Finding the propagator of massive spin-$\frac{1}{2}$ particles

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27 September 2007

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Chapter 1

Introduction

From the theory of Dirac particles we know that the numerator of the propagator of a Dirac particle is a self-adjoint projection operator (see for instance [3]). This fact is enough to find the most general form of the numerator. In the next chapters we will show how this can be done and that we indeed find the correct numerators. The begin of a solution of this problem has already been made in [1].

From now on the numerator of a propagator will be called propagator for short. We will consider three different spaces; besides the usual Minkovski space, we will also look at two other spaces: a two dimensional space with one time- and one spacelike dimension, and a four dimensional space with two time- and two spacelike dimensions. Each of these spaces has a (different) Clifford algebra associated with it. The propagators of massive spin-$\frac{1}{2}$ particles are expressed in elements of this Clifford algebra. We will calculate which (combinations of) elements are self-adjoint projection operators. The results of these calculations should be the propagators of massive spin-$\frac{1}{2}$ particles.

The following chapters deal with the propagators of the three different spaces. In each chapter we will first look at the Clifford algebra of the space. Then we will determine what equations have to be satisfied so that the elements of the Clifford algebra form a self-adjoint projection operator. Finally, when the equations are solved, we can see what the general form of the propagators is.

Chapter 2

Finding the propagator in a 1+1 dimensional space

A 1+1 dimensional space is a 2 dimensional space with 1 timelike and 1 spacelike dimension. We study this case first, because it is easier than a four dimensional space, and it shows already most of the techniques that we will use in those spaces. For finding the propagator[1] of a massive spin-$\frac{1}{2}$ particle, we need to study the Clifford algebra first. The propagator will be written out in terms of elements of the Clifford algebra. Next we will look for the equations that have to be satisfied by the propagator, also in terms of Clifford algebra elements. Then we will need to find all possible solutions. We will see that one of these solutions can be transformed to another one, with some kind of unitary transformation.

2.1 The Clifford algebra

The Clifford algebra is defined by anticommutation relations:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \cdot I \tag{2.1}$$

where $g^{\mu\nu}$ is the metrical tensor. This tensor is here:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.2}$$

meaning that there is one timelike and one spacelike dimension. We need to find a representation of the elements γ^0 and γ^1 of the Clifford algebra. One possible representation is:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

see for instance [2]. There is also an involution (Dirac conjugation)$^-$ defined by:

$$\overline{\Omega} = \gamma^0\Omega^\dagger\gamma^0 \tag{2.3}$$

[1] that is, to find the numerator of it

with the following properties:

$$\bar{i} = -i$$
$$\overline{\gamma^{\mu_0} \dots \gamma^{\mu_n}} = \gamma^{\mu_n} \dots \gamma^{\mu_0}$$

For convenience a γ^3 is also defined:

$$\gamma^3 = \gamma^0 \gamma^1 \tag{2.4}$$

2.2 The equations

A general matrix of the Clifford algebra Γ can be written in terms of the elements γ^0 and γ^1 of this algebra as:

$$\Gamma = S \cdot I + \not{V} + P\gamma^3 \tag{2.5}$$

where I is the identity and $\not{V} = V_0\gamma^0 + V_1\gamma^1$, the usual feynman slash notation. A spin-$\frac{1}{2}$ propagator is a self-adjoint projection operator, meaning that it has to satisfy the following equations:

$$\Pi\Pi - \Pi = 0, \qquad \bar{\Pi} = \Pi \tag{2.6}$$

$\bar{\Gamma} = \Gamma$ means that:

$$\overline{S \cdot I} = S \cdot I \Rightarrow \quad S \text{ real} \tag{2.7}$$
$$\overline{V_\alpha \gamma^\alpha} = V_\alpha \gamma^\alpha \Rightarrow \quad V_\alpha \text{ real} \tag{2.8}$$
$$\overline{P\gamma^3} = P\gamma^3 \quad \Leftrightarrow \overline{P} = -P \quad \Rightarrow P = iR \quad R \text{ real} \tag{2.9}$$

Another, more convenient way of writing Γ is:

$$\Gamma = \left(S + \frac{1}{2}\right) \cdot I + \not{V} + iR\gamma^3 \tag{2.10}$$

Now the equation $\Gamma^2 - \Gamma = 0$ means:

$$\begin{aligned}\Gamma^2 - \Gamma = &\left(S^2 + V_\mu V^\mu - R^2 - \frac{1}{4}\right) \cdot I \\ &+ 2SV_\alpha\gamma^\alpha \\ &+ 2iSR\gamma^3 \\ =&0\end{aligned} \tag{2.11}$$

Because all components of the matrix have to be 0, there are in fact three equations in formula 2.11. These are the equations that have to be solved.

2.3 Solutions and transformations

The equations will be solved in a systematic way, step by step. First, only S will be taken nonzero, next only $\not{V}$, etc. Then we will look at combinations of the parameters until all parameters are nonzero. Then we can see which combinations are possible solutions. The combinations are noted down by the nonzero parameters: for instance the case $S \neq 0$ means $\Gamma = \left(S + \frac{1}{2}\right) \cdot I$, etc. We will use this systematic method also in the cases of the four dimensional spaces. At one of the solutions the propagator can be transformed to a more simple type of propagator. To do that one must use a transformation of the type:

$$\begin{aligned} \Gamma_2 &= \Sigma \Gamma_1 \overline{\Sigma} \\ \text{with} \quad \Sigma\overline{\Sigma} &= 1 \end{aligned} \tag{2.12}$$

In this way, if Γ_1 is a self-adjoint projection operator, Γ_2 is too. We may do this, because this is in fact a redefinition of the (quantum field) theory. A propagator is always placed between two vertices. That means that we may call the numerator of the propagator $\Sigma\Gamma\bar{\Sigma}$ instead of Γ and absorb a factor Σ into the vertex. That is why we may call Γ_1 and Γ_2 equivalent.

2.3.1 The case $S \neq 0$

The following equation must be satisfied:

$$\begin{aligned} & S^2 - \frac{1}{4} = 0 \\ \Leftrightarrow \quad & S = \pm\frac{1}{2} \end{aligned}$$

This leads to:[2]

$$\Gamma = 0 \tag{2.13}$$

2.3.2 The case $V_\alpha \neq 0$

The following equation must hold:

$$\begin{aligned} & V_\mu V^\mu - \frac{1}{4} = 0 \\ \Leftrightarrow \quad & V_\mu V^\mu = \frac{1}{4} \end{aligned}$$

This leads to:

$$\Gamma = \frac{1}{2} \cdot I + \not{V} \tag{2.14}$$

[2] If Γ is a solution, $1 - \Gamma$ is too.

2.3.3 The case $R \neq 0$

Now we have the following equation:

$$\begin{aligned} & -R^2 - \frac{1}{4} = 0 \\ \Leftrightarrow \quad & -R^2 = \frac{1}{4} \end{aligned}$$

This is not a valid solution because $R \in \mathbb{R}$.

2.3.4 The case $S \neq 0, \quad V_\alpha \neq 0$

One of the equations that must hold is:

$$\begin{aligned} & 2SV_\alpha = 0 \\ \Leftrightarrow \quad & S = 0 \vee V_\alpha = 0 \end{aligned}$$

So this is not a valid solution.

2.3.5 The case $S \neq 0, \quad R \neq 0$

In this case, one of the equations that must be satisfied is:

$$\begin{aligned} & 2iSR = 0 \\ \Leftrightarrow \quad & S = 0 \vee R = 0 \end{aligned}$$

So this is not a valid solution.

2.3.6 The case $V_\alpha \neq 0, \quad R \neq 0$

The following equation must be satisfied:

$$\begin{aligned} & V_\mu V^\mu - R^2 - \frac{1}{4} = 0 \\ \Leftrightarrow \quad & R^2 = V_\mu V^\mu - \frac{1}{4} \end{aligned}$$

This leads to:

$$\Gamma = \frac{1}{2} \cdot I + \not{V} + iR\gamma^3 \tag{2.15}$$

This Γ can be transformed with:

$$\Sigma = \frac{1}{\sqrt{\cos(2\alpha)}} \left(\cos(\alpha) \cdot I - i\sin(\alpha) \frac{\gamma^3 \not{V}}{\sqrt{V^2}} \right) \tag{2.16}$$

where Σ has the properties defined in equation 2.12. This means that:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{\cos(2\alpha)} \left(\frac{1}{2}\cos(2\alpha) + \left(1 - \frac{R}{\sqrt{V^2}}\sin(2\alpha)\right) \not{V} + i\left(R - \sqrt{V^2}\sin(2\alpha)\right)\gamma^3 \right)$$

Choose:

$$R - \sqrt{V^2}\sin(2\alpha) = 0$$

This leads to

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I \pm \frac{1}{2}\frac{\not{V}}{\sqrt{V^2}} \tag{2.17}$$

This solution is of the same type as the propagator in formula 2.14. It means that these two propagators are equivalent.

2.3.7 The case $S \neq 0, \quad V_\alpha \neq 0, \quad R \neq 0$

The following equations must hold:

$$\begin{aligned} 2SV_\alpha = 0 \quad &\Leftrightarrow \quad S = 0 \vee V_\alpha = 0 \\ 2iSR = 0 \quad &\Leftrightarrow \quad S = 0 \vee R = 0 \end{aligned}$$

So this is not a valid solution.

2.4 Conclusion

In this space, there are three distinct solutions, of which one can be transformed to another, and one is really trivial. The remaining propagator is one with a momentum term in it. We can see that by substituting $\not{V} \to \frac{\not{p}}{2m}$ in equation 2.14:

$$\Gamma = \frac{1}{2m}\left(m + \not{p}\right) \tag{2.18}$$

so $p^2 = m^2$, as it should be. Note that there is no spin-term. We could also substitute $\not{V} \to \frac{\gamma^3 \not{s}}{2}$, because in this Clifford algebra the product of three gamma matrices is the same as one gamma matrix. Then we would have a spin term, but no momentum term. In fact, they are equivalent.

Chapter 3

Finding the propagator in Minkovski space

The Minkovski space is the usual space with one timelike and three spacelike dimensions. We will follow the same tactics as in the previous chapter. The Clifford algebra, in which we express the (numerator of) the propagator, is just the Dirac algebra. We know that the propagator of a massive spin-$\frac{1}{2}$ particle is a self-adjoint projection operator, so we need to find the equations that must be satisfied by the propagator. We will solve these equations in the same systematic way as in chapter 2. Most of the solutions can be transformed to simpler solutions. The only solutions that remain are, if the theory is right, the propagators to be expected in quantum field theory.

3.1 The Clifford algebra

This is the Minkovski metric:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.1}$$

The γ matrices of the belonging Clifford algebra, the Dirac algebra, have to satisfy:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \cdot I \tag{3.2}$$

One possible representation is:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The dirac conjugation is defined by:

$$\overline{\Omega} = \gamma^0 \Omega^\dagger \gamma^0 \tag{3.3}$$

with the following properties:

$$\bar{i} = -i$$
$$\overline{\gamma^{\mu_0} \dots \gamma^{\mu_n}} = \gamma^{\mu_n} \dots \gamma^{\mu_0}$$

Two more definitions, $\sigma^{\mu\nu}$ and γ^5:

$$\sigma^{\mu\nu} = \frac{i}{2} \left(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu \right) \tag{3.4}$$
$$\gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \tag{3.5}$$

Products of three γ matrices are usually written as $\gamma^5 \gamma^\mu$.

3.2 The equations

In Minkovski space, a Γ matrix[1] can in general be written as:

$$\Gamma = S \cdot 1 + V_\alpha \gamma^\alpha + T_{\beta\delta} \sigma^{\beta\delta} + A_\lambda \gamma^5 \gamma^\lambda + P \gamma^5 \tag{3.6}$$

Because the propagator Γ of a massive spin-$\frac{1}{2}$ particle is a projection operator, it has to satisfy:

$$\Pi\Pi - \Pi = 0, \qquad \overline{\Pi} = \Pi \tag{3.7}$$

$\bar{\Gamma} = \Gamma$ means that:

$$\overline{S \cdot I} = S \cdot I \Rightarrow \quad S \text{ real} \tag{3.8}$$
$$\overline{V_\alpha \gamma^\alpha} = V_\alpha \gamma^\alpha \Rightarrow \quad V_\alpha \text{ real} \tag{3.9}$$
$$\overline{T_{\beta\delta} \sigma^{\beta\delta}} = T_{\beta\delta} \sigma^{\beta\delta} \Rightarrow \quad T_{\beta\delta} \text{ real} \tag{3.10}$$
$$\overline{A_\lambda \gamma^5 \gamma^\lambda} = A_\lambda \gamma^5 \gamma^\lambda \Rightarrow \quad A_\lambda \text{ real} \tag{3.11}$$
$$\overline{P \gamma^5} = P \gamma^5 \quad \Leftrightarrow \overline{P} = -P \quad \Rightarrow P = iR \quad R \text{ real} \tag{3.12}$$

[1] $T_{\alpha\beta} = -T_{\beta\alpha}$, $T_{\alpha\beta}\sigma^{\alpha\beta} = T_{\beta\alpha}\sigma^{\beta\alpha}$ and $T_{\alpha\alpha} = 0$

Now, with this information and the substitution $S \to S + \frac{1}{2}$, Γ is written as follows:

$$\Gamma = \left(S + \frac{1}{2}\right) \cdot I + V_\alpha \gamma^\alpha + T_{\beta\delta}\sigma^{\beta\delta} + A_\lambda \gamma^5 \gamma^\lambda + iR\gamma^5 \quad (3.13)$$

The substitution $S \to S + \frac{1}{2}$ makes the equation of $\Gamma^2 - \Gamma = 0$ easier:

$$\begin{aligned}\Gamma^2 - \Gamma = &\left(S^2 + V_\mu V^\mu + 2T_{\mu\nu}T^{\mu\nu} - A_\mu A^\mu - R^2 - \frac{1}{4}\right) \cdot I \\ &+ \left(2SV_\alpha - 2\epsilon^{\mu\nu\rho}{}_\alpha T_{\mu\nu}A_\rho\right) \cdot \gamma^\alpha \\ &+ \left(2ST_{\beta\delta} + \epsilon^{\mu\nu}{}_{\beta\delta}(V_\mu A_\nu + T_{\mu\nu}R)\right) \cdot \sigma^{\beta\delta} \\ &+ \left(2SA_\lambda - 2\epsilon^{\mu\nu\rho}{}_\lambda V_\mu T_{\nu\rho}\right) \cdot \gamma^5\gamma^\lambda \\ &+ i \cdot (2SR - \epsilon^{\mu\nu\rho\pi}T_{\mu\nu}T_{\rho\pi}) \cdot \gamma^5 \\ =&0 \end{aligned} \quad (3.14)$$

In a similar way as in the 1+1 dimensional case, formula 3.14 are in fact five equations, because all components of Γ have to be 0. The same equations can also be found with

$$\mathrm{Tr}\left((\Gamma^2 - \Gamma)P\right) = 0 \quad (3.15)$$

where P is an arbitrary element of the Dirac algebra. One can construct two other equations with this formula that are handy in solving $\Gamma^2 - \Gamma = 0$:

$$\begin{aligned}\mathrm{Tr}\left((\Gamma^2 - \Gamma)(V_\alpha\gamma^5\gamma^\alpha - A_\lambda\gamma^\lambda)\right) &= -16SV_\mu A^\mu = 0 \quad (3.16)\\ \mathrm{Tr}\left((\Gamma^2 - \Gamma)(A_\lambda\gamma^5\gamma^\lambda - V_\alpha\gamma^\alpha)\right) &= -8S(V_\mu V^\mu + A_\nu A^\nu) = 0 \quad (3.17)\end{aligned}$$

3.3 Solutions and transformations

As said before, we will solve the equations in the same way as in chapter 2. Some of the parameters of the Γ of formula 3.13 will be taken explicitly zero and we will see whether this is a solution or not. Most of the solutions we find can be transformed to simpler solutions with the help of a transformation of the type of formula 2.12 adjusted to the now used Clifford algebra. Because the scheme of transformations is somewhat complicated, an overview has been made which can be found in appendix A on page 56. The solutions that cannot be transformed to simpler cases are called standard propagators.

3.3.1 The case $S \neq 0$

The following equation must be satisfied:

$$\begin{aligned} &S^2 - \frac{1}{4} = 0 \\ \Leftrightarrow \quad &S = \pm\frac{1}{2}\end{aligned}$$

This leads to:[2]

$$\Gamma = 0 \tag{3.18}$$

This is the trivial standard propagator.

3.3.2 The case $V_\alpha \neq 0$

Now the following equation must be satisfied:

$$V_\mu V^\mu - \frac{1}{4} = 0$$
$$\Leftrightarrow \quad V_\mu V^\mu = \frac{1}{4}$$

This leads to:

$$\Gamma = \frac{1}{2} \cdot I + \not{V} \tag{3.19}$$

This is the standard propagator with a momentum term, which can be seen with the substitution $\not{V} \to \frac{\not{p}}{2m}$, so $p^2 = m^2$.

3.3.3 The case $T_{\beta\delta} \neq 0$

Now, there are two equations:

$$2T_{\mu\nu}T^{\mu\nu} - \frac{1}{4} = 0$$
$$-\epsilon^{\mu\nu\rho\pi} T_{\mu\nu} T_{\rho\pi} = 0$$

One can find two different solutions for these equations:

1.
$$T_{\beta\delta} = \frac{1}{2}(k_\beta l_\delta - k_\delta l_\beta)$$

This leads to:

$$\Gamma = \frac{1}{2} \cdot I + i\not{k}\not{l} \tag{3.20}$$

with $k^2 l^2 = \frac{1}{4} \wedge k_\mu l^\mu = 0$. This means that $k^2 < 0$ and $l^2 < 0$. Now take the following transformation:

$$\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha)\frac{\gamma^5 \not{k}}{\sqrt{-k^2}} \tag{3.21}$$

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + i\cos(2\alpha)\not{k}\not{l} - \sqrt{-k^2}\sin(2\alpha)\gamma^5\not{l}$$

[2] If Γ is a solution, $1 - \Gamma$ is too.

Choose $\cos(2\alpha) = 0 \quad \Leftrightarrow \sin(2\alpha) = n, \quad n = \pm 1$

$$\Rightarrow \quad \Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I - n\sqrt{-k^2}\gamma^5 \not{l}$$

Define $A_\mu = -n\sqrt{-k^2} l_\mu$. This A_μ has to satisfy $A_\mu A^\mu = -\frac{1}{4}$:

$$\Rightarrow \quad A_\mu A^\mu = (-n\sqrt{-k^2})^2 l_\mu l^\mu = -k^2 l^2 = -\frac{1}{4}$$

This means that

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \gamma^5 \not{A} \tag{3.22}$$

which is the standard propagator with a spin term, which can be seen with the substitution $\not{A} \to \frac{\not{s}}{2}$. This also means $s^2 = -1$.

2.

$$T_{\beta\delta} = -\frac{1}{2} k_\mu l_\nu \epsilon^{\mu\nu}{}_{\beta\delta}$$

This leads to:

$$\Gamma = \frac{1}{2} \cdot I + \gamma^5 \not{k}\not{l} \tag{3.23}$$

with $k^2 l^2 = -\frac{1}{4} \wedge k_\mu l^\mu = 0$. This means that only one of k^2, $l^2 > 0$. Now take the following transformation:

$$\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha) \cdot \gamma^5 \not{Q} \tag{3.24}$$

with $Q^2 = -1$, $Q_\mu k^\mu = 0$ and $Q_\mu l^\mu = 0$.

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \cos(2\alpha)\gamma^5 \not{k}\not{l} + i\sin(2\alpha)\not{k}\not{l}\not{Q}$$

Choose $\cos(2\alpha) = 0 \quad \Leftrightarrow \sin(2\alpha) = n, \quad n = \pm 1$

$$\Rightarrow \quad \Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + in\not{k}\not{l}\not{Q}$$

The following equation applies:

$$\begin{aligned} \not{k}\not{l}\not{Q} &= -i\epsilon^{\mu\nu\rho\pi} k_\nu l_\rho Q_\pi \gamma^5 \gamma^\mu \\ (i\not{k}\not{l}\not{Q})^2 &= k^2 l^2 Q^2 \cdot I = \frac{1}{4} \cdot I \end{aligned}$$

This means that

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \gamma^5 \not{A} \quad \text{with} \quad A^2 = -\frac{1}{4} \tag{3.25}$$

Both solutions can be transformed to the same standard propagator. This means that they are also equivalent to each other.

3.3.4 The case $A_\lambda \neq 0$

The following equation applies:

$$
\begin{aligned}
& -A_\mu A^\mu - \frac{1}{4} = 0 \\
\Leftrightarrow \quad & A_\mu A^\mu = -\frac{1}{4}
\end{aligned}
$$

This leads to:

$$\Gamma = \frac{1}{2} \cdot I + \gamma^5 \not{A} \tag{3.26}$$

This is the standard propagator with a spin term, that we have already found in the previous section (see formulas 3.22 and 3.25).

3.3.5 The case $R \neq 0$

Now we have

$$
\begin{aligned}
& -R^2 - \frac{1}{4} = 0 \\
\Leftrightarrow \quad & R^2 = -\frac{1}{4}
\end{aligned}
$$

This is not a valid solution because $R \in \mathbb{R}$.

3.3.6 The case $S \neq 0, \quad V_\alpha \neq 0$

Now, there are two nonzero elements. From the following equation

$$
\begin{aligned}
& 2SV_\alpha = 0 \\
\Leftrightarrow \quad & S = 0 \quad \vee \quad V_\alpha = 0
\end{aligned}
$$

one can see that this is not a valid solution.

3.3.7 The case $S \neq 0, \quad T_{\beta\delta} \neq 0$

The following equation must hold:

$$
\begin{aligned}
& 2ST_{\beta\delta} = 0 \\
\Leftrightarrow \quad & S = 0 \quad \vee \quad T_{\beta\delta} = 0
\end{aligned}
$$

So this is not a valid solution.

3.3.8 The case $S \neq 0, \quad A_\lambda \neq 0$

The following equation must be satisfied:

$$\begin{aligned} & 2SA_\lambda = 0 \\ \Leftrightarrow \quad & S = 0 \quad \vee \quad A_\lambda = 0 \end{aligned}$$

So this is not a valid solution.

3.3.9 The case $S \neq 0, \quad R \neq 0$

This equation must hold:

$$\begin{aligned} & 2SR = 0 \\ \Leftrightarrow \quad & S = 0 \quad \vee \quad R = 0 \end{aligned}$$

So this is not a valid solution.

3.3.10 The case $V_\alpha \neq 0, \quad T_{\beta\delta} \neq 0$

The following equations must be satisfied:

$$\begin{aligned} & -2\epsilon^{\mu\nu\rho}{}_\lambda V_\mu T_{\nu\rho} = 0 \\ & -\epsilon^{\mu\nu\rho\pi} T_{\mu\nu} T_{\rho\pi} = 0 \\ \Leftrightarrow \quad & T_{\beta\delta} = V_\beta W_\delta - W_\beta V_\delta \end{aligned}$$

Now use:

$$\begin{aligned} & V_\mu V^\mu + 2T_{\mu\nu}T^{\mu\nu} - \frac{1}{4} = 0 \\ \Rightarrow \quad & V_\mu V^\mu + 4(V_\mu V^\mu)(W_\nu W^\nu) - 4(V_\rho W^\rho)^2 - \frac{1}{4} = 0 \end{aligned} \tag{3.27}$$

This means that $V^2 \neq 0$. So Γ can, with the help of

$$(k_\mu l_\nu - k_\nu l_\mu)\sigma^{\mu\nu} = i(\not{k}\not{l} - \not{l}\not{k}) \tag{3.28}$$

be written as:

$$\Gamma = \frac{1}{2} \cdot I + \not{V} + i(\not{V}\not{W} - \not{W}\not{V}) \tag{3.29}$$

The following substitution can always be made:

$$W_\mu \quad \rightarrow \quad W_\mu + \beta V_\mu \tag{3.30}$$

because:

$$\begin{aligned}\not{V}\not{W} - \not{W}\not{V} \to& \not{V}(\not{W} + \beta\not{V}) - (\not{W} + \beta\not{V})\not{V} \\ =& \not{V}\not{W} + \beta\not{V}\not{V} - \not{W}\not{V} - \beta\not{V}\not{V} \\ =& \not{V}\not{W} - \not{W}\not{V}\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}& V^2 + 4V^2W^2 - 4(V_\rho W^\rho)^2 - \frac{1}{4} = 0 \\ \to\quad & V^2 + 4V^2(W_\mu + \beta V_\mu)(W^\mu + \beta V^\mu) - 4(V_\rho(W^\rho + \beta V^\rho))^2 - \frac{1}{4} = 0 \\ \Leftrightarrow\quad & V^2 + 4V^2W^2 + 4\beta^2(V_\rho W^\rho)^2 + 8\beta V^2(V_\nu W^\nu) \\ & - 4\beta^2(V_\rho W^\rho)^2 - 8\beta V^2(V_\nu W^\nu) - 4(V_\mu W^\mu)^2 - \frac{1}{4} = 0 \\ \Leftrightarrow\quad & V^2 + 4V^2W^2 - 4(V_\rho W^\rho)^2 - \frac{1}{4} = 0\end{aligned} \tag{3.32}$$

Now choose:

$$\beta = -\frac{V_\mu W^\mu}{V_\nu V^\nu} \tag{3.33}$$

This means that W_ρ can be chosen in such a way that $V_\rho W^\rho = 0$:

$$V_\rho W^\rho \to V_\rho \left(W^\rho - \frac{V_\mu W^\mu}{V_\nu V^\nu} V^\rho \right) = 0$$

This means that formulas 3.29 and 3.27 can be simplified to:

$$\Gamma = \frac{1}{2} \cdot I + \not{V} + 2i\not{V}\not{W} \tag{3.34}$$

$$V^2 + 4V^2W^2 - \frac{1}{4} = 0 \tag{3.35}$$

with $V_\rho W^\rho = 0$. Now there are two cases, $V^2 > 0$ en $V^2 < 0$.

1. $V^2 > 0$. Take the following transformation:

$$\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha)\frac{\not{V}}{\sqrt{V^2}} \tag{3.36}$$

   This means that:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \not{V} + 2\sqrt{V^2}\sin(2\alpha)\not{W} + 2i\cos(2\alpha)\not{V}\not{W}$$

   Choose $\cos(2\alpha) = 0 \quad \Leftrightarrow \sin(2\alpha) = n, \quad n = \pm 1$

$$\Rightarrow \quad \Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \not{V} + 2n\sqrt{V^2}\not{W}$$

Define $U_\mu = V_\mu + 2n\sqrt{V^2}W_\mu$. This U_μ has to satisfy $U_\mu U^\mu = \frac{1}{4}$:

$$\begin{aligned}\Rightarrow \quad U_\mu U^\mu =& \left(V_\mu + 2n\sqrt{V^2}W_\mu\right)\left(V^\mu + 2n\sqrt{V^2}W^\mu\right)\\ =& V^2 + 4n\sqrt{V^2}(V_\mu W^\mu) + 4n^2V^2W^2 = V^2 + 4V^2W^2\\ =& \frac{1}{4}\end{aligned}$$

This means that

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2}\cdot I + \not{U} \tag{3.37}$$

which is the standard propagator with a momentum term (see formula 3.19 in section 3.3.2).

2. $V^2 < 0$. Take the following transformation:

$$\Sigma = \cos(\alpha)\cdot I - i\sin(\alpha)\frac{\gamma^5\not{V}}{\sqrt{-V^2}} \tag{3.38}$$

This means that:

$$\begin{aligned}\Sigma\Gamma\bar{\Sigma} =& \frac{1}{2}\cdot I + \cos(2\alpha)\not{V} + 2i\cos(2\alpha)\not{V}\not{W}\\ &- 2\sin(2\alpha)\sqrt{-V^2}\gamma^5\not{W} + i\sin(2\alpha)\sqrt{-V^2}\gamma^5\end{aligned}$$

Choose $\cos(2\alpha) = 0 \quad \Leftrightarrow \sin(2\alpha) = n, \quad n = \pm 1$

$$\Rightarrow \quad \Sigma\Gamma\bar{\Sigma} = \frac{1}{2}\cdot I - 2n\sqrt{-V^2}\gamma^5\not{W} + in\sqrt{-V^2}\gamma^5$$

Define $A_\mu = -2n\sqrt{-V^2}W_\mu$ and $R = n\sqrt{-V^2}$. A_μ and R have to satisfy $A^2 + R^2 = -\frac{1}{4}$:

$$\begin{aligned}\Rightarrow \quad A^2 + R^2 =& \left(-2n\sqrt{-V^2}\right)^2(W_\mu W^\mu) + \left(n\sqrt{-V^2}\right)^2\\ =& -4V^2W^2 - V^2 = -\frac{1}{4}\end{aligned}$$

This means that

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2}\cdot I + \gamma^5\not{A} + iR\gamma^5 \tag{3.39}$$

This propagator is the same as the one in formula 3.56 in section 3.3.15. There it will be shown that this propagator can be transformed further to a standard propagator.

3.3.11 The case $V_\alpha \neq 0, \quad A_\lambda \neq 0$

The following equation holds:

$$\begin{aligned} &\epsilon^{\mu\nu}{}_{\beta\delta} V_\mu A_\nu = 0 \\ \Leftrightarrow \quad & A_\mu = k V_\mu \quad \wedge \quad \text{at least 1 of } V_\mu \neq 0 \end{aligned}$$

Now use:

$$V_\mu V^\mu - A_\mu A^\mu - \frac{1}{4} = 0$$

This means that:

$$\Gamma = \frac{1}{2} \cdot I + \not{V} + \gamma^5 \not{A} \tag{3.40}$$

with $A_\mu = kV_\mu$ and

$$k = \pm\sqrt{1 - \frac{1}{4V_\mu V^\mu}} \quad \Leftrightarrow \quad \frac{1}{k} = \pm\sqrt{1 + \frac{1}{4A_\mu A^\mu}}$$

There are now two cases:

1. $V^2 > \frac{1}{4} \Leftrightarrow A^2 > 0$. Γ can be written as

$$\Gamma = \frac{1}{2} \cdot I + \not{V} + k\gamma^5 \not{V}$$

Take the following transformation:

$$\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha) \frac{\not{V}}{\sqrt{V^2}} \tag{3.41}$$

This means that:

$$\begin{aligned} \Sigma\Gamma\bar{\Sigma} =& \frac{1}{2} \cdot I + \not{V} + k\cos(2\alpha)\gamma^5 \not{V} + ik\sin(2\alpha)\sqrt{V^2}\gamma^5 \\ =& \frac{1}{2} \cdot I + \not{V} \pm i\sqrt{V^2 - \frac{1}{4}}\gamma^5 \quad \text{if} \quad \cos(2\alpha) = 0 \end{aligned} \tag{3.42}$$

This is the same propagator as the one of formula 3.45 in section 3.3.12. There we will show that it can be transformed further to a standard propagator.

2. $A^2 < -\frac{1}{4} \Leftrightarrow V^2 < 0$. Γ can be written as

$$\Gamma = \frac{1}{2} \cdot I + \frac{1}{k}\not{A} + \gamma^5 \not{A}$$

Take the following transformation:

$$\Sigma = \cos(\alpha)\cdot I - i\sin(\alpha)\frac{\gamma^5 \not{A}}{\sqrt{-A^2}} \tag{3.43}$$

This means that:

$$\begin{aligned}\Sigma\Gamma\bar{\Sigma} =& \frac{1}{2}\cdot I + \frac{1}{k}\cos(2\alpha)\not{A} + \gamma^5\not{A} + \frac{i}{k}\sin(2\alpha)\sqrt{-A^2}\gamma^5 \\ =& \frac{1}{2}\cdot I + \gamma^5\not{A} \pm i\sqrt{-A^2 - \frac{1}{4}}\gamma^5 \quad \text{if} \quad \cos(2\alpha) = 0 \end{aligned} \tag{3.44}$$

It is the same as the propagator of formula 3.56 in section 3.3.15 where we will show that it can be transformed further to a standard one.

3.3.12 The case $V_\alpha \neq 0, \quad R \neq 0$

The following equation must be satisfied:

$$\begin{aligned} & V_\mu V^\mu - R^2 - \frac{1}{4} = 0 \\ \Leftrightarrow \quad & R^2 = V_\mu V^\mu - \frac{1}{4} \end{aligned}$$

This leads to:

$$\Gamma = \frac{1}{2}\cdot I + \not{V} \pm i\sqrt{V_\mu V^\mu - \frac{1}{4}}\gamma^5 \tag{3.45}$$

with $V_\mu V^\mu \geq \frac{1}{4}$ because $V_\mu, R \in \mathbb{R}$. Now take the following transformation:

$$\Sigma = \frac{1}{\sqrt{\cos(2\alpha)}}\left(\cos(\alpha)\cdot I - i\sin(\alpha)\frac{\gamma^5\not{V}}{\sqrt{V^2}}\right) \tag{3.46}$$

This means that:

$$\begin{aligned}\Sigma\Gamma\bar{\Sigma} =& \frac{1}{2}\cdot I + \frac{1}{\cos(2\alpha)}\left(\not{V} - R\sin(2\alpha)\frac{\not{V}}{\sqrt{V^2}}\right) \\ & + \frac{i}{\cos(2\alpha)}\left(R - \sqrt{V^2}\sin(2\alpha)\right)\gamma^5 \end{aligned}$$

Now choose:

$$\begin{aligned} & R = \sqrt{V^2}\sin(2\alpha) \\ \Leftrightarrow \quad & \sin(2\alpha) = \frac{1}{\sqrt{V^2}}R \\ \Rightarrow \quad & \cos(2\alpha) = \pm\frac{1}{2}\frac{1}{\sqrt{V^2}} \end{aligned}$$

so Γ becomes:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \cos(2\alpha)\not{V} = \frac{1}{2} \cdot I + \frac{1}{2}\frac{\not{V}}{\sqrt{V^2}} \tag{3.47}$$

which is the standard propagator with a momentum term (see formula 3.19 in section 3.3.2).

3.3.13 The case $T_{\beta\delta} \neq 0, \quad A_\lambda \neq 0$

The following equations must be satisfied:

$$\begin{aligned} & -2\epsilon^{\mu\nu\rho}{}_{\alpha}T_{\mu\nu}A_\rho = 0 \\ & -\epsilon^{\mu\nu\rho\pi}T_{\mu\nu}T_{\rho\pi} = 0 \\ \Leftrightarrow \quad & T_{\beta\delta} = A_\beta W_\delta - W_\beta A_\delta \end{aligned}$$

Now use:

$$\begin{aligned} & 2T_{\mu\nu}T^{\mu\nu} - A_\mu A^\mu - \frac{1}{4} = 0 \\ \Rightarrow \quad & 4(A_\mu A^\mu)(W_\nu W^\nu) - 4(A_\rho W^\rho)^2 - A_\mu A^\mu - \frac{1}{4} = 0 \end{aligned} \tag{3.48}$$

This means that $A^2 \neq 0$. So Γ is now:

$$\Gamma = \frac{1}{2} \cdot I + i(\not{A}\not{W} - \not{W}\not{A}) + \gamma^5\not{A} \tag{3.49}$$

Here too W can be chosen so that $A_\mu W^\mu = 0$ is satisfied, in a similar way as with formulas 3.30 and 3.33. So formulas 3.49 and 3.48 can be simplified to:

$$\Gamma = \frac{1}{2} \cdot I + 2i\not{A}\not{W} + \gamma^5\not{A} \tag{3.50}$$

$$4A^2W^2 - A^2 - \frac{1}{4} = 0 \tag{3.51}$$

with $A_\mu W^\mu = 0$. Now there are two cases, $A^2 > 0$ en $A^2 < 0$.

1. $A^2 > 0$. Take the following transformation:

$$\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha)\frac{\not{A}}{\sqrt{A^2}} \tag{3.52}$$

This means that:

$$\begin{aligned} \Sigma\Gamma\bar{\Sigma} = & \frac{1}{2} \cdot I + 2i\cos(2\alpha)\not{A}\not{W} + 2\sqrt{A^2}\sin(2\alpha)\not{W} \\ & + \cos(2\alpha)\gamma^5\not{A} + i\sqrt{A^2}\sin(2\alpha)\gamma^5 \end{aligned}$$

and $\cos(2\alpha) = 0 \quad \Leftrightarrow \sin(2\alpha) = n, \quad n = \pm 1$

$$\Rightarrow \quad \Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + 2n\sqrt{A^2}\not{W} + in\sqrt{A^2}\gamma^5$$

Define $V_\mu = 2n\sqrt{A^2}W_\mu$ and $R = n\sqrt{A^2}$. V_μ and R have to satisfy $V^2 - R^2 = \frac{1}{4}$:

$$\begin{aligned}\Rightarrow \quad V^2 - R^2 =& \left(2n\sqrt{A^2}\right)^2 W_\mu W^\mu - \left(n\sqrt{A^2}\right)^2 \\ =& 4A^2W^2 - A^2 = \frac{1}{4}\end{aligned}$$

This leads to:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \not{V} + iR\gamma^5 \tag{3.53}$$

This is the propagator of formula 3.45 in section 3.3.12. We have shown there that it can be transformed further to a standard propagator.

2. $A^2 < 0$. Take the following transformation:

$$\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha)\frac{\gamma^5\not{A}}{\sqrt{-A^2}} \tag{3.54}$$

This means that:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + 2i\cos(2\alpha)\not{A}\not{W} + \gamma^5\not{A} - 2\sin(2\alpha)\sqrt{-A^2}\gamma^5\not{W}$$

Choose $\cos(2\alpha) = 0 \quad \Leftrightarrow \sin(2\alpha) = n, \quad n = \pm 1$

$$\Rightarrow \quad \Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \gamma^5\not{A} - 2n\sqrt{-A^2}\gamma^5\not{W}$$

Define $B_\mu = A_\mu - 2n\sqrt{-A^2}W_\mu$. This B_μ has to satisfy $B^2 = -\frac{1}{4}$:

$$\begin{aligned}\Rightarrow \quad B_\mu B^\mu =& \left(A_\mu - 2n\sqrt{-A^2}W_\mu\right)\left(A^\mu - 2n\sqrt{-A^2}W^\mu\right) \\ =& A^2 - 4n\sqrt{-A^2}(A_\nu W^\nu) - 4A^2W^2 = A^2 - 4A^2W^2 \\ =& -\frac{1}{4}\end{aligned}$$

This leads to

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \gamma^5\not{B} \tag{3.55}$$

which is the standard propagator with a spin term (see formula 3.26 in section 3.3.4).

3.3.14 The case $T_{\beta\delta} \neq 0, \quad R \neq 0$

The following equation must hold:

$$\begin{aligned} & \epsilon^{\mu\nu}{}_{\beta\delta} T_{\mu\nu} R = 0 \\ \Leftrightarrow \quad & T_{\beta\delta} = 0 \quad \vee \quad R = 0 \end{aligned}$$

So this is not a valid solution.

3.3.15 The case $A_\lambda \neq 0, \quad R \neq 0$

The following equation must be satisfied:

$$\begin{aligned} & - A_\mu A^\mu - R^2 - \frac{1}{4} = 0 \\ \Leftrightarrow \quad & R^2 = - \left(A_\mu A^\mu + \frac{1}{4} \right) \end{aligned}$$

This leads to:

$$\Gamma = \frac{1}{2} \cdot I + \gamma^5 \not{A} \pm i \sqrt{-A_\mu A^\mu - \frac{1}{4}} \gamma^5 \tag{3.56}$$

with $A_\mu A^\mu \leq -\frac{1}{4}$ because $A_\lambda, R \in \mathbb{R}$. Now take the following transformation:

$$\Sigma = \frac{1}{\sqrt{\cos(2\alpha)}} \left(\cos(\alpha) \cdot I - i \sin(\alpha) \frac{\not{A}}{\sqrt{-A^2}} \right) \tag{3.57}$$

This means that:

$$\begin{aligned} \Sigma\Gamma\bar{\Sigma} = & \frac{1}{2} \cdot I + \frac{1}{\cos(2\alpha)} \left(1 - R \sin(2\alpha) \frac{1}{\sqrt{-A^2}} \right) \gamma^5 \not{A} \\ & + \frac{i}{\cos(2\alpha)} \left(R - \sin(2\alpha) \sqrt{-A^2} \right) \gamma^5 \end{aligned}$$

Now choose:

$$\begin{aligned} & R = \sin(2\alpha) \sqrt{-A^2} \\ \Leftrightarrow \quad & \sin(2\alpha) = \frac{R}{\sqrt{-A^2}} \\ \Rightarrow \quad & \cos(2\alpha) = \pm \frac{1}{2} \frac{1}{\sqrt{-A^2}} \end{aligned}$$

so Γ becomes:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \cos(2\alpha) \gamma^5 \not{A} = \frac{1}{2} \cdot I \pm \frac{1}{2} \frac{\gamma^5 \not{A}}{\sqrt{-A^2}} \tag{3.58}$$

which is the standard propagator with a spin term (see formula 3.26 in section 3.3.4).

3.3.16 The case $S \neq 0, \quad V_\alpha \neq 0, \quad T_{\beta\delta} \neq 0$

The following equations must be satisfied:

$$\begin{array}{lclcl} 2SV_\alpha = 0 & \Leftrightarrow & S = 0 & \vee & V_\alpha = 0 \\ 2ST_{\beta\delta} = 0 & \Leftrightarrow & S = 0 & \vee & T_{\beta\delta} = 0 \end{array}$$

So this is not a valid solution.

3.3.17 The case $S \neq 0, \quad V_\alpha \neq 0, \quad A_\lambda \neq 0$

The following equations must hold:

$$\begin{array}{lclcl} 2SV_\alpha = 0 & \Leftrightarrow & S = 0 & \vee & V_\alpha = 0 \\ 2SA_\lambda = 0 & \Leftrightarrow & S = 0 & \vee & A_\lambda = 0 \end{array}$$

So this is not a valid solution.

3.3.18 The case $S \neq 0, \quad V_\alpha \neq 0, \quad R \neq 0$

One has the following equations:

$$\begin{array}{lclcl} 2SV_\alpha = 0 & \Leftrightarrow & S = 0 & \vee & V_\alpha = 0 \\ 2SR = 0 & \Leftrightarrow & S = 0 & \vee & R = 0 \end{array}$$

So this is not a valid solution.

3.3.19 The case $S \neq 0, \quad T_{\beta\delta} \neq 0, \quad A_\lambda \neq 0$

The following equations must be satisfied:

$$\begin{array}{lclcl} 2ST_{\beta\delta} = 0 & \Leftrightarrow & S = 0 & \vee & T_{\beta\delta} = 0 \\ 2SA_\lambda = 0 & \Leftrightarrow & S = 0 & \vee & A_\lambda = 0 \end{array}$$

So this is not a valid solution.

3.3.20 The case $S \neq 0, \quad T_{\beta\delta} \neq 0, \quad R \neq 0$

One has the following equations:

$$\begin{array}{rcll} 2ST_{\beta\delta} + \epsilon^{\mu\nu}{}_{\beta\delta} T_{\mu\nu} R & = & 0 & \\ \frac{R}{2S} \epsilon_{\beta\delta}{}^{\rho\pi} (2ST_{\rho\pi} + \epsilon^{\mu\nu}{}_{\rho\pi} T_{\mu\nu} R) & = & 0 & - \\ \hline 2\left(S + \frac{R^2}{S}\right) T_{\beta\delta} & = & 0 & \end{array}$$

$$\begin{array}{ll} \Leftrightarrow & (R^2 + S^2) T_{\beta\delta} = 0 \\ \Leftrightarrow & S^2 + R^2 = 0 \quad \vee \quad T_{\beta\delta} = 0 \end{array}$$

This is not a valid solution because $S, R \in \mathbb{R}$.

3.3.21 The case $S \neq 0, \quad A_\lambda \neq 0, \quad R \neq 0$

The following equations must hold:

$$\begin{array}{lcccc} 2SA_\lambda = 0 & \Leftrightarrow & S = 0 & \vee & A_\lambda = 0 \\ 2SR = 0 & \Leftrightarrow & S = 0 & \vee & R = 0 \end{array}$$

So this is not a valid solution.

3.3.22 The case $V_\alpha \neq 0, \quad T_{\beta\delta} \neq 0, \quad A_\lambda \neq 0$

The following equations must be satisfied:

$$\begin{aligned} & \epsilon^{\mu\nu}{}_{\beta\delta} V_\mu A_\nu = 0 \\ \Leftrightarrow \quad & A_\mu = kV_\mu \quad \wedge \quad \text{at least 1 of } V_\mu \neq 0 \end{aligned}$$

and:

$$\begin{aligned} & -2\epsilon^{\mu\nu\rho}{}_\lambda V_\mu T_{\nu\rho} = 0 \\ & -\epsilon^{\mu\nu\rho\pi} T_{\mu\nu} T_{\rho\pi} = 0 \\ \Leftrightarrow \quad & T_{\beta\delta} = V_\beta W_\delta - W_\beta V_\delta \end{aligned}$$

Now this equation is also satisfied:

$$-2\epsilon^{\mu\nu\rho}{}_\alpha T_{\mu\nu} A_\rho = 0$$

Use the remaining equation:

$$\begin{aligned} & V_\mu V^\mu + 2T_{\mu\nu}T^{\mu\nu} - A_\mu A^\mu - \frac{1}{4} = 0 \\ \Rightarrow \quad & (V_\mu V^\mu)(1-k^2) + 2T_{\mu\nu}T^{\mu\nu} - \frac{1}{4} = 0 \\ \Leftrightarrow \quad & 4(V_\mu V^\mu)(1 - k^2 + 4(W_\nu W^\nu)) = 1 + (4V_\rho W^\rho)^2 \end{aligned} \tag{3.59}$$

with $k \in \mathbb{R}$. This means that $V_\mu V^\mu \neq 0$. Γ now becomes:

$$\Gamma = \frac{1}{2} \cdot I + \not{V} + k\gamma^5 \not{V} + i(\not{V}\not{W} - \not{W}\not{V}) \tag{3.60}$$

Here too W can be chosen so that $V_\mu W^\mu = 0$, in a similar way as with formulas 3.30 and 3.33. So formulas 3.60 and 3.59 can be simplified to:

$$\Gamma = \frac{1}{2} \cdot I + \not{V} + k\gamma^5 \not{V} + 2i\not{V}\not{W} \tag{3.61}$$

$$4(V_\mu V^\mu)(1 - k^2 + 4(W_\nu W^\nu)) = 1 \tag{3.62}$$

with $V_\mu W^\mu = 0$. There are now two cases: $V^2 > 0$ en $V^2 < 0$.

1. $V^2 > 0$. Take the following transformation:

$$\Sigma = \cos(\alpha) \cdot I - i \sin(\alpha) \frac{\not{V}}{\sqrt{V^2}} \tag{3.63}$$

This means that:

$$\begin{aligned} \Sigma\Gamma\bar{\Sigma} =& \frac{1}{2} \cdot I + \not{V} + 2\sqrt{V^2} \sin(2\alpha) \not{W} + 2i \cos(2\alpha) \not{V} \not{W} \\ &+ k \cos(2\alpha) \gamma^5 \not{V} + ik\sqrt{V^2} \sin(2\alpha) \gamma^5 \end{aligned}$$

Choose $\cos(2\alpha) = 0 \quad \Leftrightarrow \sin(2\alpha) = n, \quad n = \pm 1$

$$\Rightarrow \quad \Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \not{V} + 2n\sqrt{V^2} \not{W} + ink\sqrt{V^2} \gamma^5$$

Define $U_\mu = V_\mu + 2n\sqrt{V^2} W_\mu$ and $R = nk\sqrt{V^2}$. U_μ and R have to satisfy $U^2 - R^2 = \frac{1}{4}$:

$$\begin{aligned} \Rightarrow \quad U^2 - R^2 =& \left(V_\mu + 2n\sqrt{V^2} W_\mu\right) \left(V^\mu + 2n\sqrt{V^2} W^\mu\right) - \left(nk\sqrt{V^2}\right)^2 \\ =& V^2 (1 - k^2 + 4W^2) = \frac{1}{4} \end{aligned}$$

This leads to

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \not{U} + iR\gamma^5 \tag{3.64}$$

This propagator is the same as the one of formula 3.45 in section 3.3.12. We have shown that it can be transformed further to a standard propagator.

2. $V^2 < 0$. Take the following transformation:

$$\Sigma = \cos(\alpha) \cdot I - i \sin(\alpha) \frac{\gamma^5 \not{V}}{\sqrt{-V^2}} \tag{3.65}$$

This means that:

$$\begin{aligned} \Sigma\Gamma\bar{\Sigma} =& \frac{1}{2} \cdot I + k\gamma^5 \not{V} + \cos(2\alpha) \not{V} + i\sqrt{-V^2} \sin(2\alpha) \gamma^5 \\ &+ 2i \cos(2\alpha) \not{V} \not{W} - 2\sqrt{-V^2} \sin(2\alpha) \gamma^5 \not{W} \end{aligned}$$

Choose $\cos(2\alpha) = 0 \quad \Leftrightarrow \sin(2\alpha) = n, \quad n = \pm 1$

$$\Rightarrow \quad \Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + k\gamma^5 \not{V} + in\sqrt{-V^2} \gamma^5 - 2n\sqrt{-V^2} \gamma^5 \not{W}$$

Define $B_\mu = kV_\mu - 2n\sqrt{-V^2}W_\mu$ and $R = n\sqrt{-V^2}$. B_μ and R have to satisfy $B^2 + R^2 = -\frac{1}{4}$:

$$\begin{aligned} \Rightarrow \quad B^2 + R^2 &= \left(kV_\mu - 2n\sqrt{-V^2}W_\mu\right)\left(kV^\mu - 2n\sqrt{-V^2}W^\mu\right) + \left(n\sqrt{-V^2}\right)^2 \\ &= V^2(k^2 - 4W^2 - 1) = -\frac{1}{4} \end{aligned}$$

This leads to

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \gamma^5 \not{B} + iR\gamma^5 \tag{3.66}$$

This propagator is the same as the one of formula 3.56 in section 3.3.15. We have seen that it can be transformed further to a standard propagator.

3.3.23 The case $V_\alpha \neq 0, \quad T_{\beta\delta} \neq 0, \quad R \neq 0$

The following equation must be satisfied:

$$\begin{aligned} & \epsilon^{\mu\nu}{}_{\beta\delta} T_{\mu\nu} R = 0 \\ \Leftrightarrow \quad & T_{\mu\nu} = 0 \quad \vee \quad R = 0 \end{aligned}$$

So this is not a valid solution.

3.3.24 The case $V_\alpha \neq 0, \quad A_\lambda \neq 0, \quad R \neq 0$

The following equation must hold:

$$\begin{aligned} & \epsilon^{\mu\nu}{}_{\beta\delta} V_\mu A_\nu = 0 \\ \Leftrightarrow \quad & A_\mu = kV_\mu \quad \wedge \quad \text{at least 1 of } V_\mu \neq 0 \end{aligned}$$

Use also:

$$V_\mu V^\mu - A_\mu A^\mu - R^2 - \frac{1}{4} = 0$$

This means that:

$$\Gamma = \frac{1}{2} \cdot I + \not{V} + \gamma^5 \not{A} + iR\gamma^5 \tag{3.67}$$

with $A_\mu = kV_\mu$ and

$$k = \pm\sqrt{1 - \frac{1}{V_\mu V^\mu}\left(R^2 + \frac{1}{4}\right)} \quad \Leftrightarrow \quad \frac{1}{k} = \pm\sqrt{1 + \frac{1}{A_\mu A^\mu}\left(R^2 + \frac{1}{4}\right)}$$

There are now two cases:

1. $V^2 > R^2 + \frac{1}{4} \quad \Leftrightarrow \quad A^2 > 0$. Γ can now be written as:

$$\Gamma = \frac{1}{2} \cdot I + \not{V} + k\gamma^5\not{V} + iR\gamma^5$$

Take the following transformation:

$$\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha)\frac{\not{V}}{\sqrt{V^2}} \tag{3.68}$$

This means that:

$$\begin{aligned} \Rightarrow \quad \Sigma\Gamma\bar{\Sigma} =& \frac{1}{2} \cdot I + \not{V} + \left(k\cos(2\alpha) - R\sin(2\alpha)\frac{1}{\sqrt{V^2}}\right)\gamma^5\not{V} \\ &+ i\left(k\sin(2\alpha)\sqrt{V^2} + R\cos(2\alpha)\right)\gamma^5 \end{aligned}$$

Now choose:

$$k\cos(2\alpha) - R\sin(2\alpha)\frac{1}{\sqrt{V^2}} = 0$$

From this follows:

$$\begin{aligned} k\sin(2\alpha)\sqrt{V^2} + R\cos(2\alpha) &= \pm\,\mathrm{sgn}\,(R)\sqrt{R^2 + k^2V^2} \\ &= \pm\,\mathrm{sgn}\,(R)\sqrt{V^2 - \frac{1}{4}} \end{aligned}$$

This means that:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \not{V} \pm i\,\mathrm{sgn}\,(R)\sqrt{V^2 - \frac{1}{4}}\gamma^5 \tag{3.69}$$

This Γ is equal to formula 3.45 in section 3.3.12. It was shown that it can be transformed further to a standard propagator.

2. $A^2 < -\left(R^2 + \frac{1}{4}\right) \quad \Leftrightarrow \quad V^2 < 0$. Γ can now be written as:

$$\Gamma = \frac{1}{2} \cdot I + \frac{1}{k}\not{A} + \gamma^5\not{A} + iR\gamma^5$$

Take the following transformation:

$$\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha)\frac{\gamma^5\not{A}}{\sqrt{-A^2}} \tag{3.70}$$

This means that:

$$\Rightarrow \quad \Sigma\Gamma\bar{\Sigma} = \frac{1}{2}\cdot I + \left(\frac{1}{k}\cos(2\alpha) - \frac{R}{\sqrt{-A^2}}\sin(2\alpha)\right) \not{A} + \gamma^5 \not{A}$$
$$+ i\left(\frac{\sqrt{-A^2}}{k}\sin(2\alpha) + R\cos(2\alpha)\right)\gamma^5$$

Now choose:

$$\frac{1}{k}\cos(2\alpha) - \frac{R}{\sqrt{-A^2}}\sin(2\alpha) = 0$$

From this follows:

$$\frac{\sqrt{-A^2}}{k}\sin(2\alpha) + R\cos(2\alpha) = \pm\,\mathrm{sgn}\,(R)\sqrt{R^2 - \frac{1}{k^2}A^2}$$
$$= \pm\,\mathrm{sgn}\,(R)\sqrt{-A^2 - \frac{1}{4}}$$

This means that:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2}\cdot I + \gamma^5\not{A} \pm i\mathrm{sgn}\,(R)\sqrt{-A^2 - \frac{1}{4}}\gamma^5 \tag{3.71}$$

This Γ is equal to formula 3.56 in section 3.3.15. It was shown that it can be transformed further to a standard propagator.

3.3.25 The case $T_{\beta\delta} \neq 0, \quad A_\lambda \neq 0, \quad R \neq 0$

The following equation must be satisfied:

$$\epsilon^{\mu\nu}{}_{\beta\delta}T_{\mu\nu}R = 0$$
$$\Leftrightarrow \quad T_{\mu\nu} = 0 \quad \vee \quad R = 0$$

So this is not a valid solution.

3.3.26 The case $S \neq 0, \quad V_\alpha \neq 0, \quad T_{\beta\delta} \neq 0, \quad A_\lambda \neq 0$

The following equation must be satisfied:

$$2ST_{\beta\delta} + \epsilon^{\mu\nu}{}_{\beta\delta}V_\mu A_\nu = 0$$
$$\Leftrightarrow \quad T_{\beta\delta} = -\frac{1}{2S}\epsilon^{\mu\nu}{}_{\beta\delta}V_\mu A_\nu \tag{3.72}$$

This equation is also satisfied:

$$-\epsilon^{\mu\nu\rho\pi}T_{\mu\nu}T_{\rho\pi} = 0$$

Fill in solution 3.72 in the following equations and use equation 3.16:

$$\begin{aligned} 2SV_\alpha - 2\epsilon^{\mu\nu\rho}{}_\alpha T_{\mu\nu}A_\rho = 0 \qquad &\Leftrightarrow \qquad (S^2 + A_\mu A^\mu)V_\alpha = 0 \\ 2SA_\lambda - 2\epsilon^{\mu\nu\rho}{}_\lambda V_\mu T_{\nu\rho} = 0 \qquad &\Leftrightarrow \qquad (S^2 - V_\mu V^\mu)A_\lambda = 0 \end{aligned}$$

This means that:

$$S^2 = V_\mu V^\mu = -A_\mu A^\mu$$

has to be satisfied. Because of equation 3.16 and $S \neq 0$:

$$V_\mu A^\mu = 0 \qquad \Leftrightarrow \quad (V_\mu A^\mu)^2 = 0$$

is also satisfied. This means:

$$2T_{\mu\nu}T^{\mu\nu} = \frac{-1}{S^2}(V_\mu V^\mu)(A_\nu A^\nu) = S^2$$

The value of S can be determined with the following equation:

$$\begin{aligned} & S^2 + V_\mu V^\mu + 2T_{\mu\nu}T^{\mu\nu} - A_\mu A^\mu - \frac{1}{4} = 0 \\ \Rightarrow \quad & 4S^2 - \frac{1}{4} = 0 \\ \Leftrightarrow \quad & S = \pm\frac{1}{4} \end{aligned}$$

With help of:

$$-\frac{1}{2}V_\mu A_\nu \epsilon^{\mu\nu}{}_{\beta\delta}\sigma^{\beta\delta} = \gamma^5 \not{V}\not{A} \qquad \text{if } V_\mu A^\mu = 0 \tag{3.73}$$

Γ can now be written as:

$$\Gamma = \frac{1}{4}\cdot I + \not{V} - 4\gamma^5\not{V}\not{A} + \gamma^5\not{A} \tag{3.74}$$

with $V_\mu A^\mu = 0$ and $V^2 = -A^2$. [3] This is a standard propagator; make the substitutions $\not{V} \to \frac{\not{p}}{4m}$ and $\not{A} \to \frac{\gamma^5\not{s}}{4}$ and formula 3.74 becomes:

$$\Gamma = \frac{1}{4m}\left(m + \not{p}\right)\left(1 + \gamma^5\not{s}\right) \tag{3.75}$$

3.3.27 The case $S \neq 0, \quad V_\alpha \neq 0, \quad T_{\beta\delta} \neq 0, \quad R \neq 0$

The following equation must hold:

$$\begin{aligned} & 2SV_\alpha = 0 \\ \Leftrightarrow \quad & S = 0 \quad \vee \quad V_\alpha = 0 \end{aligned}$$

So this is not a valid solution.

[3] $\Gamma = (S + \frac{1}{2})\cdot I + \not{V} + \frac{1}{S}\gamma^5\not{V}\not{A} + \gamma^5\not{A}$ with $S = \pm\frac{1}{4}$ and $S^2 = V^2 = -A^2$. In formula 3.74 $S = -\frac{1}{4}$ was chosen.

3.3.28 The case $S \neq 0, \quad V_\alpha \neq 0, \quad A_\lambda \neq 0, \quad R \neq 0$

The following equations must hold:

$$\begin{array}{lcl} 2SV_\alpha = 0 & \Leftrightarrow & S = 0 \quad \vee \quad V_\alpha = 0 \\ 2SA_\lambda = 0 & \Leftrightarrow & S = 0 \quad \vee \quad A_\lambda = 0 \\ 2SR = 0 & \Leftrightarrow & S = 0 \quad \vee \quad R = 0 \end{array}$$

So this is not a valid solution.

3.3.29 The case $S \neq 0, \quad T_{\beta\delta} \neq 0, \quad A_\lambda \neq 0, \quad R \neq 0$

The following equation must hold:

$$\begin{aligned} & 2SA_\lambda = 0 \\ \Leftrightarrow \quad & S = 0 \quad \vee \quad A_\lambda = 0 \end{aligned}$$

So this is not a valid solution.

3.3.30 The case $V_\alpha \neq 0, \quad T_{\beta\delta} \neq 0, \quad A_\lambda \neq 0, \quad R \neq 0$

The following equation must be satisfied:

$$\begin{aligned} & \epsilon^{\mu\nu}{}_{\beta\delta}(V_\mu A_\nu + T_{\mu\nu}R) = 0 \\ \Leftrightarrow \quad & T_{\beta\delta} = -\frac{1}{2R}(V_\beta A_\delta - V_\delta A_\beta) \end{aligned}$$

This solution also satisfies the equations:

$$\begin{aligned} & -2\epsilon^{\mu\nu\rho}{}_\alpha T_{\mu\nu}A_\rho = 0 \\ & -2\epsilon^{\mu\nu\rho}{}_\lambda V_\mu T_{\nu\rho} = 0 \\ & -\epsilon^{\mu\nu\rho\pi}T_{\mu\nu}T_{\rho\pi} = 0 \end{aligned}$$

Only the following equation has to be satisfied now:

$$\begin{aligned} & V_\mu V^\mu + 2T_{\mu\nu}T^{\mu\nu} - A_\mu A^\mu - R^2 - \frac{1}{4} = 0 \\ \Leftrightarrow \quad & V^2 + \frac{1}{R^2}(V^2A^2 - (V_\mu A^\mu)^2) - A^2 - R^2 - \frac{1}{4} = 0 \end{aligned} \tag{3.76}$$

This means that $V^2 \neq R^2$ and $-A^2 \neq R^2$. Γ now becomes:

$$\Gamma = \frac{1}{2} \cdot I + \not{V} - \frac{i}{2R}(\not{V}\not{A} - \not{A}\not{V}) + \gamma^5\not{A} + iR\gamma^5 \tag{3.77}$$

Γ can be transformed to a more simple form. There are two different transformations possible:

1. Define a fourvector Q with the following properties: $Q^2 = -1$, $\bar{\not{Q}} = \not{Q}$, $Q_\mu V^\mu = 0$ and $Q_\mu A^\mu = 0$. Such a Q can always be found. Now take the following transformation:

$$\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha)\gamma^5 \not{Q} \tag{3.78}$$

This means that:

$$\begin{aligned}\Sigma\Gamma\bar{\Sigma} =& \frac{1}{2} \cdot I + \not{V} - R\sin(2\alpha)\not{Q} - \frac{i}{2R}\not{V}\not{A} + \frac{i}{2R}\not{A}\not{V} \\ &+ i\sin(2\alpha)\not{Q}\not{A} + \cos(2\alpha)\gamma^5\not{A} + iR\cos(2\alpha)\gamma^5\end{aligned}$$

Choose $\cos(2\alpha) = 0 \quad \Leftrightarrow \sin(2\alpha) = n, \quad n = \pm 1$

$$\Rightarrow \quad \Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \not{V} - nR\not{Q} - \frac{i}{2R}(\not{V}\not{A} - \not{A}\not{V}) + in\not{Q}\not{A}$$

The following equation has to be satisfied:

$$U^2 + 4U^2W^2 - 4\,(U_\rho W^\rho)^2 - \frac{1}{4} = 0$$

with $U_\mu = V_\mu - nRQ_\mu$ and $W_\mu = -\frac{1}{2R}A_\mu$.

$$\begin{aligned}\Rightarrow \quad & (V_\mu - nRQ_\mu)\,(V^\mu - nRQ^\mu) + 4\,(V_\nu - nRQ_\nu)\,(V^\nu - nRQ^\nu)\,\frac{1}{4R^2}A_\rho A^\rho \\ & - 4\left((V_\pi - nRQ_\pi)\,\frac{-A^\pi}{2R}\right)^2 - \frac{1}{4} = 0 \\ \Leftrightarrow \quad & V^2 + \frac{1}{R^2}\left(V^2A^2 - (V_\rho A^\rho)^2\right) - A^2 - R^2 - \frac{1}{4} = 0\end{aligned}$$

This is right, so:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \not{U} + i(\not{U}\not{W} - \not{W}\not{U}) \tag{3.79}$$

This is the same propagator as formula 3.29 in section 3.3.10. We have seen that it can eventually be transformed to a standard propagator.

2. Define a fourvector Q with the following properties: $Q^2 = 1$, $\bar{\not{Q}} = \not{Q}$, $Q_\mu V^\mu = 0$ and $Q_\mu A^\mu = 0$. Such a Q can only be found if $V^2 < 0$ and $A^2 < 0$. Now take the following transformation:

$$\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha)\not{Q} \tag{3.80}$$

This means that:

$$\begin{aligned}\Sigma\Gamma\bar{\Sigma} =&\frac{1}{2}\cdot I + \cos(2\alpha)\not{V} + i\sin(2\alpha)\not{V}\not{Q} - \frac{1}{2R}\not{V}\not{A} \\ &+ \frac{i}{2R}\not{A}\not{V} + \gamma^5\not{A} + iR\cos(2\alpha)\gamma^5 - R\sin(2\alpha)\gamma^5\not{Q}\end{aligned}$$

Choose $\cos(2\alpha) = 0 \quad \Leftrightarrow \sin(2\alpha) = n, \quad n = \pm 1$

$$\Rightarrow \quad \Sigma\Gamma\bar{\Sigma} = \frac{1}{2}\cdot I + in\not{V}\not{Q} - \frac{i}{2R}(\not{V}\not{A} - \not{A}\not{V}) + \gamma^5\not{A} - nR\gamma^5\not{Q}$$

The following equation has to be satisfied:

$$4B^2W^2 - 4\left(B_\rho W^\rho\right)^2 - B^2 - \frac{1}{4} = 0$$

with $B_\mu = A_\mu - nRQ_\mu$ and $W_\mu = \frac{1}{2R}V_\mu$.

$$\begin{aligned}\Rightarrow \quad & 4\left(A_\mu - nRQ_\mu\right)\left(A^\mu - nRQ^\mu\right)\frac{1}{4R^2}V_\nu V^\nu - 4\left(\left(A_\rho - nRQ_\rho\right)\frac{1}{2R}V^\rho\right)^2 \\ & - \left(A_\mu - nRQ_\mu\right)\left(A^\mu - nRQ^\mu\right) - \frac{1}{4} = 0 \\ \Leftrightarrow \quad & V^2 + \frac{1}{R^2}\left(V^2A^2 - \left(V_\rho A^\rho\right)^2\right) - A^2 - R^2 - \frac{1}{4} = 0\end{aligned}$$

This is right, so:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2}\cdot I + i(\not{B}\not{W} - \not{W}\not{B}) + \gamma^5\not{B} \qquad (3.81)$$

The propagator is equal to the propagator of formula 3.49 in section 3.3.13. We have shown that it can eventually be transformed to a standard propagator.

3.3.31 The case $S \neq 0, \quad V_\alpha \neq 0, \quad T_{\beta\delta} \neq 0, \quad A_\lambda \neq 0, \quad R \neq 0$

The following equation applies:

$$\begin{aligned} & 2ST_{\beta\delta} + \epsilon^{\mu\nu}{}_{\beta\delta}(V_\mu A_\nu + T_{\mu\nu}R) = 0 \\ \Leftrightarrow \quad & T_{\beta\delta} = -\frac{S\epsilon^{\mu\nu}{}_{\beta\delta}V_\mu A_\nu + R(V_\beta A_\delta - V_\delta A_\beta)}{2(S^2 + R^2)}\end{aligned}$$

If this result is filled in into the following equations:

$$\begin{aligned} & 2SV_\alpha - 2\epsilon^{\mu\nu\rho}{}_\alpha T_{\mu\nu}A_\rho = 0 \\ & 2SA_\lambda - 2\epsilon^{\mu\nu\rho}{}_\lambda V_\mu T_{\nu\rho} = 0\end{aligned}$$

and $V_\mu A^\mu = 0$ is used, then this follows:

$$\begin{aligned} 2SV_\alpha(S^2 + R^2 + A_\mu A^\mu) &= 0 \\ 2SA_\lambda(S^2 + R^2 - V_\mu V^\mu) &= 0 \end{aligned}$$

This also applies:

$$\begin{aligned} & 2SR - \epsilon^{\mu\nu\rho\pi} T_{\mu\nu} T_{\rho\pi} = 0 \\ \Leftrightarrow \quad & (S^2 + R^2)^2 = (V_\mu V^\mu)^2 \end{aligned}$$

So $V_\mu A^\mu = 0, \quad V_\mu V^\mu = -A_\nu A^\nu = S^2 + R^2$. If all results are filled in into the last equation, then:

$$\begin{aligned} & S^2 + V_\mu V^\mu + 2T_{\mu\nu} T^{\mu\nu} - A_\mu A^\mu - R^2 - \frac{1}{4} = 0 \\ \Leftrightarrow \quad & S = \pm \frac{1}{4} \end{aligned}$$

This means that Γ, with help of the equations 3.28 and 3.73 can be written as:

$$\Gamma = (\frac{1}{2} + S) \cdot I + \not{V} + \frac{2S\gamma^5 \not{V}\not{A} - iR(\not{V}\not{A} - \not{A}\not{V})}{2V_\mu V^\mu} + \gamma^5 \not{A} + iR\gamma^5 \tag{3.82}$$

with $S = \pm\frac{1}{4}$ and $R^2 = V_\mu V^\mu - \frac{1}{16}$. Now take this transformation:

$$\Sigma = \frac{1}{\sqrt{\cos(2\alpha)}} \left(\cos(\alpha) \cdot I - i \sin(\alpha) \frac{\gamma^5 \not{V}}{\sqrt{V^2}} \right) \tag{3.83}$$

From this follows that:

$$\begin{aligned} \Sigma\Gamma\bar{\Sigma} = & \frac{1}{\cos(2\alpha)} \left(\frac{1}{4}\cos(2\alpha) + \left(1 - \frac{R}{\sqrt{V^2}} \sin(2\alpha)\right) \not{V} + i \left(R - \sqrt{V^2}\sin(2\alpha)\right)\gamma^5 \right. \\ & - \frac{1}{4V_\mu V^\mu} \cos(2\alpha) \gamma^5 \not{V}\not{A} + \left(1 - \frac{R}{\sqrt{V^2}} \sin(2\alpha)\right) \gamma^5 \not{A} \\ & \left. - i \left(\frac{R}{V^2} - \frac{1}{\sqrt{V^2}} \sin(2\alpha) \right) \not{V}\not{A} \right) \end{aligned}$$

Choose

$$\begin{aligned} & R - \sqrt{V^2}\sin(2\alpha) = 0 \\ \Leftrightarrow \quad & \sin(2\alpha) = \frac{R}{\sqrt{V^2}} \\ \Rightarrow \quad & \cos(2\alpha) = n\sqrt{1 - \frac{R^2}{V^2}} \quad \text{with} \quad n = \pm 1 \end{aligned}$$

This means that:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{4}\cdot I + n\frac{\not{V}}{4\sqrt{V^2}} - \frac{1}{4V^2}\gamma^5\not{V}\not{A} + n\frac{\gamma^5\not{A}}{4\sqrt{-A^2}} \tag{3.84}$$

with $V_\mu A^\mu = 0$ and $V^2 = -A^2$. This is the standard propagator with both a momentum and a spin term (see formula 3.74 in section 3.3.26).

3.4 Conclusion

We come to the conclusion that there are four different types of standard propagators in Minkovski space:

$$\Gamma = 0 \tag{3.85}$$

$$\Gamma = \frac{1}{2m}\left(m + \not{p}\right) \tag{3.86}$$

$$\Gamma = \frac{1}{2}\left(1 + \gamma^5\not{s}\right) \tag{3.87}$$

$$\Gamma = \frac{1}{4m}\left(m + \not{p}\right)\left(1 + \gamma^5\not{s}\right) \tag{3.88}$$

All these standard propagators are projection operators, but they are not the exact numerators of the dirac propagators; they differ by a factor $\frac{1}{2m}$, $\frac{1}{2}$ and $\frac{1}{4m}$.

All other possible solutions that we found in this chapter can be transformed to one of the four standard propagators. The question remains whether one of these four can be transformed to one of the others. We know that the trace of a propagator doesn't change under the transformations:

$$\mathrm{Tr}\left(\Sigma\Gamma\bar{\Sigma}\right) = \mathrm{Tr}\left(\overline{\Sigma}\Sigma\Gamma\right) = \mathrm{Tr}\left(\Gamma\right) \tag{3.89}$$

The trace of the propagator is only determined by the scalar term, so we can easily see that only propagators (3.86) and (3.87) have the same trace. We suspect that these two propagators too are not equivalent, but we will not prove it here.

Another interesting observation can be made here: when propagator 3.86 and propagator 3.87 are multiplied, this results in propagator 3.88. The physical interpretation is that propagator 3.86 specifies momentum and sums over spin-states and propagator 3.87 specifies spin and sums over momentum-states, while propagator 3.88 specifies both states. We also note that every standard propagator is commuting with every other standard propagator.

The propagators that are found in this chapter are exactly the propagators that one expects to find in quantum field theory. That means that requiring the propagators to be self-adjoint projection operators is enough for finding the correct form.

Chapter 4

Finding the propagator in a 2+2 dimensional space

Now we will turn our attention to a four dimensional space with two timelike and two spacelike dimensions. The propagator of a massive spin-$\frac{1}{2}$ particle will show some interesting differences with the one in a Minkovski space. The upbuilding of this chapter is the same as the previous two.

4.1 The Clifford algebra

A 2+2 dimensional space has the following metric:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4.1}$$

This metric has a Clifford algebra, and we can choose the following representation:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

The dirac conjugation with the usual properties can be defined as:

$$\overline{\Omega} = \left(i\gamma^2\gamma^3\right)\Omega^\dagger\left(i\gamma^2\gamma^3\right) \tag{4.2}$$

We also have the same definitions for $\sigma^{\mu\nu}$ and γ^5 (see formulas 3.4 and 3.5). In this case however, $(\gamma^5)^2 = -1$.

4.2 The equations

The general Γ-matrix, that has to be a projection operator, can be written in a convenient way as:

$$\Gamma = \left(S + \frac{1}{2}\right) \cdot I + \not{V} + T_{\beta\delta}\sigma^{\beta\delta} + \gamma^5 \not{A} + iR\gamma^5 \tag{4.3}$$

with S, V, T, A and R all real, because for self adjoint projection operators $\overline{\Pi} = \Pi$. Now from $\Pi^2 - \Pi = 0$ follows

$$\begin{aligned}\Gamma^2 - \Gamma = &\left(S^2 + V_\mu V^\mu + 2T_{\mu\nu}T^{\mu\nu} + A_\mu A^\mu + R^2 - \frac{1}{4}\right) \cdot I \\ &+ \left(2SV_\alpha + 2\epsilon^{\mu\nu\rho}{}_\alpha T_{\mu\nu}A_\rho\right) \cdot \gamma^\alpha \\ &+ \left(2ST_{\beta\delta} - \epsilon^{\mu\nu}{}_{\beta\delta}(V_\mu A_\nu + T_{\mu\nu}R)\right) \cdot \sigma^{\beta\delta} \\ &+ \left(2SA_\lambda - 2\epsilon^{\mu\nu\rho}{}_\lambda V_\mu T_{\nu\rho}\right) \cdot \gamma^5\gamma^\lambda \\ &+ i \cdot (2SR - \epsilon^{\mu\nu\rho\pi}T_{\mu\nu}T_{\rho\pi}) \cdot \gamma^5 \\ =&0\end{aligned} \tag{4.4}$$

The components of the matrix all have to be zero, so these are the five equations that have to be solved. Note that they look very similar to equations 3.14, only some plus or minus signs are different. We therefore expect that the solutions will be similar too, but not exactly the same. Also in this case we find two equations that might be useful:

$$\mathrm{Tr}\left((\Gamma^2 - \Gamma)(V_\alpha\gamma^5\gamma^\alpha - A_\lambda\gamma^\lambda)\right) = 16SV_\mu A^\mu = 0 \tag{4.5}$$

$$\mathrm{Tr}\left((\Gamma^2 - \Gamma)(A_\lambda\gamma^5\gamma^\lambda - V_\alpha\gamma^\alpha)\right) = -8S(V_\mu V^\mu - A_\nu A^\nu) = 0 \tag{4.6}$$

4.3 Solutions and transformations

We will use the same method as in the previous chapters to find all the different solutions. Most of these solutions can be transformed to less complicated ones as explained in section 2.3 on page 7. Because the (combined) transformations are somewhat complicated, an overview can be found in appendix B on page 58. The configuration of all transformations is quite different than in the Minkovski space, as can be seen clearly on the diagrams in the appendices. Also the resulting standard, i.e. non-transformable, propagators are different.

In this section we have left out all cases that give zero results. Also only the essential points in the calculations are given.

4.3.1 The case $S \neq 0$

The following equation must be satisfied:

$$
\begin{aligned}
& S^2 - \frac{1}{4} = 0 \\
\Leftrightarrow \quad & S = \pm\frac{1}{2}
\end{aligned}
$$

This leads to:

$$\Gamma = 0 \tag{4.7}$$

4.3.2 The case $V_\alpha \neq 0$

The following equation must be satisfied:

$$
\begin{aligned}
& V_\mu V^\mu - \frac{1}{4} = 0 \\
\Leftrightarrow \quad & V_\mu V^\mu = \frac{1}{4}
\end{aligned}
$$

This leads to:

$$\Gamma = \frac{1}{2} \cdot I + \not{V} \tag{4.8}$$

This is the propagator with a momentum term as can be seen easily after substituting $\not{V} \rightarrow \frac{\not{p}}{2m}$, with $p^2 = m^2$.

4.3.3 The case $T_{\beta\delta} \neq 0$

The equations

$$
\begin{aligned}
& 2T_{\mu\nu}T^{\mu\nu} - \frac{1}{4} = 0 \\
& -\epsilon^{\mu\nu\rho\pi}T_{\mu\nu}T_{\rho\pi} = 0
\end{aligned}
$$

lead to two different solutions:

$$\Gamma = \tfrac{1}{2} \cdot I + i\not{k}\not{l} \qquad \text{with} \quad k^2 l^2 = \frac{1}{4} \tag{4.9}$$

$$\Gamma = \tfrac{1}{2} \cdot I + \gamma^5\not{k}\not{l} \qquad \text{with} \quad k^2 l^2 = \frac{1}{4} \tag{4.10}$$

1. $\Gamma = \frac{1}{2} \cdot I + i\not{k}\not{l}$. There are two possibilities:

   (a) $k^2 > 0$ and $l^2 > 0$. With transformation:

   $$\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha)\frac{\not{k}}{\sqrt{k^2}} \tag{4.11}$$

   where $\cos(2\alpha) = 0$, this solution can be transformed to:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + n\sqrt{k^2}\not{l} \tag{4.12}$$

where $n = \pm 1$. This is the propagator with a momentum term (see formula 4.8 in section 4.3.2).

(b) $k^2 < 0$ and $l^2 < 0$. This propagator cannot be transformed to a simpler form.

2. $\Gamma = \frac{1}{2} \cdot I + \gamma^5 \not{k}\not{l}$. Now there are also two different solutions:

   (a) $k^2 > 0$ and $l^2 > 0$. This propagator cannot be transformed to a simpler case. It is however equivalent to solution 4.9 where $k^2 < 0$ and $l^2 < 0$. We can write this propagator in a more physical form as:

$$\Gamma = \frac{1}{2m} \left(m + \gamma^5 \not{p}\not{s} \right) \tag{4.13}$$

   with $p^2 = m^2$ and $s^2 = 1$.

   (b) $k^2 < 0$ and $l^2 < 0$. With transformation:

$$\Sigma = \cos(\alpha) \cdot I - i \sin(\alpha)\not{Q} \tag{4.14}$$

   with $Q^2 = 1$, $\not{Q} = \overline{\not{Q}}$ and $Q_\mu k^\mu = Q_\mu l^\mu = 0$, and $\cos(2\alpha) = 0$. This leads to:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + in\gamma^5 \not{k}\not{l}\not{Q} \tag{4.15}$$

   where $n = \pm 1$. This is also the propagator with a momentum term, and therefore equivalent to formula 4.12.

4.3.4 The case $A_\lambda \neq 0$

Now we have the following equation:

$$\begin{aligned} & A_\mu A^\mu - \frac{1}{4} = 0 \\ \Leftrightarrow \quad & A_\mu A^\mu = \frac{1}{4} \end{aligned}$$

This leads to:

$$\Gamma = \frac{1}{2} \cdot I + \gamma^5 \not{A} \tag{4.16}$$

Note that in this 2+2 space $A^2 > 0$ in contrast with the Minkovski space. We might interpret this propagator as one with a spin term; substitute $\not{A} \to \frac{\not{s}}{2}$ to see this. Here $s^2 = 1$.

4.3.5 The case $R \neq 0$

This is the equation:

$$\begin{aligned} & R^2 - \frac{1}{4} = 0 \\ \Leftrightarrow \quad & R^2 = \frac{1}{4} \end{aligned}$$

This leads to:

$$\Gamma = \frac{1}{2} \cdot I \pm \frac{i}{2}\gamma^5 \tag{4.17}$$

This is a new type of solution, because in Minkovski space there is no solution of this type. This solution can be transformed in two different ways:

1. The transformation

$$\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha)\frac{\not{A}}{\sqrt{A^2}} \tag{4.18}$$

with $A^2 > 0$ and $\cos(2\alpha) = 0$ leads to:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I \pm \frac{1}{2}\frac{\gamma^5 \not{A}}{\sqrt{A^2}} \tag{4.19}$$

2. The transformation

$$\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha)\frac{\gamma^5 \not{V}}{\sqrt{V^2}} \tag{4.20}$$

with $V^2 > 0$ and $\cos(2\alpha) = 0$ leads to:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I \pm \frac{1}{2}\frac{\not{V}}{\sqrt{V^2}} \tag{4.21}$$

This means that in the 2+2 space, the propagator with a momentum term (formula 4.8) and the propagator with a spin term (formula 4.16) can be transformed to each other and to propagator 4.17.

4.3.6 The case $V_\alpha \neq 0, \quad T_{\beta\delta} \neq 0$

In this case we have the following equations:

$$\begin{aligned} & -2\epsilon^{\mu\nu\rho}{}_{\lambda} V_\mu T_{\nu\rho} = 0 \\ & -\epsilon^{\mu\nu\rho\pi} T_{\mu\nu} T_{\rho\pi} = 0 \\ \Leftrightarrow \quad & T_{\beta\delta} = V_\beta W_\delta - W_\beta V_\delta \end{aligned}$$

The third equation is:

$$
\begin{aligned}
& V_\mu V^\mu + 2T_{\mu\nu}T^{\mu\nu} - \frac{1}{4} = 0 \\
\Rightarrow \quad & V_\mu V^\mu + 4(V_\mu V^\mu)(W_\nu W^\nu) - 4(V_\rho W^\rho)^2 - \frac{1}{4} = 0
\end{aligned}
$$

One can always choose $V_\mu W^\mu = 0$ because of the substitution:

$$W_\mu \quad \to \quad W_\mu - \frac{V_\nu W^\nu}{V_\rho V^\rho} V_\mu \tag{4.22}$$

This means that Γ can be written as:

$$\Gamma = \frac{1}{2} \cdot I + \not{V} + 2i\not{V}\not{W} \tag{4.23}$$

$$V^2 + 4V^2W^2 - \frac{1}{4} = 0 \tag{4.24}$$

Because $V^2 \neq 0$, there are two different cases:

1. $V^2 > 0$. Take the transformation

$$\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha)\frac{\not{V}}{\sqrt{V^2}} \tag{4.25}$$

with $\cos(2\alpha) = 0$. This leads to:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \not{V} + 2n\sqrt{V^2}\not{W} \tag{4.26}$$

with $n = \pm 1$. Define $U_\mu = V_\mu + 2n\sqrt{V^2}W_\mu$. This U_μ satisfies $U^2 = \frac{1}{4}$, so

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \not{U} \tag{4.27}$$

This propagator is the same as the one of formula 4.8 in section 4.3.2.

2. $V^2 < 0$. Define Q_μ with the following properties: $\not{Q} = \overline{\not{Q}}$, $Q^2 = 1$, $Q_\mu V^\mu = 0$ and $Q_\mu W^\mu = 0$. Now take transformation:

$$\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha)\not{Q} \tag{4.28}$$

with $\cos(2\alpha) = 0$. The result is:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + i\not{V}(2\not{W} + n\not{Q}) \tag{4.29}$$

where $n = \pm 1$. This is the propagator of formula 4.9 (see section 4.3.3) because:

$$V_\mu V^\mu (2W_\nu + nQ_\nu)(2W^\nu + nQ^\nu) = V^2(4W^2 + 1) = \frac{1}{4}$$

So:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + i\not{k}\not{l} \tag{4.30}$$

where $k^2 < 0$ and $l^2 < 0$.

4.3.7 The case $V_\alpha \neq 0, \quad A_\lambda \neq 0$

The equations are:

$$\epsilon^{\mu\nu}{}_{\beta\delta} V_\mu A_\nu = 0$$
$$V_\mu V^\mu + A_\mu A^\mu - \frac{1}{4} = 0$$

The solution is:

$$\Gamma = \frac{1}{2} \cdot I + \not{V} + k\gamma^5 \not{V} \qquad k \quad \text{real} \tag{4.31}$$
$$V^2(1 + k^2) = \frac{1}{4} \qquad \Rightarrow \quad V^2 > 0 \tag{4.32}$$

Note that in 2+2 space $V^2 > 0$ unlike this case in Minkovski space where V^2 can also be negative. Take transformation:

$$\Sigma = \cos(\alpha) \cdot I - i \sin(\alpha) \frac{\not{V}}{\sqrt{V^2}} \tag{4.33}$$

with $\cos(2\alpha) = 0$. The resulting propagator

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \not{V} \pm ik\sqrt{V^2}\gamma^5 \tag{4.34}$$

is equal to formula 4.35 in section 4.3.8 because $V_\mu V^\mu + (nk\sqrt{V^2})^2 = \frac{1}{4}$. There we will show that this propagator can be simplified with another transformation.

4.3.8 The case $V_\alpha \neq 0, \quad R \neq 0$

The equation is:

$$V_\mu V^\mu + R^2 = \frac{1}{4}$$

with solution:

$$\Gamma = \frac{1}{2} \cdot I + \not{V} \pm i\sqrt{\frac{1}{4} - V^2}\gamma^5 \tag{4.35}$$

This Γ can be transformed; there are three different cases: $V^2 > 0$, $V^2 = 0$ and $V^2 < 0$, because now the sign before R^2 has changed with regard to the same case in Minkovski space (see section 3.3.12).

1. $V^2 > 0$. Take transformation:

$$\Sigma = \cos(\alpha) \cdot I - i \sin(\alpha) \frac{\gamma^5 \not{V}}{\sqrt{V^2}} \tag{4.36}$$
$$\text{with} \quad R\cos(2\alpha) - \sin(2\alpha)\sqrt{V^2} = 0$$

   This leads to

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I \pm \frac{1}{2} \frac{\not{V}}{\sqrt{V^2}} \tag{4.37}$$

   which the propagator with a momentum term (see formula 4.8 in section 4.3.2).

2. $V^2 = 0$. Define Q_μ with the properties: $\not{Q} = \overline{\not{Q}}$, $Q^2 = 1$ and $Q_\mu V^\mu = 0$ and choose transformation:

$$\Sigma = \cos(\alpha) \cdot I - i \sin(\alpha) \gamma^5 \not{Q} \tag{4.38}$$

   with $\cos(2\alpha) = 0$. The resulting transformation is:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \not{V} + nR\not{Q} \tag{4.39}$$

   with $n = \pm 1$. This is also equal to the propagator of formula 4.8, because

$$(V_\mu + nRQ_\mu)(V^\mu + nRQ^\mu) = \frac{1}{4}$$

3. $V^2 < 0$. In this case, we take

$$\Sigma = \frac{1}{\sqrt{\cos(2\alpha)}} \left(\cos(\alpha) \cdot I - i \sin(\alpha) \frac{\gamma^5 \not{V}}{\sqrt{-V^2}} \right) \tag{4.40}$$
$$\text{with} \quad 1 + \frac{R}{\sqrt{-V^2}} \sin(2\alpha) = 0$$

   The result is

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I \pm \frac{i}{2} \operatorname{sgn}(R) \gamma^5 \tag{4.41}$$

   This propagator is the same as the propagator of formula 4.17 in section 4.3.5. We have shown that it is equivalent to "more standard" propagators.

4.3.9 The case $T_{\beta\delta} \neq 0, \quad A_\lambda \neq 0$

Now we have three equations:

$$
\begin{aligned}
&2\epsilon^{\mu\nu\rho}{}_{\alpha} T_{\mu\nu} A_\rho = 0 \\
&-\epsilon^{\mu\nu\rho\pi} T_{\mu\nu} T_{\rho\pi} = 0 \\
&2T_{\mu\nu} T^{\mu\nu} + A^2 - \frac{1}{4} = 0
\end{aligned}
$$

In analogy with sections 4.3.6 and 3.3.13 the solution is

$$
\Gamma = \frac{1}{2} \cdot I + 2i \not{A} \not{W} + \gamma^5 \not{A} \qquad (4.42)
$$

$$
\text{with} \quad A^2 + 4A^2 W^2 - \frac{1}{4} = 0
$$

and $A_\mu W^\mu = 0$. Because $A^2 \neq 0$, there are two cases:

1. $A^2 > 0$. Take transformation:

$$
\Sigma = \cos(\alpha) \cdot I - i \sin(\alpha) \frac{\not{A}}{\sqrt{A^2}} \qquad (4.43)
$$

with $\cos(2\alpha) = 0$. This leads to

$$
\Sigma \Gamma \bar{\Sigma} = \frac{1}{2} \cdot I + 2n\sqrt{A^2} \not{W} + in\sqrt{A^2} \gamma^5 \qquad (4.44)
$$

with $n = \pm 1$. This is a propagator of the same type as formula 4.35 (see section 4.3.8) because

$$
(2n\sqrt{A^2} W_\mu)(2n\sqrt{A^2} W^\mu) + (n\sqrt{A^2})^2 = 4A^2 W^2 + A^2 = \frac{1}{4}
$$

We have seen that this propagator can be simplified.

2. $A^2 < 0$. Define a Q_μ with the following properties: $\not{Q} = \overline{\not{Q}}$, $Q^2 = 1$, $Q_\mu A^\mu = 0$ and $Q_\mu W^\mu = 0$. Now take this transformation:

$$
\Sigma = \cos(\alpha) \cdot I - i \sin(\alpha) \not{Q} \qquad (4.45)
$$

with $\cos(2\alpha) = 0$. The result is

$$
\Sigma \Gamma \bar{\Sigma} = \frac{1}{2} \cdot I + i \not{A} (2\not{W} + n\not{Q}) \qquad (4.46)
$$

where $n = \pm 1$. This is the propagator of formula 4.9 (see section 4.3.3) because

$$
A_\mu A^\mu (2W_\nu + nQ_\nu)(2W^\nu + nQ^\nu) = A^2(4W^2 + 1) = \frac{1}{4}
$$

So:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + i\not{k}\not{l} \tag{4.47}$$

where $k^2 < 0$ and $l^2 < 0$.

4.3.10 The case $A_\lambda \neq 0, \quad R \neq 0$

The equation is:

$$A_\mu A^\mu + R^2 = \frac{1}{4}$$

and the solution is therefore

$$\Gamma = \frac{1}{2} \cdot I + \gamma^5 \not{A} \pm i\sqrt{\frac{1}{4} - A^2}\gamma^5 \tag{4.48}$$

This solution can be transformed to a more simple form; we distinguish three different cases, $A^2 > 0$, $A^2 = 0$ and $A^2 < 0$.

1. $A^2 > 0$. Let's take transformation:

$$\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha)\frac{\not{A}}{\sqrt{A^2}} \tag{4.49}$$
$$\text{with} \quad \sin(2\alpha)\sqrt{A^2} + R\cos(2\alpha) = 0$$

   This leads to the propagator with a spin term (see formula 4.16 in section 4.3.4):

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I \pm \frac{1}{2}\frac{\gamma^5 \not{A}}{\sqrt{A^2}} \tag{4.50}$$

2. $A^2 = 0$. Define Q_μ with the following properties: $\not{Q} = \overline{\not{Q}}$, $Q^2 = 1$ and $Q_\mu A^\mu = 0$. Now choose this transformation:

$$\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha)\not{Q} \tag{4.51}$$

   with $\cos(2\alpha) = 0$. The resulting transformation is:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \gamma^5 \not{A} - nR\gamma^5 \not{Q} \tag{4.52}$$

   with $n = \pm 1$. This propagator is also equal to formula 4.16, because we see that

$$(A_\mu - nRQ_\mu)(A^\mu - nRQ^\mu) = \frac{1}{4}$$

3. $A^2 < 0$. Now take this transformation:

$$\Sigma = \frac{1}{\sqrt{\cos(2\alpha)}} \left(\cos(\alpha) - i\sin(\alpha) \frac{\not{A}}{\sqrt{-A^2}} \right) \quad (4.53)$$
$$\text{with} \quad 1 - \frac{R}{\sqrt{-A^2}} \sin(2\alpha) = 0$$

We now find for the transformation the propagator of formula 4.17 (see section 4.3.5):

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I \pm \frac{i}{2} \mathrm{sgn}\,(R)\,\gamma^5 \quad (4.54)$$

4.3.11 The case $S \neq 0, \quad T_{\beta\delta} \neq 0, \quad R \neq 0$

In Minkovski space, this solution doesn't exist (see section 3.3.20), but here we find a nonzero result. In this case, there are three equations. First, we look at:

$$\begin{aligned} & 2ST_{\beta\delta} - \epsilon^{\mu\nu}{}_{\beta\delta} T_{\mu\nu} R = 0 \\ \Leftrightarrow \quad & (S^2 - R^2) T_{\beta\delta} = 0 \\ \Leftrightarrow \quad & S^2 = R^2 \end{aligned} \quad (4.55)$$

Now, we take for $T_{\beta\delta}$ the general form:

$$T_{\beta\delta} = \frac{1}{2}(-r\cos(\theta)\epsilon^{\mu\nu}{}_{\beta\delta} k_\mu l_\nu + r\sin(\theta)(k_\beta l_\delta - k_\delta l_\beta))$$

with $k_\mu l^\mu = 0$ and $(k^2)^2 = (l^2)^2 = 1$. Now we look at the other two equations:

$$\begin{aligned} & S^2 + 2T_{\mu\nu}T^{\mu\nu} + R^2 - \frac{1}{4} = 0 \\ \Leftrightarrow \quad & S^2 + r^2 k^2 l^2 + R^2 - \frac{1}{4} = 0 \end{aligned} \quad (4.56)$$

and

$$\begin{aligned} & 2SR - \epsilon^{\mu\nu\rho\pi} T_{\mu\nu} T_{\rho\pi} = 0 \\ \Leftrightarrow \quad & 2SR + r^2 \sin(2\theta) k^2 l^2 = 0 \end{aligned} \quad (4.57)$$

From this follows:

$$\sin(2\theta) = \frac{2SR}{S^2 + R^2 - \frac{1}{4}} = \pm \frac{S^2}{S^2 - \frac{1}{8}}$$

Because $|\sin(2\theta)| \leq 1$, this means that:

$$S^2 \leq \frac{1}{16} \quad (4.58)$$

Now we use equation 4.56 to solve r:

$$r = \sqrt{\frac{1}{k^2 l^2}\left(\frac{1}{4} - S^2 - R^2\right)}$$

Because r is real, we have:

$$k^2 l^2 = 1 \tag{4.59}$$

This leads to the following propagator:

$$\Gamma = (S + \frac{1}{2}) \cdot I - r\cos(\theta)\gamma^5 \not{k}\not{l} + ir\sin(\theta)\not{k}\not{l} + iR\gamma^5 \tag{4.60}$$

with $k^2 = l^2 = 1$ or $k^2 = l^2 = -1$. In both cases the propagator can be transformed to a more simple form.

1. $k^2 = 1$. Take the transformation:

$$\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha)\not{k} \tag{4.61}$$

   with $\cos(2\alpha) = 0$. The resulting transformation is:

$$\Sigma\Gamma\bar{\Sigma} = \left(S + \frac{1}{2}\right) \cdot I + r\sin(\theta)\not{l} - r\cos(\theta)\gamma^5 \not{k}\not{l} - R\gamma^5 \not{k} \tag{4.62}$$

   This is a propagator with both a momentum and a spin term (see formula 4.83 in section 4.3.14, which is about this propagator). Therefore we must have

$$S^2 = R^2 = \frac{1}{16}, \qquad r\cos(\theta) = \pm\frac{1}{4}, \qquad r\sin(\theta) = \pm\frac{1}{4}$$

2. $k^2 = -1$. Define a fourvector Q_μ with the following properties: $Q^2 = 1$, $\not{Q} = \overline{\not{Q}}$, $Q_\mu k^\mu = 0$ and $Q_\mu l^\mu = 0$. Now take the transformation:

$$\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha)\not{Q} \tag{4.63}$$

   with $\cos(2\alpha) = 0$. The result is:

$$\Sigma\Gamma\bar{\Sigma} = \left(S + \frac{1}{2}\right) \cdot I - inr\cos(\theta)\gamma^5 \not{k}\not{l}\not{Q} + ir\sin(\theta)\not{k}\not{l} - nR\gamma^5 \not{Q} \tag{4.64}$$

   with $n = \pm 1$. This propagator is also equal to formula 4.83, so:

$$S^2 = R^2 = \frac{1}{16}, \qquad r\cos(\theta) = \pm\frac{1}{4}, \qquad r\sin(\theta) = \pm\frac{1}{4}$$

   Now, the propagator can be written as:

$$
\begin{aligned}
\Sigma\Gamma\bar{\Sigma} =& \frac{1}{4}\cdot I + \frac{\not{V}}{4} + i\frac{q}{4}\not{k}\not{l} - \frac{n}{4}\gamma^5\not{Q} \\
=& \frac{1}{4}\cdot I + \frac{\not{V}}{4} + q\gamma^5\not{V}\not{Q} - \frac{n}{4}\gamma^5\not{Q}
\end{aligned} \tag{4.65}
$$

with $q = \pm 1$ and $V^2 = 1$.

4.3.12 The case $V_\alpha \neq 0, \quad T_{\beta\delta} \neq 0, \quad A_\lambda \neq 0$

In this case we have four independent equations:

$$
\begin{aligned}
&-\epsilon^{\mu\nu}{}_{\beta\delta}V_\mu A_\nu = 0 \\
&-2\epsilon^{\mu\nu\rho}{}_\lambda V_\mu T_{\nu\rho} = 0 \\
&-\epsilon^{\mu\nu\rho\pi}T_{\mu\nu}T_{\rho\pi} = 0 \\
&V^2 + T_{\mu\nu}T^{\mu\nu} + A^2 - \frac{1}{4} = 0
\end{aligned}
$$

The solution is

$$
\begin{aligned}
&\Gamma = \frac{1}{2}\cdot I + \not{V} + 2i\not{V}\not{W} + k\gamma^5\not{V} \qquad k \quad \text{real} \qquad (4.66)\\
&V^2(1 + 4W^2 + k^2) - \frac{1}{4} = 0 \qquad \Rightarrow \quad V^2 \neq 0
\end{aligned}
$$

with $V_\mu W^\mu = 0$. Because $V^2 \neq 0$ we see that there are two cases:

1. $V^2 > 0$. The transformation

$$
\Sigma = \cos(\alpha)\cdot I - i\sin(\alpha)\frac{\not{V}}{\sqrt{V^2}} \tag{4.67}
$$

with $\cos(2\alpha) = 0$ leads to propagator

$$
\Sigma\Gamma\bar{\Sigma} = \frac{1}{2}\cdot I + \not{V} + 2n\sqrt{V^2}\not{W} + ink\sqrt{V^2}\gamma^5 \tag{4.68}
$$

with $n = \pm 1$. This propagator is equal to formula 4.35 in section 4.3.8, because we see that

$$
(V_\mu + 2n\sqrt{V^2}W_\mu)(V^\mu + 2nW^\mu) + (nk\sqrt{V^2}) = \frac{1}{4}
$$

The propagator can be transformed to a simpler case.

2. $V^2 < 0$. Define Q_μ with $\not{Q} = \overline{\not{Q}}$, $Q^2 = 1$, $Q_\mu V^\mu = 0$ and $Q_\mu W^\mu = 0$. We can easily see that there are two different possible transformations:

(a)

$$\Sigma = \cos(\alpha) \cdot I - i \sin(\alpha)\gamma^5 \not{Q} \tag{4.69}$$

with $\cos(2\alpha) = 0$ leads to

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \not{V} + i\not{V}(2\not{W} + nk\not{Q}) \tag{4.70}$$

where $n = \pm 1$. This propagator is equal to formula 4.23 in section 4.3.6, because

$$\begin{aligned} V^2 + 4V^2X^2 &= V^2 + 4V^4\frac{1}{2}(2W_\mu + nkQ_\mu)\frac{1}{2}(2W^\mu + nkQ^\mu) \\ &= V^2(1 + k^2 + 4W^2) = \frac{1}{4} \end{aligned}$$

with $X_\mu = \frac{1}{2}(2W_\mu + nkQ_\mu)$. So the transformed propagator is

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \not{V} + 2i\not{V}\not{X} \tag{4.71}$$

This propagator can be transformed further to a simpler case.

(b)

$$\Sigma = \cos(\alpha) \cdot I - i \sin(\alpha)\not{Q} \tag{4.72}$$

with $\cos(2\alpha) = 0$ leads to

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + i\not{V}(2\not{W} + n\not{Q}) + k\gamma^5\not{V} \tag{4.73}$$

where $n = \pm 1$. This is equal to the propagator of formula 4.42 in section 4.3.9, because

$$\begin{aligned} A^2 + 4A^2X^2 &= k^2V^2 + 4k^2V^2\frac{1}{2k}(2W_\mu + nQ_\mu)\frac{1}{2k}(2W^\mu + nQ^\mu) \\ &= V^2(4W^2 + k^2 + 1) = \frac{1}{4} \end{aligned}$$

with $A_\mu = \frac{1}{k}V_\mu$ and $X_\mu = \frac{1}{2k}(2W_\mu + nQ_\mu)$. So the transformed propagator is

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + 2i\not{A}\not{X} + \gamma^5\not{A} \tag{4.74}$$

This propagator can be transformed further to a simpler case.

4.3.13 The case $V_\alpha \neq 0, \quad A_\lambda \neq 0, \quad R \neq 0$

In this case we have two different equations:

$$- \epsilon^{\mu\nu}{}_{\beta\delta} V_\mu A_\nu = 0$$
$$V^2 + A^2 + R^2 = \frac{1}{4}$$

The solution of these equations is

$$\Gamma = \frac{1}{2} \cdot I + \not{V} + k\gamma^5 \not{V} + iR\gamma^5 \qquad k \quad \text{real} \tag{4.75}$$
$$V^2(1+k^2) + R^2 = \frac{1}{4}$$

We distinguish three different cases: $V^2 > 0$, $V^2 = 0$ and $V^2 < 0$.

1. $V^2 > 0$. Take transformation

$$\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha)\frac{\not{V}}{\sqrt{V^2}} \tag{4.76}$$
$$\text{with} \quad k\cos(2\alpha) - R\sin(2\alpha)\frac{1}{\sqrt{V^2}} = 0$$

The resulting transformation

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \not{V} \pm i\,\mathrm{sgn}\,(R)\sqrt{\frac{1}{4} - V^2}\gamma^5 \tag{4.77}$$

is the propagator of formula 4.35 (see section 4.3.8). This propagator can be simplified.

2. $V^2 = 0$. Define Q_μ with the following properties: $\not{Q} = \overline{\not{Q}}$, $Q^2 = 1$ and $Q_\mu V^\mu = 0$. Now take this transformation:

$$\Sigma = \cos(\alpha) \cdot I - \sin(\alpha)\not{Q} \tag{4.78}$$

with $\cos(2\alpha) = 0$. The result is

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I - in\not{Q}\not{V} + k\gamma^5\not{V} - nR\gamma^5\not{Q} \tag{4.79}$$

with $n = \pm 1$. Define $A_\mu = kV_\mu - nRQ_\mu$ and $W_\mu = \frac{1}{2}V_\mu$. Now $A_\mu W^\mu = 0$. We can write

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + 2i\not{A}\not{W} + \gamma^5\not{A} \tag{4.80}$$

This is the same propagator as the one of formula 4.42 (see section 4.3.9). In this case also:

$$A^2 + 4A^2W^2 = R^2 = \frac{1}{4}$$

The propagator can be simplified.

3. $V^2 < 0$. Take transformation

$$\Sigma = \frac{1}{\sqrt{\cos(2\alpha)}} \left(\cos(\alpha) \cdot I - i \sin(\alpha) \frac{\not{V}}{\sqrt{-V^2}} \right) \tag{4.81}$$
$$\text{with} \quad k - \frac{R}{\sqrt{-V^2}} \sin(2\alpha) = 0$$

The resulting transformation

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + \not{V} \pm \operatorname{sgn}(R) \sqrt{\frac{1}{4} - V^2} \gamma^5 \tag{4.82}$$

is again the propagator of formula 4.35.

4.3.14 The case $S \neq 0, \quad V_\alpha \neq 0, \quad T_{\beta\delta} \neq 0, \quad A_\lambda \neq 0$

In this case we have four independent equations:

$$\begin{aligned}
&2ST_{\beta\delta} - \epsilon^{\mu\nu}{}_{\beta\delta} V_\mu A_\nu = 0 \\
&2SV_\alpha + 2\epsilon^{\mu\nu\rho}{}_{\alpha} T_{\mu\nu} A_\rho = 0 \\
&2SA_\lambda - 2\epsilon^{\mu\nu\rho}{}_{\lambda} V_\mu T_{\nu\rho} = 0 \\
&S^2 + V^2 + 2T_{\mu\nu}T^{\mu\nu} + A^2 - \frac{1}{4} = 0
\end{aligned}$$

The solution is, in analogy with section 3.3.26:

$$\Gamma = \frac{1}{4} \cdot I + \not{V} - 4\gamma^5 \not{V} \not{A} + \gamma^5 \not{A} \tag{4.83}$$

with $V^2 = A^2 = \frac{1}{16}$ [1] and $V_\mu A^\mu = 0$. We cannot transform this propagator to a more simple form. We can write this propagator in a more familiar way by substituting $\not{V} \to \frac{\not{p}}{4m}$ and $\not{A} \to \frac{\not{s}}{4}$:

$$\Gamma = \frac{1}{4m} \left(m + \not{p} \right) \left(1 + \gamma^5 \not{s} \right) \tag{4.84}$$

[1] The only difference with section 3.3.26 is that here $V^2 = A^2$, instead of $V^2 = -A^2$

4.3.15 The case $V_\alpha \neq 0, \quad T_{\beta\delta} \neq 0, \quad A_\lambda \neq 0, \quad R \neq 0$

We first look at equation

$$\begin{aligned} & -\epsilon^{\mu\nu}{}_{\beta\delta}(V_\mu A_\nu + T_{\mu\nu}R) = 0 \\ \Leftrightarrow \quad & T_{\beta\delta} = -\frac{1}{2R}(V_\beta A_\delta - V_\delta A_\beta) \end{aligned}$$

This solution for $T_{\beta\delta}$ also satisfies:

$$\begin{aligned} & 2\epsilon^{\mu\nu\rho}{}_\alpha T_{\mu\nu} A_\rho = 0 \\ & -2\epsilon^{\mu\nu\rho}{}_\lambda V_\mu T_{\nu\rho} = 0 \\ & -\epsilon^{\mu\nu\rho\pi} T_{\mu\nu} T_{\rho\pi} = 0 \end{aligned}$$

The last equation is

$$\begin{aligned} & V^2 + 2T_{\mu\nu}T^{\mu\nu} + A^2 + R^2 - \frac{1}{4} = 0 \\ \Leftrightarrow \quad & V^2 + \frac{1}{R^2}\left(V^2A^2 - (V_\mu A^\mu)^2\right) + A^2 + R^2 - \frac{1}{4} = 0 \end{aligned}$$

This means that $V^2 \neq R^2$ and $A^2 \neq R^2$. The solution now becomes:

$$\Gamma = \frac{1}{2} \cdot I + \not{V} - \frac{i}{2R}(\not{V}\not{A} - \not{A}\not{V}) + \gamma^5\not{A} + iR\gamma^5 \tag{4.85}$$

Now, we have two possibilities: $\neg(V^2 > 0 \wedge A^2 > 0) \wedge A_\mu \neq kV_\mu$ or otherwise.

1. $\neg(V^2 > 0 \wedge A^2 > 0) \wedge A_\mu \neq kV_\mu$. We choose this possibility because in this case we can take a transformation that wouldn't be possible in other cases. Define Q_μ with the properties: $Q^2 = 1$, $\not{Q} = \overline{\not{Q}}$, $Q_\mu V^\mu = 0$ and $Q_\mu A^\mu = 0$. Now choose the transformation:

$$\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha)\not{Q} \tag{4.86}$$

with $\cos(2\alpha) = 0$. The transformed propagator becomes:

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + in\not{V}\not{Q} - \frac{i}{2R}(\not{V}\not{A} - \not{A}\not{V}) + \gamma^5 - nR\gamma^5\not{Q} \tag{4.87}$$

This propagator is similar to formula 4.42 (see section 4.3.9):

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2} \cdot I + i(\not{B}\not{W} - \not{W}\not{B}) + \gamma^5\not{A} \tag{4.88}$$

with $B_\mu = A_\mu - nRQ_\mu$ and $W_\mu = \frac{1}{2R}V_\mu$. Be aware however that in this case the inner product $B_\mu W^\mu$ doesn't need to be zero. The following equation has to be satisfied:

$$\begin{aligned} & B^2(1+4W^2) - 4(B_\mu W^\mu)^2 - \frac{1}{4} = 0 \\ \Rightarrow\quad & (A_\mu - nRQ_\mu)(A^\mu - nRQ^\mu)(1 + 4\frac{1}{4R^2}V_\mu V^\mu) - 4((A_\mu - nRQ_\mu)\frac{1}{2R}V^\mu)^2 - \frac{1}{4} = 0 \\ \Leftrightarrow\quad & V^2 + \frac{1}{R^2}(V^2A^2 - (V_\mu A^\mu)^2) + A^2 + R^2 - \frac{1}{4} = 0 \end{aligned}$$

If we wish, we can choose $B_\mu W^\mu = 0$ now. The propagator can be transformed to simpler cases.

2. otherwise. We take transformation

$$\Sigma = \cos(\alpha)\cdot I - i\sin(\alpha)\frac{V}{\sqrt{V^2}} \tag{4.89}$$

with $\cos(2\alpha) = 0$. The result is

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2}\cdot I + \left(1 + n\frac{V_\mu A^\mu}{R\sqrt{V^2}}\right)\not{V} - n\frac{\sqrt{V^2}}{R}\not{A} + \gamma^5\not{A} - nR\frac{\gamma^5\not{V}}{\sqrt{V^2}} \tag{4.90}$$

with $n = \pm 1$. Next, we define

$$X_\mu = \left(1 + n\frac{V_\mu A^\mu}{R\sqrt{V^2}}\right)V_\mu - n\frac{\sqrt{V^2}}{R}A_\mu \tag{4.91}$$

$$Y_\mu = A_\mu - nR\frac{R}{\sqrt{V^2}}V_\mu \tag{4.92}$$

Propagator 4.90 should be equal to formula 4.31 (see section 4.3.7), because it has to be a projection operator. Therefore the equation $X^2 + Y^2 = \frac{1}{4}$ has to be satisfied :

$$\begin{aligned} X^2 + Y^2 =& V^2 + \frac{1}{R^2}\left(V^2A^2 - (V_\mu A^\mu)^2\right) + A^2 + R^2 - 2n\frac{R}{\sqrt{V^2}}(V_\mu A^\mu) \\ =& \frac{1}{4} \end{aligned}$$

This means that $V_\mu A^\mu = 0$, because the properties of the transformation ensure us, that propagator 4.90 is a projection operator. Now one can see as well that $Y_\mu = kX_\mu$ with $k = -n\frac{R}{\sqrt{V^2}}$. So the propagator becomes

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{2}\cdot I + \not{X} + \gamma^5\not{Y} \tag{4.93}$$

This propagator can be transformed to a simpler case (see formula 4.31).

4.3.16 The case $S \neq 0, \quad V_\alpha \neq 0, \quad T_{\beta\delta} \neq 0, \quad A_\lambda \neq 0, \quad R \neq 0$

In this case there are four independent equations:

$$
\begin{aligned}
&2ST_{\beta\delta} - \epsilon^{\mu\nu}{}_{\beta\delta}(V_\mu A_\nu + T_{\mu\nu}R) = 0 \\
&2SV_\alpha + 2\epsilon^{\mu\nu\rho}{}_\alpha T_{\mu\nu}A_\rho = 0 \\
&2SA_\lambda - 2\epsilon^{\mu\nu\rho}{}_\lambda V_\mu T_{\nu\rho} = 0 \\
&S^2 + V^2 + 2T_{\mu\nu}T^{\mu\nu} + A^2 + R^2 - \frac{1}{4} = 0
\end{aligned}
$$

In analogy with section 3.3.31 the solution is

$$
\Gamma = \left(S + \frac{1}{2}\right) \cdot I + \not{V} + \frac{2S\gamma^5\not{V}\not{A} + iR(\not{V}\not{A} - \not{A}\not{V})}{2(S^2 - R^2)} + \gamma^5\not{A} + iR\gamma^5 \tag{4.94}
$$

with $V^2 = A^2 = S^2 - R^2$,[2] $V_\mu A^\mu = 0$ and $S^2 = \frac{1}{16}$. We choose $S = -\frac{1}{4}$. One might think that $V^2 = 0$ is a valid solution, but in that case either $S = 0$ or $V_\alpha = 0$, which they shouldn't here. So there are two possibilities: $V^2 > 0$ and $V^2 < 0$.

1. $V^2 > 0$. Choose this transformation:

$$
\Sigma = \cos(\alpha) \cdot I - i\sin(\alpha)\frac{\gamma^5\not{V}}{\sqrt{V^2}} \tag{4.95}
$$
$$
\text{with} \quad -\sqrt{V^2}\sin(2\alpha) + R\cos(2\alpha) = 0
$$

   The resulting propagator is the one with both a momentum and a spin term (see formula 4.83 in section 4.3.14):

$$
\Sigma\Gamma\bar{\Sigma} = \frac{1}{4} \cdot I + \frac{n}{4}\frac{\not{V}}{\sqrt{V^2}} - \frac{1}{4V^2}\gamma^5\not{V}\not{A} + \frac{n}{4}\frac{\gamma^5\not{A}}{\sqrt{A^2}} \tag{4.96}
$$

   with $n = \pm 1$.

2. $V^2 < 0$. Now choose the transformation

$$
\Sigma = \frac{1}{\sqrt{\cos(2\alpha)}}\left(\cos(\alpha) \cdot I - i\sin(\alpha)\frac{\gamma^5\not{V}}{\sqrt{-V^2}}\right) \tag{4.97}
$$
$$
\text{with} \quad 1 + \frac{R}{\sqrt{-V^2}}\sin(2\alpha) = 0
$$

   The result is

[2] This is the main difference with the 'Minkovski' solution.

$$\Sigma\Gamma\bar{\Sigma} = \frac{1}{4} \cdot I - \frac{1}{4V^2}\gamma^5 \not{V} \not{A} + \frac{in}{4V^2}\mathrm{sgn}\,(R)\,\not{V} \not{A} + i\frac{n}{4}\mathrm{sgn}\,(R)\,\gamma^5 \qquad (4.98)$$

This propagator is equal to formula 4.60 (see section 4.3.11). It can be transformed to the propagator with both momentum and spin terms.

4.4 Conclusion

In a four dimensional space with two timelike and two spacelike dimensions, we find the following four standard propagators:

$$\Gamma = 0 \qquad (4.99)$$

$$\Gamma = \frac{1}{2m}\left(m + \not{p}\right) \qquad (4.100)$$

$$\Gamma = \frac{1}{2m}\left(m + \gamma^5 \not{p} \not{s}\right) \qquad (4.101)$$

$$\Gamma = \frac{1}{4m}\left(m + \not{p}\right)\left(1 + \gamma^5 \not{s}\right) \qquad (4.102)$$

All other solutions can be transformed to one of these four propagators. These propagators are presumably not equivalent to each other, although we don't prove it here.

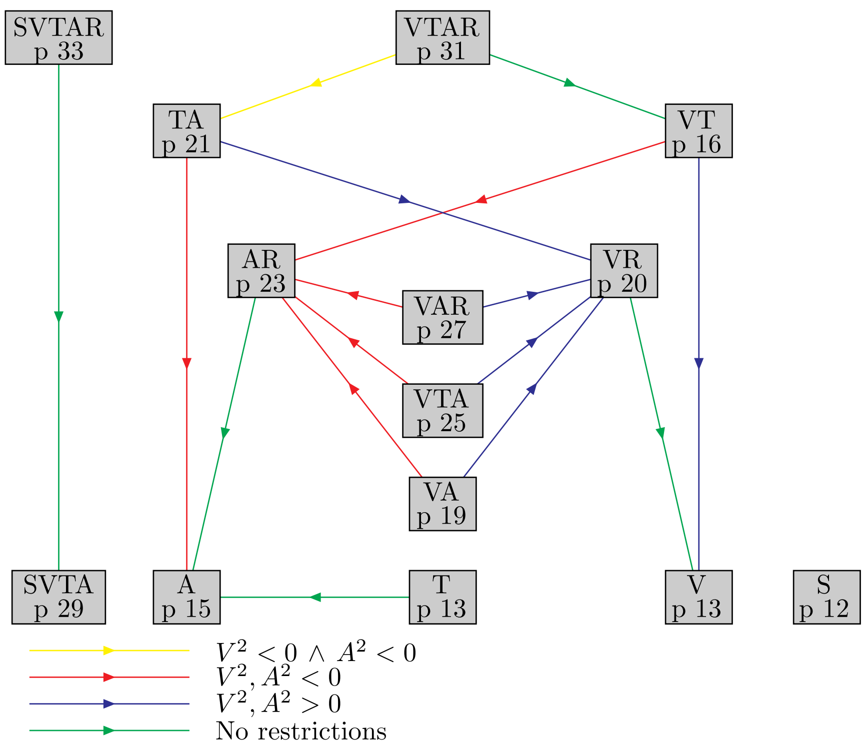
We see that in analogy with the Minkovski space when propagator 4.100 is multiplied with propagator 4.101 the result is propagator 4.102. However the physical interpretation of propagator 4.101 is somewhat more difficult. Also the standard propagators are commuting operators.

The main difference between propagators in Minkovski space and the 2+2 space is propagator 4.101. This propagator type also exists in Minkovski space, but there it can be transformed to propagator 3.87, the one with only a spin-term. In the 2+2 space this spin-term propagator (4.16) is equivalent to propagator 4.100. Another difference is the spin-term itself: in a 2+2 space the spin vector is timelike ($s^2 = 1$) while in Minkovski space, spin is spacelike ($s^2 = -1$).

Appendix A

Transformations in Minkovski space

Transformations of the propagator in Minkovski space



This picture shows all transformations of all types of projection operators in the Dirac algebra. All boxes represent a certain type of projection operator described in the cases of section 3.3. We can write in general for a projection operator:

$$\Pi = \left(S + \frac{1}{2}\right) \cdot I + \not{V} + T_{\beta\delta}\sigma^{\beta\delta} + \gamma^5 \not{A} + iR\gamma^5$$

The abbreviations in the boxes give the components in the operator. So, for instance V means the operator:

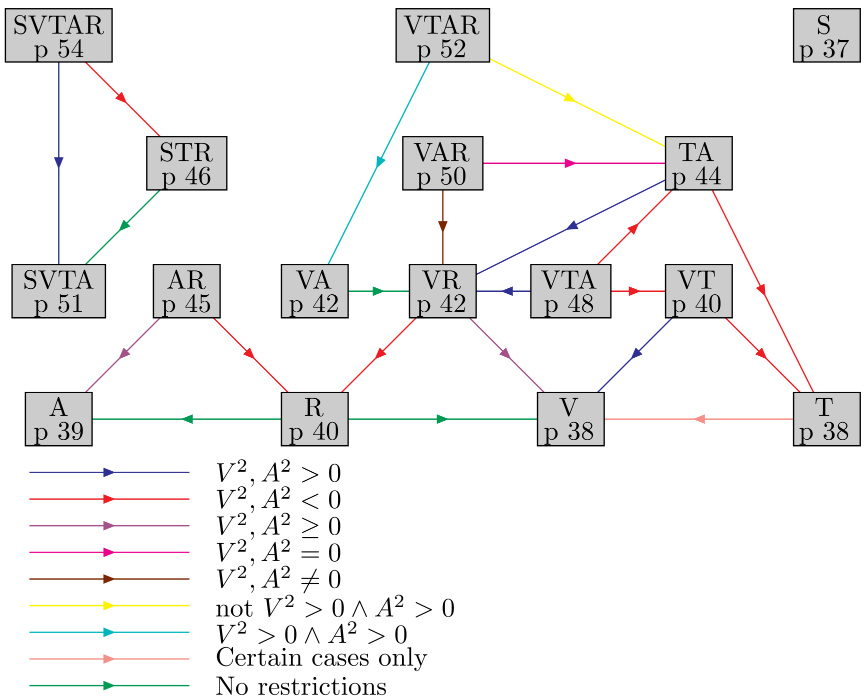
$$\Pi = \frac{1}{2} \cdot I + \not{V}$$

Some of the boxes represent more than one operator type, for instance because the operator contains a term like $\not{V}$ and $V^2 > 0$ or $V^2 < 0$. To see if this is the case one should look in section 3.3 at the corresponding case. The arrows represent transformations as described in the same section. Some transformations can only be performed when certain conditions in the projection operator are met. These extra conditions are given by the colour of the arrows and apply to the box where the arrows leave. Only when a box corresponds to more than one operator type, there can be extra conditions and one sees that more than one arrow (namely two) is leaving from the box. In the box one can also see on which page the solution and transformation of the operator type is worked out; this is given by p followed by the page number. For instance the case VA can be found on page 19.

Appendix B

Transformations in 2+2 space

Transformations of the propagator in 2+2 space



This picture shows all transformations of all types of projection operators in the Clifford algebra of the 2+2 dimensional space. This picture works in the same way as the picture of appendix A. The only difference is that there are more types of arrows, because the transformations in this algebra are more complicated.

Bibliography

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[2] P. Lounesto: Clifford algebras and spinors, 1997, Cambridge university press

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