Scale-independent scale dependence and renormalization in zero dimensions

Tim Janssen

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Prof. Dr. R.H.P. Kleiss

Abstract

In this thesis the aim is to investigate a toy model that assigns infinite terms to Feynman diagrams in zero dimensions in such a way we can renormalize it. We also wish that the scale dependence of our theory is scale-independent. In this thesis we shall derive an equation (the pCS equation) that can be used to verify this. It shall become clear that this equation shows similarities with the Callan-Symanzik equation. We shall illustrate and verify our approach by applying it to the ϕ^4 and ϕ^3 theory in four dimensions. In the zero-dimensional case we can assign divergences to diagrams in an unambiguous manner so that they can be fully absorbed in redefined parameters, and in this sense the model is 'renormalizeable'; however, we show that it can not be made to obey the pCS equation and that sense the theory is 'non-renormalizeable'. Finally we discuss the reason for this "half-renormalizeability".

Contents

1 Introduction

1.1 Preface

Renormalization is a very important tool in quantum field theory (QFT) and statistical physics. In order to better understand the underlying mechanics we study the renormalization procedure in very simple models, namely those in zero dimensions. A very important condition of renormalization is that the scale-dependence of the theory is scale-independent. In zero dimensions there is however no scale so studying renormalization is pointless in this case. We therefore wish to construct a toy model in zero dimensions that introduces a scale in zero dimensions in such a way we can renormalize the theory.

1.2 Outline

This thesis is organised as follows. In the next chapter we shall give an introduction to QFT in zero dimensions. In this chapter we shall discuss Green's functions and construct the Schwinger-Dyson equation. We shall also introduce Feynman diagrams and Feynman rules. We shall end this chapter with the effective action which shall prove to be useful later on. In chapter 3 we shall give an introduction to renormalization. We will determine the conditions for a theory to be renormalizable or not. We will then discuss regularization and show several methods how to apply it. After that we will discuss wavefunction renormalization and renormalization in general. We will end this chapter with a derivation of an equation that can be used to determine whether the scale dependence is scale-independent. In chapter 4 where we shall renormalize the ϕ^3 and ϕ^4 theory in four dimensions and verify this. In the last chapter we will then work towards the construction of a toy model in zero dimensions that is renormalizable and has a scale dependence that is scale-independent and conclude that unfortunately this last condition is not satisfied. We will end with two appendices which we will use throughout this thesis.

2 Quantum field theory

Quantum field theory is used to describe interaction between elementary particles. It is usually impossible to make exact predictions in QFT. Very precise predictions, however, have been made with the use of perturbation theory. In perturbation theory UV divergences occur. After a so-called renormalization procedure these divergences may vanish. Many physicist think renormalization is forced upon us by the fact that many calculations give rise to divergent results. This however is wrong. Even if there were no UV divergences we would still have to renormalize. The reason renormalization is unavoidable in QFT is because of the use of perturbation theory.

Since many aspects in QFT are not dependent upon the dimension we will describe QFT in zero dimensions in this chapter. I can't stress enough that the things we will do here can not always be done in higher dimensions! Most of this chapter will be based on the lecture notes of [1]. I include it for explanatory purposes.

Describing QFT in zero dimensions will simplify many things drastically. We will see that some things can be understood in higher dimensions from constructions in zero dimensions. It is therefore useful to begin with this simplification. In zero dimensions quantum fields are assignments of single random numbers. We can at most compute the collection of their moments, which are described by the Green's functions. In the following we shall study Green's functions in more detail and construct the Schwinger-Dyson equation (SDe). We will also discuss Feynman diagrams and how they can be used to organize many calculations. In the end I will show how to perform perturbation theory with Feynman diagrams and that perturbation theory results in a so-called effective action. This action will prove to be a useful tool in the chapters that follow.

2.1 The Schwinger-Dyson equation

2.1.1 Green's functions

The Green's functions are defined by

$$G_n \equiv \langle \varphi^n \rangle \equiv N \int \exp\left(-\frac{S(\varphi)}{\hbar}\right) \varphi^n d\varphi, \qquad n = 0, 1, 2, 3, ... \tag{1}$$

where φ is a quantum field and $S(\phi)$ is the action of the theory. We shall assume that G_n exist for all n, furthermore we must have $G_0 = 1$, which fixes N. We will discuss the Green's functions in terms of their generating function:

$$Z(J) = \sum_{n\geq 0} \frac{1}{n!} J^n G_n \quad . \tag{2}$$

This object is called the path integral, it can be written as

$$Z(J) = N \int \exp\left(\frac{1}{\hbar}[-S(\varphi) + J\varphi]\right) d\varphi \quad . \tag{3}$$

The number J serves here to distinguish the various Green's functions. It is basically a source. Once we know $Z(J)$ we can extract the Green's function in the following way

$$G_n = \left\lfloor \frac{\hbar^n \partial^n}{(\partial J)^n} Z(J) \right\rfloor_{J=0} \quad . \tag{4}$$

2.1.2 The Schwinger-Dyson equation for the integral

Generally the path integral is a complicated function of J. Its however easy to find an equation describing it completely. This equation is called the Swinger-Dyson equation (SDe) mentioned in the introduction of this chapter. In this section we shall construct the SDe. Let us consider the general action

$$S(\varphi) = \sum_{k\geq 1} \frac{1}{k!} \lambda_k \varphi^k \quad .$$

From equation (4) we find that

$$\left[-J + \sum_{k\geq 0} \frac{\lambda_{k+1}}{k!} \frac{\hbar^k \partial^k}{(\partial J)^k}\right] Z(J) = \\ = N \int \exp\left(\frac{1}{\hbar}[-S(\varphi) + J\varphi]\right) \left[S'(\varphi) - J\right] d\varphi \quad . \tag{5}$$

Using partial integration and the fact that the integrand vanishes at the endpoints we know that this expression must be equal to zero. We thus find

$$\left\lfloor \frac{\partial}{\partial \varphi} S(\varphi) \right\rfloor_{\varphi = \frac{\hbar \partial}{\partial J}} Z(J) = J Z(J) \quad . \tag{6}$$

This is the SDe for the path integral.

2.1.3 The field function

The path integral $Z(J)$ contains all information about the Green's function and therefore about the probability density. The same information is naturally also given by its logarithm. When we take the derivative of $\log Z(J)$ with respect to J we find the field function:

$$\phi \equiv \frac{\hbar\partial}{\partial J}\log Z(J) \quad . \tag{7}$$

From equation (4) we know that the field function is basically the expectation value of the quantum field φ. While the quantum field φ is a fluctuating random field and is therefore not computable, the field function ϕ is a well-defined function that contains all the information about the probability density of φ: once the action is given we can compute the field function. We can also construct the SDe for a field function. Using the definition of equation (7) we find

$$\frac{\hbar^p\partial^p}{(\partial J)^p}Z(J) = Z(J)\left(\phi(J) + \frac{\hbar\partial}{\partial J}\right)^p e(J) \quad , \tag{8}$$

where $e(J)$ is the unit function. We can now give the SDe in terms of the field function, this reads:

$$S'\left(\phi(J) + \frac{\hbar\partial}{\partial J}\right)e(J) = J \quad . \tag{9}$$

For a $\varphi^{3/4}$ theory given by the action

$$S(\phi) = \frac{1}{2}\mu\phi^2 + \frac{1}{3!}\lambda_3\phi^3 + \frac{1}{4!}\lambda_4\phi^4 \quad , \tag{10}$$

where λ_3 and λ_4 are called coupling constants and $\mu = \lambda_2$[1], we find

$$\begin{aligned}\phi(J) =& \frac{J}{\mu} - \frac{\lambda_3}{2\mu}\left(\phi(J)^2 + \frac{\hbar\partial}{\partial J}\phi(J)\right) \\ & - \frac{\lambda_4}{6\mu}\left(\phi(J)^3 + 3\phi(J)\frac{\hbar\partial}{\partial J}\phi(J) + \frac{\hbar^2\partial^2}{(\partial J)^2}\phi(J)\right) \quad .\end{aligned} \tag{11}$$

2.2 Feynman diagrams

Computing Green's functions can be organized using Feynman diagrams. In this section we shall introduce these diagrams and how they should be interpreted using Feynman Rules.

A Feynman diagram consist of lines and vertices. Vertices are points where one or more lines end. Lines can either be external or internal. External lines are lines with only one vertex and internal lines are lines with vertices at both ends. Feynman diagrams correspond to numbers that can be multiplied and added. They therefore form an algebra. The values of the diagram depends on the dimension, but the algebra itself does not, and that is why $d = 0$ is useful. The value of any given diagram is independent of the precise shape of the lines and the precise position of the vertices. The values of the Feynman diagrams can be computed by applying the Feynman rules. The Feynman rules depend upon the action. Let us consider the action of a $\varphi^{3/4}$ theory as given in equation (10). The corresponding Feynman rules are

[1] In higher dimensions, μ is related to the mass squared m^2

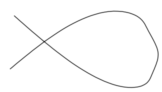
 $\leftrightarrow +\frac{\hbar}{\mu}$

 $\leftrightarrow -\frac{\lambda_3}{\hbar}$

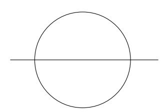
 $\leftrightarrow -\frac{\lambda_4}{\hbar}$

 $\leftrightarrow +\frac{J}{\hbar}$

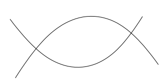
The last rule contains a J which is called the source vertex. The $\hbar$ governs the order of the diagrams, we will discuss this in more detail later. In addition to these rules we need to assign a symmetry factor to each Feynman diagram. The rule is the following: "**for every set of k lines that may be permuted without changing the diagram, there will be a factor 1/k!; for every set of m vertices that may be permuted without changing the diagram, there will be a factor 1/m!; for every set of p disjunct connected pieces that maybe interchanged without changing the diagram, there will be a factor 1/p!.**" [1]. In the following I will work out a few examples. For the first graph

 $$= -\frac{1}{2}\frac{\lambda_4 \hbar^2}{\mu^3} \quad , \tag{12}$$

the symmetry factor is $\frac{1}{2}$ because the loop can be flipped over without changing the diagram. For the next diagram

 $$= \frac{1}{6}\frac{\lambda_4^2 \hbar^3}{\mu^5} \tag{13}$$

we find a symmetry factor of 1/3! because the three internal lines are interchangeable. The last diagram

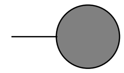
 $$= \frac{3}{2}\frac{\lambda_4^2 \hbar^4}{\mu^6} \tag{14}$$

has a symmetry factor of $\frac{1}{2}$ because the internal lines can be interchanged without changing the diagram. The factor 3 is due to the multiplicity of the diagram. If we number each external line, there are three different ways to do so.

As mentioned in the beginning of this chapter, Feynman diagrams are used to organize the calculations with perturbation theory in QFT. This suggest that a diagram must have a certain order of magnitude. If the coupling constants are in some sense a small number we know from the Feynman rules that the more vertices a diagram contains the smaller its contribution will be. Diagrams with the same number of external legs contain more coupling constants if they contain more closed loops. We therefore included the $\hbar$ in the Feynman rules in a way that the power of $\hbar$ is dependent upon the number of closed loops. A perturbation expansion is then truncated at a given number of closed loops.

2.3 Schwinger-Dyson equation as a set of diagrams

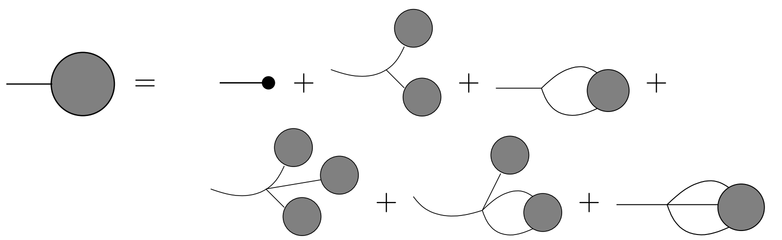
We want to construct the SDe using Feynman diagrams. In order to do this we must first make a few constructions. First lets define the object $\psi(J)$ to be the set of all connected diagrams with precisely one external line and any number of source vertices. We shall represent $\psi(J)$ with the following diagram:

$$\psi(J) = \quad$$  $$. \qquad (15)$$

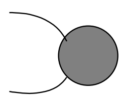
The shading indicates that all the diagrams in the blob must be connected. Algebraically we then have

$$\psi(J) = \sum_{n\geq 0} \frac{1}{n!} J^n C_{n+1} \quad , \qquad (16)$$

where C is the set of all connected graphs with no source vertices and precisely n external lines. The factor $n!$ in this equation is the additional symmetry factor for the n source vertices. When we enter the blob in $\psi(J)$ we may encounter a source. If not, we encounter a vertex. In the case of a three-point vertex the line splits in two. Taking one of these branches we may either come back to the vertex via the other branch or it may happen that the two branches end up in disjunct connected pieces of the diagram. For four-point interactions, analogous arguments holds. In the case of a $\varphi^{3/4}$ theory we then find up to second order

 (17)

We realize that

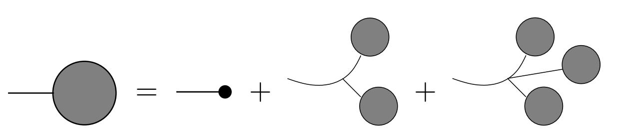
 $$= \sum_{n\geq 0} \frac{1}{n!} J^n C_{n+2} = \frac{\hbar\partial}{\partial J}\psi(J) \quad \text{, et cetera,} \qquad (18)$$

so that we find the algebraic equation for $\psi(J)$ by applying the Feynman rules. We find

$$\begin{aligned} \psi(J) = & \frac{J}{\mu} - \frac{\lambda_3}{2\mu}\left(\psi(J)^2 + \frac{\hbar\partial}{\partial J}\psi(J)\right) \\ & - \frac{\lambda_4}{6\mu}\left(\psi(J)^3 + 3\psi(J)\frac{\hbar\partial}{\partial J}\psi(J) + \frac{\hbar^2\partial^2}{(\partial J)^2}\psi(J)\right) \quad , \end{aligned} \qquad (19)$$

which is identical to equation (11). So in general we can find all connected Green's functions by summing up all the connected Feynman diagrams with no source vertices and precisely n external lines and applying the Feynman rules.

In the classical limit $\hbar \to 0$ we find the tree approximation of the SDe. In this case higher-loop diagrams become very small so the diagrammatic equation becomes

 (20)

The SDe will then take the form of

$$\psi(J) = \frac{J}{\mu} - \frac{\lambda_3}{2\mu}\psi(J)^2 - \frac{\lambda_4}{6\mu}\psi(J)^3 \to S'(\psi) = J \quad . \qquad (21)$$

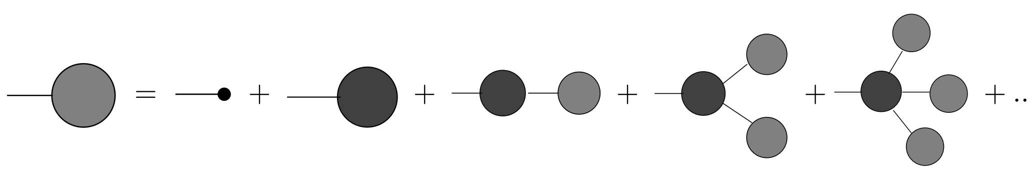
In the following section we will discuss the effective action which will prove to be very useful later in this thesis.

2.4 The effective action

The effective action ($\Gamma(\phi)$) has the property that *its* tree approximation reproduces the full field function of the original action S. Therefore we must have

$$\Gamma(\phi)' = J \quad . \tag{22}$$

In order to compute the effective action we define a new concept, namely that of a one-particle irreducible (1PI) diagram. A Feynman diagram is 1PI if it contains no internal line such that cutting that line makes the diagram disconnected. Let us denote the set of 1PI graphs with n external lines by $-\gamma_n/\hbar$. Consider now what happens if we enter the field function by way of its single external leg, as in the SDe. If we encounter a vertex, that vertex is part of a 1PI subdiagram (maybe only the vertex itself). Indicating the 1PI property with the darker blob we obtain the following equation [1]


(23)

Algebraically this reads

$$\phi(J) = \frac{J}{\mu} - \frac{1}{\mu}\left(\gamma_1 + \gamma_2\phi(J) + \frac{1}{2}\gamma_3\phi(J)^2 + \frac{1}{3!}\gamma_4\phi(J)^3 + ...\right) \quad . \tag{24}$$

This is basically equation (22), where

$$\Gamma(\phi) = \gamma_1\phi + \frac{1}{2}(\gamma_2 + \mu)\phi^2 + \frac{1}{3!}\gamma_3\phi^3 + \frac{1}{4!}\gamma_4\phi^4 + ... \quad . \tag{25}$$

Since we can also derive equation (24) without Feynman diagrams it follows we can compute the effective action without Feynman diagrams. When we know the effective action we can know all γ_n and therefore we can use the effective action as a check to make sure we have found all the 1PI diagrams with a certain number of external lines. Since the number of diagrams is independent of the dimension the effective action will also be useful when we need to find all the 1PI diagrams in higher dimensions.

2.5 Effective action for $\phi^{3/4}$ theory

Later in this thesis we will renormalize the ϕ^3 and the ϕ^4 theory in four dimensions. Also we will try to construct a toy model in zero dimensions that assigns divergences to diagrams in such a way that it is renormalizeable. We therefore need to find all connected diagrams. In this section I will therefore derive the effective action of the $\phi^{3/4}$ theory in order to check we have found all the 1PI diagrams in both the ϕ^3 and ϕ^4 theory. In order to compute the effective action let us first consider the SDe

$$\begin{aligned}\phi(J) =& \frac{J}{\mu} - \frac{\lambda_3}{2\mu}\left(\phi(J)^2 + \hbar\frac{\partial}{\partial J}\phi(J)\right) \\ &- \frac{\lambda_4}{6\mu}\left(\phi(J)^3 + 3\hbar\phi(J)\frac{\partial}{\partial J}\phi(J) + \hbar^2\frac{\partial^2}{(\partial J)^2}\phi(J)\right) \quad .\end{aligned} \tag{26}$$

In order to compute the effective action we will use a perturbation expansion

$$\Gamma(\phi) = \Gamma_0(\phi) + \hbar\Gamma_1(\phi) + \hbar^2\Gamma_2(\phi) + ... \quad , \tag{27}$$

where $\Gamma'(\phi) = J$ and therefore

$$\Gamma''(\phi)\phi' = 1 \to \phi' = \frac{1}{\Gamma''(\phi)} \quad ,$$
$$\Gamma'''(\phi)\phi'^2 + \Gamma''(\phi)\phi'' = 0 \to \phi'' = -\frac{\Gamma'''(\phi)}{\Gamma''(\phi)^3} \quad . \tag{28}$$

If we rewrite the SDe in terms of Γ we find

$$\Gamma_0'(\phi) + \hbar\Gamma_1'(\phi) + \hbar^2\Gamma_2'(\phi) + \hbar^3\Gamma_3' = \mu\phi + \frac{\lambda_3}{2}\phi^2 + \frac{\lambda_4}{6}\phi^3 + \frac{\hbar(\lambda_3 + \phi\lambda_4)}{2\Gamma_0''} - \frac{\hbar^2\lambda_4}{6\Gamma_0''^2}\frac{\Gamma_0'''}{\Gamma_0''} -$$
$$\frac{\hbar^2(\lambda_3 + \phi\lambda_4)}{2\Gamma_0''}\frac{\Gamma_1''}{\Gamma_0''} - \frac{\hbar^3(\lambda_3 + \phi\lambda_4)}{2\Gamma_0''}\frac{\Gamma_2''}{\Gamma_0''} + \frac{\hbar^3(\lambda_3 + \phi\lambda_4)}{2\Gamma_0''}\left(\frac{\Gamma_1''}{\Gamma_0''}\right)^2 -$$
$$\frac{\hbar^3\lambda_4}{6\Gamma_0''^2}\left(\frac{\Gamma_1'''}{\Gamma_0''}\right) + \frac{\hbar^3\lambda_4}{2\Gamma_0''^2}\left(\frac{\Gamma_0'''\Gamma_1''}{\Gamma_0''^2}\right) \quad . \tag{29}$$

By expanding in $\hbar$, the successive $\Gamma_n(\phi)$ can be read off. Some (computer) algebra then gives

$$\gamma_1 = \frac{1}{2}\frac{\lambda_3\hbar}{\mu} - \frac{\lambda_3\hbar^2}{12\mu^4}(5\lambda_4\mu - 3\lambda_3^2) + \frac{5\lambda_3\hbar^3}{24\mu^7}(3\lambda_3^4 + 4\lambda_4^2\mu^2 - 8\lambda_4\mu\lambda_3^2)$$
$$\gamma_2 = \mu + \frac{\hbar}{2\mu^2}(\lambda_4\mu - \lambda_3^2) - \frac{\hbar^2}{12\mu^5}(-24\lambda_4\mu\lambda_3^2 + 12\lambda_3^4 + 5\lambda_4^2\mu^2) +$$
$$\frac{5\hbar^3}{24\mu^8}(-21\lambda_3^6 - 44\lambda_4^2\mu^2\lambda_3^2 + 63\lambda_4\mu\lambda_3^4 + 4\lambda_4^3\mu^3)$$
$$\gamma_3 = \lambda_3 - \frac{\lambda_3\hbar}{2\mu^3}(3\lambda_4\mu - 2\lambda_3^2) + \frac{\lambda_3\hbar^2}{4\mu^6}(21\lambda_4^2\mu^2 - 48\lambda_4\mu\lambda_3^2 + 20\lambda_3^4\mu^2) -$$
$$\frac{5\lambda_3\hbar^3}{8\mu^9}(189\lambda_4\mu\lambda_3^4 - 56\lambda_3^6 + 36\lambda_4^3\mu^3 - 172\lambda_4^2\mu^2\lambda_3^2)$$
$$\gamma_4 = \lambda_4 - \frac{3\hbar}{2\mu^4}(\lambda_4^2\mu^2 - \lambda_4\mu\lambda_3^2 + 2\lambda_3^4) + \frac{\hbar^2}{4\mu^7}(-228\lambda_4^2\mu^2\lambda_3^2 + 340\lambda_4\mu\lambda_3^4 - 120\lambda_3^6 + 21\lambda_4^3\mu^3) -$$
$$\frac{5\hbar^3}{8\mu^{10}}(-1904\lambda_3^6\lambda_4\mu + 504\lambda_3^8 - 732\lambda_3^2\lambda_4^3\mu^3 + 2149\lambda_3^4\lambda_4^2\mu^2 + 36\lambda_4^4\mu^4) \quad . \tag{30}$$

With these equations we can determine if we have all diagrams with a certain number of legs. When we work in ϕ^3 we can also use this, we only have to take λ_4 to zero. In the case of the ϕ^4 theory we have to take λ_3 to zero and we find the effective action for that theory. In a later chapter we will use this to determine if we indeed have all the connected diagrams up to order three.

3 Renormalization

In QFT the naked parameters in the action are not the same as the ones measured in the lab. These differences arise from loop integrals of the Feynman diagrams. The loops in Feynman diagrams contain virtual particles that interact with each other. Since we can't observe these virtual particles, the naked parameters differ from the measured parameters in the lab. Renormalization establishes a relationship between the naked parameters in a theory and the physical (renormalized) parameters. In QFT this relationship often contains divergences.

In the following we will first define the Feynman rules for higher dimensions, since we need them throughout this chapter. After that we will show how to determine whether a theory is renormalizable or not. Then we will show how to regularize diagrams. Regularization is needed to tame the divergences, which is needed in order to establish a relationship between the naked and renormalized parameters. After that we will show how to rescale the field itself with wavefunction renormalization. We will then give a prescription on how to determine the renormalized parameters.

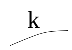
Since the naked parameters v_j $(j = 1, ..., n)$ are fixed, the renormalized parameters w_j must depend on the scale denoted by s, which contains the divergences and the scaling factor μ. Furthermore we want this dependence to be of the form

$$\frac{d}{ds} w_j \equiv \beta_j(w_1, ..., w_n) \quad . \tag{31}$$

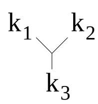
In other words we want a scale-dependence that is independent of the scale. In section 3.7 we will derive a differential equation (the pCS equation) that we can use to check if we indeed have a scale-independent scale dependence.

3.1 Euclidean Feynman rules for $\phi^{3/4}$ in d dimensions

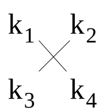
In zero dimensions we don't need regularization since applying the Feynman rules never results in divergences. In d dimensions the Feynman rules however are different than in zero dimensions. In this case the Feynman rules are



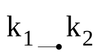
$$\leftrightarrow \frac{\hbar}{|\vec{k}|^2 + m^2}$$



$$\leftrightarrow -\frac{\lambda_3}{\hbar}(2\pi)^D \delta^D(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)$$



$$\leftrightarrow -\frac{\lambda_4}{\hbar}(2\pi)^D \delta^D(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4)$$



$$\leftrightarrow +\frac{J(\vec{k}_2)}{\hbar}(2\pi)^D \delta^D(\vec{k}_1 + \vec{k}_2)$$

At vertices, the wavevectors are considered either all incoming or all outgoing. Each internal wave vector $\vec{k}$ is to be integrated over, with integration element $d^D\vec{k}/(2\pi)^D$ [1] .

3.2 Renormalizability

In QFT we mostly like to work with dimensionless combinations of λ, $\hbar$ and m because in this case its easier to compare diagrams of different order. From the action we know that

$$\int d^d x (\partial_\mu \phi)^2 \sim \int d^d x \lambda_n \phi^n \sim \hbar \quad , \tag{32}$$

where d is the dimension, $\lambda_2 = m^2$ and $\sim$ means that the expression on the left side has the same dimensions has the expression on the right side. We have $x \sim L$ where L is a length scale. From this we know that

$$\phi^2 L^{d-2} \sim \hbar \rightarrow \phi \sim \hbar^{1/2} L^{1-d/2} \tag{33}$$

$$\lambda_n \phi^n L^d \sim \hbar \rightarrow \lambda_n \sim \hbar^{1-\frac{n}{2}} L^{-n-d+\frac{nd}{2}} \quad . \tag{34}$$

Since $m = Mc/\hbar$ is the inverse Compton wavelength, where M is the mass in kilograms, we know we should have $m \sim L^{-1}$ because of the kinetic term and when we take $n = 2$ we indeed find this. Using these results we find that the dimensionless combination is equal to (a power of)

$$\frac{\hbar^{\frac{n}{2}-1}\lambda_n}{m^{n+d-\frac{nd}{2}}} \quad . \tag{35}$$

This results in the following table

d	$n=3$	$n=4$
0	$\hbar\lambda_3^2/m^6$	$\hbar\lambda_4/m^4$
1	$\hbar\lambda_3^2/m^5$	$\hbar\lambda_4/m^3$
2	$\hbar\lambda_3^2/m^4$	$\hbar\lambda_4/m^2$
3	$\hbar\lambda_3^2/m^3$	$\hbar\lambda_4/m$
4	$\hbar\lambda_3^2/m^2$	$\hbar\lambda_4$
5	$\hbar\lambda_3^2/m$	$\hbar\lambda_4 m$
6	$\hbar\lambda_3^2$	$\hbar\lambda_4 m^2$

Dimensionless combinations for various values of n and d.

These dimensionless combinations can also explain why some theories are non-renormalizable and others are super-renormalizable or renormalizable. Take for instance the case of $n = 3$ (ϕ^3 theory) in four dimensions. In this case the dimensionless combination is proportional to $1/m^2$. Using the Feynman rules for four dimensions we can only get a $1/m^2$ with a propagator $1/(p^2+m^2)$. So basically we have the combination $\hbar\lambda_3^2/E^2$, which is very small at high energies. Thus the ϕ^3 theory in four dimensions is super-renormalizable. In a super-renormalizable theory the diagrams get less and less divergent if the loop order increases.

When we would have taken the ϕ^5 theory in four dimensions we would find a combination that is proportional to E, which blows up at higher energies. The ϕ^5 theory is therefore non-renormalizable in four dimensions. In a non-renormalizable theory there are an infinite number of superficially diverging diagrams. Adding counterterms (to be discussed later) in this case results in new divergences.

The ϕ^4 theory is renormalizable, because the combination $\hbar\lambda_4$ is already dimensionless. In this case divergences occur at all orders, but only a finite number of (sub)diagrams diverge superficially [2].

Note that the combination $\hbar^{n/2-1}\lambda_n$ can only be dimensionless if $n + d - nd/2 = 0$. This means that only the ϕ^3 in six dimensions and the ϕ^4 theory in four dimensions are renormalizable. All other purely scalar theories are either super-renormalizable or non-renormalizable.

3.3 Regularization

We must first tame the divergences in order to determine how to construct the renormalized parameter. This process is called regularization. The easiest way to tame divergences is to define a cut-off Λ, where we demand that all energies and momenta above the cut-off will not contribute. Using this cut-off, the divergences become convergent and we can continue with the calculations. At the end of the calculations we take the limit $\Lambda \to \infty$ to find the desired result. We can interpret this as only knowing physics up to a certain scale.

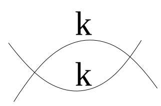
There are many other ways to regularize integrals. Two of them will be discussed in the following of this section.

The first one we will discuss is Pauli-Villars regularization [3]. With this procedure we add imaginary particles in such a way that the divergences of these particles cancel the divergences of the real particles.

The other procedure we will discuss is dimensional regularization of [4]. With this procedure we fix the number of dimensions at an arbitrary complex number. At the end of the calculation we substitute this complex number by $4-2\epsilon$, where ϵ is a fractional dimensions. By expanding in $\epsilon \to 0$ we find the divergent behavior of the integrals.

3.3.1 Pauli-Villars Regularization

Let us consider the following logarithmically divergent Feynman diagram:

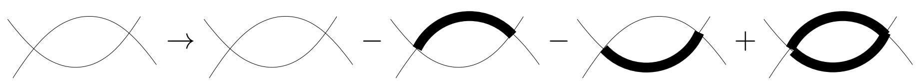
$F_1 =$ 

. (36)

Applying the Feynman rules yield

$$F_1 = \frac{3\lambda^2\hbar}{2(2\pi)^D}\int d^Dk \frac{1}{(k^2+m^2)^2} \quad . \tag{37}$$

This integral diverges for $D \geq 4$. With Pauli-Villars we add propagators in such a way it tames the diverging integral. We can represent this diagrammatically in the following way


. (38)

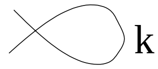
The propagator then changes to

$$\left(\frac{1}{k^2+m^2} - \frac{1}{k^2+M^2}\right)^2 = \left(\frac{M^2-m^2}{(k^2+m^2)(k^2+M^2)}\right)^2 \quad . \tag{39}$$

Note that when we take M to infinity we recover the old propagator. With Pauli-Villars we must replace each propagator in such a way that there is no momentum term in the numerator. Basically what we do is subtracting a propagator for each diverging propagator in the diagram. Substituting this new propagator for our old propagator yields

$$\begin{aligned}
F_1 &= \frac{3\lambda^2\hbar}{2(2\pi)^4}\int d^4k \left(\frac{M^2-m^2}{(k^2+m^2)(k^2+M^2)}\right)^2 \\
&= \frac{3\lambda^2\hbar}{2(2\pi)^4}\frac{\Gamma(4)}{\Gamma(2)^2}\int_0^1 dx_1\, dx_2\, \delta(x_1+x_2-1)\int d^4k \frac{x_1x_2(M^2-m^2)^2}{(k^2+x_1m^2+x_2M^2)^4} \\
&= \frac{3\lambda^2\hbar}{2(4\pi)^2}\frac{\Gamma(4)}{\Gamma(2)^2\Gamma(2)}\int_0^1 dx_1\, dx_2\, \delta(x_1+x_2-1)\Delta^2 \int ds \frac{sx_1x_2}{(s+x_1m^2+x_2M^2)^4} \\
&= \frac{3\lambda^2\hbar}{2(4\pi)^2}\Gamma(2)\int_0^1 dx_1\, dx_2\, \delta(x_1+x_2-1)x_1x_2\Delta^2(x_1m^2+x_2M^2)^{-2} \\
&= \frac{3\lambda^2\hbar}{2(4\pi)^2}\frac{1}{\Delta}\int_{m^2}^{M^2} dz \frac{(z-m^2)(M^2-z)}{z^2} \\
&\approx \frac{3\lambda^2\hbar}{2(4\pi)^2}\left[\log\left(\frac{M^2}{m^2}\right) - 2\right] \quad ,
\end{aligned} \tag{40}$$

where we used the substitutions $\Delta = M^2 - m^2$ and $z = \Delta x_2$. In the last step we used that M^2 is much larger than m^2. From this result we indeed find a logarithmic divergence. Let us now consider a quadratically divergent diagram:

$$F_2 = $$ 

$$. \tag{41}$$

Applying the Feynman rules yield

$$\frac{-\lambda\hbar}{2(2\pi)^D}\int d^D k \frac{1}{k^2+m^2} \quad . \tag{42}$$

This integral diverges for $D \geq 2$. With Pauli villars we find

$$\frac{1}{k^2+m^2} - \frac{1}{k^2+M^2} = \frac{M^2-m^2}{(k^2+m^2)(k^2+M^2)} \quad . \tag{43}$$

The integral we started with diverges in two dimensions. The modified integral still diverges in four dimensions. Apperently subtracting one propagator is not enough. We thus have to subtract a tower of propagators like in [5] where we take the limit of $M_1 = M_2$. We then find

$$\frac{1}{k^2+m^2} - \frac{A_1}{k^2+M^2} - \frac{A_2}{(k^2+M^2)^2} - \frac{A_3}{(k^2+M^2)^3} - \frac{A_4}{(k^2+M^2)^4} - \quad ... \tag{44}$$

We still have the condition that the A_n must be chosen in such a way there is no momentum in the numerator. We then find $A_n = \Delta^{n-1}$, so the propagator becomes

$$\frac{1}{k^2+m^2} - \frac{1}{k^2+M^2} - \frac{\Delta}{(k^2+M^2)^2} - \frac{\Delta^2}{(k^2+M^2)^3} - \frac{\Delta^3}{(k^2+M^2)^4} - \quad ... \tag{45}$$

Note that if we add all of the Pauli-Villars propagators we find

$$\sum_{n=0}^{\infty} \frac{\Delta^n}{(k^2+M^2)^{n+1}} = \frac{1}{k^2+m^2} \quad , \tag{46}$$

Which exactly cancels the entire propagator. In four dimensions we do not need all of these A_n, up to A_2 is enough to find something finite. Note that this is not the case if we were for instance in six dimensions. In this case diagram F_1 becomes quadratically divergent instead of logarithmic and diagram F_2 becomes quartically divergent. We will however not focus on six dimensions. In this case the propagator becomes

$$\frac{1}{k^2+m^2} - \frac{1}{k^2+M^2} - \frac{M^2-m^2}{(k^2+M^2)^2} = \frac{(M^2-m^2)^2}{(k^2+m^2)(k^2+M^2)^2} \quad . \tag{47}$$

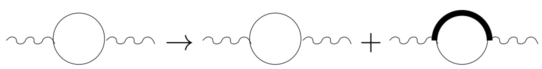
This results in the following integral

$$\begin{aligned}
F_2 &= \frac{-\lambda\hbar}{2(2\pi)^4}\int d^4 k \frac{(M^2-m^2)^2}{(k^2+m^2)(k^2+M^2)^2} \\
&= \frac{-\lambda\hbar}{2(2\pi)^4}\frac{\Gamma(3)}{\Gamma(2)}\int_0^1 dx_1\, dx_2\, \delta(x_1+x_2-1)\int d^4 k \frac{x_2\Delta^2}{(k^2+x_1 m^2+x_2 M^2)^3} \\
&= \frac{-\lambda\hbar}{2(4\pi)^2}\frac{\Gamma(3)}{\Gamma(2)\Gamma(2)}\int_0^1 dx_1\, dx_2\, \delta(x_1+x_2-1)\Delta^2 \int ds \frac{x_2 s}{(s+x_1 m^2+x_2 M^2)^3} \\
&= \frac{-\lambda\hbar}{2(4\pi)^2}\Gamma(1)\int_0^1 dx_1\, dx_2\, \delta(x_1+x_2-1)x_2\Delta^2(x_1 m^2+x_2 M^2)^{-1} \\
&= \frac{-\lambda\hbar}{2(4\pi)^2}\int_{m^2}^{M^1} dz \frac{z-m^2}{z} \\
&= \frac{-\lambda\hbar}{2(4\pi)^2}\left[\Delta - m^2 \log\left(\frac{M^2}{m^2}\right)\right] \quad ,
\end{aligned} \tag{48}$$

where we ones again used the substitutions $\Delta = M^2 - m^2$ and $z = \Delta x_2$. We find that this diagram indeed diverges quadratically. Pauli-Villars thus appears a useful method to regularize diagrams and determine their divergent behaviour. One problem with Pauli-Villars is that we do not know how many propagators to subtract. We just subtract some and notice that the result is finite. We wish that the result should not depend on the number of propagators we subtract. Of course in the case of the quadratic divergent diagram of (41) we had to subtract at least twice to find something finite. When we use this additional subtraction from the logarithmically divergent diagram of (36) we find

$$F_1 = \frac{3\lambda^2\hbar}{2(4\pi)^2}\left[\log\left(\frac{M^2}{m^2} - \frac{17}{6}\right)\right] \ , \tag{49}$$

which is not the same. Another problem with Pauli-Villars regularization is that it destroys gauge invariance in gauge theories. To show this let us consider quantum electrodynamics (QED). In quantum electrodynamics we can for instance construct the following diagram

$F =$  $+ \dots$. (50)

The curly line represents a photon whereas the other two lines are propagators with equal mass. Gauge invariance in this case depends on these internal propagators having equal mass. Using Pauli-Villars regularization we change each propagator so the two propagators may not have the same mass anymore (as show after the arrow), resulting in a violation of gauge invariance. Another disadvantage of Pauli-Villars is that it becomes really hard to do at higher order.

3.3.2 Dimensional Regularization

Let us again consider diagram (36). We will now use dimensional regularization to solve the integral of this diagram, aiming at finally ending up with $d = 4$. From appendix A we find, with $s = p^2$,

$$\begin{aligned} F_1 = \frac{3\lambda^2\hbar\mu^{4-2\omega}}{2(2\pi)^{2\omega}}\int d^{2\omega}p\frac{1}{(\mathbf{p}^2+m^2)^2} &= \frac{(\mu^2)^{2-\omega}}{(4\pi)^\omega\Gamma(\omega)}\int ds\frac{s^{\omega-1}}{(s+m^2)^2} \\ &= \frac{3\lambda^2\hbar(\mu^2)^{2-\omega}}{2(4\pi)^\omega\Gamma(\omega)}(m^2)^{\omega-2}\frac{\Gamma(\omega)\Gamma(2-\omega)}{\Gamma(2)} \\ &= \frac{3\lambda^2\hbar(\mu^2)^{2-\omega}}{2(4\pi)^\omega}(m^2)^{\omega-2}\Gamma(2-\omega) \ , \end{aligned}$$

where μ is a scaling parameter. The scaling parameter is needed to create a dimensionless coupling. In other dimensions its magnitude would be different, since the dimensions of the coupling constants are different in this case. Substituting $d = 4 - 2\epsilon$ yields

$$F_1 = \frac{3\lambda^2\hbar}{2(4\pi)^2}\left(\frac{1}{\epsilon} - L + \mathcal{O}(\epsilon)\right) \ , \tag{51}$$

where $L = \log(m^2/\mu^2) + \gamma - \log(4\pi)$, γ being Euler's constant. Note that using this substitution the table of dimensionless couplings for the ϕ^4 theory in four dimensions as introduced in section 3.2 changes to $\lambda/m^{2\epsilon}$. This mass is found in the logarithmic term. We can also define the dimensionless coupling $g = \lambda\mu^{-2\epsilon}$. This is another way to introduce μ in our equations. We can now easily deduce that μ must therefore be an arbitrary mass parameter. Let us now consider the quadratically divergent diagram from the previous section denoted by F_2. Using dimensional regularization this diagram yields

$$
\begin{aligned}
F_2 = -\frac{\lambda\mu^{4-2\omega}\hbar}{2(2\pi)^{2\omega}}\int d^{2\omega}k\frac{1}{k^2+m^2} &= -\frac{\lambda(\mu^2)^{2-\omega}\hbar}{2(4\pi)^{\omega}\Gamma(\omega)}\int ds\frac{s^{\omega-1}}{s+m^2} \\
&= -\frac{\lambda(\mu^2)^{2-\omega}\hbar}{2(4\pi)^{\omega}\Gamma(\omega)}(m^2)^{\omega-1}\frac{\Gamma(\omega)\Gamma(1-\omega)}{\Gamma(1)} \\
&= -\frac{\lambda(\mu^2)^{2-\omega}\hbar}{2(4\pi)^{\omega}}(m^2)^{\omega-1}\Gamma(1-\omega)\ . \qquad (52)
\end{aligned}
$$

After expanding around $\epsilon = 0$ this reduces to

$$
F_2 = \frac{\lambda m^2\hbar}{2(4\pi)^2}\left(\frac{1}{\epsilon}+1-L+\mathcal{O}(\epsilon)\right)\ , \qquad (53)
$$

Dimensional regularization is usually easier to compute and is useful to keep track of the divergences. This regularization is the most widely used.

3.4 Wavefunction Renormalizaton

In order to renormalize a theory we also need to rescale the quantum field to take into account the effect of interactions. This is called wavefunction renormalization. In this section we shall explain how wavefunction renormalization works and how to compute it. Let us consider the ϕ^4 theory with the Lagrangian

$$
\mathcal{L} = (\partial\phi)^2/2 - m^2\phi^2/2 - \lambda\phi^4/4!\ , \qquad (54)
$$

where $\mathcal{L}$ is written in bare quantities. In order to get physical quantities we have to rescale the field by

$$
\phi = Z_\phi^{1/2}\phi_R\ . \qquad (55)
$$

The Lagrangian in terms of the renormalized field ϕ_R is

$$
\mathcal{L} = Z_\phi(\partial\phi_R)^2/2 - m^2 Z_\phi\phi_R^2/2 - \lambda Z_\phi^2\phi_R^4/4!\ . \qquad (56)
$$

In order to compute Z_ϕ we need to take into account the external momentum. We may define the renormalized mass in the following way

$$
p^2 - m^2 - \Sigma(p^2)|_{p^2=m_R^2} \equiv 0\ , \qquad (57)
$$

where $\Sigma(p^2)|_{p^2=m_R^2}$ is the value of the Feynman diagram we perform wavefunction renormalization on at $p^2 = m_R^2$. This meets that we use the so-called *pole mass* prescription: the mass is that value for p^2 for which the denominator of the propagator vanishes. The propagator then becomes

$$
\frac{1}{p^2-m^2-\Sigma(p^2)} = \frac{1}{p^2-m_R^2-\Sigma(p^2)+\Sigma(m_R^2)}\ . \qquad (58)
$$

Performing a Taylor expansion of $\Sigma(p^2)$ around $p^2 = m^2$ yields

$$
\begin{aligned}
&\frac{1}{(p^2-m_R^2)[1-\Sigma'(m_R^2)+\frac{1}{2}(p^2-m_R^2)\Sigma''(m_R^2)+...]} = \\
&\frac{1}{(p^2-m_R^2)(1-\Sigma'(m_R^2))}\left[1-\frac{1}{2}(p^2-m_R^2)\frac{\Sigma''(m_R^2)}{\Sigma'(m_R^2)}+...\right]\ . \qquad (59)
\end{aligned}
$$

We can neglect $\Sigma''(m_R^2)$ and terms with higher order since they don't lead to poles and therefore do not affect the propagation of on-shell (external) particles. We find

$$\frac{1}{(p^2 - m_R^2)} \frac{1}{\sqrt{1 - \Sigma'(m_R^2)}} \frac{1}{\sqrt{1 - \Sigma'(m_R^2)}} = \frac{1}{(p^2 - m_R^2)} \sqrt{Z_\phi}\sqrt{Z_\phi} \ . \tag{60}$$

We can interpret this in the following way: one part of Z_ϕ goes to the beginning of a propagator and the other part to the end [6]. Note that therefore only two-leg diagrams will contribute to Z_ϕ. We will compute Z_ϕ for the sunset diagram in the next chapter. In order to renormalize a theory it is not enough to rescale the field only. We need to rescale all the parameters, including λ and μ. We can view the renormalization as

$$\begin{aligned} \mathcal{L} =& (\partial\phi_R)^2/2 - m^2\phi_R^2/2 - \lambda_R\phi_R^4/4! \\ &+ \delta Z_\phi \delta\phi_R^2/2 - \delta m_R^2\phi_R^2/2 - \delta\lambda_R\phi_R^4/4! \ . \end{aligned} \tag{61}$$

In this view the first three terms are the basic Lagrangian and the last three the counterterm Lagrangian. The counterterms $\delta Z_\phi = Z_\phi - 1$, $\delta\mu_R = \mu - \mu_R$ and $\delta\lambda_R = \lambda - \lambda_R$ are adjusted to cancel the divergences. This form is very useful when performing perturbation theory. The expansion is done in powers of the dimensionless coupling λ_R and the counterterms are expanded in infinite series where each term cancels the divergence of a collection of diagrams with a fixed number of legs up to a certain order [7]. In section 3.6 we will show how to determine the counterterms, but first we dedicate a section on constructing the renormalized parameters without the use of counterterms.

3.5 Prescription for renormalized parameters

When we have regularized the diagrams and determined the wavefunction renormalization we need a prescription for the renormalized parameters. Recalling that using dimensional regularization we ended up with an expression that contained terms of $1/\epsilon$, which were basically the divergences and we had logarithms of the scaling constant μ. With the minimal subtraction (MS) scheme one only absorbs the $1/\epsilon$ terms, but ignores μ. More widely used is the modified minimal subtraction ($\overline{\text{MS}}$)-scheme. In this scheme both the $1/\epsilon$ and μ will be absorbed, so we use L as in equation (51). In this thesis we shall mainly use the ($\overline{\text{MS}}$)-scheme. The renormalized parameters are thus constructed in the following way:

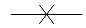
$$m_R^2 = \frac{1}{Z}\left(m^2 - \text{first order two-leg diagrams} - \text{second order two-leg diagrams} - \mathcal{O}(3)\right) \tag{62}$$

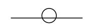
$$\lambda_{nR} = \frac{1}{\sqrt{Z}^n}\left(\lambda_n - \text{first order n leg diagrams} - \text{second order n leg diagrams} - \mathcal{O}(3)\right) \ . \tag{63}$$

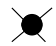
We still have to check if these renormalized parameters result in a well-defined theory. At the end of this chapter we will derive a differential equation to check this. First we shall introduce another methode to construct renormalized parameters using counterterms.

3.6 Method of Counterterms

In section 3.4 we constructed a langrangian with counterterms. These counterterms are adjusted to cancel the divergences. In this section we will explain in more detail how this works following the method of [8]. Let us consider the ϕ^4 theory. In section 3.4 we saw that in this case there are three parameters we need to renormalize, namely the field ϕ, the mass m and the coupling λ. Instead of λ we shall use the dimensionless coupling $g = \lambda\mu^{-2\epsilon}$. We thus have the counterterms c_ϕ, c_{m^2} and c_g, which have the form

 $$= (-c_{m^2})m^2 \quad , \tag{64}$$

 $$= (-c_\phi)\mathbf{p}^2 \quad , \tag{65}$$

 $$= (-c_g)g\mu^{2\epsilon} \quad , \tag{66}$$

The Lagrangian in terms of these counterterms and renormalized parameters is then given by

$$\mathcal{L} = Z_\phi \frac{1}{2}(\partial\phi)^2 - Z_{m^2}\frac{1}{2}m^2\phi^2 - Z_g\frac{\mu^{2\epsilon}g}{4!}\phi^4 \quad , \tag{67}$$

where Z_ϕ, Z_{m^2} and Z_g are the renormalization constants. The renormalization constant are defined as

$$Z_\phi \equiv 1 + c_\phi, \;\; Z_{m^2} \equiv 1 + c_{m^2}, \;\; Z_g \equiv 1 + c_g \quad . \tag{68}$$

The relation between the bare and the renormalized parameters are then given by

$$\phi_B \equiv Z_\phi^{1/2}\phi, \;\; m_B^2 \equiv \frac{Z_{m^2}}{Z_\phi}m^2, \;\; g_B \equiv \frac{Z_g}{Z_\phi^2}\mu^{2\epsilon}g \quad . \tag{69}$$

Note that if we insert this into equation (67) we retrieve the Lagrangian of the bare parameters given by equation (54).

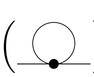
In chapter 4 we shall work out the counterterm method following the MS-scheme. The reason for this is that the counterterms in this case become *independent* of the mass m, except for a trival overall factor m^2 in c_{m^2} [9]. In chapter 4 we will see that this is indeed the case and that also the logarithmic terms containing the combinations of μ, 4π and m^2 will all vanish as well. A handy tool when using the MS-scheme is the operator $\mathcal{K}$. This operator picks out the pure pole terms of the dimensionally regularized integral:

$$\mathcal{K}\sum_{n=-k}^{\infty} A_i\epsilon^i = \sum_{n=-k}^{-1} A_i\epsilon^i = \sum_{i=1}^{k}\frac{A_{-i}}{\epsilon^i} \quad . \tag{70}$$

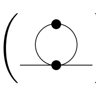
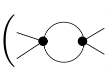
This sum stands for the general Laurent series.. It follows that

$$\mathcal{K}^2 = \mathcal{K} \quad . \tag{71}$$

Therefore $\mathcal{K}$ is by definition a projection operator. Applying $\mathcal{K}$ to a diagram means we are left with only its divergent terms. For example we find

$\mathcal{K}\Big($  $\Big)$ $$= m^2\left[\frac{g\hbar}{2(4\pi)^2}\frac{1}{\epsilon}\right] \quad . \tag{72}$$

The $\mu^{2\epsilon}$ is absorbed in m^2 in this case, in the four leg we can not do this as we shall see in the next chapter. Note that since there are no external momenta in the pole term on the right-hand side of equation (72) we have the following relation:

$\mathcal{K}\Big($  $\Big) = \mathcal{K}\Big($  $\Big) \quad .$ (73)

This property will prove to be important later on. In the next chapter we will see that using the counterterm method following the MS-scheme will result in a simpler relationship between the bare and renormalized parameters. From this relationship we wish to derive the β-function that describes the running of the coupling. This β-function should be independent of the scale and is commonly know for the scalar ϕ^4 theory. In chapter four we shall compute this β-function, but first we shall derive a differential equation that is very similar to the Callan-Symanzik equation in which the β-function is defined.

3.7 Scale Dependence: the pCS equation

Let us consider a theory with n naked parameters v^α, $\alpha = 1, 2, ..., n$, collectively denoted by v. After renormalization we will find an additional n renormalized parameters which we will denote by w^α, collectively denoted by w. From the section on dimensional renormalization we know that the renormalized parameters may contain factors like μ and ϵ. These factors are to a large extend arbitrary and are needed for respectively creating a dimensionless coupling and to take into account the divergent behavior. These factors are basically scales which from now on shall be denoted with s. The definition of the renormalized parameters is

$$w^\alpha = F^\alpha(s, v) \quad . \tag{74}$$

This must allow for an inversion

$$v^\alpha = G^\alpha(s, w) \quad . \tag{75}$$

So that

$$v^\alpha = G^\alpha(s, F(s, v)) \qquad , \qquad w^\alpha = F^\alpha(s, G(s, w)) \quad . \tag{76}$$

Denoting a partial derivative to s by $_{,0}$ and to the k-th parameter by $_{,k}$ we find

$$\begin{aligned} G^\alpha_{,\lambda}(s, w) F^\lambda_{,\beta}(s, v) = \delta^\alpha_\beta \qquad &, \qquad G^\alpha_{,0}(s, w) + G^\alpha_{,\lambda}(s, w) F^\lambda_{,0}(s, v) = 0, \\ F^\alpha_{,\lambda}(s, v) G^\lambda_{,\beta}(s, w) = \delta^\alpha_\beta \qquad &, \qquad F^\alpha_{,0}(s, v) + F^\alpha_{,\lambda}(s, v) G^\lambda_{,0}(s, w) = 0 \quad . \end{aligned} \tag{77}$$

We know that v is fixed and w must be s-dependent:

$$\frac{d}{ds} w^\alpha = F^\alpha_{,0}(s, v) \quad . \tag{78}$$

We now require that the scale drops out of these running equations when we express everything in terms of w:

$$\begin{aligned} \frac{\partial}{\partial s}\left[\frac{d}{ds} w^\alpha\right] &= \frac{\partial}{\partial s} F^\alpha_{,0}(s, G(s, w)) \\ &= F^\alpha_{,00}(s, v) + F^\alpha_{,0\lambda}(s, v) G^\lambda_{,0}(s, w) = 0 \quad . \end{aligned} \tag{79}$$

We now introduce the v-dependent inverse of $F^\alpha_{,\lambda}$

$$L^\mu_\nu(s, v) \equiv G^\mu_{,\nu}(s, F(s, v)) \quad , \tag{80}$$

so that from equation (77) we have

$$L^\mu_\nu(s, v) F^\nu_{,\rho}(s, v) = \delta^\mu_\rho \qquad , \qquad F^\mu_{,\nu}(s, v) L^\nu_\rho(s, v) = \delta^\mu_\rho \quad . \tag{81}$$

Multiplying equation (79) from the left by L we obtain the requirement

$$\begin{aligned} 0 &= L\mu_\alpha(s, v) F^\alpha_{,00}(s, v) + L^\mu_\alpha(s, v) F^\alpha_{,0\lambda}(s, v) G^\lambda_{,0}(s, w) \\ &= \frac{\partial}{\partial s}\left[L^\mu_\alpha(s, v) F^\alpha_{,0}(s, v)\right] \quad . \end{aligned} \tag{82}$$

There must therefore exist functions $h^\mu(v)$, depending on v alone, such that

$$L^\mu_\alpha(s, v) F^\alpha_{,0}(s, v) = h^\mu(v) \quad . \tag{83}$$

Multiplying by $F^\nu_{,\mu}$ we find the equation that is central to this thesis:

$$\frac{\partial}{\partial s}F^\alpha(s,v) - h^\mu(v)\frac{\partial}{\partial v^\mu}F^\alpha(s,v) = 0 \ . \tag{84}$$

This equation can be compared to the Callan-Symanzik equation (CS equation) [2]. The CS equation is given by

$$\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial \lambda} + n\gamma(\lambda)\right] G^{(n)}(\{x_i\}; M, \lambda) = 0 \ , \tag{85}$$

where β and γ are dimensionless parameters defined by

$$\beta \equiv \frac{M}{\delta M}\delta\lambda \qquad , \qquad \gamma \equiv -\frac{M}{\delta M}\delta\eta \ , \tag{86}$$

$\delta\eta$ being the shift of the field and $n = 0, 1, 2, 3, ...$ In this case M is an arbitrary mass-scale parameter. The CS equation looks much like equation (84), however its application is different. Equation (84) applies to the bare parameters and the CS equation applies to the renormalized Green's functions. Also the CS equation contains a factor γ that takes into account the wavefunction renormalization where in equation (84) the wavefunction renormalization is absorbed in the renormalized parameters we denoted with F. Note that the β-function plays a similar role as our factor h. Since equation (84) is similar to the CS equation we will call equation (84) the pre-CS equation (pCS equation).

4 Renormalizing some examples

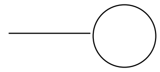
In this chapter we will renormalize the ϕ^3 and the ϕ^4 theory in four dimensions up to second order for vanishing external momenta. External momenta are, for our purposes, unimportant since they do not change the divergent behavior of the Feynman diagrams. We will start with finding all the connected diagrams up to second order for both of these theories and check the result with the effective action derived in chapter 2.5. After that we regularize the diagrams. We will distinguish between *one-loop reducible* and *one-loop irreducible* diagrams. One-loop reducible diagrams can easily be derived following the steps in 2.5. One-loop irreducible diagrams on the other hand are harder to solve by the same procedure. We will see that it is possible to derive the contribution of these diagrams from the sunset diagram. We will first regularize the one-loop reducible diagrams and will then the one-loop irreducible diagrams. In the section on one-loop irreducible diagrams we will first compute the sunset diagram. We also determine the contribution of the sunset diagram to the wavefunction renormalization. We continue to show how to derive all the remaining one-loop irreducible diagrams from the sunset diagram. In section 4.4 we will construct the renormalized parameters for the ϕ^3 theory using equations (62) and (63). After that we will use the pCS equation to check if we can indeed find a relationship between the naked and renormalized parameters that is independent of the scale. In the last section we will do the same as in section 4.4 for the ϕ^4 theory.

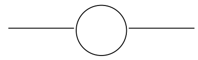
4.1 1PI Feynman diagrams

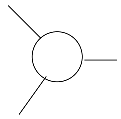
Before starting the renormalization procedure, we will first determine all the contributing Feynman diagrams for both the ϕ^3 and the ϕ^4 theory up to second order. We shall use the effective action we computed in chapter 2.5 as a check.

4.1.1 ϕ^3 theory

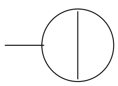
For ϕ^3 we will need all the one, two and three-leg 1PI diagrams. 1PI diagrams with more legs are only divergent if they contain divergent subgraphs. The one-loop diagrams are

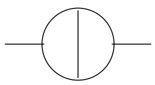
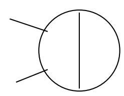
(1)  $\frac{1}{2}$

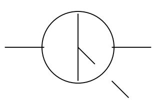
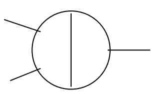
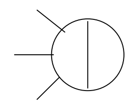
(2)  $\frac{1}{2}$

(3)  $\frac{1}{1}$,

where the last numbers denotes the symmetry factor and the multiplicity. According to the effective action the one, two and three-leg diagrams should add up to 1/2, 1/2 and 1, respectively, which they do. At two loops we find

(4)  $\frac{1}{4}$

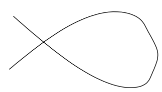
(5)  $\frac{1}{2}$ (6)  $\frac{1}{2}$

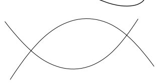
(7)  $\frac{1}{2}$ (8)  $\frac{3}{1}$ (9)  $\frac{3}{2}$.

The sum of the one-, two- and three-leg diagrams should yield 1/4, 1 and 5, respectively, which they do.

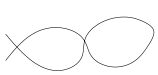
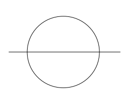
4.1.2 ϕ^4 theory

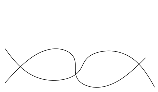
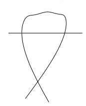
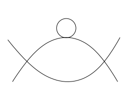
In ϕ^4 theory are no diagrams with an odd number of legs. We therefore only need the two- and the four-leg diagrams, since higher-multiplicity diagrams can only diverge because of divergent 2- or 4-leg subdiagrams. The one-loop diagrams are

(10)  $\frac{1}{2}$

(11)  $\frac{3}{2}$

in agreement with equation (30). At two loops we find

(12)  $\frac{1}{4}$ (13)  $\frac{1}{6}$

(14)  $\frac{3}{4}$ (15)  $\frac{6}{2}$ (16)  $\frac{3}{2}$.

The sum of the two- and four-leg diagrams should yield 5/12 and 21/4, respectively, which they do.

Up to first order both the diagrams of the ϕ^3 and the ϕ^4 theory are easy to compute. At second order however there are diagrams that yield integrals which are very hard to solve. Like mentioned in the introduction of this chapter we will distinguish between one-loop reducible and one-loop irreducible diagrams. One-loop reducible diagrams are basically the product of the integrals of the one-loop diagrams and are therefore still easy to solve. One-loop irreducible diagrams on the other hand can not be reduced to one-loop integrals. We can however easily derive all of the one-loop irreducible diagrams from diagram (13). Diagram (13) is called the sunset diagram. In the following two sections we will regularize all diagrams. We have verified all the results of the ϕ^4 diagrams with [8].

4.2 One-loop reducible diagrams

Using dimensional regularization as explained in section 3.3.2, we can compute the Feynman diagrams. As mentioned before we shall not compute *all* diagrams this way, since there is an easier way. For the ϕ^3 theory, we shall only compute the one-loop diagrams with the method from section 3.3.2. This yields

$$(1) = \frac{\lambda m^2 \hbar}{2(4\pi)^2}\left(\frac{1}{\epsilon} + 1 - L\right) \tag{87}$$

$$(2) = \frac{\lambda^2 \hbar}{2(4\pi)^2}\left(\frac{1}{\epsilon} - L\right) \tag{88}$$

$$(3) = -\frac{\lambda^3 \hbar}{2(4\pi)^2 m^2} \quad . \tag{89}$$

In the ϕ^4 theory the easy diagrams are (10), (11), (12), (14) and (16). For diagrams (10) and (11) we need to perform the same integrals as for diagrams (1) and (2). We find

$$(10) = \frac{\lambda m^2 \hbar}{2(4\pi)^2}\left(\frac{1}{\epsilon} + 1 - L + \epsilon\left(1 + \frac{1}{2}L^2 - L + \frac{\pi^2}{12}\right)\right) \tag{90}$$

$$(11) = \frac{3\lambda^2 \hbar}{2(4\pi)^2}\left(\frac{1}{\epsilon} - L + \epsilon\left(\frac{1}{2}L^2 + \frac{\pi^2}{12}\right)\right) \quad . \tag{91}$$

The remaining diagrams are also really easy. We realize that diagram (12) is basically the product of the loop integrals of diagram (10) and (11). In the same way the loop integral of diagram (14) is that of diagram (11) squared. It must be stressed that we are multiplying series in ϵ with one another; therefore higher-order terms in ϵ must be retained (see equations (90) and (91)). We still have to compute diagram (16), but that diagram is not that difficult either. We find

$$(12) = -\frac{\lambda^2 m^2 \hbar^2}{4(4\pi)^4}\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}(1 - 2L) + 1 - 2L + 2L^2 + \zeta(2)\right) \tag{92}$$

$$(14) = -\frac{3\lambda^3 \hbar^2}{4(4\pi)^4}\left(\frac{1}{\epsilon^2} - \frac{2L}{\epsilon} + 2L^2 + \zeta(2)\right) \tag{93}$$

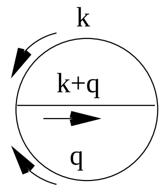
$$(16) = -\frac{3\lambda^3 \hbar^2}{2(4\pi)^4}\left(-\frac{1}{2\epsilon} + L - \frac{1}{2}\right) \quad . \tag{94}$$

4.3 One-loop irreducible diagrams

As mentioned before, the one-loop irreducible (1LI) diagrams can be derived from the sunset diagram. We therefore first derive the sunset diagram and while we are at it we also compute its contribution to the wavefunction renormalization. After that we will derive the contributions of all the remaining one-loop irreducible diagrams.

4.3.1 The Sunset diagram

Approaching the sunset diagram in the straightforward way leads to a very complicated integral which I can't solve. Luckily for us there is a different approach, namely the differential equations method of [10]. In this section we shall use this method to calculate the contribution of the sunset diagram. In the sunset diagram the momenta flow as shown in the following diagram:



The corresponding integral is

$$I(x,y) = \left(\frac{\mu^{4-2\omega}}{(2\pi)^{2\omega}}\right)^2 \int d^{2\omega}k \int d^{2\omega}q \frac{1}{(\mathbf{k}^2+x)(\mathbf{q}^2+y)[(\mathbf{k}+\mathbf{q})^2+x]}$$
$$\equiv \left(\frac{\mu^{4-2\omega}}{(2\pi)^{2\omega}}\right)^2 \int d^{2\omega}k \int d^{2\omega}q \frac{1}{D} \ , \tag{95}$$

where at the end we shall put $x = y = m^2$. Using the identities

$$0 = \int d^dk d^dq \frac{\partial}{\partial k^\mu}\left(\frac{k^\mu}{D}\right) = \int d^dk d^dq \frac{\partial}{\partial q^\mu}\left(\frac{q^\mu}{D}\right) \tag{96}$$

we can find differential equations, expanding on the somewhat concise treatment of [10], the first step is to combine the identities. We then find:

$$0 = 2dI - \int \frac{2q^2}{D(q^2+y)} - \int \frac{2k^2}{D(k^2+x)} - \int \frac{2(q+k)^2}{D[(q+k)^2+x]}$$
$$= 2(d-3)I - 2x\frac{\partial I}{\partial x} - 2y\frac{\partial I}{\partial y} \rightarrow$$
$$\frac{\partial I}{\partial x} = \frac{1}{x}\left((d-3)I - y\frac{\partial I}{\partial y}\right) \ . \tag{97}$$

Using only the first identity yields

$$0 = dI - \int \frac{2k^2}{D(k^2+x)} - \int \frac{2k(q+k)}{D[(q+k)^2+x]} \tag{98}$$
$$= dI - \int \frac{3}{D} - 2x\frac{\partial I}{\partial x} - \int \frac{y}{D(k^2+x)} + \int \frac{1}{(k^2+x)^2(q^2+x)} - \int \frac{1}{(k^2+x)^2(q^2+y)}$$
$$-2J = (y-4x)\frac{\partial I}{\partial x} + 2(d-3)I \ . \tag{99}$$

where

$$(4\pi)^d J = -2\Gamma(2-\frac{1}{2}d)\Gamma(1-\frac{1}{2}d)(x^{d-3} - x^{d/2-2}y^{d/2-1})(\mu^2)^{d-4} \ . \tag{100}$$

Using only the second identity of equation (96) yields:

$$\begin{aligned}0 &= dI - \int \frac{2q^2}{D(q^2+y)} - \int \frac{2q(q+k)}{D[(q+k)^2+x]} \\ &= dI - \int \frac{3}{D} - 2y\frac{\partial I}{\partial y} + \int \frac{1}{(k^2+x)^2(q^2+y)} - \int \frac{q^2}{D(k^2+x]} \\ &= dI - \int \frac{3}{D} - 2y\frac{\partial I}{\partial y} - \frac{1}{2}y\frac{\partial I}{\partial x} - J \\ -2xJ &= y(4x-y)\frac{\partial I}{\partial y} - (d-3)(2x-y)I \quad . \end{aligned} \tag{101}$$

In the last step we used equation (97). In order to solve this differential equation we try $I = cx^\alpha F(\Delta)$, this yields:

$$\begin{aligned} &-\frac{1}{(4\pi)^d}2x\Gamma(2-\frac{1}{2}d)\Gamma(1-\frac{1}{2}d)(x^{d-3}-x^{d/2-2}y^{d/2-1})(\mu^2)^{d-4} = \\ &= y(4x-y)\frac{\partial I}{\partial y} - (d-3)(2x-y)I, \\ &-\frac{1}{(4\pi)^d}2\Gamma(2-\frac{1}{2}d)\Gamma(1-\frac{1}{2}d)(x^{d-3}-x^{d-3}\Delta^{d/2-1})(\mu^2)^{d-4} = \\ &= (4-\Delta)\Delta c x^\alpha \frac{\partial F(\Delta)}{\partial \Delta} - (d-3)(2-\Delta)cx^\alpha F(\Delta) \quad . \end{aligned} \tag{102}$$

Take $(4\pi)^d c = -2\Gamma(2-\frac{1}{2}d)\Gamma(1-\frac{1}{2}d)(\mu^2)^{d-4}$ and $\alpha = d-3$. Solving the homogeneous equation f_0 yields:

$$\begin{aligned} 0 &= (4-\Delta)\Delta f_0'(\Delta) - (d-3)(2-\Delta)f_0(\Delta) \\ \frac{f_0'(\Delta)}{f_0(\Delta)} &= \frac{(d-3)(2-\Delta)}{(4-\Delta)\Delta} \\ f_0 &= e^{(d-3)\int_0^\Delta \frac{(4-u)-2}{(4-u)u}du} \\ &= [\Delta(4-\Delta)]^{(d-3)/2} \quad . \end{aligned} \tag{103}$$

The total solution is $f_0(\Delta)g(\Delta)$ then [10]:

$$\begin{aligned} 1-\Delta^{d/2-1} =& (4-\Delta)\Delta f_0'(\Delta)g(\Delta) + (4-\Delta)f_0(\Delta)g'(\Delta) - (d-3)(2-\Delta)f_0(\Delta)g(\Delta) \\ =& (4-\Delta)\Delta f_0(\Delta)g'(\Delta) \\ F(\Delta) =& e^{(d-3)\int_0^\Delta \frac{(4-u)-2}{(4-u)}du}\int_0^\Delta du \frac{1-\Delta^{d/2-1}}{(4-\Delta)\Delta}e^{-(d-3)\int_0^\Delta \frac{(4-u)-2}{(4-u)}du} \\ =& [\Delta(4-\Delta)]^{(d-3)/2}\times \\ & \left(\int_0^\Delta u^{(1-d)/2}(4-u)^{(1-d)/2}du - \int_0^\Delta u^{-1/2}(4-u)-(d-1)/2du\right) \\ =& \frac{2}{3-d}(4-\Delta)^{-1} + [\Delta(4-\Delta)]^{(d-3)/2}\times \\ & \left(\frac{d-1}{d-3}\int_0^\Delta u^{(3-d)/2}(4-u)^{-(d+1)/2}du - \int_0^\Delta u^{-1/2}(4-u)^{-(d-1)/2}du\right) \quad . \end{aligned} \tag{104}$$

We are interested in the case $\Delta = 1$. After expanding in $\epsilon = \frac{1}{2}(4-d)$ and the use of appendix B we then find, finally putting $x = m^2$,

$$(4\pi)^4 I = -\frac{3}{2\epsilon^2}x(1+a_1\epsilon+a_2\epsilon^2+...) \quad . \tag{105}$$

With

$$a_1 = 3 - 2L, \tag{106}$$

and

$$a_2 = 7 - 2L^2 - 2a_1 L + \zeta(2) + 8\Omega(\Delta) \ . \tag{107}$$

4.3.2 Wavefunction renormalization from the sunset diagram

The sunset diagram also contributes to the wavefunction renormalization. Since we have to take a derivative with respect to the external momentum, this external momentum cannot be put to zero from the start, but only after the derivative has been taken. In order to compute the wavefunction renormalization we will first use the Feynman Trick of appendix A:

$$\prod_{i=1}^{3} \frac{1}{q_i^2 - m^2} = \int\int\int dx_1\, dx_2\, dx_3\, \delta(x_1 + x_2 + x_3 - 1) \frac{2}{D^3} \tag{108}$$

where

$$D = x_1 q_1^2 + x_2 q_2^2 + x_3 q_3^2 - m^2 \ . \tag{109}$$

Using the substitution $q_3 = p - q_1 - q_2$ and straightforward calculus yields

$$x_1 q_1^2 + x_2 q_2^2 + x_3 q_3^2 = \alpha k_1^2 + \beta k_2^2 + \gamma p^2 \tag{110}$$

with

$$\begin{aligned} &\alpha \equiv x_1 + x_3, \qquad \beta \equiv \frac{\sigma_{123}}{x_1 + x_3}, \qquad \gamma \equiv \frac{x_1 x_2 x_3}{\sigma_{123}} \\ &k_1 \equiv q_1 + \frac{x_3}{x_1 + x_3}(q_2 - p), \qquad k_2 = q_2 - \frac{x_1 x_3}{\sigma_{123}} p \end{aligned} \tag{111}$$

where $\sigma_{123} = xy + xz + yz$. We thus have

$$\Sigma(p^2) = \frac{\lambda_4^2}{3!} \int\int\int dx_1\, dx_2\, dx_3\, \delta(x_1 + x_2 + x_3 - 1) \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \frac{2}{D^3} \tag{112}$$

where $D = \alpha k_1^2 + \beta k_2^2 + \gamma p^2 - m^2$. To compute Z we need to differentiate with respect to p^2. This yields

$$\begin{aligned} \frac{d\Sigma(p^2)}{dp^2} &= \frac{\lambda_4^2}{6} \int\int\int dx_1\, dx_2\, dx_3\, \delta(x_1 + x_2 + x_3 - 1) \times \\ &\int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \frac{-6\gamma}{[\alpha k_1^2 + \beta k_2^2 + m^2 - \gamma p^2]^4} \ . \end{aligned} \tag{113}$$

Using appendix A, B and the identities

$$\begin{aligned} \frac{6}{A^4} &= \int_0^\infty dt\, t^3 e^{-At} \\ \int \frac{d^n k}{(2\pi)^n} e^{-ctk^2} &= (4\pi ct)^{-n/2} \ , \end{aligned} \tag{114}$$

with $d = 4 - 2\epsilon$ we find

$$\frac{d\Sigma(p^2)}{dp^2} = -\frac{\lambda_4^2}{3072\pi^4} \int\int\int dx_1\, dx_2\, dx_3\, \delta(x_1 + x_2 + x_3 - 1) \times$$
$$\frac{\gamma}{(\alpha\beta)^2} \left(\frac{1}{\epsilon} + 2L + \log\left(\frac{\alpha\beta}{[1 - (p^2/m^2)\gamma]^2}\right) \right) \quad . \tag{115}$$

We solve this with Mathematica [11]. This results in

$$Z_\phi = \left. \frac{1}{1 - \frac{d\Sigma(p^2)}{dp^2}} \right|_{p^2=m^2} \approx 1 - \frac{\lambda_4^2}{6144\pi^4} \left[\frac{1}{\epsilon} + 2L - \frac{3}{2} \right] \quad . \tag{116}$$

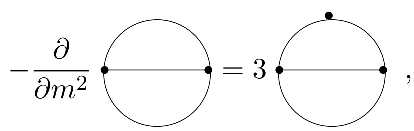
We took $p^2 = m^2$ since we are working at the leading order of perturbation theory, so that in agreement with [12] we can neglect the difference between the renormalized mass and the bare masses as a higher-order correction.

4.3.3 Remaining diagrams

We will now use the result of the sunset diagram to derive the remaining one-loop irreducible diagrams. Let us first consider the ϕ^3 diagrams, starting with diagram (4). This diagram yields the following integral

$$I = \left(\frac{\mu^{4-2\omega}}{(2\pi)^{2\omega}} \right)^2 \int d^{2\omega}k \int d^{2\omega}q \frac{1}{(\mathbf{k}^2 + x)^2(\mathbf{q}^2 + y)[(\mathbf{k} + \mathbf{q})^2 + x]} \quad . \tag{117}$$

This integral is not trivial to solve, however we realize it is irrelevant which propagator is squared. We deduce that the result of diagram (4) is basically

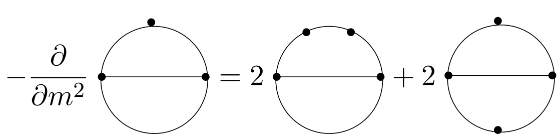


(118)

where these diagrams only denote the integrals, so no symmetry factors or couplings are included. Note that the ϵ expansion will change as well, since taking the derivative will change the gamma functions. We find

$$(4) = -\frac{\lambda^3\hbar^2}{8(4\pi)^4} \left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}(1 - 2L) + 1 - 2L + 2L^2 + \zeta(2) + 8\Omega(1) \right) \quad . \tag{119}$$

For the two-leg there are two diagrams, namely diagrams (5) and (6). Solving these diagram independently is hard, fortunately we are only interested in their sum. We can easily deduce that

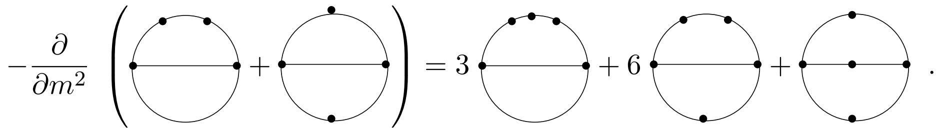


(120)

which has the correct ratio as far as the symmetry factors are concerned. We thus find

$$(5) + (6) = \frac{\lambda^4\hbar^2}{8(4\pi)^4 m^2} \left(\frac{2}{\epsilon} - 4L + 2 \right) \quad . \tag{121}$$

At three-leg we have diagrams (7), (8) and (9), which again we can not solve independently. However we can deduce that



(122)

This expression has yet again the correct ratio to solve the sum

$$(7) + (8) + (9) = -\frac{\lambda^5\hbar^2}{8(4\pi)^4 m^4}\left(\frac{2}{\epsilon} - 4L + 6\right) \quad . \tag{123}$$

The only diagram left that needs solving is diagram (15) from the ϕ^4 theory. We however realize that we can compute this diagram in the same way as diagram (4). We find

$$(15) = -\frac{3\lambda^3\hbar^2}{2(4\pi)^4}\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}(1-2L) + 1 - 2L + 2L^2 + \zeta(2) + 8\Omega(1)\right) \tag{124}$$

We have now regularized all diagrams. In the following two sections we will use these results to renormalize ϕ^3 and ϕ^4, respectively.

4.4 The pCS equation for ϕ^3 theory

In this section we will renormalize the ϕ^3 theory. Using the results from the previous sections 4.2 and 4.3 we can construct the renormalized parameters with equations (62) and (63). We find

$$\begin{aligned}\tau_R \equiv F^1 =& \tau - \frac{\lambda m^2\hbar}{2(4\pi)^2}\left(\frac{1}{\epsilon} + 1 - L\right) + \\ & \frac{\lambda^3\hbar^2}{8(4\pi)^4}\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}(1-2L) + 1 - 2L + 2L^2 + \zeta(2) + 8\Omega(1)\right) \qquad (125)\\ m_R^2 \equiv F^2 =& m^2 - \frac{\lambda^2\hbar}{2(4\pi)^2}\left(\frac{1}{\epsilon} - L\right) - \frac{\lambda^4\hbar^2}{8(4\pi)^4 m^2}\left(\frac{2}{\epsilon} - 4L + 2\right) \qquad (126)\\ \lambda_R \equiv F^3 =& \lambda + \frac{\lambda^3\hbar}{2(4\pi)^2 m^2} + \frac{\lambda^5\hbar^2}{8(4\pi)^4 m^4}\left(\frac{2}{\epsilon} - 4L + 6\right). \qquad (127)\end{aligned}$$

Inserting this in the pCS equation yields

$$\begin{aligned} h_1^1 &= -\frac{\lambda m^2}{2(4\pi^2)\tau}, & h_1^2 &= -\frac{\lambda^3}{8(\pi)^4\tau}, \\ h_2^1 &= -\frac{\lambda^2}{2(4\pi^2)m^2}, & h_2^2 &= 0, \\ h_3^1 &= 0, & h_3^2 &= 0. \end{aligned} \tag{128}$$

We have now renormalized the ϕ^3 theory in four dimensions and have shown that the scale dependence is independent of the scale. Note that $\lambda_R = \lambda$ in ϕ^3 theory in four dimensions. We could have expected this since diagram (3) is finite. In six dimensions however this is not the case. In this case we expect to find $\lambda_R \neq \lambda$.

One may wonder why no wavefunction renormalization is involved. The reason for this is that the two-leg diagrams up to second order yield at most a $\frac{1}{\epsilon}$ divergence. Computing the wavefunction renormalization means including an external momentum p and differentiation with respect to p^2. After differentiation the integral becomes less divergent. So the wavefunction renormalizion is finite up to second order. We can absorb these finite parts in the functions of h and therefore the wavefunction part at this order does not contribute to the renormalized parameters.

As a final check we can compute the β-function of the ϕ^3 theory. The scale in this β-function should drop out. From section 3.7 we know that the β-function can be computed with following equation

$$\beta_v = \frac{\partial}{\partial s} F^v \bigg|_B \quad . \tag{129}$$

We must therefore first determine the bare parameters in terms of the renormalized parameters. Also we must note that however $\lambda_R = \lambda_B$, this does not hold for the dimensionless coupling, since $g = \lambda/m$. We will therefore also find a nontrivial β_g. We can use equation (129) and find

$$\beta_\tau = -\frac{\lambda m^2}{32\pi^2}\hbar - \frac{\lambda^3}{2048\pi^4}\hbar^2 \quad , \tag{130}$$

$$\beta_{m^2} = -\frac{\lambda^2}{32\pi^2}\hbar \tag{131}$$

$$\beta_g = \frac{g^3}{64\pi^2}\hbar \quad . \tag{132}$$

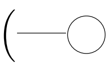
In the next section we shall determine the β-function using counterterms in the hope we might find an explanation for this.

4.5 Counterterms for the ϕ^3 theory

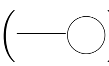
In this section we will apply the counterterm method we introduced in section 3.6 following the steps of [8]. We will determine the counterterms order by order. Since in the ϕ^3 theory there are no leading-order divergences at the three-point vertex function we know that c_λ is zero at all orders (we will show this explicitly). Also the wavefunction renormalization up to second order is finite, so the only corrections we need are corrections for the tadpole and the mass. We expect that there are only a finite number of overall divergences since the ϕ^3 is super-renormalizable. We will see that there are indeed only a few overall divergences. For simplicity we shall denote all counterterms with a black square in this case, we thus define

 $= -\tau c_\tau,$  $= -m^2 c_{m^2}$. (133)

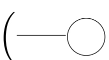
We shall now start to renormalize that tadpole vertex function $\bar{\Gamma}^{(1)}$, which yields the following expression

$\bar{\Gamma}^{(1)} = \tau - \Big($  $+$  $+ \mathcal{O}(\hbar^2)\Big)$ (134)

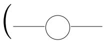
In order to cancel this we find

 $= -\tau c_\tau^1 = -\mathcal{K}\Big($  $\Big) = -\frac{g m^3 \hbar}{2(4\pi)^2}\frac{1}{\epsilon}$, (135)

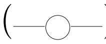
where the superscript denotes the order of the approximation. The tadpole vertex function is now finite up to first order in $\hbar$ and reads

$\bar{\Gamma}^{(1)} = \tau - \Big($  $- \mathcal{K}\Big($  $\Big) + \mathcal{O}(\hbar^2)\Big)$. (136)

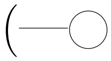
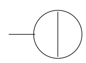
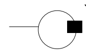
We continue with the two-point vertex function at first order in $\hbar$. Written in diagrammatical terms we find

$\bar{\Gamma}^{(2)} = m^2 - \Big($  $+$  $+ \mathcal{O}(\hbar^2)\Big) \quad ,$ (137)

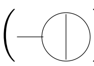
we thus have

 $= -m^2 c^1_{m^2} = -\mathcal{K}\Big($  $\Big) = -m^2 \frac{g^2\hbar}{2(4\pi)^2}\frac{1}{\epsilon} \quad .$ (138)

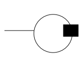
The two-point vertex function is now also finite. At first order nothing needs to be done about the three-point vertex function, since the only diagram contributing to this function is finite. We will therefore continue with the tadpole vertex function at second order in $\hbar$.

$\bar{\Gamma}^{(1)} = \tau - \Big($  $+$  $+$  $+$  $\Big) \quad .$ (139)

The third diagram in this expression has been computed in the previous chapter, taking its pole terms yields

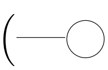
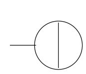
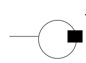
$\mathcal{K}\Big($  $\Big) = -\frac{g^3 m^3 \hbar^2}{8(4\pi)^4}\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}(1-2L)\right) \quad .$ (140)

We see that this expression still contains a logarithm involving the mass, while we expected from section 3.6 that these terms would cancel. We will see that they indeed cancel because of the last diagram in equation (139). This is a diagram with a counterterm in it, we shall refer to such diagrams as counterterm diagrams. We compute this diagram in the following way

 $= -\frac{-g m^3 c^1_{m^2}\hbar}{2(4\pi)^2}\left(\frac{1}{\epsilon} + 1 - L\right)$

$$\frac{g^3 m^3 \hbar^2}{4(4\pi)^2}\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}(1-L))\right] \quad . \tag{141}$$

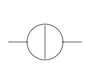
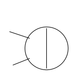
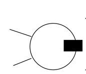
This expression does again contain a logarithm from the integral part of the diagram. The counterterm $c^1_{m^2}$ on the other does not contain a logarithm, but multiplied with this diagram it does. We find that the logarithm does indeed drop out. Our second order tadpole counterterm thus becomes

 $= -\mathcal{K}\Big($  $+$  $+$  $\Big)$ (142)

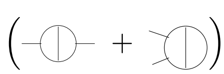
Algebraically this reads

$$\tau(c^1_\tau + c^2_\tau) = \tau\left[\frac{g^2\hbar}{2(4\pi)^2\tau}\frac{1}{\epsilon} + \frac{g^3 m^3 \hbar^2}{8(4\pi)^4\tau}\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\right)\right] \tag{143}$$

We have now renormalized the tadpole vertex function up to second order in $\hbar$. We shall continue to renormalize the two-point vertex function up to second order in $\hbar$. We find

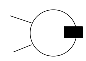
$\bar{\Gamma}^{(2)} = m^2 - \Big($  $+$  $+$  $+$  $+$  $\Big) \quad .$ (144)

We know the sum of the two-leg diagrams up to second order from calculations in the previous section. Taking only the pole terms yield

$\mathcal{K}\Big($  $\Big)$

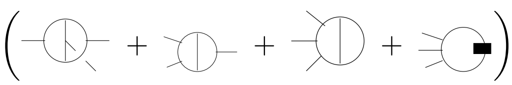
$$= \frac{g^4 m^2 \hbar^2}{4(4\pi)^4}\frac{1}{\epsilon} \quad . \tag{145}$$

The counterterm diagram yields

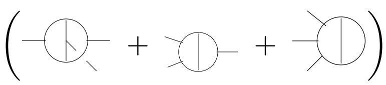


$$= \frac{-g^2 m^2 c^1_{m^2}\hbar}{2(4\pi)^2}\frac{1}{\epsilon} - \frac{g^4 m^2 \hbar^2}{4(4\pi)^4}\frac{1}{\epsilon} \quad , \tag{146}$$

which is enough to make the two-point vertex function finite. Our second order mass term is therefore zero. We expect the same thing to happen for the three-point vertex function at second order in $\hbar$. For the sake of argument we will show this. The three-point vertex function (assuming $c_g = 0$) reads

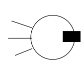
$\bar{\Gamma}^{(3)} = g\mu^{2\epsilon} - \Big($  $\Big) \quad . \qquad (147)$

The pole terms of the three-leg diagrams read

$\mathcal{K}\Big($  $\Big)$

$$= -\frac{g^5 m \hbar^2}{4(4\pi)^4}\frac{1}{\epsilon} \tag{148}$$

and the counterterm diagram reads



$$= -\frac{-3g^3 m c^1_{m^2}\hbar}{6(4\pi)^2} \; \frac{g^5 m \hbar^2}{4(4\pi)^4}\frac{1}{\epsilon} \quad . \tag{149}$$

The counterterm diagram does indeed cancel the $1/\epsilon$ terms. So our assumption for c_g turns out to be true. The renormalization constants are thus given by

$$Z_\tau = 1 + \left[\frac{g m^3 \hbar}{2(4\pi)^2 \tau}\frac{1}{\epsilon} + \frac{g^3 m^3 \hbar^2}{8(4\pi)^4 \tau}\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\right)\right] \tag{150}$$

$$Z_{m^2} = 1 + \frac{g^2 \hbar}{2(4\pi)^2}\frac{1}{\epsilon} \quad . \tag{151}$$

The bare parameters expressed in the renormalized parameters are thus given by

$$\tau_B = \tau + \left[\frac{g m^3 \hbar}{2(4\pi)^2}\frac{1}{\epsilon} + \frac{g^3 m^3 \hbar^2}{8(4\pi)^4}\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\right)\right] \tag{152}$$

$$m_B^2 = m^2 + \frac{g^2 m^2 \hbar}{2(4\pi)^2}\frac{1}{\epsilon} \quad . \tag{153}$$

In order to compute the β-function it is easier to work with λ instead of g. The renormalized parameters in terms λ can be computed from the bare parameters when we substitute back $\lambda = g/m$. Note that we also need g_R in terms of g_B, we then find

$$\tau_R = \tau - \frac{\lambda m^2 \hbar}{2(4\pi)^2}\frac{1}{\epsilon} + \frac{\lambda^3 \hbar^2}{8(4\pi)^4}\left(\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\right) \tag{154}$$

$$m_R^2 = m^2 - \frac{\lambda^2 \hbar}{2(4\pi)^2}\frac{1}{\epsilon} \tag{155}$$

$$g_R = g + \frac{\hbar g^3}{64\pi^2}\frac{1}{\epsilon} + \frac{3\hbar^2 g^5}{8192\pi^4}\frac{1}{\epsilon^2} \quad . \tag{156}$$

We can now determine the β-function by inserting our renormalized parameters in equation (129) and take $1/\epsilon$ as the scale, this yields

$$\beta_\tau = -\frac{\lambda m^2}{32\pi^2}\hbar - \frac{\lambda^3}{2048\pi^4}\hbar^2 \quad , \tag{157}$$

$$\beta_{m^2} = -\frac{\lambda^2}{32\pi^2}\hbar \tag{158}$$

$$\beta_g = \frac{g^3}{64\pi^2}\hbar \quad . \tag{159}$$

We find that the scale does indeed drop out. We also find that these β-functions are the same as in section 4.4. Unfortunately I have not been able to find any literature on the β-functions for the scalar ϕ^3 theory in four dimensions. There is however much literature to be found on the β-function of the scalar ϕ^4 theory. In section 4.7 we shall see that using the counterterm procedure does indeed result in the correct β-functions for the scalar ϕ^4 theory in four dimensions.

We also want to know if the renormalized parameters we constructed satisfy the pCS equation. We then find

$$\begin{aligned} h_1^1 &= -\frac{\lambda m^2}{2(4\pi^2)\tau}, \qquad h_1^2 = -\frac{\lambda^3}{8(\pi)^4\tau}, \\ h_2^1 &= -\frac{\lambda^2}{2(4\pi^2)m^2}, \qquad h_2^2 = 0, \\ h_3^1 &= 0, \qquad h_3^2 = 0. \end{aligned} \tag{160}$$

Note that the functions h has the same factors as the β-function with the exception of h_3. The last one however accounts for λ and $\lambda_R = \lambda_B$, so this should be zero. The β-function on the other hand we computed for $g = \lambda/m$ so this explains the difference. Also note that using the counterterms model we found the exact same results as with the other method. We can also insert the renormalized parameters with the dimensionless coupling constant. In this case we find

$$\begin{aligned} h_1^1 &= -\frac{g m^3}{2(4\pi^2)\tau}, \qquad h_1^2 = -\frac{g^3 m^3}{8(\pi)^4\tau}, \\ h_2^1 &= -\frac{g^2}{2(4\pi^2)}, \qquad h_2^2 = 0, \\ h_3^1 &= \frac{g^3}{64\pi^2}, \qquad h_3^2 = 0. \end{aligned} \tag{161}$$

We find that h_g is thus also similar to β_g. In the next section we will follow the same procedure but then for the ϕ^4 theory.

4.6 The pCS equation for ϕ^4 theory

In this section we renormalize the ϕ^4 theory in four dimensions. Using the results from section 4.2 and 4.3 we can construct the renormalized parameters with equations (62) and (63). We find

$$\begin{aligned} m_R^2 \equiv F^1 =& m^2 - \frac{\lambda m^2 \hbar}{2(4\pi)^2}\left(\frac{1}{\epsilon}+1-L\right) - \frac{\lambda^2 m^2 \hbar^2}{24(4\pi)^4}\left(\frac{1}{\epsilon}-2L-J\right) + \\ & \frac{\lambda^2 m^2\hbar^2}{2(4\pi)^4}\left(\frac{1}{\epsilon^2}+\frac{2}{\epsilon}(1-L)+4-4L+2L^2+\zeta(2)+4\Omega(1)\right) \end{aligned} \tag{162}$$

$$\begin{aligned} \lambda_R \equiv F^2 =& \lambda - \frac{3\lambda^2\hbar}{2(4\pi)^2}\left(\frac{1}{\epsilon}-L\right) - \frac{\lambda^3\hbar^2}{12(4\pi)^4}\left(\frac{1}{\epsilon}-2L-J\right) + \frac{3\lambda^3\hbar^2}{2(4\pi)^4}\left(-\frac{1}{2\epsilon}+L-\frac{1}{2}\right) + \\ & \frac{3\lambda^3\hbar^2}{2(4\pi)^4}\left(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}(1-2L)+1-2L+2L^2+\zeta(2)+8\Omega(1)\right) + \\ & \frac{3\lambda^3\hbar^2}{4(4\pi)^4}\left(\frac{1}{\epsilon^2}-\frac{2L}{\epsilon}+2L^2+\zeta(2)\right) \quad . \end{aligned} \tag{163}$$

Inserting this in the pCS equation yields

$$\begin{aligned} & -\frac{\lambda m^2\hbar}{2(4\pi)^2} + \frac{\lambda^2 m^2\hbar^2}{2(4\pi)^4}\left(\frac{2}{\epsilon}+2-2L\right) - \\ & \frac{\lambda^2\hbar^2}{24(4\pi)^4} - (\hbar h_1^{(1)} + \hbar^2 h_1^{(2)})\left(m^2 - \frac{\lambda m^2\hbar}{2(4\pi)^2}\left(\frac{1}{\epsilon}-L\right)\right) + \\ & (\hbar h_2^{(1)} + \hbar^2 h_2^{(2)})\left(\frac{\lambda m^2\hbar}{2(4\pi)^2}\left(\frac{1}{\epsilon}+1-L\right)\right) = 0 \quad , \end{aligned} \tag{164}$$

and

$$\begin{aligned} & -\frac{3\lambda^2\hbar}{2(4\pi)^2} + \frac{3\lambda^3\hbar^2}{2(4\pi)^4}\left(\frac{2}{\epsilon}+1-2L\right) + \frac{3\lambda^3\hbar^2}{4(4\pi)^4}\left(\frac{2}{\epsilon}-2L\right) - \frac{3\lambda^3\hbar^2}{4(4\pi)^4} - \frac{\lambda^3\hbar^2}{12(4\pi)^4} - \\ & (\hbar h_1^{(1)} + \hbar^2 h_1^{(2)})\left(0 + \frac{3\lambda^2\hbar}{2(4\pi)^2}\right) - (\hbar h_2^{(1)} + \hbar^2 h_2^{(2)})\left(\lambda - \frac{3\lambda^2\hbar}{(4\pi)^2}\left(\frac{1}{\epsilon}-L\right)\right) = 0 \quad , \end{aligned} \tag{165}$$

where we choose $\frac{1}{\epsilon} = s$. At first order in $\hbar$ we find

$$-\frac{\lambda m^2}{2(4\pi)^2} - h_1^{(1)} m^2 = 0 \rightarrow h_1^{(1)} = -\frac{\lambda}{2(4\pi)^2} \tag{166}$$

$$-\frac{3\lambda^2}{2(4\pi)^2} - h_2^{(1)}\lambda = 0 \rightarrow h_2^{(1)} = -\frac{3\lambda}{2(4\pi)^2} \quad . \tag{167}$$

For second order in $\hbar$ we find

$$\begin{aligned} & \frac{\lambda^2 m^2}{2(4\pi)^4}\left(\frac{2}{\epsilon}+2-2L\right) + \frac{\lambda^2 m^2}{24(4\pi)^4} - \\ & h_1^{(2)} m^2 - \frac{\lambda^2 m^2}{4(4\pi)^4}\left(\frac{1}{\epsilon}-L\right) - \frac{3\lambda^2 m^2}{4(4\pi)^4}\left(\frac{1}{\epsilon}+1-L\right) = 0 \rightarrow h_1^{(2)} = \frac{5\lambda^2}{24(4\pi)^4} \end{aligned} \tag{168}$$

and

$$\begin{aligned} & \frac{3\lambda^3}{2(4\pi)^4}\left(\frac{2}{\epsilon}+1-2L\right) + \frac{3\lambda^3}{4(4\pi)^4}\left(\frac{2}{\epsilon}-2L\right) - \frac{3\lambda^3}{4(4\pi)^4} + \frac{\lambda^3}{12(4\pi)^4} - \\ & h_2^{(2)}\lambda - \frac{3\lambda^3}{4(4\pi)^2} - \frac{9\lambda^3}{2(4\pi)^2}\left(\frac{1}{\epsilon}-L\right) = 0 \rightarrow h_2^{(2)} = -\frac{\lambda^2}{12(4\pi)^4} \quad . \end{aligned} \tag{169}$$

Note that if we would multiply the second order contributions by a factor ρ we would find

$$\rho\left(\frac{\lambda^2m^2}{2(4\pi)^4}\left(\frac{2}{\epsilon}+2-2L\right)+\frac{\lambda^2m^2}{24(4\pi)^4}\right)-h_1^{(2)}m^2-\frac{\lambda^2m^2}{4(4\pi)^4}\left(\frac{1}{\epsilon}-L\right)-\frac{3\lambda^2m^2}{4(4\pi)^4}\left(\frac{1}{\epsilon}+1-L\right)=$$
$$(\rho-1)\left(\frac{\lambda^2m^2}{2(4\pi)^4}\left(\frac{2}{\epsilon}+2-2L\right)+\frac{\lambda^2m^2}{24(4\pi)^4}\right) \tag{170}$$

and

$$\rho\left(\frac{3\lambda^3}{2(4\pi)^4}\left(\frac{2}{\epsilon}+1-2L\right)+\frac{3\lambda^3}{4(4\pi)^4}\left(\frac{2}{\epsilon}-2L\right)-\frac{3\lambda^3}{4(4\pi)^4}+\frac{\lambda^3}{12(4\pi)^4}\right)-h_2^{(2)}\lambda-$$
$$\frac{3\lambda^3}{4(4\pi)^2}-\frac{9\lambda^3}{2(4\pi)^2}\left(\frac{1}{\epsilon}-L\right)=$$
$$(\rho-1)\left(\frac{3\lambda^3}{2(4\pi)^4}\left(\frac{2}{\epsilon}+1-2L\right)+\frac{3\lambda^3}{4(4\pi)^4}\left(\frac{2}{\epsilon}-2L\right)-\frac{3\lambda^3}{4(4\pi)^4}+\frac{\lambda^3}{12(4\pi)^4}\right) \quad . \tag{171}$$

We can therefore predict the scale-dependence at order O from order $O-1$.

As a final check we can compute the β-function of the ϕ^4 theory, to see if the pCS equation really corresponds with the CS equation. In order to do this we first need to find the inverse of F^1 and F^2: From section 3.7 we know that we should now be able to find the β-function with equation (129). So we have to differentiate F with respect to s and substitute m^2 and λ by the inverse of F. From section 3.7, we know this to be G, which represents the bare parameters expressed in the renormalized parameters. Using Maple [13] we then find

$$\begin{aligned}\beta_{m^2} &= -\frac{\lambda m^2}{32\pi^2}\hbar+\frac{17\lambda^2m^2}{6144\pi^4}\hbar^2\\ \beta_\lambda &= -\frac{3\lambda m^2}{32\pi^2}\hbar+\frac{\lambda^2m^2}{384\pi^4}\hbar^2 \quad .\end{aligned} \tag{172}$$

The scale does drop out which is a good thing, but this β-function is not the same as in the literature. The most important difference is the minus sign. We can explain this because we took the scale to be $1/\epsilon$. The scale should be in the same direction as the cutoff parameter Λ. This means we should have taken $s=-1/\epsilon$. However even then we have different factors and there is only one β-function. There is also another way to compute the β-function, which is shown in the in the book of [8]. Using the method of [9] it is shown that when using the minimal subtraction scheme the β-function for λ should be

$$\beta(\lambda_R)=-\epsilon\lambda_R\left(\frac{\mathrm{d}\log(\lambda)}{\mathrm{d}\log(\lambda_R)}\right)^{-1} \quad . \tag{173}$$

Using this equation we find for both the MS-scheme and the $\overline{\text{MS}}$-scheme that

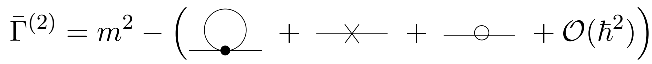
$$\beta(\lambda_R)=\frac{3}{16}\frac{g^2}{\pi^2}\hbar-\frac{17}{768}\frac{g^3}{\pi^4}\hbar^2+\mathcal{O}(\hbar^2) \ , \tag{174}$$

which is in correspondence with [8], [10] and [14]. In both the pCS equation and in the CS equation we find expressions that are independent of the scale. So it seems that the pCS equation does indeed say something about the CS equation.

We still want to understand why our own prescription to find the β-function does not work. In order to answer this we will first determine the β-function following a different approach, namely using counterterms.

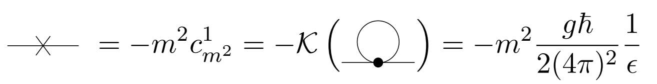
4.7 Counterterms for the ϕ^4 theory

In this section we will apply the counterterm method we introduced in section 3.6 following the steps of [8]. We will determine the counterterms order by order. We will start adding counterterms to make the two-point vertex function $\bar{\Gamma}^{(2)}$ finite. We then find



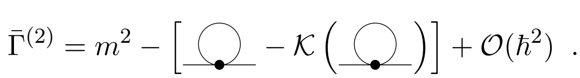
(175)

The line with the cross is chosen to cancel the pole term of the two-leg diagram. We then find



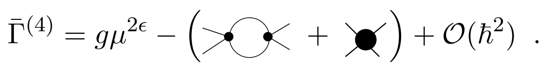
(176)

The two-point vertex function is now finite, so we do not need a counterterm c_ϕ at first order in $\hbar$. We will therefore set $c_\phi = 0$. The renormalized two-point vertex function up to first order in $\hbar$ is thus given by



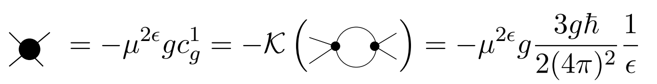
(177)

We shall continue to add counterterms to make the vertex function $\bar{\Gamma}^{(4)}$ finite up to first order in $\hbar$. We find



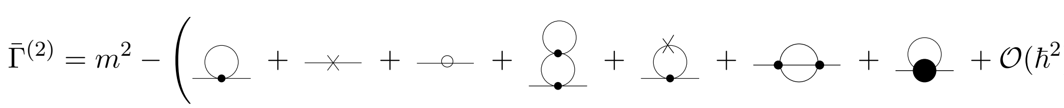
(178)

In order for this to be finite we find



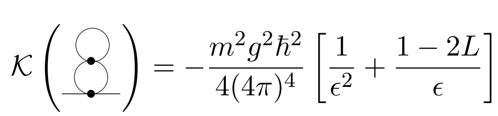
(179)

We shall move on to the two-loop calculation. At second order in $\hbar$ the two-point vertex function with counterterms has the following expression



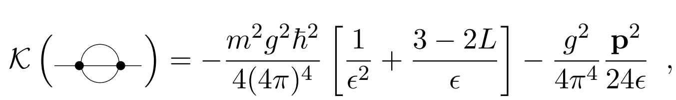
(180)

There are only two second-order diagrams. The pole terms of these diagrams are given by



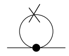
(181)

and



(182)

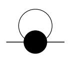
where the last term of the sunset diagram is due to wavefunction renormalization. Note that there are still logarithmic terms in these expressions. In section 3.6 we however said that there would be no such terms. We will see that in the end they will indeed be cancelled. This is due to the so-called counterterm diagrams:



$$= \quad -\frac{-m^2 c^1_{m^2} g\hbar}{2(4\pi)^2}\left[\frac{1}{\epsilon} - L\right] \tag{183}$$

$$\frac{m^2 g^2\hbar^2}{4(4\pi)^4}\left[\frac{1}{\epsilon^2} - \frac{L}{\epsilon} + \mathcal{O}(\epsilon^0)\right] \tag{184}$$

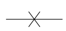
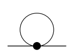
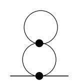
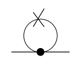
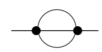
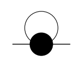
and



$$= \quad -\frac{-m^2 c^1_{g} g\hbar}{2(4\pi)^2}\left[\frac{1}{\epsilon} + 1 - L\right] \tag{185}$$

$$\frac{3m^2 g^2\hbar^2}{4(4\pi)^4}\left[\frac{1}{\epsilon^2} + \frac{1-L}{\epsilon} + \mathcal{O}(\epsilon^0)\right] \quad . \tag{186}$$

These diagrams also contain logarithms and are multiplied by c_{m^2} and c_g respectively. The counterterms c_{m^2} and c_g do not contain logarithms, but the counterterm diagrams does in such a way all the logarithms cancell. Therefore will be no logarithms in the relation between the bare and renormalized parameters. We still need to determine the second order counterterms. These are given by

 + $= -\mathcal{K}\Big[$  +  +  +  +  $\Big]$ (187)

$$= -\left[\frac{m^2 g\hbar}{2(4\pi)^2}\frac{1}{\epsilon} + \frac{g^2\hbar^2}{(4\pi)^4}\left(\frac{m^2}{2\epsilon^2} - \frac{m^2}{4\epsilon} - \frac{\mathbf{p}^2}{24\epsilon}\right)\right] \quad . \tag{188}$$

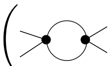
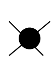
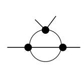
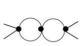
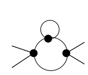
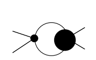
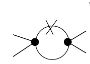
The pole term proportional to m^2 should be absorbed by the mass counterterms:

$$m^2(c^1_{m^2} + c^2_{m^2}) = m^2\left[\frac{g\hbar}{2(4\pi)^2}\frac{1}{\epsilon} + \frac{g^2\hbar^2}{2(4\pi)^4}\left(\frac{1}{\epsilon^2} - \frac{1}{2\epsilon}\right)\right] \quad . \tag{189}$$

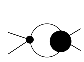
The remaining pole term is then cancelled in the following way

$$\mathbf{p}^2 c^2_\phi = -\mathbf{p}^2\frac{g^2\hbar^2}{24(4\pi)^4}\frac{1}{\epsilon} \tag{190}$$

Lastly we turn to the two-loop renormalization of the four-point vertex function. This reads

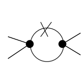
$\bar{\Gamma}^{(4)}(\mathbf{k}_i) = g\mu^{2\epsilon} - \Big($  +  +  +  +  +  +  $\Big)$ (191)

By now it should be clear that we only take the pole terms of the diagrams and how this works. We shall therefore not write these down explicity. We will work out the counterterm diagrams, these are given by



$$= \mu^{2\epsilon} g\frac{9g^2\hbar^2}{2(4\pi)^4}\left(\frac{1}{\epsilon^2} - \frac{L}{\epsilon}\right) \tag{192}$$

and



$$= -\mu^{2\epsilon} g\frac{3g^2\hbar^2}{4(4\pi)^4}\frac{1}{\epsilon} \quad . \tag{193}$$

All pole terms containing logarithms do indeed cancel each other. The counterterm of the coupling becomes

$$\mu^{2\epsilon}g(c_g^1+c_g^2)=\mu^{2\epsilon}g\left[\frac{3g\hbar}{2(2\pi)^2}\frac{1}{\epsilon}+\frac{g^2\hbar^2}{(4\pi)^4}\left(\frac{9}{4\epsilon^2}-\frac{3}{2\epsilon}\right)\right] \tag{194}$$

The renormalization constants are thus given by

$$Z_\phi=1-\frac{g^2\hbar^2}{24(4\pi)^4}\frac{1}{\epsilon}\ , \tag{195}$$

$$Z_{m^2}=1+\frac{g\hbar}{2(4\pi)^2}\frac{1}{\epsilon}+\frac{g^2\hbar^2}{(4\pi)^4}\left(\frac{1}{2\epsilon^2}-\frac{1}{4\epsilon}\right)\ , \tag{196}$$

$$Z_g=1+\frac{3g\hbar}{2(4\pi)^2}\frac{1}{\epsilon}+\frac{g^2\hbar^2}{(4\pi)^4}\left(\frac{9}{4\epsilon^2}-\frac{3}{2\epsilon}\right)\ . \tag{197}$$

The bare parameters in terms of the renormalized parameters are then given by

$$m_B^2=m^2+\frac{gm^2\hbar}{32\pi^2}\frac{1}{\epsilon}+\frac{g^2m^2\hbar^2}{256\pi^4}\left(\frac{1}{2\epsilon^2}-\frac{5}{24\epsilon}\right)\ , \tag{198}$$

$$g_B=g\mu^{2\epsilon}+\frac{3g\mu^{2\epsilon}\hbar}{32\pi^2}\frac{1}{\epsilon}+\frac{g^3\mu^{2\epsilon}\hbar^2}{256\pi^4}\left(\frac{9}{4\epsilon^2}-\frac{17}{12\epsilon}\right)\ . \tag{199}$$

We can invert this to obtain the renormalized parameters in terms of the bare parameters

$$m_R^2=m^2-\frac{m^2g\hbar}{32\pi^2\mu^{2\epsilon}}\frac{1}{\epsilon}+\frac{g^2m^2\hbar^2}{512\pi^4\mu^{4\epsilon}}\left(\frac{1}{\epsilon^2}+\frac{5}{12\epsilon}\right)\ , \tag{200}$$

$$g_R=g-\frac{3g^2\hbar}{32\pi^2\mu^{2\epsilon}}\frac{1}{\epsilon}+\frac{g^3\hbar^2}{1024\pi^4\mu^{4\epsilon}}\left(\frac{9}{\epsilon^2}+\frac{17}{3\epsilon}\right)\ . \tag{201}$$

From section 3.7 we know that we should now be able to find the β-function with equation (129). Inserting our renormalized parameters in this equation and take $\log(\mu)$ as the scale yields

$$\beta_{m^2}=\frac{m^2g}{16\pi^2}\hbar-\frac{5m^2g^2}{1536\pi^4}\hbar^2\ , \tag{202}$$

$$\beta_g=\frac{3g^2}{16\pi^2}\hbar-\frac{17g^3}{768\pi^4}\hbar^2\ , \tag{203}$$

which are the correct β-functions. So now we should ask ourselves why we do find the correct β-functions now. Note that we used a different prescription here for the renormalized parameters. In this case we used the counterterms and in the previous section we did not. When using counterterms we basically decoupled the mass terms from the coupling terms. In our other prescription of the renormalized parameters they are not decoupled. Note for instance that diagram (16) is a diagram with a mass correction. Using the counterterm procedure this term is cancelled completely by a counterterm diagram. So it does not appear anywhere in the renormalized parameters. This is even more apparent for the renormalized mass. In this case all the higher order diagrams contain corrections for the coupling constant, but this part is removed by counterterm diagrams so it does not appear in the relation between the renormalized mass and bare mass.

We also wish for our renormalized parameters to satisfy the pCS equation. Inserting our renormalized parameters in the pCS equation at first order in $\hbar$ results in the following expressions for h:

$$m^2h_{m^2}^{(1)}=\frac{m^2\lambda}{16\pi^2}\qquad,\qquad\lambda h_\lambda^{(1)}=\frac{3\lambda^2}{16\pi^2}\ . \tag{204}$$

At second order in $\hbar$ we find

$$-\frac{m^2g^2}{128\pi^4}\frac{1}{\epsilon} - \frac{5m^2g^2}{1536\pi^4} - m^2h^{(2)}_{m^2} + \frac{m^2g^2}{128\pi^4}\frac{1}{\epsilon} \to m^2h^{(2)}_{m^2} = -\frac{5m^2g^2}{1536\pi^4} \tag{205}$$

and

$$-\frac{9g^3}{256\pi^4}\frac{1}{\epsilon} - \frac{17g^3}{768\pi^4} - gh^{(2)}_g + \frac{9g^3}{256\pi^4}\frac{1}{\epsilon} \to gh^{(2)}_g = -\frac{17g^3}{768\pi^4} \quad . \tag{206}$$

We see that the function h is in this case extremely similar to the β-function. They both have the same signs and same factors, but h is expressed in the bare parameters and the β-function in the renormalized parameters. In [1] it is shown that for the case of one parameter we should have expected this. In our case we have however two parameters, however in the case of g there is no mass involved so we can understand that β_g is indeed the same as h_g. That we also find that h_m and β_m have the same factors can not be understood from this, but it hints that it might actually be true for more parameters as well. Note that we also found this similarity for the ϕ^3 theory. Note that the wavefunction renormalization up to second order in $\hbar$ is not required to find a scale-independent scale dependence. In this case the bare parameters expressed in the renormalized parameters are given by

$$m_B^2 = m^2 + \frac{gm^2\hbar}{32\pi^2}\frac{1}{\epsilon} + \frac{g^2m^2\hbar^2}{256\pi^4}\left(\frac{1}{2\epsilon^2} - \frac{1}{4\epsilon}\right) \quad , \tag{207}$$

$$g_B = g\mu^{2\epsilon} + \frac{3g\mu^{2\epsilon}\hbar}{32\pi^2}\frac{1}{\epsilon} + \frac{g^3\mu^{2\epsilon}\hbar^2}{256\pi^4}\left(\frac{9}{4\epsilon^2} - \frac{3}{2\epsilon}\right) \quad . \tag{208}$$

The inverse of this yields

$$m_R^2 = m^2 - \frac{m^2g\hbar}{32\pi^2\mu^{2\epsilon}}\frac{1}{\epsilon} + \frac{g^2m^2\hbar^2}{512\pi^4\mu^{4\epsilon}}\left(\frac{1}{\epsilon^2} + \frac{1}{2\epsilon}\right) \quad , \tag{209}$$

$$g_R = g - \frac{3g^2\hbar}{32\pi^2\mu^{2\epsilon}}\frac{1}{\epsilon} + \frac{g^3\hbar^2}{1024\pi^4\mu^{4\epsilon}}\left(\frac{9}{\epsilon^2} + \frac{6}{\epsilon}\right) \tag{210}$$

Inserting this in equation (129) yields the following β-functions

$$\beta_{m^2} = \frac{m^2g}{16\pi^2}\hbar - \frac{m^2g^2}{256\pi^4}\hbar^2 \quad , \tag{211}$$

$$\beta_g = \frac{3g^2}{16\pi^2}\hbar - \frac{3g^3}{128\pi^4}\hbar^2 \quad , \tag{212}$$

Although we do not find the correct β-functions it is indeed scale independent. The reason that the wavefunction renormalization is not needed to cancel the scale is because it only contains a $1/\epsilon$ at order $\hbar$. Differentiating this with respect to the scale is finite and thus only contributes to the finite part. At higher order however the wavefunction possibly contains a $1/\epsilon^n$ term with $n > 1$. In this case we do need to include the wavefunction renormalization to find a scale-independent scale dependence.

When we insert the renormalized parameters in the pCS equation we also find again that the β-functions are similar to our functions h. Let us investige why these two functions are similar. First we note that the factors of the β-functions are not the same as the factors in h when using our prescription without counterterm. It apparently thus has something to do with the method of counterterms. In the method of counterterms using the MS-scheme there are no finite terms. The only finite terms thus arise when we differentiate the renormalized parameters with respect to the scale. This differentiation occurs in both the pCS equation and in the equation for the β-function. When looking at the pCS equation we find that this finite term can never be affected since all other derivatives yield again something proportional to $1/\epsilon$ terms. In the equation for the β-function (129) this is also the case, but it is harder to see why. Let us again consider the renormalized parameters with wavefunction. Differentiating these parameters with respect to the scale yields

$$\frac{\partial}{\partial s} m_R^2 = \frac{m^2 g\hbar}{16\pi^2\mu^{2\epsilon}} - \frac{g^2 m^2 \hbar^2}{128\pi^4\mu^{4\epsilon}}\frac{1}{\epsilon} - \frac{5g^2 m^2\hbar^2}{1536\pi^4\mu^{4\epsilon}} \ , \tag{213}$$

$$\frac{\partial}{\partial s} g_R = \frac{3g^2\hbar}{16\pi^2\mu^{2\epsilon}} - \frac{9g^3\hbar^2}{256\pi^4\mu^{4\epsilon}}\frac{1}{\epsilon} - \frac{17g^3\hbar^2}{768\pi^4\mu^{4\epsilon}} \tag{214}$$

Inserting the bare paremeters expressed in the renormalized parameters only contributes up to first order since higher order terms yields something of order $\hbar^3$. We thus find that all bare parameters become renormalized parameters and we get an extra term from the first order term of the bare paremeters inserted in the first order term of the renormalized parameters. Since the first order bare parameter term only contains a $1/\epsilon$ term by construction it can not affect the finite term. The finite term is thus the β-function following this method. That is why the β-function is similar to the functions of h in this case.

5 Renormalization in zero dimensions

As mentioned before no momenta can occur in zero dimensions. Without momenta there are no loop integrals to yield infinity. In this chapter we will however construct a toy model by modifying the Feynman rules in order to assign infinities to certain loops. This way we introduce (by hand) a scale which we need to absorb in renormalized parameters. We desire that the renormalized parameters of the toy model satisfy the pCS equation. In the previous chapter we have seen how this works for the ϕ^3 and ϕ^4 theory in four dimensions. In this chapter we will try to renormalize the same theories but then with a toy model in zero dimensions.

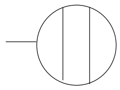
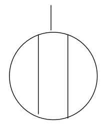
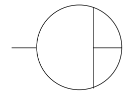
Since there is no guarantee that our toy model will work we wish to verify it at a higher order than in four dimensions. We therefore start with a section to find all the Feynman diagrams up to three loops. After that we describe a very simple toy model called the *dot model* and apply it to the ϕ^4 theory. This model was introduced in [1]; in this thesis we study it in greater detail. We will find that renormalized parameters of the ϕ^4 theory using this toy model do not satisfy the pCS equation. This is actually a quite curious situation: we can absorb all the infinities in the renormalized parameters, so formally the model is renormalizable. Nevertheless the scale dependence is scale dependent. The reason for this seems to be the sunset diagram. In the section that follows we will show a solid proof that even with changing the sunset diagram in any way you want we still can not get the dot model to work. We will then apply the dot model on the super-renormalizable ϕ^3 theory in the hope it might work in this case. We will find that it can easily be renormalized, but again the renormalized parameters do not satisfy the pCS equation. In the last section we will thus construct a more sophisticated model in the hope that such a model does work.

5.1 Third order diagrams

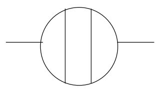
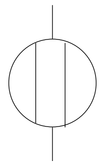
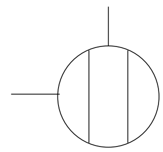
At three loops there are diagrams that are not planar. They are also numerous and because of this it is easy to overlook some. We will not look at the first and second order diagrams, since we have already determined these in previous chapters.

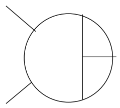
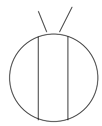
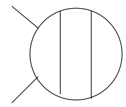
5.1.1 ϕ^3 theory

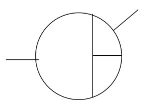
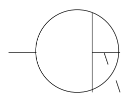
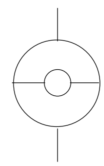
For the ϕ^3 theory we are interested in the one, two and three-leg diagrams. For the one-leg diagrams at third order we find

(17)  $\frac{1}{4}$ (18)  $\frac{1}{8}$ (19)  $\frac{1}{4}$.

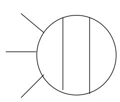
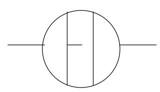
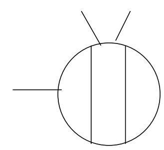
This adds up to $\frac{5}{8}$, as predicted by equation (30). For the two-leg graphs we find

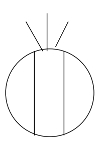
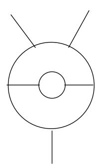
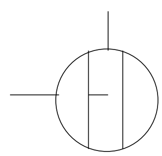
(20)  $\frac{1}{2}$ (21)  $\frac{1}{8}$ (22)  $\frac{2}{2}$

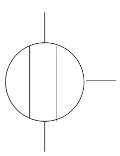
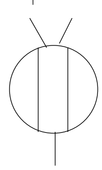
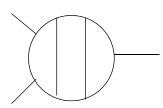
(23)  $\frac{1}{2}$ (24)  $\frac{1}{4}$ (25)  $\frac{1}{2}$

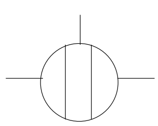
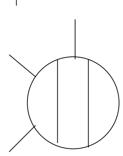
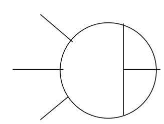
(26)  $\frac{1}{1}$ (27)  $\frac{1}{4}$ (28)  $\frac{1}{4}$.

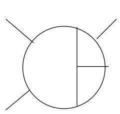
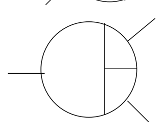
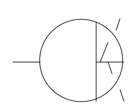
Summing this up yields $\frac{35}{8}$, which agrees with equation (30). For the three-leg diagrams we find

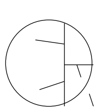
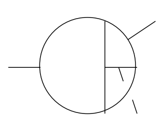
(29)  $\frac{3}{2}$ (30)  $\frac{3}{2}$ (31)  $\frac{6}{2}$

(32)  $\frac{3}{4}$ (33)  $\frac{3}{2}$ (34)  $\frac{3}{2}$

(35)  $\frac{3}{2}$ (36)  $\frac{3}{4}$ (37)  $\frac{3}{1}$

(38)  $\frac{3}{1}$ (39)  $\frac{6}{2}$ (40)  $\frac{3}{2}$

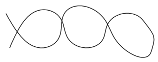
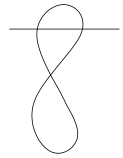
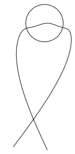
(41)  $\frac{6}{1}$ (42)  $\frac{1}{1}$ (43)  $\frac{3}{2}$

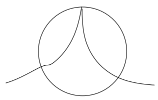
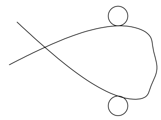
(44)  $\frac{1}{1}$ (45)  $\frac{3}{1}$.

This should add up to 35 according to equation (30), which it does.

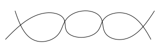
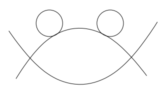
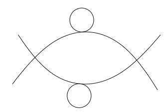
5.1.2 ϕ^4 theory

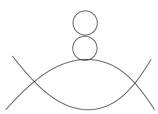
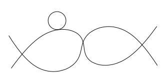
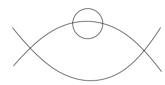
For the ϕ^4 we are interested in the two- and four-leg diagrams. The two-leg diagrams at third order are

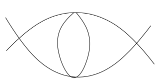
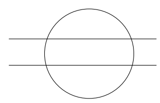
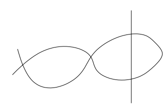
(46)  $\frac{1}{8}$ (47)  $\frac{1}{4}$ (48)  $\frac{1}{12}$

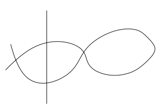
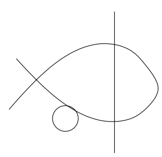
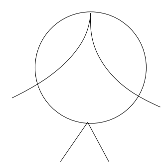
(49)  $\frac{1}{4}$ (50)  $\frac{1}{8}$.

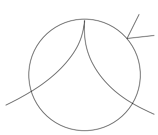
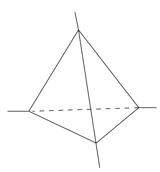
According to the effective action they should add up to $\frac{5}{6}$, which they do. For the four-leg we find

(51)  $\frac{3}{8}$ (52)  $\frac{3}{4}$ (53)  $\frac{3}{8}$

(54)  $\frac{3}{4}$ (55)  $\frac{6}{4}$ (56)  $\frac{3}{6}$

(57)  $\frac{3}{4}$ (58)  $\frac{3}{2}$ (59)  $\frac{6}{4}$

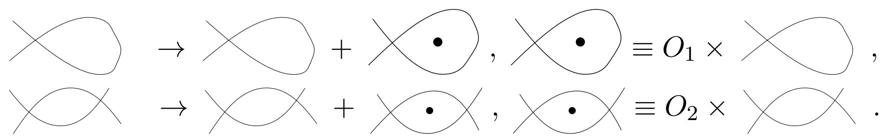
(60)  $\frac{6}{2}$ (61)  $\frac{12}{4}$ (62)  $\frac{6}{4}$

(63)  $\frac{12}{2}$ (64)  $\frac{1}{1}$.

This adds up to $\frac{45}{2}$, which corresponds with equation (30). The last diagram in this list is non-planar and was therefore hard to find. Also the multiplicity factors in some of these cases are not trivial.

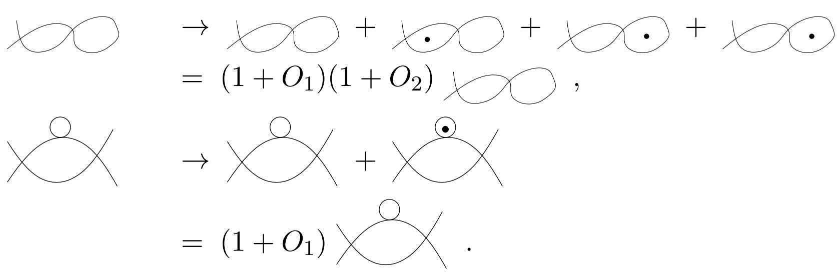
5.2 The dot model

When we discussed renormalization in chapter 3 it was mentioned that the reason we use renormalization is because we use perturbation theory and has nothing to do with the divergent loop integrals. However these divergences do make the need to do something about it all the more urgent. We have seen that loop divergences basically arise from internal degrees of freedom of Feynman diagrams. In zero dimensions we however have no such degrees of freedom, and all diagrams are finite. In this section we will introduce a toy model following [1] called the dot model that does assign infinity to certain diagrams. In order to do this we introduce a new Feynman rule: we shall apply a factor $1 + O_1$ or $1 + O_2$ to every closed loop that contains one or two propagators, respectively. Loops that contain more propagators remain unaffected. The numbers O_1 and O_2 may depend on the parameters of the theory, or on other parameters. We shall envisage that $O_{1,2} \to \infty$ at some stage, which is how we introduce infinities in our toy model. Basically we thus duplicate Feynman diagrams in the following way



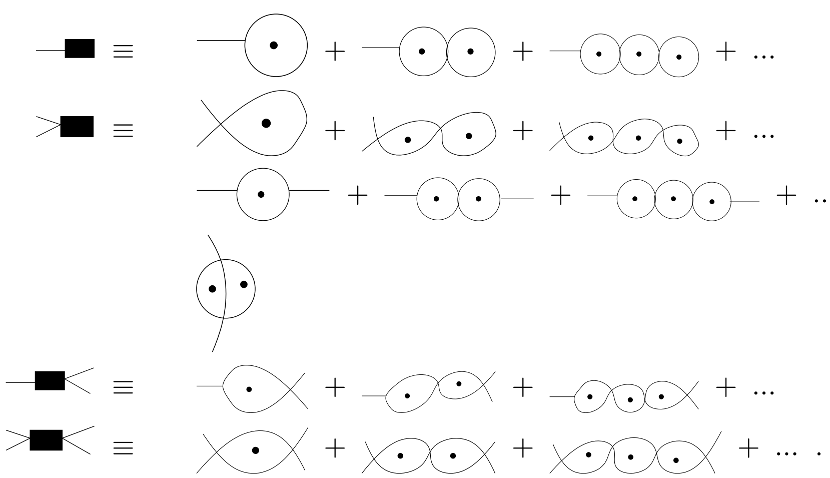
(215)

Applying the dot model on the following two-loop diagrams yield



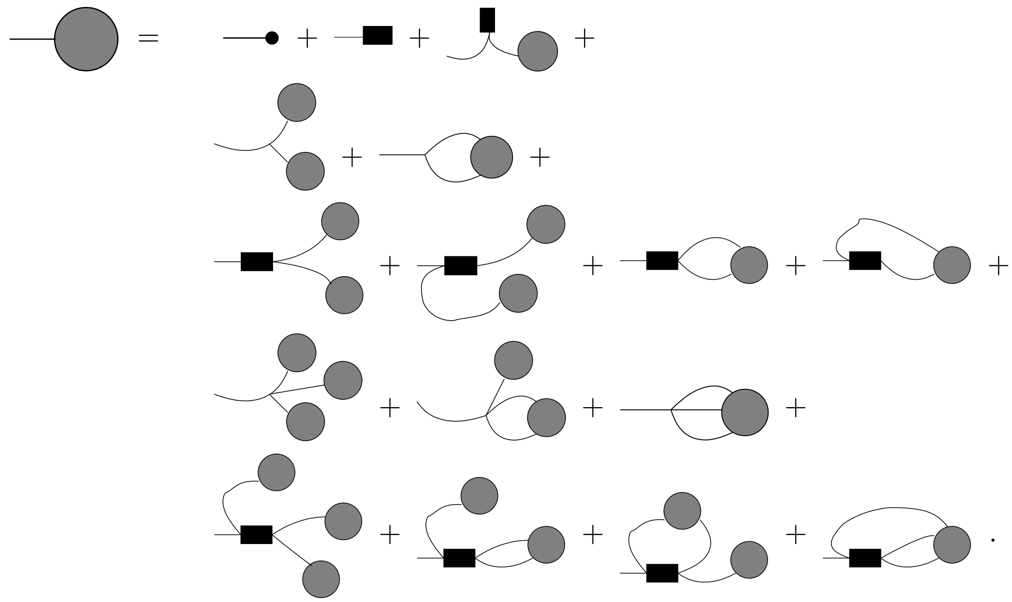
(216)

Feynman diagrams are governed by the SDe. Our new rule must therefore be implemented into a modified SDe. In order to find this modified SDe we define



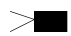
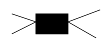
(217)

Note that the only diagram that does not carry a 'tower' of loops on its back is the last diagram in the two-point dotted series. This diagram basically contains three loops instead of two (there are two inner loops and the diagram is basically a loop itself). We can now rewrite the SDe:



(218)

We can write

 $= B_1$,  $= B_2$,  $= B_3$,  $= B_4$. (219)

When we work out the graphs in the order they are displayed we then find

$$
\begin{aligned}
\phi =& \frac{J}{\mu} - \frac{B_1}{\mu} - \frac{B_2}{\mu}\phi - \frac{\lambda_3}{2\mu}\phi^2 - \frac{\hbar\lambda_3}{2\mu}\phi' \\
& - \frac{B_3}{2\mu}\phi^2 - \frac{B_3}{\mu}\phi^2 - \frac{\hbar B_3}{2\mu}\phi' - \frac{\hbar B_3}{\mu}\phi' \\
& - \frac{\lambda_4}{6\mu}\phi^3 - \frac{\hbar\lambda_4}{2\mu}\phi\phi' - \frac{\hbar^2\lambda_4}{6\mu}\phi'' \\
& - \frac{B_4}{2\mu}\phi^3 - \frac{\hbar B_4}{2\mu}\phi\phi' - \frac{\hbar B_4}{\mu}\phi\phi' - \frac{\hbar^2 B_4}{2\mu}\phi'' \ .
\end{aligned} \tag{220}
$$

This is the 'dotted-loop' modified SDe, we can rewrite this as

$$
\begin{aligned}
(\mu + B_2)\phi =& (J - B_1) - (\lambda_3 + 3B_3)(\phi^2 + \hbar\phi') \\
& - (\lambda_4 + 3B_4)(\phi^3 + 3\hbar\phi\phi' + \hbar^2\phi'') \ .
\end{aligned} \tag{221}
$$

This is exactly the SDe belonging to the action

$$
S(\varphi) = B_1\varphi + \frac{1}{2}(\mu + B_2)\varphi^2 + \frac{1}{6}(\lambda_3 + 3B_3)\varphi^3 + \frac{1}{24}(\lambda_4 + 3B_4)\varphi^4 \ . \tag{222}
$$

We thus find that the bare parameters μ, λ_3 and λ_4 only occur in the combinations $\mu + B_2$, $\lambda_3 + B_3$ and $\lambda_4 + B_4$. This means that the values of $B_{2,3,4}$ will automatically be finite if the experimental quantities in which they enter are finite. It is therefore possible to choose the action's parameters in such a way that all Green's functions will come out finite. We can therefore always absorb the remaining B_1 into a linear term of the bare action. In our dotted model we can thus compensate the infinite loop corrections by infinite bare parameters [1].

5.3 Application of the dot model to ϕ^4 theory

Applying the dot model up to second two loops results in the following renormalized parameters

$$m_R^2 = m^2 + \frac{\hbar\lambda}{2m^2}(1+O_1) + \frac{\hbar^2\lambda^2}{4m^6}(1+O_1+O_2+O_1O_2) + \frac{\hbar^2\lambda^2}{6m^6}(1+3O_2+O_2^2) \tag{223}$$

$$\lambda_R = \lambda - \frac{3\hbar\lambda^2}{2m^4}(1+O_2) - \frac{3\hbar^2\lambda^3}{4m^8}(1+2O_2+O_2^2) - \frac{3\hbar^2\lambda^3}{m^8}(1+O_2) - \frac{3\hbar^2\lambda^3}{2m^8}(1+O_1) \ . \tag{224}$$

Inserting this in the pCS equation up to first order in $\hbar$ yields

$$\frac{\lambda}{2m^2}O_1' = m^2 h_1^{(1)} \quad , \quad \frac{-3\lambda^2}{2m^4}O_2' = \lambda h_2^{(1)} \ . \tag{225}$$

We thus find that $O_{1,2}$ must be linear functions in s. This was to be expected since in the ϕ^4 theory in four dimensions we found with dimensional regularization they diverge with $1/\epsilon$, which we choose as s. We can thus write

$$O_1 = as + b \quad , \quad O_2 = es + f \tag{226}$$

Let us first insert λ_R in the pCS equation up to second order in $\hbar$. We then find

$$-\frac{3\lambda^3}{4m^8}(2e^2s + 2ef + 2e) - \frac{3\lambda^3}{m^8}e - \frac{3\lambda^3}{2m^8}a = \lambda h_2^{(2)} + \frac{3\lambda^3}{2m^8}a(1+es+f) + \frac{9\lambda^2}{2m^8}e(1+es+f) \ . \tag{227}$$

We therefore must have $e = -\frac{1}{4}a$ in order to cancel the dependence of the scale. Inserting m_R^2 in the pCS equation then yields

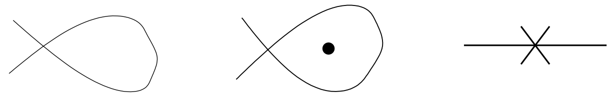
$$\frac{\lambda^2}{4m^6}(2aes + af + eb + a + e) + \frac{\lambda^2}{6m^6}(2e^2s + 2ef + 3e) = m^2h_1^{(2)} - \frac{\lambda^2}{4m^6}a(1+as+b) - \frac{3\lambda^2}{4m^6}e(1+as+b) \ . \tag{228}$$

In order for this expression to be scale independent we must have

$$\frac{5}{4}ae + \frac{1}{4}a^2 + \frac{1}{3}e^2 = 0 \ . \tag{229}$$

With $e = -\frac{1}{4}a$ we find $-\frac{2}{3}e^2 = 0$. So at second order our renormalized parameters do no longer satisfy the pCS equation. We can understand this since in four dimensions the integral of the sunset diagram times its symmetry factor yields $\frac{1}{4}$. In zero dimensions we do not have an integral and we therefore find a factor $\frac{1}{6}$. Since all other divergences of the two-leg diagrams behave the same as in four dimensions, the divergences do not cancel because of this difference of the sunset diagram. It is however also a bit unclear what the divergence of the sunset diagram is. Naively we would say O_2^2, but we also know from four dimensions this is not really true. By taking the scale to infinity we will prove in the next section however that whatever we choose for the sunset diagram, using this toy model we will not be able to find renormalized parameters that satisfy the pCS equation.

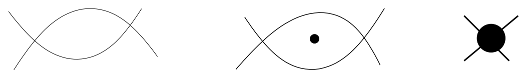
Before we move on to the next section let us first try to renormalize this toy model using counterterms. Let us first take a look at the two-leg diagrams including counterterms to first order in $\hbar$, we find



We can solve this algebraically:

$$-\frac{\hbar\lambda}{m^2}(1+O_1) - m^2 c^1_{m^2} = 0 \to c^1_{m^2} = -\frac{\hbar\lambda}{m^4}(1+O_1) \ . \tag{230}$$

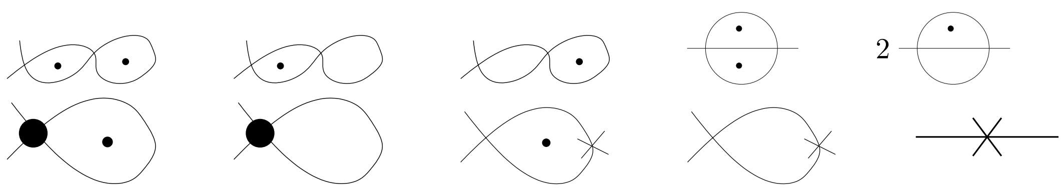
For the four-leg diagrams we find



This yields

$$\frac{3\hbar\lambda^2}{2m^4}(1+O_2) - \lambda c^1_\lambda = 0 \to c^1_\lambda = \frac{3\hbar\lambda}{2m^4}(1+O_2) \ . \tag{231}$$

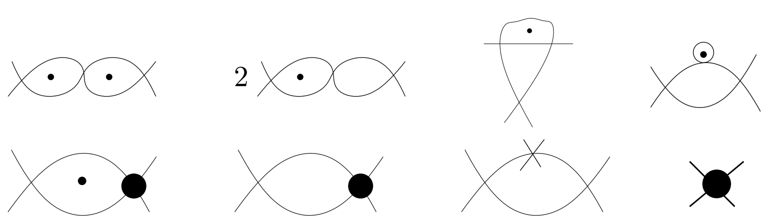
In the following we will for simplicity only write down divergent diagrams, since we are only interested in the cancellation of the diverging parts. Since at order $\hbar$ the divergences are already cancelled we shall no longer write down these contributions. At order $\hbar^2$ we then find the following diagrams



This yields

$$\begin{aligned}
&\frac{\hbar^2\lambda^2}{4m^6}(1+O_1+O_2+O_1O_2) + \frac{\hbar^2\lambda^2}{6m^6}(1+2O_2+O_2^2) - \\
&\frac{3\hbar^2\lambda^2}{4m^6}(1+O_1+O_2+O_1O_2) - \frac{\hbar^2\lambda^2}{4m^6}(1+O_1+O_2+O_1O_2) - m^2 c^2_{m^2} = 0 \to \\
&c^2_{m^2} = -\frac{\hbar^2\lambda^2}{m^8}\left(\frac{7}{12} + \frac{3}{4}O_1 + \frac{5}{12}O_2 + \frac{3}{4}O_1O_2 - \frac{1}{6}O_2^2\right)
\end{aligned} \tag{232}$$

We can do the same for the four-leg diagrams, we then find



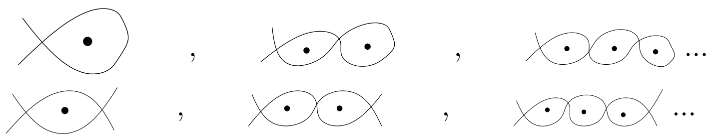
This results in

$$\begin{aligned}
&-\frac{3\hbar^2\lambda^3}{4m^8}(1+2O_2+O_2^2) - \frac{3\hbar^2\lambda^3}{m^8}(1+O_2) - \frac{3\hbar^2\lambda^3}{4m^8}(1+O_1) + \\
&\frac{9\hbar^2\lambda^3}{2m^8}(1+2O_2+O_2^2) + \frac{3\hbar^3\lambda^3}{4m^8}(1+O_1) - \frac{\hbar^2}{m^4}c^2_\lambda = 0 \to \\
&c^2_\lambda = -\frac{3\hbar^2\lambda^2}{4m^8}\left(5O_2^2 + 6O_2 + 1\right)
\end{aligned} \tag{233}$$

Although the results of the counterterms are not particularly elegant we can cancel all the divergences this way. Sadly our renormalized parameters do not satisfy the pCS equation. We can understand this because of the sunset diagram, we will prove however in the next section that it is not only because of this diagram.

5.4 Scale to infinity

If the renormalized parameters agree with the pCS equation then changing the scale should not change anything. In this section we will see what happens if we take the scale to infinity. Let us again consider the ϕ^4 theory. Using the dot model at leading order we can easily deduce that the only diagrams that contribute are



with the exception of the sunset diagram. In four dimensions we saw that these these diagrams are all one-loop reducible diagrams and are therefore product of the integrals from diagrams (10) and (11). The sunset diagram however is different. We shall therefore denote its divergent behavior with O_3. Summing up all contributions yield the following result:

$$\mu_R = \mu \left(1 + \frac{gO_1(s)}{1 + gO_2(s)} - 2g^2 O_3(s)\right) \tag{234}$$

$$\lambda_R = \lambda \left(1 - \frac{3gO_2(s)}{1 + gO_2(s)}\right) \quad , \tag{235}$$

where $g \equiv \hbar\lambda/2\mu^2$. The pCS equation for λ_R is

$$\frac{\partial}{\partial s}\lambda_R = \left(h_\mu(g)\mu\frac{\partial}{\partial\mu} + h_\lambda(g)\lambda\frac{\partial}{\partial\lambda}\right) \quad . \tag{236}$$

We therefore must have

$$h_\mu(g) = \alpha g + \rho(g)g^2 + ... \tag{237}$$

$$h_\lambda(g) = \beta g + \sigma(g)g^2 + ... \quad . \tag{238}$$

It follows that without loss of generality $O_2 = \beta s$. Taking $s \to \infty$ then yields $\lambda_R = -2\lambda$, so

$$\frac{\partial}{\partial s}\lambda_R \sim \frac{1}{s} \quad . \tag{239}$$

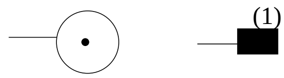
Inserting equation (235) in equation (236) then yields

$$\begin{aligned} -\frac{3gO_2'}{(1+gO_2)^2} &= h_\lambda(g)\left(1 - \frac{3gO_2}{1+gO_2} - \frac{3gO_2}{(1+gO_2)^2}\right) + h_\mu(g)\left(\frac{6gO_2}{(1+gO_2)^2}\right) \\ -\frac{3g\beta}{(1+g\beta s)^2} &= h_\lambda(g)\left(\frac{1-2g\beta s}{1+g\beta s} - \frac{3g\beta s}{(1+g\beta s)^2}\right) + h_\mu(g)\left(\frac{6g\beta s}{(1+g\beta s)^2}\right) \\ -3g\beta &= h_\lambda(g)\left(1 + 4g\beta s - 2g^2\beta^2 s^2\right) + h_\mu(g)(6g\beta s) \quad . \end{aligned} \tag{240}$$

We find that the left side is independent of s, while the right side does have an s dependence. We have now proven that the dot model does indeed not work as we have already witnessed. We also see that even if we change the sunset diagram this toy model still does not satisfy our desired conditions. In the next section we check if the dot model does maybe work on the ϕ^3 theory. Since the ϕ^3 theory is super-renormalizable it might work in this case.

5.5 Dot model in ϕ^3

Let us first try to renormalize the ϕ^3 theory using counterterms. For simplicity we shall write down only the divergent terms, since we are only interested in getting rid of the scale anyway. For the one-leg diagrams at first order we find



. (241)

This results in

$$-\frac{\hbar\lambda}{2m^2}O_1 - \tau c_\tau^1 = 0 \rightarrow c_\tau^1 = -\frac{\hbar\lambda}{2m^2\tau}O_1 \quad . \tag{242}$$

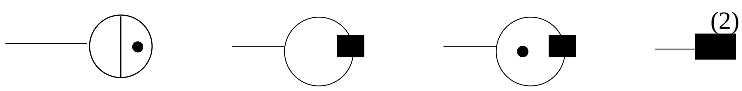
For the two-leg diagrams we find

. (243)

This yields

$$\frac{\hbar\lambda^2}{2m^4}O_2 - m^2 c_{m^2}^1 = 0 \rightarrow m^2 c_{m^2}^1 = \frac{\hbar\lambda^2}{2m^6}O_2 \quad . \tag{244}$$

At first order there are no diverging three-leg diagrams, meaning that in ϕ^3 third theory we do not have to correct the parameter λ. So the next thing we do is to cancel the divergences of the one-leg diagrams at second order.

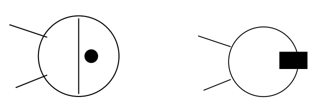


. (245)

The first two diagrams cancel each other out. The last two diagrams yield:

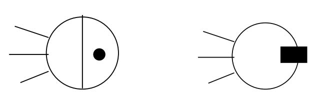
$$\frac{\hbar^2\lambda^3}{4m^8}O_2^2 - \tau c_\tau^2 = 0 \rightarrow c_\tau^2 = \frac{\hbar^2\lambda^3}{4m^8\tau}O_2^2 \quad . \tag{246}$$

At two leg we find



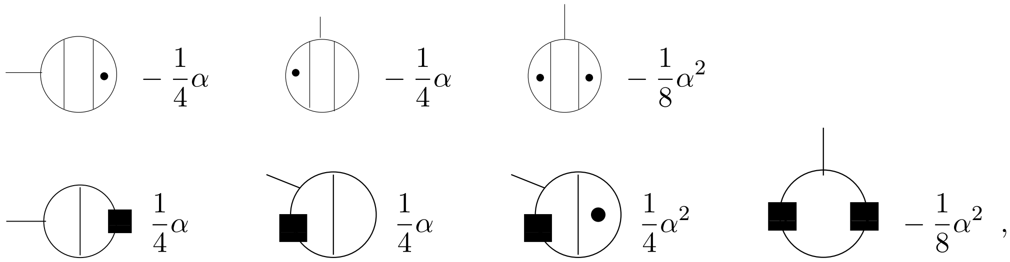
. (247)

These diagrams cancel each other out, meaning there is no second order correction for μ. For the three-leg diagrams we find



. (248)

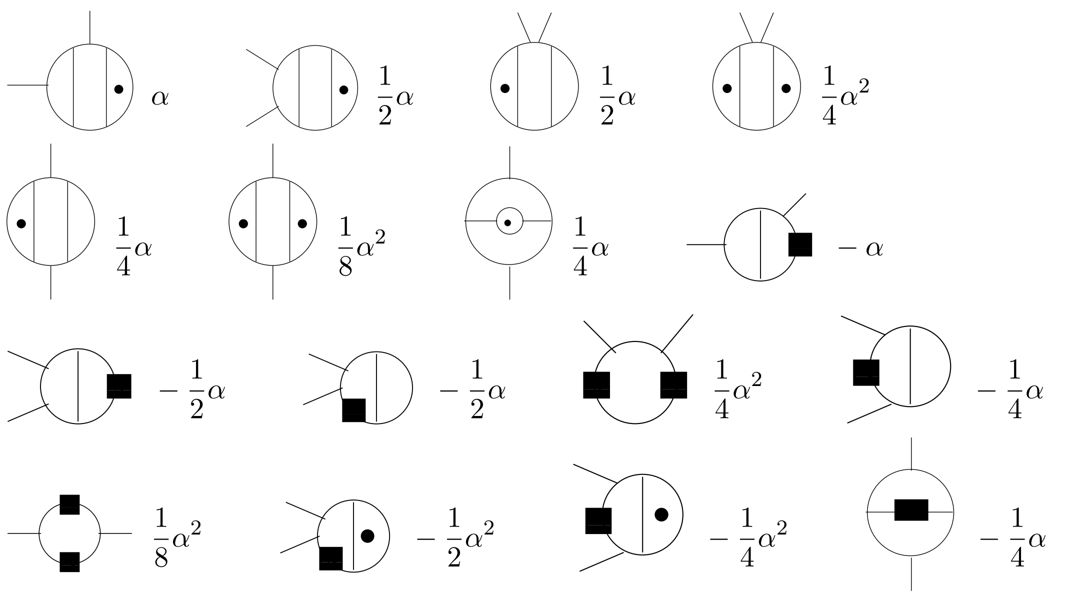
These cancel each other as was to be expected, since there were no diverging three-leg diagrams at first order. From now on we shall not write down all the contributions explicitly, because the only difference is due to combinatorics, symmetry factors and the divergent behaviour. Since there are no O_1 either, but only O_2, we will denote everything in terms of a number α. In this case a dot yields an α and a square a $-\alpha$. We also take into account whether a diagram yields a positive or a negative contribution. We will now apply this to the tadpoles. The contributing diagrams are



where we wrote its divergent behavior to the right. Summing this up yields

$$
\begin{aligned}
&-\frac{1}{4}\alpha-\frac{1}{4}\alpha-\frac{1}{8}\alpha^2\\
&+\frac{1}{4}\alpha+\frac{1}{4}\alpha+\frac{1}{4}\alpha^2-\frac{1}{8}\alpha^2=0\ .
\end{aligned}
\tag{249}
$$

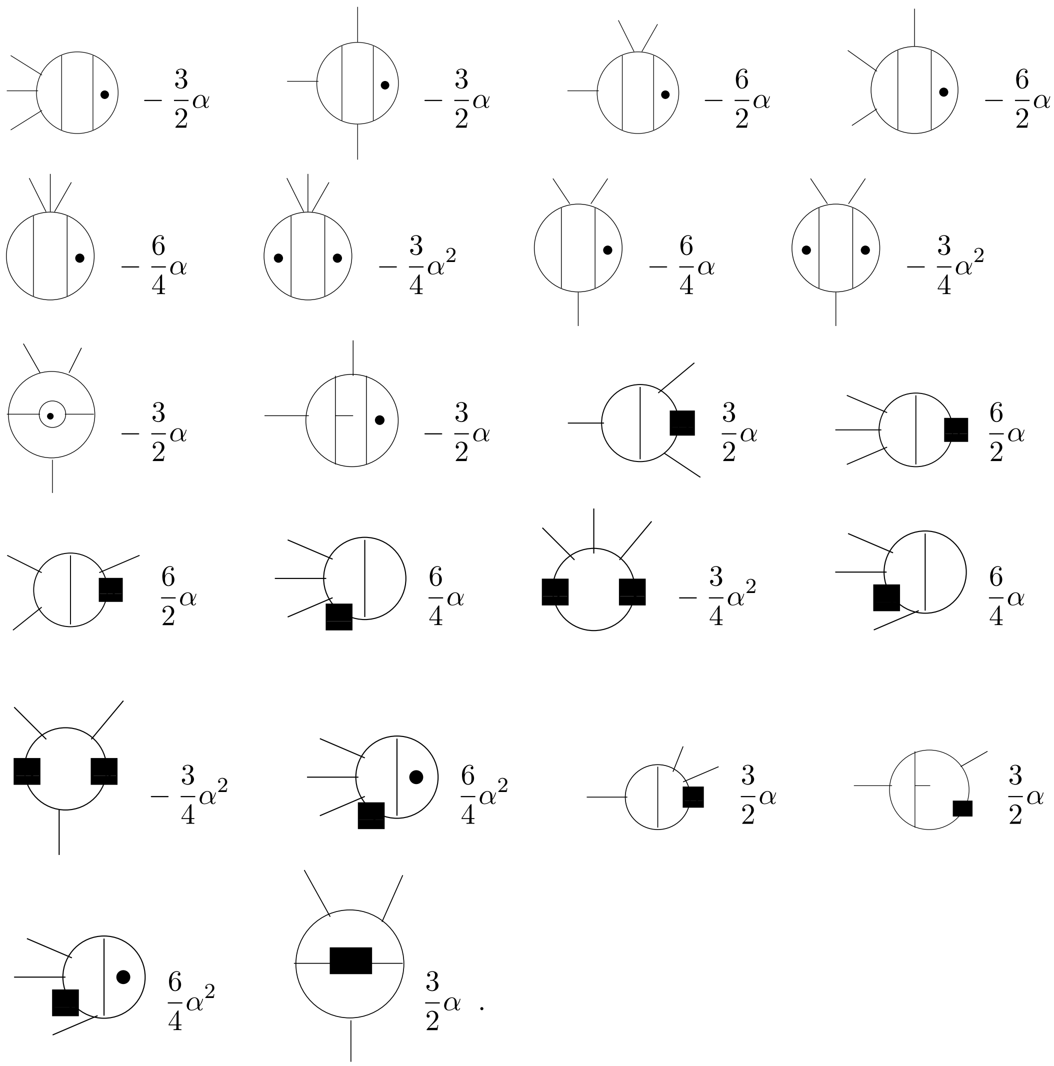
We see that everything is cancelled. For the two-leg we find



Adding this up results in

$$
\begin{aligned}
\alpha+\frac{1}{2}\alpha+\frac{1}{2}\alpha+\frac{1}{4}\alpha^2+\frac{1}{4}\alpha+\frac{1}{8}\alpha^2+\frac{1}{4}\alpha\\
-\alpha-\frac{1}{2}\alpha-\frac{1}{2}\alpha+\frac{1}{4}\alpha^2-\frac{1}{4}\alpha+\frac{1}{8}\alpha^2-\frac{1}{2}\alpha^2-\frac{1}{4}\alpha^2-\frac{1}{4}\alpha=0\ .
\end{aligned}
\tag{250}
$$

Lastly for the three-leg diagrams we find



This adds up to

$$\begin{aligned}
&-\frac{3}{2}\alpha-\frac{3}{2}\alpha-\frac{6}{2}\alpha-\frac{6}{2}\alpha-\frac{6}{4}\alpha-\frac{3}{4}\alpha^2-\frac{6}{4}\alpha-\frac{3}{4}\alpha^2-\frac{3}{2}\alpha\\
&\frac{3}{2}\alpha+\frac{3}{2}\alpha+\frac{6}{2}\alpha+\frac{6}{2}\alpha+\frac{6}{4}\alpha-\frac{3}{4}\alpha^2+\frac{6}{4}\alpha\\
&-\frac{3}{4}\alpha^2+\frac{6}{4}\alpha^2+\frac{6}{4}\alpha^2+\frac{3}{2}\alpha=0 \quad .
\end{aligned} \tag{251}$$

We see that there are only a few divergent terms that need a counterterm, after that everything vanishes automatically. This was to be expected since ϕ^3 is super-renormalizable. The bare parameters expressed in the renormalized parameters are then given by

$$\begin{aligned}
\tau_B &= \tau - \frac{\hbar\lambda}{2m^2}O_1 + \frac{\hbar^2\lambda^3}{4m^8}O_2^2\\
m_B^2 &= m^2 + \frac{\hbar\lambda^2}{2m^4}O_2 \quad .
\end{aligned} \tag{252}$$

Note how similar this is to the four-dimensional case. In both cases the bare tadpole is divergent up to second order in $\hbar$ and the bare mass only up to first order in $\hbar$. Also we find that the renormalized coupling equals the bare coupling. Using our other prescription we find the renormalized parameters in terms of the bare parameters

$$\tau_R \equiv F^1 = \tau - \frac{\lambda\hbar}{2m^2}(1+O_1) - \frac{\lambda^3\hbar^2}{4m^8}(1+O_2) \tag{253}$$

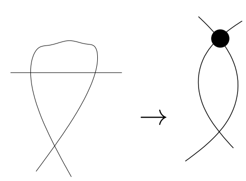
$$m_R^2 \equiv F^2 = m^2 + \frac{\lambda^2\hbar}{2m^4}(1+O_2) + \frac{\lambda^4\hbar^2}{2m^{10}}(2+O_2) \ . \tag{254}$$

Note that in this equation $m^2 = m_B^2$ and $\tau = \tau_B$, while in equation (252) $m^2 = m_R^2$ and $\tau = \tau_R$. In both cases however we find that they only satisfy the pCS equation up to order $\hbar$. In the four-dimensional case we found that using the prescription without counterterms also holds up to second order in $\hbar$. We can easilly see why it is not working in this zero dimensional case. In order to understand this let us consider the renormalized mass from equation (254). When we insert this in the pCS equation we get something finite on the right-hand side, since the renormalized mass up to second order is of order s. Since the renormalized mass is independent of τ the only term that can contain s is the term where we differentiate with respect to the mass. This is sadly also the case, so it is impossible to cancel the scale in this case.

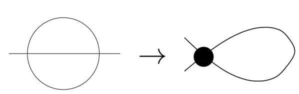
Note that the scale in the ϕ^3 theory in four dimensions we discussed in section 4.4 did drop out, while the divergent behavior was very similar. The reason for this was because the renormalized mass was proportional to the logarithm of mass at first order in $\hbar$. Differentiating this with respect to the mass resulted in an expression that is independent of s, which was fortunate since the derivative of the renormalized mass to the second order of $\hbar$ did not contain a factor s either. In the following we will construct a more sophisticated model in the hope that the renormalized parameters do satisfy the pCS equation.

5.6 A more sophisticated toy model

In the dot model we assumed that only loops connected with one or two vertices diverge. However with diagram 15 from the previous chapter we already saw this is not really true. One may wonder why diagram 15 contains a factor $1/\epsilon^2$. Intuitively we can understand this when we take the momentum in the upper loop to infinity. In this case that loop becomes a point and we are left with



Note that this dot in this case is *not* a counterterm. We see that we find something that we can duplicate with a factor O_2. So in this case we find a scale dependence of order two, like in four dimensions. From now on we will use this to assign more infinities to Feynman diagrams. In the case of the sunset diagram we now find



Note that there are three loop-momenta we can take to infinity to find this result, we therefore find $3O_1O_2$. With the symmetry factor of $\frac{1}{6}$ we thus find a factor of $\frac{1}{2}$. Let us now include counterterms and see if we can absorb all the infinities. On first order we obviously get the same results but for the two-leg on second order we now find:

$$\frac{\hbar^2\lambda^2}{4m^6}(1+O_1+O_2+O_1O_2)+\frac{\hbar^2\lambda^2}{2m^6}(1+O_2+O_1O_2)-$$
$$\frac{3\hbar^2\lambda^2}{4m^6}(1+O_1+O_2+O_1O_2)-\frac{\hbar^2\lambda^2}{4m^6}(1+O_1+O_2+O_1O_2)-m^2c_{m^2}^2=0\rightarrow$$
$$c_{m^2}^2=-\frac{\hbar^2\lambda^2}{4m^8}\left(O_1O_2+3O_1+O_2+1\right) \tag{255}$$

For the four-leg diagrams up to second order we find

$$-\frac{3\hbar^2\lambda^3}{4m^8}(1+2O_2+O_2^2)-\frac{3\hbar^2\lambda^3}{m^8}(1+O_2+O_2^2)-\frac{3\hbar^2\lambda^3}{4m^8}(1+O_1)+$$
$$\frac{9\hbar^2\lambda^3}{2m^8}(1+2O_2+O_2^2)+\frac{3\hbar^2\lambda^3}{4m^8}(1+O_1)-\lambda c_\lambda^2=0\rightarrow$$
$$c_\lambda^2=-\frac{3\hbar^2\lambda^2}{4m^8}\left(O_2^2+6O_2+1\right) \tag{256}$$

We find again that the divergences can be absorbed in the renormalized parameters. The counterterms in this more sophisticated model look a little better than in the dot model, however this does not really matter much. What we want is for our renormalized parameters to satisfy the pCS equation. From the counterterms we can construct the bare parameters in terms of the renormalized parameters, this yields

$$m_B^2=m^2-\frac{\hbar\lambda}{2m^2}(1+O_1)-\frac{\hbar^2\lambda^2}{4m^6}\left(O_1O_2+3O_1+O_2+1\right)$$
$$\lambda_B=\lambda+\frac{3\hbar\lambda^2}{2m^4}(1+O_2)-\frac{3\hbar^2\lambda^2}{4m^8}\left(O_2^2+6O_2+1\right)\quad . \tag{257}$$

Using the prescription without counterterms yields

$$m_R^2=m^2+\frac{\hbar\lambda}{2m^2}(1+O_1)+\frac{\hbar^2\lambda^2}{4m^6}(1+O_1+O_2+O_1O_2)+\frac{\hbar^2\lambda^2}{2m^6}(1+O_2+O_1O_2) \tag{258}$$
$$\lambda_R=\lambda-\frac{3\hbar\lambda^2}{2m^4}(1+O_2)-\frac{3\hbar^2\lambda^3}{4m^8}(1+2O_2+O_2^2)-\frac{3\hbar^2\lambda^3}{m^8}(1+O_2+O_2^2)-$$
$$\frac{3\hbar^2\lambda^3}{2m^8}(1+O_1)\quad . \tag{259}$$

They both do not satisfy the pCS equation. We shall show this for the last case. Inserting equation (259) in the pCS equation yields in first order of $\hbar$ yields

$$\frac{\lambda}{2m^2}O_1'=m^2h_1^{(1)}\qquad,\qquad\frac{-3\lambda^2}{2m^4}O_2'=\lambda h_2^{(1)}\quad, \tag{260}$$

just like in the dot model. We again find that $O_{1,2}$ should be linear functions in s. Let us again take

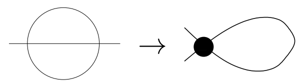
$$O_1=as+b\qquad,\qquad O_2=es+f\quad . \tag{261}$$

Inserting the renormalized parameters up to second order in $\hbar$ in the pCS equation then results in

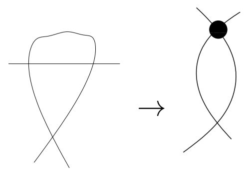
$$\frac{\lambda^2}{4m^6}(6aes+3af+3eb+3e+a)=m^2h_1^{(2)}-\frac{\lambda^2}{4m^6}a(1+as+b)-\frac{3\lambda^2}{4m^6}e(1+as+b) \tag{262}$$
$$-\frac{3\lambda^3}{4m^8}(10e^2s+10ef+6e+2a)=\lambda h_2^{(2)}+\frac{3\lambda^3}{2m^8}a(1+es+f)+\frac{9\lambda^2}{2m^8}e(1+es+f)\quad, \tag{263}$$

for m_R^2 and λ_R, respectively. We require $9e = -a$ and $8e = -a$, respectively. So sadly even with this more sophisticated model our renormalized parameters do not satisfy the pCS equation.

We can change this model even further so it is more similar to the four dimensional case. When we take one of the loop-momenta in the sunset diagram to infinity we found



We could do this in three different ways so we found $\frac{1}{6}3 = \frac{1}{2}$. The diagram we are left with has a symmetry factor of $\frac{1}{2}$. Multiplying these two results yields $\frac{1}{4}$, just like in the four dimensional case. With diagram 15 we found



There was only one way to get to this diagram and we started with a symmetry factor of $\frac{1}{2}$. The diagram we are left with has a symmetry factor and multiplicity of $\frac{6}{2}$. Multiplying these two results yields $\frac{3}{2}$, again just like in the four dimensional case. The divergences are now nearly the same as in four dimensions, however because of the different mass dependence the renormalized parameters will not satisfy the pCS equation. Note that if we use the dimensionless parameter $g = \hbar\lambda/m^4$ we find the same mass dependence as in four dimensions. Of course this is not really true, since our g is mass dependent unlike λ. If we would change λ to g in the pCS equation we would find a scale-independent scale dependence up to second order. However we are not allowed to do this.

6 Conclusion

In this thesis we derive the pCS equation that shows similarities with the CS equation. This equation can be used to verify if the scale dependence of the theory is scale-independent. We found that the function h in the pCS equation shows many similarities with the β-function of the CS equation. We also find that when our theory satisfies the pCS equation it does also satisfy the CS equation and vica versa.

We define two ways to construct the renormalized parameters, one with the use of counterterms and another one without the use of counterterms. We find that in the second method we do find a scale dependence that is independent of the scale and therefore a β-function that is scale independent, however it is not the same β-function as in the literature. This is due to the fact that when using counterterms we decouple the corrections of the parameters. Without counterterms these corrections are mixed and we therefore find different results. We do however find the correct β-function for both theories from the bare parameters expressed in the renormalized parameters. This suggests that the method without the counterterms is not completely wrong, but needs adjustment.

We try to construct a zero dimensional toy model that assign infinities in such a way the theory is renormalizable and has a scale that is independent. An interesting result is that in all of our attempts the theories are renormalizable, but none of them has a scale-independent scale dependence. We think that it is not possible to construct a toy model in zero dimensions that satisfies both conditions since in zero dimensions, since we basically add Feynman rules in order to get a similar divergent behaviour as in four dimensions, while the mass dimensions are completely different from the four dimensional case. Another problem is the occurence of one-loop irreducible diagrams, since it is very hard to determine what their divergent behaviour should be.

A Important Tools

In the following we use that the dimensionality equals $d = 2\omega$ and we denote the engineering dimension by μ. We shall use this in sections 3.3.1 and 3.3.2 to simplify the integrals.

A.1 The Feynman trick

The Feynman trick is used to rewrite the denominators in such a way that we can decouple the momenta. The Feynman trick reads

$$\begin{aligned}\frac{1}{D_1^{\alpha_1}...D_n^{\alpha_n}} &= \int_0^\infty dz_1...dz_n z_1^{\alpha_1-1}...z_n^{\alpha_n-1}\frac{\exp(-z_1D_1...-z_nD_n)}{\prod\Gamma(\alpha)}\\ &\times d\eta\delta(z_1+...+z_n-\eta)\\ &\times dx_1...dx_n\delta\left(x_1-\frac{z_1}{\eta}\right)...\delta\left(x_n-\frac{z_n}{\eta}\right)\\ &= \int_0^\infty dx_1...dx_ndz_1...dz_n\\ &\times\delta(z_1-\eta x_1)...\delta(z_1-\eta x_n)\eta^n\\ &\times x_1^{\alpha_1-1}...x_n^{\alpha_n-1}\eta^{\sum\alpha-n}\\ &\times d\eta\delta(x_1+...+x_n-1)\eta^{-1}\\ &\times\frac{\exp(\eta(x_1D_1+...x_nD_n))}{\prod\Gamma(\alpha)}\\ &= \frac{\Gamma(\sum_j\alpha_j)}{\prod_j\Gamma(\alpha_j)}\int_0^\infty dx_1...dx_n\frac{x_1^{\alpha_1-1}...x_n^{\alpha_n-1}}{(x_1D_1+...+x_nD_n)^{\sum_j\alpha_j}}\quad.\end{aligned}\tag{264}$$

A.2 The angular integral

When the Feynmantrick is performed we end up with an integral over x_i. This integral can be switched with the one over the momenta, which is now easier to solve than the original one. We can simplify the momentum integral with the angular integral

$$\begin{aligned}Q_d(s) &\equiv \frac{\mu^{4-d}}{(2\pi)^d}\int d^dp\delta(|\vec{p}|^2-s)\\ &= \frac{\mu^{4-d}}{(2\pi)^d}2^d\int_0^\infty dp_1...dp_d\delta(p_1^2+...+p_d^2-s)\\ &= \frac{\mu^{4-d}}{(2\pi)^d}s^{\omega-1}\int_0^\infty dz_1...dz_dz_1^{-1/2}...z_d^{-1/2}\delta(z_1+...z_d-1)\\ &= \frac{\mu^{4-d}}{(2\pi)^d}s^{\omega-1}\frac{\Gamma(1/2)^d}{\Gamma(\omega)}\\ &= \frac{(\mu^2)^{2-\omega}s^{\omega-1}}{(4\pi)^\omega\Gamma(\omega)}\quad.\end{aligned}\tag{265}$$

A.3 The mass integral

The angular integral yields an integral we can solve with the mass integral, this results in

$$\begin{aligned}\int_0^\infty ds \frac{s^\alpha}{(s+A)^\beta} &= A^{\alpha+1-\beta} \int_0^\infty dt \frac{t^\alpha}{(1+t)^\beta} \\ &= A^{\alpha+1-\beta} \int_1^\infty du \frac{(u-1)^\alpha}{u^\beta} \\ &= A^{\alpha+1-\beta} \int_0^1 d\nu \nu^{\beta-\alpha-2}(1-\nu)^\alpha \\ &= A^{\alpha+1-\beta} \frac{\Gamma(\alpha+1)\Gamma(\beta-\alpha-1)}{\Gamma(\beta)} \quad . \end{aligned} \tag{266}$$

We then end up with an integral over x_i alone, which are easier to solve than the integral we started with.

B Gamma functions

The Gamma function $\Gamma(z)$ is defined by

$$\Gamma(z) = \int_0^\infty dt t^{z-1} e^- t \quad . \tag{267}$$

This integral has poles for integer $z \leq 0$. Partial integration yields

$$\Gamma(z+1) = z \int_0^\infty dt t^{z-1} e^- t = z\Gamma(z) \quad . \tag{268}$$

In dimensional regularization we need expansions of the Gamma function in ϵ. In order to do this we use [15]:

$$\psi^{(n)}(z) = \frac{d^n}{dz^n}\psi(z) = \frac{d^{n+1}}{dz^{n+1}} \log \Gamma(z) \quad , \tag{269}$$

where

$$\psi^{(n)}(1) = (-1)^{n+1} n! \zeta(n+1) \quad . \tag{270}$$

We can now solve $\Gamma(\epsilon)$ with these identities and (268). We then find

$$\Gamma(\epsilon) = \frac{\Gamma(1+\epsilon)}{\epsilon} \quad . \tag{271}$$

Taylor expansion gives

$$\begin{aligned}\Gamma(1+\epsilon) &= \Gamma(1)\left[1 + \epsilon\frac{\Gamma(1)'}{\Gamma(1)} + \frac{1}{2}\epsilon^2 \frac{\Gamma(1)''}{\Gamma(1)} + ...\right] \\ &= 1 + \epsilon\psi(1) + \frac{1}{2}[\epsilon^2(\psi(1)' + \psi(1)^2)] + ... \end{aligned} \tag{272}$$

So we find that

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + \frac{1}{2}\epsilon\left[\frac{\pi^2}{6} + \gamma^2\right] \quad . \tag{273}$$

We can handle all Γ function in a similar way because of identity (268).

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