{-V}e^{i\Lambda}D_\alpha(e^{-i\Lambda^\dagger}e^{V}e^{i\Lambda}e^{-i\Lambda}\Phi) \\ &\overset{\Lambda\text{ chiral}}{\widehat{=\!=}} e^{-i\Lambda}e^{-V}D_\alpha(e^{V}\Phi) \\ &= e^{-i\Lambda}\nabla_\alpha\Phi\end{aligned} \tag{142}$$

We can also determine the operator $\nabla^2\Phi = \nabla^\alpha\nabla_\alpha\Phi$:

$$\nabla^2\Phi = e^{-V}D^\alpha(e^{V}e^{-V}D_\alpha(e^{V}\Phi)) = e^{-V}D^\alpha D_\alpha(e^{V}\Phi) = e^{-V}D^2(e^{V}\Phi) \tag{143}$$

Since $\nabla_\alpha\Phi$ is chiral this means $\nabla^2\Phi$ is also chiral. So in the end using ∇_α instead of D_α makes super gauge invariance manifest. The super field strength could then be written as:

$$G_\alpha = -\frac{1}{8}\bar{D}^2\nabla_\alpha \tag{144}$$

Since we are after a fully supersymmetric version for a gauge theory we wish to include dynamics for the gauge field V but especially its vector component V_μ. We argue in the following way: since the super field strength is defined by the vector superfield as a gauge field we can write it in components. In Wess-Zumino gauge and conventions from [16] it is given by:

$$G_\alpha = \lambda_\alpha + \theta_\alpha D(x) + \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha F_{\mu\nu} + i\theta^2(\sigma^\mu\nabla_\mu\bar{\lambda})_\alpha \tag{145}$$

where ∇_μ is the gauge covariant derivative given by the usual expression $\nabla_\mu = \partial_\mu - igV_\mu$ with V_μ the vector component of the super gauge field V. Observe the presence of the ordinary field strength defined by $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + ig[V_\mu, V_\nu]$. This leads us to think that $G^\alpha G_\alpha$ will contain at least the term $F_{\mu\nu}F^{\mu\nu}$ and is thus a suitable part for the action of a gauge field. Using these insights we can develop a fully supersymmetric and super gauge invariant action in $\mathcal{N} = 1$ superspace.

6.2.5 Supersymmetric actions 2: non-abelian gauge theory

Let us go back to (127) and implement all the adjustments that we have just made in order to promote this in to a non-abelian gauge theory. Recall from the previous section where we have argued that the Kähler potential should take the form $K(\Phi, \Phi^\dagger, V) = \Phi^\dagger e^V \Phi$ equipped with the non-abelian transformation laws (130) and (132). The next term was the superpotential which we just copy, $W(\Phi_i)$ and add its conjugate to keep the Lagrangian real. The FI-term breaks non-abelian gauge symmetry and needs to vanish. Finally we have to include a kinetic term for the gauge field V. Following the arguments from quantum field theory a kinetic term for a gauge field looks like $\mathcal{L} \sim F^2 = (dA)^2$. Using (145) we conclude that $\text{Tr}(G^\alpha G_\alpha)$ will contain the Yang-Mills action (27).

We therefore write down the following part of the Lagrangian: $\mathcal{L} \sim \text{Tr}(G^\alpha G_\alpha)|_F$. We take the F-term because the way that we have defined G_α allows us to actually take the $\bar{D}^2$ and absorb it in a integration over $\bar{\theta}$, since integration coincides with differentiation. The term is therefore actually a D-term. Observe how super gauge invariance is built in by the cyclic property of traces. The full action is then:

$$S = \int d^4x d^2\theta d^2\bar{\theta}\ \Phi_i^\dagger (e^V)^i_j \Phi^j + \int d^4x d^2\theta\ (W(\Phi_i) + \text{h.c.}) + \int d^4x d^2\theta\ (\text{Tr}(G^\alpha G_\alpha) + \text{h.c.}) \tag{146}$$

To finish this section let us consider the action:

$$S = \frac{1}{4}\int d^4x\ d^2\theta\ \text{Tr}(G^\alpha G_\alpha) + \text{h.c.} \tag{147}$$

where we have included the factor 1/4 by convention. In components this will take the form [16]:

$$S = \int d^4x\ \text{Tr}(\frac{1}{2}D^2(x) + i\lambda\bar{\sigma^\mu}\nabla_\mu\bar{\lambda} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{8}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}) \tag{148}$$

We note the similarity to ordinary gauge theories with one gauge field V_μ coupled to a fermion λ. The D field is an auxilliary field that is not dynamical and can be integrated out by its equations of motion. This will is the step up to Yang-Mills theory.

The last generalization that can be made is by realising that in ordinary gauge theory there sometimes is a gauge coupling in front of the field strength squared term. This term is ambiguous in the sense that it can be absorbed in the definion of the gauge field or put it in front of the kinetic term. In the rest of the thesis we choose to put it in front of the kinetic term. To achieve this one may include a factor κ in front of the trace part. This function can in principle be anything. To recover the well-known supersymmetric Yang-Mills theory one has $\kappa = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$ with g the Yang-Mills coupling and θ the so called θ-parameter. For a theory with just one chiral super field we arrive at the so called $\mathcal{N} = 1$ supersymmetric Yang-Mills action:

$$S_{SYM}^{\mathcal{N}=1}[\Phi, \bar{\Phi}, V] = \frac{1}{g^2} \int d^8 z \ \mathrm{Tr}(\bar{\Phi} e^V \Phi) + \frac{1}{2g^2} \int d^6 z \ \mathrm{Tr}(G^\alpha G_\alpha) \tag{149}$$

where now every field belongs to the adjoint representation of the gauge group and we adopted the notation (91). After we have introduced a supersymmetric way of doing field theory we will now turn to supersymmetric Yang-Mills theory.

7 Supersymmetric Yang-Mills theory

In this section we will come in contact with the famous $\mathcal{N} = 4$ SYM theory. Our goal is to find the β-function for a supersymmetric Yang-Mills theory using functional renormalisation group approach explained in section 3. We start with the $\mathcal{N} = 1$ SYM theory in $\mathcal{N} = 1$ superspace formalism and then apply the gauge fixing. Using the Wetterich equation (13) will yield the β-function for the gauge coupling g in $\mathcal{N} = 1$ and $\mathcal{N} = 4$ supersymmetry. The calculation presented closely follows follow [2, 3].

7.1 The super Yang-Mills action

We first present the full $\mathcal{N} = 1$ SYM action (149) where the gauge field is coupled to a chiral superfield Φ. The the action is:

$$S_{\mathrm{SYM}}^{\mathcal{N}=1}[\Phi, \bar{\Phi}, V_s] = \frac{1}{g^2} \int d^8 z \ \mathrm{tr}(\bar{\Phi} e^{V_s} \Phi) + \frac{1}{2g^2} \int d^6 z \ \mathrm{tr}(G^\alpha G_\alpha) \tag{150}$$

Here Φ and $\bar{\Phi}$ are chiral superfields, V_s the vector superfield and G^α is the super field strength (135). Furthermore we shall use $[T^a, T^b] = if^{abc}T^c$, with f^{abc} antisymmetric structure constants of the gauge group G. All fields in a vector supermultiplet transform in the adjoint representation [16], so that the chiral superfield, vector superfield and the field strength transform equally under gauge transformations. The generators of the gauge group are normalised as $\mathrm{tr}(T^a T^b) = \delta^{ab}$ as in [2]. The action has an obvious $U(1)$ R-symmetry and we have already used the fact that [2]:

$$\int d^6 z \ \mathrm{tr}(G^\alpha G_\alpha) = \int d^6 \bar{z} \ \mathrm{tr}(\bar{G}_{\dot{\alpha}} \bar{G}^{\dot{\alpha}}) \tag{151}$$

In order to proceed we will first consider the pure Yang-Mills sector i.e. the second term of the action and finally the chiral sector which is a lot easier to deal with.

7.1.1 Pure Yang-Mills

Let us begin with the Yang-Mills action:

$$S_{\mathrm{YM}} = \frac{1}{2g^2} \int d^6 z \ \mathrm{tr}(G^\alpha G_\alpha) \tag{152}$$

In this thesis we will work with the **background field** method. This means that the gauge field is decomposed in a background and fluctuating part. We split the full vector superfield V_s according to

$$e^{V_s} = e^{\Omega} e^{V} e^{\bar{\Omega}} \tag{153}$$

with Ω and $\bar{\Omega}$ background superfields and V the quantum fluctuation gauge field. We will only use gauge covariant derivatives (140). These allow us to introduce background covariant derivatives. By replacing the full vector superfield V_s by the fluctuation field V we introduce the full- and background derivatives given in the background chiral representation [3, 2]

$$\mathcal{D}_\alpha = e^{-V_s} D_\alpha e^{V_s} = e^{-\bar{\Omega}} e^{-V} \nabla_\alpha e^{V} e^{\bar{\Omega}} \tag{154}$$

$$\bar{\mathcal{D}}_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}} = e^{-\bar{\Omega}} \bar{\nabla}_{\dot{\alpha}} e^{\bar{\Omega}} \tag{155}$$

being the full super gauge covariant derivatives whereas,

$$\nabla_\alpha = e^{-\Omega} D_\alpha e^{\Omega} \tag{156}$$

$$\bar{\nabla}_{\dot{\alpha}} = e^{\bar{\Omega}} \bar{D}_{\dot{\alpha}} e^{-\bar{\Omega}} \tag{157}$$

are the background gauge covariant derivatives. Using these we will now construct the second variation $S^{(2)}$ for each term of the action that is involved. We will need the action up to second order in the fluctuation fields as introduced in section 3.

The algebra in appendix C is used to show that the full super field strength is given by:

$$G_\alpha = \frac{1}{8}[\bar{\mathcal{D}}^{\dot{\alpha}}, \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}\}] \tag{158}$$

Analogously we define the background field strength

$$W_\alpha = \frac{1}{8}[\bar{\nabla}^{\dot{\alpha}}, \{\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}\}] \tag{159}$$

We expand the full field strength G_α using above definitions and the D-algebra from appendix C:

$$G_\alpha = \frac{1}{8}[\bar{\mathcal{D}}^{\dot{\alpha}}, \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}\}] = e^{-\bar{\Omega}}(W_\alpha + \frac{1}{8}\bar{\nabla}^2 e^{-V}\nabla_\alpha e^V)e^{\bar{\Omega}} \tag{160}$$

It is then straightforward to rewrite the action in terms of the background field strength:

$$\begin{aligned} S_{\text{YM}} &= \frac{1}{2g^2}\int d^6z \ \text{tr}(G^\alpha G_\alpha) \qquad (161)\\ &= \frac{1}{2g^2}\int d^6z \ \text{tr}((W^\alpha + \frac{1}{8}\bar{\nabla}^2 e^{-V}\nabla^\alpha e^V)(W_\alpha + \frac{1}{8}\bar{\nabla}^2 e^{-V}\nabla_\alpha e^V)) \\ &= S_{\text{Background}} + S_{\text{Fluctuation}} \end{aligned}$$

with:

$$S_{\text{Background}} = \frac{1}{g^2}\int d^6z \ \frac{1}{2}\text{tr}(W^\alpha W_\alpha) \tag{162}$$

$$\begin{aligned} S_{\text{Fluctuation}} &= -\frac{1}{g^2}\int d^8z \ \text{tr}(\frac{1}{2}W^\alpha(e^{-V}\nabla_\alpha e^V) \qquad (163)\\ &+ \frac{1}{32}(e^{-V}\nabla^\alpha e^V)\bar{\nabla}^2(e^{-V}\nabla_\alpha e^V)) \end{aligned}$$

Here we used the cyclicity of the trace and the differentiation property on superspace $\int d^2\bar{\theta} = -\frac{1}{4}\bar{\nabla}^2$ so that we can lift the expression to an integral over full superspace.

In order to continue we fix the gauge to Wess-Zumino as in (124), such that $V^n = 0$ for $n > 2$. We also will omit terms linear in V since these are irrelevant when computing the flow of g. We are now going to expand (163) up to second order in V. In order to do so recall that $e^{\pm V} = 1 \pm V + \frac{1}{2}V^2$. Therefore:

$$\begin{aligned} e^{-V}\nabla_\alpha e^V &= (1 - V + \frac{1}{2}V^2)\nabla_\alpha(1 + V + \frac{1}{2}V^2) \\ &= \nabla_\alpha V + \frac{1}{2}\nabla_\alpha V^2 - V\nabla_\alpha V \\ &= \nabla_\alpha V - \frac{1}{2}[V, \nabla_\alpha V] \end{aligned} \tag{164}$$

Plugging this right into the action yields:

$$\begin{aligned} S_{\text{Fluctuation}} &= -\frac{1}{g^2}\int d^8z \ \mathrm{tr}\Big(\frac{1}{2}W^\alpha(\nabla_\alpha V - \frac{1}{2}[V, \nabla_\alpha V]) \\ &+ \frac{1}{32}(\nabla^\alpha V - \frac{1}{2}[V, \nabla^\alpha V])\bar{\nabla}^2(\nabla_\alpha V - \frac{1}{2}[V, \nabla_\alpha V])\Big) \\ &= \frac{1}{g^2}\int d^8z \ \mathrm{tr}\Big(\frac{1}{4}W^\alpha[V, \nabla_\alpha V] \\ &- \frac{1}{32}(\nabla^\alpha V)\bar{\nabla}^2(\nabla_\alpha V)\Big) \end{aligned} \tag{165}$$

Keeping in mind that all fields are Lie algebra valued we can remove the commutators by writing the fields as vectors. Furthermore, using partial integration we can rewrite this action in a nicer form. This also removes the trace part over the gauge indices, these are contracted with the generators of the Lie group [2]:

$$\begin{aligned} S_{\text{Fluctuation}} &= \frac{1}{g^2}\int d^8z(\frac{1}{4}VW^\alpha\nabla_\alpha V) + \frac{1}{32}V(\nabla^\alpha\bar{\nabla}^2\nabla_\alpha V) \\ &= \frac{1}{2g^2}\int d^8z \ V(\frac{1}{2}W^\alpha\nabla_\alpha + \frac{1}{16}\nabla^\alpha\bar{\nabla}^2\nabla_\alpha)V \\ &= \frac{1}{32g^2}\int d^8z \ V(8W^\alpha\nabla_\alpha + \nabla^\alpha\bar{\nabla}^2\nabla_\alpha)V \end{aligned} \tag{166}$$

This is our final result for the quadratic action for the vector superfield. We will continue with the gauge fixing procedure.

7.2 Gauge fixing and ghosts

As discussed in section 4.2 we need a gauge fixing condition in order to restrict the path integral over physical inequivalent configurations. Recall that the action for a pure gauge theory is indeed gauge invariant under local super gauge transformations. We will therefore copy the procedure explained in section 4.2 and start by writing down the gauge fixing action as used by [14, 2, 3].

$$S_{\text{Gauge fixing}} = \frac{1}{32\alpha g^2} \int d^8z\ V\{\nabla^2, \bar{\nabla}^2\}V \tag{167}$$

With α a gauge parameter that can be chosen for later convenience. The gauge fixing condition is [14] $\nabla^2 V$ and $\bar{\nabla}^2 V = 0$. We can now without ghost sector complete the calculation of the full gauge action from the previous section. We will use the definition of the super d'Alembertian[2]:

$$\Box_V \equiv \nabla^a \nabla_a - W^\alpha \nabla_\alpha + W_{\dot{\alpha}} \bar{\nabla}^{\dot{\alpha}} \tag{168}$$

$$= -\frac{1}{8}\nabla^\alpha \bar{\nabla}^2 \nabla_\alpha + \frac{1}{16}\{\nabla^2, \bar{\nabla}^2\} - W^\alpha \nabla_\alpha - \frac{1}{2}\nabla^\alpha W_\alpha \tag{169}$$

where we used appendix C. The last term vanishes since it is the equation of motion for the super field strength[2, 14]. Combining the fluctuation action together with the gauge fixing action yields the full gauge action:

$$S_{\text{Full}} = \frac{1}{32g^2} \int d^8z\ V(8W^\alpha \nabla_\alpha + \nabla^\alpha \bar{\nabla}^2 \nabla_\alpha)V \tag{170}$$

$$+ \frac{1}{32\alpha g^2} \int d^8z\ V\{\nabla^2, \bar{\nabla}^2\}V \tag{171}$$

$$= \frac{1}{32g^2} \int d^8z\ V(-8\Box_V + (\frac{1}{2} + \frac{1}{\alpha})\{\nabla^2, \bar{\nabla}^2\})V \tag{172}$$

It is convenient choice to take $\alpha = -2$ which corresponds to a Fermi-Feynman gauge [2]. We will adopt this choice and find:

$$S_{\text{Full}} = -\frac{1}{4g^2} \int d^8z\ V\Box_V V \tag{173}$$

When implementing the gauge fixing condition, we get a Jacobian factor from the δ-function known as the Faddeev-Popov determinant (30). Let us call the gauge fixing function $F = \bar{\nabla}^2 V$. The Faddeev-Popov determinant Δ_{FP}

is then given by the following expression consisting of four Faddeev-Popov ghosts $c, c', \bar{c}, \bar{c}'$:

$$\Delta_{FP} = \int \mathcal{D}c\mathcal{D}c'\mathcal{D}\bar{c}\mathcal{D}\bar{c}' \exp\Big(\int d^6z \,\mathrm{tr}(c'(\frac{\delta F}{\delta c}c + \frac{\delta F}{\delta \bar{c}}\bar{c})) + \int d^6\bar{z}\,\mathrm{tr}(\bar{c}'(\frac{\delta \bar{F}}{\delta c}c + \frac{\delta \bar{F}}{\delta \bar{c}}\bar{c}))\Big) \tag{174}$$

with the Faddeev-Popov ghosts being the chiral superfields satisfying $\bar{\nabla}_{\dot{\alpha}}c = \nabla_\alpha \bar{c} = 0$ acting as gauge parameters. The gauge fixing condition $\bar{\nabla}^2 V = 0$ depends on the background fields Ω and $\bar{\Omega}$ (156). This phenomenon we have not encoutered earlier in the usual Yang-Mills sector but we have to take care of this time. We need to introduce a second kind of ghosts called the Nielsen-Kallosh ghost[2, 14] b and $\bar{b}$. Therefore weighing the whole path integral by an extra contribution with:

$$S_{\mathrm{NK}} = -\frac{1}{g^2}\int d^8z\ \mathrm{tr}(\bar{b}b) \tag{175}$$

Let us now continue with the Faddeev-Popov sector to find the action of the Faddeev-Popov ghosts. Using the gauge fixing condition together with the transformation rule of a vector superfield (132) one finds that:

$$\frac{\delta F}{\delta c}c = \frac{\delta F}{\delta \bar{c}}\bar{c} = i\bar{\nabla}^2\delta V \tag{176}$$

$$\frac{\delta \bar{F}}{\delta c}c = \frac{\delta \bar{F}}{\delta \bar{c}}\bar{c} = i\nabla^2\delta V \tag{177}$$

where δV has to be determined. From the transformation rule $e^V \to e^{-i\bar{c}}e^V e^{ic}$ we can derive that [2]:

$$\delta V = V' - V \tag{178}$$

$$= \mathfrak{L}_{\frac{V}{2}}(-i(\bar{c}+c) + i\coth \mathfrak{L}_{\frac{V}{2}}(\bar{c}-c)) \tag{179}$$

with $\mathfrak{L}_{\frac{V}{2}}X$ denoting the Lie-derivative of X defined by $\mathfrak{L}_{\frac{V}{2}}X = [\frac{V}{2}, X]$ and $\coth(X)$ is understood as its power series. Observe that this expansion terminates by itself using the Wess-Zumino gauge.

We can now rewrite the Faddeev-Popov determinant entirely. First we use the expression of the gauge fixing function F to push the covariant derivatives into the integral so both become a full superspace integral. The factor δV is common in every term so we get modulo constants:

$$\Delta_{FP} = \int \mathcal{D}c\mathcal{D}c'\mathcal{D}\bar{c}\mathcal{D}\bar{c}' \exp\Big(-\int d^8z\,\mathrm{tr}((c'+\bar{c}')\mathfrak{L}_{\frac{V}{2}}((\bar{c}+c) - \coth\mathfrak{L}_{\frac{V}{2}}(\bar{c}-c)))\Big) \tag{180}$$

So that finally:

$$S_{\text{Ghost}} = -\frac{1}{g^2}\int d^8z\ \text{tr}((c' + \bar{c}')\mathfrak{L}_{\frac{V}{2}}((\bar{c}+c) - \coth \mathfrak{L}_{\frac{V}{2}}(\bar{c}-c) + \bar{b}b)) \qquad (181)$$

This can be put in a much more compact form by writing out the power series for $\coth(x)$ which is given by: $\coth(x) = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45}\ldots$. The Faddeev-Popov term turns out to be just $(c' + \bar{c}') \cdot (c - \bar{c})$. Writing out the brackets yields:

$$-\int d^8z\ \text{tr}((c' + \bar{c}') \cdot (c - \bar{c})) = -\int d^8z\ \text{tr}(c'c - c'\bar{c} + \bar{c}'c - \bar{c}'\bar{c}) \qquad (182)$$

Making use of the fact that the ghosts are in fact background chiral under ∇ and $\bar{\nabla}$ i.e. $\bar{\nabla}c', c = 0$. We arrive at the final ghost action:

$$S_{\text{Ghost}} = -\frac{1}{g^2}\int d^8z\ \text{tr}(\bar{c}'c - c'\bar{c} + \bar{b}b) \qquad (183)$$

$\frac{1}{g^2}$ in front of the Faddeev-Popov action has been included for later convenience. In the next section we will introduce the action for a general chiral superfield and derive its second functional variation. We can then apply this procedure to the above ghost action since they are chiral.

7.2.1 The chiral super field

Let us consider the Kähler kinetic term of background chiral superfields Φ and $\bar{\Phi}$, satisfying $\nabla_\alpha\bar{\Phi} = 0$ and the same for Φ. The action looks rather simple and already has the form "Field × operator × Field" which will make it easy to take the second functional variation needed for the Wetterich equation. The $\mathcal{N} = 1$ Kähler action for a chiral superfield is given by (134).

$$S_{\text{Chiral}}[\Phi, \Phi, V] = \frac{1}{g^2}\int d^8z\ \text{tr}_{\mathcal{R}}(\bar{\Phi}e^V\Phi) \qquad (184)$$

In order to compute its second functional variation we observe that the operator left will be off-diagonal. Using the definition of functional variation of a chiral super field[2, 14]:

$$\frac{\delta\Phi(z)}{\delta\Phi(z')} = \frac{1}{4}\bar{\nabla}^2\delta^{(8)}(z, z') \qquad (185)$$

with $\delta^{(8)}(z,z') = \delta^{(4)}(x,x')\delta^{(2)}(\theta,\theta')\delta^{(2)}(\bar{\theta},\bar{\theta}')$ the Dirac δ-function on superspace. Then we can extract the second functional variation of the chiral action:

$$S^{(2)}_{\text{Chiral}}[\Phi] = \frac{1}{16g^2}\nabla^2\bar{\nabla}^2 \tag{186}$$

Here we have included the trace over gauge indices and have set all fluctuation values to zero. Observe how we can simply use this functional variation for the ghosts since (184) at $V=0$ corresponds to form of (183). We now have computed all terms that enters the effective average action $\Gamma_k[\Phi,\bar{\Phi},V]$.

7.3 The $\mathcal{N}=1$ SYM β-function

To continue the calculation we now plug in every part of the Wetterich equation. Recall that the full action at this point is given by the following action up to second order in the fluctuation field V:

$$S_{\text{Total}} = S_{\text{Full}} + S_{\text{Ghost}} + S_{\text{Background}} + S_{\text{Chiral}} \tag{187}$$

The ansatz for the effective average action $\Gamma_k[\Phi,\bar{\Phi},V]$ is then analogously:

$$\begin{aligned}\Gamma_k^{\text{Total}}[\Phi,\bar{\Phi},c',\bar{c}',V] &= \Gamma_k^{\text{Full}}[\Phi,\bar{\Phi},V] + \Gamma_k^{\text{Ghost}}[c',\bar{c}',b] \\ &+ \Gamma_k^{\text{Chiral}}[\Phi,\bar{\Phi},V] + \Gamma_k^{\text{Background}}\end{aligned} \tag{188}$$

where we promoted the coupling constant g to be k-dependent. We use a renormalisation constant Z_k. g_k is then given by:

$$g_k^2 \equiv \frac{g^2}{Z_k} \tag{189}$$

Furthermore we introduce the anomalous dimension defined by:

$$\eta \equiv -\frac{\partial_t Z_k}{Z_k} \tag{190}$$

The β-function is then given by:

$$\beta(g_k^2) = \partial_t g_k^2 = g^2\partial_t\frac{1}{Z_k} = \eta g_k^2 \tag{191}$$

The Wetterich equation requires the second functional variation with respect to each field. In order to compute this we first take a look at each part of the above effective average action.

7.3.1 The pure Yang-Mills contribution

The first contribution that we derived was the vector superfield contribution including the gauge fixing action. We found that:

$$S_{\text{Full}} = -\frac{1}{4g^2} \int d^8z \; V \Box_V V \tag{192}$$

We can turn this in to the effective action by using the mean fields v:

$$\Gamma_k^{\text{Full}}[\Phi, \bar{\Phi}, V] = -\frac{1}{4g_k^2} \int d^8z \; v \Box_V v = -\frac{Z_k}{4g^2} \int d^8z \; v \Box_V v \tag{193}$$

The second functional variation with respect to the fluctuation field v is given by:

$$\Gamma_k^{\text{Full},(2)} = -\frac{Z_k}{2g^2} \Box_V \tag{194}$$

The relevant operator will be simply $\Box_V$. Next will be the contribution from the ghost fields.

7.3.2 The ghost contribution

We derived the ghost action for the six ghosts. Four of which are Faddeev-Popov and two Nielsen-Kallosh ghosts. Using the ghosts anti-commutativity and the general result for a chiral super field it is straightforward to see that after also defining the mean fields for the ghosts that the effective action becomes:

$$\Gamma_k^{\text{Ghost}}[c', \bar{c}', b] = -\frac{Z_k}{g^2} \int d^8z \; \text{tr}(\bar{c}' c - c' \bar{c} + \bar{b} b) \tag{195}$$

Every term in this action contributes the same factor to the second functional variation and thus the second functional variation for each term reads:

$$\Gamma_k^{\text{Ghost},(2)} = -\frac{Z_k}{16g^2} \nabla^2 \bar{\nabla}^2 \tag{196}$$

Keep in mind though that the action is off-diagonal and we therefore account for every term because of this. This will be important when we write out the trace contributions.

7.3.3 The chiral super field contribution

For the inclusion of the chiral superfield action we have already seen how to find the second functional variation, even with a non trivial Kähler potential. We found (186):

$$\Gamma_k^{\mathrm{Chiral},(2)} = \frac{Z_k}{16g^2}\nabla^2\bar{\nabla}^2 \tag{197}$$

This is all we will need from the actions. In the next sections we will use heat kernel techniques to calculate the traces of the operators needed for the Wetterich equation.

7.3.4 The computation of the traces

We now use the Wetterich equation (13) to compute the β-function for g_k. Techniques introduced in section 3 allow us to calculate the left-hand-side of the Wetterich equation and then project onto corresponding operators on the right-hand-side evaluated at zero fluctuation. Our ansatz for Γ_k from (188) decomposes in to a vector, ghost and chiral sector.

$$\begin{aligned}\partial_t\Gamma_k^{\mathrm{Total}}[\Phi,\bar{\Phi},c',\bar{c}',V] = {} & \frac{1}{2}\mathrm{Tr}\Big((\Gamma_k^{\mathrm{Vector},(2)} + R_{k,\mathrm{Vector}})^{-1}\partial_t R_{k,\mathrm{Vector}}\Big) \\ & - 3\mathrm{Tr}\Big((\Gamma_k^{\mathrm{Ghost},(2)} + R_{k,\mathrm{Ghost}})^{-1}\partial_t R_{k,\mathrm{Ghost}}\Big) \\ & + N_c\mathrm{Tr}\Big((\Gamma_k^{\mathrm{Chiral},(2)} + R_{k,\mathrm{Chiral}})^{-1}\partial_t R_{k,\mathrm{Chiral}}\Big)\end{aligned} \tag{198}$$

N_c denotes the number of chiral superfields in the theory. For $N_c = 0$ we get pure $\mathcal{N} = 1$ SYM theory. By increasing the number of chiral superfields we can in principle increase the amount of supersymmetry because we change the field content of the theory. It is possible to make the theory $\mathcal{N} = 2$ or $\mathcal{N} = 4$ supersymmetric by choosing only N_c appropriately since we have already fixed the coupling constant in front of the chirals to be the same as the coupling in front of the pure gauge action. The factors 3 and N_c arise because we have a total of 6 ghosts that have an antisymmetric action and therefore $6 \times \frac{1}{2} = 3$ and the same argument holds for the chiral superfields. The factor is $\frac{N_c}{2}$ but gets multiplied by 2 because the chirals are also complex fields. The trace includes integration over full superspace and gauge indices. In the last step of the calculation this will yield an extra factor of N, the size of the adjoint representation of $SU(N)$.

Keep in mind that the right hand side of the equation is calculated with a truncation where fluctuations ~ 0 i.e. $v = 0$. We therefore need to take the same truncation at the left hand side of the equation which is a choice one can make in order to project onto the correct terms that we will find in the trace calculation. This corresponds to setting v, the ghosts and the chirals to zero. One is left with a left hand side of just the background action.

$$\Gamma_k^{\text{Background}} = \frac{Z_k}{2g^2} \int d^6z \; \text{tr}(W^\alpha W_\alpha) \tag{199}$$

Therefore, the left-hand-side reads:

$$\partial_t \Gamma_k^{\text{Background}} = \frac{\partial_t Z_k}{2g^2} \int d^6z \; \text{tr}(W^\alpha W_\alpha) \tag{200}$$

The β-function for g_k is obtained by extracting this kinetic term from the right-hand-side as well. One then compares coefficients on both sides.

The next step will choose regulators R_k for each field. It is of great importance to observe that the kinetic operators for the ghosts actually coincide with the chirals. Their regulators will therefore be chosen to be the same. The kinetic operator for the vector is $\Box_V$ so this will have a different operator trace. We pick the exponential regulator (11):

$$R^{(0)}(r) = \frac{r}{\exp r - 1} \tag{201}$$

with $r = \Box_V$ for the vector superfield or $r = \nabla^2 \bar{\nabla}^2$ for the chiral superfield. Explicitly,

$$\begin{aligned} R_{k,\text{Vector}} &= \frac{Z_k}{2g^2} k^2 R^{(0)}\Big(\frac{-\Box_V}{k^2}\Big) \\ R_{k,\text{Ghost}} &= -\frac{Z_k}{16g^2} k^2 R^{(0)}\Big(\frac{\nabla^2 \bar{\nabla}^2}{k^2}\Big) \\ R_{k,\text{Chiral}} &= \frac{Z_k}{16g^2} k^2 R^{(0)}\Big(\frac{\nabla^2 \bar{\nabla}^2}{k^2}\Big) \end{aligned} \tag{202}$$

The couplings are included such that they cancel inside the trace against the factors in the second variations (194)(196) (186), then only Z_k is left. The minus sign in front of the ghost regulator is included because of the sign of (196). Substituting all results, the right-hand-side of the Wetterich equation is:

$$\partial_t \Gamma_k^{\text{Total}}[\Phi, \bar{\Phi}, c', \bar{c}', V] = \frac{1}{2}\text{Tr}\Big[\Big(-\Box_V + k^2 R^{(0)}\Big(\frac{-\Box_V}{k^2}\Big)\Big)^{-1}\Big(\frac{\partial_t (Z_k k^2 R^{(0)}\big(\frac{-\Box_V}{k^2}\big))}{Z_k}\Big)\Big] \tag{203}$$

$$+ (N_c - 3)\text{Tr}\Big[\Big(\nabla^2\bar{\nabla}^2 + k^2 R^{(0)}\Big(\frac{\nabla^2\bar{\nabla}^2}{k^2}\Big)\Big)^{-1}\Big(\frac{\partial_t (Z_k k^2 R^{(0)}\big(\frac{\nabla^2\bar{\nabla}^2}{k^2}\big))}{Z_k}\Big)\Big]$$

We now calculate the traces on the right hand side of the Wetterich equation using the heat kernel techniques [2].
Let us start by introducing the heat kernel to find the traces. We define:

$$W(r) \equiv (\Gamma^{(2)}(r) + R_k(r))^{-1}\partial_t R_k(r) \tag{204}$$

with r as in (201). We are therefore generally interested in finding $\text{Tr}[W(r)]$. The Fourier transform takes the form $W(r) = \int ds\ \tilde{W}(s)e^{-isr}$. To find the trace of $W(r)$ we thus need to find the trace of e^{-isr}. One defines the kernel $K(s) \equiv e^{-isr}$ and its expansion in Seeley-deWitt coefficients[15][3]:

$$i\text{Tr}(K(s)) = \int d^8 z \sum_{n=0}^{\infty} \text{Tr}(a_n(z))(is)^{\frac{n-4}{2}} \tag{205}$$

where the $a_n(z)$ are real but furthermore unconstrained super fields. Conventions may vary between literature but we adopt this expansion. Therefore:

$$\text{Tr}[W(r)] = \frac{1}{i}\int ds\ \tilde{W}(s) \int d^8 z \sum_{n=0}^{\infty} \text{Tr}(a_n(z))(is)^{\frac{n-4}{2}} \tag{206}$$

In principle it is our task now to expand the kernel $K(s)$ with the exponential and the corresponding operator. However the calculations have been done and we will cite the results in this thesis for both operators $\Box_V$ and $\nabla^2\bar{\nabla}^2$. The first contribution that we wish to calculate is the one with $r = \Box_V$. It is shown in [2] that the first non-vanishing coefficient in the heat kernel expansion using our convention is actually $a_8(z)$. However, this coefficient

[15] In four space time dimensions

contributes a factor of $W^2\bar{W}^2$ to the action and is not part of our truncation; which only has W^2 terms. It thus does not contribute to the flow of g_k. This in turn means that the whole vector part will not contribute to the flow equation.
The next trace we need to consider is the ghost and chiral contribution with $r = \nabla^2\bar{\nabla}^2$. One can show that the lowest contribution is of order $\mathcal{O}(s^0)$ which corresponds to Seeley-deWitt coeffecient $a_4(z)$. We find therefore that:

$$\mathrm{Tr}(W(\nabla^2\bar{\nabla}^2)) = \int ds\ \tilde{W}(s) \int d^8z\ \mathrm{Tr}(a_4(z)) + \dots \tag{207}$$

The coefficient $a_4(\nabla^2\bar{\nabla}^2)$, is given by [2, 3]:

$$\int d^8z\ \mathrm{Tr}(a_4(\nabla^2\bar{\nabla}^2)) = \frac{i}{2}\frac{N}{16\pi^2}\int d^6z\ \mathrm{tr}(W^\alpha W_\alpha) \tag{208}$$

where the factor N comes from the functional trace over $a_4(z)$. It remains to show what $\int ds\ \tilde{W}(s)$ is. We have that $W(r) = \int ds\ \tilde{W}(s)e^{-isr}$. Thus,

$$\int ds\ \tilde{W}(s) = W(0) \tag{209}$$

and we finally arrive at:

$$\mathrm{Tr}(W(\nabla^2\bar{\nabla}^2)) = \frac{N}{2}\frac{1}{16\pi^2}W(0)\int d^6z\ \mathrm{tr}(W^\alpha W_\alpha) \tag{210}$$

Which is exactly the projection which we needed.

7.3.5 The result

Before we go further we need to calculate $W(0)$:

$$\begin{aligned} W(0) &= W(r=0) \\ &= (R_k(0))^{-1}\partial_t R_k(0) = (k^2R^{(0)}(0))^{-1}\frac{\partial_t(k^2Z_kR^{(0)}(0))}{Z_k} \\ &= \frac{1}{k^2R^{(0)}(0)Z_k}\Big(2k^2Z_kR^{(0)}(0) + k^2R^{(0)}(0)\partial_t Z_k + k^2Z_k\partial_t R^{(0)}(0)\Big) \\ &= (2-\eta) \end{aligned} \tag{211}$$

where we used that $\partial_t = k\partial_k$, $R^{(0)}(0) = 1$ and the anomalous dimension from (190). Note that this result is completely independent of the choice of the regulator. We can now extract η by comparing the coefficients in front of the operator $\int d^6z\ \mathrm{tr}(W^\alpha W_\alpha)$ appearing on the left- and right-hand-side of the Wetterich equation. Plugging (211) and (210) in to (203) gives:

$$\frac{\partial_t Z_k}{g^2} = (N_c - 3)N\frac{1}{16\pi^2}(2-\eta) \tag{212}$$

On the left hand side it is possible to construct η by dividing both sides by Z_k and using the definition of $g_k^2 = \frac{g^2}{Z_k}$:

$$\eta = (3 - N_c)\frac{Ng_k^2}{16\pi^2}(2-\eta) \tag{213}$$

Solving the above equation for η then yields:

$$\eta = \frac{2N(3-N_c)}{16\pi^2}\frac{g_k^2}{1+\frac{Ng_k^2(3-N_c)}{16\pi^2}} \tag{214}$$

Using (191) we arrive at the expression for the β-function for g_k:

$$\beta(g_k^2) = \frac{N(3-N_c)}{8\pi^2}\frac{g_k^4}{1+\frac{Ng_k^2(3-N_c)}{16\pi^2}} \tag{215}$$

This is our main result and agrees with literature sources [3]. Let us now discuss the properties by considering the case of $\mathcal{N} = 1$ SYM theory without chiral superfields i.e. $N_c = 0$. The β-function becomes:

$$\beta(g_k^2)_{\mathcal{N}=1} = \frac{3N}{8\pi^2}\frac{g_k^4}{1+\frac{3Ng_k^2}{16\pi^2}} \tag{216}$$

This is in agreement with [3]. We observe that the theory has only a trivial fixed point $\beta(g_\star) = 0$ for $g_\star = 0$. Assuming g_k is very small the β-function up to order $\mathcal{O}(g_k^4)$ becomes:

$$\beta(g_k^2)_{\mathcal{N}=1} = \frac{3N}{8\pi^2}g_k^4 + \mathcal{O}(g_k^6) \tag{217}$$

From this we can find the flow for the $\mathcal{N} = 1$ SYM coupling constant by using $\beta(g_k^2) = k\partial_k g_k^2$. The solution to the differential equation $k\partial_k g_k^2 = \frac{3N}{8\pi^2}g_k^4$ is:

$$g_k^2 = \frac{1}{A - \frac{3N}{8\pi^2}\log(k)} \tag{218}$$

for some constant A. We can find A by taking a boundary condition that $g_k(k = k_0) = g_0$ as a reference scale. This fixes A uniquely to $A = \frac{1}{g_0^2} + \frac{3N}{8\pi^2}\log(k_0)$ and finally the running of the coupling constant up to first order is given by:

$$g_k^2 = g_0^2 \frac{1}{1 - \frac{3Ng_0^2}{8\pi^2}\log(\frac{k}{k_0})} \tag{219}$$

which is the result that we first intended to derive and agrees with [3]. We are however convinced that (215) should contain a global minus-sign since the high-energy regime of the theory should be described by an asymptotically free theory, i.e. $g_k \to 0$ for $k \to \infty$. Our result is the exact opposite and we are not convinced by our result.

As mentioned earlier we can increase the amount of supersymmetry by including the chiral super fields with the appropriate coupling constant. We derived the flow equation and β-function for SYM theory coupled to N_c chiral super fields.

7.4 The $\mathcal{N} = 4$ SYM β-function

At the level of field content $\mathcal{N} = 4$ SYM can be represented as:

$$\Big[\mathcal{N} = 4 \text{ SYM}\Big] = \Big[\mathcal{N} = 1 \text{ Vector}\Big] \oplus \Big[3 \times \mathcal{N} = 1 \text{ Chiral}\Big] \tag{220}$$

This follows from the field content that we found in 5.3.2. The theory above then corresponds to the choice $N_c = 3$. Plugging this in to the general result we recover the well-known result:

$$\beta(g_k^2)_{\mathcal{N}=4} = 0 \tag{221}$$

An astonishing all order result was obtained using super field techniques. The used formalism made sure that supersymmetry invariance was realised at each step in the calculation and kept calculations short. We also required the Yang-Mills coupling constant g to appear in front of the Faddeev-Popov ghosts (183), this was needed to make sure the $\mathcal{N} = 4$ SYM β-function vanishes exactly. This can be used as an argument as why to put the coupling there. What we still want to mention at this point is that we managed to find a zero β-function because we did not project on to higher order contributions of the vector contribution. The contribution from the ghost and chiral sector

cancels exactly for $N_c = 3$ but we would still have to calculate the complete trace from the vector.
This concludes our calculation for the β-function. In the next part we will devote sections on conformal field theory and string theory and see how the result $\beta(g_k^2) = 0$ is implemented in the AdS/CFT correspondence.

Part III
String theory and AdS/CFT

8 Super conformal field theory

8.1 Conformal transformations and the conformal algebra

$\mathcal{N} = 4$ SYM theory serves as our example of a theory with a vanishing β-function. This suggests that the theory is invariant under rescalings. Typically, invariance under scale transformations yields invariance with respect to a larger group, the **conformal group**. In this section we will introduce conformal transformations and basic features of conformal field theory (CFT) in order to make statements on the AdS/CFT correspondence. Furthermore, conformal field theory is also a very important feature in string theory as we will see later. Let us start with a definition of a conformal transformation[19]:

Definition 1. *A* ***conformal*** *transformation is a transformation* $x \to \tilde{x}(x)$ *such that the metric* g *on a manifold* M *changes by a rescaling i.e.* $g \to Og$ *where* O *is called the conformal factor.*

As a consequence, a conformal transformation preserves angles and direction of a curve through a point on the manifold. It is common to use a **Weyl** transformation:

$$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu} = e^{2\omega(x)} g_{\mu\nu} \tag{222}$$

A conformal transformation is then a diffeomorphism $x \to \tilde{x}(x)$ followed by a Weyl rescaling of the metric. If these two are each others inverses the corresponding theory is said to be conformally invariant. It is straightforward to show that for infinitesimal Weyl transformations we have that:

$$\begin{aligned} g_{\mu\nu} &\to e^{2\omega(x)} g_{\mu\nu} = g_{\mu\nu} + 2\omega(x) g_{\mu\nu} + \mathcal{O}(\omega^2) \\ &\Rightarrow \delta g_{\mu\nu} = 2\omega(x) g_{\mu\nu} \end{aligned} \tag{223}$$

We are now going to derive the conformal algebra in d dimensions on $\mathbb{R}^{p,q}$ with $p + q = d > 2$. Consider an infinitesimal coordinate transformation $x \to \tilde{x}(x) = x + \epsilon$. The metric changes as:

$$g_{\mu\nu} \to g_{\mu\nu} - \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu + \mathcal{O}(\epsilon^2) \tag{224}$$

$$\delta g_{\mu\nu} \Rightarrow -\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu \tag{225}$$

In d dimensions we therefore find that $\omega(x) = \frac{-1}{d}\partial \cdot \epsilon$ is a solution to (223). The above equation reduces to:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} g_{\mu\nu} \partial \cdot \epsilon \tag{226}$$

This is called the conformal Killing equation. It can be generalised to curved space times by using a covariant derivative. One now seeks vectors $\epsilon_\mu(x)$ such that this equation is satisfied and the transformation is conformal. By contracting the equation with two derivatives, symmetric and anti-symmetric ones, one can show that the solution of $\epsilon_\mu(x)$ is of order $\mathcal{O}(x^2)$. The ansatz:

$$\epsilon_\mu(x) = a_\mu + \omega_{\mu\nu} x^\nu + \lambda x_\mu + b_\mu x^2 - 2(b \cdot x) x_\mu \tag{227}$$

solves equation (226). Here a_μ, $\omega_{\mu\nu}$, λ, and b_μ correspond to translations, rotations, scalings and special conformal transformations, respectively. The special conformal transformations correspond to the composition of an inversion $x \to \tilde{x}^\mu = x^\mu / x^2$ and a translation. Here we introduced inversions which are also conformal transformations [15].
With (227) at hand we can give the infinitesimal generators of the translations, rotations, scalings, often called dilatations, and special conformal transformations as differential operators as [15]:

$$\begin{aligned} P_\mu &= -i\partial_\mu \\ M_{\mu\nu} &= -i(x_\mu \partial_\nu - x_\nu \partial_\mu) \\ D &= i x^\mu \partial_\mu \\ K_\mu &= -i(x^2 \partial_\mu - 2 x_\mu x \cdot \partial) \end{aligned} \tag{228}$$

These generators form a Lie algebra called the conformal algebra. We give the algebra in terms of its commutators [15]:

$$\begin{aligned} &[P_\mu, P_\nu] = [K_\mu, K_\nu] = [M_{\mu\nu}, D] = 0 \\ &[P_\mu, D] = iP_\mu \\ &[K_\mu, D] = -iK_\mu \\ &[K_\mu, P_\nu] = 2i(g_{\mu\nu} D + M_{\mu\nu}) \\ &[M_{\mu\nu}, K_\rho] = i(g_{\mu\rho} K_\nu - g_{\nu\rho} K_\mu) \\ &[M_{\mu\nu}, P_\rho] = i(g_{\mu\rho} P_\nu - g_{\nu\rho} P_\mu) \\ &[M_{\mu\nu}, M_{\rho,\sigma}] = i(g_{\mu\rho} M_{\nu\sigma} - g_{\mu\sigma} M_{\nu\rho} + g_{\nu\sigma} M_{\mu\rho} - g_{\nu\rho} M_{\mu\sigma}) \end{aligned} \tag{229}$$

The last two lines together with $[P_\mu, P_\nu] = 0$ span the Poincaré algebra we introduced in (46). This means Poincaré transformations are conformal transformations. This was expected since Poincaré transformations are translations, boosts and rotations. This shows how the conformal algebra arises purely from the Poincaré algebra extended with special conformal transformations and dilatations.
In fact, we rewrite the above generators in the following way [15]:

$$J_{\mu\nu} \equiv M_{\mu\nu} \tag{230}$$

$$J_{\mu,d+1} \equiv \frac{1}{2}(P_\mu - K_\mu) \tag{231}$$

$$J_{\mu,d} \equiv \frac{1}{2}(P_\mu + K_\mu)$$

$$J_{d,d+1} \equiv D$$

These new generators satisfy the algebra:

$$[J_{mn}, J_{pq}] = i(g_{mq}J_{np} + g_{np}J_{mq} - g_{mp}J_{nq} - g_{nq}J_{mp}) \tag{232}$$

This appears to be exactly the Lorentz algebra $\mathfrak{so}(2, d)$ provided we make the dd component of the metric a timelike direction whereas the $d + 1$ becomes spatial. So in the original Minkowskian setting $\mathbb{R}^{1,d-1}$ we found the Lorentz algebra $\mathfrak{so}(2, d)$. This generalises to the case $\mathbb{R}^{p,q}$ where the conformal algebra is $\mathfrak{so}(p+1, q+1)$. We can quickly check that the conformal algebra we derived has d translations $\frac{d(d-1)}{2}$ rotations, d special conformal transformations and 1 scalar dilatation. These add to a total of $\frac{(d+2)(d+1)}{2}$ generators. This is indeed the number of generators for the group $SO(d + 2)$[16]. Since we have seen that dilatations are part of the conformal algebra we can now argue that indeed if a theory is scale independent, which is given by the vanishing of the β-function, it is a conformal field theory. We can thus conclude that $\mathcal{N} = 4$ SYM is a conformal field theory.

8.2 A word on conformal field theory

We now discuss a field theory that is invariant under these transformations. For a theory with action S and Lagrangian $\mathcal{L}$ we can directly derive the

[16] $SO(N)$ has $\frac{N(N-1)}{2}$ generators.

following requirement for it to be conformally invariant. Let the metric transform like $\delta g_{\mu\nu} = 2\omega g_{\mu\nu}$. Then [19]:

$$0 = \delta S = \int d^D x \, \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} \delta g_{\mu\nu} \sim \int d^D x \, T^{\mu\nu}(2\omega g_{\mu\nu}) \sim \int d^D x \, T^{\mu}_{\mu} \tag{233}$$

Thus the stress energy tensor of the theory must be traceless, i.e. $T^{\mu}_{\mu} = 0$. Furthermore we can say that under the action of the Poincaré group, the coordinates may be transformed by $x \to x' = g \cdot x$ with g an element of the Poincaré group. A quantum field ϕ of the theory then transforms as [15]: $\phi^A(x) \to \bar{\phi}^A(x') = M^A_B(g)\phi^B(x)$ with A a spin index and $M^A_B(g)$ a representation of the group element g.
Under dilatations we have $x \to x' = \lambda x$ for some number λ. In a D dimensional field theory we have $d^D x \to d^D x' = \lambda^{-D} d^D x$. Now consider a scalar quantum field $\phi(x)$. The action contains $(\partial_\mu \phi)^2$. Only the derivatives change and contribute a factor λ^2. Thus a scalar kinetic term is only conformally invariant if $\lambda^{2-D} = 1$, i.e. $D = 2$.
Let us for all general purposes now consider the theory with all possible operators Φ and impose the following general transformation rule supplementing the previous two:

$$\Phi^A \to \bar{\Phi}^A = \lambda^{-\Delta}\Phi^A \tag{234}$$

Here Δ is called the scaling dimension of the operator Φ. Δ can be thought of as the eigenvalue of the dilatation operator D. The above example can then be restated by saying that we can cancel the λ^{2-D} factor if $\Phi \to \bar{\Phi} = \lambda^{\frac{D-2}{2}}\Phi$. This means $\Delta_\Phi = \frac{D-2}{2}$. We call this the classical scaling dimension of the operator Φ. When we go to the quantum case there might be an anomalous dimension η_Φ present due to operator renormalisations. The complete scaling dimension in such quantum theories is then $\Delta = \Delta_\Phi + \eta_\Phi$. Using the above transformation property one can derive n-point functions of the conformal field theory. They are an important ingredient in finding the correct field content on both sides of the AdS/CFT correspondence explained later. In the next section we will see how conformal invariance can be combined together with supersymmetry.

8.3 The super conformal algebra

In this this section we combine the two main symmetries that we have introduced in this thesis, supersymmetry and conformal symmetry.

It turns out that fusing supersymmetry with conformal symmetry is possible only for $D \leq 6$ [20][17]. One can show that special conformal transformations K do not commute with the supersymmetry generators Q. This gives rise to new generators S which are also spinorial and called special supersymmetry generators. The new algebra that one gets is called the supersymmetric conformal algebra, or super conformal algebra. Since the representations of the conformal algebra do differ from dimension to dimension we will not give the full super conformal algebra here but refer to [21].
Focusing on the theory of four-dimensional $\mathcal{N} = 4$ SYM we have the conformal symmetry group $SO(2,4)$ and the R-symmetry group $SU(4) \simeq SO(6)$. Regarding the former as the bosonic subgroup [15] of a new group called the super conformal group we use $SO(2,4) \simeq SU(2,2)$ and find the super conformal group $SU(2,2|4)$. This group depends on the dimension, because one can not always identify an isomorphism between $SO(p+1,q+1)$ and an $SU(N)$ group in any spacetime dimension. An example would be the case of $D = 2+1$ dimensions. The conformal group is $SO(2,3)$. There exist isomorphism $SO(2,3) \simeq Sp(4,\mathbb{R})$ and the super conformal group in that case is $OSp(2,\mathbb{R}|\mathcal{N})$[22]. The rest of the thesis we will refer to $\mathcal{N} = 4$ SYM as a super conformal theory. In the next section we will switch to a more geometrical part which will introduce anti-de Sitter spacetime and how it relates to the conformal algebra.

9 Anti de Sitter spacetime

In this section we will introduce anti-de Sitter (AdS) space, which together with string theory, is needed to formulate the AdS/CFT correspondence that we wish to adress within this thesis. AdS spacetime is a maximally symmetric Lorentzian manifold with constant negative curvature. One could say it is the Lorentzian analogue of a sphere. We embed the d-dimensional AdS space in a $d+1$ ambient $\mathbb{R}^{2,d-1}$. Analogous to a sphere we define AdS_{d+1} as the space with line element:

$$ds^2 = -dX_0^2 - dX_d^2 + \sum_{i=0}^{d-1} dX_i^2 \tag{235}$$

[17] The case $D = 6$ very special. It also turns out that there exist a six dimensional conformal field theory with $\mathcal{N} = 2$ supersymmetry.

The coordinates satisfy the relation:

$$-X_0^2 - X_d^2 + \sum_{i=0}^{d-1} X_i^2 = -L^2 \tag{236}$$

with L the AdS radius. This definition shows that AdS_{d+1} has an $SO(2, d-1)$ isometry.
Note that we have seen this symmetry group before, namely in the context of the conformal algebra. It can be written as a coset manifold like Minkowski space as $SO(2, d-1)/SO(2, d-2)$, very similar to the $d-1$-sphere. From now on we will use so-called Poincaré coordinates for AdS which are given by the coordinate transformation [15]:

$$\begin{aligned} X_0 &= \frac{1}{2y}(1 + y^2(L^2 + \vec{x}^2 - t^2)) \\ X_d &= Lyt \\ X_{d-1} &= \frac{1}{2y}(1 - y^2(L^2 - \vec{x}^2 + t^2)) \\ X_i &= Lyx_i \end{aligned} \tag{237}$$

with $\vec{x}^2 = \sum_{i=1}^{d-2} x_i^2$. Substituting $u = \frac{L^2}{y}$ we can write the line-element of AdS_{d+1} in the particulary simple form:

$$ds^2 = \frac{L^2}{u^2}(du^2 + \eta_{\mu\nu}dx^\mu dx^\nu) \equiv \frac{L^2}{u^2}(-dt^2 + du^2 + \delta_{ij}dx^i dx^j) \tag{238}$$

In this form it can be manifestly seen that this line element is invariant under rescaling $u \to \lambda u$ and $x^\mu \to \lambda x^\mu$. It is also worth pointing out that this metric only covers half of AdS_{d+1} since $y > 0 \to u > 0$. As we have seen in the previous section these are examples of conformal transformations. We can derive the scalar curvature of AdS_{d+1} in absence of matter by contracting the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0 \tag{239}$$

with the metric to find:

$$R = 2\frac{d+1}{d-1}\Lambda \tag{240}$$

Using that for AdS space we have $R_{\mu\nu} = -\frac{d}{L^2} g^{\text{AdS}}_{\mu\nu}$ and $\Lambda = -\frac{d(d-1)}{2L^2}$ gives:

$$R = -\frac{d(d+1)}{L^2} \tag{241}$$

which is constant and negative, as expected. Also observe the apparent singularity at $u = 0$ in (238) corresponding to the regime where $y \to \infty$. This is a coordinate singularity. We can introduce conformal coordinates in which it becomes apparent that the subspace satisfying $u = 0$ defines a boundary of AdS_{d+1} called the conformal boundary ∂AdS_{d+1}. In these coordinates the induced metric on AdS_d reads:

$$ds^2 = \frac{1}{\cos^2(\beta)}(-dt^2 + d\Omega^2_{d-1}(\beta)) \tag{242}$$

with $\beta \in [0, \pi/2)$ and $d\Omega^2_d$ the line element of the d-sphere given by $d\Omega^2_d(\beta) = d\beta^2 + \sin^2(\beta) d\Omega^2_{d-1}$. In these coordinates we see that AdS_d is conformally equivalent to the cylinder $\mathbb{R} \times S^d$. Slices of constant u yield d-dimensional Minkowksi spacetime. We can write Minkowksi space as the product $\mathbb{R} \times \mathbb{R}^{d-1}$. Then recall that the one-point compactification of $\mathbb{R}^n$ gives an n-sphere S^n. Hence Minkowski space is also conformally equivalent to a cylinder $\mathbb{R} \times S^{d-1}$ if compactified. Therefore the identification made earlier makes sense and the boundary ($u = 0$) of AdS space is given by Minkowksi space.
However, what is the boundary space then exactly? Since we cover only half of AdS_d with this metric we use the fact that the spatial part of AdS_d, which is a half-sphere, is topologically equivalent to the disc. The spatial boundary is then given by the boundary of the half sphere (the disc) hence we find again another sphere, yet one dimension lower. This means that $\partial\text{AdS}_{d+1} \simeq \mathbb{R} \times S^{d-1}$ with metric $ds^2 = (-dt^2 + d\Omega^2_{d-1})$ is Minkowksi space. This will play an important role in the AdS/CFT correspondence.
We will now start a detailed discussion on string theory. Starting with its original form, bosonic string theory. Thereafter we will introduce the superstring.

10 String theory

Since the early 1970s string theory has evolved in one of the biggest areas of research on quantum gravity. The idea of string theory was to replace every point particle in the standard model by a one-dimensional string-like object. The exact consequences were not known by that time. We try to give a short overview over different string theories by starting out with the original bosonic string and later we will see how fermions are imported.

10.1 String theory

Let us start string theory by looking at an first attempt of writing down the action for an object that is one dimensional in space, i.e. ,a string. One of the first attempts done by Nambu and Goto [19] was the Nambu-Goto (NG) action S_{NG}. It is very similar to the action of a pointparticle of mass m, which is $S_{\text{point}} = -m \int ds$. However, the integral along a world line ds gets replaced by the intgral over an area, the **world sheet**:

$$S_{\text{NG}} = -T \int d^2\sigma \, \sqrt{-G} \tag{243}$$

with $G = \det(G_{ab}), G_{ab} = \partial_a X^\mu \partial_b X_\mu$. Where the $\mu \in \{0, \cdots, D-1\}$ index is contracted via the D-dimensional spacetime metric and $a, b = 1, 2$ are worldsheet coordinates. We have defined the two-dimensional area element $d^2\sigma = d\tau d\sigma$, where $\tau \in (-\infty, \infty)$ the proper time, σ a spatial coordinate that parametrises the string and T is the string tension. We call $X^\mu(\tau, \sigma)$ the embedding coordinates of the worldsheet manifold Σ in to the D-dimensional space time manifold $\mathcal{M}$. They can be understood as scalar[18] maps: $X^\mu : \Sigma \longrightarrow \mathcal{M}$. In It is clear that $\tau \in (-\infty, \infty)$. Depending on whether the string is open or closed boundary conditions may vary. For a closed string the boundary conditions are somewhat trivial. Since it closes we have a periodicity condition $X^\mu_{\text{cl}}(\tau, \sigma) = X^\mu_{\text{cl}}(\tau, \sigma + 2\pi)$ where we have taken $\sigma \in [0, 2\pi]$. An open string has $\sigma \in [0, \pi]$ but requires an extra boundary condition explained below. The observation that the embedding fields occur under a square root has let physicists to think about the quantisation problems this action will bring. It is therefore costumary to rewrite the action in the following way:

$$S_{\text{P}} = \frac{T}{2} \int_\Sigma d^2\sigma \, \sqrt{h} h^{ab} \partial_a X^\mu \partial_b X_\mu \tag{244}$$

S_{P} is called the Polyakov action. The product $X^\mu X_\mu = X^\mu X^\nu g_{\mu\nu}$, with some target space metric $g_{\mu\nu}$ and h_{ab} acts as a worldsheet metric.

[18] Note that this theory therefore only contains scalars, fermions and other type of particles have to be added later. The theory is therefore not supersymmetric.

The Polyakov action has three main symmetries:

1. Poincaré transformations, transformations of the form $\delta X^\mu = \omega^\mu_\nu X^\nu + a^\mu$ leave the action invariant provided that $\delta h^{ab} = 0$
2. Reparametrizations, also known as local diffeomorphisms $\sigma^a \to f(\sigma^a)$ leave it invariant. Here $\sigma^a = (\tau, \sigma)$.
3. Weyl transformations, transformations that are of the form $h_{\alpha\beta} \to \Omega^2(\sigma) h_{ab}$ give factors inside the action that cancel, this is therefore also a symmetry of the Polyakov action. Note that this only happens for a two-dimensional worldsheet. For higher dimensional objects this is no symmetry.

Using the reparametrisation invariance of the action it is possible for us to fix a gauge in which the worldsheet metric becomes a flat Minkowski metric η_{ab}. This local invariance needs gauge fixing. One follows the Faddeev-Popov procedure like in the Yang-Mills case and finds a gauge-fixed action with two ghosts usually called b and c. It is now possible to find equations of motion for the scalars X^μ. In thesis we will not do this but we refer to [19] for more information. As mentioned, open strings require an additional boundary condition: $\partial_\sigma X^\mu \delta X_\mu|_{\sigma=0,\pi} = 0$. This can be accomplished by setting $\partial_\sigma X^\mu|_{\sigma=0,\pi} = 0$ or $\delta X_\mu|_{\sigma=0,\pi} = 0$. The first case is called Neumann (N) and the second Dirichlet (D) boundary condition, both have to be satisfied at the endpoints of the open string, for which we have.

$$\mathrm{N} : \partial_\sigma X^\mu|_{0,\pi} = 0 \tag{245}$$

$$\mathrm{D} : \delta X_\mu|_{0,\pi} = 0 \tag{246}$$

Before discussing implications of these boundary conditions let us see more general results. Equations of motion that follow from the variation of the Polyakov action (244) with respect to $X^\nu(\tau, \sigma)$. Solving the equations of motion with given boundary conditions results in the following mode expansion for the closed string[19]:

$$X^\mu(\tau, \sigma) = x^\mu + \frac{1}{\pi T} p^\mu \tau + \frac{i}{\sqrt{\pi T}} \sum_{m \neq 0} \frac{1}{m} \alpha^\mu_m \exp(-im\tau) \cos(m\sigma) \tag{247}$$

[19] This expansion is the sum of a left- and right moving expansion X and $\bar{X}$. The fact that we have a left and right moving solution lies in the equations of motion. For more see [19]

where: x^μ is the center of mass position, p^μ the center of mass momentum. The mode functions α_m^μ are independent of spacetime and are related to creation and annihilation operators seen in quantum field theory. One defines so called Virasoro generators in the quantum[20] case $L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{m-n}^\mu \alpha_{n\mu}:$. These operators are generators of the conformal[21] algebra on the two-dimensional worldsheet and satisfy the Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{248}$$

where c is a central charge of the theory. The Virasoro generators actually satisfy two copies of the conformal algebra because we also have left- and right moving modes X and $\bar{X}$. It is clear that the generator L_0 is ambiguous due to the normal ordering. One fixes this by imposing the physical state condition: $(L_0 - a)|\text{physical state}\rangle = 0$ for some normal ordering constant a. From this condition it is possible to calculate the mass spectrum of the open string. By definition one has $M^2 = -p^2$. One finds:

$$M^2_{\text{open}} = 2\pi T(N - a) \tag{249}$$

wheareas:

$$M^2_{\text{closed}} = 8\pi T(N - a) \tag{250}$$

Here N is the number operator defined as L_0 without normal ordering. A calculation which we shall not present here reveals that $a = \frac{D-2}{24}$. One can deduce the field content for the levels $N = 0$ and $N = 1$. The $N = 0$ level for both cases means that $M^2 < 0$. This state is called the Tachyon. The next level is $N = 1$. By acting with the correct creation operators on the vacuum state one finds that there are $(D-2)^2$ states in this level. These can not fit into a massive representation of $SO(D-1)$. They may be put into the little group $SO(D-2)$. Representations of the little group are massless. This means that the first level has massless modes. If we say $M^2(N = 1) = 0$ we can solve the equations for D. We find:

$$0 = 1 - \frac{D-2}{24} \tag{251}$$

[20]This is called quantum because we use the normal ordering of operators denoted by ::

[21]For now conformal means that it is invariant under rescalings of coordinates. We will come back to it later.

or $D = 26$. This means that $N = 1$ contains 576 massless states[22]. It can be decomposed in the 24×24 dimensional representation of $SO(24)$ as:

$$\text{traceless symmetric} \oplus \text{anti-symmetric} \oplus \text{trace} \tag{252}$$

The corresponding massless fields:

$$\begin{aligned} &G_{\mu\nu}(X) \text{ Graviton} \\ &B_{\mu\nu}(X) \text{ Kalb-Ramond} \\ &\Phi(X) \text{ Dilaton} \end{aligned} \tag{253}$$

The Kalb-Ramond field also goes by the name two-form. This reduces the symmetry group of the system. Lastly, observe that we found a symmetric two-tensor in the massless spectrum. This will be regarded as the metric of spacetime, and is thus a dynamical field with its own equations of motion. As we will see later, in superstring theory the tachyon will vanish from the spectrum.

Let us return to the boundary conditions for open and closed strings. It is possible for a string to have boundary conditions that are pure Neumann or Dirichlet, but is also possible to have mixed boundary conditions ND, or DN. The Neumann boundary condition means physically that no momentum is flowing through the endpoints of the string since we interpret the differentiation as momentum. Dirichlet boundary conditions imply that $X^\mu|_{0,\pi} \sim$ constant. This condition, if fulfilled for all μ, maximally breaks Poincaré invariance and has therefore not been studied in the early days of string theory. This means that the full Lorentz group is broken according to:

$$SO(1, D-1) \to SO(1,p) \times SO(D-1-p) \tag{254}$$

if p directions are affected by Dirichlet boundary conditions. A more recent interpretation is that of a hypersurface to which the endpoints attach. These hypersurfaces are called Dp-branes. Here D stands for Dirichlet and p for the spatial dimension of the brane.

10.2 D- branes

We have just seen the notion of a Dp-brane. These objects are extended along $p + 1$ dimensions. Note that a $D1$-brane is different from a fundamental string since an open fundamental string has to end on a Dp-brane.

[22] This is because we have two copies of the creation and annihilation operators α^μ

This is not the case for a D1-brane since it can be anywhere in spacetime. Furthermore, a D0-brane is a one-dimensional object in spacetime and is also known as a D-instanton as it behaves like a domain wall in time[23]. It has long been unclear whether these objects are dynamical and have their own equations of motion. Studies have shown that this is in fact the case [23][24].

It can of course occur that only a part of the full spacetime is filled by a Dp-brane or even the presence of multiple non-intersecting D-branes. This means that if both endpoints of an open string do not end on the same Dp-brane we have $X^I|_0 = c^I$ and $X^J|_\pi = \tilde{c}^J$. With I, J taking values in time direction and the respective dimensions in which the certain Dp-brane is present. One can also make mixed boundary condtions like the ND case, then we write in a somewhat strange fashion that $\mu = a \oplus I$. With this we mean that some directions are Neumann and the others are Dirichlet.

The next thing that we consider because it gives rise to relevant notions is a so called brane stack. A brane stack is nothing else than a stack of Dp-branes that might be coincident, or separated by some non-zero distance. When we consider the two endpoints of a string separated by a distance L one gets an extra contribution to the mass spectrum given by $M^2_{\text{open}} \sim L^2T^2 + N\text{terms} + \text{oscillators}$. Hence we can make extra massless modes depending on L. Massless modes correspond to a $U(1)$ gauge symmetry. For a stack of N coincident Dp-branes this symmetry is enhanced to $U(N)$. Because open strings must end on D-branes one introduces the notion of a new "quantum number" $i = 1, \ldots, N$ called a Chan-Paton factor which labels on which D-brane the string ends. These can in principle be non commuting parameters and the strings now belong to the N^2-dimensional adjoint representation of $U(N)$. This gives a first hint at the fact why we are interested in looking into string theory. The gauge degrees of freedom living on D-branes correspond to gauge vector bosons of the field theory living on its world volume. In the next section we look at the closed bosonic string and have a first look at compactification.

10.3 String compactifications and T-duality

As we have seen string theory requires, at least in the bosonic case, 26 dimensions of spacetime to be Lorentz invariant and self-consistent. However, we only observe four. An idea that Kaluza and Klein [23] developed in order to unify electrodynamics and gravity was to start in $D = 5$ and **compactify** the extra dimension to $D = 4$ on a circle S^1. Let us try out the same for a closed string and see what we get. Again, we will not give the full derivation. Consider the 25-component of the scalars X^μ. Compactifying this field on S^1 requires the condition that:

$$X^{25}(\tau, \sigma + 2\pi) = x^{25}(\tau, \sigma) + 2\pi w R \tag{255}$$

where R is the radius of the compact 25th dimension and w is called the winding number. This also means that the mode expansion of this scalar will change.
The mode expansion of X^{25} looks like[23][19]:

$$X^{25}(\tau, \sigma) \sim x^{25} + \frac{1}{\pi T} p^{25}\tau + 2wR\sigma \cdots \tag{256}$$

There is one more thing we can say about the momentum. In Kaluza-Klein reduction of a scalar field in $D = d + 1$ dimensions where the $(d + 1)$th dimension gets compactified the momentum of that direction gets quantised[24] according to:

$$p^{25} = \frac{n}{R}, n \in \mathbb{N} \tag{257}$$

Using the expansion of the closed string [19] one finds the mass formula for the closed compactified with $T = 1$ string:

$$M^2_{\text{closed,comp.}} = \Big((\frac{n}{R})^2 + \big(wR\big)^2 \Big) + 2N_L + 2N_R - 4 \tag{258}$$

Here $N_{L,R}$ just denotes the number operator of the left- and right moving modes respectively. It becomes apparent that this mass formula has one more symmetry that is very exciting. Define the map

$$T_R : R \to \frac{1}{R}, w \leftrightarrow n \tag{259}$$

Observe how $T_R M^2_{\text{closed,comp.}} = M^2_{\text{closed,comp.}}$, i.e., the same physics is described by a theory with momentum and winding numbers exchanged when described on a circle with radius R or $\frac{1}{R}$. This symmetry goes by name **T-duality** and

[23]This is again the full mode expansion i.e. the sum of the left- and right moving expansion

[24]This follows directly by plugging in the mode expansion of the scalar field in to the constraint that $\phi(x + 2\pi) = \phi(x) + 2\pi w R$ with x the extra dimension.

is not specific for this theory. Next we look at what happens to an open string under T-duality. The mode expansions of the open string that under a T-duality the open string mode expansion looks like:

$$T_R X^{25}(\tau,\sigma) = \tilde{X}^{25} = \tilde{x}^{25} + p\sigma + \sum_{m\neq 0} \frac{1}{m}\alpha_m \exp(-in\tau)\sin(n\sigma) \tag{260}$$

Observe that there is no dependence of the momentum in the 25-direction i.e. there is only oscillatory motion. The most important result is that at the endpoints, the expansion is fixed. This is exactly the behaviour of a string with Dirichlet boundary conditions. Notice that the boundary conditions also yield exactly the description of a Dp-brane, with $p = 24$. We conclude that: **Under S^1 T-duality an open bosonic string with Neumann boundary conditions gets mapped in to a dual open bosonic string with Dirichlet boundary conditions ending on a D24-brane.** This reasoning can be carried out even more generally. Considering a compactification of an n-torus, also called toroidal compactification, by $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}$ we have Dirichlet boundary conditions in n directions and the T-dual theory is defined on a D$(25-n)$-brane. We conclude that T-duality reduces the dimension of the Dp-brane in the dual theory by each compactified dimension. We can use this duality to describe open strings and closed strings in the same context since these two types of theories are related under T-duality. We will see T-duality again in the section on superstrings.

10.4 The bosonic string coupled to a background

It is also possible to relate the world sheet action of a string to a spacetime action. One then generally talks about coupling the massless spectrum of the bosonic string to a background. In order to couple the bosonic string to the target space we return to our massless representations of the closed string. The various modes then interact with $G_{\mu\nu}$, the two-form field $B_{\mu\nu}$ and the dilaton Φ. The full action reads [19]:

$$S = \frac{T}{2}\int_\Sigma d^2\sigma\ \sqrt{h}h^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \frac{T}{2}\int_\Sigma d^2\sigma\ \Big(\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu} - \frac{1}{2\pi T}\sqrt{h}R\Phi\Big) \tag{261}$$

Here R is the $2d$ Ricci scalar of the worldsheet with induced metric $h_{\alpha\beta}$ Defining a field strength $H_{(3)} = dB_{(2)}$ for the Kalb-Ramond field we can write down an action that will derive the field equations.[25] This action is given by a spacetime action [19]:

$$S_{\text{Background}} = \int d^{26}x \, \sqrt{G} e^{-2\Phi} \Big(R - \frac{1}{12} H_{(3)}^2 - 4(\nabla\Phi)^2 \Big) \tag{262}$$

This action is referred to as a low-energy action for bosonic string theory because its equations of motion are β-functions up to one loop order. The exponential of the dilaton appears because if it is constant the last term in the worldsheet action is a topological invariant given by the Euler characteristic. It is referred to as the string coupling $g_s = e^{\Phi}$. We will see this kind of spacetime actions later when we introduce the superstring.

11 Superstring theory

In this section we will take a new look on the string theory that we have just introduced. Thus far we have only encountered the bosonic theory which is reflected in the fact that the embedding coordinates $X^\mu(\sigma) : \Sigma \to \mathcal{M}$ are bosonic. As we observe nature we see that the world around us also constitutes of fermions, particles with half-integer spin. These particles, also described by fields, have a long history are are described by an action called the Dirac action. Its massless Lagrangian is:

$$\mathcal{L}_{\text{Dirac}} = i\bar{\Psi}\gamma^\mu \partial_\mu \Psi \tag{263}$$

with γ^μ the usual gamma matrices,Ψ a Dirac spinor and $\bar{\Psi} = \Psi^\dagger \gamma^0$ the Dirac adjoint spinor. If we replace the gamma matrices by their two-dimensional equivalent and take the spinor to be a two component Weyl spinor we have the Weyl Lagrangian [23]. Let us take ψ to be chiral Weyl spinors. To extend the bosonic string theory discussed earlier with fermions we add the following worldsheet action to the Polyakov action (244):

$$S_{\text{Fermion}} = \frac{i}{4\pi} \int_\Sigma d^2\sigma \, \sqrt{h} h^{ab} \bar{\psi}_{\alpha,\mu} (\gamma_a)^{\alpha\beta} \partial_b \psi_{\beta,\nu} \eta^{\mu\nu} \tag{264}$$

[25] Here the field equations amount to three β-functions for each field G, B and Φ. These beta functions should vanish since we want to maintain conformal invariance throughout the background coupling. This then implies Einstein equations in vacuum.

where a, b are worldsheet coordinates and α, β are spinor indices. The worldsheet metric is again h_{ab} and spacetime is flat Minkowskian. The missing string tension factor comes due to the mass dimension of a fermionic field.

11.1 Super string theory action and field content

The complete action is then known as the Ramond-Neveu-Schwarz (RNS) action [23] and is given by the worldsheet action:

$$S_{\text{RNS}} = \frac{1}{2}\int_{\Sigma} d^2\sigma\ \sqrt{h}h^{ab}\big(T\partial_a X^\mu \partial_b X_\mu + \frac{i}{2\pi}\bar{\psi}^\mu_\alpha(\gamma_a)^{\alpha\beta}\partial_b\psi_{\beta,\mu}\big) \tag{265}$$

As one can show by explicit calculation this theory exhibits a chiral supersymmetry and local diffeomorphism invariance [23][24]. This means that the theory has an $\mathcal{N}=1$ supersymmetry for the Weyl spinor ψ and another one for $\bar{\psi}$. This is usually denoted by a chiral $\mathcal{N}$ =(Left,Right) $\rightarrow \mathcal{N} = (1,1)$ supersymmetry. This theory has worldsheet supersymmetry in the sense that the supersymmetry parameter ϵ is it itself a worldsheet spinor [23] and has its own equations of motion. This means that we are talking about a spacetime dependent supersymmetry parameter rather that a constant Grassmann parameter. In this way we localize the supersymmetry and the theory is sometimes referred to as **supergravity**. This is however not manifest in our notation and instead requires the introduction of another field to be the superpartner of the metric. Because we will only introduce string theory we will not further elaborate on equations of motion and constraints. The gauge fixing procedure of the RNS action (265) again requires introduction of two ghosts, usually denoted by γ and β. These are accompanied by the ghosts from bosonic theory. The requirement of conformal invariance on the worldsheet then fixes the critical dimension of this so called **super string theory** to be $D = 10$. So string theory containing fermions is only consistent in ten-dimensional spacetime.
We next discuss the field content of this new string theory. The bosonic theory gives rise to a massless spectrum consisting of the dilaton, two-form $B_{\mu\nu}$ and the metric $G_{\mu\nu}$. The fermions have their own respective equations of motion which look like the Dirac equation in two dimensions. These satisfy their own boundary conditions on the worldsheet. The **closed**, also called Type II, string spinors must satisfy [23, 24]: $\psi(\sigma) = \pm\psi(\sigma + l)$ with l the length of the string. Since we have two possible chiralities we have four sectors to

which each spinor can belong. We can distinguish between two scenarios:

$$\psi(\sigma + l) = e^{2\pi i k}\psi(\sigma) \tag{266}$$

We call boundary conditions with $k = 0$ the Ramond (R) sector, and boundary conditions with $k = \frac{1}{2}$ the Neveu-Schwarz (NS) sector. The four independent sectors are then given by: (R,R), (NS,R), (R,NS) and (NS,NS) where we denote (L,R) with left and right movers and every left and right mover itself can have positive or negative chirality. The mode expansions for each sector look like:

$$\text{Ramond: } \psi^\mu_\pm(\tau,\sigma) = \sqrt{\frac{2\pi}{l}}\sum_{n\in\mathbb{Z}} b^\mu_n e^{\frac{-2\pi i}{l}n(\tau\pm\sigma)} \tag{267}$$

$$\text{Neveu-Schwarz: } \psi^\mu_\pm(\tau,\sigma) = \sqrt{\frac{2\pi}{l}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_r e^{\frac{-2\pi i}{l}r(\tau\pm\sigma)} \tag{268}$$

where $\pm$ denotes chirality and $b^\mu_{r/n}$ are mode functions with μ a spacetime index.

The **open**, also called Type I, string spinors have Neumann (N) or Dirichlet (D) boundary conditions. The R sector belongs to boundary conditions where the mode functions are summed over integers while the NS sector are boundary conditions with half-integer mode functions like in (267) but replacing 2π by π. DD boundary conditions are found by flipping signs of the scalars and by supersymmetry also the sign of the fermions at both boundaries. mixed, ND/DN, boundary conditions are gained by flipping sign only at one of the boundaries $\sigma = 0, l$ instead of both. In the rest we will only consider Type II strings but for a detailed discussion on other string theories like Type I, Heterotic string theories etcetera we refer the reader to [24]. We can now construct the full string spectrum of the NS and R sectors. To do this it is important to note that we can again build up all states from the vacua given by:

$$\alpha^\mu_m|0\rangle_{\text{NS}} = b^\mu_r|0\rangle_{\text{NS}} = 0 \tag{269}$$

$$\alpha^\mu_m|0\rangle_{\text{R}} = b^\mu_n|0\rangle_{\text{R}} = 0 \tag{270}$$

The NS vacuum is a unique vacuum and is a spacetime scalar. A consequence thereof is that all states in the NS sector are spacetime bosons [23]. The R vacuum is however not unique because $b^\mu_0|0\rangle_{\text{R}} \neq 0$ while it is zero for higher modes. The mode functions satisfy $\{b^\mu_m, b^\nu_n\} = \eta^{\mu\nu}\delta_{m,-n}$ [23, 24]. This reveals that b^μ_0 satisfies its own anticommutation relation. In turn this implies that the R groundstate is in fact part of the ten-dimensional Clifford algebra and is thus a ten-dimensional spinor. It turns out that the NS vacuum has again negative mass squared operator and is thus a tachyon. Using a technique called GSO projection [23, 24] removes the tachyon from the spectrum and leaves us with only physical states. This also divides the Type II theory in two other subtheories called Type IIA and Type IIB string theory. In Type IIA the left and right moving states have opposite chiralities while Type IIA has the same chirality for left and right movers. The Type IIB is therefore said to be a chiral theory. We will only present the massless spectrum of the Type IIB theory since this is the relevant theory for this thesis. Like in the bosonic string theory case the massless spectrum of the string should transform in the little group $SO(D-2)$ of the D-dimensional Lorentz group which in this case is $SO(8)$. The states in the massless level of the (NS,NS), (NS,R), (R,NS) and (R,R) sectors respecting equal chirality are then organised into representations of the little group. The (NS,NS) contains spacetime bosons and corresponds to the field content of bosonic string theory. The mixed (R,NS) and (NS,R) sector consist of spacetime fermions which are superpartners of the bosonic fields in (NS,NS) and (R,R). The (R,R) sector consist of tensor representations. The massless bosonic spectrum looks like:

(NS,NS): Dilaton scalar ϕ, antisymmetric two-form $B_{\mu\nu}$ and metric $G_{\mu\nu}$.

(R,R): Axion zero-form (scalar) $C^{(0)}$, antisymmetric two-form $C^{(2)}$ and four-form $C^{(4)}$ with self-dual field strength.

Self-duality implies that for some field strength $F = \star F$. We see a theory that consist of certain type of p-form potentials. In analogue to conventional Yang-Mills theory one can schematically introduce associated field strengths $G^{(p+1)} = dC^{(p)}$. These field strengths are obviously gauge invariant under $C^{(p)} \to C^{(p)} + dV^{(p-1)}$ and have a canonical action $S \sim \int G^{(p+1)} \wedge \star G^{(p+1)}$. The self-duality constraint of the four-form is then imposed at the level of equations of motion for the associated five-form field strength $G^{(5)}$. The bosonic sector of Type IIB, (R,R) and (NS,NS), can again be put in a low energy effective action like (262). Since the (NS,NS) sector for Type IIA and IIB and even the bosonic string theory are the same, the actions will also be the same, only integrated over ten-dimensional target space. Without going in to further detail, the bosonic part of the ten-dimensional space time action is given by [23]:

$$S_{\text{Type IIB}} = S_{\text{NS}} + S_{\text{R}} + S_{\text{CS}} \tag{271}$$

$$S_{\text{NS}} = \int e^{-2\phi}(R + 4(d\phi)^2 - \frac{1}{12}|H^{(3)}|^2)d\text{Vol} \tag{272}$$

$$S_{\text{R}} = -\int (\frac{1}{2}|G^{(1)}|^2 + \frac{1}{2\cdot 3!}|G^{(3)}|^2 + \frac{1}{2\cdot 5!}|G^{(5)}|^2)d\text{Vol} \tag{273}$$

$$S_{\text{CS}} = -\int C^{(4)} \wedge H^{(3)} \wedge dC^{(2)} \tag{274}$$

here ϕ is the dilaton, R the Ricci scalar and we also denote $|F^{(p+1)}|^2 d\text{Vol} = \langle F, F\rangle d\text{Vol} = F^{(p+1)} \wedge \star F^{(p+1)}$. Furthermore:

$$\begin{aligned} G^{(1)} &= dC^{(0)} \\ G^{(3)} &= dC^{(2)} - C^{(0)}H^{(3)} \\ H^{(3)} &= dB^{(2)} \\ G^{(5)} &= dC^{(4)} - \frac{1}{2}dB^{(2)} \wedge C^{(2)} + \frac{1}{2}dC^{(2)} \wedge B^{(2)} \end{aligned} \tag{275}$$

This action can not be written in a manifest covariant way because it lacks the equation of motion constraint of the four-form. The constraint $G^{(5)} = \star G^{(5)}$ must be imposed by hand at the level of the equations of motion and it is also $\mathcal{N} = 2$ supersymmetric.

Now that we have the (bosonic)[26] field content and the action of the Type IIB theory we can take a closer look at the other dynamical objects in the theory that we have called Dp-branes earlier.

11.2 D-branes revisited

The discussion of T-duality from section 10.3 can be continued in a sense that it maps NN to DD boundary conditions under compactification. The presence of DD boundary conditions implies the presence of D-branes. The same discussion might be carried out for the Type IIA and Type IIB theories. One concludes that Type IIA gets mapped to Type IIB under compactification on

[26] We omit fermions but the fermionic sector consist of and equal number of degrees of freedom. It contains the superpartners of all bosonic fields.

a circle and vice versa [23]. Regarding the Type IIB theory with NN boundary conditions, or with a spacetime filling D9-brane one finds that under compactification the theory is T-dual to Type IIA with D8-branes. Continuing this argument one finds that Type IIA consist of all even-numbered Dp-branes, $p = 0, 2, 4, 6, 8$ while Type IIB has all odd-numbered Dp-branes, $p = 1, 3, 5, 7, 9$ plus an D-instanton.
What we want to look at next is the way the various Dp-branes appear as dynamical objects in these theories. Recall that the simplest gauge theory, namely abelian Yang-Mills theory, with a one-form potential $A^{(1)} = A_\mu dx^\mu$, couples naturally to a one-dimensional line. In the same fashion a p-form $A^{(p)}$ couples to a p-dimensional submanifold Σ_p of the target space. Therefore we replace:[15, 24] :

$$q_{\text{electromagnetic}} \int_{\text{worldline}} A^{(1)} \to S_{\text{CS}} \sim q \int_{\Sigma_p} A^{(p)} \tag{276}$$

with q its charge. This action is diffeomorphism and gauge invariant and serves as a coupling between the potential and higher dimensional objects. Actions of this type are called **Chern-Simons** action. Higher order curvature terms may be included by T-duality [23] just like in non-abelian Yang-Mills theory. It has been shown by Polchinski [23] [24] by considering two parallel D-branes and interchanging string interactions that D-branes do actually couple to the (R,R) sector potentials. It is therefore necessary to include a term like above to the complete action. The RR potentials couple to the charges carried by the various branes according to (276). We conclude that whatever theory is under consideration implies automatically the presence of the corresponding D-branes. As an example, the two-forms couple to 1-branes and the four-form therefore to a 3-brane. This also justifies the presence of the odd-numbered branes in the IIB theory explained earlier. In Type II theories the full Lorentz group $SO(1,9)$ breaks down to $SO(1, p-1)$ on the brane and full rotational invariance in the transverse space with $SO(10-p)$ in the presence of a p-form potential. Note that this corresponds to (254) if we have a D$(p+1)$-brane [15, 23].
It is furthermore possible to generalise the action for a fundamental string to a $p+1$-dimensional object. Because the Polyakov action (244) appears to be hard to generalise we can go back to the Nambu-Goto action (243). We then simply include a p-dimensional tension T_p in front. This part of the action comes from the gravitational part of the (NS,NS) sector and contains the dilaton, two-form and the metric. For a $p+1$-dimensional object (243)

generalises to:

$$S_{p+1} = T_p \int_{\Sigma_{p+1}} d^{p+1}\sigma \, \sqrt{G} \tag{277}$$

with $G = \det(G_{ab})$ the metric of the **worldvolume** Σ_{p+1}. We wish to include the full (NS,NS) spectrum. First we include the dilaton. One does this by observing that the dilaton acts as the string coupling g_s. The next step is to include the two-form $B^{(2)}$ given by the worldsheet two-form B_{ab}. It is given by the pullback of the spacetime two-form $B_{\mu\nu}$ under the embedding fields X^μ.

$$B_{ab} = \partial_a X^\mu \partial_b X^\nu B_{\mu\nu} \tag{278}$$

However, when including it under the square root we break gauge invariance [23, 24]. We can repair this broken symmetry by introducing the Yang-Mills field strenght $F_{\mu\nu}$ and transform the one-form potential. This term may be added as a Chern-Simons term because the open string worldsheet has a one-dimensional boundary and thus couples to a one-form. The transformation laws look like:

$$\delta B_{\mu\nu} = \partial_\mu \sigma_\nu - \partial_\nu \sigma_\mu \tag{279}$$

$$\delta A_\mu = -T\sigma_\mu \tag{280}$$

with T the string tension. The full invariant D-brane action then takes the form [23]:

$$S_{\text{DBI}} = T_p \int_{\Sigma_{p+1}} d^{p+1}\sigma \; e^{-\phi} \sqrt{G + \mathcal{F}} \tag{281}$$

where $G + \mathcal{F}$ is the absolute value of the determinant of $G_{ab} + \mathcal{F}_{ab}$ and $\mathcal{F}_{ab} = F_{ab} + TB_{ab}$ is the gauge invariant combination. The above action is the (NS,NS) sector action and is called the Dirac-Born-Infeld (DBI) action. The complete action is then given by:

$$S_{\text{brane}} = S_{\text{CS}} + S_{\text{DBI}} \tag{282}$$

This action is invariant under $U(1)$ gauge symmetry. We say that D-branes carry a $U(1)$ symmetry and produce gauge theories. We have seen this already in the bosonic string. When we first encountered D-branes we have seen that a stack of N D-branes gives a $U(N)$ gauge theory. The appearance of the Yang-Mills field strength in the DBI action (281) hints at the presence of a gauge theory living on the worldvolume of the D-brane. Expanding

(282), which amounts to expanding the square root, up to some order in the field strength F_{ab}, we can find a $U(N)$ gauge theory with action [24]:

$$S_{\text{Expanded}} \sim \frac{T_p}{4T^2}\int d^{p+1}\sigma \; e^{-\phi} \; \mathrm{Tr}(F_{ab}F^{ab} + 2D^aX^iD_aX_i + [X^i, X^j][X_i, X_j]) \tag{283}$$

which is a Yang-Mills type theory coupled to scalar fields X^a with gauge coupling $\frac{1}{g^2} = \frac{T_p}{4T^2}g_s^{-1}$. A careful investigation of above statement [23] leads to the conclusion that one can in fact construct the gauge group $U(3)\times U(2)\times U(1)$. This can be done by taking intersecting D-branes with a stack of 3 D-branes, 2 D-branes and a single D-brane. Under some considerations given in [23] the worldvolume theory on their intersection then yields the correct gauge group of the standard model $SU(3)\times SU(2)\times U(1)$. Reconstructing the standard model is called **string phenomenology**.
Let us now discuss an important consequence of the presence of Dp-branes in the geometry of the target space. From the IIB action (271) we can derive the equation of motion for every bosonic field. For the moment the dilaton and p-form fields are not of interest to us but will come back in the last section of this thesis. The interesting solution is the metric. If we turn on a p-form potential this implies the presence of a D$(p-1)$-brane. The metric solution of (271) then resembles the geometry around this D-brane. A conventional ansatz is to split spacetime in to coordinates parallel, and transverse to the Dp-brane. The solution for the metric is then given by:

$$ds^2 = H_p(r)^{-\frac{1}{2}}dx^a dx_a + H_p(r)^{\frac{1}{2}}d\vec{y}\cdot d\vec{y} \tag{284}$$

with $r^2 = \vec{y}\cdot\vec{y}$ the radial distance from the D-brane and a labels the parallel directions. The function H_p is a harmonic function. Requiring flat space for $r\to\infty$ constrains the function H to be:

$$H_p(r) = 1 + \left(\frac{L}{r}\right)^{7-p} \tag{285}$$

with L a constant of dimension length. The solution depends on p which means that the metric changes depending on which Dp-brane is present.

Let us recap what we have learned in this section. We have introduced superstrings as a supersymmetric extension to the bosonic string theory. On the massless level the field content is also extended by extra fermions that are required by supersymmetry. Furthermore we have got p-form potentials.

We have seen that in Type IIB string theory, with only closed strings, we can T-dualise the theory to obtain open strings. The p-form potentials are generalised gauge fields that couple to higher dimensional objects called Dp-branes. The IIB theory specifically has odd-numbered branes, these branes carry gauge theories on their world volume and we have found the metric solution corresponding to the low energy limit of string theory called supergravity.

12 The AdS/CFT correspondence

We have now every ingredient to give a short but detailed overview of the AdS/CFT correspondence. In previous sections on string theory, conformal field theory and the anti-de Sitter spaces we have seen that they share common features, especially their symmetry groups agree. Let us first discuss symmetries of $\mathcal{N} = 4$ SYM and AdS spaces. The $\mathcal{N} = 4$ supersymmetric Yang-Mills theory has a full $SU(2,2|4)$ super conformal symmetry which is the $\mathcal{N} = 4$ super Poincaré algebra. One can identify its bosonic subgroup as $SO(2,4) \times SU(4)$. The group factors into the conformal group $SO(2,4)$ in $1+3$ dimensions and the R-symmetry group $SU(4)_R$. On the other hand we have seen that AdS$_{d+1}$ has isometry group $SO(2, d-1)$. Since we consider four-dimensional SYM theory we see that for $d = 5$ on the AdS side we will get the correct symmetry group. This suggests a space AdS$_5$ in order to match the global symmetries. Now recall the construction of $\mathcal{N} = 4$ SYM from a ten-dimensional $\mathcal{N} = 1$ SYM action by compactifying on a six-torus. Because of this the relation $SU(4) \simeq SO(6)$ is more intuitive by saying we rotate the six compact dimensions. Notably $SO(6)$ is actually the isometry group of a five-sphere. The claim is therefore that we need a S^5 on the AdS side. This together with AdS$_5$ will add up to ten dimensions.
We have now seen that there exists a correspondence between the symmetries of $\mathcal{N} = 4$ SYM theory and a space AdS$_5 \times S^5$. Let us consider for the purpose of this discussion Type IIB string theory sourced with N Dp-branes discussed in section 11.2. We have already seen in (283) that these branes will give rise to a supersymmetric $U(N)$ gauge theory living on the world volume. In section 11.1 we have seen that the four-form $C^{(4)}$ is part of the massless spectrum of the IIB string and there is a D3-brane charged under it. This leads to further investigation of D3-branes. For a stack of N D3-branes the

charge of the D3-branes is then given by the flux[27] of the five-form through the five-sphere surrounding it, i.e $N = \int_{S^5} G^{(5)}$. This is because the presence of the D-branes is implicit in the IIB theory since the four-form is the source for the D-brane charge. So having N units of five-form flux is identical to N coincident D3-branes. We furthermore have a constant dilaton $e^{-2\phi_0}$ as part of the equations of motion following from (271). The solution of the ten-dimensional metric (284) is, for $p = 3$, given by:

$$ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-\frac{1}{2}} d\vec{x}^2 + \left(1 + \frac{L^4}{r^4}\right)^{\frac{1}{2}} d\vec{y}^2 \tag{286}$$

where we denoted $\vec{x} = \{x_0, \ldots x_3\}$ as the longitudinal directions and $\vec{y} = \{y_1, \ldots y_6\}$ the transverse directions to the brane. L is the characteristic length scale and we also define the radial distance r from the brane $r^2 = \vec{y} \cdot \vec{y}$. The brane has a four-dimensional world volume and therefore six transverse directions. The key observation made by Maldacena [4] was that this geometry in the limit where $r \ll L$ reduces as follows:

$$ds^2_{r \to 0} = \frac{r^2}{L^2} d\vec{x}^2 + \frac{L^2}{r^2} d\vec{y}^2 \tag{287}$$

Let us parametrise the six-dimensional ambient space as $d\vec{y}^2 = dy^2 + y^2 d\Omega_5^2$. Defining $\rho^2 = \frac{L^4}{r^2}$ we find the following line element for the D3-brane geometry embedded in ten dimensions:

$$ds^2_{r \ll L} = \frac{\rho^2}{L^2}(d\vec{x}^2 + d\rho^2) + L^2 d\Omega_5^2 \tag{288}$$

Comparing this to (238) this is the space $\mathrm{AdS}_5 \times S^5$ and describes the near-brane geometry. Note that L serves as both the radius of the anti-de Sitter space and the sphere. We have used the near horizon limit $r \ll L$ to find a symmetric space solution. Also observe that for $r \to \infty$ we find a flat space. This means that the asymptotics of the metric is governed by two maximally symmetric spaces, the flat Minkowski space at infinity and an anti-de Sitter space at the origin. Recall that the conformal boundary of AdS is at $r = 0$. In this situation this corresponds to a four-dimensional Minkowksi space $\mathbb{R} \times S^3$ which is the space that lives near the brane. Because of (283) we thus find a four-dimensional conformal theory on the boundary of AdS_5.
Therefore Maldacena conjectured the following [4]:

[27] The language comes from Stokes' theorem.

"Type IIB string theory on $AdS_5 \times S^5$ is dual to a gauge theory with super conformal group $SU(2,2|4)$"

By looking at the symmetry group we can in principle identify the gauge theory already by $SU(N)$ $\mathcal{N}=4$ SYM. Requiring $\mathcal{N}=4$ super Poincaré invariance identifies the conformal theory as $\mathcal{N}=4$ SYM. Here we observe the subtlety that we have an $SU(N)$ theory rather than $U(N)$. The extra $U(1)$ factor from $SU(N)\times U(1) = U(N)$ becomes irrelevant when considering large N values. There are furthermore arguments that tell us the $U(1)$ factor carries center-of-mass dynamics of the brane and does not change the worldvolume theory [23]. The last symmetry which is also present in both theories is the discrete $SL(2,\mathbb{Z})$ group acting on the coupling constant g_{YM} in Yang-Mills while on the string theory side it can be shown to act on the combination $\tau = C^{(0)} + ie^{2\phi}$. From this one can infer that $g_s \sim g_{\text{YM}}^2$ since $g_s \sim e^{2\phi}$, where the dilaton ϕ is constant in the presence of a D3-brane. The full conjecture is therefore:

"Type IIB string theory on $AdS_5 \times S^5$ with N units of five-form flux is dual to four- dimensional $SU(N)$ $\mathcal{N}=4$ SYM."

This is referred to as the AdS/CFT conjecture or correspondence. In order to make this statement precise, one needs to deliver a "dictionary" relating quantities on both sides of the correspondence. We will not give any details on how to deduce these relations. It is however worth mentioning that the field content of $\mathcal{N}=4$ SYM is fixed by supersymmetry. On the string theory side is is possible to find a relation between all states living on AdS$_5$ and operators that generate states on the side of Yang-Mills theory.
A detailed classification of Yang-Mills operators together with their supergravity dual can be found in [25]. Furthermore, we need to specify the regime where the correspondence is valid. The string theory could live in a spacetime where the curvature is rather small, or large. In order to give a quick exposition of these regimes recall the general action for a D-brane as a non-linear σ model (281):

$$S \sim T_p \int_{\Sigma_{p+1}} d^{p+1}\sigma \; e^{-2\phi}\sqrt{G} \tag{289}$$

with G the worldvolume metric on the brane given by the usual expression:

$$G_{ab} = \frac{\partial x^\mu}{\partial \sigma^a}\frac{\partial x^\nu}{\partial \sigma^b} G_{\mu\nu} \tag{290}$$

Where we split the coordinates μ, ν in to longitudinal and transversal parts along the brane. If we plug in the expression for the background geometry and the Dp-tension one can express the curvature radius L of AdS$_5$ in terms of the string coupling g_s.

$$L^4 = \frac{N g_s}{2T_3} \tag{291}$$

The factor N comes from the five-form flux through the five-sphere [21]. From this relation we can distinguish two regimes of validity of the duality. First one defines the effective coupling constant[28] $\lambda = g_s N = g_{\text{YM}}^2 N$. Now we look at fixed values of this coupling constant while changing the string tension. One needs to consider the regimes with small or a large curvature radius for the AdS$_5$. This will then in turn correspond to a weakly coupled or strongly coupled theory. The only relevant length scale in string theory is the string length $l_s \sim T^{-1}$. Therefore we consider $L \ll l_s$ and $L \gg l_s$. Small length scales correspond to high energies and we have that $L \ll l_s$ corresponds to full Type IIB string theory. If on the other hand $L \gg l_s$ the system may be studied in Type IIB supergravity corresponding to the low energy regime. Because it appears to be extremely complicated to quantising string theory on AdS$_5 \times S^5$ one focuses on the supergravity limit. However it is still conjectured that the duality holds for all values of L. A weaker conjecture is found when we make the following approximation; assume we take λ a fixed value but take $N \to \infty$. This in turn corresponds to $g_s \to 0$ and thus a weakly coupled string theory. Taking $\lambda \gg 1$ we are in the non-perturbative classical version of string theory. This corresponds to large values of L. On the side of SYM this "large N" scenario corresponds to perturbation theory where $\lambda = g_{\text{YM}} N \ll 1$[7]. This is why the AdS/CFT correspondence is also called a strong-weak duality. To summarise this section, what we have found is that there exists a duality between string theory on anti-de Sitter space-times and a conformal field theory living on the boundary. Therefore the correspondence is a realisation of the **holographic** principle. Furthermore it is a duality in the sense of strong$\leftrightarrow$weak coupling constants between the regarded theories on either side of the correspondence. The described setting is the best known example of the correspondence.

It has been conjectured that any string theory[29] on AdS$_n \times X^p$ with

[28] This coupling is known in the literature as the 't Hooft coupling [7].
[29] This includes M-theory in eleven dimensions with various branes.

$n + p = D$ and X some compact space has a conformal field theory dual description. Examples are[22, 26]:

> "*M-theory on* $AdS_4 \times S^7$ *is dual to a three-dimensional* $\mathcal{N} = 6$ *Chern-Simons theory*"

and

> "*M-theory on* $AdS_7 \times S^4$ *is dual to a six-dimensional* $\mathcal{N} = 2$ *conformal field theory*"

The latter is the famous six-dimensional $\mathcal{N} = (2, 0)$ theory and has as far as we know no Lagrangian description [26]. This concludes the section on the AdS/CFT correspondence. In what follows we will introduce a new class of SYM theories and see that we can formulate an AdS/CFT correspondence for these theories as well.

Part IV

Marginal deformations of $\mathcal{N} = 4$ super Yang-Mills theory

13 $\mathcal{N} = 4$ super Yang-Mills theory

In section 7.4 we have derived the vanishing of the β-function using super field formalism and identified the $\mathcal{N} = 4$ theory by using $\mathcal{N} = 1$ super fields for 1 loop. There exist more general proofs for up to 5-loop calculations. In general it is expected that $\mathcal{N} = 4$ SYM is finite at all loop orders. If one starts at the most general $\mathcal{N} = 4$ action one can include a superpotential that is cubic in the three chiral super fields. This was left out earlier since it is of higher order in the interactions and the chiral fluctuations will be set to zero after taking the second functional variation. There is also the non-renormalisation theorem [16] which states that the coupling in front of the superpotential gets no corrections. The full $\mathcal{N} = 4$ SYM action in $\mathcal{N} = 1$ superspace is uniquely fixed by supersymmetry and the gauge group [2, 27]:

$$S_{\mathcal{N}=4} = \mathrm{tr}\Big(\frac{1}{g^2}\int d^8z\ e^{-V}\bar{\Phi}^i e^V \Phi_i + \frac{1}{2g^2}\int d^6z\ G^\alpha G_\alpha + \Big(ig\int d^6z\ \epsilon^{ijk}[\Phi_i, \Phi_j]\Phi_k + c.c\Big)\Big) \tag{292}$$

with $i = 1, 2, 3$ and ϵ^{ijk} is fully anti-symmetric Levi-Civita symbol with $\epsilon^{123} = 1$. All fields belong to the adjoint representation of the gauge group. Besides the $\mathcal{N} = 1$ supersymmetry the theory itself has another $SU(3)$ symmetry which acts on the chiral super fields Φ_i [14]. The theory has an $SU(4)$ R-symmetry we denoted by $SU(4)_R$ which becomes apparent in the supersymmetry transformations[30]. The theory contains a vector, four fermions and six scalars which can already be seen in the superfield formulation given. The component version of this theory may be recovered by compactifying a ten-dimensional SYM theory on a six-torus T^6. The ten-dimensional theory has an $\mathcal{N} = 1$ action schematically given by [21]:

$$S_{10d} \sim \int d^{10}x\ \mathrm{tr}(-\frac{1}{4}F^{MN}F_{MN} + \frac{i}{2}\bar{\lambda}\Gamma^M D_M\lambda) \tag{293}$$

with F_{MN} the ten-dimensional field strength of the gauge field A_M, λ a spinor in ten dimensions and Γ^M are 32×32 gamma matrices[31]. Compactifying

[30] In this expression the manifest symmetry looks more like $SU(4) = SU(3) \times U(1)$. In which the $SU(3)$ represents the symmetry of the chiral super fields and the $U(1)$ is so to say the residual $\mathcal{N} = 1$ R-symmetry because we express the action in $\mathcal{N} = 1$ super space.

[31] The Clifford algebra consist of $2^{D/2} = 2^5 = 32$-dimensional matrices

this theory on a six-torus means neglecting field dependence on the 6 extra directions. Since these directions can be arbitrarily rotated in the compact space this yields a global $SO(6) \simeq SU(4)$. We recover the full R-symmetry of $\mathcal{N} = 4$ SYM in four spacetime dimensions. The theory is also invariant under discrete $SL(2, \mathbb{Z})$ transformations acting on the coupling constant [21] as we have seen earlier.

14 Deformations of $\mathcal{N} = 4$ SYM and the Lunin-Maldacena solution

$\mathcal{N} = 4$ SYM given by (292) admits a dual supergravity description given by the low energy limit of Type IIB super string theory in the geometry of a stack of N D3-branes. The rest of this thesis will be dedicated to give an overview over so called **deformations** of the $\mathcal{N} = 4$ theory. By deformations we mean changes to the action that deform the theory in such a way that it does no longer correspond to $\mathcal{N} = 4$ SYM, retains super conformal invariance but is less restricted in the sense of supersymmetry. These deformed theories do also have a supergravity description where the geometrical setting corresponds to a deformed geometry of $\text{AdS}_5 \times S^5$. The first task is to classify the relevant kind of deformations that one is allowed to introduce to the action (292). Subsequently the analysis can be carried further, formulating a description of the deformed theory in the context of a geometrical interpretation via the AdS/CFT correspondence. In the following sections we describe the classification of different deformations, discuss the renormalisation group flow of these deformations in some detail and finally come to the supergravity solution found by Lunin and Maldacena [6].

14.1 Leigh-Strassler Deformations

The objective of our analysis is to include additional terms to the bare $\mathcal{N} = 4$ action that break a certain amount of supersymmetry but maintain super conformal invariance at the quantum level, i.e. $\beta(g) = 0$. Requiring super conformal invariance it should be clear that we may only add terms that will not change the scaling behaviour of the theory. From general quantum field theory we know that operators that add to the action do have their own respective properties under rescalings. For a D dimensional field theory consisting of operators with scaling dimension Δ we have that the operators [27, 28]:

1. $\underline{\Delta > D}$ are called irrelevant and their contribution to the renormalisation group flow goes to zero with decreasing the renormalisation scale Λ.

2. $\underline{\Delta < D}$ are called relevant and their contribution to the renormalisation group flow grows with decreasing the renormalisation scale Λ.

3. $\underline{\Delta = D}$ are called marginal and their contribution to the renormalisation group flow depends logarithmically on the scale Λ.

This simple observation is enough to conclude that we are interested in marginal operators. If we deform the $\mathcal{N} = 4$ theory with marginal operators that come with their own coupling constant we can maintain super conformal invariance but will break supersymmetry partially. If one adds a term $\delta\mathcal{L} \sim h\mathcal{O}$ to the Lagrangian with $\mathcal{O}$ a marginal operator a line of fixed points will be added to the new theory. We break supersymmetry from the $\mathcal{N} = 4$ theory to a new theory down to $\mathcal{N} = 1$ along a line parametrised by the coupling h. The deformed theory is still $\mathcal{N} = 1$ manifest supersymmetric since it is formulated in $\mathcal{N} = 1$ superspace. The super conformal invariance needs to be restored by imposing the constraint that $\beta_g(g, h)$ and $\beta_h(g, h)$ vanishes. By looking at the most general form of the β-function (191) it can be easily seen that for this theory conformal invariance is restored when the anomalous dimension of the chiral super field vanishes. In principle the two β-functions $\beta_g(g, h)$ and $\beta_h(g, h)$ can depend on each other. The constraint $\beta_g(g, h) = \beta_h(g, h) = 0$ then defines a curve in theory space (g, h) on which fixed points lie and the theory is in its super conformal phase. Leigh and Strassler [5] have shown that there exist two different terms that may be added to the theory to maintain super conformal invariance. These two terms may come with a coupling h and h'. Therefore the full constraint reads:

$$\eta_\Phi(g, h, h', q) = 0 \tag{294}$$

where q is another coupling constant that will be introduced later and acts as a free parameter. It defines a three dimensional surface in theory space. By solving the above constraint they found that within this three dimensional space there is a whole class of finite $\mathcal{N} = 1$ theories and the original $\mathcal{N} = 4$ theory is represented as a line in the three dimensional theory space of these

$\mathcal{N} = 1$ theories.
Let us now discuss the possible marginal operators that we can add to the action that will preserve the super conformal invariance. The usual superpotential in the $\mathcal{N} = 4$ theory has the form:

$$W(\Phi) = \frac{1}{3} f^{ijk} \mathrm{Tr}(\Phi_i \Phi_j \Phi_k) \tag{295}$$

with f^{ijk} totally anti-symmetric and $i, j, k = 1, 2, 3$. Leigh and Strassler found in an extensive analysis [5] of different extended supersymmetric theories that there is only a three-parameter family of marginal superpotentials that will preserve $\mathcal{N} = 1$ supersymmetry.

The deformed superpotential in its most general form can be written as:

$$W(\Phi) = d^{ijk} \mathrm{Tr}(\Phi_i \Phi_j \Phi_k) \tag{296}$$

where d^{ijk} is now fully symmetric. For a symmetric object with three indices there are $\frac{3(3+1)(3+2)}{3!} = 10$ independent components. So we add 10 independent terms to the superpotential. Three different kind of terms will be generated, schematically:

$$W(\Phi) \sim d_1 \Phi\Phi\Phi + d_2 \Phi^3 + d_3 \Phi^2 \Phi \tag{297}$$

In [29] it is shown using the requirement for exact marginality that only the first two terms are to be considered because the last one spoils the fact that $\beta_{d_3}(g, h, h', q, d_1, d_2, d_3) = 0$. We consider the $\mathcal{N} = 1$ theory with deformed superpotential:

$$W(\Phi) = d_1(\Phi_1\Phi_2\Phi_3 + \Phi_1\Phi_3\Phi_2) + d_2(\Phi_1^3 + \Phi_2^3 + \Phi_3^3) \tag{298}$$

Conveniently renaming the coupling constants we write the deformed action:

$$\begin{aligned} S_{\mathrm{DEF}} = \mathrm{tr}\Big(& \frac{1}{g^2} \int d^8 z\ e^{-V} \bar{\Phi}^i e^V \Phi_i + \frac{1}{2g^2} \int d^6 z\ G^\alpha G_\alpha \\ & + \Big(ih \int d^6 z\ (q \Phi_1 \Phi_2 \Phi_3 - \frac{1}{q} \Phi_1 \Phi_3 \Phi_2) + c.c \Big) \\ & + \Big(i \frac{ih'}{3} \int d^6 z\ (\Phi_1^3 + \Phi_2^3 + \Phi_3^3) + c.c \Big) \Big) \end{aligned} \tag{299}$$

It can be checked [27] that indeed the requirement $\beta(g, hq, \frac{h}{q}, h') = 0$ for all couplings leads to the vanishing of the anomalous dimension η_Φ. This means that according to the systematic analysis of Leigh and Strassler the action above supplemented with $\eta_\Phi = 0$ does in fact represent the most general exact marginal deformation of the $\mathcal{N}=4$ SYM theory. We can recover the original theory by setting $h=g$, $h'=0$ and $q=1$.

The symmetries of (299) are not that obvious but there is of course a manifest $U(1)_R$ symmetry that we get by breaking $\mathcal{N}=4$ to $\mathcal{N}=1$. Also, the original $SU(3)$ symmetry of rotating the chiral super fields is lost, there is a mere $\mathbb{Z}_3 \times \mathbb{Z}_3$ residual symmetry left that acts on both terms of the deformed super potential as cyclic permutations of Φ_i's on the first term and $(\Phi_1, \Phi_2, \Phi_3) \to (\Phi_1, \omega\Phi_2, \omega^2\Phi_3)$ with $\omega = \exp(\frac{\pi i}{3})$. There is also symmetry under exchange of Φ_i with Φ_j and under q with $-\frac{1}{q}$.

At this point we restrict ourselves to the case $h' = 0$. In this scenario the $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry is enhanced to a global $U(1)_1 \times U(1)_2$. The permutations and the symmetry $(\Phi_1, \Phi_2, \Phi_3) \to (\Phi_1, \omega\Phi_2, \omega^2\Phi_3)$ is still present but we gain a global $U(1)_1 \times U(1)_2$ which acts as:

$$\begin{aligned} U(1)_1 &: (\Phi_1, \Phi_2, \Phi_3) \to (\Phi_1, \exp(i\alpha)\Phi_2, \exp(-i\alpha)\Phi_3) \\ U(1)_2 &: (\Phi_1, \Phi_2, \Phi_3) \to (\exp(-i\gamma)\Phi_1, \exp(i\gamma)\Phi_2, \Phi_3) \end{aligned} \tag{300}$$

After setting $h' = 0$ the action is called the β-deformed $\mathcal{N}=4$ SYM theory and is given by:

$$\begin{aligned} S_{\text{DEF}} = \text{tr}\Big(&\frac{1}{g^2}\int d^8z\; e^{-V}\bar{\Phi}^i e^V \Phi_i + \frac{1}{2g^2}\int d^6z\; G^\alpha G_\alpha \\ &+ \Big(ih\int d^6z\; (q\Phi_1\Phi_2\Phi_3 - \frac{1}{q}\Phi_1\Phi_3\Phi_2) + c.c\Big)\Big) \end{aligned} \tag{301}$$

where $q \equiv e^{i\pi\beta}$. This $\mathcal{N}=1$ theory is not super conformally invariant. We next will give a brief outline of the constraint $\eta_\Phi(g,h,q) = 0$.

14.2 The vanishing of $\eta_\Phi(g,h,q)$

So far we have not yet specified whether β is real or complex. In the following we will assume that $\beta \in \mathbb{R}$. If β is real we have that $q\bar{q} = 1$. Or in our notation $q \times \frac{1}{q} = 1$ [27]. It turns out that perturbatively the Feynman diagrams will be independent of q. Also, since we are after the β-function for the chiral superfields one has to determine the n-loop contribution to the Φ-propagato. This propagator gets loop corrections from the super potential three point interaction but also loops from the vector super field have to be included.

The observation is that since the diagrams will be independent of q the only degree of freedom is choosing h properly such that $\beta_h(g, h, q) = 0$. This is proportional to the anomalous dimension of the renormalisation constant of Φ_i. The different loop graphs can only cancel if we require:

$$h^2 = g^2 \tag{302}$$

hence, $h^2 = g^2$ represents the super conformal phase of the deformed $\mathcal{N} = 1$ theory. An equally strong conclusion may be drawn by observing that since this result is independent of q it holds for any deformed theory and we have a two-parameter family of $\mathcal{N} = 1$ super conformal field theories parametrized by the line $h^2 = g^2$ in theory space. The special case of $\mathcal{N} = 4$ SYM is obtained by only setting $q = 1$ ($\beta = 0$).

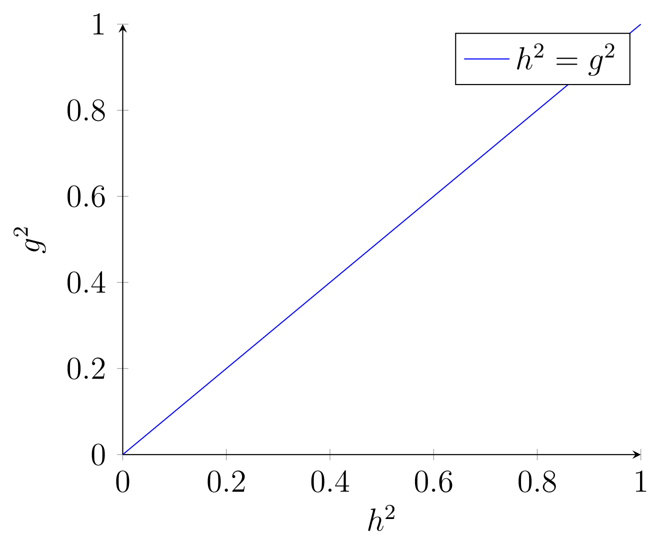

Figure 1: The line of finite $\mathcal{N} = 1$ super conformal field theories deformed by a parameter q.

The line graphically shown in Figure 1 represents all finite $\mathcal{N} = 1$ super conformal field theories including the one where $q = 1$ ($\beta = 0$). Thus $\mathcal{N} = 4$ SYM belongs to a larger class of deformed theories and this thus means that the β-function for h vanishes identically on this line. The theory which lives on this line has the action:

$$S_{\text{DEF}} = \text{tr}\Big(\frac{1}{g^2}\int d^8z\; e^{-V}\bar{\Phi}^i e^V \Phi_i + \frac{1}{2g^2}\int d^6z\; G^\alpha G_\alpha \\ + \Big(ig\int d^6z\;(q\Phi_1\Phi_2\Phi_3 - \frac{1}{q}\Phi_1\Phi_3\Phi_2) + c.c\Big)\Big) \tag{303}$$

with $q = \exp(i\pi\beta)$ and is the most general β-deformed finite $\mathcal{N} = 1$ super conformal field theory. The theory does still have the global $U(1)_1 \times U(1)_2$ symmetry from (300).

14.3 The first step to a solution

One can rewrite the action (303) in terms of a newly introduced product, lets call it $\star$-product defined for two functions F and G by:

$$F \star G = e^{i\pi\beta(Q_F^1 Q_G^2 - Q_F^2 Q_G^1)} FG \tag{304}$$

where Q_A^i with $i = 1, 2$ and $A = F, G$ denote the $U(1)_1 \times U(1)_2$ charges. The action then reads:

$$S_{\text{DEF}} = \text{tr}\Big(\frac{1}{g^2}\int d^8z\; e^{-V}\bar{\Phi}^i e^V \Phi_i + \frac{1}{2g^2}\int d^6z\; G^\alpha G_\alpha \\ + \Big(ig\int d^6z\;(\Phi_1\star\Phi_2\star\Phi_3 - \Phi_1\star\Phi_3\star\Phi_2) + c.c\Big)\Big) \tag{305}$$

This introduces the notion of a non-commutative field theory [30] even though the resulting theory is a ordinary field theory [6]. Furthermore, this theory admits an $SL(2,\mathbb{R})$ symmetry in the sense that the complex structure κ associated with the two $U(1)$ symmetries[32] transforms under $SL(2,\mathbb{R})$. The theory is invariant if also β transforms like a modular form:

$$\kappa \to \kappa' = \frac{a\kappa + b}{c\kappa + d} \\ \beta \to \beta' = \frac{\beta}{c\kappa' + d} \tag{306}$$

i.e. β "lives" on the two-torus. Since it was shown that this theory is in fact is a super conformal field theory one may conjecture that this theory does

[32]The complex structure here refers to the two-torus of the two $U(1)$ symmetries and also the general coupling constant g.

have a supergravity dual description. In the next section we motivate the existence of a dual description on the gravity side by comparing symmetries and present the arguments given by [27, 29, 6].

14.4 The Lunin-Maldacena solution

Because of the AdS/CFT correspondence relating Type IIB superstring theory and $\mathcal{N} = 4$ SYM we will be considering the Type IIB sector only. Recall the low energy effective action for Type IIB string theory (271). From this action, by varying it with respect to each field, one can derive the equations of motion. The self-duality constraint $G^{(5)} = \star G^{(5)}$ of the five-form has to be added by hand. It has been shown by [31] that this theory exhibits a global $SU(1,1) \simeq SL(2,\mathbb{R})$ symmetry. It acts on the complex field $\tau = ie^{2\phi}+C^{(0)}$ as a modular transformation and the fields B, C as a doublet[33]. Let us assume that we place this theory in a background which has two extra $U(1)$ symmetries that are realised geometrically. In other words the geometry contains a two-torus. This argument comes from the fact that the original IIB theory has a metric on $\mathrm{AdS}_5 \times S^5$. However, one should keep the anti-de Sitter solution because of supersymmetry arguments in the theory but is able to deform the S^5. The two-torus is then embedded in the line element for the S^5. Therefore the IIB theory in ten dimensions has another $SL(2,\mathbb{R})$ symmetry given by the one acting on the complex structure of this two-torus.
Recall that the string couples on the worldsheet effectively to a background matrix $E = g + B$ introduced in (261) but likewise for the superstring. One can define another modular form which transforms under yet another $SL(2,\mathbb{R})$. This is called the Kähler modulus [32] and is given by $\rho = B+i\sqrt{g}$, where $\sqrt{g}$ is the volume of the two-torus. The full symmetry of the IIB supergravity side is then given by an $SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$. Here we used that the $SL(3,\mathbb{R})$ is the combined two $SL(2,\mathbb{R})$'s from the non-geometric Kähler modulus and the strong-weak duality. The last $SL(2,\mathbb{R})$ is the geometric symmetry as a result of the transformation of the geometric two-torus. The important $SL(2,\mathbb{R})$ transformation is the non-geometric one acting on the Kähler modulus ρ. If we now compactify the ten-dimensional theory on a torus T^2 we are left with an eight-dimensional theory with symmetry $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ [6], where the first acts on the complex structure of

[33] In the process of quantisation of the full string theory this symmetry becomes $SL(2,\mathbb{Z})$ due to charge quantisation. This means this symmetry relates the strong coupling limit of the string theory to the weak coupling of supergravity.

the torus and the other one on the Kähler modulus. Taking a fixed element from $SL(2,\mathbb{R})$ and letting it act on ρ then, proposed by [6] and developed by [33], generates new solutions to the original IIB supergravity action (271). Explicitly,

$$\rho \to \rho' = \frac{1}{1+\beta\rho} \tag{307}$$

can be seen as a solution generating operation[34]. To complete the argument we need to go back to the geometric setting with two $U(1)$ symmetries and recall that in string theory the objects that give rise to $U(1)$ symmetries are D-branes. If we have a D-brane in the original geometry that is invariant under both global $U(1)$ symmetries one can perform the transformation above and describe the brane in the new background. As it turns out [6] we will have the same world volume theory living on the brane after the transformation up to a phase. This phase is exactly given by the star-product given in (304) up to a non-commutativity parameter β. It is therefore conjectured that the new world volume theory on the brane will be an $\mathcal{N}=4$ SYM theory with an extra phase in the original Lagrangian given by the star product that we introduced in the action (305). It has been shown by Witten and Seiberg [30] that an open string theory with non-vanishing B-field can be described as an non-commutative field theory with an extra parameter Θ that transforms the open string metric G under T-duality like $G+\Theta = \left(\frac{1}{g+B}\right)$, which is exactly ρ^{-1}. If we identify the non commutativity parameter $\Theta = \beta$ we complete the argument. Hence we get the marginally β-deformed theory as the new world volume theory on a D-brane after an $SL(2,\mathbb{R})$ transformation of the Kähler modulus of the original ten- dimensional IIB theory compactified on a two-torus. The resulting deformed supergravity solution after the $SL(2,\mathbb{R})$ transformation on ρ is explicitly given in [6, 27]. As a note: The supergravity dual of a β-deformed theory with a superpotential of the form $W \sim \sum_i^3 \Phi_i^3$ has yet to be found [27].

14.5 The FRGE of the deformed theory

A question resulting from the previous section is if the non-perturbative functional renormalisation group techniques developed in 7 are capable of

[34] As a side note, this transformation can be seen as performing a T-duality on one of the S^1, changing coordinates and followed by another T-duality. Where T-duality acts according to the Buscher rules[24]

deriving the results found in [27]? Even though the superspace formalism was used it is stated clearly how perturbative methods were fundamental. One would want to include a non perturbative approach to the calculation $\eta_\Phi = 0$ and see rather different constraints come up during the full calcuation. Recall however that one still needs to truncate the action at some point in order to use heat kernel techniques to calculate the traces. The relevant part of this calculation would be a projection onto the operators that belongs to the superpotential of the three chiral superfields. We suppose that it should be possible to reproduce the super conformal phase where $h^2 = g^2$. The actual calculation has not been done but could be of great interest for future research in this area. We believe that such calculation could also lead to new insights of the supergravity theory.

15 Summary and outlook

String theory is regarded as a theory of quantum gravity which is inherently supersymmetric. One of the first ideas of string theory was to resolve the non-renormalisability of general relativity. We have accepted string theory as a quantum theory of gravity in the sense that it is able to reproduce Einsteins field equations by requiring conformal invariance. String theory contains more than just one-dimensional fundamental strings, it is a theory of higher dimensional dynamical objects. This contains hypersurfaces called D-branes on which open strings must end. Together with these D-branes come extra symmetries and one can show that a stack of N D-branes carries a $U(N)$ gauge symmetry. Together with other arguments one can show that there exists a connection between certain types of string theory with D-branes and these $U(N)$ gauge theories. This connection goes by the name of AdS/CFT correspondence and it is a framework in which one can compute strongly coupled conformal field theory correlators from a weakly coupled gravitational dual theory.

In this thesis we investigated the well-known theory, $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. For this theory it has long been advocated that it is a supersymmetric conformal field theory which has a dual description in terms of Type IIB string theory on $\text{AdS}_5 \times S^5$. With this in our mind we showed conformal invariance of the $\mathcal{N} = 4$ super Yang-Mills theory by showing that its β-function vanishes. Because we are motivated by AdS/CFT which is a strong/weak duality we based our computation of the renormalisation group

flow of the super Yang-Mills gauge coupling g on the Wetterich equation which is non-perturbative and therefore applies to small and large coupling constants. Furthermore, in order to keep supersymmetry at each step manifest we used the superspace formulation of the $\mathcal{N}=4$ theory. We reviewed the general action for an $\mathcal{N}=1$ supersymmetric Yang-Mills theory in the superspace formalism and computed together with the gauge fixing action the effective action up to quadratic order in the vector superfield. The actual computation was based on heat kernel techniques to calculate the trace on the right hand side of the Wetterich equation and found, up to a sign, for some proper truncation of the action and projection on to the background field strength the β-function for $\mathcal{N}=1$ super Yang-Mills theory. The ghost action was found to require a factor $\frac{1}{g^2}$ in order to cancel the chiral super field contribution. We expect that this discrepancy can be attributed to an inconsistency in the conventions used in our computation, though we are not exactly sure where it might come from. The analysis of the $\mathcal{N}=4$ theory used that its matter content can be arranged in one $\mathcal{N}=1$ vector supermultiplet and three $\mathcal{N}=1$ chiral supermultiplets. We found that this in combination with three ghost indeed leads to a zero β-function.
We conclude that $\mathcal{N}=4$ SYM theory is a super conformal field theory. We believe that our techniques apply to other theories in superspace formalism. A point in the calculation that is worth coming back to is the fact that the left and right hand side of the Wetterich equation only combines to a zero β-function if the vector part goes to zero. In this vision it was very illuminating to consider the superfield formulation of the super Yang-Mills theory in order to reproduce well-known results and see how we can make the Wetterich equation compatible with said formalism.

As an extension to the original statement of the AdS/CFT correspondence we elaborated the idea of partial supersymmetry breaking by deforming the superpotential of the original theory and finding the constraints that come by requiring super conformal invariance. These calculations have so far only been done perturbatively and we argued that a non perturbative treatment using the techniques developed in this thesis might contribute to a better understanding of the new supergravity theory that will be the dual description of this so called β-deformed $\mathcal{N}=4$ theory.

16 Nederlandstalige samenvatting voor niet-fysici

In dit master verslag is gewerkt op het grensvlak tussen twee van de meest belangrijke fundamentele theorieën die de moderne natuurkunde op dit moment heeft. Het draait hier bij om de twee pilaren waar elk natuurkundig verschijnsel op gebaseerd is; de theorie van het allerkleinste genaamd de **kwantum mechanica** en de theorie van het allergrootste genaamd de **algemene relativiteits theorie**. Het centrale probleem van de moderne natuurkunde, die beschreven wordt door of de kwantum mechanica of de algemene relativiteitstheorie is het feit dat deze twee theorieën tot op heden niet in één geunificeerde theorie van alles beschreven kunnen worden. Een van de redenen hiervoor zal zijn dat deze theorieën gebaseerd zijn op twee volledig verschillende concepten en zelfs filosofieën. Als we kijken naar de wereld van het allerkleinste dat wij kennen dan valt op gegeven moment op dat wij niet verder kunnen kijken dan onze ogen zelf kunnen waarnemen. De vraag hierbij is, is er dan niet meer dat wij niet kunnen zien met het blote oog. Het antwoord hier op is ja, er is een hele wereld die wij niet kunnen waarnemen. Dit werd bevestigd door talloze metingen en experimenten al jaren geleden. Het begon met moleculen, atomen en zo kwam men er vrij snel achter dat zelfs een atoom splitsbaar was in nog kleinere sub-atomaire deeltjes. Dit kunnen bijvoorbeeld protonen, neutronen of elektronen zijn. Weer een hele tijd later vond men echter dan zelfs het proton uit weer kleinere deeltjes bestaat, dit zijn zogenaamde quarks. Nou is het de taak van een natuurkundige om zelfs deze deeltjes zo nauwkeurig mogelijk te beschrijven. Het probleem is hier bij, zoals aangegeven, dat wij dit niet meer kunnen waarnemen met het blote oog. Denk hierbij aan afstanden kleiner dan 10^{-15} meter[35]. Dit maakt de beschrijving van de bewegingen etcetera erg lastig en onnauwkeurig. Rond 1900 kwam de revolutie, Max Planck vond de kwantum mechanica uit in de zin dat hij ontdekte dat energie van deze "kwantum deeltjes" in discrete pakketjes kwam, vandaar de naam kwantum wat zoveel betekent als een ondeelbare grootheid. Dit leidde in de daar op volgende jaren tot een steeds groter veld van de theoretische natuurkunde waarin bijvoorbeeld Erwin Schrödinger liet zien dat de deeltjes beschreven kunnen worden als een soort golven. Het probleem van de hele kwantum mechanica was dat

[35]Oftewel 0.000000000000001 meter. Ter vergelijking, de diameter van een menselijk haar is 0.1 mm. Dat is 0.0001 meter.

zij baseert op een waarschijnlijkheid. Niks is zeker in de kwantum mechanica, alles wordt bepaald door de waarschijnlijkheid van een gebeurtenis. Zo is het bijvoorbeeld nooit zeker of een deeltje daadwerkelijk op een bepaalde plek is. Er is enkel een waarschijnlijkheid die aangeeft dat het deeltje zich op die positie bevindt. Samen met kansen komen onzekerheden, zo heeft bijvoorbeeld Werner Heisenberg laten zien dat er een onzekerheid tussen de snelheid en de positie van een deeltje bestaat. Dit houdt in dat men nooit tegenlijkertijd kan meten hoe snel het deeltje is of waar het zich bevindt. Een gevolg hier van zijn enorme fluctuaties in bijvoorbeeld de snelheid als men héél erg exact weet waar het deeltje zich bevindt. Kortom, de wereld van de kwantum mechanica is een heen en weer vliegende wereld waar in niks met 100% zekerheid te zeggen is. Er zijn in totaal drie verschillende krachten die relevant zijn op de schaal van de sub-atomaire deeltjes. Dit is de welbekende elektromagnetische kracht die verteld hoe elektrisch geladen deeltjes elkaar kunnen aantrekken bijvoorbeeld, de zwakke kernkracht welke verantwoordelijk is voor radioactief verval en de sterke kernkracht die er voor zorgt dat atoomkernen niet uit elkaar vliegen door de positieve protonen. Alle verschijnselen die gevoelig zijn voor bovengenoemde drie natuurkrachten zijn onderdeel van een geheel dat tegenwoordig onder de naam **het standaard model van de deeltjesfysica** valt.

Er is nog een andere fundamentele natuurkracht in het universum. Deze kracht is voornamelijk relevant voor hele zware objecten, de zwaartekracht. Laten we nou even naar de andere kant van de medaille kijken, alle grote en massieve objecten in ons universum, denk bijvoorbeeld aan sterren zoals de zon worden beschreven door de zwaartekracht. De krachten die op subatomair niveau belangrijk zijn zijn ineens heel erg onbelangrijk op deze schaal, net als dat de zwaartekracht irrelevant is op de schaal van de sub-atomaire deeltjes. Al ruim 400 jaar geleden begrepen wetenschappers zoals Isaac Newton hoe de zwaartekracht werkte in combinatie met waarnemingen van de hemellichamen zoals de draaiing van de aarde om de zon. Een lange tijd later werd steeds meer bekend vanuit de wiskunde van gekromde oppervlakken. Denk hierbij aan een bol, het oppervlak van een bol is gekromd, terwijl het oppervlak van een blaadje papier op de tafel vlak is. Stel nou eens een rubberen oppervlak voor dat gespannen is zodat het vlak is als het blaadje papier. Leggen we nou een massief object op dit rubberen oppervlak zal dit gaan indeuken oftewel, krommen. Door het toevoegen van een massief object hebben wij een ruimte van vlak naar krom vervormd. Gooien we nu een ander kleiner massief object op dit rubberen gekromd vlak zal men zien dat dit

object niet meer in een rechte baan er over heen rolt maar een andere baan volgt. Het waren ideeën zo als deze die in 1915 Albert Einstein ertoe aanzette om op precies deze manier de zwaartekracht uit te leggen. Hij bedacht dat ons universum dat drie ruimte- en een tijds dimensie heeft gewoon een vierdimensionaal rubberen medium is dat men kan krommen door er massa aan toe te voegen en de planeten banen uitgelegd worden door de kromming te volgen die bijvoorbeeld de zon veroorzaakt. Verdere berekeningen en later zelfs metingen lieten zien dat dit een hoognauwkeurige aanname is geweest die tot op heden de beste voorspellingen geeft van alles dat wij zien op die schaal. Het is echter niet zinvol om deze manier van denken te gebruiken in het dagelijkse leven, hiervoor zou een object zoals een auto de ruimte veel te weinig krommen. Deze theorie staat bekend als de algemene relativiteitstheorie van Einstein.
Wanneer we nou even terug gaan naar het stuk over de kwantum mechanica ziet men al vrij snel dat deze twee manieren van denken volledig anders zijn. Met waarschijnlijkheden en onzekerheden aan de ene kant en gekromde ruimten aan de andere kant. Het zijn nou kleine dingen zoals de tijd vlak na de getheoretiseerde oerknal als het onstaan van het universum dat men kan aannemen dat kwantum mechanica en de algemene relativiteitstheorie één geheel moeten zijn omdat in die situatie alle vier natuurkrachten een centrale rol speelden. Ook heeft het esthetische redenen dat men op zoek is naar een theorie waar in deze twee theorieën samenvinden.
We gaan een aantal jaar vooruit richting de 1970er jaren. Het was rond deze tijd dat theoretici een nieuwe theorie begonnen de bedenken die dit probleem zou moeten oplossen. Een van de meest veelbelovende theorieën, genaamd de **snaartheorie** is een kwantum theorie, maar met als grote verschil dat alle sub-atomaire deeltjes nu beschreven worden als microscopische trillende eendimensionale snaartjes en dat men kan laten zien dat deze ook zwaartekrachtsinteractie hebben. Deze verandering zou voor op dat moment onbekende nieuwe inzichten leiden en een geheel nieuwe tak van onderzoek openen die tegenwoordig bekend staat onder de naam **kwantumzwaartekracht**. De snaartheorie komt met volledig absurde nieuwe ideeën zoals een tiendimensionale ruimte waarin de zes extra dimensies op nog kleinere schaal samengerold zouden zijn. Het bleek later zo te zijn dat de snaartheorie niet enkel een theorie zou zijn van eendimensionale snaren, maar eigenlijk in zijn geheel van hogerdimensionale voorwerpen. Zij komt met ob-

jecten genaamd D-branen[36] wat objecten zijn waar op een snaar die open is op moet eindigen. Al deze inzichten uit de snaartheorie zijn gebruikt in dit master verslag. Wij hebben gewerkt aan een berekening die heeft laten zien dat een bepaald onderdeel uit het standaard model van de deeltjesfysica relevant onderdeel is van de snaartheorie en dat het ene niet zonder het andere kan. Dit houdt in dat deze twee theorieën gemeenschappelijke onderdelen hebben en dat zij niet onafhankelijk zijn. Bovendien is gebruik gemaakt van de zogenaamde AdS/CFT dualiteit welke voor het eerst in 1997 door Juan Maldacena is aangetoond binnen de snaartheorie. Deze dualiteit bezegd dat men voor een bepaalde gekromde ruimte, wat dus een zwaartekrachtstheorie is, een kwantum mechanische theorie kan vinden welke aan bepaalde voorwaarden moet voldoen. Op deze manier is het mogelijk een berekening die heel lastig blijkt te zijn aan de ene kant uit de voeren aan de andere kant. Dit maakt erg veel berekeningen een heel stuk gemakkelijker. Het master verslag concludeerd uiteindelijk dat het mogelijk is om deze dualiteit te gebruiken op een hele specifieke manier zonder benaderingen te gebruiken in de kwantum theorie.

A Spacetime and spinor Conventions

In this Appendix we list a few conventions used throughout the thesis. We start with spacetime and spinor conventions. Furthermore we will be working in natural units and set $c = \hbar = G = 1$. Spacetime as a target manifold $\mathcal{M}$ has a background metric $g_{\mu\nu}$. This metric is used to raise and lower indices on bosonic coordinates $x_\mu = g_{\mu\nu}x^\nu$. Spinors are vectors of the group $SL(2,\mathbb{C}) \simeq SO(1,3)$[37]. We denote spinors mostly by ψ, χ or λ. We will be using two-component Weyl spinors that can be left- or righthanded. These are denoted with lefthanded indices α or righthanded indices $\dot{\alpha}$. The Levi-Civita symbol $\epsilon_{\alpha\beta}$ is an invariant element of the group $SL(2,\mathbb{C})$ and is used to raise and lower spinor indices. Define the fully antisymmetric $\epsilon_{12} = 1 = -\epsilon_{21}$.

[36] Het woord "braan" is kort voor membraan.

[37] This is a homeomorphism, not an isomorphism [13]

We then define:

$$\psi\chi = \psi^\alpha\chi_\alpha \tag{308}$$

$$\psi^2 = \psi^\alpha\psi_\alpha = \epsilon^{\alpha\beta}\psi_\beta\psi_\alpha = -\epsilon^{\beta\alpha}\psi_\beta\psi_\alpha = -\psi_\alpha\psi^\alpha \tag{309}$$

$$\bar{\psi}^2 = \bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}\bar{\psi}^{\dot{\alpha}} = -\epsilon_{\dot{\beta}\dot{\alpha}}\bar{\psi}^{\dot{\beta}}\bar{\psi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}} \tag{310}$$

Furthermore define a vector $\sigma^\mu = (1, \vec{\sigma})$ a four-vector[38] of Pauli matrices. Then $\bar{\sigma}^\mu = (1, -\vec{\sigma})$ and we can define generators of the Lorentz algebra: $(\sigma^{\mu\nu})_\alpha^\beta = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_\alpha^\beta$. σ^μ satisfies the $2^{D/2}$ dimensional Clifford algebra $\{\sigma^\mu, \sigma^\nu\} = 2g^{\mu\nu}1$. It follows that for two Weyl spinors ψ and χ:

$$\psi\chi = \chi\psi \tag{311}$$

$$\psi\sigma^{\mu\nu}\chi = -\chi\sigma^{\mu\nu}\psi \tag{312}$$

Even though spinor components are Grassmann numbers, i.e. $\psi_1\psi_2 = -\psi_2\psi_1$.

B Susy transformations

We can use all spinor techniques available and determine the transformations of the various parts of a general superfield $S(z)$. For a given superfield:

$$\begin{aligned} S(z) =& \phi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta^2 M(x) + \bar{\theta}^2 N(x) + \theta\sigma^\mu\bar{\theta}V_\mu(x) \\ &+ \theta^2\bar{\theta}\bar{\lambda}(x) + \bar{\theta}^2\theta\rho(x) + \theta^2\bar{\theta}^2 D(x) \end{aligned} \tag{313}$$

[38]In four dimensions, otherwise we tensor together these matrices to obtain higher dimensional analogues.

Now using $\delta_{\epsilon,\bar{\epsilon}}S(z) = i(\epsilon Q + \bar{\epsilon}\bar{Q})S(z)$ we list all supersymmetry variations:

$$\begin{aligned}
\delta\phi &= \epsilon\psi + \bar{\epsilon}\bar{\chi} \qquad (314)\\
\delta\psi &= 2\epsilon M + \sigma^\mu\bar{\epsilon}(i\partial_\mu\phi + V_\mu)\\
\delta\bar{\chi} &= 2\bar{\epsilon}N - \epsilon\sigma^\mu(i\partial_\mu\phi - V_\mu)\\
\delta M &= \bar{\lambda} - \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar{\epsilon}\\
\delta N &= \epsilon\rho + \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\bar{\chi}\\
\delta V_\mu &= \epsilon\sigma_\mu\bar{\lambda} + \rho\sigma_\mu\bar{\epsilon} + \frac{i}{2}(\partial^\nu\psi\sigma_\mu\bar{\sigma}_\nu\epsilon - \bar{\epsilon}\bar{\sigma}_\nu\sigma_\mu\partial^\nu\bar{\chi})\\
\delta\bar{\lambda} &= 2\bar{\epsilon}D + \frac{i}{2}(\bar{\sigma}^\nu\sigma^\mu\bar{\epsilon})\partial_\mu V_\nu + i\bar{\sigma}^\mu\epsilon\partial_\mu M\\
\delta\rho &= 2\epsilon D - \frac{i}{2}(\sigma^\nu\bar{\sigma}^\mu\epsilon)\partial_\mu V_\nu + i\sigma^\mu\bar{\epsilon}\partial_\mu N\\
\delta D &= \frac{i}{2}\partial_\mu(\epsilon\sigma^\mu\bar{\lambda} - \rho\sigma^\mu\bar{\epsilon})
\end{aligned}$$

Where we observe the total derivative of the D-term.

C D-algebra

The background covariant chiral derivatives ∇_α, given by:

$$\nabla_\alpha = e^{-\Omega}D_\alpha e^{\Omega} \qquad (315)$$

$$\bar{\nabla}_{\dot{\alpha}} = e^{\bar{\Omega}}\bar{D}_{\dot{\alpha}}e^{-\bar{\Omega}} \qquad (316)$$

with Ω a background vector superfield and D_α the conventional supersymmetric derivatives, satisfy following algebra:

$$\begin{aligned}
\nabla_a &\equiv -\frac{i}{4}\bar{\sigma}_a{}^{\dot{\alpha}\alpha}\{\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}\} = -\frac{1}{2}\bar{\sigma}_a{}^{\dot{\alpha}\alpha}\nabla_{\alpha\dot{\alpha}}\\
[\nabla_\alpha, \nabla_{\beta\dot{\beta}}] &= 2i\epsilon_{\alpha\beta}\bar{W}_{\dot{\beta}}\\
[\bar{\nabla}_{\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] &= 2i\epsilon_{\dot{\alpha}\dot{\beta}}W_\beta
\end{aligned}$$

The total gauge covariant derivatives used:

$$\mathcal{D}_\alpha = e^{-V_s}D_\alpha e^{V_s} = e^{-\bar{\Omega}}e^{-V}\nabla_\alpha e^V e^{\bar{\Omega}} \qquad (317)$$

$$\bar{\mathcal{D}}_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}} = e^{-\bar{\Omega}}\bar{\nabla}_{\dot{\alpha}}e^{\bar{\Omega}} \qquad (318)$$

satisfy the same algebra but W gets replaced by the full field strength G.

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