Diagrammatic derivation of the BCJ identity

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June 2021

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Abstract

In this masters thesis, color-ordered decompositions of tree amplitudes into *partial amplitudes* will be discussed. Furthermore, various identities for these *partial amplitudes* will be reviewed. Next, the way the Bern, Carrasco and Johansson identity, also known as the BCJ identity [1], was originally found is explained. Furthermore, examples are given on how to apply the BCJ identity in some specific cases. Finally, the BCJ identity will be investigated and our search for a diagrammatic proof is elaborated.

Contents

1 Introduction

The purpose of this thesis is to provide a diagrammatic derivation of the identity between *partial amplitudes* originally conjectured by Bern, Carrasco and Johansson [1]. This identity consists of relations between *partial amplitudes* of an n-point Yang-Mills amplitude. Yang-Mills amplitudes have a far simpler structure than would be expected just from the Feynman rules alone. In particular, at tree level, Yang-Mills amplitudes can be stripped of their *color structure*. The *partial amplitudes*, which are left after stripping Yang-Mills amplitudes of their color, obey various relations and identities, which significantly speeds up computing all *partial amplitudes* for a given Yang-Mills amplitude. This in turn greatly speeds up the computation of Yang-Mills tree amplitudes. A reader might now already wonder what *color structure* or *partial amplitudes* exactly are. For such readers, and others who need a refresher on Yang-Mills theory, a preliminary chapter (chapter two) is included in which all of these concepts are explained and applied.

Furthermore in this work, the previously known *cyclic*, *reversal*, $photon-decoupling$ and *Kleiss-Kuijf* identities are discussed beforehand to give a clearer picture of identities for *partial amplitudes*. Derivations of the first three of these identities mentioned are also included. Next, the Bern-Carrasco-Johansson identity, also known as the BCJ identity for short, is introduced. This identity was first conjectured in 2008, and has been proven since in 2011 by Chen et al.. [2] In this thesis, the focus lies on a diagrammatic derivation of the BCJ identity, as this has not been accomplished yet to our knowledge. Such a derivation would hopefully provide a more intuitive argument for the validity of the BCJ identity and further knowledge on the structure of *partial amplitudes*.

The *cyclic*, *reversal*, $photon-decoupling$ and *Kleiss-Kuijf* are discussed in chapter three. The discussion about the original derivation and conjecture by Bern, Carrasco Johansson will be done in chapter four, complemented with some example applications of the BCJ identity. Next, in chapter five, identities between Yang-Mills *numerators* are discussed, which are important for the BCJ identity. Finally, in chapter six we will explore the BCJ identity using the *Britto-Cachazo-Feng-Witten (BCFW) identity*, and its diagrammatic interpretation.

2 Preliminaries

To be able to discuss any theory regarding the BCJ identity, a theoretical framework must first be established which can be built upon. In this chapter, a variety of preliminaries are discussed, which are relevant or even required to be able to discuss further work in this thesis. As the BCJ identity only applies to pure Yang-Mills tree amplitudes, the focus of this discussion will be on pure Yang-Mills theory, and any loop contributions will be dropped. Some basic knowledge on Quantum Field Theory is assumed, and our discussion starts at the Feynman rules for pure Yang-Mills theory. The Feynman rules for pure Yang-Mills theory will not be derived explicitly, however the derivation can be found in most introductory books on quantum field theory. Continuing, color structure is discussed in general for Yang-Mills theory, ways to decompose Yang-Mills amplitudes are considered and spinor helicity formalisms are introduced. In this thesis the sign convention $(+,-,-,-)$ is used for the metric tensor, as is usual within quantum field theory (QFT).

2.1 Yang-Mills theory

Yang-Mills theory is a gauge theory based on the special unitary group $SU(N)$. In particular, Yang-Mills theory describes Quantum Chromodynamics (QCD) for $N = 3$. The Lie group $SU(N)$ consists of all unitary matrics (as the name implies), with determinant 1. This group is non-abelian and has a dimension of $N^2 - 1$. Yang-Mills theory is invariant under local $SU(N)$ transformations. To achieve this, structure constants of the Lie algebra $su(N)$ are incorporated in the Yang-Mills lagrangian. The elements of the Lie algebra $su(N)$ are the matrices

$$Lie(SU(N)) = su(N) = \{X \in M(n,\mathbb{C}) \mid e^{tX} \in SU(N)\} \tag{1}$$

Mathematically, as the Lie group $SU(N)$ is compact and connected, for a given representation R of the lie group $SU(N)$, any element M of $SU(N)$ may be written as $M = exp(i\alpha^j T_j)$, with T_j the generators of the Lie group. Note that the elements T^j are self-adjoint and traceless by virtue of being elements of $su(N)$. For a reader unfamiliar with Lie theory, the main thing to remember is that these elements T_j now form a representation of the Lie algebra $su(N)$ and obey the following relations

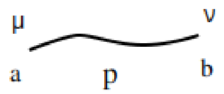
$$Tr(T^a) = 0 \tag{2}$$

$$T^{a\dagger} = T^a \tag{3}$$

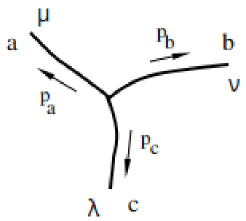
$$Tr(T^a T^b) = \frac{1}{2}\delta^{ab} \tag{4}$$

$$[T^a, T^b] = f^{abc} T^c \tag{5}$$

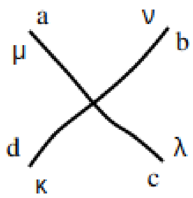
Where summation over c is implied in equation (5) (c runs from 1 to $N^2 - 1$). The elements f^{abc} will henceforth be known as 'structure constants'. These are a key ingredient to discuss the Feynman rules for Yang-Mills theory, as the structure constants f^{abc} appear within the Feynman rules. As only pure Yang-Mills amplitudes are considered, straight lines will be used instead of the usual coiled lines for the Feynman diagrams for ease of reading.



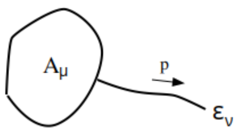
$$\Leftrightarrow \quad i\hbar \frac{1}{p^2 + i\eta}(g^{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2})\delta^{ab}$$



$$\Leftrightarrow \quad i(2\pi)^4 \delta^{(4)}(p_a + p_b + p_c)\frac{ig}{\hbar} f^{abc}\Big\{ g^{\mu\nu}(p_a - p_b)^\lambda + g^{\nu\lambda}(p_b - p_c)^\mu + g^{\lambda\mu}(p_c - p_a)^\nu \Big\}$$



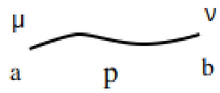
$$\Leftrightarrow \quad i(2\pi)^4 \delta^{(4)}(p_a + p_b + p_c + p_d)\frac{ig^2}{\hbar}\Big\{ g^{\mu\nu} g^{\lambda\kappa}\big(f^{adn} f^{ncb} + f^{acn} f^{ndb}\big) + g^{\mu\lambda} g^{\nu\kappa}\big(f^{adn} f^{nbc} + f^{abn} f^{ncb}\big) + g^{\mu\kappa} g^{\nu\lambda}\big(f^{abn} f^{ncd} + f^{acn} f^{nbd}\big)\Big\}$$



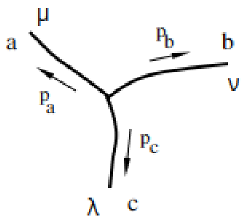
$$\Leftrightarrow \quad A_\mu g^{\mu\nu} \varepsilon_\nu(p)$$

Feynman rules for pure Yang-Mills theory. For every internal momentum p, an integral $\int d^4 p$ is added. From now on, Feynman gauge will be used, hence $\xi = 1$.

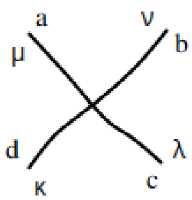
The BCJ identity only applies to tree amplitudes, which are diagrams without any loops. In tree diagrams, all internal momenta are fixed by integrating over the delta functions $\delta^{(4)}$. Hence, to simplify our expressions, a factor of $i(2\pi)^4\delta^{(4)}$ is dropped out from our vertices, together with all integrals over internal momenta. Doing this leaves us with the set of Feynman rules which will be used throughout this thesis. Note that the structure constants f are still present within our vertices, and in the next section they will be separated from the amplitude, leaving us with colored and colorless components.



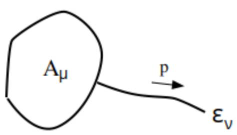
$$\Leftrightarrow \quad i\hbar\frac{1}{p^2}g^{\mu\nu}\delta^{ab}$$



$$\Leftrightarrow \quad \frac{ig}{\hbar}f^{abc}\Big\{g^{\mu\nu}(p_a-p_b)^\lambda + g^{\nu\lambda}(p_b-p_c)^\mu + g^{\lambda\mu}(p_c-p_a)^\nu\Big\}$$



$$\Leftrightarrow \quad \frac{ig^2}{\hbar}\Big\{g^{\mu\nu}g^{\lambda\kappa}\big(f^{adn}f^{ncb}+f^{acn}f^{ndb}\big) + g^{\mu\lambda}g^{\nu\kappa}\big(f^{adn}f^{nbc}+f^{abn}f^{ncb}\big) + g^{\mu\kappa}g^{\nu\lambda}\big(f^{abn}f^{ncd}+f^{acn}f^{nbd}\big)\Big\}$$



$$\Leftrightarrow \quad A_\mu g^{\mu\nu}\varepsilon_\nu(p)$$

2.2 Color structure

As mentioned previously, $su(N)$ structure constants are present within the Feynman rules for Yang-Mills theory. In this thesis a general number of colors N is considered, as it does not make the discussion much more involved. From now on, these will be known as color factors. Let us introduce 'color matrices', which obey the following rules

$$Tr(T^a) = 0 \tag{6}$$

$$T^{a\dagger} = T^a \tag{7}$$

$$Tr(T^a T^b) = \frac{1}{2}\delta^{ab} \tag{8}$$

$$[T^a, T^b] = f^{abc}T^c \tag{9}$$

An observant reader might notice that these color matrices are exactly the same as the $su(N)$ generators discussed earlier. Next, it will be discussed how to rewrite color factors, and how to combine color matrices into forms more easily workable. This discussion is based on the discussion given in [3] and use similar conventions. Our eventual aim will be to write the amplitude as a sum of objects with their color structure stripped, multiplied by objects that encode the color structure. Our aim would be to obtain something in the following form:

$$A^n_{tree} = g^{n-2}\sum_i B_i C_i \tag{10}$$

Where the objects B contain all momenta and polarizations, but no color and the objects C encodes the color dependencies of the amplitude. The coupling constant is also extracted, as every diagram contributing to the n-gluon tree amplitude will carry the same power of the coupling constant g

2.2.1 Color algebra

First let us consider how to write color factors in terms of color matrices, C_i in equation (10) may be taken as a trace of color matrices. To achieve this, the color factors f^{abc} need to be rewritten into traces of color matrices, and products of traces such as $Tr(AT)Tr(BT)$ need to be combined into a single trace somehow.

For the former, make use of equations (8) and (9). Combining these, color factors can be written in the following from

$$f^{abc} = 2Tr(T^aT^bT^c - T^aT^cT^b) \tag{11}$$

Let us now deduce some simple properties of f^{abc}. The cyclic property of traces $Tr(ABC) = Tr(CAB) = Tr(BCA)$ implies that f^{abc} is cyclic in its indices. Furthermore there also exists antisymmetry in exchange of any two indices, as is apparent when using the following notation

$$f^{abc} = 2Tr(T^a[T^b,T^c]) = 2Tr(T^b[T^c,T^a]) = 2Tr(T^c[T^a,T^b]) \tag{12}$$

Where the commutator is to be understood, as usual, as

$$[T^a,T^b] = T^aT^b - T^bT^a \tag{13}$$

As constantly writing T for every single color matrix tends to clutter calculations quite a bit, and the only matrices used will be color matrices to begin with, the following shorthand notation is introduced

$$Tr(T^{a_1}T^{a_2}...T^{a_n}) = Tr(a_1\,a_2\ldots a_n) \tag{14}$$

Thus our expression for f^{abc} turns into:

$$f^{abc} = 2Tr(a[b,c]) = 2Tr(b[c,a]) = 2Tr(c[a,b]) \tag{15}$$

Secondly, it is necessary to have a method of expressing products of color factors in terms of a single color matrix trace. For instance, let us look at the following expression

$$f^{abn}f^{ncd} = 4Tr(n\,[a,b])Tr(n\,[c,d]) \tag{16}$$

To combine the traces in equation (16), an identity known as the 'Fierz identity' is required. Deriving the Fierz identity is done in the following way. Firstly, as the color matrices T^j are generators for $su(N)$, they are linearly independent. secondly, all color matrices T^j are traceless. As there are N^2-1 independent generators, and there are N^2 complex entries in $M(N,\mathbb{C})$, the generators together with the identity matrix span $M(N,\mathbb{C})$ over $\mathbb{C}$. Simply put any $N\times N$ matrix M may be written in terms of a basis of color matrices T^j and the identity matrix $\mathbb{1}$

$$M = a_0\mathbb{1} + \sum_i a_iT^i \tag{17}$$

equation (17) may now be combined with $Tr(T^jT^k) = \frac{1}{2}\delta^{jk}$ to obtain the coefficients a_i for the matrix M

$$M = \frac{1}{N}Tr(M)\mathbb{1} + 2\sum_i Tr(T^iM)T^i \tag{18}$$

In terms of matrix components this yields

$$M_c^d \delta_b^c \delta_d^a = \frac{1}{N} M_c^d \, \delta_d^c \, \delta_b^a + 2 \sum_i (T^i)_d^c M_c^d \, (T^i)_b^a \tag{19}$$

From which M_c^d is extracted to obtain the Fierz identity

$$\frac{1}{2}\left(\delta_b^c \delta_d^a - \frac{1}{N}\delta_d^c \delta_b^a\right) = \sum_i (T^i)_d^c \, (T^i)_b^a \tag{20}$$

Equation (20) is now contracted with either $A_a^b B_c^d$ or $A_a^d B_b^c$ to obtain the two following key identities

$$\begin{aligned} Tr(T^i A)Tr(T^i B) &= \frac{1}{2}\left(Tr(AB) - \frac{1}{N}Tr(A)Tr(B)\right) \\ Tr(T^i A T^i B) &= \frac{1}{2}\left(Tr(A)Tr(B) - \frac{1}{N}Tr(AB)\right) \end{aligned} \tag{21}$$

Where summation over i is implied. As A and B can be any matrix, these equations can now be simplified using equation (16). Taking $A = [T^a, T^b]$ and $B = [T^c, T^d]$

$$f^{abn} f^{ncd} = 4Tr(n[a,b])Tr(n[c,d]) = 2\left(Tr([a,b][c,d]) - \frac{1}{N}Tr([a,b])Tr([c,d])\right) \tag{22}$$

This is approaches the formula we initially set out to derive closely, namely a way to write color factors in terms of a single color matrix trace $Tr(a_1 a_2 .... a_n)$. The only issue now is the term leading in 1 over N. Fortunately, this term vanishes as can be shown easily

$$Tr([a,b])Tr([c,d]) = Tr([a,b])Tr(cd - dc) = Tr([a,b])Tr(cd - cd) = 0 \tag{23}$$

More generally, the 1 over N term will always drop out when considering tree diagrams. Say $f^{a_1 a_2 a_3}$ is multiplied with other color factors already written out into a sum of traces (And hence not containing the 1 over N contributions). Each of these traces will contain T^{a_1}, and due to the cyclic property of traces, T^{a_1} can always be brought in front of each individual trace. Let us collect the other color factors into $Tr(T^{a_1}\Gamma)$. Then, multiplying with $f^{a_1 a_2 a_3}$ yields

$$\begin{aligned} & Tr(a_1\Gamma) f^{a_1 a_2 a_3} \\ & = 2Tr(a_1\Gamma)Tr(a_1[a_2, a_3]) \\ & = Tr(\Gamma[a_2, a_3]) - \frac{1}{N}Tr(\Gamma)Tr([a_2, a_3]) \\ & = Tr(\Gamma[a_2, a_3]) \end{aligned} \tag{24}$$

Where once again the 1 over N component drops out as it is taking the trace of a commutator. Through induction, it can be shown that the 1 over N component will drop out in general, as long as a new color factor shares at most one index with color matrices already present in the trace. In tree diagrams this is certainly the case, hence the following shortened equation to evaluate color factors in terms of color matrix traces in tree diagrams may be used

$$Tr(T^i A)Tr(T^i B) = \frac{1}{2}Tr(AB) \tag{25}$$

To finish off this section, a new notation for color factors is introduced. When working with tree diagrams, in many cases it is more convenient to write color factors into 'color brackets'

$$f^{abc} = [abc] \tag{26}$$

$$[a_1...a_n b][bc_1...c_k] = [a_1...a_n c_1...c_k] \tag{27}$$

These are very useful for expressing some combinations of color factors, for instance

$$\begin{aligned} f^{a_1a_2n_1}f^{n_1a_3a_4} &= [a_1a_2a_3a_4] \\ f^{a_1a_2n_1}f^{n_1a_3n_2}f^{n_2a_4a_5} &= [a_1a_2a_3a_4a_5] \\ f^{a_1a_2n_1}f^{n_1a_3n_2}...f^{n_{k-3}a_{k-1}a_k} &= [a_1a_2a_3...a_{k-1}a_k] \end{aligned} \tag{28}$$

However for other combinations they are less convenient

$$f^{a_1a_2n_1}f^{a_3a_4n_2}f^{a_5a_6n_3}f^{n_1n_2n_3} = [a_1a_2n_3a_3a_4][n_3a_5a_6] \tag{29}$$

Note that this cannot be simplified further, as color brackets do not have cyclic symmetry. The following identities hold for color brackets

$$\begin{aligned} [a_1a_2...a_{n-1}a_n] &= -[a_2a_1...a_{n-1}a_n] \\ [a_1a_2...a_{n-1}a_n] &= -[a_1a_2...a_na_{n-1}] \\ [a_1a_2...a_{n-1}a_n] &= (-1)^l[a_na_{n-1}...a_2a_1] \end{aligned} \tag{30}$$

Where in the last equation, l is the length of the bracket and the bracket is completely reversed. Color brackets may also be expressed in terms of color traces in a convenient way. First recall that $[abc] = f^{abc} = 2Tr([a,b]c)$. Now using induction a general color bracket $[a_1...a_n]$ is written in terms of color traces in the following way

$$[a_1...a_{n-1}] = 2Tr([...[[a_1,a_2],a_3],...]a_{n-1}) \tag{31}$$

Then, as $[a_1...a_n] = [a_1...a_{n-2}b]f^{ba_{n-1}a_n}$

$$\begin{aligned}
[a_1...a_n] &= 4Tr([...[[a_1,a_2],a_3],...]b)Tr(b[a_{n-1},a_n]) \\
&= 2Tr([...[[a_1,a_2],a_3],...][a_{n-1},a_n]) \\
&= 2Tr([...[[a_1,a_2],a_3],...]a_{n-1}a_n]) \\
&- 2Tr(a_{n-1}[...[[a_1,a_2],a_3],...]a_n]) \\
&= 2Tr([[...[[a_1,a_2],a_3],...],a_{n-1}]a_n)
\end{aligned} \tag{32}$$

This completes our induction step, showing that formula (31) holds. The final identity derived in this section is the Jacobi identity for color factors and color brackets. This identity is key to the BCJ identity which will be discussed later. Using $[T^a,T^b] = f^{abc}T^c$ the next result follows

$$\begin{aligned}
0 &= [[T^j,T^k],T^l] + [[T^k,T^l],T^j] + [[T^l,T^j],T^k] \\
&= f^{jkn}[T^n,T^l] + f^{kln}[T^n,T^j] + f^{ljn}[T^n,T^k] \\
&= f^{jkn}f^{nlm}T^m + f^{kln}f^{njm}T^m + f^{ljn}f^{nkm}T^m
\end{aligned} \tag{33}$$

Hence the following identity holds for color brackets

$$[jklm] + [kljm] + [ljkm] = 0 \tag{34}$$

This is known as the color Jacobi identity, which will be used extensively throughout this thesis.

2.3 Amplitude decompositions

The techniques discussed in section 2.2 allow us to rewrite the full amplitude in terms of colored and colorless contributions as proposed in equation (10). At first glance, there appear to be two obvious ways to do this: In terms of color brackets or in terms of color traces.

2.3.1 Decomposition in terms of color brackets

The first decomposition, in terms of color brackets, is rather straightforward. As Feynman rules already carry their color information in terms of color brackets, the amplitude will naturally fall apart in the desired components. Most importantly every tree diagram containing only three-gluon vertices will correspond to a unique color bracket, and vice versa every unique color bracket corresponds to a unique diagram containg only three-gluon vertices. From now on, diagrams containing only three-gluon vertices will be called 'cubic diagrams'. All other diagrams, which necessarily contain at least one four-gluon vertex, will be known as 'quartic diagrams'. The decomposition of the full amplitude, containing both three- and four-gluon vertices will be written in the following form

$$A^n_{tree} = g^{n-2} \sum_j \frac{n_j c_j}{\prod_i p^2_{ij}} \tag{35}$$

The sum runs across all nontrivial permutations of color brackets (Trivial permutations being listed in (30)). g is the coupling constant, which will always be raised to a power of $n-2$ in a n-gluon amplitude. Henceforth, n_j will be known as a 'numerator'. Furthermore, p_{ij} are the internal momenta within the cubic diagram corresponding to c_i. This then contains all products of momenta such that n_j does not contain any singularities. It is not instantly obvious how contributions from quartic diagrams have been absorbed into n_j, and in fact each quartic diagram is divided over multiple numerators. As an example, let us consider a five-gluon amplitude, with $c_j = [12345]$. There are three diagrams which contribute to the numerator $n_j = n(12345)$.

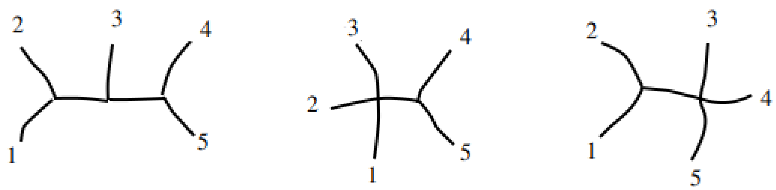


Figure 1: *The three diagrams contributing to the five-gluon numerator* $n(12345)$ *corresponding to* $[12345]$

The cubic diagram contributes

$$ig^3\hbar\sqrt{\hbar}\,\frac{Y(p_1,\varepsilon_1;p_2,\varepsilon_2;-p_1-p_2,\mu)Y(p_1+p_2,\mu;p_3,\varepsilon_3;p_4+p_5,\nu)}{(p_1+p_2)^2(p_4+p_5)^2} \times Y(-p_4-p_5,\nu;p_4,\varepsilon_4;p_5,\varepsilon_5)[12345] \tag{36}$$

In this formula, the function Y is defined as

$$Y(p_1,\nu;p_2,\lambda;p_3,\mu) = (p_1-p_2)^\mu g^{\nu\lambda} + (p_2-p_3)^\nu g^{\lambda\mu} + (p_3-p_1)^\lambda g^{\mu\nu} \tag{37}$$

It should be clear that this way of writing out the contribution from a diagram is quite unwieldy. As stated previously, equation (36) may be written more compactly the values of the internal momenta are implied due to momentum conservation. Furthermore using the common definition $(p_a + p_b)^2 = 2p_a \cdot p_b = s_{ab}$, turns equation (36) into

$$ig^3\hbar\sqrt{\hbar}\,\frac{Y(1,2,\mu)Y(\mu,3,\nu)Y(\nu,4,5)}{s_{12}s_{45}}[12345] \tag{38}$$

The two quartic diagrams do not solely contribute to $n(12345)$, rather they contribute to other numerators as well.

$$\begin{aligned}-ig^3\hbar\sqrt{\hbar}\,\frac{Y(1,2,\mu)[12n]}{s_{12}} \times \Big\{&\varepsilon_{5\mu}(\varepsilon_3\cdot\varepsilon_4)\left([n345]+[n435]\right)\\ &+\varepsilon_{3\mu}(\varepsilon_4\cdot\varepsilon_5)\left([n543]+[n453]\right)\\ &+\varepsilon_{4\mu}(\varepsilon_5\cdot\varepsilon_3)\left([n534]+[n354]\right)\Big\}\end{aligned} \tag{39}$$

$$\begin{aligned}-ig^3\hbar\sqrt{\hbar}\,\frac{Y(\nu,4,5)[n45]}{s_{45}} \times \Big\{&\varepsilon_{3\nu}(\varepsilon_1\cdot\varepsilon_2)\left([n123]+[n213]\right)\\ &+\varepsilon_{1\nu}(\varepsilon_2\cdot\varepsilon_3)\left([n321]+[n231]\right)\\ &+\varepsilon_{2\nu}(\varepsilon_3\cdot\varepsilon_1)\left([n312]+[n132]\right)\Big\}\end{aligned} \tag{40}$$

Next, making use of the rules for color brackets as previously stated in (27) and (30), the relevant factors may be extracted from equations (39) and (40)

$$-\,ig^3\hbar\sqrt{\hbar}\,\frac{Y(1,2,\mu)\Big(\varepsilon_{5\mu}(\varepsilon_3\cdot\varepsilon_4)-\varepsilon_{4\mu}(\varepsilon_3\cdot\varepsilon_5)\Big)[12345]}{s_{12}} \tag{41}$$

$$-\,ig^3\hbar\sqrt{\hbar}\,\frac{\Big(\varepsilon_{1\nu}(\varepsilon_2\cdot\varepsilon_3)-\varepsilon_{2\nu}(\varepsilon_3\cdot\varepsilon_1)\Big)Y(\nu,4,5)[12345]}{s_{45}} \tag{42}$$

Combining equations (38), (41) and (42) yields the numerator $n(12345)$ that was our original aim to find. It has to be kept in mind that common factors of g^3, $[12345]$ and $\frac{1}{s_{12}s_{45}}$ have been factored out, as stated previously.

$$\begin{aligned}n(12345) =\;& i\hbar\sqrt{\hbar}\,Y(1,2,\mu)Y(\mu,3,\nu)Y(\nu,4,5)\\ &-\; i\hbar\sqrt{\hbar}\,Y(1,2,\mu)\Big(\varepsilon_{5\mu}(\varepsilon_3\cdot\varepsilon_4)-\varepsilon_{4\mu}(\varepsilon_3\cdot\varepsilon_5)\Big)s_{45}\\ &-\; i\hbar\sqrt{\hbar}\,\Big(\varepsilon_{1\nu}(\varepsilon_2\cdot\varepsilon_3)-\varepsilon_{2\nu}(\varepsilon_3\cdot\varepsilon_1)\Big)Y(\nu,4,5)s_{12}\end{aligned} \tag{43}$$

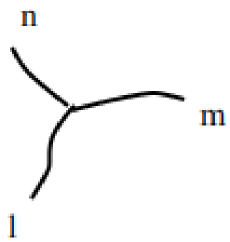
Coincidentally, as all color contributions are of the form $[12345]$ in a five-gluon amplitude, the general expression for $n(abcde)$ corresponding to $[abcde]$ is also found, by simply substituting a to e for 1 to 5 in equation (43). There are a few advantages and disadvantages of decomposing the amplitude in terms of color brackets. Advantages are the simplicity of finding the numerators, and in addition, the numerator has the same symmetries as the corresponding color bracket. Disadvantages are that in this decomposition, the numerator is not gauge invariant, or even unique to begin with (there are hidden degrees of freedom within the numerators). These problems will be discussed later on in this thesis, but for now the focus lies on another method of decomposing amplitudes: In terms of color traces.

2.3.2 Decomposition in terms of color traces

In this section, amplitudes will be decomposed in terms of color traces. This is the standard way gluonic, or even general Yang-Mills tree amplitudes are stripped of their color structure. In the BCJ identity, the objects known as 'partial amplitudes' are simply the coefficients corresponding to these color traces. Otherwise the coefficients of color traces might also be known as 'color ordered amplitudes'. In this thesis, partial amplitudes will be used. The decomposition is given as follows

$$A_n^{tree} = g^{n-2} \sum_{P(2,3,\ldots,n)} A_n^{part}(1,2,3,\ldots,n) Tr(T^{a_1} T^{a_2} T^{a_3} \ldots T^{a_n}) \tag{44}$$

Here A_n^{part} are the n-leg partial amplitudes. The sum runs over all noncyclic permutations of legs 1 to n, which is equivalent to all permutations holding leg 1 fixed. For the formal definition, each color matrix T^j is fully written out. Once again, henceforth the T in these color matrices will be dropped and only the indices will be written down for simplicity. Using previously discussed information from section 2.2, the amplitude can certainly be written in this way, as all products of color factors can be written into a sum of color traces. What is less obvious is which diagrams contribute to A_n^{part}. In order to find the diagrams contributing to this partial amplitude, it is required to first write the contributions from vertices in terms of color traces. When looking the following cubic vertex

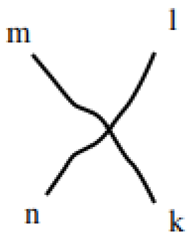


$$\begin{aligned} \Leftrightarrow \quad & \frac{2i}{\hbar} g\, Y(n,m,l)\, Tr(n\,[m,l]) \\ = & \frac{2i}{\hbar} g\, Y(n,m,l)\, Tr(n\,m\,l) \\ - & \frac{2i}{\hbar} g\, Y(n,m,l)\, Tr(n\,l\,m) \end{aligned} \tag{45}$$

This means that the relation between color traces and cubic vertices is quite straightforward.

$$Tr(n\,[m\,l]) \Leftrightarrow Y(n,m,l) \tag{46}$$

For the quartic vertices, however, things are not as straightforward. Let us recap the structure of a quartic vertex



$$\begin{aligned} \Leftrightarrow \quad & i\frac{g^2}{\hbar} X(m,n,l,k) \\ = \; & i\frac{g^2}{\hbar} \Big\{ g^{\mu\nu} g^{\lambda\kappa} ([mlkn]+[mkln]) \\ & + g^{\mu\lambda} g^{\nu\kappa} ([mnkl]+[mknl]) \\ & + g^{\mu\kappa} g^{\nu\lambda} ([mnlk]+[mlnk]) \Big\} \end{aligned} \tag{47}$$

In this equation Latin letters are used for the colors and Greek letters are used for their respective indices ($m \sim \mu$ etc.). The aim is now to extract, say, the terms that have a factor of $Tr(m\,n\,l\,k)$. This essentially comes down to the question: which color brackets contain $Tr(m\,n\,l\,k)$? Naturally, it follows that $[abcd] = 2Tr([a,b][c,d])$, and hence naively only $[mnlk]$ and its permutations contribute. This, however, would be incomplete as cyclic permutations also have to be considered. Cyclic permutation gives $Tr(m\,n\,l\,k) = Tr(n\,l\,k\,m)$, hence $[nlkm]$ also contributes. Comparing this to equation (47) yields

$$X(m,n,l,k) \mid_{Tr(m\,n\,l\,k)} = g^{\mu\kappa} g^{\nu\lambda} - 2g^{\mu\lambda} g^{\nu\kappa} + g^{\mu\nu} g^{\lambda\kappa} \tag{48}$$

Now, an observant reader may wonder: what about quartic interactions with the legs ordered differently. It so happens, that the quartic interaction vertex is the same no matter then ordering of the legs, and hence the contribution remains the same. Generally speaking

$$Tr(mnlk) \iff g^{\mu\kappa}g^{\nu\lambda} - 2g^{\mu\lambda}g^{\nu\kappa} + g^{\mu\nu}g^{\lambda\kappa} \tag{49}$$

In a small example is now provided in order to illustrate this approach. Let us consider $A_4^{part}(1,2,3,4)$. Either $Tr(1234)$, $Tr(n[1,2])Tr(n[3,4])$ or $Tr(n[2,3])Tr(n[4,1])$ are required to be present as a color factor for a diagram. The diagrams which contain these traces are shown in figure 2.

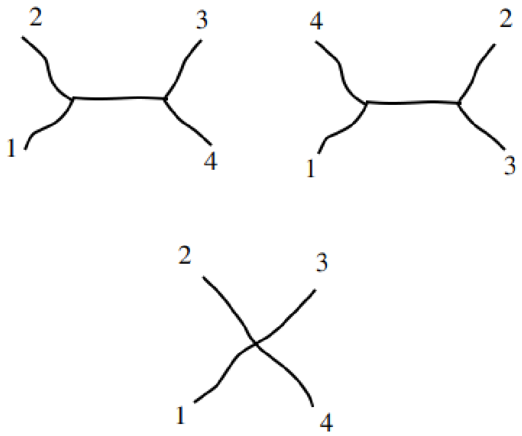


Figure 2: *The three diagrams contributing to the partial amplitude* $A_4^{part}(1,2,3,4)$

From the quartic diagram, keeping in mind that it does not contribute in its entity, we find

$$i\hbar \left\{ (\varepsilon_1 \cdot \varepsilon_4)(\varepsilon_2 \cdot \varepsilon_3) - 2(\varepsilon_1 \cdot \varepsilon_3)(\varepsilon_2 \cdot \varepsilon_4) + (\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \varepsilon_4) \right\} \tag{50}$$

The other two diagrams are simply cubic contributions of the following form

$$-i\hbar \frac{Y(1,2,\mu)Y(\mu,3,4)}{s_{12}} \tag{51}$$

$$-i\hbar \frac{Y(2,3,\mu)Y(\mu,4,1)}{s_{23}} \tag{52}$$

The sum of the three equations (50), (51) and (52) give $A_4^{part}(1,2,3,4)$ as required. This concludes our example. Partial amplitudes may be constructed using analogs to Yang-Mills Feynman rules, known as 'Color ordered Feynman rules', which is explained in [4]. In this thesis partial amplitudes will be derived from the numerators described in the previous section, as will be discussed in chapter five.

It turns out that these partial amplitudes have far better properties than the numerators considered previously, when decomposing in terms of color brackets. Not only are partial amplitudes unique, they are also gauge invariant. A proof of this gauge invariance is given below. The gauge invariance of partial amplitudes will be especially relevant going forward in this thesis. Let us first quickly recap what gauge invariance is. Consider a full amplitude A_n^{tree}, and one specific gluon with momentum k. Let $\varepsilon(k)$ denote the polarization vector of this gluon. The full amplitude may now be denoted as

$$A_n^{tree} = A(k)^{\mu}\varepsilon(k)_{\mu} \tag{53}$$

Now, if a gauge transformation is performed, the polarization vector may change as $\varepsilon(k)_{\mu} \to \varepsilon(k)_{\mu} + \alpha k_{\mu}$. Gauge invariance hence implies

$$A(k)^{\mu}k_{\mu} = 0 \tag{54}$$

This is also known as the Ward identity, and an amplitude or any object is gauge invariant whenever replacing any of the polarizations within with their respective momenta will cause the amplitude to vanish.

$$A_n^{tree} \mid_{\varepsilon_j \to p_j} = 0 \tag{55}$$

Replacing a polarization with its respective momentum is known as 'applying the handlebar', which will be discuss in more detail in section 2.6. Our claim now is that for partial amplitudes, this holds as well

$$A_n^{part} \mid_{\varepsilon_j \to p_j} = 0 \tag{56}$$

The easiest way to obtain this gauge invariance, would be to project the color traces out of the full amplitude. Recall that $A^{tree} = \sum_i C_i A_i^{part}$, so if our color elements from an orthonormal basis, $C_i C_j^* = \delta_{ij}$, then the partial amplitude may be projected out of the full amplitude through $A_i^{part} = A^{tree} C_i^*$. Furthermore as A^{tree} is gauge invariant as discussed earlier, and C_i certainly is as well, gauge invariance of A_i^{part} would immediately follow. As defined previously

$$C_i = Tr(a_1\, a_2 \ldots a_n) \tag{57}$$

Similarly the other color trace equals

$$C_j = Tr(b_1\, b_2 \ldots b_n) \tag{58}$$

Where both $a_1 \dots a_n$ and $b_1 \dots b_n$ are permutations of $1 \dots n$. If C_i and C_j are cyclic permutations of each other, they correspond to the same partial amplitude as would be expected because they are identical through the cyclic property of traces. Let us examine our statement on gauge invariance more closely. As the partial amplitudes A^{part} are stripped of color, they are certainly independent of the number of colors in our theory N. This implies, that even if $C_i C_j^* = \delta_{ij} + \mathscr{O}\left(\frac{1}{N}\right)$, gauge invariance may be deduced

$$A^{tree} C_i^* = A_i^{part} + \mathscr{O}\left(N^{-1}\right) \tag{59}$$

As A_i^{part} is independent of N, it still must be gauge invariant if $A^{tree} C_i^* = A_i^{part}$ is. Successively, it will be shown that $C_i C_j^* = \delta_{ij} + \mathscr{O}\left(\frac{1}{N}\right)$.

$$\begin{aligned} C_i C_j^* &= Tr(a_1 a_2 \dots a_n) Tr(b_1 b_2 \dots b_n)^* \\ &= Tr(a_1 a_2 \dots a_n) Tr(b_n \dots b_2 b_1) \end{aligned} \tag{60}$$

As color matrices are Hermitian, and as both $a_1 \dots a_n$ and $b_1 \dots b_n$ are permutations of $1 \dots n$, there is a b_i such that $b_i = a_1$. Let us order the color trace, and bring b_i in the first position, and relabel the other indices b from k_1 to k_{n-1}

$$\begin{aligned} C_i C_j^* &= Tr(a_1 a_2 \dots a_n) Tr(a_1 a_{k_1} \dots a_{k_{n-1}}) \\ &= \frac{1}{2} Tr(a_2 \dots a_n a_{k_1} \dots a_{k_{n-1}}) - \frac{1}{2N} Tr(a_2 \dots a_n) Tr(a_{k_1} \dots a_{k_{n-1}}) \end{aligned} \tag{61}$$

As stated previously, only leading order contributions are relevant. As $\frac{1}{2N}$ is of subleading order, the corresponding term is dropped. Let us now get rid of a_n: there is some k_j for which $k_j = n$, hence

$$C_i C_j^* = \frac{1}{2} Tr(a_n A\, a_n B) + \mathscr{O}\left(N^{-1}\right) \tag{62}$$

Here the cyclic property of traces was used to bring a_n in the first position. Please note that A or B may equal the identity matrix. This equation is now simplified using the color algebra as shown in equation (21). This yields

$$C_i C_j^* = \frac{1}{4} Tr(A) Tr(B) + \mathscr{O}\left(N^{-1}\right) \tag{63}$$

A key observation can now be made: If either A or B are the identity matrix, then an additional term N appears in our product. If neither are, then the resulting product will be at least of an order of N smaller. Hence, to obtain the behaviour in the leading order of N, assume either A or B equals the identity matrix. Say, $A = \mathbb{1}$

$$C_i C_j^* = \frac{N}{4} Tr(B) + \mathscr{O}(1) \tag{64}$$

This process may now be repeated with a_{n-1}, to obtain $Tr(B) = Tr(a_{n-1}\,C\,a_{n-1}\,D)$, and once again either C or D has to equal the identity matrix. Continuing the process until only $Tr(a_2 a_2)$ is left yields

$$C_i C_j^* = \frac{1}{2}\left(\frac{N}{2}\right)^{n-2} Tr(a_2\,a_2) + \mathscr{O}\left(N^{n-3}\right) \tag{65}$$

Simplifying this final expression yields

$$Tr(a_2\,a_2) = \frac{1}{2}Tr(\mathbb{1})Tr(\mathbb{1}) + \frac{1}{2N}Tr(\mathbb{1}) = \frac{N^2}{2} + \mathscr{O}(1) \tag{66}$$

Giving us, under the assumption that at every step, at least one matrix equals the identity

$$C_i C_j^* = \left(\frac{N}{2}\right)^n + \mathscr{O}\left(N^{n-2}\right) \tag{67}$$

To find the original color traces C_i and C_j, this process has to be reversed. If in equation (62) $A = \mathbb{1}$, then both a_n were adjacent. As one of the matrices equaled the identity in every step, all pairs a_k were adjacent at that step. Notice that $a_n\,a_n\,a_{n-1}\,a_{n-1}$ or anything like it cannot appear in the trace, as that would imply either a_n or a_{n-1} appears twice in one of the color traces C_i or C_j, which is not the case by the assumption that each index appears exactly once. Combining this, it follows that equation (62) must have the following structure

$$C_i C_j^* = \frac{1}{2}Tr(a_2 \ldots a_{n-1}\,a_n\,a_n\,a_{n-1} \ldots a_2) \tag{68}$$

Next, going back one more step

$$C_i C_j^* = Tr(a_1\,a_2 \ldots a_{n-1}\,a_n)Tr(a_1\,a_n\,a_{n-1} \ldots a_2) \tag{69}$$

Hence C_j is a cyclic permutation of C_i. This means that if and only if C_i is a cyclic permutation of C_j

$$C_i C_j^* = \left(\frac{N}{2}\right)^n \delta_{ij} + \mathscr{O}\left(N^{n-2}\right) \tag{70}$$

It follows that A_n^{part} is gauge invariant. Furthermore, this result can also be used to deduce that A_n^{part} is independent of the gauge-fixing parameter ξ which was set to 1 in our Feynman rules. Finally, equation (70) also shows that A_n^{part} is unique in the sense that there are not multiple choices for the partial amplitudes which produce the same full amplitude A_n^{tree}. Because of these properties, when stripping Yang-Mills theory of its color structure, the amplitude is generally decomposed into partial amplitudes and color traces instead of numerators and color brackets. This, however, does not mean that decomposing the amplitude into numerators is completely useless, as the numerators are a much easier way to decompose the full amplitude in the first place. In turn, numerators can be used to express partial amplitudes which is very convenient, as is used further down this thesis.

2.4 Spinors and polarization vectors

In this section, conventions for spinors and polarization vectors used in this thesis are introduced and discussed. In this thesis we will need to calculate polarization vectors and spinor products in order to verify our results numerically. Weyl spinors are used in this section, however a derivation of their properties are not explicitly given, and it is assumed that the reader knows what they are. If otherwise, the reader might refer to introductory material on quantum field theory such as [3].

In this section, the definitions and conventions in [3] for spinors and polarization vectors are followed. In a short recap, the Dirac matrices are given by

$$\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \ \gamma^\mu = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \ , \ \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{71}$$

where σ_k are the Pauli matrices. The Clifford elements $\omega_\pm = \frac{1}{2}(1 \pm \gamma^5)$ are defined using Dirac matrices. These appear in the key requirement for Weyl spinors

$$u_\pm = \omega_\pm u_\pm \tag{72}$$

Successively, it can be shown that, for some $\not{p} = \gamma^\mu p_\mu$

$$u_+ \bar{u}_+ = \omega_+ \not{p} \tag{73}$$

And similarly for the $-$ state. The positive helicity spinor corresponding to p is known as $u_+(p)$. This momentum will always be real and have positive energy. Next, the following important quantities known as 'spinor products' are defined

$$\begin{gathered} s_\pm(p,q) = \bar{u}_\pm(p) u_\mp(q) \\ s_\pm(p,q) = -s_\pm(q,p) = -s_\mp(p,q)^* = s_\mp(q,p)^* \\ s_\pm(p,q) s_\mp(q,p) = 2(p \cdot q) \end{gathered} \tag{74}$$

Usually in literature, $s_+(p,q)$ is written as $[pq]$ and $s_-(p,q)$ is written as $\langle pq\rangle$, and these shorthand notations are used in this thesis as well. For massless spinors, it generally is convenient to work in the standard form for helicity spinors. Choose two basis vectors k_0 and k_1, for which $k_0^2 = 0$, $k_1^2 = -1$ and $k_0 \cdot k_1 = 0$. A basis spinor can be defined as

$$\begin{aligned} u_-(k_0) &= u_0 \\ u_+(k_0) &= \not{k}_1 u_0 \end{aligned} \tag{75}$$

All other massless spinors are defined relative to the basis spinor as

$$u_\pm(p) = \frac{1}{\sqrt{2p\cdot k_0}} \not{p}\, u_\mp(k_0) \tag{76}$$

Next, for massless vector particles (such as the gluon), polarization vectors are defined through

$$\varepsilon^\mu_\lambda = \lambda \frac{\bar{u}_\lambda(p)\gamma^\mu u_\lambda(r)}{\sqrt{2} s_{-\lambda}(p,r)} \tag{77}$$

Here r is the gauge vector corresponding to the polarization ε^μ_λ. This vector may be chosen arbitrarily as long as it is not parallel to p. Let us now verify that this polarization vector has a norm of -1. To do so, the ability to remove repeated indices from a product of spinors is required, which is given as follows

$$\gamma_\mu u_\lambda(r)\bar{u}_\lambda(p)\gamma^\mu = -2u_{-\lambda}(p)\bar{u}_{-\lambda}(r) \tag{78}$$

Using the equation above, it is quite simple to show that for example

$$\begin{aligned} \varepsilon^\mu_+ \bar{\varepsilon}_{+\mu} &= \frac{1}{4p\cdot r}\bar{u}_+(p)\gamma^\mu u_+(r)\bar{u}_+(r)\gamma_\mu u_+(p) \\ &= \frac{-1}{2p\cdot r}\bar{u}_+(p)u_-(r)\bar{u}_-(r)u_+(p) \\ &= -1 \end{aligned} \tag{79}$$

Similarly, the following quantities may be derived

$$\begin{aligned} q\cdot\varepsilon_{1+} &= -\frac{[q\,p_1]\langle q\,r_1\rangle}{\sqrt{2}\langle p_1\,r_1\rangle} \\ q\cdot\varepsilon_{2-} &= \frac{\langle q\,p_2\rangle[q\,r_2]}{\sqrt{2}[p_2\,r_2]} \\ \varepsilon_{1+}\cdot\varepsilon_{2-} &= -\frac{[p_1\,r_2]\langle p_2\,r_1\rangle}{\langle p_1\,r_1\rangle[p_2\,r_2]} \end{aligned} \tag{80}$$

Here p_1 is the momentum vector and r_1 is the gauge vector for ε_1. p_2 and r_2 are defined similarly for ε_2

$$p_\mu = \frac{1}{2}(\bar{u}_+(p)+\bar{u}_-(p))\gamma_\mu(u_+(p)+u_-(p)) \tag{81}$$

Let us now consider how to apply all of this knowledge to generate valid polarization vectors. Instantly a problem arises: All momenta have been chosen as outgoing in chapter two, hence there must be at least one momentum with a negative energy. There, however, is no spinor corresponding to a negative-energy momentum. To avoid this problem, a new spinor w is introduced, such that $u\bar{w} = \omega_{\pm}\not{p}$. It turns out that in terms of standard spinors, this is not very complicated using basis spinor u_0

$$\begin{aligned}\bar{w}_{+}(p) &= \frac{1}{\sqrt{2p\cdot k_0}}\bar{u}_0\not{p} \\ \bar{w}_{-}(p) &= \frac{1}{\sqrt{2p\cdot k_0}}\bar{u}_0\not{k}_1\not{p}\end{aligned} \tag{82}$$

This definition even works for complex momenta p. Notice that in this definition, $u \neq w$. The key difference is, that if p has negative energy or is complex, the terms $\sqrt{2p\cdot k_0}$ are no longer real. This means that our spinor products no longer satisfies the property $s_{+} = -s_{-}^{*}$. Let us now consider how to actually numerically compute polarization vectors and spinor products from these formulae. For spinor products, choose $k_0 = (1,1,0,0)$ and $k_1 = (0,0,1,0)$. Then s_{+} and s_{-} can be calculated as follows

$$\begin{aligned}s_0(p,q) &= (p^2+ip^3)(q^0-q^1)-(q^2+iq^3)(p^0-p^1) \\ s_{+}(p,q) &= \frac{s_0(p,q)}{\sqrt{p^0-p^1}\sqrt{q^0-q^1}} \\ s_{-}(p,q) &= -\frac{s_0(p,q)^{*}}{\sqrt{p^0-p^1}\sqrt{q^0-q^1}}\end{aligned} \tag{83}$$

for $p = (p^0,p^1,p^2,p^3)$ and $q = (q^0,q^1,q^2,q^3)$. Polarization vectors ε are generated numerically as well. Consider a polarization vector with momentum p and gauge vector n.

$$\begin{aligned}f^{\mu} &= (p\cdot k_0)n^{\mu} - (p\cdot n)k_0^{\mu} + (n\cdot k_0)p^{\mu} - i\varepsilon^{\mu\nu\lambda\kappa}k_{0\nu}p_{\lambda}n_{\kappa} \\ \varepsilon_{+}^{\mu} &= \frac{f^{\mu}}{s_{-}(p,r)\sqrt{2(p\cdot k_0)(n\cdot k_0)}} \\ \varepsilon_{-}^{\mu} &= -\frac{f^{\mu}}{s_{+}(p,r)\sqrt{2(p\cdot k_0)(n\cdot k_0)}}\end{aligned} \tag{84}$$

Notice that the object f^{μ} has all properties of ε^{μ}, but simply lacks normalization, which is added after. Partial amplitudes may now be evaluated numerically, using equations (83) and (84). However in order to check our partial amplitudes, we need something to check them against numerically. In the next section, some analytical results for specific helicity configurations are considered in order to verify our partial amplitudes.

2.5 MHV amplitudes

Using the knowledge gained in the previous section on polarization vectors, some general statements on partial amplitudes can be made. Firstly, lets consider a case where all gluons have the same helicity. As partial amplitudes are gauge invariant, a gauge vector may be chosen freely, for each gluon without affecting the result as long as the gauge is not perpendicular to the momentum. The key to evaluate partial amplitudes by hand in an even remotely reasonable way, is to pick these gauge vectors in such a way, that as many terms within the partial amplitude vanish as possible. Let us consider the partial amplitude

$$A_n^{part}(1^+,\dots,n^+) \tag{85}$$

As the gauge vectors of polarization vectors $\varepsilon_1,\dots,\varepsilon_n$ may be picked freely, the same gauge vector is simply picked for all. Evaluating the following product of polarization vectors using equation (78) produces

$$\varepsilon^{\mu}_{i+}\varepsilon_{j+\mu} \sim \bar{u}_+(p_i)\gamma^{\mu}u_+(r)\bar{u}_+(p_j)\gamma_{\mu}u_+(r) = -2[p_i\, p_j]\langle r\, r\rangle = 0 \tag{86}$$

As a partial amplitude does not have an open index such as μ, all indices must be contracted internally. Furthermore a diagram contributing to A_n^{part} contains at most $n-2$ cubic vertices. As each external gluon adds a polarization vector ε and each cubic vertex adds an internal momenta q to contract with, there are $n-(n-2)=2$ polarization vectors contracted with each other. Combining equations (85) and (86) then yields

$$A_n^{part}(1^+,\dots,n^+) = 0 \tag{87}$$

A similar case is a partial amplitude which contains one negative-helicity gluon while all other helicities are positive. The gauge vectors of all positive-helicity gluons can be chosen to equal to p_-, the momentum corresponding to the negative-helicity gluon. This once again makes all products of polarization vectors vanish and

$$A_n^{part}(1^+,\dots,i^-,\dots,n^+) = 0 \tag{88}$$

The type of non-vanishing partial tree amplitudes with, for instance, two negative helicity gluons and all others positive, are known as 'MHV amplitudes'. MHV comes from Maximally Helicity Violating, as these amplitudes violate helicity conservation to the highest degree possible without vanishing. For MHV amplitudes, the usual choice of gauge vectors is as follows: For the positive helicity gluon, pick as gauge vector one of the momenta of the negative helicity gluons, and vice versa. The polarization vectors then have the following form

$$\varepsilon^{\mu}_{j+} = \frac{\bar{u}_{+}(p_j)\gamma^{\mu}u_{+}(p_{-})}{\sqrt{2}s_{-}(p_j,p_{-})} \tag{89}$$

$$\varepsilon^{\mu}_{j-} = -\frac{\bar{u}_{-}(p_j)\gamma^{\mu}u_{-}(p_{+})}{\sqrt{2}s_{+}(p_j,p_{+})} \tag{90}$$

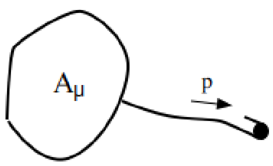
This choice ensures that if, say, $p_{-} = p_i$ is picked, that only the products $\varepsilon_j \cdot \varepsilon_k \neq 0$, where k is any gluon except i, or the gluon corresponding to p_{+}. In 1980, Parke and Taylor conjectured [5] that for MHV amplitudes the following formula holds

$$A_n^{part}(1^{+},\ldots,i^{-},\ldots,j^{-},\ldots,n^{+}) = ig^{n-2}\frac{\langle i\, j\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle} \tag{91}$$

This result was later proven in [6]. Equation (91) is a great result, as partial amplitudes may be derived analytically, and calculated numerically using polarization vectors and spinor products from section 2.4 and then checked against equation (91).

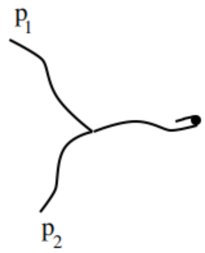
2.6 Handlebars

The final preliminary topic which needs attention are handlebars. As already shortly introduced, replacing a polarization vector with its respective momentum is known as ’applying the handlebar’. In this section we will investigate handlebars more extensively. Partial amplitudes vanish under this operation by the Ward identity, but the operation also simplifies many other structures. In particular, anywhere a line is multiplied by the momentum flowing through it, will be known as applying the handlebar. Diagrammatically, the handlebar is represented as follows



$$\Leftrightarrow \quad A_{\mu}p_{\mu}$$

Let us now investigate how the handlebar simplifies calculations. In many instances, handlebars may be isolated from more complicated calculations. Important is the following structure



$$\Leftrightarrow \quad \varepsilon_{1\mu}\varepsilon_{2\nu}Y(p_1,\mu;p_2,\nu;-p_1-p_2,\lambda)(-p_1-p_2)_\lambda \tag{92}$$

Here the black dot with the line attached to it signifies the 'handlebar'. To find what this equals, let us first simplify

$$\begin{aligned}&\varepsilon_{1\mu}\varepsilon_{2\nu}Y(p_1,\mu;p_2,\nu;-p_1-p_2,\lambda)(-p_1-p_2)_\lambda = \\ &(p_2+p_1+p_2)^\mu\varepsilon_{1\mu}\varepsilon_2^\lambda + (-p_1-p_2-p_1)^\nu\varepsilon_{2\nu}\varepsilon_1^\lambda + (p_1-p_2)^\lambda(\varepsilon_1\cdot\varepsilon_2) = \\ &2(p_2\cdot\varepsilon_1)\varepsilon_2^\lambda - 2(p_1\cdot\varepsilon_2)\varepsilon_1^\lambda + (p_1-p_2)^\lambda(\varepsilon_1\cdot\varepsilon_2)\end{aligned} \tag{93}$$

Contracting this expression with $(-p_1-p_2)_\lambda$, it clearly vanishes. This was of course to be expected, as equation (92) is nothing but the handle bar on a three-gluon partial amplitude, which vanishes by the Ward identity. Furthermore, another useful result is

$$Y(1,2,\mu) = 2(p_2\cdot\varepsilon_1)\varepsilon_2^\mu - 2(p_1\cdot\varepsilon_2)\varepsilon_1^\mu + (p_1-p_2)^\mu(\varepsilon_1\cdot\varepsilon_2) \tag{94}$$

Which makes it easy to write out any terms of $Y(a,b,\mu)$. This result is used extensively throughout this thesis, as it makes calculating three-point terms far easier. Lets now look at a more general handlebar identity, which will prove also be useful later in this thesis. Consider

$$\begin{aligned}Y(p,\mu;q\nu;r,\lambda)r_\lambda &= (p-q)\cdot rg^{\mu\nu} + (q-r)^\mu r^\nu + (r-p)^\nu r^\mu \\ &= -(p-q)\cdot(p+q)g^{\mu\nu} + q^\mu r^\nu - p^\nu r^\mu \\ &= -(p^2-q^2)g^{\mu\nu} - q^\mu q^\nu + p^\mu p^\nu \\ &= K(p)^{\mu\nu} - K(q)^{\mu\nu}\end{aligned} \tag{95}$$

Define the following quantity

$$K(p)^{\mu\nu} = p^\mu p^\nu - p^2 g^{\mu\nu} \tag{96}$$

For a gluon with momentum p and polarization ε this has the following properties

$$\begin{aligned}K(p)^{\mu\nu}p_\mu &= 0 \\ K(p)^{\mu\nu}\varepsilon_\mu &= 0\end{aligned} \tag{97}$$

Finally, this concludes our short elaboration on handlebars, and with it this chapter on preliminaries.

2.7 Recap of chapter two

In chapter two, various preliminary concepts have been considered, which will be applied in the next chapters. Starting out, the Feynman rules for Yang-mills theory were listed. Continuing, a review of Yang-mills color structure was completed. Next, we considered ways to decompose Yang-Mills into colored and colorless components. In the last three sections, ways to calculate partial amplitudes, analytical results for partial amplitudes and handlebars were considered. These concepts were explained as they are required to understand the material in the rest of this thesis. In particular, decompositions of the full amplitude in terms of partial amplitudes are very important.

3 Identities for partial amplitudes

In this chapter relations between partial amplitudes which were known before the discovery of the BCJ identity are investigated. In total there were four main identities known. All four of these identities are discussed in the three sections below as they are relevant for reducing the total number of independent partial amplitudes similar to how the BCJ identity is used. Firstly, the cyclic and reversal identities are examined, which are quite simple. Secondly the photon-decoupling identity is discussed, where the derivation in [4] is followed. Finally the Kleiss-Kuijf identity [7] is explained. The main aim of this chapter is to show how the total number of independent partial amplitudes is reduced as much as possible, starting with the cyclic identity and ending with the Kleiss-Kuijf identity. Let us quickly recap the decomposition into partial amplitudes before continuing our investigation.

$$A_n^{tree} = g^{n-3} \sum_{P(a_2,a_3,\ldots,a_n)} A_n^{part}(1,a_2,a_3,\ldots,a_n)Tr(T^1 T^{a_2} T^{a_3} \ldots T^{a_n}) \quad (98)$$

With the sum running over all noncyclic permutations of the legs.

3.1 Cyclic and reversal identities

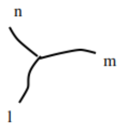
The first property, the cyclic property, follows directly from the cyclic property of traces. As $Tr(a_1 a_2 \ldots a_n) = Tr(\sigma(a_1 a_2 \ldots a_n))$, where σ denotes cylic permutation, it immediately follows that

$$\begin{aligned} & A_n^{part}(a_1,a_2,\ldots,a_n) \\ &= A_n^{tree}\,|_{Tr(a_1 a_2 \ldots a_n)} \\ &= A_n^{tree}\,|_{Tr(\sigma(a_1 a_2 \ldots a_n))} \\ &= A_n^{part}(\sigma(a_1,a_2,\ldots,a_n)) \end{aligned} \quad (99)$$

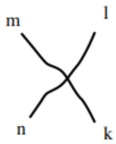
This directly explains why in equation (98), the sum runs over partial amplitude while keeping leg 1 fixed. Naturally, another convention could also be chosen by summing over all permutations and then multiplying by $\frac{1}{n}$, but then we would simply be introducing redundancy into or formulas, without any particular reason. Using the cyclic property, it can be seen that there are at most $(n-1)!$ independent partial amplitudes. The cyclic property is usually used to order partial amplitudes, bringing index 1 to the first position, as has been done in this thesis prior in section 2.3.2. Let us now discuss a slightly less trivial symmetry: The reversal identity.

$$A_n^{part}(a_1, a_2, \ldots, a_n) = (-1)^n A_n^{part}(a_n, \ldots, a_2, a_1) \tag{100}$$

If the color-ordered Feynman rules are considered, which are used to build A_n^{part} using color traces, as described in section 2.3.2, the result becomes clear



$$|_{Tr(nml)} = Y(n,m,l) \tag{101}$$



$$|_{Tr(mnlk)} = g^{\mu\kappa} g^{\nu\lambda} - 2g^{\mu\lambda} g^{\nu\kappa} + g^{\mu\nu} g^{\lambda\kappa} \tag{102}$$

Clearly the three-gluon color-ordered Feynman rule (101) is antisymmetric under exchange of either of its indices. The four-gluon color-ordered Feynman rule (102) is symmetric, but this is as expected, as two three-gluon vertices are also symmetric under exchange of their indices. This means that, as the partial amplitude is built up out of these Feynman rules, it will exhibit the same symmetry. If the partial amplitude has n external gluons, then it will contain a diagram with $n-2$ cubic interactions, and hence obtain a relative factor of $(-1)^{n-2} = (-1)^n$ upon reversal of its indices. The reversal identity reduces the total number of independent partial amplitudes to $\frac{(n-1)!}{2}$, as it pairs each partial amplitude with its reflection. This is not a bad result, however it can be optimized far further. In the next chapter we consider the Photon-decoupling identity.

3.2 The photon-decoupling identity

The next identity discussed is the photon-decoupling identity along the discussion in [4]. It arises as our decomposition (98) is valid for a $U(N)$ gauge theory just as much as an $SU(N)$ gauge theory. As mathematically $U(N) \cong SU(N) \times U(1)$, a $U(1)$ generator has to be added to our generators T^i for $SU(N)$ to obtain a set of generators for $U(N)$. $U(1)$ is simply the circle group, and its generator is proportional to $\mathbb{1}$. This generator is henceforth known as T_0. In order to preserve the desired normalization $Tr(T^aT^b) = \frac{1}{2}\delta^{ab}$, take $T_0 = \frac{1}{\sqrt{2N}}\mathbb{1}$. Next, the structure constant f^{ab0} is investigated. First, notice

$$[T^a, T^b] = f^{abc}T^c \tag{103}$$

But as the identity matrix commutes with every other generator, it easily follows that $f^{ab0} = 0$ for all a and b. Cyclic symmetry in f then requires that any structure constant containing T^0 vanishes. In theory, a $U(1)$ particle is usually known as a 'photon' or a 'colorless gluon'. Note that this is not necessarily the photon known in QED. In any case, $f^{ab0} = 0$ implies that colored gluons do not interact with colorless gluons, as the structure constant is present in all interaction vertices. Next, this implies that any amplitude containing one colorless and $n-1$ colored gluons should vanish, as the colorless gluon cannot interact and cannot disappear into thin air either. Finally, the decomposition in equation (98) still has to be valid for a $U(N)$ theory. For the $SU(N)$ case, this is true as shown in section 2.3.2. For $U(N)$, following the same idea, a 'Fierz identity' is obtained

$$\frac{1}{2}\delta^c_b\delta^a_d = \sum_i (T^i)^c_d(T^i)^a_b \tag{104}$$

Naturally, this will allow us to combine traces of color matrices into single traces: $Tr(T^iA)Tr(T^iB) = \frac{1}{2}Tr(AB)$, and all products of color factors can once again be expressed in terms of sums of color traces. Furthermore, the partial amplitudes will clearly have the same structure as before, as the $\frac{1}{N}$ term that appeared in the Fierz identity was cancelled out in the $SU(N)$ case anyways. Let us write out the full amplitude and study the effect of inserting a $U(1)$ colorless gluon.

$$0 = A_n^{tree} = \sum_{P(a_2,a_3,\ldots,a_n)} A_n^{part}(1, a_2, a_3, \ldots, a_n)Tr(T^1T^{a_2}T^{a_3}\ldots T^{a_n}) \tag{105}$$

Take the first particle in our amplitude to be the colorless gluon, and hence $T^1 = T_0 = \frac{1}{\sqrt{2N}}\mathbb{1}$. The sum has to equal zero, and as A_n^{part} does not contain any color information, all partial amplitudes that are coefficients of the same color trace must sum up to zero. As $T^1 \sim \mathbb{1}$, it can be dropped out of the trace in (105) to obtain

$$\sum_{\sigma(a_2,a_3,\ldots,a_n)} A_n^{part}(1, a_2, a_3, \ldots, a_n) = 0 \tag{106}$$

Where σ denotes cyclic permutation. This identity is known as the 'photon decoupling identity', as it arises from the fact that the photon does not interact with gluons, or also as the 'subcyclic identity'. Using this identity, $(n-2)!$ partial amplitudes can be written in terms of others through

$$\begin{aligned} A_n^{part}(1,2,b_1,\dots,b_n) = &-A_n^{part}(1,b_1,b_2,\dots,2) \\ &-A_n^{part}(1,b_2,b_3,\dots,b_1) \\ &-\dots \\ &-A_n^{part}(1,b_n,2,\dots,b_{n-1}) \end{aligned} \tag{107}$$

Without the reversal identity, this reduces the total number of independent partial amplitudes to $(n-1)!-(n-2)!$. Keeping in mind that the reverse of a cyclic permutation is still a cyclic permutation, both identities together give us $\frac{(n-1)!-(n-2)!}{2}$ independent partial amplitudes. We can, however, still do better than this. The Kleiss-Kuijf identity will reduce the number of independent partial amplitudes to $(n-2)!$.

3.3 Kleiss-Kuijf identity

In this section, the Kleiss-Kuijf identity is introduced. This identity was first theorised in [7] and proven in [8]. The proof of this identity is not discussed in this thesis, as it is quite long and beyond the scope of this work. Nevertheless, what Kleiss and Kuijf noticed about the photon-decoupling identity is that it can, in a way, be extended to reduce the total number of independent partial ampitudes to $(n-2)!$. For this, a 'meshing' of two sets has to be introduced, as it is used in the identity. Say there are two sets $\{\alpha\} = \{\alpha_1,\dots,\alpha_n\}$ and $\{\beta\} = \{\beta_1,\dots,\beta_n\}$. A 'meshing' now is a set which contains all elements of both $\{\alpha\}$ and $\{\beta\}$, but maintains their respective orderings. A few examples of meshings of $\{\alpha\}$ and $\{\beta\}$ would be

$$\begin{aligned} &\{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n\} \\ &\{\alpha_1,\beta_1,\alpha_2,\beta_2,\dots,\alpha_n,\beta_n\} \\ &\{\beta_1,\dots,\beta_j,\alpha_1,\dots,\alpha_n,\beta_{j+1},\dots,\beta_n\} \\ &\{\beta_1,\dots,\beta_j,\alpha_1,\dots,\alpha_k,\beta_{j+1},\dots,\beta_n,\alpha_{k+1}\dots,\alpha_n\} \end{aligned} \tag{108}$$

Because, if these sets are read from left to right, the elements of α_i and β_i are encountered in the same order from 1 to n as in the original sets. Meshings are sometimes also referred to as 'ordered permutations' in other works. The Kleiss-Kuijf identity is now given as follows

$$A_n^{part}(1,\{\alpha\},n,\{\beta\}) = (-1)^{n_\beta} \sum_{\{\sigma\}_i\in M(\{\alpha\},\{\beta^T\})} A_n^{part}(1,\{\sigma\}_i,n) \tag{109}$$

Where $\{\beta^T\}$, which is the reverse of $\{\beta\}$, and n_β is the number of elements of the set $\{\beta\}$. This is the sum over all the meshings of $\{\alpha\}$ and $\{\beta^T\}$. These relations are proven in [8]. As an example, let us consider how the Kleiss-Kuijf identity decomposes the five-gluon amplitude $A_n^{part}(1,2,5,4,3)$, where $n = 5$. By the definitions of $\{\alpha\}$ and $\{\beta\}$, $\{\alpha\} = \{2\}$ and $\{\beta\} = \{3,4\}$. Applying the Kleiss-Kuijf identity hence yields

$$\begin{aligned} A_n^{part}(1,2,5,4,3) &= A_n^{part}(1,2,4,3,5) \\ &= A_n^{part}(1,4,2,3,5) \\ &= A_n^{part}(1,4,3,2,5) \end{aligned} \tag{110}$$

The reversal identity is embedded into the Kleiss-Kuijf identity. Take $\{\alpha\}$ to be empty, which instantaneously gives

$$A_n^{part}(1,n,\{\beta\}) = (-1)^{n_\beta} A_n^{part}(1,\{\beta^T\},n) \tag{111}$$

Which is exactly the reversal identity, as $n_\beta = n-2$ in this case. Furthermore, this identity also implies the photon-decoupling identity: if the set β is taken to contain a single element, the photon-decoupling identity follows instantly. To reproduce equation (107) simply observe

$$\begin{aligned} A_n^{part}(2,\{\alpha\},b_n,1) =& (-1) \sum_{\{\sigma\}_i \in M(\{\alpha\},\{1\})} A_n^{part}(2,\{\sigma\}_i,b_n) \\ =& -A_n^{part}(2,1,b_1,\dots,b_n) \\ & -A_n^{part}(2,b_1,1,b_2,\dots,b_n) \\ & -\dots \\ & -A_n^{part}(2,b_1,\dots,b_{n-1},1,b_n) \end{aligned} \tag{112}$$

By the cyclic property, this can easily be seen to equal equation (107). Notice that the Kleiss-Kuijf identity allows us to write any partial amplitude in terms of $A_n^{part}(n,1,\{\sigma\}_i)$, and hence in terms of $A_n^{part}(1,2,\dots)$, by redefining our gluons. As the Kleiss-Kuijf identity implies the reversal identity and photon-decoupling identity, it follows there are $(n-2)!$ independent partial amplitudes in total when considering all four identities discussed in this chapter. Summarizing, we investigated identities for partial amplitudes known prior to the BCJ identity. The Kleiss-Kuijf identity is, together with the BCJ identity, necessary to provide a basis of $(n-3)!$ independent partial amplitudes. The BCJ will be considered further on in the next chapter.

4 The BCJ identity

In this and the following chapters a discussion and investigation of the new relations for partial amplitudes conjectured by Z. Bern, J. J. M. Carrasco and H. Johansson in [1] is given. In this chapter, the original paper is investigated, and the original conjecture of the BCJ identity is explained. From the onset, four-gluon partial amplitudes were considered by Bern, Carrasco and Johansson, and an already well-known identity was expressed in a new way. After this, the extension to an identity between partial amplitudes with five and then more gluons is discussed, which is known as the BCJ identity.

4.1 The four-gluon BCJ identity

The BCJ identity was derived by first considering four-gluon partial amplitudes. It turns out that there is only one independent four-gluon partial amplitude. Furthermore Bern, Carrasco and Johansson found a new identity for the kinematic poles of the Manelstam variables. In this section, the BCJ identity for four-gluon partial amptlitudes is discussed. Let us consider what is already known about the four-gluon partial amplitudes A_4^{part}. Both the Kleiss-Kuijf identity and the photon-decoupling identity, imply that there are a total of two independent partial amplitudes. Partial amplitudes can be written as rational functions of momenta, polarization vectors and spinors. The photon-decoupling identity implies

$$A_4^{part}(1,2,3,4)+A_4^{part}(1,4,2,3)+A_4^{part}(1,3,4,2)=0 \qquad (113)$$

It should be clear, as partial amplitudes are gauge invariant, that this must hold regardless of gauge choice for our gluons. Key is that partial amplitudes may depend on the Mandelstam variables, which are defined as usual for 4-particle amplitudes as $s=(p_1+p_2)^2$, $t=(p_1+p_3)^2$ and $u=(p_1+p_4)^2$. Also known is a key identity for Mandelstam variables:

$$s+t+u=p_1^2+p_2^2+p_3^2+p_4^2 \qquad (114)$$

In the four-gluon case, this implies $s+t+u=0$. As equation (113) cannot rely on specific polarizations, the cancellation must entirely arise from the cancellation in the mandelstam variables. This implies that

$$\begin{aligned} &A_4^{part}(1,2,3,4)+A_4^{part}(1,4,2,3)+A_4^{part}(1,3,4,2) \\ &\quad=(s+t+u)\chi=0 \end{aligned} \tag{115}$$

BCJ concluded that as $A_4^{part}(1,2,3,4)=A_4^{part}(4,1,2,3)$, the partial amplitude $A_4^{part}(1,2,3,4)$ must treat the Mandelstam variables s and u equally. This can be seen as any factor multiplied s in $A_4^{part}(1,2,3,4)$ will be multiplied by u in $A_4^{part}(4,1,2,3)$. This then implies that

$$A_4^{part}(1,2,3,4)=t\,\alpha_1+(s+u)\,\alpha_2 \tag{116}$$

However, as $s+t+u=0$, this simply means that $A_4^{part}(1,2,3,4)=t\,\chi_1$. The same argument may be used for $A_4^{part}(1,4,2,3)$ and $A_4^{part}(1,3,4,2)$ to give the following equations

$$\begin{aligned} A_4^{part}(1,2,3,4)&=t\,\chi_1 \\ A_4^{part}(1,4,2,3)&=s\,\chi_2 \\ A_4^{part}(1,3,4,2)&=u\,\chi_3 \end{aligned} \tag{117}$$

Finally using (115) it can be seen that $\chi_1=\chi_2=\chi_3=\chi$. This means that in fact, all four-gluon partial amplitudes are a function of a single shared factor. This immediately implies that there is only one indendent four-gluon partial amplitude. Let us express the other two partial amplitudes in terms of $A_4^{part}(1,2,3,4)$

$$\begin{aligned} A_4^{part}(1,4,2,3)&=\frac{s}{t}A_4^{part}(1,2,3,4)=\frac{s_{12}}{s_{13}}A_4^{part}(1,2,3,4) \\ A_4^{part}(1,3,4,2)&=\frac{u}{t}A_4^{part}(1,2,3,4)=\frac{s_{14}}{s_{13}}A_4^{part}(1,2,3,4) \end{aligned} \tag{118}$$

Where as usual $s_{ij}=(p_i+p_j)^2$ and should not be confused with the Mandelstam variable s. Let us now consider a decomposition of a partial amplitude into Mandelstam variable poles. This should certainly be possible, as the Mandelstam variables are squares of the only internal momenta present within the Feynman diagrams contributing to the full four-gluon amplitude A_4^{tree}. Such a decomposition is given as follows

$$A_4^{part}(1,2,3,4)=\frac{f_s}{s}+\frac{f_t}{t}+\frac{f_u}{u} \tag{119}$$

Where f is a function of polarizations and momenta which contains no poles. From our previous argument, $A_4^{part}(1,2,3,4)=u\chi$. As f_u contains at most a square of a momentum, it follows that $f_u=0$. Furthermore, this same decomposition may be done for the other two partial amplitudes in equation (116), and combining their decompositions with the photon-decoupling identity (113) yields

$$\begin{aligned}\frac{1}{2}A_4^{part}(1,2,3,4) &= \frac{n_s}{s} - \frac{n_t}{t}\\ \frac{1}{2}A_4^{part}(1,3,4,2) &= \frac{n_u}{u} - \frac{n_s}{s}\\ \frac{1}{2}A_4^{part}(1,4,2,3) &= \frac{n_t}{t} - \frac{n_u}{u}\end{aligned} \tag{120}$$

Here quartic contributions have been adsorbed into the numerators, in a currently unspecified way. Every partial amplitude is defined with a relative factor of $\frac{1}{2}$, as this will give us a nicer result for the full amplitude in terms of the numerators. These numerators are non-unique as there is ambiguity in how to adsorb the quartic contributions as will be discussed further on in chapter five. The first equation in (118) and (120) can now be combined in order to find

$$\frac{n_t}{t} - \frac{n_u}{u} = \frac{s}{u}\frac{n_s}{s} - \frac{s}{u}\frac{n_t}{t} \tag{121}$$

Multiplying by u, and using $s+t+u=0$ yields

$$n_t + n_u + n_s = 0 \tag{122}$$

This new identity for four-gluon numerators was found by Bern, Carrasco and Johansson. The derivation they gave, which was just discussed here, is independent of any degrees of freedom within n_s, n_u or n_t. This means that in fact it does not matter how the quartic contributions are adsorbed into the numerators. Notice how the factor $\frac{1}{2}$ drops out due to equation (120). It should be noted that these numerators are identical to the coefficients of color brackets discussed in section 2.3.1.

$$A_4^{tree} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right) \tag{123}$$

With

$$\begin{aligned}c_s &= [1234]\\ c_t &= [1342]\\ c_u &= [1432]\end{aligned} \tag{124}$$

Now from the jacobi identity for color brackets, Bern, Carrasco and Johansson found a duality

$$\begin{gathered}c_s + c_t + c_u = 0\\ \Leftrightarrow\\ n_s + n_t + n_u = 0\end{gathered} \tag{125}$$

This duality is key to deriving the BCJ identity. What Bern, Carrasco and Johansson conjectured, is that this identity will also hold for numerators beyond four gluons, at least for some choice of numerators n. Using this, Bern, Carrasco and Johansson derived new relations between five-gluon partial amplitudes and conjectured a general new identity which reduces the total number of independent partial amplitudes to the order of $(n-3)!$.

4.2 The five-gluon BCJ identity

In this section, the result Bern, Carrasco and Johansson derived for five-gluon partial amplitudes is presented. In this discussion about their result, the same definitions and conventions as in [1] for color factors and numerators are used to make sure their idea is followed as closely as possible. Note that in general, color factors and numerator may be redefined as $c_i \to -c_i$ and $n_i \to -n_i$ whenever it is convenient without changing either the full amplitude or any of the partial amplitudes, save for a relative sign of within the partial amplitudes. Assume the following correspondence holds

$$c_i - c_j + c_k = 0 \quad \Rightarrow \quad n_i - n_j + n_k = 0 \tag{126}$$

The first aspect to consider is to define which color bracket and numerator corresponds to which diagram.

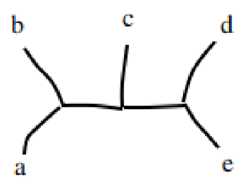


Figure 3: *The structure of cubic diagrams at the five-point level.*

Cearly, there are 15 unique ways to arrange five external gluons on this diagram, as there are five choices for the middle gluon and three ways arrange the remaining four gluons. This implies that there are 15 unique five-gluon diagrams containing only cubic vertices and 15 numerators n_i and 15 color factors c_i corresponding to said diagrams. Remember that all quartic contributions are absorbed into the numerators, and do not have to be considered on their own here. Continuing, all color factors and the structure of each corresponding numerator is given explicitly. Here each numerator does not have to

i	c_i	n_i	i	c_i	n_i
1	[12345]	n(1 2 3 4 5)	9	[13425]	n(1 3 4 2 5)
2	[23451]	n(2 3 4 5 1)	10	[42513]	n(4 2 5 1 3)
3	[34512]	n(3 4 5 1 2)	11	[51342]	n(5 1 3 4 2)
4	[45123]	n(4 5 1 2 3)	12	[12435]	n(1 2 4 3 5)
5	[51234]	n(5 1 2 3 4)	13	[35124]	n(3 5 1 2 4)
6	[14325]	n(1 4 3 2 5)	14	[14235]	n(1 4 2 3 5)
7	[32514]	n(3 2 5 1 4)	15	[13245]	n(1 3 2 4 5)
8	[25143]	n(2 5 1 4 3)			

Table 1: *The 15 color brackets and numerators corresponding to unique cubic five-gluon diagrams*

be found explicitly, the main relevant information are the internal symmetries $n(a\,b\,c\,d\,e) = -n(b\,a\,c\,d\,e) = n(b\,a\,c\,e\,d) = -n(e\,d\,c\,b\,a)$. An example for a possible choice of five-gluon numerators can be found in section 2.3.1. As with the four-gluon case, once again the full amplitude can be written in terms of numerators and color brackets:

$$\begin{aligned} A_5^{tree} = g^3 \Big(& \frac{c_1 n_1}{s_{12}s_{45}} + \frac{c_2 n_2}{s_{23}s_{51}} + \frac{c_3 n_3}{s_{34}s_{12}} + \frac{c_4 n_4}{s_{45}s_{23}} + \frac{c_5 n_5}{s_{51}s_{34}} \\ & + \frac{c_6 n_6}{s_{14}s_{25}} + \frac{c_7 n_7}{s_{32}s_{14}} + \frac{c_8 n_8}{s_{25}s_{43}} + \frac{c_9 n_9}{s_{13}s_{25}} + \frac{c_{10} n_{10}}{s_{42}s_{13}} \\ & + \frac{c_{11} n_{11}}{s_{51}s_{42}} + \frac{c_{12} n_{12}}{s_{12}s_{35}} + \frac{c_{13} n_{13}}{s_{35}s_{24}} + \frac{c_{14} n_{14}}{s_{14}s_{35}} + \frac{c_{15} n_{15}}{s_{13}s_{45}} \Big) \end{aligned} \tag{127}$$

Let us now look at the partial amplitudes A_5^{part}. The Kleiss-Kuijf relations imply that there are $(5-2)! = 6$ independent five-gluon partial amplitudes. BCJ picked an independent basis of partial amplitudes and expressed them in terms of five-gluon numerators. The easiest way to find the numerators that contribute to each partial amplitude is to find the color traces which contain $Tr(a\,b\,c\,d\,e)$, and sum the corresponding numerators while keeping the sign of $Tr(a\,b\,c\,d\,e)$ in the color trace in mind. It turns out that the only color factors which contribute are $[abcde]$ and its cyclic permutations. In terms of numerators, BCJ found the following six Kleiss-Kuijf amplitudes:

$$\frac{1}{2}A_5^{part}(1,2,3,4,5) = \frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{51}s_{34}} \tag{128}$$

$$\frac{1}{2}A_5^{part}(1,4,3,2,5) = \frac{n_6}{s_{14}s_{25}} + \frac{n_5}{s_{43}s_{51}} + \frac{n_7}{s_{32}s_{14}} + \frac{n_8}{s_{25}s_{43}} + \frac{n_2}{s_{51}s_{32}} \tag{129}$$

$$\frac{1}{2}A_5^{part}(1,3,4,2,5) = \frac{n_9}{s_{13}s_{25}} - \frac{n_5}{s_{34}s_{51}} + \frac{n_{10}}{s_{42}s_{13}} - \frac{n_8}{s_{25}s_{34}} + \frac{n_{11}}{s_{51}s_{42}} \tag{130}$$

$$\frac{1}{2}A_5^{part}(1,2,4,3,5) = \frac{n_{12}}{s_{12}s_{35}} + \frac{n_{11}}{s_{24}s_{51}} - \frac{n_3}{s_{43}s_{12}} + \frac{n_{13}}{s_{35}s_{24}} - \frac{n_5}{s_{51}s_{43}} \tag{131}$$

$$\frac{1}{2}A_5^{part}(1,4,2,3,5) = \frac{n_{14}}{s_{14}s_{35}} - \frac{n_{11}}{s_{42}s_{51}} - \frac{n_7}{s_{23}s_{14}} - \frac{n_{13}}{s_{35}s_{42}} - \frac{n_2}{s_{51}s_{23}} \tag{132}$$

$$\frac{1}{2}A_5^{part}(1,3,2,4,5) = \frac{n_{15}}{s_{13}s_{45}} - \frac{n_2}{s_{32}s_{51}} - \frac{n_{10}}{s_{24}s_{13}} - \frac{n_4}{s_{45}s_{32}} - \frac{n_{11}}{s_{51}s_{24}} \tag{133}$$

These partial amplitudes are exactly the same as the ones chosen in the original paper on the BCJ identity. Notice the cyclic symmetry in the poles contributing to each partial amplitude. The second piece of the puzzle to eliminate as many numerators as possible are the color and numerator Jacobi identities. In total there are 15 color factors, and each color bracket is part of two Jacobi identities. Together with the Jacobi identity involving three color brackets, there are ten Jacobi identities, which are listed in table 2. Assuming the color-numerator Jacobi

$c_3 - c_5 + c_8 = 0$	$\Rightarrow$	$n_3 - n_5 + n_8 = 0$
$c_4 - c_1 + c_{15} = 0$	$\Rightarrow$	$n_4 - n_1 + n_{15} = 0$
$c_5 - c_2 + c_{11} = 0$	$\Rightarrow$	$n_5 - n_2 + n_{11} = 0$
$c_8 - c_6 + c_9 = 0$	$\Rightarrow$	$n_8 - n_6 + n_9 = 0$
$c_{10} - c_{11} + c_{13} = 0$	$\Rightarrow$	$n_{10} - n_{11} + n_{13} = 0$
$c_3 - c_1 + c_{12} = 0$	$\Rightarrow$	$n_3 - n_1 + n_{12} = 0$
$c_4 - c_2 + c_7 = 0$	$\Rightarrow$	$n_4 - n_2 + n_7 = 0$
$c_7 - c_6 + c_{14} = 0$	$\Rightarrow$	$n_7 - n_6 + n_{14} = 0$
$c_{10} - c_9 + c_{15} = 0$	$\Rightarrow$	$n_{10} - n_9 + n_{15} = 0$
$c_{13} - c_{12} + c_{14} = 0$	$\Rightarrow$	$n_{13} - n_{12} + n_{14} = 0$

Table 2: *All ten color Jacobi identities for five-gluon color brackets and the implied 'numerator Jacobi identity' corresponding to each color Jacobi identity.*

identity duality, a set of nontrivial relations is obtained for the kinematic numerators. Next, what needs to be considered is: how many independent numerators are there under these relations. Due to the color-numerator duality, the number of independent numerators will equal the number of independent color brackets under the jacobi identity. This turns out to equal $(n-2)!$, as is discussed later. For now the only aspect required to be known is that there are six independent five-gluon numerators. In this case, the numerators n_1 to n_6 were picked as basis numerators, and all other numerators were expressed in terms of these. This can be done using the relations in table 2 and yields the following equations

$$
\begin{aligned}
n_7 &= n_2 - n_4 \\
n_8 &= n_5 - n_3 \\
n_9 &= n_3 - n_5 + n_6 \\
n_{10} &= n_3 - n_1 + n_4 - n_5 + n_6 \\
n_{11} &= n_2 - n_5 \\
n_{12} &= n_1 - n_3 \\
n_{13} &= n_1 + n_2 - n_3 - n_4 - n_6 \\
n_{14} &= n_4 - n_2 + n_6 \\
n_{15} &= n_1 - n_4
\end{aligned} \tag{134}
$$

An observant reader might have noticed that in fact there are only nine numerator equations here, while there were ten before. The reason was that one was lost in the system of numerator equations in table 2. One equation turns out to be dependent on the others. This is logical as there are only 6 independent numerators and 15 numerators in total, leaving at most 9 independent equations between them. Next, two of our independent numerators are expressed in terms of two partial amplitudes combined with the other numerators. This strategy was employed in the original paper to ascertain that the choices made for the numerators are automatically consistent with all partial amplitudes. Define the following for n_5 and n_6

$$n_5 = s_{51}s_{34}\left(\frac{1}{2}A_5^{part}(1,2,3,4,5) - \frac{n_1}{s_{12}s_{45}} - \frac{n_2}{s_{23}s_{51}} - \frac{n_3}{s_{34}s_{12}} - \frac{n_4}{s_{45}s_{23}}\right) \tag{135}$$

$$n_6 = s_{14}s_{25}\left(\frac{1}{2}A_5^{part}(1,4,3,2,5) - \frac{n_5}{s_{43}s_{51}} - \frac{n_7}{s_{32}s_{14}} - \frac{n_8}{s_{25}s_{43}} - \frac{n_2}{s_{51}s_{32}}\right) \tag{136}$$

The first equation is completely in terms of n_1, n_2, n_3 and n_4, however the second equation is not. Fortunately, the relations in equation (134) can be employed to eliminate all unwanted numerators in equation (136)

$$
\begin{aligned}
n_6 = {} & \frac{1}{2}A_5^{part}(1,4,3,2,5)s_{14}s_{25} - \frac{1}{2}A_5^{part}(1,2,3,4,5)(s_{15}+s_{25})s_{14} \\
& + n_1\frac{s_{14}(s_{15}+s_{25})}{s_{12}s_{45}} + n_2\frac{s_{23}+s_{35}}{s_{23}} \\
& + n_3\frac{s_{14}}{s_{12}} + n_4\frac{s_{14}s_{15}+s_{14}s_{25}+s_{25}s_{45}}{s_{23}s_{45}}
\end{aligned} \tag{137}
$$

Now the following question arises: what can be done with numerators n_1, n_2, n_3 and n_4? The partial amplitudes are gauge invariant, and the remaining numerators generally are not. It could be also expected for the partial amplitudes to be independent of them. This is where a brilliant realisation from the original paper can be used. Remember that kinematic numerators are in fact not at all unique, and degrees of freedom can be introduced in the following way

$$\begin{aligned} n'_1 &= n_1 + \alpha_1 s_{12} s_{45} \\ n'_2 &= n_2 + \alpha_2 s_{23} s_{51} \\ n'_3 &= n_3 + \alpha_3 s_{34} s_{12} \\ n'_4 &= n_4 + \alpha_4 s_{45} s_{23} \end{aligned} \tag{138}$$

Through the way the other numerators are set up in equation (134), the numerator Jacobi identities will be preserved. The change of n_5 and n_6 is also clear using their definitions in terms of the other numerators, which equals

$$n'_5 = n_5 - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) s_{51} s_{34} \tag{139}$$

$$\begin{aligned} n'_6 = n_6 + &\alpha_1 s_{14}(s_{51} + s_{25}) + \alpha_2 s_{51}(s_{23} + s_{35}) \\ &+ \alpha_3 s_{34} s_{14} + \alpha_4 (s_{14} s_{15} + s_{14} s_{25} + s_{25} s_{45}) \end{aligned} \tag{140}$$

Now BCJ found that in fact, all partial amplitudes are invariant under this shift. As an example, consider $A_5^{part}(1,3,4,1,5)$. Let us define

$$\Delta A_5^{part}(1,3,2,4,5) = A_5^{\prime part}(1,3,2,4,5) - A_5^{part}(1,3,2,4,5) \tag{141}$$

$$\Delta n_i = n'_i - n_i \tag{142}$$

Combining these with equation (133) yields

$$\Delta A_5^{part}(1,3,2,4,5) = \frac{\Delta n_{15}}{s_{13} s_{45}} - \frac{\Delta n_2}{s_{32} s_{45}} - \frac{\Delta n_{10}}{s_{24} s_{14}} - \frac{\Delta n_4}{s_{45} s_{32}} - \frac{\Delta n_{11}}{s_{51} s_{24}} \tag{143}$$

This equation, together with the way shifts in the numerators are defined in (138), and way the other numerators are defined through equation (134) gives

$$\begin{aligned} \Delta A_5^{part}(1,3,2,4,5) = &\alpha_1 \frac{s_{12}}{s_{13}} - \alpha_4 \frac{s_{23}}{s_{13}} - \alpha_2 - \alpha_3 \frac{s_{45} s_{12}}{s_{24} s_{13}} + \alpha_1 \frac{s_{34} s_{12}}{s_{24} s_{13}} \\ &- \alpha_4 \frac{s_{45} s_{23}}{s_{24} s_{13}} - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) \frac{s_{51} s_{34}}{s_{24} s_{13}} \\ &- \alpha_1 \frac{s_{14}(s_{51} + s_{25})}{s_{24} s_{13}} - \alpha_2 \frac{s_{51}(s_{23} + s_{35})}{s_{24} s_{13}} \\ &- \alpha_3 \frac{s_{34} s_{14}}{s_{24} s_{13}} - \alpha_4 \frac{s_{14} s_{15} + s_{14} s_{25} + s_{25} s_{45}}{s_{24} s_{13}} \\ &- \alpha_4 - \alpha_2 \frac{s_{23}}{s_{24}} - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) \frac{s_{34}}{s_{24}} \end{aligned} \tag{144}$$

This expression turns out to vanish using simple conservation of momentum. Remember that the momenta are defined such that $p_1+p_2+p_3+p_4+p_5=0$, and hence $s_{1i}+s_{2i}+s_{3i}+s_{4i}+s_{5i}=0$ with i being any of the five momenta. Another important relation is $s_{12}=s_{34}+s_{45}+s_{53}$. For instance, the sum of contributions of all terms leading by α_1 equal

$$\frac{s_{12}}{s_{13}}+\frac{s_{45}s_{12}}{s_{24}s_{13}}-\frac{s_{51}s_{34}}{s_{24}s_{13}}-\frac{s_{14}(s_{51}+s_{25})}{s_{24}s_{13}}-\frac{s_{34}}{s_{24}} \tag{145}$$

Bringing most terms under one denominator and using $s_{51}+s_{13}=s_{14}+s_{12}$, equation (145) maybe simplified to

$$\frac{s_{12}}{s_{13}}+\frac{s_{45}s_{12}+s_{34}s_{12}+s_{14}(s_{34}-s_{51}-s_{25})}{s_{24}s_{13}} \tag{146}$$

Next, $s_{34}=s_{12}+s_{51}+s_{25}$ and $s_{45}+s_{34}+s_{14}+s_{24}=0$ yields

$$\frac{s_{12}}{s_{13}}+\frac{(s_{45}+s_{34}+s_{14})s_{12}}{s_{24}s_{13}}=0 \tag{147}$$

Using similar techniques, the same can be shown for terms leading in α_2, α_3 and α_4, which means that indeed $\Delta A_5^{part}(1,3,2,4,5)=0$. For the other shifts in partial amplitudes, the same can be done and they also equal zero. Concluding, five-gluon partial amplitudes are indeed invariant under the degrees of freedom in equation (138). As originally, there were no constraints on α_i in (138), they may simply be chosen such that they cancel out the numerators n_i completely, and $n'_1=n'_2=n'_3=n'_4=0$. Doing so leaves us with

$$n'_5=\frac{1}{2}A_5^{part}(1,2,3,4,5)s_{51}s_{34} \tag{148}$$

$$n'_6=\frac{1}{2}A_5^{part}(1,4,3,2,5)s_{14}s_{25}-\frac{1}{2}A_5^{part}(1,2,3,4,5)(s_{15}+s_{25})s_{14} \tag{149}$$

As all partial amplitudes are functions of these numerators and only these numerators (the others being zero), it is certain that all other partial amplitudes can be written in terms of these two. Doing some simple simplifications, Bern, Carrasco and Johansson found the following new identities for five-gluon partial amplitudes:

$$\begin{aligned}
A_5^{part}(1,3,4,2,5)&=\frac{-s_{12}s_{45}A_5^{part}(1,2,3,4,5)+s_{14}(s_{24}+s_{25})A_5^{part}(1,4,3,2,5)}{s_{13}s_{24}}\\
A_5^{part}(1,2,4,3,5)&=\frac{-s_{14}s_{25}A_5^{part}(1,4,3,2,5)+s_{45}(s_{12}+s_{24})A_5^{part}(1,2,3,4,5)}{s_{35}s_{24}}\\
A_5^{part}(1,4,2,3,5)&=\frac{-s_{12}s_{45}A_5^{part}(1,2,3,4,5)+s_{25}(s_{14}+s_{24})A_5^{part}(1,4,3,2,5)}{s_{35}s_{24}}\\
A_5^{part}(1,3,2,4,5)&=\frac{-s_{14}s_{25}A_5^{part}(1,4,3,2,5)+s_{12}(s_{24}+s_{45})A_5^{part}(1,2,3,4,5)}{s_{13}s_{24}}
\end{aligned} \tag{150}$$

This means that similarly to the four-gluon case, the total number of independent five-gluon partial amplitudes is reduced to $(n-3)!$. This is clearly better than the $(n-2)!$ independent partial amplitudes under the Kleiss-Kuijf relations. It would now be interesting to see if this pattern holds for amplitudes beyond five gluons. To do this, will now look at BCJ's conjecture on the extension of this identity to amplitudes with more than five gluons.

4.3 The n-gluon BCJ identity

4.3.1 General argument

The idea behind for the n-gluon BCJ identity is very similar to the four- and five-gluon cases. Firstly take a basis of color factors c_i, then identify numerators n_i with them. Secondly, assume a duality between color and numerator Jacobi identities to hold. Through this duality all numerators may be expressed in terms of basis of $(n-2)!$ basis numerators. Later on in this section it is shown that indeed there are $(n-2)!$ independent color factors under the color Jacobi identity. Next, every partial amplitude may be decomposed into a sum of kinematic numerators. Finally, fix $(n-3)!$ numerators through an basis of $(n-3)!$ partial amplitudes and introduce degrees of freedom in all other numerators to set them to zero. An outline to derive the BCJ identity for n-gluon partial amplitudes is given as follows

1. Decompose the full amplitude into numerators and color factors in the usual way: $A_n^{tree} = \sum_i \frac{c_i n_i}{\prod_m p_{mi}^2}$. Here p_{mi} represents the internal momenta of the cubic diagram associated with n_i.
2. Impose the conjectured Jacobi identity duality between color factors and numerators. Next, pick a basis of $(n-2)!$ numerators which are independent under the numerator Jacobi identity. Then, express all other numerators in terms of these basis numerators.
3. Express all partial amplitudes in the form $A_n^{part}(a_1,\ldots,a_n) = \sum_j \frac{n_j}{\prod_m p_{mj}^2}$. This may certainly be done as all color factors c_i can be written into a sum

of traces of color matrices $c_i = \sum_j Tr(a_1 \dots a_n)$, and hence all color factors c_i can be projected onto $Tr(a_1 a_2 \dots a_n)$.

4. Choose a basis of $(n-3)!$ partial amplitudes which are independent under the Kleiss-Kuijf identity. Independence under the Kleiss-Kuijf identity is key as otherwise one of the basis amplitudes can be expressed in terms of the others, which would be a contradiction.

5. Fix $(n-3)!$ numerators to our basis amplitudes, similar to how n_5 was fixed in equation (135).

6. Finally, BCJ conjectured that the remaining $(n-2)! - (n-3)!$ numerators which are not anchored to a partial amplitude may be set to zero while every partial amplitude remains invariant. This then allows us to substitute the remaining $(n-3)!$ numerators for partial amplitudes, as they are expressed purely into partial amplitudes at this point, yielding us $(n-3)!$ independent partial amplitudes

4.3.2 Various counts for the BCJ identity

In this section, a discussion about the various formulae given for various counts which are relevant to the BCJ identity is provided. In the original paper, BCJ listed these counts up to eight gluons. These quantities are tabulated in table 3, Let us now discuss how to derive these quantities through theory. As this will in some cases require knowledge beyond what might be expected of the reader, a reference is provided to a more thorough discussion on said derivation to the reader in those cases. There are five quantities discussed in this section: the total number of independent color factors (A), the total number of independent color factors under the Jacobi identity (B), the number of numerators per partial amplitude (C), the total number of numerator equations (D) and finally the total number of independent numerator equations (E).

A The Total number of independent color factors

As discussed previously in section 4.2, the total number of independent n-gluon color factors (without considering the Jacobi identity) equals the total number of independent n-gluon cubic tree diagrams. Now, a derivation of a formula for the latter is given. As the total number of cubic tree diagrams is sought, which

gluons	3	4	5	6	n
independent color factors	1	3	15	105	$(2n-5)!!$
Jacobi-independent color factors	1	2	6	24	$(n-2)!$
numerators per partial amplitude	1	2	5	14	$\frac{2^{n-2}(2n-5)!!}{(n-1)!}$
numerator equations	0	1	10	105	$\frac{n-3}{3}(2n-5)!!$
independent numerator equations	0	1	9	81	$(2n-5)!!-(n-2)!$
Kleiss-Kuijf amplitudes	1	2	6	24	$(n-2)!$
independent BCJ amplitudes	1	1	2	6	$(n-3)!$

Table 3: *Various relevant counts of amplitudes, numerators and equations as a function of the total number of external gluons in the full tree amplitude. In the second row the total number of independent color factors without considering the color Jacobi identity is listed. The third row states the total number of independent color factors under the Jacobi identity. In the fourth row the total the total number of numerators contained in a single partial amplitude is enumerated. In the fifth row, the total number of numerator equations implied by the Jacobi identity is stated. Next, the sixth row lists the number of nontrivial numerator Jacobi identities. Finally, the seventh and eight rows should speak for themselves and are not further detailed.*

is known as ' diagram counting', is employed following the method described in the section on diagram counting in [3]. Notice that the exact structure of the amplitude does not really matter when considering the total number of unique diagrams. For this reason the total number of unique n-gluon cubic diagrams tree will be the same as the number of unique n-particle zero-dimensional scalar tree diagrams. This implies that simply a theory may be used where all propagators and vertices are set to unity. Notice that symmetry factors are not relevant in tree diagrams and do not have to be considered. Examine the cubic action

$$S(\phi) = \frac{1}{2}\phi^2 - \frac{1}{6}\phi^3 \tag{151}$$

Plugging this into the Schwinger-Dyson equation ($\frac{\delta S}{\delta \phi} = J$) yields

$$\phi = J + \frac{1}{2}\phi^2 \tag{152}$$

Here J is our source. Now notice that if the field ϕ is expanded in terms of J, the amplitude for ϕ with n sources is the coefficient of $\frac{J^n}{n!}$ in this expansion. The number of diagrams with $n+1$ external legs is then equal to this coefficient, as each unique diagram contributes exactly 1, and for $n=0$ there is one leg.

Inverting the previous equation yields

$$\phi = 1 - \sqrt{1-2J} \tag{153}$$

Which has the following Taylor expansion around $J = 0$

$$\phi = \sum_{n=0}^{\infty} \frac{(2n-3)!!}{n!} J^n \tag{154}$$

Next, this implies there are $(2n-3)!!$ cubic diagrams with $n+1$ external legs, hence $(2n-5)!!$ cubic diagrams with n external legs as required.

B Jacobi-independent color factors

To calculate the total number of color factors independent under the Jacobi identity, color factors need to be expressed in a diagrammatic way. As only tree amplitudes are under consideration, each string of f^{abc} factors can be represented by a cubic tree diagram. Many proofs of this nature are given in [9]. Let us define the following

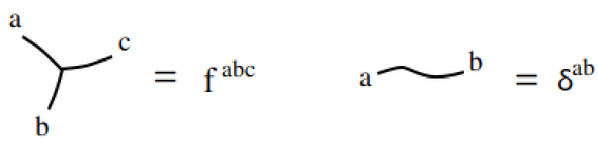


Figure 4: *The diagrammatic rules for color factors*

Now recall the Jacobi identity, which implies $f^{abn}\delta^{nm}f^{mcd} = f^{acn}\delta^{nm}f^{mdb} - f^{adn}\delta^{nm}f^{mcb}$. Diagrammatically implying

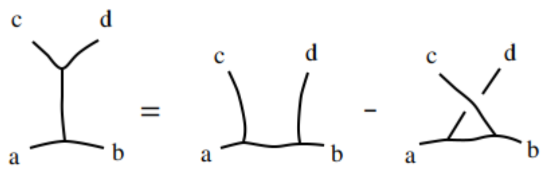
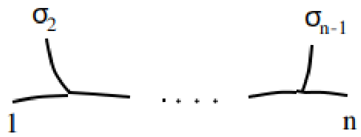


Figure 5: *The diagrammatic representation of the Jacobi identity*

Next, let us pick the colors corresponding to two gluons, 1 and n, and examine how the diagram can be 'straightened' in between. Using the Jacobi identity a branch can be removed in favor of two smaller branches, as shown in figure (6).

Figure 6: *Straightening out a branch between* 1 *and* n

Successively, this process may be continued until all diagrams have the following form



Clearly there are $n-2$ unspecified branches here. There are $(n-2)!$ possible configurations of the gluons 2 to $n-1$. As all other color configurations can be brought into this form through the Jacobi identity, there are $(n-2)!$ independent color factors.

C Numerators per partial amplitude

The derivation of the number of numerators contained within each partial amplitude is approached through a counting argument. Recall that each string of color factors may be expressed in terms of matrix traces. In particular, every factor f introduces a commutator into the trace. This implies that a trace of n factors will decompose into a sum of 2^n traces. Now, as none of these traces will equal each other through cyclic symmetry, and every single trace corresponds to a partial amplitude $A_n^{part}(1,\dots)$. Furthermore, as all partial amplitudes have the same structure, they will all contain the same number of numerators. Hence there are 2^{n-2} numerators per color factor, divided over $(n-1)!$ partial amplitudes, implying the number of numerators per partial amplitude equals

$$\frac{2^{n-2}(2n-5)!!}{(n-1)!} \tag{155}$$

D Numerator equations

The count of numerator equations is equivalent to the number of Jacobi identities each color factor is present in multiplied by the number of unique color factors. Notice that if a color factor consists out of m factors f, there are $m-1$ possible ways to choose two adjacent f to be part of a Jacobi identity. As every Jacobi identity themselves involves three numerators, and a color factor with n gluons consists of $n-2$ factors of f, the total number of Jacobi identities equals

$$\frac{n-3}{3}(2n-5)!! \tag{156}$$

E Independent numerator equations

The number of independent numerator equations directly follows from linear algebra, as in total there are $(2n-5)!!$ independent numerators without considering the color-numerator Jacobi identity duality, and $(n-2)!$ if do, the number of independent numerator equations equals $(2n-5)!!-(n-2)!$.

4.3.3 n-gluon BCJ formula

In this section, the n-gluon BCJ formula proposed in the original paper [1] will be introduced. This formula is quite complicated, however it will be explained fully, and some examples on its usage are be given. Remember that using the Kleiss-Kuijf identity, all partial amplitudes may be written in terms of a basis of $A_n^{part}(1,2,\mathscr{P}(3,\ldots,n))$. Bern, Carrasco and Johansson also conjectured that the basis of $(n-3)!$ independent partial amplitudes may be chosen arbitrarily, as long as the basis elements are independent under the Kleiss-Kuijf identity. A basis of amplitudes of the form $A_n^{part}(1,2,3,\mathscr{P}(4,\ldots,n))$ was chosen. Using the Kleiss-Kuijf identity, any amplitude may be written in terms of basis amplitudes of the following form

$$A_n^{part}(1,2,\{\alpha\},3,\{\beta\}) \tag{157}$$

Let us now define a 'Partially ordered permutation' of $\{\alpha\}$ and $\{\beta\}$. This concept is quite similar to that of a meshing which was introduced earlier in section 3.3, however for a partially ordered permutation, only the elements of $\{\beta\}$ have to remain ordered. For instance, consider $\{\alpha\}=\{1,2\}$ and $\{\beta\}=\{3,4,5\}$, then the following lists the partially ordered permutation $POP(\{\alpha\},\{\beta\})$ of $\{\alpha\}$ and

$\{\beta\}$

$$\begin{aligned}&\{1,2,3,4,5\},\{2,1,3,4,5\},\{1,3,2,4,5\},\{2,3,1,4,5\},\\&\{3,1,2,4,5\},\{3,2,1,4,5\},\{3,1,4,2,5\},\{3,2,4,1,5\},\\&\{3,4,1,2,5\},\{3,4,2,1,5\},\{3,4,1,5,2\},\{3,4,2,5,1\},\\&\{3,4,5,1,2\},\{3,4,5,2,1\},\{1,3,4,2,5\},\{2,3,4,1,5\},\\&\{1,3,4,5,2\},\{2,3,4,5,1\},\{3,1,4,5,2\},\{3,2,4,5,1\}\end{aligned} \tag{158}$$

Bern, Carrasco and Johansson extrapolated the structure of solutions to their new identity up to 8 external gluons, and conjectured the BCJ formula. The formula is given as follows

$$\begin{aligned}A_n^{part}(1,2,\{\alpha\},3,\{\beta\}) = &\sum_{\{\sigma\}_j\in POP(\{\alpha\},\{\beta\})} A_n^{part}(1,2,3,\{\sigma\}_j)\\&\times\prod_{k=4}^{m+3}\frac{F(3,\{\sigma\}_j,1\mid k)}{s_{2,4,\ldots,k}}\end{aligned} \tag{159}$$

Here m equals the number of elements of set $\{\alpha\}$, and $s_{2,4,\ldots,k} = (p_2+p_4+\cdots+p_k)^2$. The function F is defined as follows

$$\begin{aligned}&F(3,\{\sigma\}_j,1\mid k) = F(\{\rho\}\mid k) =\\&\left\{\sum_{l=t_k}^{n-1} G(k,\rho_l)\ \text{ if }\ t_{k-1}<t_k,\ -\sum_{l=1}^{t_k} G(k,\rho_l)\ \text{ if }\ t_{k-1}>t_k\right\}+\\&\left\{s_{2,4,\ldots,k}\ \text{ if }\ t_{k-1}<t_k<t_{k+1},\ -s_{2,4,\ldots,k}\ \text{ if }\ t_{k-1}>t_k>t_{k+1}\right\}\end{aligned} \tag{160}$$

Here t_k is the position of leg k in the set $\{\rho\}$, except for t_3 and t_{m+1}, which are defined through

$$\begin{aligned}t_3 &= t_5\\t_{m+1} &= 0\end{aligned} \tag{161}$$

Finally $G(i,j) = s_{ij}$ if and only if $i<j$ or $j=1$ or 3. Otherwise G equals zero. As the implementation of this formula might not be obvious at first sight, some example calculations are given next. Using the BCJ formula, $A_6^{part}(1,2,4,5,6,3)$ can be expressed in terms of basis amplitudes. Clearly $\{\alpha\}=\{4,5,6\}$, $\{\beta\}=\emptyset$. The partially ordered permutations of $\{\alpha\}$ are now simply all permutations of $\{\alpha\}$ itself. Lets now evaluate the factor corresponding to $A_6^{part}(1,2,3,4,5,6)$ in equation (159). The product in equation (159) has to be evaluated first

$$\prod_{k=4}^{6}\frac{F(3,4,5,6,1\mid k)}{s_{2,4,\ldots,k}} \tag{162}$$

Using the definition of t_k

$$t_3=3,\ \ t_4=2\ \ ,t_5=3,\ \ t_6=4,\ \ t_7=0 \tag{163}$$

Now F can be calculated using this

$$
\begin{aligned}
F(3,4,5,6,1\,|\,4) &= -\sum_{l=1}^{2} G(4,\rho_l) = -G(4,3) - G(4,4) = -s_{34} \\
F(3,4,5,6,1\,|\,5) &= \sum_{l=3}^{5} G(5,\rho_l) + s_{245} = s_{245} + s_{56} + s_{15} \\
F(3,4,5,6,1\,|\,6) &= \sum_{l=4}^{5} G(6,\rho_l) = s_{16}
\end{aligned}
\tag{164}
$$

Notice the extra factor of s_{245} originating from the third line in equation (160). Combining these terms means that $A_6^{part}(1,2,3,4,5,6)$ obtains a coefficient of

$$
-\frac{s_{34}(s_{245}+s_{56}+s_{15})s_{16}}{s_{24}s_{245}s_{2456}} \tag{165}
$$

As a second example, let us examine the factor corresponding to $A_6^{part}(1,2,3,5,4,6)$. Now $F(3,5,4,6,1\,|\,k)$ is needed. This time, t_k is given as follows

$$
t_3 = 2,\ \ t_4 = 3\ \ ,t_5 = 2,\ \ t_6 = 4,\ \ t_7 = 0 \tag{166}
$$

Once again calculate F

$$
\begin{aligned}
F(3,5,4,6,1\,|\,4) &= \sum_{l=3}^{5} G(4,\rho_l) = s_{46} + s_{41} \\
F(3,5,4,6,1\,|\,5) &= -\sum_{l=1}^{2} G(5,\rho_l) = -s_{53} \\
F(3,5,4,6,1\,|\,6) &= \sum_{l=4}^{5} G(6,\rho_l) = s_{61}
\end{aligned}
\tag{167}
$$

So $A_6^{part}(1,2,3,5,4,6)$ obtains a factor of

$$
-\frac{(s_{46}+s_{41})s_{53}s_{61}}{s_{24}s_{245}s_{2456}} \tag{168}
$$

This calculation is now done for all other permutations of $\{\alpha\}$, obtaining the full expansion of $A_6^{part}(1,2,4,5,6,3)$ in terms of BCJ amplitudes

$$
\begin{aligned}
A_6^{part}(1,2,4,5,6,3) = &-A_6^{part}(1,2,3,4,5,6)\frac{s_{34}(s_{245}+s_{56}+s_{15})s_{16}}{s_{24}s_{245}s_{2456}} \\
&+A_6^{part}(1,2,3,4,6,5)\frac{s_{34}s_{51}(s_{2456}+s_{36})}{s_{24}s_{245}s_{2456}} \\
&+A_6^{part}(1,2,3,6,4,5)\frac{(s_{34}+s_{46})s_{51}(s_{2456}+s_{36})}{s_{24}s_{245}s_{2456}} \\
&-A_6^{part}(1,2,3,5,4,6)\frac{(s_{46}+s_{41})s_{53}s_{61}}{s_{24}s_{245}s_{2456}} \\
&-A_6^{part}(1,2,3,5,6,4)\frac{s_{14}s_{35}s_{16}}{s_{24}s_{245}s_{2456}} \\
&+A_6^{part}(1,2,3,6,5,4)\frac{s_{14}(s_{245}+s_{35}+s_{56})(s_{2456}+s_{36})}{s_{24}s_{245}s_{2456}}
\end{aligned} \tag{169}
$$

The original derivation of the BCJ identity has now been discussed, together with an example application on how to apply the BCJ formula to express partial amplitudes in terms of a basis of $(n-3)!$ basis amplitudes.

In this chapter, we have considered the BCJ identity. Using this BCJ identity, any partial amplitude can be written in terms of a basis of $(n-3)!$ independent partial amplitudes. This is a strong reduction from the $(n-2)!$ result by Kleiss and Kuijf, and allows for a far more efficient calculations of gluonic tree amplitudes. Let us now start investigating a diagrammatic way of deriving this identity. This investigation starts by trying to find numerators which satisfy the numerator Jacobi identity in the next chapter.

5 Solving numerator identities

The aim in this chapter is to find the numerators previously discussed which admit the Jacobi identity. Furthermore, our aim is to do it in such a way that the numerators may be described in some kind of a diagrammatic way, as our aim is to prove the BCJ identity in this a way. As a quick recap, numerators n_i are the coefficients of color factors c_i in the following expansion

$$A^n_{tree} = g^{n-2} \sum_j \frac{n_j c_j}{\prod_i p^2_{ij}} \tag{170}$$

Where p_{ij} denote the internal momenta of the cubic diagram corresponding to c_i. Keep in mind that all quartic contributions have been contracted into the cubic numerators as well. As discussed previously in section 4.2, numerators are not unique. When calculating a numerator from Feynman diagrams, however it is needed to have some way to achieve a consistent result. For this reason the color Jacobi identity will be ignored at first when calculating numerators from diagrams, as this will produce an unique result. Our aim is to identify a pattern in the choice of numerators which satisfy the numerator Jacobi identity, and give a diagrammatic representation of this pattern. Our expectation then is to extrapolate from this pattern and prove that the numerator Jacobi identity holds generally.

5.1 Degrees of freedom within numerators

As hinted at previously, the numerators n_i are in fact not unique. This property was used extensively in the original derivation of the BCJ identity, however first it would be convenient to understand what degrees of freedom a numerator n_i actually admits to. Let us consider the decomposition into color factors c_i. Known is that $\{c_i\}$ is a basis for all color factors present in the full amplitude A^{tree}_n, but through the Jacobi identity it is dependent. In chapter four, however, it has been shown that there exists a basis, say $\{c_\alpha\}$ which is independent. Now, projecting A^{tree}_n onto this basis will produce us with an unique decomposition of coefficients. Through this it will be seen that the only degrees of freedom in the numerators n_i arise through the Jacobi identity. Examine what form these degrees of freedom have. Consider

$$A^n_{tree} = A^n_{tree} + c_i + c_j + c_k \tag{171}$$

Where c_i, c_j and c_k are picked such that they obey the Jacobi identity. As $c_i + c_j + c_k = 0$, the sum may multiplied by any factor whatsoever, and added to A_n^{tree} without changing the full amplitude.

$$A^n_{tree} = A^n_{tree} + (c_i + c_j + c_k)\Delta \tag{172}$$

Next, consider how this changes our numerators. A_n^{tree} is clearly invariant under the aforementioned change. In each of the numerators, however, a new term is picked up. A degree of freedom appears in numerators n_i, n_j and n_k in the following way

$$\begin{aligned} n_i' &= n_i + \Delta \prod_{q_i} q_i^2 \\ n_j' &= n_j + \Delta \prod_{q_j} q_j^2 \\ n_k' &= n_k + \Delta \prod_{q_k} q_k^2 \end{aligned} \tag{173}$$

Where q_i are the internal momenta of the cubic diagram corresponding to numerator n_i. In total, for each color jacobi identity, a degree of freedom Δ is obtained through the process described above. Each color factor is connected to $n-3$ such degrees of freedom, which are captured into its corresponding numerator n_i. Let us now consider the degrees of freedom obtained in our simple example (173). It appears that the ways in which changes in the numerators can be made are still constrained by the products of momenta $\prod_m p_m^2$. Naturally, a redefinition could be made

$$\Delta' = \frac{\Delta}{\prod_{q_i} q_i^2} \tag{174}$$

but this would only shift the problem around. Instead it is beneficial to examine the structure of the numerators n_i, n_j and n_k. Consider the Jacobi identity on the color factors c_i, c_j and c_k. As these obey the Jacobi identity, somewhere within them there must be terms hidden such as

$$[nmlk] + [nkml] + [nlkm] = 0 \tag{175}$$

Furthermore, if this Jacobi identity was to be the core of $c_i + c_j + c_k = 0$, then all other terms must be shared between them, so a common term may be extracted such as

$$C^{nmlk}[nmlk] = c_i \tag{176}$$

Where C^{nmlk} contains all other factors f. Now, as only tree diagrams are considered, in particular C^{nmlk} must have the structure of

$$F_1^n F_2^m F_3^l F_4^k = C^{nmlk} \tag{177}$$

Returning to the cubic diagrams corresponding to the numerators n_i, n_j and n_k, it can be seen that they all must have the following structure as shown in figure 7

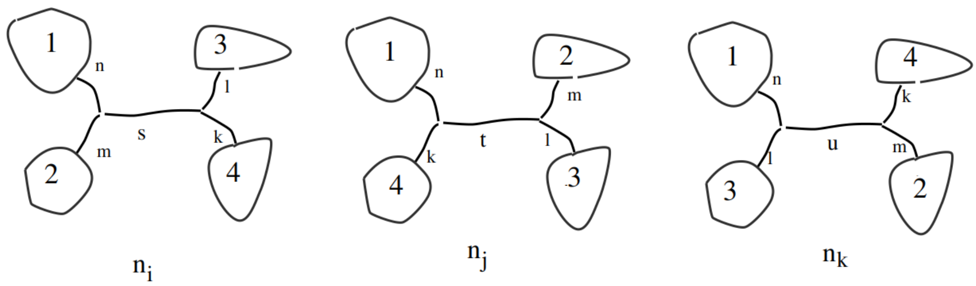


Figure 7: *The internal structure of cubic diagrams corresponding to the numerators n_i, n_j and n_k. Note that blobs with the same number are the same in all three diagrams. External gluons have not been drawn on the blobs but they are implied.*

Notice how the cubic diagrams are almost identical and share the vast majority of their structure between them. The only thing differentiating is the internal momentum running across a single internal line. In this case, define $s = (p_n + p_m)^2$, $t = (p_n + p_k)^2$ and $u = (p_n + p_l)^2$ as analog to the Mandelstam variables. These are the only internal momenta different between the diagrams, and hence the factor Δ may be redefined in equation (173) to divide out all other internal momenta. Doing so leaves the following structure for our degree of freedom:

$$\begin{aligned} n_i' &= n_i + \Delta s \\ n_j' &= n_j + \Delta t \\ n_k' &= n_k + \Delta u \end{aligned} \tag{178}$$

From now on when referring to a 'degree of freedom corresponding to a momentum', the implication should be a degree of freedom Δ in a numerator n_i which arises from color a Jacobi identity. The specific momentum is not shared by the other numerators corresponding to the color factors in the Jacobi identity.

For instance, in (178) Δ corresponds to s in n_i. As every internal momentum in a cubic diagram is connected to two cubic vertices, which together carry a color bracket $[abcd]$, each internal momentum will give rise to a degree of freedom.

$$n_i' = n_i + \sum_{q \in Q_i} q^2 \Delta_{q,i} \tag{179}$$

Furthermore, consider the degrees of freedom obtained in a 'numerator jacobi identity'. Combining previous results gives, for the numerators defined in figure 7

$$\begin{aligned} n_i' + n_j' + n_k' =& n_i + n_j + n_k + \Delta(s+t+u) \\ &+ \sum_{q \in Q_i \cap Q_j} q^2 (\Delta_{q,i} + \Delta_{q,j} + \Delta_{q,k}) \end{aligned} \tag{180}$$

As all internal momenta are shared except for one, each shared internal momentum picks up three degrees of freedom. One from each numerator. Notice that as s, t and u are analogs of the Mandelstam variables, hence $s+t+u = p_n^2 + p_m^2 + p_l^2 + p_k^2$. This means that in fact all degrees of freedom in this sum of numerators is multiplied by a shared internal momenta.

One final check has to be performed now. It is of key importance that the partial amplitudes A_n^{part} are invariant under the degrees of freedom introduced in the numerators. Let us assume n_i is present within a partial amplitude A_n^{part}. A color factor c_i may then be decomposed as

$$c_i = F_1^n F_2^m F_3^l F_4^k [nmlk] \tag{181}$$

Let us define $F_j^n = 2Tr(T^n G_j)$, where G_j is a sum of products of color matrices T. Using this, and $[nmlk] = 2Tr([n,m][l,k])$ gives

$$c_i = 32\, Tr(T^n G_1)\, Tr(T^m G_2)\, Tr(T^l G_3)\, Tr(T^k G_4)\, Tr([n,m][l,k]) \tag{182}$$

Next, without loss of generality, the assumption may be made that the trace which corresponds to A_n^{part} is contained in $Tr(G_1 G_2 G_3 G_4)$. Let us now examine if this trace is also present in c_j or c_k. For c_j the calculation gives

$$\begin{aligned} c_j =& 32\, Tr(T^n G_1)\, Tr(T^m G_2)\, Tr(T^l G_3)\, Tr(T^k G_4)\, Tr([n,k][m,l]) \\ =& 2Tr(G_1 G_4 G_2 G_3) - 2Tr(G_4 G_1 G_2 G_3) \\ &+ 2Tr(G_1 G_4 G_3 G_2) - 2Tr(G_4 G_1 G_3 G_2) \end{aligned} \tag{183}$$

And for c_k it yields

$$\begin{aligned} c_k =& 32\, Tr(T^n G_1)\, Tr(T^m G_2)\, Tr(T^l G_3)\, Tr(T^k G_4)\, Tr([n,l][k,m]) \\ =& 2Tr(G_1 G_3 G_4 G_2) - 2Tr(G_3 G_1 G_4 G_2) \\ &+ 2Tr(G_1 G_3 G_2 G_4) - 2Tr(G_3 G_1 G_2 G_4) \end{aligned} \tag{184}$$

Comparing both c_j and c_k to permutations of $Tr(G_1 G_2 G_3 G_4)$, it can easily be seen that this trace only appears with a factor of -2 in c_j, and does not appear at all in c_k. Please note that even though the factors G_j are sums of products of color matrices, no permutation of the color matrices internally can produce the correct ordering in the full trace if the factors G_j are not ordered like $1,2,3,4$. If numerators n_i and n_j are extracted from the partial amplitude, the following expression is obtained

$$\frac{1}{2}A_n^{part} = \cdots + \frac{n_i}{\prod_{q\in Q_i} q^2} - \frac{n_j}{\prod_{q\in Q_j} q^2} + \dots \tag{185}$$

Introducing our degree of freedom Δ yields us with

$$\frac{1}{2}A_n'^{part} = \cdots + \frac{n_i}{\prod_{q\in Q_i} q^2} + \frac{\Delta s}{\prod_{q\in Q_i} q^2} - \frac{\Delta t}{\prod_{q\in Q_j} q^2} - \frac{n_j}{\prod_{q\in Q_j} q^2} + \dots \tag{186}$$

As c_i and c_j obey a Jacobi identity, by definition all momenta except s and t are shared between the cubic diagrams lying at the root of n_i and n_j, hence

$$s \prod_{q\in Q_j} q^2 = t \prod_{q\in Q_i} q^2 \tag{187}$$

Which means that the freedom Δ drops out in fact $A_n'^{part} = A_n^{part}$ as required.

What has been found so far in this section is the following:

1. The only degrees of freedom that leave the full amplitude A_n^{tree} invariant are ones that are introduced through the color Jacobi identity.
2. In the numerators, these changes can be represented as a degree of freedom, which may be chosen arbitrarily, multiplied by an internal momentum of the cubic diagram corresponding to the numerator. Each internal momentum will produce such a degree of freedom
3. Each degree of freedom is shared by three numerators
4. These degree of freedom leave the partial amplitudes A_n^{part} invariant

Finally, let us consider some constraints on Δ, if any sort of diagrammatic representation is to be obtained for such a degree of freedom. Firstly, consider a general diagrammatic 'fix' to the numerators such that they obey the numerator

Jacobi identity. This implies each degree of freedom arising from a class Jacobi identities must have the same structure. Imagine

$$\Delta_{abcde}([abcde]+[abecd]+[abdec])=0 \tag{188}$$

This clearly fixes a whole class of Δ arising from many Jacobi identites. Secondly, color factors have various symmetries, and degrees of freedom Δ would need to reflect this property. Equation 188) can be changed to, for instance, the following

$$\Delta_{abcde}(-[bacde]-[baecd]-[badec])=0 \tag{189}$$

Finally, to ensure a diagrammatic description, $\Delta_{bacde} = -\Delta_{abcde}$ and similar symmetries are required. Notice that any symmetries a numerator n_i before it was changed, will be preserved. These requirements may seem very constraining, and indeed they are, but to obtain any kind of diagrammatic description for the BCJ numerators which satisfies $n_i+n_j+n_k=0$ it is needed to impose them. Let us now apply what was discussed in this section to four-point numerators.

5.2 Numerator identity for four gluons

As shown already in the section on the derivation of the BCJ identity, the numerator Jacobi identity for four point amplitudes holds. Nevertheless, let us calculate the numerators explicitly and explicitly show that the identity is fulfilled. As the four-gluon amplitude consists out of four diagrams only, as shown in figure 8, the easiest way to find the numerators is to simply write out the full amplitude, and read off the numerators as they appear.

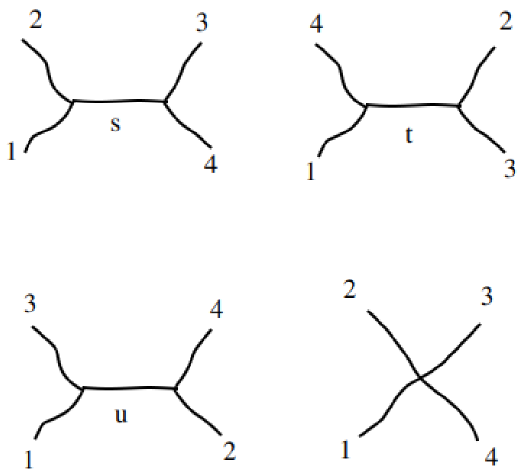


Figure 8: *The four diagrams contributing to the full four gluon amplitude*

Using the Feynman rules, the full amplitude may be constructed in a straightforward manner. Keep in mind that as usual in four-gluon amplitudes, $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$ and $u = (p_1 + p_4)^2$. Note that once again, momenta are always taken to be outgoing.

$$\begin{aligned} A_4^{tree} = & i\hbar g^2 [1234] \frac{Y(1,2,\mu)Y(\mu,3,4)}{s} \\ & + i\hbar g^2 [1423] \frac{Y(1,4,\mu)Y(\mu,2,3)}{u} \\ & + i\hbar g^2 [1342] \frac{Y(1,3,\mu)Y(\mu,4,2)}{t} \\ & - i\hbar g^2 [1234] \Big((\varepsilon_1 \cdot \varepsilon_4)(\varepsilon_2 \cdot \varepsilon_3) - (\varepsilon_1 \cdot \varepsilon_3)(\varepsilon_2 \cdot \varepsilon_4) \Big) \\ & - i\hbar g^2 [1423] \Big((\varepsilon_1 \cdot \varepsilon_3)(\varepsilon_4 \cdot \varepsilon_2) - (\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_4 \cdot \varepsilon_3) \Big) \\ & - i\hbar g^2 [1342] \Big((\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \varepsilon_4) - (\varepsilon_1 \cdot \varepsilon_4)(\varepsilon_3 \cdot \varepsilon_2) \Big) \end{aligned} \tag{190}$$

Successively, define $n(1234)$ to be the numerator corresponding to $[1234]$, this then allows the extraction of the numerator $n(1234)$

$$n(1234) = i\hbar \Big(Y(1,2,\mu)Y(\mu,3,4) + s_{12}(\varepsilon_1 \cdot \varepsilon_4)(\varepsilon_2 \cdot \varepsilon_3) - s_{12}(\varepsilon_1 \cdot \varepsilon_3)(\varepsilon_2 \cdot \varepsilon_4) \Big) \tag{191}$$

Notice how the quartic terms in $n(1234)$ have picked up a factor of s_{12} due to pulling them into the numerator. From now on, as the aim is to show that the sum of three numerators equals zero, any common terms such as $i\hbar$ will be dropped.

As $[1234]+[1423]+[1342]=0$, the numerator identity which has to be shown is

$$n(1234)+n(1423)+n(1342)=0 \tag{192}$$

Let us first look at the sum of the cubic contributions which appear in the numerators. For this a result obtained previously for $Y(a,b,\mu)$ is. Formally,

$$Y(p_1,\mu;p_2,\nu;p_3,\lambda)=(p_1-p_2)^{\lambda}g^{\mu\nu}+(p_2-p_3)^{\mu}g^{\nu\lambda}+(p_3-p_1)^{\nu}g^{\lambda\mu} \tag{193}$$

Now, using momentum conservation and the Lorenz condition $p_i\cdot\varepsilon_i=0$, to simplify this for our particular case

$$Y(a,b,\mu)=2(p_b\cdot\varepsilon_a)\varepsilon_b^{\mu}-2(p_a\cdot\varepsilon_b)\varepsilon_a^{\mu}+(p_a-p_b)^{\mu}(\varepsilon_a\cdot\varepsilon_b) \tag{194}$$

This specific result makes evaluating amplitudes a lot easier, and will be used from now on. Examine the following expression

$$\begin{aligned}
&Y(1,2,\mu)Y(\mu,3,4)+Y(1,4,\mu)Y(\mu,2,3)+Y(1,3,\mu)Y(\mu,4,2)\\
&\quad=\Big(2(p_2\cdot\varepsilon_1)\varepsilon_2^{\mu}-2(p_1\cdot\varepsilon_2)\varepsilon_1^{\mu}+(p_1-p_2)^{\mu}(\varepsilon_1\cdot\varepsilon_2)\Big)\\
&\quad\times\Big(2(p_4\cdot\varepsilon_3)\varepsilon_{4\mu}-2(p_3\cdot\varepsilon_4)\varepsilon_{3\mu}+(p_3-p_4)_{\mu}(\varepsilon_3\cdot\varepsilon_4)\Big)\\
&\quad+\Big(2(p_4\cdot\varepsilon_1)\varepsilon_4^{\mu}-2(p_1\cdot\varepsilon_4)\varepsilon_1^{\mu}+(p_1-p_4)^{\mu}(\varepsilon_1\cdot\varepsilon_4)\Big)\\
&\quad\times\Big(2(p_3\cdot\varepsilon_2)\varepsilon_{3\mu}-2(p_2\cdot\varepsilon_3)\varepsilon_{2\mu}+(p_2-p_3)_{\mu}(\varepsilon_2\cdot\varepsilon_3)\Big)\\
&\quad+\Big(2(p_3\cdot\varepsilon_1)\varepsilon_3^{\mu}-2(p_1\cdot\varepsilon_3)\varepsilon_1^{\mu}+(p_1-p_3)^{\mu}(\varepsilon_1\cdot\varepsilon_3)\Big)\\
&\quad\times\Big(2(p_2\cdot\varepsilon_4)\varepsilon_{2\mu}-2(p_4\cdot\varepsilon_2)\varepsilon_{4\mu}+(p_4-p_2)_{\mu}(\varepsilon_4\cdot\varepsilon_2)\Big)
\end{aligned} \tag{195}$$

This expression is rather complicated, but fortunately through internal symmetry, we only have to solve for this expression for a few terms. All other terms can then be found using aforementioned internal symmetries. Consider what factor $(\varepsilon_1\cdot\varepsilon_2)$ is multiplied by in this sum. Extracting this term yields

$$\begin{aligned}
(\varepsilon_1\cdot\varepsilon_2)\times\Big\{&2(p_1-p_2\cdot\varepsilon_4)(p_4\cdot\varepsilon_3)-2(p_1-p_2\cdot\varepsilon_3)(p_3\cdot\varepsilon_4)\\
&+4(p_1\cdot\varepsilon_4)(p_2\cdot\varepsilon_3)-4(p_1\cdot\varepsilon_3)(p_2\cdot\varepsilon_4)\\
&+(p_1-p_2)\cdot(p_3-p_4)(\varepsilon_3\cdot\varepsilon_4)\Big\}
\end{aligned} \tag{196}$$

Where an additional set of brackets has been omitted for terms such as $(p_1-p_2\cdot\varepsilon_3)$. They are to be be understood as $(p_1-p_2)\cdot\varepsilon_3$. Next, using momentum conservation states $p_1 = -p_2 - p_3 - p_4$ hence (196) can be simplified further

$$(\varepsilon_1 \cdot \varepsilon_2) \times \Big\{ 2(p_1 + p_2 \cdot \varepsilon_4)(p_4 \cdot \varepsilon_3) - 2(p_1 + p_2 \cdot \varepsilon_3)(p_3 \cdot \varepsilon_4) \\ - 4(p_2 \cdot \varepsilon_4)(p_2 \cdot \varepsilon_3) + 4(p_2 \cdot \varepsilon_3)(p_2 \cdot \varepsilon_4) \\ + (p_1 - p_2) \cdot (p_3 - p_4)(\varepsilon_3 \cdot \varepsilon_4) \Big\} \tag{197}$$

The second line in equation (197) clearly vanishes, and the first line vanishes in the same way by using momentum conservation once again. The only term left is now simply $(p_1 - p_2) \cdot (p_3 - p_4)(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \varepsilon_4)$. Cyclic symmetry is now employed to evaluate the full sum in equation (195). This, combined with all quartic contributions produces

$$\begin{aligned} n(1234) + n(1432) + n(1342) = \\ &(p_1 - p_2) \cdot (p_3 - p_4)(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \varepsilon_4) \\ &+ (p_1 - p_4) \cdot (p_2 - p_3)(\varepsilon_1 \cdot \varepsilon_4)(\varepsilon_2 \cdot \varepsilon_3) \\ &+ (p_1 - p_3) \cdot (p_4 - p_2)(\varepsilon_1 \cdot \varepsilon_3)(\varepsilon_4 \cdot \varepsilon_2) \\ &- (s_{13} - s_{14})(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \varepsilon_4) \\ &- (s_{12} - s_{13})(\varepsilon_1 \cdot \varepsilon_4)(\varepsilon_2 \cdot \varepsilon_3) \\ &- (s_{14} - s_{12})(\varepsilon_1 \cdot \varepsilon_3)(\varepsilon_4 \cdot \varepsilon_2) \end{aligned} \tag{198}$$

This equals zero, as for instance $(p_1 - p_2) \cdot (p_3 - p_4) = p_1 \cdot p_3 - p_1 \cdot p_4 - p_2 \cdot p_3 + p_2 \cdot p_4 = s_{13} - s_{14}$. As was the original aim in this section, it has been established analytically that the numerator Jacobi identity holds for a four-point case. Continuing, what can be said about the degree of freedom which is produced by the color Jacobi identity $[1234] + [1423] + [1342] = 0$? Following the discussion in section 5.1 gives the following expression

$$\begin{aligned} n'(1234) + n'(1423) + n'(1342) = &n(1234) + n(1423) + n(1342) \\ &+ (s_{12} + s_{13} + s_{14})\Delta \\ = &0 + (p_1^2 + p_2^2 + p_3^2 + p_4^2)\Delta \\ = &0 \end{aligned} \tag{199}$$

Hence, by virtue of the momenta being massless, in fact our degree of freedom does not change the numerator Jacobi identity at all, and may be chosen arbitrarily. As a short introduction has now been completed on working with numerator Jacobi identities, let us move on to the five-gluon case where the numerator Jacobi identity does not follow as clearly as in this section.

5.3 Numerator identity for five gluons

In this section, the five-gluon numerator Jacobi identity is to be investigated. Firstly, the numerators themselves actually have to be computed. Fortunately, as in the four-gluon case, there is only one class of color factors which is to be considered. All color factors are of the form $f^{abn}f^{ncm}f^{mde}$ or $[abcde]$. In chapter two, the five-gluon numerator corresponding to $[abcde]$ has already been derived. In a quick recap, the diagrams which contribute to the five-gluon numerator are shown in figure 9

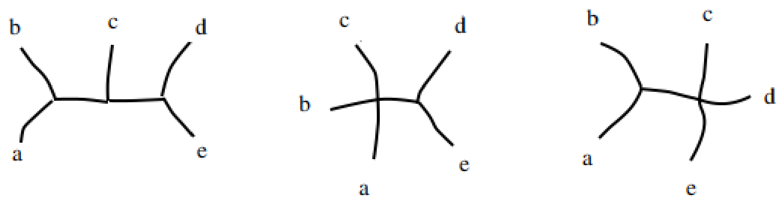


Figure 9: *The three diagrams contributing to the five-gluon numerator $n(abcde)$ corresponding to $[abcde]$*

As discussed in section 2.3.1, these diagrams produce the following numerator:

$$\begin{aligned} n(abcde) = & \; i\hbar\sqrt{\hbar}\, Y(a,b,\mu)Y(\mu,c,\nu)Y(\nu,d,e) \\ & - i\hbar\sqrt{\hbar}\, Y(a,b,\mu)\Big(\varepsilon_{e\mu}(\varepsilon_c\cdot\varepsilon_d) - \varepsilon_{d\mu}(\varepsilon_c\cdot\varepsilon_e)\Big)s_{de} \\ & - i\hbar\sqrt{\hbar}\,\Big(\varepsilon_{a\nu}(\varepsilon_b\cdot\varepsilon_c) - \varepsilon_{b\nu}(\varepsilon_c\cdot\varepsilon_a)\Big)Y(\nu,d,e)s_{ab} \end{aligned} \tag{200}$$

Continuing, the partial amplitude $A_5^{part}(a,b,c,d,e)$ may be constructed from these numerators by projecting each color factor onto $Tr(a\,b\,c\,d\,e)$. This usually is far more convenient than calculating the partial amplitude directly form color-orderd Feynman rules. The only color brackets contributing to $Tr(a\,b\,c\,d\,e)$ are

$$[abcde], [eabcd], [deabc], [cdeab], [bcdea] \tag{201}$$

Summing over the contributions of these color brackets and keeping in mind a relative factor of 2 yields

$$\begin{aligned} \frac{1}{2}A_5^{part}(a,b,c,d,e) = & \frac{n(abcde)}{s_{ab}s_{de}} + \frac{n(bcdea)}{s_{bc}s_{ea}} \\ & + \frac{n(cdeab)}{s_{cd}s_{ab}} + \frac{n(deabc)}{s_{de}s_{bc}} + \frac{n(eabcd)}{s_{ea}s_{cd}} \end{aligned} \tag{202}$$

It has been numerically checked that amplitude (202) constructed from numerators (200) is gauge invariant, and that it equals the Parke-Taylor result in an MHV case with a general choice of gauge vectors. This assures us that the partial amplitude constructed is indeed correct, and the expression for the numerator $n(abcde)$ is a valid choice of numerator. Next, examine in which Jacobi identities the color factor $[abcde]$ is present. There are two ways to split a four-color color bracket off, and so there are two color Jacobi identities to be considered

$$\begin{aligned}
[abcde] + [bcade] + [cabde] &= 0 \\
[abcde] + [abdec] + [abecd] &= 0
\end{aligned} \tag{203}$$

Notice that the first identity may be rewritten by using the antisymmetry in the first two and last two indices:

$$[abcde] + [bcade] + [cabde] = -[decab] - [deabc] - [debca] = 0 \tag{204}$$

This clearly is just a relabeled version of the second equation in (203), hence without loss of generality only one color Jacobi identity may be considered. For instance, $[abcde] + [abdec] + [abecd]$. The next step is to now solve the numerator Jacobi identity corresponding to this color Jacobi identity. Firstly, a naieve calculation is given using the choices of numerators stated previously. Then, using the degrees of freedom within each numerator, an attempt is made to fix the numerator Jacobi identity to zero.

$$J := n(abcde) + n(abdec) + n(abecd) \tag{205}$$

The aim is to show $J = 0$. As with the four-gluon case, the terms arising from cubic interactions are considered first. Furthermore, any common factors such as $i\hbar\sqrt{\hbar}$ are once again dropped out of J for simplicity. Consider

$$S_{=}Y(a,b,\mu)\left\{\sum_{C(cde)} Y(\mu,c,\nu)Y(\nu,d,e)\right\} \tag{206}$$

where C denotes cyclic permutation. Firstly, the sum term has to be calculated, hence will for the moment drop out the common term of $Y(a,b,\mu)$. If $Y(\mu,c,\nu)Y(\nu,d,e)$ is first expanded, the following expression is obtained

$$\begin{aligned}
Y(\mu,c,\nu) =& Y(q,\mu;p_c,\varepsilon_c;-q-p_c,\nu) \\
&= \underset{\text{I}}{(q+2p_c)_\mu \varepsilon_{c\nu}} - \underset{\text{II}}{(2q\cdot\varepsilon_c)g_{\mu\nu}} + \underset{\text{III}}{\varepsilon_{c\mu}(q-p_c)_\nu}
\end{aligned} \tag{207}$$

Where $q = -p_c - p_d - p_e$. Note that as stated previously, momenta are always taken to be outgoing. In equation (207) each term has been labeled as I, II or II, and these terms are treated individually when contracted with $Y(\nu, d, e)$

$$\begin{aligned}(q+2p_c)_\mu \varepsilon_{c\nu} Y(\nu,d,e) = & \; (q+2p_c)_\mu (p_d - p_e \cdot \varepsilon_c)(\varepsilon_d \cdot \varepsilon_e) \\ & + (q+2p_c)_\mu (p_e - p_c \cdot \varepsilon_d)(\varepsilon_e \cdot \varepsilon_c) \\ & + (q+2p_c)_\mu (p_c - p_d \cdot \varepsilon_e)(\varepsilon_c \cdot \varepsilon_d) \\ & - (q+2p_c)_\mu (q \cdot \varepsilon_d)(\varepsilon_e \cdot \varepsilon_c) \\ & + (q+2p_c)_\mu (q \cdot \varepsilon_e)(\varepsilon_c \cdot \varepsilon_d)\end{aligned} \tag{208}$$

The benefit of writing this expression in this way becomes clear when calculating the sum over c, d and e of term I in (207) contracted with $Y(\nu, d, e)$. Performing this calculation produces

$$\begin{aligned}\sum_{C(cde)} (q+2p_c)_\mu Y(\nu,d,e) = & (3q+2p_c+2p_d+2p_e)_\mu Y_3(c,d,e) \\ & +2(p_c - p_d)_\mu (q \cdot \varepsilon_e)(\varepsilon_c \cdot \varepsilon_d) \\ & +2(p_d - p_e)_\mu (q \cdot \varepsilon_c)(\varepsilon_d \cdot \varepsilon_e) \\ & +2(p_e - p_c)_\mu (q \cdot \varepsilon_d)(\varepsilon_e \cdot \varepsilon_c)\end{aligned} \tag{209}$$

Where the useful quantity Y_3 is defined as follows

$$Y_3(c,d,e) = (p_d - p_e \cdot \varepsilon_c)(\varepsilon_d \cdot \varepsilon_e) + (p_e - p_c \cdot \varepsilon_d)(\varepsilon_e \cdot \varepsilon_c) + (p_c - p_d \cdot \varepsilon_e)(\varepsilon_c \cdot \varepsilon_d) \tag{210}$$

Notice that this is exactly the interaction which is obtained in a separate three-gluon diagram. For term II, some simiplification is first required

$$\begin{aligned}-(2q \cdot \varepsilon_c) g^{\mu\nu} Y(\nu,d,e) = & -2(q \cdot \varepsilon_c)(p_d - p_e)_\mu (\varepsilon_d \cdot \varepsilon_e) \\ & -2(q \cdot \varepsilon_c)(p_e - p_c \cdot \varepsilon_d)\varepsilon_{e\mu} \\ & -2(q \cdot \varepsilon_c)(p_c - p_d \cdot \varepsilon_e)\varepsilon_{d\mu} \\ & +2(q \cdot \varepsilon_c)(q \cdot \varepsilon_d)\varepsilon_{e\mu} \\ & -2(q \cdot \varepsilon_e)(q \cdot \varepsilon_c)\varepsilon_{d\mu}\end{aligned} \tag{211}$$

Once again summing over permutations of c, d and e, the contribution from II equals

$$
\begin{aligned}
-\sum_{C(cde)}(2q\cdot\varepsilon_c)g^{\mu\nu}Y(\nu,d,e) = &-2(p_c-p_d)_\mu(q\cdot\varepsilon_e)(\varepsilon_c\cdot\varepsilon_d)\\
&-2(p_d-p_e)_\mu(q\cdot\varepsilon_c)(\varepsilon_d\cdot\varepsilon_e)\\
&-2(p_e-p_c)_\mu(q\cdot\varepsilon_d)(\varepsilon_e\cdot\varepsilon_c)\\
&-2(q\cdot\varepsilon_c)(p_e-p_c\cdot\varepsilon_d)\varepsilon_{e\mu}-2(q\cdot\varepsilon_c)(p_c-p_d\cdot\varepsilon_e)\varepsilon_{d\mu}\\
&-2(q\cdot\varepsilon_d)(p_c-p_d\cdot\varepsilon_e)\varepsilon_{c\mu}-2(q\cdot\varepsilon_d)(p_d-p_e\cdot\varepsilon_c)\varepsilon_{e\mu}\\
&-2(q\cdot\varepsilon_e)(p_d-p_e\cdot\varepsilon_c)\varepsilon_{d\mu}-2(q\cdot\varepsilon_e)(p_e-p_c\cdot\varepsilon_d)\varepsilon_{c\mu}
\end{aligned}
\tag{212}
$$

Finally, consider contributions from term III in (207). As with term II, the term is first simplified

$$
\begin{aligned}
\varepsilon_{c\mu}(q-p_c)_\nu Y(\nu,d,e) = &\,2\varepsilon_{c\mu}q_\nu Y(\nu,d,e)\\
= &\,2\varepsilon_{c\mu}(p_d-p_e)\cdot q(\varepsilon_d\cdot\varepsilon_e)\\
&+2\varepsilon_{c\mu}(p_e-p_c\cdot\varepsilon_d)(\varepsilon_e\cdot q)\\
&+2\varepsilon_{c\mu}(p_c-p_d\cdot\varepsilon_e)(\varepsilon_d\cdot q)\\
&-2\varepsilon_{c\mu}(\varepsilon_d\cdot q)(\varepsilon_e\cdot q)+2\varepsilon_{c\mu}(\varepsilon_e\cdot q)(\varepsilon_d\cdot q)
\end{aligned}
\tag{213}
$$

Here the handlebar $(p_d+p_e)_\nu Y(\nu,d,e)=0$ has been used. Summing then yields

$$
\begin{aligned}
\sum_{C(cde)}\varepsilon_{c\mu}(q-p_c)_\nu Y(\nu,d,e) = &\\
&2(q\cdot\varepsilon_c)(p_e-p_c\cdot\varepsilon_d)\varepsilon_{e\mu}+2(q\cdot\varepsilon_c)(p_c-p_d\cdot\varepsilon_e)\varepsilon_{d\mu}\\
&+2(q\cdot\varepsilon_d)(p_d-p_d\cdot\varepsilon_e)\varepsilon_{c\mu}+2(q\cdot\varepsilon_d)(p_d-p_e\cdot\varepsilon_c)\varepsilon_{e\mu}\\
&+2(q\cdot\varepsilon_e)(p_d-p_e\cdot\varepsilon_c)\varepsilon_{d\mu}+2(q\cdot\varepsilon_e)(p_e-p_c\cdot\varepsilon_d)\varepsilon_{c\mu}\\
&+2p_{c\mu}(p_d-p_e)\cdot q(\varepsilon_d\cdot\varepsilon_e)\\
&+2p_{d\mu}(p_e-p_c)\cdot q(\varepsilon_e\cdot\varepsilon_c)\\
&+2p_{e\mu}(p_c-p_d)\cdot q(\varepsilon_c\cdot\varepsilon_d)
\end{aligned}
\tag{214}
$$

Now S_μ can be calculated easily by summing the contributions from I, II and III. In fact, almost all terms drop out, leaving just

$$
\begin{aligned}
S_\mu = Y(a,b,\mu)\left\{\sum_{C(cde)}Y(\mu,c,\nu)Y(\nu,d,e)\right\} = &\,q_\mu Y_3(c,d,e)\\
&+2\varepsilon_{c\mu}(p_d-p_e)\cdot q(\varepsilon_d\cdot\varepsilon_e)\\
&+2\varepsilon_{d\mu}(p_e-p_c)\cdot q(\varepsilon_e\cdot\varepsilon_c)\\
&+2\varepsilon_{e\mu}(p_c-p_d)\cdot q(\varepsilon_c\cdot\varepsilon_d)
\end{aligned}
\tag{215}
$$

Now let us calculate $Y(a,b,\mu)S_\mu$. As $q_\mu = p_a + p_b$, $Y(a,b,\mu)q_\mu = 0$. Furthermore, if other contributions are rewritten, the following expression is obtained

$$\begin{aligned} Y(a,b,\mu)S_\mu = Y(a,b,\mu)\Big\{ & s_{de}(\varepsilon_c \cdot \varepsilon_d \varepsilon_{e\mu} - \varepsilon_c \cdot \varepsilon_e \varepsilon_{d\mu}) \\ & + s_{ec}(\varepsilon_d \cdot \varepsilon_e \varepsilon_{c\mu} - \varepsilon_d \cdot \varepsilon_c \varepsilon_{e\mu}) \\ & + s_{cd}(\varepsilon_e \cdot \varepsilon_c \varepsilon_{d\mu} - \varepsilon_e \cdot \varepsilon_d \varepsilon_{c\mu}) \Big\} \end{aligned} \tag{216}$$

This is a key result required to calculate the numerator identity considered previously in this section. As all contributing terms are now considered, quartic contributions are incorporated. Keeping in mind that any common factors such as $-i\hbar\sqrt{\hbar}$ have been divided out previously

$$\begin{aligned} J := n(abcde) + n(abdec) + n(abecd) = & \\ & Y(ab\mu)\left\{ \sum_{C(c,d,e)} Y(\mu c \nu) Y(\nu d e) \right\} \\ & - Y(ab\mu)\left\{ \sum_{C(c,d,e)} s_{de}(\varepsilon_c \cdot \varepsilon_d \varepsilon_{e\mu} - \varepsilon_c \cdot \varepsilon_e \varepsilon_{d\mu}) \right\} \\ & - s_{ab} \sum_{C(c,d,e)} Y(de\mu)(\varepsilon_c \cdot \varepsilon_b \varepsilon_{a\mu} - \varepsilon_c \cdot \varepsilon_a \varepsilon_{b\mu}) \end{aligned} \tag{217}$$

It should be clear that the first two terms cancel out against each other. The third term may be written in a slightly more compact way

$$\begin{aligned} J &= -s_{ab}(\varepsilon_{a\mu} \cdot \varepsilon_{b\nu} - \varepsilon_{a\nu} \cdot \varepsilon_{b\mu}) \sum_{C(c,d,e)} \varepsilon_{c\nu} Y(de\mu) \\ &= -s_{ab}[a,b]_{\mu\nu} \sum_{C(c,d,e)} \varepsilon_{c\nu} Y(de\mu) \end{aligned} \tag{218}$$

Where the commutator $[a,b]_{\mu\nu}$ is to be understood as $\varepsilon_{a\mu} \cdot \varepsilon_{b\nu} - \varepsilon_{a\nu} \cdot \varepsilon_{b\mu}$. As the reader might now notice, this term does not quite equal zero. It turns out that the numerator Jacobi identity is not automatically satisfied in amplitudes beyond four gluons. Our previously derived degrees of freedom Δ are now required in order to 'shift' the numerators such that they do obey the numerator Jacobi identity. As stated previously, each degree of freedom Δ must have the same symmetries as the color Jacobi identity underlying it to ensure a diagrammatic description. Degrees of freedom are produced by the following color Jacobi identity

$$[abcde] + [bcade] + [cabde] = 0 \tag{219}$$

Hence on the degree of freedom corresponding to this, the following symmetries are required

$$\Delta_{abcde} = -\Delta_{bacde} = \Delta_{abecd} = -\Delta_{baecd} = \Delta_{abdec} = -\Delta_{badec} \tag{220}$$

To represent these symmetries intuitively, from now on these degrees of freedom are written as $\Delta_{[ab](cde)}$. As stated before, each color factor is present in two Jacobi identities. This means each numerator picks up two degrees of freedom Δ. For $n(abcde)$ these are $s_{de}\Delta_{[ab](cde)}$ and $s_{ab}\Delta_{[de](abc)}$. Specifically

$$
\begin{aligned}
n(abcde)' &= n(abcde) + s_{de}\Delta_{[ab](cde)} - s_{ab}\Delta_{[de](abc)} \\
n(abecd)' &= n(abecd) + s_{cd}\Delta_{[ab](cde)} - s_{ab}\Delta_{[cd](abe)} \\
n(abdec)' &= n(abdec) + s_{ec}\Delta_{[ab](cde)} - s_{ab}\Delta_{[ec](abd)}
\end{aligned}
\tag{221}
$$

If J' is then defined in terms of the shifted numerators n' and set it to zero the degrees of freedom must obey the following equation

$$
\begin{aligned}
s_{ab}[a,b]_{\mu\nu} \sum_{C(cde)} \varepsilon_{c\nu} Y(de\mu) = \; & s_{de}\Delta_{[ab](cde)} - s_{ab}\Delta_{[de](abc)} \\
& + s_{cd}\Delta_{[ab](cde)} - s_{ab}\Delta_{[cd](abe)} \\
& + s_{ec}\Delta_{[ab](cde)} - s_{ab}\Delta_{[ec](abd)} \\
& = s_{ab}\Delta_{[ab](cde)} - s_{ab}\Delta_{[de](abc)} \\
& - s_{ab}\Delta_{[cd](abe)} - s_{ab}\Delta_{[ec](abd)}
\end{aligned}
\tag{222}
$$

An educated guess for Δ is now made to make this calculation easier. Notice that Δ must contain each polarization ε exactly once. To satisfy the required symmetries make a simple guess on the structure of Δ

$$
\begin{aligned}
\Delta_{[ab](cde)} &= [a,b]_{\mu\nu} \sum_{C(c,d,e)} \big\{ \alpha(\varepsilon_c \cdot p_d)\varepsilon_{d\mu} + \beta(\varepsilon_d \cdot p_c)\varepsilon_{c\mu} + \\
& \qquad\qquad \gamma(p_c - p_d)_\mu(\varepsilon_c \cdot \varepsilon_d) \big\} \varepsilon_{e\nu} \\
&= [a,b]_{\mu\nu} \sum_{C(c,d,e)} Z(\mu,c,d)\varepsilon_{e\nu}
\end{aligned}
\tag{223}
$$

For simplicity the term within the sum is referred to as $Z(c,d,\mu)\varepsilon_{e\nu}$. This gives us the following equation to solve for α, β and γ

$$
\begin{aligned}
& [a,b]_{\mu\nu} \sum_{C(c,d,e)} \big\{ Y(c,d,\mu) - Y'(c,d,\mu) \big\} \varepsilon_{e\nu} \\
& \quad + \sum_{C(c,d,e)} [d,e]_{\mu\nu} \sum_{C(a,b,c)} Y'(a,b,\mu)\varepsilon_{c\nu} = 0
\end{aligned}
\tag{224}
$$

Let us first examine terms containing $(\varepsilon_b \cdot \varepsilon_c)(\varepsilon_d \cdot \varepsilon_e)$ in equation (224). This term appears twice in the first term of (224) and four times in the second term of (224). It gives us the following coefficient

$$
\begin{aligned}
& (\varepsilon_a \cdot p_d - p_e) - \gamma(\varepsilon_a \cdot p_d - p_e) - \alpha(\varepsilon_a \cdot p_b) \\
& + \beta(\varepsilon_a \cdot p_e) + \alpha(\varepsilon_a \cdot p_b) - \beta(\varepsilon_a \cdot p_d)
\end{aligned}
\tag{225}
$$

As this has to equal zero, $1-\gamma=-\beta$. Let us now look at terms containing $(\varepsilon_b \cdot \varepsilon_c)(\varepsilon_a \cdot \varepsilon_e)$. This yields

$$\begin{aligned}&(\varepsilon_d \cdot p_a)(\gamma+\alpha)+(\varepsilon_d \cdot p_b)(\gamma-\beta)\\+&(\varepsilon_d \cdot p_c)(2+\beta-\gamma)+(\varepsilon_d \cdot p_e)(2-\alpha-\gamma)=0\end{aligned} \tag{226}$$

Using $(p_a+p_b+p_c+p_e \cdot \varepsilon_d)=-(p_d \cdot \varepsilon_d)=0$, another requirement can be found, namely $\alpha=-\beta$. The final component to be checked is anything with a factor of $(\varepsilon_a \cdot \varepsilon_b)$. This does not appear at all in the first term of (224), hence leaving us with

$$\begin{aligned}&(p_a-p_b)_\mu \sum_{C(c,d,e)} \varepsilon_{c\mu}[d,e]_{\mu\nu}=\\&(p_a-p_b)_\mu \sum_{C(c,d,e)} (\varepsilon_c \cdot \varepsilon_d)\varepsilon_{e\mu}-(\varepsilon_e \cdot \varepsilon_c)\varepsilon_{d\mu}=0\end{aligned} \tag{227}$$

Which means that equation (224) is satisfied. Now note that through cyclic symmetry in c, d and e, and antisymmetry in a and b, all terms will vanish. This means leftover degrees of freedom α remain in our problem which may set arbitrarily. The final result obtained, with additional degree of freedom α has been checked in MAPLE with an independent choice of gauge vectors. The final result may be written down as

$$\begin{aligned}\Delta_{[ab](c,d,e)}=[a,b]_{\mu\nu}\sum_{C(cde)}\Big\{&\alpha(\varepsilon_c \cdot p_d)\varepsilon_{d\mu}-\alpha(\varepsilon_d \cdot p_c)\varepsilon_{c\mu}\\&+(1-\alpha)(p_c-p_d)_\mu(\varepsilon_c \cdot \varepsilon_d)\Big\}\varepsilon_{e\nu}\end{aligned} \tag{228}$$

To find the BCJ numerators which satisfy the numerator Jacobi identity, all one has to do is to enter equation (228) into equation (221). This shows that indeed, the numerator Jacobi identity holds for the five-point case. There, however, is no clear pattern in the required Δ in the four- and five- point cases. To continue our search for a pattern, in the next section an attempt to solve the numerator Jacobi identity for six gluons is outlined.

5.4 Numerator identity for six gluons

For six gluons the problem becomes a lot more complicated: Not only are there more diagrams with four-gluon contact terms but also a new class of diagrams has appeared, which will be called 'mercedes' diagrams for reasons which will become clear soon. Let us consider the following color factor $[abcdef]$. As $n-3=3$, there are three color Jacobi identitites to consider.

$$\begin{aligned}[abcdef]+[bcadef]+[cabdef]&=0\\ [abcdef]+[abcefd]+[abcfde]&=0\\ [abcdef]-[abdcef]-[abnef][ncd]&=0\end{aligned} \tag{229}$$

Notice that in the last of these Jacobi identities, a color factor has appeared which cannot be written in a very nice way using color brackets. Let us calculate the numerators corresponding to both of these types of color factors. For $[abcdef]$ the contributing diagrams are shown in figure 10

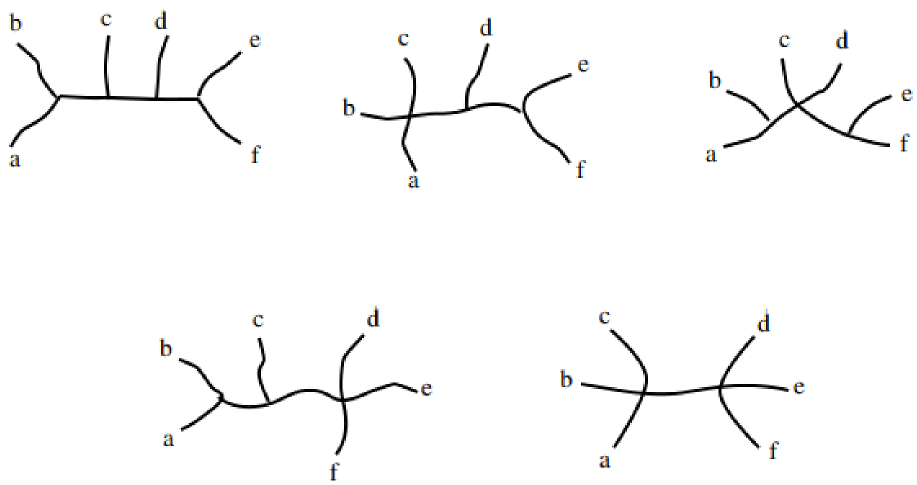


Figure 10: *The five diagrams contributing to the six-gluon numerator n(abcdef) corresponding to* $[abcdef]$

Working through all of these diagrams yields a contribution of

$$\begin{aligned}n(abcde)=&-i\hbar^2 Y(a,b,\mu)Y(\mu,c,\nu)Y(\nu,d,\lambda)Y(\lambda,e,f)\\ &+i\hbar^2 s_{ab}(\varepsilon_b\cdot\varepsilon_c\varepsilon_{a\mu}-\varepsilon_a\cdot\varepsilon_c\varepsilon_{b\mu})Y(\mu,d,\lambda)Y(\lambda,e,f)\\ &+i\hbar^2 s_{abc}Y(a,b,\mu)(\varepsilon_c\cdot\varepsilon_d g^{\mu\lambda}-\varepsilon_{d\mu}\varepsilon_{c\lambda})Y(\lambda,e,f)\\ &+i\hbar^2 s_{ef}Y(a,b,\lambda)Y(\lambda,c,\mu)(\varepsilon_d\cdot\varepsilon_e\varepsilon_{f\mu}-\varepsilon_d\cdot\varepsilon_f\varepsilon_{e\mu})\\ &-i\hbar^2 s_{ab}s_{ef}(\varepsilon_b\cdot\varepsilon_c\varepsilon_{a\mu}-\varepsilon_a\cdot\varepsilon_c\varepsilon_{b\mu})(\varepsilon_d\cdot\varepsilon_e\varepsilon_{f\mu}-\varepsilon_d\cdot\varepsilon_f\varepsilon_{e\mu})\end{aligned} \tag{230}$$

Here s_{abc} is defined as $(p_a+p_b+p_c)^2$. This expression is quite lengthy, but can be simplified by identifying common factors in the first four terms of the numerator. Doing so yields the slightly more compact expression

$$n(abcdef) = -i\hbar^2 \Big\{ Y(a,b,\mu)Y(\mu,c,\nu) - s_{ab}(\varepsilon_b \cdot \varepsilon_c \varepsilon_{a\mu} - \varepsilon_a \cdot \varepsilon_c \varepsilon_{b\mu}) \Big\} \times \Big\{ Y(\nu,d,\lambda)Y(\lambda,e,f) - s_{ef}(\varepsilon_d \cdot \varepsilon_e \varepsilon_{f\mu} - \varepsilon_d \cdot \varepsilon_f \varepsilon_{e\mu}) \Big\} + i\hbar^2 s_{abc} Y(a,b,\mu)(\varepsilon_c \cdot \varepsilon_d g^{\mu\lambda} - \varepsilon_{d\mu}\varepsilon_{c\lambda})Y(\lambda,e,f) \tag{231}$$

As there is also a second type of color factor, a whole additional class of numerators has to be considered. Let us denote these numerators with 'm', where $m(abcdef)$ corresponds to $[abnef][ncd]$, or $f^{nab}f^{mcd}f^{kef}f^{nmk}$. Notice that this numerator will also have completely different symmetries than $n(abcdef)$. For the numerator m, only four diagrams contribute, shown in 11

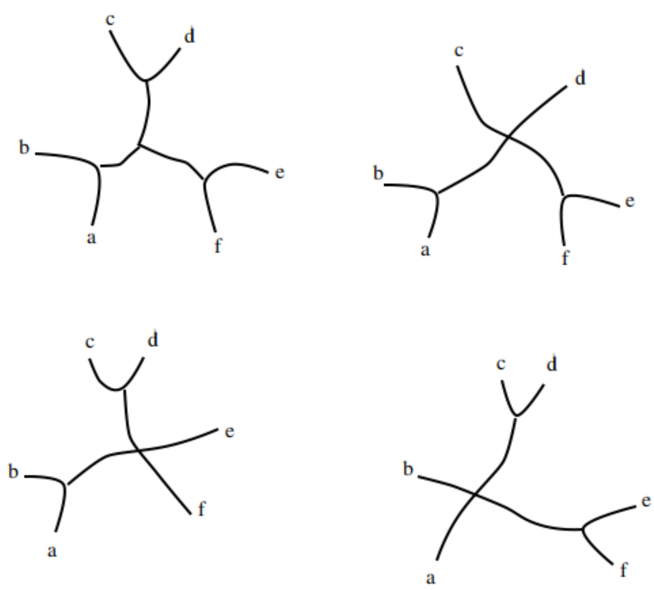


Figure 11: *The four diagrams which contain the color factor $[abnef][ncd]$, and contribute to $m(abcdef)$*

The contribution from each of these diagrams is now explicitly evaluated. Doing so yields

$$\begin{aligned} m(abcdef) = &- i\hbar^2 Y(a,b,\mu)Y(c,d,\nu)Y(e,f,\lambda)Y(\mu,\nu,\lambda) \\ &+ i\hbar^2 s_{cd} Y(a,b,\mu)(\varepsilon_{d\mu}\varepsilon_{c\nu} - \varepsilon_{c\mu}\varepsilon_{d\nu})Y(\nu,e,f) \\ &+ i\hbar^2 s_{ef} Y(c,d,\mu)(\varepsilon_{f\mu}\varepsilon_{e\nu} - \varepsilon_{e\mu}\varepsilon_{f\nu})Y(\nu,a,b) \\ &+ i\hbar^2 s_{ab} Y(e,f,\mu)(\varepsilon_{b\mu}\varepsilon_{a\nu} - \varepsilon_{a\mu}\varepsilon_{b\nu})Y(\nu,c,d) \end{aligned} \tag{232}$$

Similarly to the five-point case, a partial amplitude is constructed out of the numerators in (231) and (232) to check whether or not our expressions are valid. For the six-point case, when color factors are expanded into color traces, the following expression is obtained

$$\begin{aligned}\frac{1}{2}A_6^{part}(a,b,c,d,e,f) = &\frac{n(abcdef)}{s_{ab}s_{abc}s_{ef}} + \frac{n(abcfed)}{s_{cb}s_{abc}s_{ef}} + \frac{n(cbadef)}{s_{ab}s_{abc}s_{ed}} + \frac{n(cbafed)}{s_{cb}s_{abc}s_{ed}} \\ &+ \frac{n(bcdefa)}{s_{bc}s_{bcd}s_{fa}} + \frac{n(bcdafe)}{s_{bc}s_{bcd}s_{fe}} + \frac{n(dcbefa)}{s_{dc}s_{bcd}s_{fa}} + \frac{n(dcbafe)}{s_{dc}s_{bcd}s_{fe}} \\ &+ \frac{n(cdefab)}{s_{cd}s_{cde}s_{ab}} + \frac{n(cdebaf)}{s_{cd}s_{cde}s_{af}} + \frac{n(edcfab)}{s_{ed}s_{cde}s_{ab}} + \frac{n(edcbaf)}{s_{ed}s_{cde}s_{af}} \\ &+ \frac{m(abcdef)}{s_{ab}s_{cd}s_{ef}} + \frac{m(bcdefa)}{s_{bc}s_{de}s_{fa}}\end{aligned} \tag{233}$$

This result can be verified by projecting color factors on $Tr(abcdef)$. This partial amplitude has been checked for gauge invariance in MAPLE. When choosing the polarization vectors such that an MHV aplitude is produced, the result agrees with the Parke-Taylor formula, ensuring that our expressions for the numerators n and m are valid. Consider the color Jacobi identities in (229). The first two equations can be written into each other using known symmetries of the color brackets. As stated previously, there are two numerator Jacobi identities to consider

$$n(abcdef) + n(abcfde) + n(abcfde) = 0 \tag{234}$$

$$n(abcdef) - n(abdcef) - m(abcdef) = 0 \tag{235}$$

Once again, similar to the five point case, these do not equal zero. As a small example, lets calculate the first numerator Jacobi identity.

$$\begin{aligned}\sum_{C(d,e,f)} n(abcdef) = &\\ &-i\hbar^2\Big\{Y(a,b,\mu)Y(\mu,c,\nu) - s_{ab}(\varepsilon_b\cdot\varepsilon_c\varepsilon_{a\mu} - \varepsilon_a\cdot\varepsilon_c\varepsilon_{b\mu})\Big\}\times \\ &\qquad\sum_{C(d,e,f)}\Big\{Y(\nu,d,\lambda)Y(\lambda,e,f) - s_{ef}(\varepsilon_d\cdot\varepsilon_e\varepsilon_{f\mu} - \varepsilon_d\cdot\varepsilon_f\varepsilon_{e\mu})\Big\} \\ &+i\hbar^2 s_{abc}\sum_{C(d,e,f)} Y(a,b,\mu)(\varepsilon_c\cdot\varepsilon_d g^{\mu\lambda} - \varepsilon_{d\mu}\varepsilon_{c\lambda})Y(\lambda,e,f)\end{aligned} \tag{236}$$

In the five-point discussion, the sum in the first equation in (236) has already been derived, and this result is substituted into our equation. Doing this, however, will leave us with a factor of $q = p_a + p_b + p_c$ to be contracted with the first term in equation (236). Hence, let us calculate this first. Particularly useful for this is the handlebar, as it allows us to write

$$Y(p_a+p_b,\mu;p_c,\varepsilon_c;-q,\nu)q_\nu = -K(p_a+p_b)^{\mu\rho}\varepsilon_{c\rho} + K(p_c)^{\mu\rho}\varepsilon_{c\rho} \tag{237}$$

Where K is defined as in equation (95). Naturally, the last term vanishes, and again through the handlebar $Y(a,b,\mu)(p_a+p_b)_\mu = 0$, evaluating the full first term of (236) yields

$$\begin{aligned} Y(a,b,\mu)Y(\mu,c,\nu)q_\mu - s_{ab}(\varepsilon_b\cdot\varepsilon_c\varepsilon_{a\mu} - \varepsilon_a\cdot\varepsilon_c\varepsilon_{b\mu})q_\mu = \\ s_{ab}\{(p_b-p_c\cdot\varepsilon_a)(\varepsilon_b\cdot\varepsilon_c) \\ +(p_c-p_a\cdot\varepsilon_b)(\varepsilon_c\cdot\varepsilon_a) \\ +(p_a-p_b\cdot\varepsilon_c)(\varepsilon_a\cdot\varepsilon_b)\} \\ = s_{ab}Y_3(a,b,c) \end{aligned} \tag{238}$$

This numerator identity can now be evaluated, and it certainly does not vanish if considering, for instance, terms of $(\varepsilon_a\cdot\varepsilon_b)(\varepsilon_c\cdot\varepsilon_d)(\varepsilon_e\cdot\varepsilon_f)$. As can be seen in the expression below, this term will carry a factor of $i\hbar^2(p_a-p_b)\cdot(p_e-p_f)$

$$\begin{aligned} \sum_{C(d,e,f)} n(abcdef) = \\ -i\hbar^2 s_{ab}Y_3(a,b,c)Y_3(d,e,f) \\ +i\hbar^2 s_{abc}\sum_{C(d,e,f)} Y(a,b,\mu)(\varepsilon_c\cdot\varepsilon_d g^{\mu\lambda} - \varepsilon_{d\mu}\varepsilon_{c\lambda})Y(\lambda,e,f) \end{aligned} \tag{239}$$

The same is done for the second numerator Jacobi identity (235). The explicit calculation to evaluate (235) will not be given, as it is quite long and involved from scratch. The reader can reproduce this result using the general formula given in the next section if they wish to verify the formula stated below

$$\begin{aligned} n(abcdef) - n(abdcef) - m(abcdef) = \\ -i\hbar^2 s_{ab}\Big\{(\varepsilon_b\cdot\varepsilon_d\varepsilon_{a\nu} - \varepsilon_a\cdot\varepsilon_d\varepsilon_{b\nu})Y(\nu,c,\lambda) \\ -(\varepsilon_b\cdot\varepsilon_c\varepsilon_{a\nu} - \varepsilon_a\cdot\varepsilon_c\varepsilon_{b\nu})Y(\nu,d,\lambda) \\ +(\varepsilon_{b\lambda}\varepsilon_{a\nu} - \varepsilon_{a\lambda}\varepsilon_{b\nu})Y(\nu,c,d)\Big\}Y(\lambda,e,f) \\ -i\hbar^2 s_{ef}\Big\{(\varepsilon_e\cdot\varepsilon_d\varepsilon_{f\nu} - \varepsilon_f\cdot\varepsilon_d\varepsilon_{e\nu})Y(\nu,c,\lambda) \\ -(\varepsilon_e\cdot\varepsilon_c\varepsilon_{f\nu} - \varepsilon_f\cdot\varepsilon_c\varepsilon_{e\nu})Y(\nu,d,\lambda) \\ +(\varepsilon_{e\lambda}\varepsilon_{f\nu} - \varepsilon_{f\lambda}\varepsilon_{e\nu})Y(\nu,c,d)\Big\}Y(\lambda,a,b) \\ -i\hbar^2 s_{ab}s_{ef}[a,b]_{\mu,\nu}[c,d]_{\nu,\lambda}[e,f]_{\lambda,\mu} \end{aligned} \tag{240}$$

Here the commutator $[a,b]_{\mu,\nu}$ is once again defined as $\varepsilon_{a\mu}\varepsilon_{b\nu} - \varepsilon_{b\mu}\varepsilon_{a\nu}$. Both of these results for the numerator identities have been verified in MAPLE and found to be correct. Both are also generally nonzero. This means that, once again, degrees of freedom Δ are to be found such that the numerator Jacobi identities are satisfied. As there are two distinct color Jacobi identities, there are two distinct degrees of freedom, unlike in section 5.3. Nevertheless, using the same conventions as at five points, the following degrees of freedom are obtained

$$\begin{aligned} &\Delta_{[ab]c(def)}\Big([abcdef]+[abcfde]+[abcefd]\Big)\\ &\Delta_{[ab][cd][ef]}\Big([abcdef]+[abdcef]+[abnef][ncd]\Big) \end{aligned} \tag{241}$$

Note that the second factor also has the additional antisymmetry in $[ab] \leftrightarrow [ef]$. This antisymmetry of Δ is not denoted separately, as the symmetry indicators on Δ already distinguish them. Let us examine how these degrees of freedom appear in our numerators. Starting at

$$n'(abcdef) = n(abcdef) + s_{ef}\Delta_{[ab]c(def)} + s_{ab}\Delta_{[de]f(abc)} + s_{abc}\Delta_{[ab][cd][ef]} \tag{242}$$

$$m'(abcdef) = m(abcdef) - s_{cd}\Delta_{[ab][cd][ef]} - s_{ab}\Delta_{[ef][ab][cd]} - s_{ef}\Delta_{[cd][ef][ab]} \tag{243}$$

Pay close attention to the sign of the degrees of freedom in m, as follows from the color Jacobi identity. From these, using the numerator Jacobi identities, two equations are found which the degrees of freedom Δ have to satisfy. Note that the common terms have been simplified using conservation of momentum using conservation of momentum.

$$\begin{aligned} &i\hbar^2 s_{ab} Y_3(a,b,c)Y_3(d,e,f)\\ &-i\hbar^2 s_{abc}\sum_{C(d,e,f)} Y(a,b,\mu)(\varepsilon_c\cdot\varepsilon_d g^{\mu\lambda} - \varepsilon_{d\mu}\varepsilon_{c\lambda})Y(\lambda,e,f)\\ &= s_{ab}\Big\{\Delta_{[de]f(abc)} + \Delta_{[ef]d(abc)} + \Delta_{[fd]e(abc)}\Big\}\\ &+s_{abc}\Big\{\Delta_{[ab][cd][ef]} + \Delta_{[ab][cf][de]} + \Delta_{[ab][ce][fd]} + \Delta_{[ab]c(def)}\Big\} \end{aligned} \tag{244}$$

Here we have already combined the coefficients of $\Delta_{[ab]c(def)}$, $s_{ab}\, s_{bc}$ and c_{ca} into s_{abc}. The second equation the degrees of freedom Δ are required to satisfy is

$$
\begin{aligned}
& i\hbar^2 s_{ab}\Big\{(\varepsilon_b\cdot\varepsilon_d\varepsilon_{a\nu}-\varepsilon_a\cdot\varepsilon_d\varepsilon_{b\nu})Y(\nu,c,\lambda) \\
& \quad -(\varepsilon_b\cdot\varepsilon_c\varepsilon_{a\nu}-\varepsilon_a\cdot\varepsilon_c\varepsilon_{b\nu})Y(\nu,d,\lambda) \\
& \quad +(\varepsilon_{b\lambda}\varepsilon_{a\nu}-\varepsilon_{a\lambda}\varepsilon_{b\nu})Y(\nu,c,d)\Big\}Y(\lambda,e,f) \\
& +i\hbar^2 s_{ef}\Big\{(\varepsilon_e\cdot\varepsilon_d\varepsilon_{f\nu}-\varepsilon_f\cdot\varepsilon_d\varepsilon_{e\nu})Y(\nu,c,\lambda) \\
& \quad -(\varepsilon_e\cdot\varepsilon_c\varepsilon_{f\nu}-\varepsilon_f\cdot\varepsilon_c\varepsilon_{e\nu})Y(\nu,d,\lambda) \\
& \quad +(\varepsilon_{e\lambda}\varepsilon_{f\nu}-\varepsilon_{f\lambda}\varepsilon_{e\nu})Y(\nu,c,d)\Big\}Y(\lambda,a,b) \\
& +i\hbar^2 s_{ab}s_{ef}[a,b]_{\mu,\nu}[c,d]_{\nu,\lambda}[e,f]_{\lambda,\mu} \\
& = s_{ab}\Big\{\Delta_{[ef]d(abc)}-\Delta_{[ef]c(abd)}+\Delta_{[ab][cd][ef]}+\Delta_{[ef][ab][cd]}\Big\} \\
& + s_{ef}\Big\{\Delta_{[ab]c(def)}-\Delta_{[ab]d(cef)}+\Delta_{[ab][cd][ef]}+\Delta_{[cd][ef][ab]}\Big\}
\end{aligned}
\tag{245}
$$

It should be instantly clear that these two (sets of) equations are far more difficult to solve than the one equation obtained at the five-gluon level. Furthermore, it might appear as if parts which share a common factor of momentum products s can easily be identified with eachother, however this also is not the case. Notice that it could reasonably be true that, for instance in equation (244), the second-to-last line provides a term of $+s_{ab}s_{abc}\ldots$ and the last line provides $-s_{ab}s_{abc}\ldots$. At first an attempt was made to solve these equations by making educated guesses and attempting to solve for the unknown variables by plugging them into (244) and (245), however this did not provide any solution. Then, the problem was split up into two 'component classes'. Notice that each term in the left-hand sides of (244) and (245) has one of the two following forms

$$
\begin{aligned}
&(q_1\cdot q_2)(\varepsilon\cdot\varepsilon)(\varepsilon\cdot\varepsilon)(\varepsilon\cdot\varepsilon) \\
&(q_1\cdot\varepsilon)(q_2\cdot\varepsilon)(\varepsilon\cdot\varepsilon)(\varepsilon\cdot\varepsilon)
\end{aligned}
\tag{246}
$$

By considering these separately, we first tried to solve the terms of the first kind ($q^2(\varepsilon\cdot\varepsilon)^3$). By collecting all of these terms from (244) and (245), and dropping a factor of $-i\hbar^2$, the following more manageable expressions were obtained

$$
\begin{aligned}
& s_{abc}\sum_{C(d,e,f)}(p_a-p_b)\cdot(p_e-p_f)(\varepsilon_a\cdot\varepsilon_b)(\varepsilon_c\cdot\varepsilon_d)(\varepsilon_e\cdot\varepsilon_f) \\
& = s_{ab}\Big\{\Delta_{[de]f(abc)}+\Delta_{[ef]d(abc)}+\Delta_{[fd]e(abc)}\Big\} \\
& + s_{abc}\Big\{\Delta_{[ab][cd][ef]}+\Delta_{[ab][cf][de]}+\Delta_{[ab][ce][fd]}+\Delta_{[ab][cd][ef]}\Big\}
\end{aligned}
\tag{247}
$$

And

$$
\begin{aligned}
&2s_{ef}(p_a-p_b)\cdot(p_e+p_f)(\varepsilon_a\cdot\varepsilon_b)[c,d]^{\mu\nu}[e,f]^{\mu\nu}\\
&-2s_{ab}(p_a+p_b)\cdot(p_e-p_f)(\varepsilon_e\cdot\varepsilon_f)[c,d]^{\mu\nu}[a,b]^{\mu\nu}\\
&+s_{ab}s_{ef}[a,b]_{\mu,\nu}[c,d]_{\nu,\lambda}[e,f]_{\lambda,\mu}\\
&=s_{ab}\Big\{\Delta_{[ef]d(abc)}-\Delta_{[ef]c(abd)}+\Delta_{[ab][cd][ef]}+\Delta_{[ef][ab][cd]}\Big\}\\
&+s_{ef}\Big\{\Delta_{[ab]c(def)}-\Delta_{[ab]d(cef)}+\Delta_{[ab][cd][ef]}+\Delta_{[cd][ef][ab]}\Big\}
\end{aligned}
\tag{248}
$$

Then a choice was made to simplify the second equation by assuming that all parts carrying s_{ef} have to add up to zero, and similarly for s_{ab}. This requires the $s_{ab}s_{ef}$ mixing term has to be split evenly over both due to its internal antisymmetry in $[ab] \leftrightarrow [ef]$. This results in

$$
\begin{aligned}
&2(p_a-p_b)\cdot(p_e+p_f)(\varepsilon_a\cdot\varepsilon_b)[c,d]^{\mu\nu}[e,f]^{\mu\nu}\\
&+\frac{1}{2}s_{ab}s_{ef}[a,b]_{\mu,\nu}[c,d]_{\nu,\lambda}[e,f]_{\lambda,\mu}\\
&=\Big\{\Delta_{[ab]c(def)}-\Delta_{[ab]d(cef)}+\Delta_{[ab][cd][ef]}+\Delta_{[cd][ef][ab]}\Big\}
\end{aligned}
\tag{249}
$$

Once again, at first trials were made to guess solutions to see if a solution could be obtained quickly, however this once again was without any result. Then a different approach was employed: Proceeding to pick the degrees of freedom Δ as generally as possible. Consider, for instance, that a term of $q_1\cdot q_2(\varepsilon_a\cdot\varepsilon_b)(\varepsilon_c\cdot\varepsilon_e)(\varepsilon_d\cdot\varepsilon_f)$ occurring in $\Delta_{[ab][cd][ef]}$. As there exists an antisymmetry in a and b, for instance, without loss of generality, $q_1=(p_a-p_b)$. Furthermore this also implies

$$
\begin{aligned}
(p_a-p_b)\cdot q'(\varepsilon_a\cdot\varepsilon_b)(\varepsilon_c\cdot\varepsilon_f)(\varepsilon_e\cdot\varepsilon_d)&\in\Delta_{[ab][ce][fd]}\\
(p_a-p_b)\cdot q''(\varepsilon_a\cdot\varepsilon_b)(\varepsilon_c\cdot\varepsilon_d)(\varepsilon_f\cdot\varepsilon_e)&\in\Delta_{[ab][cf][de]}
\end{aligned}
\tag{250}
$$

Notice how two terms q' and q'' appear, which may or may not be the same depending on the momenta contained within them. By working through all possible factors, of which there are 15 in total, and by considering the required sums through equations (247) and (249), a unique solution to this part of the six-point numerator Jacobi identity was found. Much of the tabulating of possible shapes of the terms in the degrees of Δ was done in a spreadsheet. One noteworthy finding is that in order to find a solution, the equations in (247) in terms of s_{ab} and s_{abc} could not be separated. In fact, a cross term appears in the solution where

$$s_{abc}\Big\{\Delta_{[ab][cd][ef]}+\Delta_{[ab][cf][de]}+\Delta_{[ab][ce][fd]}+\Delta_{[ab][cd][ef]}\Big\}=$$
$$Cs_{ab}s_{abc} \tag{251}$$
$$+\,s_{abc}\sum_{C(d,e,f)}(p_a-p_b)\cdot(p_e-p_f)(\varepsilon_a\cdot\varepsilon_b)(\varepsilon_c\cdot\varepsilon_d)(\varepsilon_e\cdot\varepsilon_f)$$

And

$$s_{ab}\Big\{\Delta_{[de]f(abc)}+\Delta_{[ef]d(abc)}+\Delta_{[fd]e(abc)}\Big\}=-Cs_{ab}s_{abc} \tag{252}$$

Where C is a common factor. The solution found for this problem is now given as follows. Let us define a special momentum $k=p_a+p_b-\frac{8}{3}p_c$

$$\begin{aligned}\Delta_{[ab]c(def)}=&\frac{3}{10}(p_a+p_b)\cdot k[a,b]_{\mu\nu}\varepsilon_{c\lambda}\sum_{C(d,e,f)}\varepsilon_{d\mu}\varepsilon_{e\nu}\varepsilon_{f\lambda}\\&+\frac{4}{10}\Big\{(\varepsilon_a\cdot\varepsilon_b)\varepsilon_{c\nu}(p_a-p_b)_\lambda+[a,b]_{\mu\nu}\varepsilon_{c\mu}(p_a+p_b)_\lambda\Big\}\times\\&\sum_{C(d,e,f)}(p_e-p_f)_\lambda\varepsilon_{d\nu}(\varepsilon_e\cdot\varepsilon_f)\end{aligned} \tag{253}$$

$$\begin{aligned}\Delta_{[ab][cd][ef]}=&\frac{3}{10}\Big\{(\varepsilon_a\cdot\varepsilon_b)[c,d]_{\mu\nu}[e,f]_{\mu\nu}(p_a-p_b)\cdot(p_e+p_f)\\&-(\varepsilon_e\cdot\varepsilon_f)[c,d]_{\mu\nu}[a,b]_{\mu\nu}(p_e-p_f)\cdot(p_a+p_b)\Big\}\\&+\frac{2}{10}(p_a+p_b)(p_c-p_d)\Big\{[a,b]_{\mu\nu}\{c,d\}_{\nu\lambda}[e,f]_{\lambda\mu}\\&+\frac{1}{2}[a,b]_{\mu\nu}(\varepsilon_c\cdot\varepsilon_d)[e,f]_{\mu\nu}\Big\}\end{aligned} \tag{254}$$

In equation (254) the anticommutator is defined as $\{a,b\}=\varepsilon_a\cdot\varepsilon_b+\varepsilon_b\cdot\varepsilon_a$. Clearly the solutions in equations (253) and (254) are quite complicated, explaining why simple guesses were not able to find them. The quantities found for the factors Δ were checked in MAPLE and indeed make the numerators n' and m' satisfy the numerator Jacobi identity for terms of the form $(q_1\cdot q_2)(\varepsilon\cdot\varepsilon)(\varepsilon\cdot\varepsilon)(\varepsilon\cdot\varepsilon)$. Now the other half of the problem remains: terms of the form $(q_1\cdot\varepsilon)(q_2\cdot\varepsilon)(\varepsilon\cdot\varepsilon)(\varepsilon\cdot\varepsilon)$. One advantage is that in this case, we know with certainty that all terms which have a common factor of, say, s_{ab} must equal eachother. Once again, an attempt was made to apply the same method as used previously to the 45 different terms that appear. However, unfortunately, no suitable expressions for the degrees of freedom Δ were found. On the other hand, the existence of a solution cannot be excluded, even though none was found. This is quite dissatisfying, as it completely ruins our approach to finding a pattern in the degrees of freedom Δ which can be extrapolated. On the other hand, looking at the solution for terms of the form $(q_1 \cdot q_2)(\varepsilon \cdot \varepsilon)(\varepsilon \cdot \varepsilon)(\varepsilon \cdot \varepsilon)$, it would almost seem that no diagrammatic pattern exists, given the strange and non-obvious form of these terms. This implies that the assumptions made on symmetry requirements on our degrees of freedom Δ are too restrictive. However, if these requirements are too restrictive, no diagrammatic representation of a 'fix' of numerators exists such that they satisfy the numerator Jacobi identity. In the next section a different approach is employed and proof is given that the numerator identity is always satisfied for MHV numerators.

5.5 Numerator identity for MHV amplitudes

Let us consider MHV amplitudes. As is known, in some cases, a 'minimal gauge' may be picked. This is discussed in section 2.5, to greatly simplify calculations on Yang-Mills amplitudes. A general form for purely gluonic MHV partial amplitudes was derived by Parke and Taylor in [5]. There is, however, one problem with picking a gauge when considering numerators. In section 2.2, it was discussed that numerators are generally not gauge invariant, hence a numerator identity has to hold in any gauge to be useful. This line of thinking has one major flaw: let us consider a relation between partial amplitudes which relies on the numerator Jacobi identity. Now assume that for a single choice of gauge vectors, the numerator Jacobi identity vanishes, and hence for this single gauge choice our relation between partial amplitudes holds. Partial amplitudes are gauge invariant, and hence a relation between gauge invariant quantities must also be gauge invariant. This implies that any relation between gauge invariant quantities derived from a gauge dependent identity, will hold for a general gauge, even if the underlying identity does not. As the BCJ identity certainly is gauge invariant, the numerator identity only has to hold for a single gauge. If it is possible to show that the numerator identity holds for a single gauge, it may be used in a general way when working with gauge invariant identities resulting from it. For MHV numerators, this is particularly interesting as using the techniques discussed in section 2.5, they may be simplified them greatly. Let us first consider a general numerator Jacobi identity. In the section 5.1 the following numerators were considered

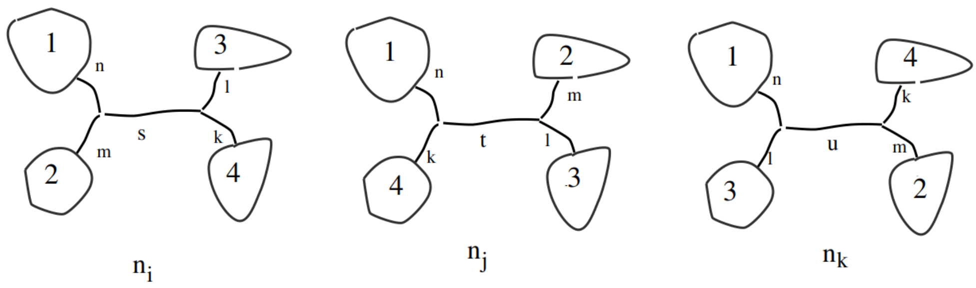


Figure 12: *The internal structure of cubic diagrams corresponding to the numerators n_i, n_j and n_k.*

Corresponding to color factors

$$
\begin{aligned}
c_i &= F_1^n F_2^m F_3^l F_4^k [nmlk] \\
c_j &= F_1^n F_2^m F_3^l F_4^k [nkml] \\
c_k &= F_1^n F_2^m F_3^l F_4^k [nlkm]
\end{aligned}
\tag{255}
$$

Similarly to previous sections, let us consider the cubic diagrams corresponding to the numerators n_i, n_j and n_k, which are listed in figure 12. The diagrammatic contribution of blob 1 is denoted by D_1^ν, and similarly for the other blobs. The cubic contributions in the numerators are as follows:

$$
\begin{aligned}
n_i &= D_1^\nu D_2^\mu D_3^\lambda D_4^\kappa Y(\nu,\mu,\delta) Y(\delta,\lambda,\kappa) \\
n_j &= D_1^\nu D_2^\mu D_3^\lambda D_4^\kappa Y(\nu,\kappa,\delta) Y(\delta,\mu,\lambda) \\
n_k &= D_1^\nu D_2^\mu D_3^\lambda D_4^\kappa Y(\nu,\lambda,\delta) Y(\delta,\kappa,\mu)
\end{aligned}
\tag{256}
$$

Let us now consider the numerator identity $n_i + n_j + n_k$. Disregarding the common factors D the following expression is found

$$
Y(\nu,\mu,\delta)Y(\delta,\lambda,\kappa) + Y(\nu,\lambda,\delta)Y(\delta,\kappa,\mu) + Y(\nu,\kappa,\delta)Y(\delta,\mu,\lambda) \tag{257}
$$

By investigating this expression, clearly it is seen this sum will contain terms such as $g^{\mu\nu}g^{\lambda\kappa}$, and terms such as $p^\mu q^\nu g^{\lambda\kappa}$ together with permutations on the indices. Furthermove, this expression is antisymmetric under exchange of any two indices. Let us now identify an outgoing momentum p_i with each component D_i. Terms of the form $g^{\mu\nu}g^{\lambda\kappa}$ give a total contribution of

$$
g^{\mu\nu}g^{\lambda\kappa}(p_1 - p_2)(p_3 - p_4) \tag{258}
$$

These terms behave similarly to quartic interactions and are discussed later on together with the other quartic interactions later. For terms of the form $p^\nu q^\mu g^{\lambda\kappa}$, let us look at

$$\begin{aligned} &(p_1+2p_2)_\nu(p_3-p_4)_\mu \\ +&(p_1+2p_3)_\nu(-p_2-2p_4)_\mu \\ +&(p_1+2p_4)_\nu(p_2+2p_3)_\mu \\ +&(p_3-p_4)_\nu(-p_2-2p_1)_\mu \end{aligned} \tag{259}$$

Now combine the first three terms

$$\begin{aligned} &p_{1\nu}(p_3-p_4)_\mu \\ +&(p_1+2p_3)_\nu p_{1\mu} \\ -&(p_1+2p_4)_\nu p_{1\mu} \\ +&(p_3-p_4)_\nu(-p_2-2p_1)_\mu \end{aligned} \tag{260}$$

Next, combining the last three terms yields

$$p_{1\nu}(p_3-p_4)_\mu - p_{2\mu}(p_3-p_4)_\nu \tag{261}$$

Using antisymmety under the exchange of any two indices allows us to calculate equation (257). In total this results in

$$\begin{aligned} &Y(\nu,\mu,\delta)Y(\delta,\lambda,\kappa)+Y(\nu,\lambda,\delta)Y(\delta,\kappa,\mu)+Y(\nu,\kappa,\delta)Y(\delta,\mu,\lambda) \\ &\quad = p_{1\nu}Y(\mu,\lambda,\kappa)-p_{2\mu}Y(\nu,\lambda,\kappa)+p_{3\lambda}Y(\kappa,\nu,\mu)-p_{4\kappa}Y(\lambda,\nu,\mu) \\ &\quad + g^{\mu\nu}g^{\lambda\kappa}(p_1-p_2)(p_3-p_4) \\ &\quad + g^{\mu\lambda}g^{\kappa\nu}(p_1-p_3)(p_4-p_2) \\ &\quad + g^{\mu\kappa}g^{\nu\lambda}(p_1-p_4)(p_2-p_3) \end{aligned} \tag{262}$$

This form is very interesting. Firstly, when contracted with D_1^ν, D_2^μ, D_3^λ and D_4^κ, the resulting expression basically consists of applying a handlebar on each of these in turn. Furthermore, $g^{\mu\nu}g^{\lambda\kappa}(p_1-p_2)(p_3-p_4)+g^{\mu\lambda}g^{\kappa\nu}(p_1-p_3)(p_4-p_2)+g^{\mu\kappa}g^{\nu\lambda}(p_1-p_4)(p_2-p_3)$ cancels out fully against the quartic contribution to the numerators shown in figure 13

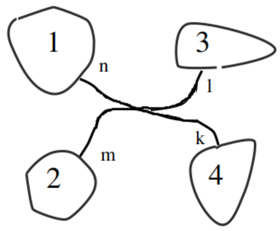


Figure 13: *The quartic contribution relevant to the considered numerator identity*

To summarize

$$n_i + n_j + n_k = D_1^\nu D_2^\mu D_3^\lambda D_4^\kappa \Big\{ p_{1\nu} Y(\mu,\lambda,\kappa) - p_{2\mu} Y(\nu,\lambda,\kappa) \\ + p_{3\lambda} Y(\kappa,\nu,\mu) - p_{4\kappa} Y(\lambda,\nu,\mu) \Big\} \tag{263}$$

For the cubic contributions in the numerators, this identity may also be expressed diagrammatically. Using the usual definition of the handlebar

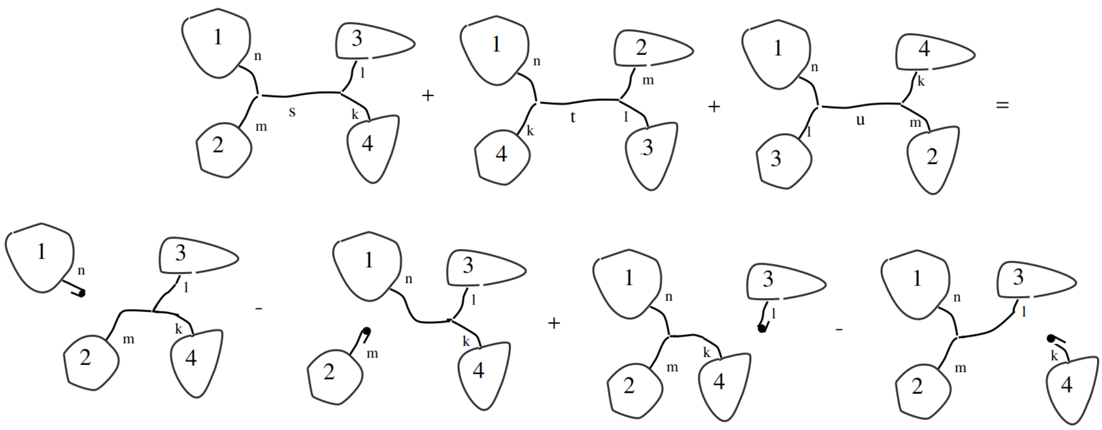


Figure 14: *Diagrammatic equation for the cubic components of the numerator Jacobi identity*

Notice that in figure 14, the signs in the last two diagrams are flipped, as the blobs are ordered differently than in equation (263). Using this identity, the numerator Jacobi identity for MHV amplitudes can finally be investigated. As discussed previously in section 2.5, for MHV amplitudes, the minimal gauge is picked

$$\varepsilon_{j+}^\mu = \frac{\bar{u}_+(p_j)\gamma^\mu u_+(p_-)}{\sqrt{2}s_-(p_j,p_-)} \tag{264}$$

$$\varepsilon^{\mu}_{j-} = -\frac{\bar{u}_{-}(p_j)\gamma^{\mu}u_{-}(p_{+})}{\sqrt{2}s_{+}(p_j,p_{+})} \tag{265}$$

Key is now that without loss of generality, we may now work with two gluons having positive helicity, say a and b, and all others having negative helicity. Using aforementioned definitions, say $p_a = p_+$. This then means that the only non-vanishing products of polarizations can be $(\varepsilon_b \cdot \varepsilon_j)$. As there is only one polarization ε_b, all terms with more than one factor of $(\varepsilon \cdot \varepsilon)$ must vanish. Now, through a simple counting argument, this will imply that only cubic diagrams contribute to the MHV numerator identity. Notice that each cubic vertex introduces a term of p into the numerator, which is either contracted with a polarization or another momentum. In a n-point cubic diagram, there are n polarizations, and $n-2$ cubic vertices. This implies, that as the complete diagram does not contain any open indices such as μ or ν, there must be $n-2$ factors of $(\varepsilon \cdot p)$ and 1 factor of $(\varepsilon \cdot \varepsilon)$. Similarly, in a quartic diagram there are only $n-4$ momenta to contract against. This means at least two factors of $(\varepsilon \cdot \varepsilon)$ exist, making it vanish in the MHV case. In figure 14, a cubic diagram is multiplied by a handlebarred blob in all cases on the right hand side. The cubic diagram will contain a factor of $(\varepsilon \cdot \varepsilon)$, and hence if the handlebarred blob also contains a factor of $(\varepsilon \cdot \varepsilon)$, the whole numerator identity will vanish. Next will be shown that a handlebarred blob will always contain a term of $(\varepsilon \cdot \varepsilon)$.

By another counting argument, if the blob has m external lines, it contains $m-1$ polarizations and $m-2+1$ momenta. This means for the contribution to be non-vanishing, all terms will need to be of the form $(\varepsilon \cdot p)$. Let us expand the blob D_1. If the blob is trivial, and only contains one polarization then $D_1^{\nu} p_{\nu} = 0$, hence assume it is not. As only cubic interactions are under consideration, the following expansion may be performed

$$D_1^{\nu} p_{\nu} = E_1^{\alpha} E_2^{\beta} Y(p_a, \alpha; p_b, \beta; -p_1, \nu) p_{1\nu} \tag{266}$$

Where E_1 and E_2 are two new blobs. Next, as discussed in section 2.6

$$Y(p_a, \alpha; p_b, \beta; -p_1, \nu) p_{1\nu} = K(p_a)^{\alpha\beta} - K(p_b)^{\alpha\beta} \tag{267}$$

With, as defined previously

$$K(p_a)^{\alpha\beta} = p_a^{\alpha} p_a^{\beta} - p_a^2 g^{\alpha\beta} \tag{268}$$

As stated before, all terms in $D_1^{\nu} p_{\nu}$ must be of the form $(\varepsilon \cdot p)$. This means that a p^2 contribution will vanish for the MHV numerator identity. Now observe that this implies

$$D_1^\nu p_\nu = E_1^\alpha E_2^\beta Y(p_a,\alpha;p_b,\beta;-p_1,\nu)p_{1\nu} = E_1^\alpha p_{a\alpha} E_2^\beta p_{a\beta} - E_1^\alpha p_{b\alpha} E_2^\beta p_{b\beta} \tag{269}$$

Hence once again terms are obtained containing handlebarred blobs. This procedure can be continued at most $m-2$ times, at which point the diagram must be of the form $\varepsilon_j \cdot p_j = 0$. Diagramatically this argument is expressed in figure 15.

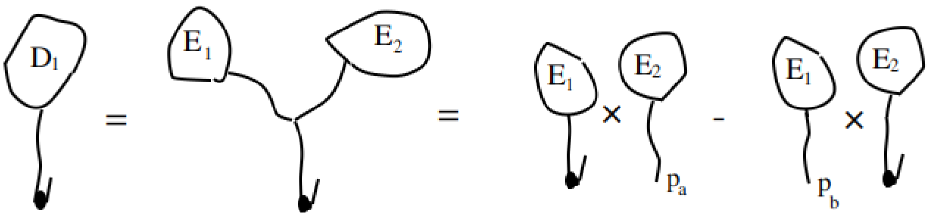


Figure 15: *Diagrammatic representation of the inductive argument on handlebarred blobs. Momenta at the end of a leg represent a contraction of the open index on the blob with that momentum.*

By induction $D_1^\nu p_\nu = 0$ for MHV numerator identities, and hence the numerator Jacobi identity holds for a minimal gauge. This means that the numerator identity may be assumed to be true when deriving the BCJ identity for MHV partial amplitudes. Now a question arises: Is there a way to prove that the BCJ identity holds in general for MHV partial amplitudes? It should be noticed that in the original paper, two properties were conjectured. Firstly, the numerator Jacobi identity which was just shown to hold for MHV partial amplitudes. Secondly, that by picking a basis of $(n-3)!$ numerators and fixing them to partial amplitudes, all other numerators can be freely set to zero without changing any of the partial amplitudes. This was not possible to show specifically for MHV amplitudes, as it is a property that cannot be based on a specific gauge. It rather is related to the number of numerators which have to be changed at the very least to preserve invariance of the partial amplitudes under the change of another numerator. Notice that the numerator Jacobi identity allows us to write any partial amplitude as a function of $(n-2)!$ independent numerators

$$A_n^{part} = f(n_1 \dots n_{(n-2)!}) \tag{270}$$

Known is that the partial amplitude is invariant under numerator changes as described in the first section of this chapter, and hence we could follow the following procedure

1. Set the degrees of freedom Δ such that numerator Jacobi holds
2. Express all partial amplitudes in terms of $(n-2)!$ independent numerators
3. Set the degrees of freedom Δ to zero

At the end of this procedure, as the partial amplitudes are invariant under numerator change, the expressions in terms of numerators n_1 to $n_{(n-2)!}$ still hold. The problem is, however, that it is not known how the degrees of freedom Δ correlate exactly to each numerator, as they are not the same as the 'freedoms' Bern, Carrasco and Johannson used. In the next chapter the focus lies on deriving the BCJ identity directly from partial amplitudes, as has been done previously by Chen et al. [2], and giving a diagrammatic interpretation of this proof.

6 Proof of the BCJ identity

In this section, the proof of the BCJ identity resulting directly from partial amplitudes A_n^{part} is outlined. In order to do so, prof. Kleiss pointed out to me that in fact, partial amplitudes may be written as sums of other partial amplitudes using the BCFW identity [10]. In this chapter the BCFW identity will briefly introduced, but not derived. Next, a diagrammatic interpretation discussed in [11] of the BCFW identity is considered, and used to prove a simple variant of the BCJ identity diagrammatically. Finally, the general proof for the BCJ identity given by Chen et al. in [2] is discussed, and using [11] a diagrammatic interpretation of this proof is given.

6.1 The BCFW identity

The BCFW identity applies to tree level partial amplitudes A_n^{part}. The identity arises through considering partial amplitudes A_n^{part} in the complex plane. Notice that a partial amplitude is a rational function of polarizations and momenta, and hence when considered in the complex plane, is a meromorphic function. The BCFW identity is derived through considering two momenta p_i and p_j, and deforming them with a complex parameter

$$\begin{aligned} p_i &\to \hat{p}_i(z) = p_i + zq \\ p_j &\to \hat{p}_j(z) = p_j - zq \end{aligned} \tag{271}$$

With the further requirement that $p_i \cdot q = 0$ and $p_j \cdot q = 0$. Notice that this leaves the sum over all momenta invariant. This can only be done if q is complex, but this is not really an issue as the original momenta are being deformed with a complex parameter anyways. Substituting $p_{i,j}$ for $\hat{p}_{i,j}$ will now make our partial amplitude a function of the complex variable z. Remember that $A_n^{part}(z)$ is a rational function, and hence is certainly meromorphic in z. This allows the use of the Cauchy residue theorem on this complex partial amplitude $A_n^{part}(z)$. From the residue theorem the following is obtained

$$\frac{1}{2\pi i}\oint_C \frac{A_n^{part}(z)}{z} dz = \sum_{poles\, z_k} Res\left(\frac{A_n^{part}(z)}{z}, z_k\right) \tag{272}$$

In [10], the contour $C = \{Re^{i\theta} \mid \theta \in [0,2\pi]\}$ was chosen, with $R \to \infty$. This ensures all poles $A_n^{part}(z)$ admits in the complex plane are enclosed in the contour. One pole should be obvious: for $z = 0$ the residu simply equals the orignal real partial amplitude $A_n^{part}(0) = A_n^{part}$. If it is now possible to argue that the left hand side of equation (272) vanishes, then an identity would be established between a partial amplitude A_n^{part} and the residues of the poles of $A_n^{part}(z)$. As discussed in the original paper [10], if this is the case, the following identity is obtained

$$A^{part} = \sum_{\{a_n \dots a_m\} \in \mathscr{P}} \sum_{\lambda} A^{part}(a_n \dots a_m, -P^{\lambda}(z)) \frac{1}{s_{n \dots m}} A^{part}(P^{-\lambda}(z), a_{m+1} \dots a_{n-1}) \tag{273}$$

Next, a few quantities are to be defined. Firstly $\{a_n \dots a_m\} \in \mathscr{P}$ if gluon i is in $\{a_n \dots a_m\}$ and gluon j is not. Secondly $s_{n \dots m} = (p_n + p_{n+1} + \dots + p_m)^2$. Note that this means that $s_{n \dots m}(z)$ is indeed dependent on z when combined with the the way $\mathscr{P}$ is defined. Finally, a sum is taken over two possible helicities $\lambda = \pm$. The identity (273) is the famous BCFW recursion identity, and is the tool required to prove the BCJ identity. The original aim of this thesis was to find a diagrammatic argument for the validity of the BCJ identity. Fortunately, the BCFW identity has been proven by Kleiss et. al. in [11], which means that the BCFW identity identity provides a diagrammatic argument to prove the BCJ identity. Furthermore, this approach shows that the $z \to \infty$ contour integral vanishes, as the diagrammatic proof in [11] shows that all terms connected to the contour integral cancel out. Let us now discuss a simple way to derive a simple BCJ identity using the BCFW recursion relations.

6.2 Proof of a simple BCJ identity

Let us return to the BCJ identity. In most cases, the BCJ identity is rather complicated and introduces many factors of momenta in the coefficient of each partial amplitude in the $(n-3)!$ base. Let us once again look at the BCJ formula

$$\begin{aligned} A_n^{part}(1,2,\{\alpha\},3,\{\beta\}) = &\sum_{\{\sigma\}_j \in POP(\{\alpha\},\{\beta\})} A_n^{part}(1,2,3,\{\sigma\}_j) \\ &\times \prod_{k=4}^{m+3} \frac{F(3,\{\sigma\}_j,1 \mid k)}{s_{2,4,\dots,k}} \end{aligned} \tag{274}$$

The simplest case of the BCJ identity would be the one where $m = 1$, hence $\{\alpha\}$ only contains a single element. Furthermore, in this case $POP(\{\alpha\},\{\beta\})$ is also simple and consists out of only $n-2$ elements. Without loss of generality, let us take $\{\alpha\} = \{4\}$ and $\{\beta\} = \{5,6,\ldots,n\}$, as gluons can be redefined in our final result to get other identities. In this case the following expansion is obtained

$$\begin{aligned}
A_n^{part}(1,2,4,3,5,\ldots,n) =& A_n^{part}(1,2,3,4,5,6,\ldots,n)\frac{s_{14}+s_{n4}+\cdots+s_{64}+s_{54}}{s_{24}} \\
&+A_n^{part}(1,2,3,5,4,6,\ldots,n)\frac{s_{14}+s_{n4}+\cdots+s_{64}}{s_{24}} \\
&+\ldots \\
&+A_n^{part}(1,2,3,5,6,\ldots,n,4)\frac{s_{14}}{s_{24}}
\end{aligned} \tag{275}$$

Which will be called the 'simple BCJ identity'. The terms in this equation can now be rearranged in a clever way, and using $s_{24} = -\sum_j s_{j4}$ with j all momenta except 2 and 4, the simple BCJ identity turns into

$$\begin{aligned}
0 =& A_n^{part}(1,2,4,3,5,6,\ldots,n)(s_{14}+s_{n4}+\cdots+s_{64}+s_{54}+s_{43}) \\
&+A_n^{part}(1,2,3,4,5,6,\ldots,n)(s_{14}+s_{n4}+\cdots+s_{64}+s_{54}) \\
&+A_n^{part}(1,2,3,5,4,6,\ldots,n)(s_{14}+s_{n4}+\cdots+s_{64}) \\
&+\ldots \\
&+A_n^{part}(1,2,3,5,6,\ldots,n,4)(s_{14})
\end{aligned} \tag{276}$$

A clear pattern can be observed here, and it can be used to generalize this simple BCJ identity with other permutations of gluons. Remember that using the Kleiss-Kuijf identity, two legs can be put next to eachother, consequently only amplitudes with legs n and 1 adjacent will be considered. The following identity is obtained

$$\begin{aligned}
0 =& A_n^{part}(1,r,a_3,a_4,a_5,a_6,\ldots,n)(s_{rn}+\cdots+s_{ra_5}+s_{ra_4}+s_{ra_3}) \\
&+A_n^{part}(1,a_3,r,a_4,a_5,a_6,\ldots,n)(s_{rn}+\cdots+s_{ra_5}+s_{ra_4}) \\
&+A_n^{part}(1,a_3,a_4,r,a_5,a_6,\ldots,n)(s_{rn}+\cdots+s_{ra_5}) \\
&+\ldots \\
&+A_n^{part}(1,a_3,a_4,a_5,a_6,\ldots,r,n)(s_{rn})
\end{aligned} \tag{277}$$

Here gluon r 'runs' across the partial amplitude, and each partial amplitude is multiplied by the sum of s_{rj} with j all gluons that are after r in the respective partial amplitude. Naturally, by using momentum conservation the identity may also be 'flipped around' as follows

$$
\begin{aligned}
0 = & -A_n^{part}(1,r,a_3,a_4,a_5,a_6,\dots,n)(s_{r1}) \\
& -A_n^{part}(1,a_3,r,a_4,a_5,a_6,\dots,n)(s_{r1}+s_{ra_3}) \\
& -A_n^{part}(1,a_3,a_4,r,a_5,a_6,\dots,n)(s_{r1}+s_{ra_3}+s_{ra_4}) \\
& -\dots \\
& -A_n^{part}(1,a_3,a_4,a_5,a_6,\dots,r,n)(s_{r1}+s_{ra_3}+s_{ra_4}+\dots+s_{ra_{n-1}})
\end{aligned} \tag{278}
$$

Combinations of both equations (277) and (278) are also valid. Our aim is now to prove this identity using the BCFW identity. Clearly, as amplitudes are being split up into smaller pieces, this relation has to be recursive. Next, BCFW deformations are performed on momenta 1 and n on all partial amplitudes simultaneously. As a simple example consider the identity

$$
\begin{aligned}
F = & A_6^{part}(1,2,3,4,5,6)(s_{26}+s_{25}+s_{24}+s_{23}) \\
& +A_6^{part}(1,3,2,4,5,6)(s_{26}+s_{25}+s_{24}) \\
& +A_6^{part}(1,3,4,2,5,6)(s_{26}+s_{25}) \\
& +A_6^{part}(1,3,4,5,2,6)(s_{26})
\end{aligned} \tag{279}
$$

To show that the simple BCJ identity holds, F has to vanish. This will be shown using the BCFW identity. Consider the following deformation

$$
\begin{aligned}
\frac{F(z)}{z} = & \frac{A_n^{part}(z)(\hat{1},2,3,4,5,\hat{6})}{z}(s_{2\hat{6}}+s_{25}+s_{24}+s_{23}) \\
& +\frac{A_6^{part}(z)(\hat{1},3,2,4,5,\hat{6})}{z}(s_{2\hat{6}}+s_{25}+s_{24}) \\
& +\frac{A_6^{part}(z)(\hat{1},3,4,2,5,\hat{6})}{z}(s_{2\hat{6}}+s_{25}) \\
& +\frac{A_6^{part}(z)(\hat{1},3,4,5,2,\hat{6})}{z}(s_{2\hat{6}})
\end{aligned} \tag{280}
$$

Successively examine the function $F(z)$ and its contour integral. Notice that firstly the contour integral over $\frac{F(z)}{z}$ will equal a sum of contour integrals over $\frac{A(z)}{z}f(z)$, where f is a linear function in z. Let us consider this contour integral

$$
\oint \frac{A^{part}}{z} f(z)dz \tag{281}
$$

If equation (281) vanishes, the BCFW identity is known to be satisfied. In [11], a diagrammatic proof has been given which shows that the BCFW identity holds for Yang-Mills amplitudes. This means that indeed equation (281) vanishes.

$$
\oint \frac{A^{part}}{z} f(z)dz = 0 \quad \Rightarrow \quad \oint \frac{F(z)}{z}dz = 0 \tag{282}
$$

Next, $\frac{F(z)}{z}$ is decomposed in terms of its poles. Notice now that the pole at $z = 0$ is simply F, and the other poles consist out of the BCFW identities applied to the right hand side of equation (280). Using [11], the expansion into BCFW splittings once again is valid diagrammatically for each partial amplitude individually. The reader might wonder why this convoluted approach is taken in the first place. Can we simply not expand the BCJ identity straight away? The answer is no, as the coefficients of each partial amplitude need to be dependent on z to actually apply a lower-order BCJ identity, and not only the partial amplitude itself, as will become clear in expression (283)

$$\begin{aligned}
F = F(0) = &-(s_{2\hat{1}})A^{part}(\hat{1},2,-P)\frac{1}{s_{12}}A^{part}(P,3,4,5,\hat{6}) \\
&+A^{part}(\hat{1},3,-P)\frac{1}{s_{13}}A^{part}(P,2,4,5,\hat{6})(s_{2\hat{6}}+s_{25}+s_{24}) \\
&+A^{part}(\hat{1},3,-P)\frac{1}{s_{13}}A^{part}(P,4,2,5,\hat{6})(s_{2\hat{6}}+s_{25}) \\
&+A^{part}(\hat{1},3,-P)\frac{1}{s_{13}}A^{part}(P,4,5,2,\hat{6})(s_{2\hat{6}}) \\
&-(s_{2\hat{1}})A^{part}(\hat{1},2,3,-P)\frac{1}{s_{123}}A^{part}(P,4,5,\hat{6}) \\
&-(s_{23}+s_{2\hat{1}})A^{part}(\hat{1},3,2,-P)\frac{1}{s_{123}}A^{part}(P,4,5,\hat{6}) \\
&+A^{part}(\hat{1},3,4,-P)\frac{1}{s_{134}}A^{part}(P,2,5,\hat{6})(s_{2\hat{6}}+s_{25}) \\
&+A^{part}(\hat{1},3,4,-P)\frac{1}{s_{134}}A^{part}(P,5,2,\hat{6})(s_{2\hat{6}}) \\
&-(s_{2\hat{1}})A^{part}(\hat{1},2,3,4,-P)\frac{1}{s_{56}}A^{part}(P,5,\hat{6}) \\
&-(s_{23}+s_{2\hat{1}})A^{part}(\hat{1},3,2,4,-P)\frac{1}{s_{56}}A^{part}(P,5,\hat{6}) \\
&-(s_{24}+s_{23}+s_{2\hat{1}})A^{part}(\hat{1},3,4,2,-P)\frac{1}{s_{56}}A^{part}(P,5,\hat{6}) \\
&+A^{part}(\hat{1},3,4,5,-P)\frac{1}{s_{26}}A^{part}(P,2,\hat{6})(s_{2\hat{6}})
\end{aligned} \tag{283}$$

Let us consider this expression line by line. In the first line a term of $s_{2\hat{1}}A^{part}(\hat{1},2,-P)$ appears. This is exactly the BCJ identity for a three-point partial amplitude, and hence should vanish. As P is on-shell, and $s_{2\hat{1}} = P^2 = 0$, it is seen that the term $s_{2\hat{1}}A^{part}(\hat{1},2,-P)$ does indeed vanish as required. For the next three lines, the following expression is obtained when extracting common factors

$$\begin{aligned} &A(P,2,4,5,\hat{6})(s_{2\hat{6}}+s_{25}+s_{24}) \\ +&A(P,4,2,5,\hat{6})(s_{2\hat{6}}+s_{25}) \\ +&A(P,4,5,2,\hat{6})(s_{2\hat{6}}) \end{aligned} \tag{284}$$

This clearly is a five-point BCJ identity with complex momenta. When setting up some sort of inductive argument, the five-point BCJ identity allows us to set this term to zero. Notice that it is important to have the BCJ identity vanish for complex momenta also. To easily see that this assumption is valid, replace the denominator z in (282) by $z-w$ (Where w is not a pole of $F(z)$) to show that our derivation here is also valid to expand $F(w)$ in terms of the exact same BCFW poles. The next two lines in equation (283) form a four-point BCJ identity, as do the two after that. Next is another five-point BCJ identity and finally a term of $A(P,2,\hat{6})(s_{2\hat{6}})$ which vanishes. This means that $F=0$, and this simple BCJ identity holds. Let us set up an inductive proof of the simple BCJ identity (277). The following induction basis is used

$$s_{ab}A^{part}(a,b,c)=0 \tag{285}$$

As mentioned before, this follows simply through momentum conservation. Now assume equation (277) holds for $(n-1)$ gluons, and through the same argument as used before

$$\oint \frac{F(z)}{z}dz=0 \tag{286}$$

Now split (286) into $F(0)$, which is the identity we want to prove, and the sum over residues of the other poles in F. F consists out of partial amplitudes A^{part}, and the residues of the other poles are given by the BCFW identity (273). Let us consider the contribution from a random splitting between positions j and $j+1$

in A^{part}. These contributions are given as follows

$$
\begin{aligned}
&-(s_{r1})A_n^{part}(1,r,a_3,a_4,\dots,a_{j-1},-P)\frac{1}{s_{a_j,\dots,a_n}}A^{part}(P,a_j,\dots,n)\\
&-(s_{ra_3}+s_{r1})A_n^{part}(1,a_3,r,a_4,\dots,a_{j-1},-P)\frac{1}{s_{a_j,\dots,a_n}}A^{part}(P,a_j,\dots,n)\\
&-\dots\\
&-(s_{ra_{j-1}}+s_{ra_3}+s_{r1})A_n^{part}(1,a_3,a_4,\dots,a_{j-1},r,-P)\frac{1}{s_{a_j,\dots,a_n}}A^{part}(P,a_j,\dots,n)\\
&+A_n^{part}(1,a_3,a_4,\dots,a_{j-1},-P)\frac{1}{s_{a_1,\dots,a_{j-1}}}A^{part}(P,r,a_j,\dots,n)(s_{ra_n}+\dots+s_{ra_j})\\
&+\dots\\
&+A_n^{part}(1,a_3,a_4,\dots,a_{j-1},-P)\frac{1}{s_{a_1,\dots,a_{j-1}}}A^{part}(P,a_j,\dots,r,n)(s_{ra_n})
\end{aligned}
\tag{287}
$$

Now the first half of this equation is a j-point simple BCJ identity, and the second half is a $n-j$-point simple BCJ identity, which by the induction hypothesis hold. As the split is performed in a generic position, all residues produced in the BCFW identity will add up to zero, and hence this means that indeed $F(0)=0$ as required. This shows that the 'simple BCJ identity' holds. This is of course nice, however it only represents one specific case in the general BCJ formula. Using the simple BCJ identity, some other cases of the BCJ identity can be proven, however it makes more sense to discuss the beautiful proof of the general BCJ identity by Chen et al. presented in [2]. This proof will also be interpreted in a diagrammatic way through the diagrammatic proof of the BCFW identity. This means that [11] allows us to conclude that the proof given in [2] can be interpreted as a diagrammatic proof.

6.3 Proof of the 'general BCJ identity' by Chen et al.

In this section, the 'general BCJ identity' is introduced. This is not the same as the identity introduced in the original paper [1], however it also implies that there are only $(n-3)!$ independent n-point partial amplitudes, thus serving the same purpose. The general BCJ identity was first derived through string theory in [12] and later proven in [2]. It can be used to derive the expansion given in section 4.3.2, however this derivation is quite long and not very instructive, and will not be discussed in this thesis. The reader may refer to [2] for the aforementioned explicit derivation. Let us now define $\{\alpha\} = \{\alpha_1, \ldots, \alpha_r\}$ and $\{\beta\} = \{\beta_1, \ldots, \beta_s\}$ through a partial amplitude $A_n^{part}(1, \{\alpha\}, \{\beta\}, n)$. The general BCJ identity is given as follows

$$\sum_{\{\sigma\} \in M(\{\alpha\}, \{\beta\})} \sum_{i=1}^{r} \sum_{a_j < a_{\alpha_i}} s_{\alpha_i j} \, A_n^{part}(1, \{\sigma\}, n) \tag{288}$$

Let us first explain this formula. Firstly, $M(\{\alpha\}, \{\beta\})$ is a meshing of $\{\alpha\}$ and $\{\beta\}$, defined in exactly the same way as in the Kleiss-Kuijf identity, and it consists out of all sets which contain all elements of both $\{\alpha\}$ and $\{\beta\}$ which leave their respective elements in their place. Secondly, a_n is the position of element n in the partial amplitude $A_n^{part}(1, \{\sigma\}, n)$. From now on, the index n and the *part* label in our partial amplitudes are suppressed to avoid our equations from becoming cluttered. Notice that if $\{\alpha\}$ contains only one element, this identity equals the simple BCJ identity which was discussed previously. As an instructive example, consider the simplest non-trivial case, which is also discussed in [2]. Consider $\{\alpha\} = \{2, 3\}$ and $\{\beta\} = \{4, 5\}$. The general BCJ identity then gives

$$\begin{aligned}
0 = {} & A(1,2,3,4,5,6)(s_{21} + s_{31} + s_{32}) \\
& + A(1,2,4,3,5,6)(s_{21} + s_{31} + s_{32} + s_{34}) \\
& + A(1,2,4,5,3,6)(s_{21} + s_{31} + s_{32} + s_{34} + s_{35}) \\
& + A(1,4,2,3,5,6)(s_{21} + s_{24} + s_{31} + s_{34} + s_{32}) \\
& + A(1,4,2,5,3,6)(s_{21} + s_{24} + s_{31} + s_{34} + s_{32} + s_{35}) \\
& + A(1,4,5,2,3,6)(s_{21} + s_{24} + s_{25} + s_{31} + s_{34} + s_{32} + s_{35})
\end{aligned} \tag{289}$$

Continuing, in [2] this specific identity was then proven using the BCFW identity. Key is that in [2], the contour integral is shown to evaluate to zero through the specific structure of the terms in the contour integral. We, on the other hand, will use [11] to conclude that terms arising from contour integrals must vanish through a diagrammatic argument. For this purpose, set equation (289) equal to $F(z)$ after deforming momenta 1 and 6. Using the same argument as before

$$\oint \frac{F(z)}{z} dz = 0 \tag{290}$$

For a sufficiently large contour. This successively allows us to express $F(0)$ in terms of all of its BCFW poles. The aim is now to show that $F(0)$ vanishes. Using the BCFW expansion, this yields exactly the same expansion as found in

[2], which are the following

$$
\begin{aligned}
F(0) = {} & A(\hat{1},2 \mid 3,4,5,\hat{6})(s_{2\hat{1}}+s_{3\hat{1}}+s_{32}) \\
& + A(\hat{1},2 \mid 4,3,5,\hat{6})(s_{2\hat{1}}+s_{3\hat{1}}+s_{32}+s_{34}) \\
& + A(\hat{1},2 \mid 4,5,3,\hat{6})(s_{2\hat{1}}+s_{3\hat{1}}+s_{32}+s_{34}+s_{35}) \\
& + A(\hat{1},4 \mid 2,3,5,\hat{6})(s_{2\hat{1}}+s_{24}+s_{3\hat{1}}+s_{34}+s_{32}) \\
& + A(\hat{1},4 \mid 2,5,3,\hat{6})(s_{2\hat{1}}+s_{24}+s_{3\hat{1}}+s_{34}+s_{32}+s_{35}) \\
& + A(\hat{1},4 \mid 5,2,3,\hat{6})(s_{2\hat{1}}+s_{24}+s_{25}+s_{3\hat{1}}+s_{34}+s_{32}+s_{35}) \\
& + A(\hat{1},2,3 \mid 4,5,\hat{6})(s_{2\hat{1}}+s_{3\hat{1}}+s_{32}) \\
& + A(\hat{1},2,4 \mid 3,5,\hat{6})(s_{2\hat{1}}+s_{3\hat{1}}+s_{32}+s_{34}) \\
& + A(\hat{1},2,4 \mid 5,3,\hat{6})(s_{2\hat{1}}+s_{3\hat{1}}+s_{32}+s_{34}+s_{35}) \\
& + A(\hat{1},4,2 \mid 3,5,\hat{6})(s_{2\hat{1}}+s_{24}+s_{3\hat{1}}+s_{34}+s_{32}) \\
& + A(\hat{1},4,2 \mid 5,3,\hat{6})(s_{2\hat{1}}+s_{24}+s_{3\hat{1}}+s_{34}+s_{32}+s_{35}) \\
& + A(\hat{1},4,5 \mid 2,3,\hat{6})(s_{2\hat{1}}+s_{24}+s_{25}+s_{3\hat{1}}+s_{34}+s_{32}+s_{35}) \\
& + A(\hat{1},2,3,4 \mid 5,\hat{6})(s_{2\hat{1}}+s_{3\hat{1}}+s_{32}) \\
& + A(\hat{1},2,4,3 \mid 5,\hat{6})(s_{2\hat{1}}+s_{3\hat{1}}+s_{32}+s_{34}) \\
& + A(\hat{1},2,4,5 \mid 3,\hat{6})(s_{2\hat{1}}+s_{3\hat{1}}+s_{32}+s_{34}+s_{35}) \\
& + A(\hat{1},4,2,3 \mid 5,\hat{6})(s_{2\hat{1}}+s_{24}+s_{3\hat{1}}+s_{34}+s_{32}) \\
& + A(\hat{1},4,2,5 \mid 3,\hat{6})(s_{2\hat{1}}+s_{24}+s_{3\hat{1}}+s_{34}+s_{32}+s_{35}) \\
& + A(\hat{1},4,5,2 \mid 3,\hat{6})(s_{2\hat{1}}+s_{24}+s_{25}+s_{3\hat{1}}+s_{34}+s_{32}+s_{35})
\end{aligned} \tag{291}
$$

Here a definition from [2] has been employed to simplify writing down the split amplitudes resulting from the BCFW identity. Define

$$A(\hat{1},2 \mid 3,4,5,\hat{6}) = A(\hat{1},2,P)\frac{1}{s_{\hat{1}2}}A(-P,3,4,5,\hat{6}) \tag{292}$$

Next, using the general BCJ identity inductively, expression (291) can be shown to vanish. Consider, for instance, all terms with the BCFW splitting in the middle. The first terms obtained are $A(\hat{1},2,3 \mid 4,5,\hat{6})(s_{2\hat{1}}+s_{3\hat{1}}+s_{32})$. Notice that the momentum multiplying this term actually equals $s_{\hat{1}23}$, and as the momentum flowing across the split is on-shell, $A(\hat{1},2,3,P)s_{\hat{1}23}=0$. The same concept holds for the term containing $A(\hat{1},4,5 \mid 2,3,\hat{6})$, which vanishes similarly. There are four other terms to consider, which are divided out cleverly in the following way

$$\begin{aligned} & A(\hat{1},2,4,P_{\hat{1}24})\Big\{A(-P_{\hat{1}24},3,5,\hat{6})(s_{P_{\hat{1}24}3}) \\ & \qquad +A(-P_{\hat{1}24},5,3,\hat{6})(s_{P_{\hat{1}24}3}+s_{53})\Big\} \\ & +A(\hat{1},4,2,P_{\hat{1}24})\Big\{A(-P_{\hat{1}24},3,5,\hat{6})(s_{P_{\hat{1}24}3}) \\ & \qquad +A(-P_{\hat{1}24},5,3,\hat{6})(s_{P_{\hat{1}24}3}+s_{53})\Big\} \\ & +A(-P_{\hat{1}24},3,5,\hat{6})\Big\{A(\hat{1},2,4,P_{\hat{1}24})(s_{\hat{1}2}) \\ & \qquad +A(\hat{1},4,2,P_{\hat{1}24})(s_{\hat{1}2}+s_{24})\Big\} \\ & +A(-P_{\hat{1}24},5,3,\hat{6})\Big\{A(\hat{1},2,4,P_{\hat{1}24})(s_{\hat{1}2}) \\ & \qquad +A(\hat{1},4,2,P_{\hat{1}24})(s_{\hat{1}2}+s_{24})\Big\} \end{aligned} \tag{293}$$

It should be clear that the four terms in the curly brackets are all four-point general BCJ identities. This means that when using an inductive argument, all of these terms vanish. The other splittings in expression (291) may also be decomposed in similar ways, and they will also vanish through lower-point BCJ identities. This means that $F(0) = 0$ as required inductively.

Let us now discuss the proof of the general BCJ identity given in [2]. The induction basis is once again $s_{ab}A(a,b,c) = 0$. Now, let us consider the induction step. Similarly to the example that was given previously in this section, the argument will once again start out at $\oint \frac{F(z)}{z}dz = 0$ through the diagrammatical proof of the BCFW identity. Besides this deviation, the proof in [2] is followed

$$F(0) = \sum_{\{\sigma\}\in M(\{\alpha\},\{\beta\})} \sum_{i=1}^{r} \sum_{a_j<a_{\alpha_i}} s_{\alpha_i j} \sum_{all\ splittings} A(\hat{1},\{\sigma_L\} \mid \{\sigma_R\},\hat{n}) \tag{294}$$

Where $\{\sigma_L\}$ are all elements in $\{\sigma\}$ that end up to the left of the splitting and vice versa. Now, as $\{\sigma\}$ is built up out of $\{\alpha\}$ and $\{\beta\}$, Chen et al. called the elements of $\{\alpha\}$ that end up in $\{\sigma_L\}$ $\{\alpha_L\}$, and similarly defined $\{\alpha_R\}$, $\{\beta_L\}$ and $\{\beta_R\}$. Notice now that for a given BCFW splitting, the set $\{\sigma_L\}$ will be taken from the meshing $M(\{\alpha_L\},\{\beta_L\})$. Fixing the splitting and $\{\sigma_L\}$, the next component to study is

$$\sum_{\{\sigma_L\}\in M(\{\alpha_L\},\{\beta_L\})} \sum_{\{\sigma_R\}\in M(\{\alpha_R\},\{\beta_R\})} \sum_{i=1}^{r} \sum_{a_j<a_{\alpha_i}} s_{\alpha_i j} A(\hat{1},\{\sigma_L\} \mid \{\sigma_R\},\hat{n}) \tag{295}$$

Defining r_L as the number of elements in $\{\alpha_L\}$, the previous expression may now be expanded in the following clever way

$$\begin{aligned}
&\left\{\sum_{\{\sigma_L\}\in M(\{\alpha_L\},\{\beta_L\})}\sum_{i=1}^{r_L}\sum_{a_j<a_{\alpha_i}} s_{\alpha_i j}\, A(\hat{1},\{\sigma_L\},-P)\right\}\frac{1}{s_{1,\dots,\sigma_{r_L}}}\times \\
&\left\{\sum_{\{\sigma_R\}\in M(\{\alpha_R\},\{\beta_R\})} A(P,\{\sigma_R\},\hat{n})\right\} \\
&\left\{\sum_{\{\sigma_L\}\in M(\{\alpha_L\},\{\beta_L\})} A(\hat{1},\{\sigma_L\},-P)\right\}\frac{1}{s_{1,\dots,\sigma_{r_L}}}\times \\
&\left\{\sum_{\{\sigma_R\}\in M(\{\alpha_R\},\{\beta_R\})}\sum_{i=r_L+1}^{r}\sum_{1\le a_j<a_{\alpha_i}} (s_{\alpha_i j}+s_{\alpha_i P})\, A(P,\{\sigma_R\},\hat{n})\right\}
\end{aligned} \tag{296}$$

Let us explain how this result is obtained. The first term in (296) is produced by cutting off the terms $s_{\alpha_i j}$ in the original expression at $i = r_L$, ensuring that α_i lies within $\{\alpha_L\}$. The second term consists out of the other terms, where $i \geq r_L + 1$. Here the terms s both result from within $A(P,\{\sigma_R\},\hat{n})$ and $A(\hat{1},\{\sigma_L\},-P)$. Choosing a specific index i yields

$$s_{\alpha_i 1} + s_{\alpha_i \sigma_1} + \cdots + s_{\alpha_i \sigma_{r_L}} + s_{\alpha_i \sigma_{r_L+1}} + \dots \tag{297}$$

Notice now that through momentum conservation, $p_1 + \cdots + p_{\sigma_{r_L}} = P$. This means that indeed the required form $\sum_{1\le a_j<a_{\alpha_i}}(s_{\alpha_i j}+s_{\alpha_i P})$ is obtained as stated in (296). Next, examine

$$\sum_{\{\sigma_L\}\in M(\{\alpha_L\},\{\beta_L\})}\sum_{i=1}^{r_L}\sum_{a_j<a_{\alpha_i}} s_{\alpha_i j}\, A(\hat{1},\{\sigma_L\},-P) \tag{298}$$

Notice this is simply the n_L+2-point general BCJ identity, where n_L is the number of elements in $\{\sigma_L\}$. Similarly examine

$$\begin{aligned}
&\sum_{\{\sigma_R\}\in M(\{\alpha_R\},\{\beta_R\})}\sum_{i=r_L+1}^{r}\sum_{1\le a_j<a_{\alpha_i}} (s_{\alpha_i j}+s_{\alpha_i P})\, A(P,\{\sigma_R\},\hat{n}) = \\
&\sum_{\{\sigma_R\}\in M(\{\alpha_R\},\{\beta_R\})}\sum_{i=r_L+1}^{r}\sum_{a_j<a_{\alpha_i}} (s_{\alpha_i j})\, A(P,\{\sigma_R\},\hat{n})
\end{aligned} \tag{299}$$

This is simply the $n-n_L$-point general BCJ identity. $\{\sigma_L\}$ contains at least one element and at most $n-3$ elements, hence both n_L+2 and $n-n_L$ are less than n. This means that inductively, both expressions vanish and so does equation (296). This then implies that the sum over all splittings in equation (294) vanishes, hence indeed $F(0) = 0$ as required. Through [11], the proof outlined here can be interpreted diagrammatically, as the only tool used was the BCFW identity interpreted in a diagrammatic way. The final piece of the puzzle is to show that indeed the general BCJ identity can be used to write all partial amplitudes in terms of a $(n-3)!$-partial amplitude basis. To show this, pick $\{\alpha\} = \{\alpha_1, \ldots, 2\}$. Applying the general BCJ identity once will allow us to express $A(1,\ldots,2,\ldots,n)$ in terms of partial amplitudes with the number 2 shifted one place to the left. Then, applying the general BCJ identity another time on each of these terms yields an expansion with 2 shifted to the left once again. Continuing this process, eventually the original partial amplitude $A(1,\ldots,2,\ldots,n)$ is written in terms of a basis of partial amplitudes $A(1,2,\ldots,n)$. This process exactly reproduces the orignal BCJ expansion from [1] as is proven in [2]

$$\begin{aligned} A_n^{part}(1,2,\{\alpha\},3,\{\beta\}) = & \sum_{\{\sigma\}_j \in POP(\{\alpha\},\{\beta\})} A_n^{part}(1,2,3,\{\sigma\}_j) \\ & \times \prod_{k=4}^{m+3} \frac{F(3,\{\sigma\}_j,1 \mid k)}{s_{2,4,\ldots,k}} \end{aligned} \tag{300}$$

Combining the proof by Chen et al. with the diagrammatic proof of the BCFW identity by Kleiss et. al. in [11] hence completes our original intention of providing a diagrammatic derivation of a BCJ identity.

7 Summary

In this thesis we started by decomposing the full n-point Yang-Mills amplitude into color ordered partial amplitudes. Successively, we completed our review of known identities on these partial amplitudes. In the next step, the BCJ identity was introduced which had not been proven diagrammatically yet. We discussed how the BCJ identity was originally derived and explained the reasoning behind its general formula. Furthermore, some examples were given on the application of the BCJ identity to partial amplitudes in order to write partial amplitudes in terms of a $(n-3)!$-partial amplitude basis.

In this thesis, the main aim was to give a diagrammatic derivation of the BCJ identity. The first approach to come to a diagrammatic proof was to derive the conjectured 'numerator Jacobi identity', and next following the same strategy as in the original paper [1]. In order to do so, an attempt was made to find a diagrammatic representation of n-point numerators which satisfy this numerator Jacobi identity. First we discussed the nature of numerators more closely, and secondly we proceeded to apply this knowledge in order to solve the numerator Jacobi identity.

$$c_i + c_j + c_k = 0 \;\; \Leftrightarrow \;\; n_i + n_j + n_k = 0 \tag{301}$$

This approach was quite successfully for four- and five-point numerators. With this approach, we were unfortunately unable to solve the numerator Jacobi identity for six-gluon numerators. Furthermore, the result we did find for six-gluon numerators strongly implied that there is no general diagrammatic representation for numerators which satisfy this Jacobi identity. As another approach, we considered the numerators corresponding to a MHV amplitudes. This did yield a good result, from which we were able to conclude that numerators for MHV amplitudes satisfy the numerator Jacobi identity for at least one gauge choice. If the numerator identity is used to relate gauge-invariant quantities, for MHV amplitudes, it may be assumed to hold. This approach, however, did not allow us to derive the BCJ identity, as this required a specific re-arrangement of degrees of freedom in order to set $(n-2)! - (n-3)!$ numerators to zero.

In a second attempt to prove the BCJ identity diagrammatically, we considered the BCFW identity [10]. While the proof of the BCFW identity is not diagrammatic in itself, combining it with a diagrammatic proof [11] allows us to interpret it in a diagrammatic way. We succesfully considered a simple BCJ identity, and derived this simple BCJ identity inductively using the BCFW identity. Finally, we discussed the proof of the 'general BCJ identity' by Chen et al. [2], which simultaneously also implied our simple BCJ identity. In the review of the general BCJ identity, we used the diagrammatic proof of the BCFW identity to show that this proof of the general BCJ identity can be interpreted in a diagrammatic way. As the BCJ identity reduces the number of independent partial amplitudes to $(n-3)!$, we were able to accomplish what we originally aimed to do, namely to prove diagrammatically that all partial amplitudes may be written in terms of a $(n-3)!$-partial amplitude basis, as the BCJ identity implied.

8 Conclusion

In this thesis, two different approaches were tried to provide a diagrammatic proof for the BCJ identity. Firstly, an attempt was made to prove the BCJ identity diagrammatically by giving diagrammatic arguments for every step conducted in the original paper by Bern, Carrasco and Johansson. Secondly, we considered a previously known proof of the BCJ identity based on the BCFW identity.

Our first approach did not allow us to prove the BCJ identity, however we found multiple interesting properties on the structure of gluonic numerators. Furthermore, we were able to prove that the conjectured 'numerator Jacobi identity' may be assumed to hold for MHV amplitudes, when working with gauge-invariant quantities. In addition, we were able to verify the numerator Jacobi identity explicitly for four- and five-gluon amplitudes, however the approach we employed broke down for amplitudes with six or more gluons.

The second approach combined an existing proof of the BCJ identity based on the BCFW identity, with a diagrammatic interpretation of the BCFW identity. Using this approach, we were able to show that a diagrammatic argument holds for an analog of the BCJ identity introduced by Bern, Carrasco and Johansson. Specifically, the diagrammatic argument outlined in this thesis shows that an independent basis of $(n-3)!$ independent partial amplitudes exists as conjectured in the original paper on the BCJ identity. In conclusion, the original aim of this thesis has been accomplished, and a diagrammatic argument for the BCJ identity has been successfully provided.

9 Discussion and Outlook

Our initial approach to a possible diagrammatic derivation of the BCJ identity was not successful. While originally the BCJ identity was conjectured through an argument considering gluon numerators, these did not seem to provide an actual concise way to prove the BCJ identity. Consider the numerator identity on its own: If it is valid we are able to quickly determine that there are at most $(n-2)!$ independent partial amplitudes, however this is simply the same number as can be found through the Kleiss-Kuijf identity. In the original paper by Bern, Carrasco and Johansson, the argument to make $(n-2)!-(n-3)!$ additional numerators vanish is not very clear. In the five-point case it can be shown by hand that the procedure by Bern, Carrasco and Johansson leaves all partial amplitudes invariant, but we were not able to extrapolate this property at all for amplitudes with more external gluons. We have searched for ways to argue that the degrees of freedom we have in our numerators allows us to do this, but were not able to find any such argument. Conversely, it might be argued that if we have only $(n-3)!$ independent partial amplitudes, we should be able to fix $(n-3)!$ numerators to these partial amplitudes, and have all others vanish. It should also be noted that the 'general BCJ identity' introduced in [12] is far more concise than the original identity introduced by Bern, Carrasco and Johansson. Perhaps we would have been more successful if we had tried to derive a diagrammatic proof by using a diagrammatic representation of partial amplitudes and considering the general BCJ identity. A fact we have realised while investigating the BCJ identity using the BCFW identity, is that the BCJ identity almost appears as a natural result of the BCFW identity. Considering that the diagrammatic proof of the BCFW identity is quite involved, it should not come as a surprise that it is hard to prove the BCJ identity diagrammatically, as it follows almost naturally from the BCFW identity.

A number of interesting questions have arisen in this thesis:

- Is there a way to set the degrees of freedom Δ we have introduced to make $(n-2)!-(n-3)!$ basis numerators vanish?
- Is there a simpler way to prove the BCJ identity directly from diagrams, rather than using a detour through the BCFW identity?

- Are there further dependencies between partial amplitudes, which reduce the total number of independent partial amplitudes below $(n-3)!$?

Answering these questions could provide new insights into the structure of partial amplitudes, and with it the structures of amplitudes in general.

References

[1] Z. Bern, J.J.M. Carrasco and H. Johansson. “New relations for gauge-theory amplitudes”. In: *Phys. Rev.* D78 (2008). URL: https://arxiv.org/pdf/0805.3993.pdf.

[2] Y. Chen, Y. Du and B. Feng. “A Proof of the Explicit Minimal-basis Expansion of Tree Amplitudes in Gauge Field Theory”. In: *JHEP* 1102 (2011). URL: https://arxiv.org/pdf/1101.0009.pdf.

[3] R.H.P. Kleiss. *Pictures, Paths, Particles, Processes*. 2021.

[4] L. Dixon. *CALCULATING SCATTERING AMPLITUDES EFFICIENTLY*. 1996. URL: https://arxiv.org/pdf/hep-ph/9601359.pdf.

[5] S. J. Parke and T. R. Taylor. “Amplitude for n-Gluon Scattering”. In: *Phys. Rev. Lett.* 56 (1986). DOI: https://doi.org/10.1103/PhysRevLett.56.2459.

[6] F. A. Berends and W. T. Giele. “Recursive calculations for processes with n gluons”. In: *Nucl. Phys.* B306 (1988).

[7] R. Kleiss and H. Kuijf. “Multigluon cross sections and 5-jet production at hadron colliders”. In: *Nucl. Phys.* B312 (1989).

[8] V. Del Duca, L. Dixon and F. Maltoni. “New Color Decompositions for Gauge Amplitudes at Tree and Loop Level”. In: *Nucl. Phys.* B571 (2000). URL: https://arxiv.org/pdf/hep-ph/9910563.pdf.

[9] P. Cvitanovic and E. Gyldenkerne. *Field Theory*. 1983.

[10] R. Britto et al. “Direct Proof Of Tree-Level Recursion Relation In Yang-Mills Theory”. In: *Phys. Rev. Lett.* 94 (2005). URL: https://arxiv.org/pdf/hep-th/0501052.pdf.

[11] P. Draggiotis et al. “Diagrammatic proof of the BCFW recursion relation for gluon amplitudes in QCD”. In: *Eur.Phys.J.* C46 (2006). URL: https://arxiv.org/pdf/hep-ph/0511288.pdf.

[12] N. E. J. Bjerrum-Bohr, P. H. Damgaard and P. Vanhove. “Minimal Basis for Gauge Theory Amplitudes”. In: *Phys. Rev. Lett.* 103 (2009). URL: https://arxiv.org/pdf/0907.1425.pdf.