Fermions in the Asymptotic Safety Program

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Abstract

For the asymptotic safety mechanism to be able to operate the existence of a suitable renormalization group fixed point is crucial. This fixed point governs the high energy behavior of theory and renders the theory well-defined and predictive. In order to be able to analyze phenomenologically interesting gravity-matter systems one first needs to properly incorporate fermions. Presently it is an open question under which conditions gravity coupled to Dirac fermions possesses a non-Gaussian fixed point suitable for asymptotic safety. This thesis clarifies this picture. Critical to understand the fixed point structure is the inclusion of a gravitational Yukawa-like interaction term into the truncation ansatz. This interaction is singled out based on a "smart truncation building principle". The resulting renormalization group flow possesses two families of interacting renormalization group fixed points. The first family exhibits an upper bound on the number of fermions for which the fixed points could provide a phenomenologically viable high-energy completion via the asymptotic safety mechanism. The second family comes without such a bound. Strong regulator-dependence of the fixed point structure reported in earlier literature is also clarified. The computations serve as a benchmark for dealing with fermions on a curved background within the asymptotic safety program. Notably this is the first time all ingredients of the flow equation are computed on the same background spacetime. This thesis also emphasizes the importance of the inclusion of matter into asymptotic safety and offers some additional perspectives on this topic.

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1 Introduction

Our current understanding of nature is based on four fundamental forces. The electromagnetic, strong and weak force are described by the standard model of particle physics. The other force, gravity, currently only has a classical description, general relativity (GR). If general relativity is quantized along the same lines as the other fundamental forces the resulting quantum theory fails [1, 2]. The reason is that at higher order perturbations the theory encounters an increasing number of new types of divergences that all must be absorbed by suitable counter terms. This feature of the emerging 'infinite number of infinities' is rephrased as perturbative non-renormalizability. The precise reason that this property of a theory is worrying is that each counter term comes with an undetermined coefficient, that has to be fixed by experiment. This means that an infinite number of measurements would be needed before the theory would become predictive. While this does imply that the resulting quantum theory cannot be a fundamental theory of nature, it can still be viewed as an effective field theory. This effective field theory of general relativity then does make unambiguous predictions for certain leading quantum corrections. The energy scale up to which these corrections are viable is known as the Plank scale.

The loss of predictivity at high energy provides a strong motivation for the search for a fundamental quantum theory of gravity. This is even further supported by the appearances of singularities within the classical theory of gravity. Many proposals have been put forward. Motivated by the failure of the standard quantum field theory approach it has been commonly argued that additional features need to be included. These features range from additional symmetries to extra dimensions and can even include new principles such as holography. By contrast, other approaches retain the quantum field theory framework without including new features. For example asymptotic safety, the main focus of this thesis, does not bring such such additions into the theory and instead abandons the perturbative techniques employed in the standard quantum field theories. It is fair to say, that so far no conclusions can be made as to what the correct approach would entail. The limiting factor here is the lack of experimental guidance in realms where the quantum gravitational effects truly become apparent. What is very clear however is that any theory of quantum gravity should describe, or at least be consistent with the existence of standard model matter.

The standard model of particle physics has been strikingly successful in describing the known particle content of nature. In fact, for some physicists its success has become a source of frustration. For quantum gravity however, the possible desert of no new physics between the current energy scales of the LHC and the Planck scale could provide an unimpeded view om some Planck scale phenomena. Anyhow, the matter sector of any quantum gravity theory should minimalistically comprise the matter content of the standard model. It is also conceivable that consistency of the theory constrains the admissible matter sectors. Thus, making contact with the low energy world could provide a crucial test on the quantum gravity theory, possibly falsifying it. For many approaches performing this test is still out of reach due to technical complexity. For others, it may become possible in the not too distant future. Asymptotic safety [3, 4] is a good example of this. The reason is that this program lives on so-called theory space comprising all actions which can be constructed from a given field content and are compatible with postulated symmetry requirements. The asymptotic safety condition then restricts the available matter sectors by requiring the existence of a non-Gaussian renormalization group fixed point which could provide the high-energy completion of the theory. Through this fixed point the theory gains its predictive power.

A significant hurdle in analyzing phenomenologically interesting gravity-matter systems in this framework originates from the inclusion of fermionic matter fields. Presently, it is an open question whether gravity coupled to N_f Dirac fermions possesses a fixed point suitable for asymptotic safety. In particular, the fluctuation field computation carried out in [6] concluded that there is no upper bound $N_f^{\rm crit}$ [6] while [5, 7] reported that $N_f^{\rm crit} \simeq \mathcal{O}(10)$. Moreover, as soon as one moves to a curved background spacetime the answer to this question seems to depend on (supposedly unphysical) details in the regularization procedure [8, 9]. From a phenomenological viewpoint this situation is rather unsatisfactory, since the number of fermions contained in the standard model of particle physics, $N_f^{\rm SM} = 22.5$, exceeds typical values for $N_f^{\rm crit}$ and only the inclusion of minimally coupled gauge fields renders the model suitable for a high-energy completion via the asymptotic safety mechanism. This thesis provides an answer to this puzzle as well as insights into the strong regularization dependence.

The outline of the work is as follows. First the core ideas of the asymptotic safety program will be explained in section 2. Then fermions in flat space will be briefly touched upon and their generalization to curved spacetimes will be discussed in section 3. With these ingredients at hand the inclusion of fermions into the asymptotic safety program will be analyzed and resolved in section 4. Finally, in 5 some additional perspectives on the inclusion of matter will be discussed. The computations are covered in the appendices in the following way. Firstly the variations and their appearances in the flow equation are discussed in appendices A and B, respectively. The main computation is outlined in appendices C and D.

2 Asymptotic Safety

Over the past few decades asymptotic safety, reviewed in [10–16], has gone from an exotic possibility to a serious contender of quantum gravity theories. It takes a conservative approach, still adhering to the framework of quantum field theories. Asymptotic safety builds on the insight of Wilson that the renormalizability of a quantum field theory hinges on the existence of a suitable renormalization group fixed point. If the fixed point is non-interacting the theory is asymptotically free while for interacting fixed points it is termed asymptotically safe. The key ingredient in the gravitational asymptotic safety program is a non-Gaussian fixed point governing the flow of couplings at high energy. At such a point the theory exhibits an enhanced symmetry, so-called quantum scale invariance [17]. An RG trajectory whose high-energy behavior is controlled by a NGFP is free from unphysical ultraviolet (UV) divergencies and termed "asymptotically safe". By definition, these trajectories span the UV-critical hypersurface S^{UV} of the fixed point. Typically not all RG trajectories in the vicinity of a NGFP are within S^{UV} as some may be repelled along an unstable direction. Selecting one specific trajectory within S^{UV} then requires specifying $\dim(S^{\text{UV}})$ parameters. All other couplings are a prediction of the theory and can be expressed in terms of these "relevant parameters", see [18, 19] for explicit examples of such relations. For some recent developments see [20–26], and most importantly the form factor program in [27, 28]. For a detailed bibliography and current open problems we refer to [29]. We will now provide an introduction into asymptotic safety explaining the core points and principles going into the construction.

2.1 Ingredients

To give the above statements a precise meaning we will go over their ingredients one by one. We will start with reviewing the effective action at the core of quantum field theory. Then we will introduce the effective average action as well as the flow equation. Here the asymptotic safety mechanism will come in. We will mainly follow along the lines of [3] and [4]. The main idea of asymptotic safety will be discussed in In subsection 2.1.4.

2.1.1 The Effective Average Action

The Effective Average Action (EAA) plays a prominent role within asymptotic safety. In most of the works related to asymptotic safety it is this object that is under intense study. For example in computations regarding the existence of fixed points (or lack thereof), the (truncated) EAA is the essential starting point. In order to talk about the EAA, one first needs to understand the effective action.

In quantum field theories the effective action is an important tool to study, among other things, the perturbative renormalizability of said theories. It is particularly useful in the case of theories where hidden symmetries are present. It also provides key insides in understanding spontaneous symmetry breaking as well as the backreaction of the vacuum fluctuations on the classical background. Furthermore it provides a powerful method of calculation, significantly reducing the complexity of calculating correlation functions. We will thus first focus our attention on the effective action, following along the lines of [30, 31]

Consider the bare action of a scalar field in Euclidean signature, $S[\phi]$. It is the bare action that governs the dynamics of this scalar field.Typically this action would be given in terms of the Klein-Gordon Lagrangian plus possible interaction terms, though no particular form is assumed in the following. It should also be noted that it is due to simplicity that the discussion here is limited to a scalar field, and that extensions are conceptually straightforward. The expectation value of an observable $\mathcal{O}$ in terms of the path integral formulation is given by

$$\langle \mathcal{O} \rangle \, = \frac{1}{N} \int \mathcal{D}\phi \, \mathcal{O}(\phi) \exp\{-S[\phi]\}. \tag{1}$$

Here N serves as a normalization factor with $N = \int \mathcal{D}\phi \exp\{-S[\phi]\}$. A prototypical example is the case where $\mathcal{O}$ is given by the field value at a certain point in spacetime, $\mathcal{O} = \phi(x)$. A convenient tool for computing these expectation values is the generating functional $Z[J]$ defined as

$$Z[J] \coloneqq \frac{1}{N} \int \mathcal{D}\phi \exp\Big\{ - S[\phi] + \int d^d y \phi(y) J(y) \Big\}. \tag{2}$$

Here one introduces a source function $J(x)$ that serves the purpose of a useful book keeping device. The generating functional is defined as One can now compute vacuum expectation values as

$$\langle \phi(x) \rangle \, = \frac{\delta Z[J]}{\delta J(x)} \bigg|_{J=0}. \tag{3}$$

Indeed,

$$\begin{aligned} \frac{\delta Z[J]}{\delta J(x)} \bigg|_{J=0} &= \frac{1}{N} \int \mathcal{D}\phi \frac{\delta}{\delta J(x)} \Big[\int d^d y \phi(y) J(y) \Big] \bigg|_{J=0} \exp\Big\{ - S[\phi] \Big\} \\ &= \frac{1}{N} \int \mathcal{D}\phi \, \phi(x) \exp\Big\{ - S[\phi] \Big\}. \end{aligned} \tag{4}$$

This procedure can be extended quite trivially to higher-order correlation functions, meaning a string of field values all evaluated at specific points in space, by simply taking higher order functional derivatives. Computing expectation values in this way avoids doing many functional integrations, at the cost of doing functional differentiations, usually a worthy tradeoff. Additionally one can define another functional related to this one via

$$e^{W[J]} \coloneqq Z[J]. \tag{5}$$

In Lorentzian signature this functional would be the vacuum energy as a function of the external source. In analogy to statistical mechanics, the functional $W[J]$ would correspond to the Helmholtz free energy, and J would be the analogue of the external magnetic field. Additionally we have

$$\frac{\delta W[J]}{\delta J[x]} = \frac{\delta}{\delta J(x)} \mathrm{log(Z[J])} = \frac{\int \mathcal{D}\phi \, \phi(\mathrm{x}) \exp\big\{ - \mathrm{S}[\phi] + \int \mathrm{d^d y} \phi(\mathrm{y}) \mathrm{J(y)} \big\}}{\int \mathcal{D}\phi \exp\big\{ - \mathrm{S}[\phi] + \int \mathrm{d^d y} \phi(\mathrm{y}) \mathrm{J(y)} \big\}}, \tag{6}$$

which automatically takes care of the normalization. It is thus precisely this quantity that gives the expectation value of the field, in the presence of the source J. We denote this source dependent quantity as the classical field,

$$\phi_{cl}(x) \coloneqq \frac{\delta W[J]}{\delta J[x]}. \tag{7}$$

It is precisely a weighted average over all possible fluctuations.

It is at this point that we are finally ready to introduce the *effective action*. It is defined as the Legendre transform of $W[J]$:[1]

$$\Gamma[\phi_{cl}] \coloneqq -W[J] + \int d^d y J(y) \phi_{cl}(y). \tag{8}$$

[1] This is for the scenario where the functional relationship $\phi = \phi[J]$ can be solved for the source to yield a relation of the form $J = \mathcal{J}(\phi)$. If this is not the case then one should take the supremum over all functions $J(x)$. Conceptually this changes nothing.

Note first of all that this functional is not a functional of J. This can be seen straightforwardly, as taking the functional derivative of Γ with respect to J yields zero. Notably, it is a general feature of the Legendre transform. One can now compute

$$\begin{aligned}\frac{\delta\Gamma[\phi_{cl}]}{\delta\phi_{cl}(x)} &= -\frac{\delta W[J]}{\delta\phi_{cl}(x)} + \int d^dy \frac{\delta J(y)}{\delta\phi_{cl}(x)}\phi_{cl}(y) + J(x) \\ &= -\int d^dy \frac{\delta J(y)}{\delta\phi_{cl}(x)}\frac{\delta W[J]}{\delta J(y)} + \int d^dy \frac{\delta J(y)}{\delta\phi_{cl}(x)}\phi_{cl}(y) + J(x) \\ &= J(x),\end{aligned} \tag{9}$$

using the functional chain rule in the second step. Setting the source function to zero, this implies

$$\frac{\delta\Gamma[\phi_{cl}]}{\delta\phi_{cl}(x)} = 0. \tag{10}$$

Thus, minimizing the effective action yields the values $\langle\phi(x)\rangle$, which are the expectation values of the field, or the stable quantum states of the theory. Note that in classical theories, one usually obtains these by minimizing the potential that is present in the bare action. In quantum field theories however, the classical value can be altered by quantum corrections. It is of course desirable to have a function whose minimization yields the expectation values, in the full quantum theory. This justifies the above definition, as the effective action is constructed to do exactly that. It gives rise to quantum equations of motion whose solution yields the expectation values of the field. It should be stressed that if one knows the effective action one can compute all expectation values with it. This means that if the effective action is known, one knows all there is to know about the theory, at least in principle.

We now illustrate how the effective action is computed using perturbation theory. The starting point to compute the effective action is the classical field equation (to lowest order)

$$\frac{\delta S}{\delta\phi}\Big|_{\phi=\phi_{cl}} = J(x). \tag{11}$$

One then decomposes the field according to $\phi(x) = \phi_{cl}(x) + \eta(x)$, and expands the action $S[\phi_{cl} + \eta]$ appearing in (2) up to quadratic order in η. The linear term in $\eta(x)$ vanishes due to equation (11). The integral is thus a Gaussian integral, where the $O(\eta^3)$ terms give perturbative corrections. The Gaussian integral yields a functional determinant. Performing the Legendre transform (8), after a bit of a calculation, one obtains

$$\Gamma[\phi_{cl}] = S[\phi_{cl}] + \frac{1}{2}\log\det\Big[\frac{\delta^2\mathcal{L}}{\delta\eta\delta\eta}\Big|_{\eta=0}\Big] \tag{12}$$

It is clear that the effective action is closely related to the bare action. The functional determinant is given by the product over all the eigenvalues of the operator. This will contain divergencies that have to be dealt with via renormalization, by absorbing them into the (unobservable) parameters of the bare action. It is for this reason that the effective action is stated to be 'the action plus quantum corrections build in', precisely because of the form it takes as well as that it satisfies relation (10).

The derivation of the one-loop effective action manifestly uses perturbation theory. In a setting where one does not assume quantum corrections are small, this method does not work. Of course also the methods of renormalization, clearly needed in this formalism, are altered significantly in the non-perturbative setting. Here we will use the Wilsonian idea of renormalization. In contrast to the perturbative approach where a loop integrates out all quantum fluctuations in one stroke, the Wilsonian approach integrates the momentum modes out shell-by-shell.

We will now introduce the effective *average* action. Similarly to how the effective action Γ started out from the functional $W[J]$, the effective average action will start out from the functional $W_k[J]$. This functional is constructed from the path integral in (2), where the bare action is supplemented by an additional cutoff action $\Delta S_k[\phi]$

$$\exp\{W_k[J]\} := \int \mathcal{D}\phi \exp\Big\{ - S[\phi] - \Delta S_k[\phi] + \int d^4x\phi(x)\, J(x)\Big\} \tag{13}$$

The purpose of the cutoff action is to suppress the IR modes of $\phi(x)$. That is, the low momentum, or equivalently large wavelength Fourier modes should be suppressed by ΔS_k. 'Small' and 'large' should be interpreted relative to the scale k, which serves as a variable mass scale. Thus the Fourier modes with momentum $p^2 < k^2$ should contribute only with a (highly) suppressed weight. The modes $p^2 > k^2$ should contribute without any suppression. In practice this is implemented by choosing the cutoff action (in momentum space representation) as

$$\Delta S_k[\phi] = \frac{1}{2}\int \frac{d^dp}{(2\pi)^d}\mathcal{R}_k(p^2)\phi_p^2. \tag{14}$$

Here $\mathcal{R}_k(p^2)$ is called the cutoff function, or regulator [2]. So, it is clear that for $p^2 > k^2$, the cutoff function should quickly go to zero. Altogether it should satisfy the following criteria

1. It must be continuous and monotonically decreasing in both p^2 and k^2;
2. For $p^2 > k^2$ it should quickly go to zero;
3. For $p^2 \ll k^2$ the cutoff function must be approximately k^2;
4. For $k \to 0$, the cutoff function must go to zero.

In perturbation theory, the modified action then leads to the modified propagator $\left(p^2+m^2\right)^{-1} \mapsto \left(p^2+m^2+\mathcal{R}_k(p^2)\right)^{-1}$. By requirement (3), it is clear than that indeed k^2 acts indeed as a mass type IR cutoff. Also in the case where $k^2 \ll m^2$, the presence of the cutoff plays no role since the mass itself already acts as an IR cutoff.

Note that the cutoff function is thus left arbitrary for a large extent. Examples of cutoff functions satisfying all criteria stated above are the exponential cutoff given by

$$\mathcal{R}_k(p^2) = \frac{p^2}{\exp(\frac{p^2}{k^2})-1}, \tag{15}$$

or the Litim cutoff

$$\mathcal{R}_k(p^2) = k^2\big(1-\frac{p^2}{k^2}\big)\Theta\big(1-\frac{p^2}{k^2}\big), \tag{16}$$

where Θ is the Heaviside step function. In this thesis we will use the Litim regulator only.

Returning to the construction of the EAA, we then proceed along the same lines leading to the effective action. At fixed k we compute the Legendre transform of the functional $W_k[J]$, according to

$$\tilde{\Gamma}_k[\phi] \coloneqq -W_k[J] + \int d^dx\,\phi(x)J(x). \tag{17}$$

Here we used the notation $\phi(x) = \langle\phi(x)\rangle$, denotes the (now k-dependent) expectation value of the field. The EAA is then defined as

$$\Gamma_k[\phi] \coloneqq \tilde{\Gamma}_k[\phi] - \frac{1}{2}\int d^dx\,\phi(x)\mathcal{R}_k(\Gamma)\phi(x). \tag{18}$$

The new notation of the action cutoff is to avoid having to resort to Fourier transforms of the field. The reason this term needs to be subtracted will become clear in a moment. Analogous to (10), it is the EAA that satisfies the effective field equation

$$\frac{\delta\Gamma_k[\phi]}{\delta\phi} + \mathcal{R}_k()\phi(x) = J(x). \tag{19}$$

The EAA satisfies the following properties

- $\lim_{k\to 0}\Gamma_k = \Gamma$;

[2] In position space the regulator is an operator of the form $\mathcal{R}_k(\Box)$, where $\Box$ is the cutoff operator that typically has eigenvalues closely related to the momenta of the field. In a flat spacetime a canonical choice is $\Box = -\partial^2$.

- $\lim_{k\to\infty}\Gamma_k = S$.

The first property can be seen straightforwardly by equation (13) and property (2) of the cutoff function. The cutoff action vanishes and one recovers the unmodified path integral generating the effective action. It is thus in the $k \to 0$ limit where the physics is residing. The second limit can be understood from the observation that for $k \to \infty$ the regulator suppresses all fluctuations in the path-integral. As a result nothing is integrated out and $\Gamma_{k\to\infty}$ agrees with the bare action, also see [61] for a detailed discussion.

2.1.2 The Flow Equation

At the core of the majority of the computations within asymptotic safety, and many more branches of physics, is the Wetterich equation, first derived in [32]. It also known as the functional renormalization group equation (FRGE), and reads

$$k\partial_k\Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)}[\phi] + \mathcal{R}_k\right)^{-1} k\partial_k\mathcal{R}_k\right]. \tag{20}$$

This equation encodes how the EAA changes if the cutoff scale is varied, meaning more or fewer momentum modes are integrated out. Alternatively formulated, Wetterich equation provides a powerful tool to investigate what happens to the EAA when integrating out quantum fluctuations of momenta $p^2 \approx k^2$.

This equation contains many aspects, subtleties, and furthermore, it introduces some new objects. We will go over them one by one, making some remarks along the way.

- Often one uses the shorthand notation $\partial_t = k\partial_k$, to denote differentiation with respect to what is called RG 'time'. Here $t \equiv \ln(\frac{k}{k_0})$, where k_0 is a value of the cutoff at an arbitrary reference scale. It has no physical meaning. Thus the RHS of (20) encodes the evolution of the EAA as the RG time passes.

- The functional trace present in the flow equation also appears in quantum field theories and non-relativistic quantum mechanics. It is defined with the help of a complete orthonormal set of states $\{e_n\}$ of the underlying Hilbert space. This set can be both discrete and continuous. The trace of an operator A is then defined as

$$\mathrm{Tr}[A] = \sum_n \langle e_n|\, A\, |e_n\rangle, \tag{21}$$

  where $\langle\, ,\,\rangle$ denotes the inner product on that Hilbert space and the sum (or integral) over n is implied. Note that this trace yields a number that is independent on the choice of basis inserted. Indeed, if $\{f_m\}$ is another such set, by making use of the completeness relation twice we can show

$$\begin{aligned}\sum_n \langle e_n\,|A|\,e_n\rangle &= \sum_{n,m} \langle e_n\,|A|\,f_m\rangle \langle f_m\,|\,e_n\rangle \\ &= \sum_{n,m} \langle f_m\,|\,e_n\rangle \langle e_n\,|A|\,f_m\rangle \\ &= \sum_m \langle f_m\,|A|\,f_m\rangle.\end{aligned} \tag{22}$$

- The argument of the functional trace contains several objects appear. Here, for compactness, we use deWitt condensed notation where matrix multiplication sums over both continuous and discrete index sets. One often uses the position space representation. Here, the Hessian $\Gamma_k^{(2)}$ contains two functional derivatives with respect to the field and has as matrix element $\Gamma_k^{(2)}(x,y) = \frac{\delta^2\Gamma_k}{\delta\phi(x)\delta\phi(y)}$. In computations often the first step consists of computing the Hessian - a task that is far from trivial. The cutoff operator has the matrix elements $\mathcal{R}_k(-\partial_x^2)\delta(x-y)$. In this case the functional trace corresponds to an integral $\int d^dx$, as the complete set of states (in this case Dirac delta functions) from (21) are already taken into account in the matrix elements.

- To make the EAA well-defined one needs to formally introduce a UV cutoff to discretize the path integral. So far this UV cutoff was implicit, but at the level of the FRGE this implicit UV cutoff is no longer needed meaning it can be removed trivially. The reason for this is the trace is already well convergent in the UV, simply because $\partial_t \mathcal{R}_k(z)$ quickly approaches zero for $z > k^2$.

We are now in a position to give a derivation of the FRGE. To do so it is important to distinguish the field $\phi(x)$, and its expectation value $\bar{\phi} \equiv \langle \phi(x) \rangle$. We start by taking the k-derivative of equation (17). Although the expectation value of the field is now clearly k-dependent, $\tilde{\Gamma}_k[\bar{\phi}]$ views its arguments k and $\bar{\phi}$ as independent variables (we realize this by working with partial derivatives). This way, the only k-dependence is inside the function $W_k[J]$. Thus we have

$$\partial_t \tilde{\Gamma}_k[\bar{\phi}] = -\partial_t W_k[J]. \tag{23}$$

The source term J should be understood as a functional of $\tilde{\phi}(x)$. Working this out using (13) and the form of the cutoff action in (18) gives

$$\begin{aligned}
\partial_t \tilde{\Gamma}_k[\bar{\phi}] &= -e^{-W_k[J]} \int \mathcal{D}\phi \, \partial_t \, \exp\Big\{ - S[\phi] - \Delta S_k[\phi] + \int d^d z \phi(z) J(z) \Big\} \\
&= -e^{-W_k[J]} \int \mathcal{D}\phi (-\frac{1}{2}) \int d^d x \phi(x) \big[\partial_t \mathcal{R}_k(-\partial_x^2)\phi(x)\big] \exp\Big\{ ... \Big\} \\
&= \frac{1}{2} e^{-W_k[J]} \int \mathcal{D}\phi \int d^d x \int d^d y \phi(x)\phi(y) \partial_t \mathcal{R}_k(x,y) \exp\Big\{ ... \Big\} \\
&= \frac{1}{2} \int d^d x \int d^d y \langle \phi(x)\phi(y) \rangle \partial_t \mathcal{R}_k(x,y).
\end{aligned} \tag{24}$$

Here we inserted $\phi(x) = \int d^d y \, \phi(y) \, \delta(x-y)$ to obtain the matrix element (x, y) of the regulator and $\exp\{...\}$ refers to the exponent of the functionals as in the first line of (24). The brackets $\langle \cdot \rangle$ denote the normalized expectation value similar to the RHS of equation (6), that is both dependent on J as well as k. We want to cast (24) into a closed equation for the EAA. For this purpose we compute

$$\begin{aligned}
\frac{\delta^2 W_k[J]}{\delta J(x) \delta J(y)} &= \frac{\delta}{\delta J(x)} \Big[e^{-W_k[J]} \frac{\delta Z_k[J]}{\delta J(y)} \Big] \\
&= -e^{-W_k[J]} \frac{\delta W_k[J]}{\delta J(x)} \frac{\delta Z_k[J]}{\delta J(y)} + e^{-W_k[J]} \frac{\delta^2 Z_k[J]}{\delta J(x) \delta J(y)} \\
&= -\tilde{\phi}(x) \, \tilde{\phi}(y) + \langle \phi(x) \, \phi(y) \rangle.
\end{aligned} \tag{25}$$

In other words

$$\langle \phi(x) \, \phi(y) \rangle = \frac{\delta^2 W_k[J]}{\delta J(x) \delta J(y)} + \tilde{\phi}(x) \, \tilde{\phi}(y) \tag{26}$$

Next we note a general feature of Legendre transforms, namely that functionals related by a Legendre transform (17) satisfy the relation

$$\int d^d z \frac{\delta^2 W_k[J]}{\delta J(x) \delta J(z)} \frac{\delta^2 \tilde{\Gamma}_k[\bar{\phi}]}{\delta \bar{\phi}(z) \delta \bar{\phi}(y)} = \delta(x-y). \tag{27}$$

This means, by definition, that the two objects are inverses of each other: $W_k^{(2)} = \big[\tilde{\Gamma}_k^{(2)}\big]^{-1}$. Substituting (26) back into (24) yields

$$\begin{aligned}
\partial_t \tilde{\Gamma}[\tilde{\phi}] &= \frac{1}{2} \int d^d x \int d^d y \frac{\delta^2 W_k[J]}{\delta J(x) \delta J(y)} \partial_t \mathcal{R}(x,y) + \frac{1}{2} \int d^d x \, \bar{\phi}(x) \bar{\phi}(y) \partial_t \mathcal{R}_k(x,y) \\
&= \frac{1}{2} \mathrm{Tr} \Big[W_k^{(2)} \partial_t \mathcal{R} \Big] + \frac{1}{2} \int d^d x \, \bar{\phi}(x) \partial_t \mathcal{R}(-\partial_x^2) \bar{\phi}(x),
\end{aligned} \tag{28}$$

where we wrote down the explicit form of the trace in position space representation first, and then converted to general notation. By relation (18) the second term is then canceled when computing $\partial_t \Gamma_k$.

This is the main reason as to why this additional term was included in the definition of the EAA. Thus we have

$$\partial_t \Gamma_k = \frac{1}{2}\mathrm{Tr}\Big[\big(\tilde{\Gamma}_k^{(2)}\big)^{-1}\partial_t \mathcal{R}_k\Big]. \tag{29}$$

Finally, we have $\tilde{\Gamma}_k^{(2)} = \Gamma_k^{(2)} + \mathcal{R}_k$, which follows from functionally differentiating (18). Substituting $\big(\tilde{\Gamma}_k^{(2)}\big)^{-1} = \big(\Gamma_k^{(2)} + \mathcal{R}_k\big)^{-1}$, then yields the flow equation (20).

It is important to mention that we have not made any specifications of the (effective) action whatsoever. Given the effective average action at some scale k the FRGE determines the EAA at another scale $\tilde{k}$. In particular, it gives the form of the EAA in the limit $k \to 0$, this is the effective action that we would actually like to know.

2.1.3 Theory Space and Beta Functions

The FRGE can be applied to a vast range of physics problems, ranging from phase transitions in condensed matter physics to confinement of quarks in quantum field theory. In this work we will focus on its applications in the context of quantum gravity. The EAA is a functional, where its functional dependence is on quantum fields (such as the scalar field discussed previously). Natural questions for constructing such a quantum gravity theory are 'What type of quantum fields are allowed?' and 'What are admissible terms for this action?'.

For this purpose theory space, referred to as $\mathcal{T}$ is introduced. This is done in such a way that all average effective actions, effective actions as well bare actions are treated on the same footing. Theory space contains all possible action terms that satisfy a priori defined symmetry requirements. For example, it could be that the only admissible action terms are invariant under (the proper representations of) Lorentz transformations acting on the field. The elements of theory space are called action monomials, abbreviated by $A[\cdot]$. Here the field content is specified in the square brackets. For the case of a scalar field, the space $\mathcal{T}$ could consist of all functionals that can occur in a derivative expansion, meaning arbitrary field monomials, integrated over spacetime, containing any number of fields and derivatives. Concrete examples of admissible action monomials are

$$\int d^dx\phi(-\partial^2)\phi, \quad \int d^dx\phi^{12}, \quad \int d^dx\phi^2\partial_\mu\phi\partial^\mu\phi\partial_\nu\phi\partial^\nu\phi. \tag{30}$$

This is the general philosophy of theory space. One considers all possible one could come up with, that satisfy a priori defined symmetry conditions. Each possible action term is then 'weighted' by a real number, called a coupling. This shows that theory space has the structure of a vector space. The addition of two admissible action monomials is again an admissible action monomial. This leads us to the notion of 'basis functionals'. From the examples, where one usually expands in terms of the field powers and respective derivatives acting on it, it should be clear that a generic $A \in \mathcal{T}$ can be given in terms of the basis functionals

$$A[\cdot] = \sum_\alpha \bar{u}^\alpha I_\alpha[\cdot]. \tag{31}$$

The $I_\alpha[\cdot]$ could take the form of the examples given in (30), and the action monomial $A[\cdot]$ would be given by their sum, where each term is weighted by the coefficients, or dimensionful couplings $\bar{u}^\alpha$. The couplings can carry dimension, and the bar precisely indicates that these are the dimensionful couplings. The set of coefficients $\{\bar{u}^\alpha\}$ can be thought of as the components of $A[\cdot]$ with respect to the basis $\{I_\alpha[\cdot]\}$. Although this is not at a complete level of mathematical rigor, it should be clear that in any case there are infinitely many basis functionals.

With this terminology introduced we can now look at the FRGE in component form. Given that every action monomial can be expanded in terms of basis functionals, we have that for each value of the cutoff scale k the EAA can be written as

$$\Gamma_k[\cdot] = \sum_\alpha \bar{u}^\alpha(k) I_\alpha[\cdot]. \tag{32}$$

Since the EAA presumably takes on a different from at different values of k, the couplings are now k-dependent. It is precisely this k-dependence that we would like to determine. This can be done by means of the FRGE (20), computing the LHS yields

$$\partial_t \Gamma_k[\cdot] = \sum_\alpha \partial_t \bar{u}^\alpha(k) I_\alpha[\cdot]. \tag{33}$$

In order to solve for $\partial_t \bar{u}^\alpha(k)$, we need to project the right hand side of the flow equation. Given that this is again an action monomial it too has a basis expansion. Expanding in terms of the same basis gives new expansion coefficients

$$\frac{1}{2}\mathrm{Tr}\Big[\Big(\sum_\alpha \bar{u}^\alpha(k) I_\alpha^{(2)}[\cdot] + \mathcal{R}_k\Big)^{-1} \partial_t \mathcal{R}_k\Big] = \sum_\alpha \bar{b}^\alpha(\bar{u}(k);k) I_\alpha[\cdot]. \tag{34}$$

Thus, by comparing to (33) we can equate the coefficients in front of each basis functional, arriving at

$$\partial_t \bar{u}^\alpha(k) = \bar{b}^\alpha(\bar{u}(k);k) \tag{35}$$

For convenience, we now switch to the dimensionless couplings

$$u^\alpha := k^{-d_\alpha}\bar{u}^\alpha, \quad b_\alpha := k^{-d_\alpha}\bar{b}_\alpha, \tag{36}$$

where d_α is the canonical mass dimension of the coupling $\bar{u}^\alpha$, in order to make the combination $\bar{u}^\alpha I_\alpha[\phi]$ dimensionless. The scale-dependence of the running couplings is encoded in the beta functions

$$\beta^\alpha(u) = -d_\alpha u^\alpha + b^\alpha(u), \tag{37}$$

satisfying

$$\partial_t u^\alpha = \beta^\alpha(u(t)). \tag{38}$$

The complete set of beta functions contains all the information on the EAA. By far the most difficult challenge to obtain the beta functions is to evaluate the RHS of the flow equation, that is computing the expansion coefficients in equation (34).

2.1.4 Fixed Points and Predictive Power

In section 2.1.2, we derived the FRGE satisfied by the effective average action. Given initial condition for Γ_k, at a certain value of k, one has all information needed to determine the EAA at any other scale k. In particular once can obtain the effective action by looking at the $k \to 0$ limit. Alternatively formulated one can look at trajectories in theory space. That is, given an initial condition for the couplings $u^\alpha(k)$ at a certain scale k, one can by means of the beta functions determine the values of the couplings $u^\alpha(\tilde{k})$ at another scale $\tilde{k}$. This then gives rise to a trajectory in theory space where each point on the trajectory, parametrized by k, is given by the coordinates $u^\alpha(k)$. It is then the FRGE that gives you all the possible renormalization group trajectories possible. Note that for a generic point in theory space there is only one RG trajectory that goes through it, given that the partial differential equations are first order. Thus what is so far missing, is a piece of extra input supplying physically acceptable initial conditions. Based on these conditions, we could then compute the effective action, which would mean we have completely solved the theory. This is where fixed points of the RG flow play a crucial role.

The definition of a fixed point entails that at this point all beta functions vanish simultaneously, that is

$$\beta^\alpha(u^\star) = 0 \quad \forall\alpha. \tag{39}$$

What this means is that all couplings will stop running, if the RG scale is increased. Thus the EAA will not change form any longer. Recall that increasing the RG scale k means further suppressing modes with momenta $p^2 < k^2$. Now if gradually increasing k, and thus suppressing additional modes does not change anything, that means then that these newly suppressed modes never contributed anything in the first place. It is therefore that we are interested in fixed points: a theory where above some cutoff value the high-momentum modes do not contribute (significantly) to the path integral any longer, has an RG trajectory that necessarily runs into a fixed point. This line of reasoning is certainly correct when only considering dimensionless couplings. For dimensionful ones however it are the dimensionless counterparts that stop running. Notably, observables are constructed from the dimensionless couplings. If all of them are finite, any observable should be finite as well. Note that the RG trajectory cannot hit the fixed point at any finite value of k. This is because, if it were to hit the fixed point at some finite value, then under the inverse flow it could never escape the fixed point, precisely because all beta functions vanish. Thus our attention lies at RG trajectories that approach a UV fixed point as $k \to \infty$. The fixed point then ensures that the theory is free from UV divergencies.

The existence of a suitable fixed point is then crucial for realizing the asymptotic safety mechanism. .[3] We are however not done. We still have to identify uniquely on what RG trajectory we are. So far, we have eliminated all trajectories that do not run into a fixed point.

We know, by assumption, that for $k \to \infty$ the RG trajectory realized in nature runs into the fixed point. Let us take a further look at where our trajectory is infinitesimally close to the fixed point, that is where we can linearize the flow.

$$\partial_t u^\alpha(k) = \sum_\gamma B^\alpha_\gamma \big(u^\gamma(k) - u^\gamma_\star\big) + \mathcal{O}\big((u^\gamma(k) - u^\gamma_\star)^2\big). \tag{40}$$

Here the $u^\beta_\star$ are the fixed point values of the couplings, given as the solutions of equations (39), and the zeroth order term vanished precisely because of (39). The stability matrix B^α_β is given by

$$B^\alpha_\gamma = \frac{\partial}{\partial u^\gamma}\beta^\alpha\Big|_{u=u_\star} \tag{41}$$

It should be noted that the stability matrix defined in (41) has no reason to be symmetric. This means that this matrix does not necessarily possess a complete system of eigenvectors. Furthermore its eigenvalues do not have to be real. In all investigations done so far however (within truncations that is), the matrix B did admit a complete system of right eigenvectors. We will therefore assume in the following that this is the case in the complete theory. The eigenvalues $(-\theta_J)$ and eigenvectors V_J of this stability matrix, satisfy the relation

$$\sum_\gamma B^\alpha_\gamma V^\gamma_J = -\theta_J V^\alpha_J. \tag{42}$$

If the eigenvectors $\{V_J\}$ form a complete system, the linearized flow (40) has the solution

$$u^\alpha(k) = u^\alpha_\star + \sum_J C_J V^\alpha_J \left(\frac{k_0}{k}\right)^{\theta_J}. \tag{43}$$

The θ_J can in general turn out to be complex. In this case the C_J, which are constants of integration determined by the initial condition, should arrange themselves such that all imaginary parts within the sum of (43) cancel.

Suppose we are on the trajectory given by (43). we can then see what happens to the component of $u(k) - u_\star$ in the direction of a vector V_J, if we increase k. If $\mathrm{Re}(\theta_\mathrm{J}) > 0$, the component will decrease, meaning that along this direction the solution indeed does approach the fixed point. In this case the direction V_J is called UV-attractive, because it runs into the fixed point by itself. Analogously, a direction is called UV-repulsive if $\mathrm{Re}(\theta_\mathrm{J}) < 0$, and in this case the component runs away from the fixed point. As argued above, we are only interested in RG trajectories eventually running into the fixed point. The only way to achieve this for the UV-repulsive directions, is to set the corresponding $C_J = 0$. The same holds true for the case where $\mathrm{Re}(\theta_\mathrm{J}) = 0$, because in this case the component along the direction V_J is purely oscillatory. For completeness we note that eigendirections with $\mathrm{Re}(\theta_\mathrm{J}) > 0$, $\mathrm{Re}(\theta_\mathrm{J}) < 0)$, $\mathrm{Re}(\theta_\mathrm{J}) = 0$ are called relevant, irrelevant and marginal respectively. The θ_J's are called critical exponents.

[3] We will not go into detail on what suitable refers to here but basically it needs to be non-Gaussian. Owed to the negative mass-dimension of Newton's coupling, the free fixed point is UV repulsive for theories with non-vanishing G and thus can not provide the high-energy completion of the RG flow.

Thus, the number of undetermined initial conditions precisely equals the number of θ_J that have a positive real part. This means that for the theory to become predictive, one first has to do

$$\dim \mathcal{S}_{\mathrm{UV}} = \#\{\theta_{\mathrm{J}} \,|\, \mathrm{Re}(\theta_{\mathrm{J}}) > 0\} \tag{44}$$

measurements. Computations based on the FRGE suggest that this number is both finite and small. This means that only a few parameters need to be fixed by experiment, the others are a prediction of the theory.

2.1.5 Truncations

Establishing the validity of the asymptotic safety scenario requires checking whether a suitable fixed point exists and if this fixed point possesses finitely many UV-attractive directions. The flow equation however, is actually a beast to solve. Therefore it might even be impossible to solve it including all action monomials. In component form, if we had access to the beta functions it would still be a highly difficult task to decompose the fixed point structure. However the actual task of finding the beta functions, or rather computing the expansion coefficients in (34) is by far the most difficult challenge. To make progress, it is thus necessary to resort to approximation methods. The best tool currently available is the 'truncation of theory space'. Here one truncates the effective average action to a finite number of basis functionals. One then evaluates the exact FRGE on this subspace to obtain finitely many beta functions. Given the usual complexity of these beta functions, one then resorts to numerical methods to find the underlying fixed points.

Concretely one makes the ansatz for the EAA as

$$\Gamma_k[\phi] = \sum_{i=1}^{N} \bar{u}^i(k) I_i[\phi], \tag{45}$$

where one 'picks out' some of the basis functionals $I_i[\phi]$. Note that plugging this into the flow equation would yield an inconsistent result. The reason is that the RHS

$$\frac{1}{2}\mathrm{Tr}\Big[\Big(\sum_{i=1}^{N} \bar{u}^i(k) I_i^{(2)}[\phi] + \mathcal{R}_k\Big)^{-1} \partial_t \mathcal{R}_k\Big] \tag{46}$$

typically contains terms outside of the span of $I_1, ..., I_N$. The truncation provides a good approximation to the full solution if the components of the flow perpendicular to the span $\mathrm{I}_1, I_n arenegligible.Thusinthetruncation$
$\frac{1}{2}\mathrm{Tr}\Big[\Big(\sum_{i=1}^{N} \bar{u}^i(k) I_i^{(2)}[\phi] + \mathcal{R}_k\Big)^{-1} \partial_t \mathcal{R}_k\Big]\Big|_{I_i[\phi]}$, (47)for $i = 1, ..., N$.

This approximation scheme might seem rather crude. Furthermore it is complicated to estimate the order of the error made. Although there are some tests to judge the reliability of a given truncation, they typically provide qualitative insights only. Still, given our knowledge on low-energy physics so far one could expect that

$$\int d^4x \sqrt{g} R, \tag{48}$$

which is the curvature term in the Einstein-Hilbert action would be much more relevant than, say, the action monomial

$$\int d^4x \sqrt{g} R^{100}. \tag{49}$$

One could then start with the Einstein-Hilbert action, and gradually and systematically include more terms in the truncation, further exploring the uncharted theory space. It is then conceivable that after including 'enough' terms, it could be theoretically checked that including even more terms does not notably alter any previously obtained results. If this is the case, and it is checked systematically for sufficiently many terms, it would indicate that the truncation is becoming more and more reliable. It is technical complexity that slows down this progress. Over the years however, a significant number of truncations have been worked out in detail. All of them gave further support that the fixed point found in the Einstein-Hilbert truncation persists if one includes higher curvature terms.

It should be emphasized that the Einstein-Hilbert action plays no special role within asymptotic safety. The low energy physics should come out of the fixed point realized in nature together with the measurements needed to render the theory predictive. In this sense all action monomials are treated completely on equal footing. However, it is of course clear that the Einstein-Hilbert action certainly should be included in the truncation scheme, since we already know that these terms are crucial for a valid description of gravitational physics observed at macroscopic scales. It was for this reason very natural to start with the truncation of the Einstein-Hilbert action. The fixed point found here, now known as the Reuter fixed point [33], constitutes the original and main piece of motivation for asymptotic safety. For this reason, and to illustrate how computations within asymptotic safety go about, we will go through some of the essentials of this computation in section 2.2. Before we do this though, we first make a few comments on the inclusion of matter degrees of freedom.

2.1.6 The Inclusion of Matter

Although our understanding of pure gravity is not yet complete, it seems reasonable to already start towards including matter degrees of freedom. It should go without saying that an asymptotically safe theory of gravity alone is not sufficient. A realistic theory has to incorporate matter degrees of freedom as well. Not only is it insufficient to show consistency of the asymptotic safety program disregarding matter, it is also unnecessary. Additionally it seems way more practical to think of physical observables in the presence of matter than for pure gravity. Not only is this because in pure gravity there do not exist any local observables due to diffeomorphism invariance, it is also because all experimental situations involve matter in one way or another. Of course the detectors themselves are made out of matter, but more importantly because in fact everything we currently know about spacetime comes from probing it with matter. Also, the scattering of gravitons alone, certainly seems further away from experiment than the gravitational scattering of standard model particles. In short, the inclusion of matter might be the only practical way to eventually make contact with observation, through practically testable predictions. An example of where this might occur is the field of cosmology, in particular inflationary theory. There are some tentative hints that quantum-gravity effects might strongly constrain the inflationary potential that is usually introduced in a rather ad-hoc manner [34]. It also seems plausible that general scalar potentials, including the Higgs potential, are driven towards flatness as a result of quantum gravity theories [35–37]. The hope also is that quantum-gravity effects that really kick in at the Planck scale, might leave an imprint on low energy physics, that could be tested in for instance collider experiments.

To confirm a theory, or rather provide evidence for it, new predictions made by the theory need to be confirmed. In order to rule out a theory, more ways become available. For instance a theory of quantum gravity might make a prediction of an observable quantity that has already been measured. Although this does not count as a new prediction, it certainly could rule out the theory if the prediction and the measured result do not match. On the other hand, if they do match up, it would be a good hint that one is indeed working towards a phenomenologically viable theory. For the asymptotic safety program it seems likely that such a scenario will occur. This is because quantum fluctuations present at the NGFP can turn a power-counting marginal coupling into an irrelevant one, thereby fixing the parameter. Examples where such a mechanism of enhanced predictive power might be operative include the Higgs mass [36,38], the fine-structure constant [39,40], and ratios among quark masses [41]. Note that these predictions still rely on the (mostly) untested viability of the truncation schemes involved, as well as a critical assumption that these results carry over from Euclidean to Lorentzian signature. Nevertheless the references above clearly provide a proof-of-principle for the occurrences of such mechanisms. Although much work needs to be done it is highly encouraging that asymptotically safe gravity might actually be ruled out observationally in the low-energy regime. Conversely, if it could be shown that asymptotic safety results in the standard model of particle physics supplemented by general relativity then this would be a major breakthrough in quantum gravity research.

The above reasoning provides significant motivation for including matter into the asymptotic safety program. Although this is still very much work in progress some significant advances have already been made. For a recent review of gravity-matter systems in asymptotic safety see [15]. Conceptually, the addition of matter fields into the asymptotic safety program is relatively straightforward. This is due to the field-theoretic nature of the approach. Practically, the first thing one could do is to include minimally coupled matter, and see if it spoils the fixed point found in the Einstein-Hilbert truncation.[4] A first interesting thing to check would then be to include standard model type matter. That is, one includes N_S, N_V, N_f minimally coupled scalar, vector and spin-$\frac{1}{2}$ fields, respectively. One could then see if a fixed point exists for $N_S = 4$, $N_V = 12$, $N_f = 22.5$, which are the values for the standard model of particle physics (one could also include right-handed neutrinos, in that case the value of N_f would be $N_f = 24$). Results for this are encouraging, though not completely without problems. Next, one would include the anomalous dimensions via the wave function renormalizations (these are the inessential couplings appearing in front of the kinetic terms of the matter fields), and see if this induces any significant changes. In this way one would gradually include the entire standard model, including all its couplings and the Higgs mechanism. A fixed point in this scenario, with the Einstein-Hilbert truncation on the gravitational side, would provide a key piece of motivation for the asymptotic safety scenario. This would also be the smallest truncation that could make contact with observational physics. Extensions of the standard model via additional fields are phenomenologically interesting as well. Before this can be (gradually) achieved, a lot of work still needs to be done. This thesis contributes to this large-scale program by focusing specifically on the inclusion of (non-minimally coupled) fermions in a curved background spacetime. This is the first time where such computations are done consistently, taking all relevant curvature terms into account

2.2 The Einstein-Hilbert Truncation

In this section we will work with the background-field method. This means one makes a linear split of the spacetime metric $g_{\mu\nu}$,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{50}$$

Here $\bar{g}_{\mu\nu}$ is the background field, and $h_{\mu\nu}$ is called the fluctuation field. It should be emphasized that equation (50) should not be read as an expansion of any sort, nor are the values of the fluctuation fields assumed to be small compared to the background field. What this refers to is simply splitting the field into a non-dynamical and dynamical part, where the non-dynamical part may be thought of as a classical expectation value. Furthermore, the background EAA for a vanishing expectation value of the dynamical field contains the entire physical content of the theory. This method is very intuitive, and it will become clear how one should deal with this particular computational technique. As a final remark it should be said that quantities carrying an overbar, e.g. $\bar{R}$, are constructed from the background metric only.

The Einstein Hilbert action in Euclidean space is given by

$$S_{EH} = \frac{1}{16\pi G}\int d^dx\sqrt{g}\big[-R+2\Lambda\big]. \tag{51}$$

It involves both the cosmological as well as Newton's constant. At the level of the EAA, we then promote both constants to k-dependent couplings G_k and $\bar{\lambda}_k$.[5] The action monomials spanning our truncation are

$$\int d^dx\sqrt{g}, \quad \int d^dx\sqrt{g}R. \tag{52}$$

Additionally, gauge fixing and ghost terms are needed for a proper truncation ansatz. These do not introduce any new couplings but do contribute to the Hessian $\Gamma_k^{(2)}$. Our starting point will be the full truncation ansatz

$$\Gamma_k[g,\bar{g}] = \frac{1}{16\pi G_k}\int d^dx\sqrt{g}\big[-R+2\bar{\lambda}_k\big] + \frac{1}{32\pi G_k}\int d^dx\sqrt{\bar{g}}\bar{g}^{\mu\nu}\big(\mathcal{F}_\mu{}^{\alpha\beta}g_{\alpha\beta}\big)\big(\mathcal{F}_\nu{}^{\rho\sigma}g_{\rho\sigma}\big) + S_{\text{gh}}[\bar{g},h,C,\bar{C}] \tag{53}$$

[4]For the case of minimally coupled action monomials, one already includes the contributions the action monomial would make to the gravitational sector, but does not allow for this term to come with a coupling. Although of course this should always be done in principle, the minimally coupled case already gives first insides about the matter contributions to the gravitational beta functions.

[5]These "running couplings" must not be confused with the renormalized couplings appearing in the effective action which are scale-independent [27].

Here the second term comes from the gauge fixing term with

$$\mathcal{F}_\mu^{\ \alpha\beta} g_{\alpha\beta} = \bar{D}^\nu h_{\mu\nu} - \frac{1}{2}\bar{D}_\mu \bar{g}^{\alpha\beta} h_{\alpha\beta}, \tag{54}$$

working in standard harmonic gauge. For this choice of gauge the ghost action specifically reads

$$S_{\text{gh}}[\bar{g}, h, C, \bar{C}] = -\sqrt{2}\int d^dx\sqrt{\bar{g}}\,\bar{C}_\mu\,\mathcal{M}[g,\bar{g}]^\mu_{\ \nu}\,C^\nu, \tag{55}$$

where the Faddeev-Popov operator $\mathcal{M}$ is given by means of both g and $\bar{g}$ as

$$\mathcal{M}[g,\bar{g}]^\mu_{\ \nu} = \bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\,\bar{D}_\lambda\left(g_{\rho\nu}D_\sigma + g_{\sigma\nu}D_\rho\right) - \bar{g}^{\rho\sigma}\bar{g}^{\mu\lambda}\,\bar{D}_\lambda \bar{g}_{\sigma\nu}D_\rho. \tag{56}$$

We will go to the computation (skipping some details) setting out several steps needed in order to go to the beta functions. A full derivation can be found in [3].

1. Compute the LHS of the FRGE
This is the easiest step. One simply has to evaluate $\partial_t\Gamma_k$ at zeroth order in the fluctuation field

$$\partial_t\Gamma_k\Big|_{h=0} = \frac{1}{16\pi G_k^2}\partial_t G_k\int d^dx\sqrt{\bar{g}}\bar{R} + \frac{1}{8\pi}\partial_t\Big(\frac{\bar{\lambda}_k}{G_k}\Big)\int d^dx\sqrt{\bar{g}}. \tag{57}$$

This indicates that the information of the flow of G_k and $\bar{\lambda}_k$ can be read off from the coefficients multiplying the basis elements. Thus, this is to the structure $\int d^dx\sqrt{g}$ and $\int d^dx\sqrt{g}R$. Introducing the dimensionless couplings

$$g_k := k^{-2}G_k \quad \lambda_k := k^2\lambda_k \tag{58}$$

one then needs to solve for $\partial_t g_k$ and $\partial_t\lambda_k$ to obtain the full beta functions. The rest of the steps show how to do the projection of the RHS of the truncated FRGE.

2. Expand the EAA
In (20) the Hessian appears within the functional trace. In general this object is nontrivial to compute, and is the starting point of any computation involving the FRGE. Note that for the background field method the variations should be taken with respect to the fluctuation field(s). In order to do this one should first expand the EAA to second order in fluctuation fields, keeping only terms quadratic in them. In this case that means one needs the following formulae

$$\begin{aligned}
\delta g_{\mu\nu} &= h_{\mu\nu}\\
\delta g^{\mu\nu} &= -h^{\mu\nu}\\
\delta\sqrt{g} &= \frac{1}{2}\sqrt{\bar{g}}\bar{g}^{\mu\nu}h_{\mu\nu}\\
\delta R &= -\bar{R}^{\mu\nu}h_{\mu\nu} + \bar{D}_\beta\bar{D}_\alpha h^{\alpha\beta} - \bar{D}^2 h^\alpha_{\ \alpha}
\end{aligned} \tag{59}$$

for the first order variations and

$$\begin{aligned}
\delta^2\sqrt{g} &= \frac{1}{2}\sqrt{\bar{g}}\big(\frac{1}{2}\bar{g}^{\mu\nu}\bar{g}^{\rho\sigma}h_{\mu\nu}h_{\rho\sigma} - h^{\mu\nu}h_{\mu\nu}\big)\\
\delta^2 R \ &= \bar{R}_{\beta\mu}h^{\beta\gamma}h^\mu_{\ \gamma} - \bar{R}_{\alpha\beta\gamma\rho}h^{\beta\gamma}h^{\alpha\rho} - 3h^{\beta\gamma}\bar{D}_\gamma\bar{D}_\alpha h^\alpha_{\ \beta} + 2h^{\beta\gamma}\bar{D}_\beta\bar{D}_\gamma h^\alpha_{\ \alpha}\\
&\quad + 2h_{\beta\gamma}\bar{D}^2h^{\beta\gamma} - h^{\beta\gamma}\bar{D}_\alpha\bar{D}_\beta h^\alpha_{\ \gamma} - \big(\bar{D}_\alpha h^{\beta\gamma}\big)\big(\bar{D}_\beta h^\alpha_{\ \gamma}\big)\\
&\quad + \frac{3}{2}\big(\bar{D}_\lambda h_{\beta\gamma}\big)\big(\bar{D}^\lambda h^{\beta\gamma} - 2\big(\bar{D}_\gamma h^{\beta\gamma}\big)\big(\bar{D}_\alpha h^\alpha_{\ \beta}\big) + 2\big(\bar{D}_\beta h^{\beta\gamma}\big)\big(\bar{D}_\gamma h^\alpha_{\ \alpha}\big)\\
&\quad - \frac{1}{2}\big(D_\lambda h^\gamma_{\ \gamma}\big)\big(\bar{D}^\lambda h^\alpha_{\ \alpha}\big).
\end{aligned} \tag{60}$$

Here indices are raised and lowered with the background metric $\bar{g}_{\mu\nu}$.

Using these formulae the second order expansions of (53) collected in Γ_k^{quad} reads

$$\Gamma_k^{quad}[h;\bar{g}] = \frac{1}{32\pi G_k}\int d^dx\sqrt{\bar{g}}h_{\mu\nu}\Big[K^{\mu\nu}{}_{\rho\sigma}\bar{D}^2 + U^{\mu\nu}{}_{\rho\sigma}\Big]h^{\rho\sigma}, \tag{61}$$

where

$$K^{\mu\nu}{}_{\rho\sigma} = \frac{1}{4}\big[\delta^\mu_\rho\delta^\nu_\sigma + \delta^\mu_\sigma\delta^\nu_\rho - \bar{g}^{\mu\nu}\bar{g}_{\rho\sigma}\big] \tag{62}$$

and

$$\begin{aligned} U^{\mu\nu}{}_{\rho\sigma} =& \frac{1}{4}\big[\delta^\mu_\rho\delta^\nu_\sigma + \delta^\mu_\sigma\delta^\nu_\rho - \bar{g}^{\mu\nu}\bar{g}_{\rho\sigma}\big](\bar{R} - 2\bar{\lambda}_k) + \frac{1}{2}\big[\bar{g}^{\mu\nu}\bar{R}_{\rho\sigma} + \bar{g}_{\rho\sigma}\bar{R}^{\mu\nu}\big] \\ &- \frac{1}{4}\big[\delta^\mu_\rho\bar{R}^\nu_\sigma + \delta^\mu_\sigma\bar{R}^\nu_\rho + \delta^\nu_\rho\bar{R}^\mu_\sigma + \delta^\nu_\sigma + \delta^\nu_\sigma\bar{R}^\mu_\rho\big] - \frac{1}{2}\big[\bar{R}^\nu{}_\rho{}^\mu{}_\sigma + \bar{R}^\nu{}_\sigma{}^\mu{}_\rho\big]. \end{aligned} \tag{63}$$

3. Diagonalize the quadratic form

To compute the functional trace in the flow equation the inverse of the Hessian (with added regulator) is needed. This step is facilitated by diagonalizing the quadratic form using a simple field decomposition. In the present case, it is sufficient to decompose the full fluctuation field into a traceless, and trace part:

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \frac{1}{d}\bar{g}_{\mu\nu}h, \quad \bar{g}^{\mu\nu}\hat{h}_{\mu\nu} = 0. \tag{64}$$

Here h is given by the trace of the fluctuation field $h = \bar{g}^{\mu\nu}h_{\mu\nu}$. Expressed in this way (61) becomes

$$\begin{aligned} \Gamma^{quad}_k[h;\bar{g}] = \frac{1}{32\pi G_k}\int d^dx\sqrt{\bar{g}}\Big\{ & \frac{1}{2}\hat{h}_{\mu\nu}\big[-\bar{D}^2 - 2\bar{\lambda}_k + \bar{R}\big]\hat{h}^{\mu\nu} \\ &- \frac{d-2}{4d}h\big[-\bar{D}^2 - 2\bar{\lambda}_k + \frac{d-4}{d}\bar{R}\big]h \\ &- \bar{R}_{\mu\nu}\hat{h}^{\nu\rho}\hat{h}^\mu{}_\rho + \bar{R}_{\alpha\beta\nu\mu}\hat{h}^{\beta\nu}\hat{h}^{\alpha\mu} + \frac{d-4}{d}h\bar{R}_{\mu\nu}\hat{h}^{\mu\nu}\Big\}. \end{aligned} \tag{65}$$

Note that in above form Γ^{quad}_k is not diagonalized completely. The non-diagonal terms contain at least one power of the background curvature and can be diagonalized by making a convenient choice for $\bar{g}$. This will be done in the next step.

4. Specify background

Since in the end no results should depend on the choice of background, it is of course clever to pick one that simplifies the computations. Having said this, with the choice of background it is possible to loose certain pieces of information. For instance, if we were to opt for a flat space background we could not project the RHS onto $\int d^dx\sqrt{\bar{g}}\bar{R}$, simply because in flat space $\bar{R} = 0$ everywhere. A canonical background for the present computation is the maximally symmetric background. On these spaces the Riemann and Ricci tensor satisfy the following identities

$$\begin{aligned} \bar{R}_{\mu\nu\rho\sigma} &= \frac{1}{d(d-1)}[\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}]\bar{R}, \\ \bar{R}_{\mu\nu} &= \frac{1}{d}\bar{g}_{\mu\nu}\bar{R}. \end{aligned} \tag{66}$$

A concrete example of such a space is the familiar family of d-spheres characterized by their radius. Furthermore on a d-sphere there is another important simplification, namely

$$\bar{D}_\mu\bar{R} = 0. \tag{67}$$

On this background the quadratic action (65) reads

$$\begin{aligned} \Gamma^{quad}_k[h;\bar{g}] = \frac{1}{2}\frac{1}{32\pi G_k}\int d^dx\sqrt{\bar{g}}\Big\{ & \hat{h}_{\mu\nu}\big[-\bar{D}^2 - 2\bar{\lambda}_k + C_T\bar{R}\big]\hat{h}^{\mu\nu} \\ &- \frac{d-2}{2d}h\big[-\bar{D}^2 - 2\bar{\lambda}_k + C_S\bar{R}\big]h\Big\}, \end{aligned} \tag{68}$$

where the dimension-dependent constants are given by

$$C_T = \frac{d(d-3)+4}{d(d-1)}, \quad C_S = \frac{d-4}{d}. \tag{69}$$

Note that for the case $d = 4$, we have $C_S = 0$, simplifying calculations even further. The quadratic action is now fully diagonalized, It should be noted that in practise a background is often specified earlier. In these cases the insights of how to simplify most efficiently are already gained through experience.

Given everything is now brought to diagonal form the Hessian explicitly written out in matrix form as,

$$\Gamma_k^{(2)}(x,y) = \begin{pmatrix} \frac{\delta^2\Gamma_k}{\delta\hat{h}_{\mu\nu}(x)\delta\hat{h}_{\rho\sigma}(y)} & \frac{\delta^2\Gamma_k}{\delta\hat{h}_{\mu\nu}(x)\delta h(y)} \\ \frac{\delta^2\Gamma_k}{\delta h(x)\delta\hat{h}_{\mu\nu}(y)} & \frac{\delta^2\Gamma_k}{\delta h(x)\delta h(y)}, \end{pmatrix} \tag{70}$$

can now be computed straightforwardly. Given that the unit on the space of traceless symmetric tensors is given by $(\mathbb{1} - P)$, with

$$\mathbb{1}^{\mu\nu}{}_{\rho\sigma} = \frac{1}{2}(\delta^\mu_\rho\delta^\nu_\sigma + \delta^\mu_\sigma\delta^\nu_\rho), \quad P^{\mu\nu}{}_{\rho\sigma} = \frac{1}{d}\bar{g}^{\mu\nu}\bar{g}_{\rho\sigma}, \tag{71}$$

we have

$$\frac{\delta\hat{h}^{\mu\nu}(x)}{\delta\hat{h}^{\rho\sigma}(y)} = (\mathbb{1} - P)^{\mu\nu}{}_{\rho\sigma}\delta(x-y), \quad \frac{\delta h(x)}{\delta h(y)} = \delta(x-y). \tag{72}$$

The non-zero components of the Hessian $\Gamma_k^{(2)}$ then read

$$\begin{aligned} \frac{\delta^2\Gamma_k}{\delta\hat{h}_{\mu\nu}(x)\delta\hat{h}_{\rho\sigma}(y)} &= \frac{1}{32\pi G_k}\big[-\bar{D}^2 - 2\bar{\lambda}_k + C_T\bar{R}\big](\mathbb{1} - P)^{\mu\nu}{}_{\rho\sigma}\,\delta(x-y), \\ \frac{\delta^2\Gamma_k}{\delta h(x)\delta h(y)} &= -\frac{1}{32\pi G_k}\frac{d-2}{2d}\big[-\bar{D}^2 - 2\bar{\lambda}_k + C_S\bar{R}\big]\,\delta(x-y). \end{aligned} \tag{73}$$

These are supplemented by the relevant entry in the ghost sector yielding

$$\frac{\delta^2\Gamma_k}{\delta\bar{C}_\mu\delta C_\nu} = \sqrt{2}\,\delta^{\mu\nu}\big[D^2 + \frac{1}{d}R\big] \tag{74}$$

5. Specify regulator

Given our cutoff insertion in (18), the precise form of $\mathcal{R}_k$ should be constructed in such a way that it satisfies the following rule: For the case when

$$\left(\Gamma_k^{(2)}\right)_{ij} = f_{ij}(-\bar{D}^2), \tag{75}$$

where f_{ij} can depend on the couplings as well as on k directly, the cutoff operator $\mathcal{R}_k(-\bar{D}^2)$ should be constructed in such a way that is satisfies

$$\left(\Gamma_k^{(2)} + \mathcal{R}(-\bar{D}^2)\right)_{ij} = f_{ij}\Big(-\bar{D}^2 + k^2R^{(0)}(-\frac{\bar{D}^2}{k^2})\Big). \tag{76}$$

Here $R^{(0)}$ is the cutoff shape function that should satisfy the criterion outlined below equation (14). Although it can already be specified now, usually one would like to do this as late as possible to do the calculations for more than one shape function. (This can also provide some tests to check the reliability of the truncation used.) The indices i, j range over the independent fields considered. In our case we then have

$$\begin{aligned} \mathcal{R}_k^{\hat{h}\hat{h}}(x,y)^{\mu\nu}{}_{\rho\sigma} &= \frac{1}{32\pi G_k}(\mathbb{1} - P)^{\mu\nu}{}_{\rho\sigma}k^2R^{(0)}(-\frac{\bar{D}^2}{k^2})\,\delta(x-y), \\ \mathcal{R}_k^{hh}(x,y) &= -\frac{d-2}{2d}\frac{1}{32\pi G_k}k^2R^{(0)}(-\frac{\bar{D}^2}{k^2})\,\delta(x-y) \\ \mathcal{R}_k^{\text{gh}}(x,y)^\mu{}_\nu &= -\sqrt{2}\,\delta^\mu{}_\nu k^2\mathcal{R}^{(0)}(-\frac{D^2}{k^2})\,\delta(x-y). \end{aligned} \tag{77}$$

The off-diagonal components of the regulator vanish, simply because they also vanish for the Hessian.

6. Identify relevant diagrams

In this step, one should determine the terms on the RHS of the FRGE that are proportional to the basis elements identified in step one. In this case this step is rather trivial however. For a different, more involved computation the reader is referred to appendix B.

In the case at hand, we have to compute the inverse of the matrix $\big(\Gamma_k^{(2)}+\mathcal{R}_k\big)$, which is again a diagonal matrix with the entries

$$\begin{aligned}\Big(\Gamma_k^{(2)}+\mathcal{R}_k\Big)^{-1}_{\hat h\hat h} &= 32\pi G_k\big[-\bar D^2+k^2R^{(0)}(-\frac{\bar D^2}{k^2})-2\bar\lambda_k+C_T\bar R\big]^{-1}(\mathbb{1}-P)^{\mu\nu}{}_{\rho\sigma}\\ \Big(\Gamma_k^{(2)}+\mathcal{R}_k\Big)^{-1}_{hh} &= -\frac{2d}{d-2}32\pi G_k\big[-\bar D^2+k^2R^{(0)}(-\frac{\bar D^2}{k^2})-2\bar\lambda_k+C_S\bar R\big]^{-1},\end{aligned} \tag{78}$$

and for the propagator of the ghost fields we have

$$\Big(\Gamma_k^{(2)}+\mathcal{R}_k\Big)^{-1}_{\rm gh}=\big[-\bar D^2+k^2R^{(0)}(-\frac{\bar D^2}{k^2})-\frac{1}{d}\bar R\big]^{-1}. \tag{79}$$

Here we made use of the fact that the inverse of $(\mathbb{1}-P)^{\mu\nu}{}_{\rho\sigma}$ is itself, simply because it is the unit matrix. We also went back to an arbitrary representation, as opposed to working in position space. Thus the functional traces that now need to be computed are

$$T_1=\frac12{\rm Tr}_T\Big[\Big(\Gamma_k^{(2)}+\mathcal{R}_k\Big)^{-1}_{\hat h\hat h}\partial_t\mathcal{R}_k^{\hat h\hat h}\Big],\quad T_2=\frac12{\rm Tr}_0\Big[\Big(\Gamma_k^{(2)}+\mathcal{R}_k\Big)^{-1}_{hh}\partial_t\mathcal{R}_k^{hh}\Big], \tag{80}$$

from the gravitational sector. The ghost sector gives rise to the contribution

$$T_3=-{\rm Tr}_V\Big[\Big(\Gamma_k^{(2)}+\mathcal{R}_k\Big)^{-1}_{\rm gh}\partial_t\mathcal{R}_k^{\rm gh}\Big], \tag{81}$$

where the minus sign originates from the Grassmann nature of the ghost fields. The subscript T of the first functional trace indicates that the trace has to be taken over a full set of traceless symmetric two-tensors. Similarly the subscript zero means the trace is simply over scalar functions.

7. Employ the heat kernel technique

The heat kernel technique [59] can be used to compute the functional traces in (80). It is a powerful method to compute traces involving second-order differential operators which does not require any input on the eigenfunctions of the operator inside. This method is well-known and used in many areas of physics, so we will just give the result here. For a function W satisfying suitable fall-off conditions we have[6]

$${\rm Tr}\big[W(-D^2)\big]=\frac{{\rm tr}(I)}{(4\pi)^{d/2}}\Big\{Q_{d/2}[W]\int d^dx\sqrt{g}+\frac16 Q_{d/2-1}[W]\int d^dx\sqrt{g}R\Big\}+\mathcal{O}(R^2) \tag{82}$$

The I denotes the identity matrix in the space over which the trace is taken, for scalars, vectors and traceless symmetric two tensors we have $tr(I)=1,d,\frac12(d-1)(d-2)$ respectively. The Q_n functionals are given by

$$\begin{aligned}Q_0[W]&=W(0),\\ Q_n[W]&=\frac{1}{\Gamma(n)}\int_0^\infty dz z^{n-1}W(z)\qquad n>0,\\ Q_n[W]&=(\frac{d}{dz})^{-n}W(z)\Big|_{z=0}\qquad n<0.\end{aligned} \tag{83}$$

Finally, $\Gamma(n)$ is the gamma function, which for positive integer values of n reads $\Gamma(n)=(n-1)!$.

[6]The overbar is left out here since the validity of the heat kernel formula is by far not limited to just the sphere.

8. Solve for the beta functions

At this stage, all the hard work has been done. The explicit form of the functions W is given by the trace arguments appearing in (76) (80). All that needs to be done is the projection to $\bar{R}^0, \bar{R}$, which can be done by employing standard Taylor series. Then, together with step one and the FRGE the beta functions can be found by solving for $\partial_t g_k$ and $\partial_t \lambda_k$. To do this it is convenient to introduce the anomalous dimension

$$\eta_N := -\frac{1}{G_k}\partial_t G_k, \tag{84}$$

which originates from $\partial_t \mathcal{R}_k$. In order to be able to work with a general cutoff, it is then convenient to further introduce the dimensionless threshold functions

$$\begin{aligned} \Phi(w)^p_n &:= \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}\frac{R^{(0)}(z) - zR^{(0)'}(z)}{[z+R^{(0)}(z)+w]^p}, \\ \tilde{\Phi}(w)^p_n &:= \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}\frac{R^{(0)}(z)}{[z+R^{(0)}(z)+w]^p}, \end{aligned} \tag{85}$$

In this way one obtains the final form of the beta function for the dimensionless Newton coupling reads,

$$\beta_g(g_k,\lambda_k) = \big(d-2+\eta_N(g_k,\lambda_k)\big)\, g_k. \tag{86}$$

whereas the one for λ_k is given by

$$\begin{aligned} \beta_\lambda(g_k,\lambda_k) =& -(2-\eta_N(g_k,\lambda_k))\lambda_k \\ &+ \frac{1}{2}g_k(4\pi)^{1-d/2}\big[2d(d+1)\Phi^1_{d/2}(-2\lambda_k) - 8d\Phi^1_{d/2}(0) \\ &\qquad - d(d+1)\eta_N(g_k,\lambda_k)\tilde{\Phi}^1_{d/2}(-2\lambda_k)\big]. \end{aligned} \tag{87}$$

Here the anomalous dimension is solved for as

$$\eta_N(g_k,\lambda_k) = \frac{g_k B_1(\lambda_k)}{1-g_k B_2(\lambda_k)}, \tag{88}$$

where

$$\begin{aligned} B_1(\lambda_k) =& \frac{1}{3}(4\pi)^{1-d/2)}\Big[d(d+1)\Phi^1_{d/2-1}(-2\lambda_k) - 6d(d-1)\Phi^2_{d/2}(-2\lambda_k) \\ &- 4d\Phi^1_{d/2-1}(0) - 24\Phi^2_{d/2}(0)\Big], \\ B_2(\lambda_k) =& -\frac{1}{6}(4\pi)^{1-d/2}\Big[d(d+1)\tilde{\Phi}^1_{d/2-1}(-2\lambda_k) - 6d(d-1)\tilde{\Phi}^2_{d/2}(-2\lambda_k)\Big]. \end{aligned} \tag{89}$$

Together, equation (86) and (87) determine the scale dependence of G and $\bar{\lambda}$ in the Einstein-Hilbert truncation.

9. Searching for non-Gaussian fixed points

To proceed further we specify the cutoff shape as the Litim shape function (16), $R^{(0)}(z) = (1-z)\Theta(1-z)$, and going to the phenomenologically most interesting case $d=4$, the expressions reduce to

$$\begin{aligned} \beta_g &= (2+\eta_N)g_k \\ \beta_\lambda &= (\eta_N - 2)\lambda_k + \frac{g}{8\pi}\left[\left(20 - \frac{10}{3}\eta_N\right)\frac{1}{1-2\lambda_k} - 16\right]. \end{aligned} \tag{90}$$

with B_1 and B_2 given by

$$B_1^{\rm grav} = \frac{1}{3\pi}\left[-\frac{9}{(1-2\lambda_k)^2} + \frac{5}{1-2\lambda_k} - 7\right], \quad B_2^{\rm grav} = \frac{1}{12\pi}\left[\frac{6}{(1-2\lambda_k)^2} - \frac{5}{1-2\lambda_k}\right]. \tag{91}$$

Finally, solving the system

$$\beta_g = 0, \quad \beta_\lambda = 0 \tag{92}$$

yields the non-trivial solution

$$g_\star = 0.707 \quad \lambda_\star = 0.193. \tag{93}$$

The critical exponents can also be computed using (42). Their real parts are both positive and given by

$$\theta^1 = 1.941 \quad \theta^2 = 3.147. \tag{94}$$

The fact that the system (92) possesses a non-Gaussian solution with both eigendirections being relevant, showing the existence of a suitable fixed point, constitutes one of the main pieces of support for the asymptotic safety scenario. For an in-depth analysis on this fixed point see [33]. This concludes the computation of the Einstein-Hilbert truncation.

3 Dirac Fermions

We will now turn our attention to fermions[7]. First we will shortly review the essentials in flat space, with a Lorentzian signature and then look at the generalization of this formalism to curved spacetime in Euclidean signature. We will need this in order to study the inclusion of fermions into the asymptotic safety program, focusing in particular on the fixed point structure.

3.1 Fermions in Minkowski Space

Fermions are an important building block in the standard model of particle physics. While interacting with the bosonic force carriers it are the fermions that make up all the structures in the everyday world. Remarkably, only the electron and up and down quark directly build up these structures. Other fermions, e.g. charm, strange, bottom and top quark as well as muon and tau particles and their corresponding neutrino's together with the antiparticles have also been observed. The interactions of these particles with the gauge fields mediating the fundamental forces are dictated by local gauge symmetries. The spin-$\frac{1}{2}$ particles are described by the Dirac equation. This equation has had many successes and forms the basis to one of the most accurate predictions in theoretical physics to date, namely the electrons magnetic moment.

Spinors are vectors carrying multiple degrees of freedom. The Dirac equation consists of a differential operator acting on these spinors comprising the matter field. The representation theory of the Lorentz group indicates that Dirac fermions should have four components. Explicitly, the Dirac equation reads

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0. \tag{95}$$

Here for the mass term we used a slight abuse of notation, $m\psi = m\,\mathbb{1}_\gamma \psi$. This notation is used throughout. The matrices γ^μ are constant four by four matrix in Dirac space, satisfying the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,\eta^{\mu\nu}\mathbb{1}_\gamma. \tag{96}$$

From this relation one can derive several trace identities, the two most frequently used in this thesis are

$$\begin{aligned} &\mathrm{tr}\big(\gamma^{\mu_1}....\gamma^{\mu_{2N+1}}\big) = 0, \\ &\mathrm{tr}\big(\gamma^\mu\gamma^\nu\big) = \frac{1}{2}\mathrm{tr}\big(\{\gamma^\mu,\gamma^\nu\}\big) = 4\,\eta^{\mu\nu}. \end{aligned} \tag{97}$$

The propagator (modulo a factor of i) is obtained via the standard 'trick'

$$\begin{aligned} (i\gamma^\mu\partial_\mu - m)^{-1} &= \big(i\gamma^\mu\partial_\mu - m\big)^{-1}\big(i\gamma^\nu\partial_\nu + m\big)^{-1}\big(i\gamma^\alpha\partial_\alpha + m\big) \\ &= \frac{i\gamma^\alpha p_\alpha + m}{-\partial^2 - m^2}, \end{aligned} \tag{98}$$

[7] From now on the phrase 'fermions' will always refer to Dirac fermions.

which, after Fourier transforming, and including a factor of i reduces to

$$\frac{i(\not{p}+m)}{p^2-m^2}. \tag{99}$$

Here the common slash notation is used: $\not{a} := \gamma^\mu a_\mu$. In the same way one can show that if $\psi(x)$ satisfies the Dirac equation it also satisfies the Klein-Gordon equation

$$(\partial^2+m^2)\psi(x)=0. \tag{100}$$

This must be the case for any relativistic wave equation and is directly tied to the denominator p^2-m^2 of the propagator. Equation (100) is also referred to as the dispersion relation.

The transformation of a Dirac spinor with respect to the Poincare group is

$$\psi'(x') \equiv S(\Lambda)\psi(x), \tag{101}$$

where $S(\lambda)$ is a matrix in Dirac space that denotes how the Poincaré transformations act on spinor space, satisfying $S(\Lambda_1\Lambda_2)=S(\Lambda_1)S(\Lambda_2)$ and $S(\Lambda^{-1})=S^{-1}(\Lambda)$. Requiring that the Dirac equation is relativistically invariant, i.e. if it holds in one inertial system it holds in all inertial systems, one can deduce

$$S^{-1}(\Lambda)\gamma^\mu S(\Lambda)=\Lambda^\alpha{}_\beta\,\gamma^\beta, \tag{102}$$

which is the standard transformation behavior of a four-vector. It is then possible to find the explicit form of $S(\Lambda)$ using the requirements specified above. First in infinitesimal form, and then by performing an infinitesimal transformation an infinite number of times in a controlled way, thus yielding finite transformations. Using this to work out the explicit form of the spin operator, being the generator of rotations, one can look at the eigenvalue of the spin operator squared. From this eigenvalue one can determine the spin of the Dirac fields. From this one is led to conclude that the Dirac equation is a relativistic wave equation giving rise to spin-$\frac{1}{2}$ particles.

In order to write down a Lagrangian for Dirac fermions, one needs to come up with a way to produce a real-valued scalar out of these spinorial objects. The obvious guess using a form $\psi^\dagger\psi$ does not work, because this does not transform as a scalar under Lorentz transformations. The reason for this is that the representation of the Poincaré group onto spinor space is not unitary. The solution is to define

$$\bar{\psi} := \psi^\dagger\gamma^0. \tag{103}$$

Using relation (102) one can show that the object $\bar{\psi}\psi$ does not transform under Lorentz transformations. The Lorentz-invariant Dirac Lagrangian is then given by

$$\mathcal{L}_{\text{Dirac}}=\bar{\psi}\big(i\,\gamma^\mu\partial_\mu-m\big)\psi \tag{104}$$

Indeed, one can show that in this case the Euler-Lagrange equations give rise to the Dirac equation.

Going back to Dirac space, a useful tool for proving several identities can be gained by employing an explicit basis. Indeed, in this sixteen dimensional space for any matrix $A\in \text{Mat}_{\text{d}_\gamma\text{xd}_\gamma}(\mathbb{C})$ there exist complex coefficients $c_a, a=1,...,16$, such that

$$A=c_a\Gamma^a. \tag{105}$$

Here the matrices Γ^a span a basis on Dirac space and are constructed in such a way that they satisfy

$$tr(\Gamma^a\Gamma^b)=4\,\delta^{ab}. \tag{106}$$

The basis matrices Γ^a can be given completely in terms of the gamma matrices,

$$\big\{\Gamma^a\big\}_{a=1,\dots,16}=\big\{\mathbb{1},\gamma^\mu,\frac{i}{2}\big[\gamma^\mu,\gamma^\nu\big],\gamma^5,\gamma^\mu\gamma^5\big\}, \tag{107}$$

Here γ^5 is the fifth gamma matrix, $\gamma^5 := i\,\gamma^0\gamma^1\gamma^2\gamma^3$, obeying $(\gamma^5)^2=\mathbb{1}_\gamma$. By making use of this explicit basis one can for example derive the Fierz reordering formula [30]

$$\big(\bar{u}_1\Gamma^a u_2\big)\big(\bar{u}_3\Gamma^b u_4\big)=\frac{1}{16}\sum_{c,d}tr(\Gamma^a\Gamma^b\Gamma^c\Gamma^d)\big(\bar{u}_1\Gamma^c u_4\big)\big(\bar{u}_3\Gamma^d u_2\big), \tag{108}$$

where the u_i are arbitrary Dirac spinors.

In the later sections of this thesis we will also need the projection of $\bar{\psi}A\psi$, onto say $\bar{\psi}\Gamma^a\psi$. To do this we simply expand the matrix A and find

$$\bar{\psi}A\psi = \bar{\psi}\Big(\frac{1}{4}\sum_b \mathrm{tr}\big(A\Gamma^b\big)\,\Gamma^b\Big)\psi \rightrightarrows \frac{1}{4}\mathrm{tr}\big(A\Gamma^a\big)\bar{\psi}\Gamma^a\psi \tag{109}$$

Here $\rightrightarrows$ indicates the projection. The Fierz reordering comes into play when one want to to project the structure $\big(\bar{u}_1\,u_2\big)\,\big(\bar{u}_3\,B\,u_4\big)$, onto $\bar{u}_3u_2$. For this we have

$$\begin{aligned}\bar{u}_1\,u_2\,\bar{u}_3\,B\,u_4 &= \frac{1}{4}\sum_b \bar{u}_1\,u_2\,\bar{u}_3\,tr(B\Gamma^b)\,\Gamma^b\,u_4 \\ &\rightrightarrows \frac{1}{4}\sum_b tr(B\Gamma^b)\,\frac{1}{16}\,\sum_c tr(\Gamma^b\Gamma^c)\big(\bar{u}_1\Gamma^c\,u_4\big)\bar{u}_3\,u_2 \\ &= \frac{1}{4}\big(\bar{u}_1\,\frac{1}{4}\,\sum_b tr(B\Gamma^b)\,\Gamma^b\,u_4\big)\,\bar{u}_3\,u_2 \\ &= \frac{1}{4}\big(\bar{u}_1\,B\,u_4\big)\,\bar{u}_3\,u_2.\end{aligned} \tag{110}$$

When canonically quantizing the fermionic fields demanding that the energy spectrum be bounded from below dictates that the spinor fields must obey canoncial anticommutation relations. In the framework of functional quantization, one mostly works with classical fields. In order to be able to make contact with the anticommutation relations as opposed to the commutation relations encountered for bosonic degrees of freedom, one needs to work classically anticommuting fields. That means one needs the components of the classical spinor fields to be Grassmann valued,

$$\bar{\psi}_i\psi^j = -\,\psi^j\bar{\psi}_i. \tag{111}$$

We will work with Grassmann valued spinor fields for the remainder of this thesis. This, for instance, gives rise to an overall minus sign in the Fierz reordering formula (108).

3.2 Fermions in Curved Euclidean Space

The example of the Einstein-Hilbert truncation given in section 2.2 shows that, in order to incorporate fermions in the Asymptotic Safety program, we need to work with fermions on a curved, Euclidean background spacetime. This requires generalizing various objects including the generalization of the partial derivative in the Dirac equation, to a covariant spin derivative, a generalization of the Dirac algebra for an arbitrary metric, as well as the introduction of a spin metric to write down the Dirac adjoint. This, as well as the transition for Lorentzian to Euclidean signature will give rise to some subtleties. In this section we follow along the lines of [42,43], reviewed in [44], and will start with the Clifford algebra through the Vierbein formulation.

3.2.1 The Vierbein Formulation and Clifford Algebra

A helpful tool to extend certain properties of spinors and Dirac matrices in flat space to a general spacetime is the Vierbein formulation. Since this formulation underlies many derivations, such as the transformation behavior of the spin connection, as well as a solid foundation to the Clifford algebra, it will shortly be reviewed here.

Let $\{x^\mu\}$ be local coordinates on a Riemannian space M, then $\{dx^\mu\}$ provides a coordinate induced basis of one-forms for the cotangent space. The line element in this basis can then be expressed as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu. \tag{112}$$

It will prove fruitful to make a connection with the local orthonormal frame at a point in M, that is always guaranteed to exist. It has a basis of one-forms $\{w^a(x)\}$ that satisfy

$$ds^2 = \delta_{ab}w^a(x)w^b(x). \tag{113}$$

To distinguish tensorial components in the local orthonormal frame from those in the coordinate induced basis we will use Latin indices. Since both $\{dx^\mu\}$ and $\{w^a(x)\}$ span there must exist coefficients $e^a_{\ \mu}(x)$, and $e^\mu_{\ a}(x)$ such that

$$w^a(x) = e^a_{\ \mu}(x)dx^\mu, \quad dx^\mu = e^\mu_{\ a}(x)w^a(x). \tag{114}$$

These coefficients are known as the vierbeins. Combining this with the line element (113) we have

$$g_{\mu\nu}(x) = e^a_{\ \mu}(x)e^b_{\ \nu}(x)\delta_{ab}. \tag{115}$$

It can easily be seen that the covectors $w^a(x)$, or equivalently the vierbeins, are not unique. They are determined up to local Euclidean Lorentz transformations. Requiring invariance under these transformations serve as important guiding principles for constructing various objects.

We will now use the vierbeins to construct the gamma matrices that will satisfy the Clifford algebra in general spacetimes

$$\big\{\gamma^\mu(x), \gamma^\nu(x)\big\} = 2g^{\mu\nu}(x). \tag{116}$$

Note that since the metric is now coordinate dependent, the gamma matrices necessarily must be too. We can start in flat space, where the gamma matrices $\bar\gamma^\mu$, satisfying $\{\bar\gamma^\mu, \bar\gamma^\nu\} = 2\,\delta^{\mu\nu}$, are known to exist. Given this relation, together with (115), we have

$$e^\mu_{\ a}(x)e^\nu_{\ b}(x)\big\{\bar\gamma^a, \bar\gamma^b\big\} = 2\,e^\mu_{\ a}(x)e^\nu_{\ b}(x)\,\delta^{\mu\nu} = 2\,g^{\mu\nu}(x) \tag{117}$$

Thus defining $\gamma^\mu(x) := e^\mu_{\ a}(x)\bar\gamma^a$, means we have found matrices in Dirac space that satisfy (116). It is thus through the local orthonormal frame that we can make progress here. Furthermore the x-dependence of the curved Dirac matrices comes in via the vierbeins. With this information one can show later on that the Dirac matrices are (spin-)covariantly constant.

3.2.2 Spin Connection and Spin Metric

To make sure that the Dirac equation satisfies the right transformation behavior the partial derivative ∂_μ, must be replaced by a covariant derivative ∇_μ. Also, the matrix γ^0 will no longer do the job of giving the Dirac adjoint ensuring the scalar transformation behavior of the Dirac Lagrangian. For this one needs to introduce a spin-metric h. The Dirac adjoint is then defined as

$$\bar\psi := \psi^\dagger h. \tag{118}$$

In order to avoid introducing any scale between ψ and $\bar\psi$ the spin metric has to obey $|\det(h)| = 1$. We then make the following assumptions on the spin metric and covariant derivative

$$\begin{aligned}
(i)\,\text{linearity}: &\quad \nabla_\mu(\psi_1 + \psi_2) = \nabla_\mu\psi_1 + \nabla_\mu\psi_2 \\
(ii)\,\text{product rule}: &\quad \nabla_\mu(\psi\bar\psi) = (\nabla_\mu\psi)\bar\psi + \psi(\nabla_\mu\bar\psi) \\
(iii)\,\text{metric compatibility}: &\quad \nabla_\mu\bar\psi = \overline{\nabla_\mu\psi} \\
(iv)\,\text{Covariance I}: &\quad \nabla_\mu\psi \to \nabla'_\mu\psi' = \frac{\partial x^\nu}{\partial(x')^\mu}\,S\,\nabla_\nu\psi \\
(v)\,\text{Covariance II}: &\quad \nabla_\mu(\bar\psi\gamma^\nu\psi) = D_\mu(\bar\psi\gamma^\nu\psi).
\end{aligned} \tag{119}$$

Here S is a similarity transformation, also commonly referred to as a spin-base transformation. These transformations are closely tied to representations of local Lorentz transformations, since the vierbeins are unique exactly up to these transformations. We will now show that the spin connection can carry a connection piece using above assumptions. Using the product rule and the second covariance requirement we compute

$$\begin{aligned}
\partial_\mu(\bar\psi\chi) &= (\partial_\mu\bar\psi)\chi + \bar\psi(\partial_\mu\chi) \\
&= \nabla_\mu(\bar\psi\chi) \\
&= (\nabla_\mu\bar\psi)\chi + \bar\psi(\nabla_\mu\chi).
\end{aligned} \tag{120}$$

Thus the only remaining freedom we have is

$$\nabla_\mu \psi = \partial_\mu \psi + \Gamma_\mu \psi, \quad \nabla_\mu \bar{\psi} = \partial_\mu \bar{\psi} - \bar{\psi}\Gamma_\mu \tag{121}$$

For some (possibly coordinate dependent) matrix in Dirac space. This matrix is the connection piece of the spin covariant derivative. From the demanded transformation behavior of $\nabla_\mu \psi$, it can be shown that Γ_μ should transform as a rank one tensor under diffeomorphisms. We will only work with a torsion free connection, which has the consequence that the connection piece must be traceless

$$\mathrm{tr}(\Gamma_\mu) = 0. \tag{122}$$

From these assumptions one can also derive the following important commutator identity

$$\big[\nabla_\mu, \nabla_\nu\big] = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + \big[\Gamma_\mu, \Gamma_\nu\big] = \frac{1}{8} R_{\mu\nu\alpha\beta}\big[\gamma^\alpha, \gamma^\beta\big]. \tag{123}$$

It can also be shown, by employing the vierbeins that

$$D_\mu \gamma^\nu = \partial_\mu \gamma^\nu + \Gamma^\nu{}_{\mu\lambda}\gamma^\lambda = \big[\gamma^\nu, \Gamma_\mu\big]. \tag{124}$$

Notably this is not zero. Using this however we can show that the gamma matrices are spin-covariantly constant

$$\begin{aligned}
\nabla_\mu(\gamma^\nu \psi) &= (D_\mu \gamma^\nu)\psi + \gamma^\nu (D_\mu \psi) + \Gamma_\mu \gamma^\nu \psi \\
&= \big[\gamma^\nu, \Gamma_\mu\big] + \gamma^\nu(\partial_\mu \psi) + \Gamma_\mu \gamma^\nu \psi \\
&= \gamma^\nu \Gamma_\mu \psi + \gamma^\nu(\partial_\mu \psi) \\
&= \gamma^\nu \nabla_\mu \psi.
\end{aligned} \tag{125}$$

We are now in a position to derive the well-known Lichnerowicz formula [45]

$$-\nabla\!\!\!/^2 \psi = (\Delta_\psi + \frac{R}{4})\psi, \tag{126}$$

where $\Delta_\psi := -\nabla_\mu \nabla^\mu$. We do this using the commutator relation (123) as well as the general identity $\frac{1}{8} R_{\mu\nu\sigma\rho}\big[\gamma^\sigma, \gamma^\rho\big] = \frac{R}{48}\big[\gamma_\mu, \gamma_\nu\big]$, that can be derived from the Clifford algebra and the symmetries of the Riemann tensor.

$$\begin{aligned}
-\nabla\!\!\!/^2 \psi &= -\gamma^\mu \gamma^\nu \nabla_\mu \nabla_\nu \psi \\
&= -\frac{1}{2}\Big(\big\{\gamma^\mu, \gamma^\nu\big\} + \big[\gamma^\mu, \gamma^\nu\big]\Big)\nabla_\mu \nabla_\nu \psi \\
&= -\nabla_\mu \nabla_\mu \psi - \frac{1}{2}\gamma^\mu \gamma^\nu \big[\nabla_\mu, \nabla_\nu\big]\psi \\
&= \Delta_\psi \psi - \frac{1}{2}\gamma^\mu \gamma^\nu \, \frac{R}{48}\,\big[\gamma_\mu, \gamma_\nu\big]\psi
\end{aligned} \tag{127}$$

Evaluating the last term will result in equation (126).

3.2.3 Reality Conditions

Next to the conditions imposed in (119), some further requirements can be imposed.These are called reality conditions. They are tied very closely to the form of the fermionic action, as they are there to guarantee that the action is real. For example since the mass term is written as

$$\int d^4x \sqrt{g}\, m\, \bar{\psi}\psi, \tag{128}$$

the corresponding reality condition would mean that the combination $\bar{\psi}\psi$ must be real, for any spinor ψ. This has an immediate consequence on the spin metric, namely that it must be anti-Hermitian. To see this first work out

$$\big(\bar{\psi}\psi\big)^\star = \big(\psi^\dagger\, h\, \psi\big)^\star = \psi^T\, h^\star\, \psi^\star = \big(\psi^T\, h^\star\, \psi^\star\big)^T = -\psi^\dagger\, h^\dagger\, \psi. \tag{129}$$

Thus the reality condition

$$(\bar{\psi}\psi)^{\star} = \bar{\psi}\psi, \tag{130}$$

can only hold if

$$h^{\dagger} = -h. \tag{131}$$

It is evident that this minus sign is a direct consequence of the Grassmann nature of the fermion fields that comes in when taking the transpose.

When constructing an action in Euclidean signature one has to make sure that the correct dispersion relation is satisfied. This means that the field $\psi(x)$ that satisfies the equation of motion also has to satisfy

$$(\Delta_{\psi} + m^2)\psi(x) = 0. \tag{132}$$

Note that with respect to Euclidean signature, the Lorentzian counterpart had a relative minus in the two terms. Naturally when constructing the action going from flat to curved spacetimes one makes the replacements

$$\int d^4x \mapsto \int d^4x\sqrt{g}, \quad \partial_{\mu} \mapsto \nabla_{\mu}. \tag{133}$$

In order to guarantee (132) it is most natural to take the action

$$S^{\text{ferm}} = \int d^4x\sqrt{g}\,\bar{\psi}(\nabla\!\!\!/ + m)\psi, \tag{134}$$

removing the i in front of the kinetic term. This amounts to the reality conditions that the two objects

$$\bar{\psi}\nabla\!\!\!/\psi, \quad \bar{\psi}\psi \tag{135}$$

are real for any spinor ψ. It turns out, that the requirements made so far are in *contradiction* with each other. That is, the reality conditions (135), the Lichnerowicz formula (126) and the statement that Δ_{ψ} only has positive eigenvalues are inconsistent with each other. The latter is a well known fact since the eigenvalues of Δ_{ψ} should be the value of 'the momentum' squared. The proof showing this contradiction is simple: Let θ be an eigenspinor[8] of the Dirac operator satisfying

$$\nabla\!\!\!/\theta = \lambda\,\theta. \tag{136}$$

Then by the reality conditions we have that $\lambda \in \mathbb{R}$, meaning $\lambda^2 > 0$. Then by (126) we have

$$-\nabla\!\!\!/^2\theta = (\Delta_{\psi} + \frac{R}{4})\theta \;\Rightarrow\; \Delta_{\psi}\theta = -(\lambda^2 + \frac{R}{4})\theta. \tag{137}$$

Thus in a spacetime with $R > 0$ we have constructed an eigenmode of the Laplacian with a negative eigenvalue. Thus the conventions advocated in (134) will not be consistent with our other requirements. One could try to reconcile this problem by absorbing the factor of i into the gamma matrices, which would then satisfy the Anti-Clifford algebra. Here however one runs into the problem of not being able to obtain the right dispersion relation (132). The solution to this problem is to take for the fermionic action

$$S^{\text{ferm}} = \int d^4x\sqrt{g}\,\bar{\psi}(i\nabla\!\!\!/ + m\gamma^5)\psi. \tag{138}$$

Here γ^5 is the fifth gamma matrix satisfying $\gamma^{\mu}\gamma^5 = -\gamma^5\gamma^{\mu}$ and $\left(\gamma^5\right)^2 = \mathbb{1}_{\gamma}$. In order to stay as close as possible to the Lorentzian counterpart one would like to not introduce additional graviton-matter vertices owed to the γ^5. Precisely this means that the variations of this matrix with respect to the metric should vanish. This can simply be done by taking $\gamma^5 := i\bar{\gamma}^0\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^3$, where $\bar{\gamma}^{\mu}$ are the flat space gamma matrices. Alternatively, one could also construct a matrix by contracting gamma matrices with a fully anti-symmetric tensor. In this way, the variations with respect to the metric vanish as well but the proof for this is nontrivial. With the action (138), the reality conditions (3.2.3), the Lichnerowicz formula (126) and the Dirac algebra (116) are mutually consistent, and the correct dispersion also follows from the equation of motion.

[8] One can certainly think of spaces where the Dirac operator indeed does possess (non-zero) eigenmodes. Examples of this are flat space as well as the sphere.

3.2.4 Eigenspinors on a Sphere

An efficient strategy for studying RG flows associated with non-minimally coupled Dirac fermions extends the background field method to spinors. For our purpose, we need to do this in a curved spacetime which, in the present setting, is again conveniently chosen as the four-sphere. The simplest but sufficient choice turns out to be the four-sphere. One would then need to study the spectral properties of spinors on the sphere in order to choose the background spinor in a way that leads to the most effective simplifications. Typical candidates for this would be eigenmodes of the Dirac operator. Here one needs to take into account that the Dirac operator on a sphere does **not** possess a zero mode [46]. Thus, even when one does not need to project onto derivative terms, one still cannot work with covariantly constant spinors because on a sphere they simply do not exist. It turns out that the eigenvalues of $\not\nabla$ on a unit d-sphere are given by $\pm i(n+\frac{d}{2})$ for nonnegative integer values of n. Here the radial dependence can be restored by dimensionality. The formula

$$\frac{1}{r} = \sqrt{\frac{R}{d(d-1)}} \tag{139}$$

shows that on a sphere curvature and momentum are inseparable since $\not\nabla\,\theta \propto R\,\theta$. This also has the consequence that the background field method on the sphere will not be able to distinguish between higher-derivative terms like $\bar{\psi}\not\nabla^2\psi$ and curvature coupled to spinor bilinears.

The lowest momentum mode corresponding to $n = 0$ turns out to be special in the sense that it satisfies a generalized eigenvalue equation for the covariant derivative. This equation reads

$$\nabla_\mu \theta = i\sqrt{\frac{R}{48}}\gamma_\mu\,\theta. \tag{140}$$

One can verify this by constructing an explicit representation of the gamma matrices as well as the spin connection and evaluating both sides. Using [46] it is possible to perform the calculations on the two-sphere, and use their inductive setup to conclude it also must hold in $d = 4$. The reason why the lowest eigenmode of $\not\nabla$ is special, is because this is the *only* mode that satisfies such a generalized eigenvalue equation. This can be seen by employing the commutation relation (123), onto a spinor satisfying $\nabla_\mu\chi = ic\,\gamma_\mu\chi$. By the commutator relation of the covariant derivative we have on a sphere

$$[\nabla_\mu, \nabla_\nu]\,\psi = \frac{R}{48}[\gamma_\mu, \gamma_\nu]\,\psi, \tag{141}$$

for any spinor ψ, whereas for our specific spinor χ we have

$$\begin{aligned} [\nabla_\mu, \nabla_\nu]\,\psi &= ic\nabla_\mu\,\gamma_\nu\psi - ic\nabla_\nu\,\gamma_\mu\psi \\ &= -c^2[\gamma_\nu, \gamma_\mu]\,\psi. \end{aligned} \tag{142}$$

Thus, the only admissible value of the coefficient c is

$$c = \sqrt{\frac{R}{48}}. \tag{143}$$

This proofs that only the lowest momentum mode of $\not\nabla$ can satisfy such a generalized eigenvalue relation.

We will end this section by deriving the corresponding generalized eigenvalue equation to (140) for the adjoint spinor $\bar{\theta}$. To do so we define for a matrix M in spinor space $\bar{M} := h^{-1}M^\dagger\,h$, and make use of the metric compatibility condition in (119). We work out

$$\begin{aligned} \nabla_\mu\bar{\theta} = \overline{\nabla_\mu\theta} &= \left(ic\,\gamma_\mu\theta\right)^\dagger h \\ &= -ic\,\theta^\dagger\gamma_\mu^\dagger\,h \\ &= -ic\,\bar{\theta}\,h^{-1}\gamma_\mu^\dagger\,h \\ &= -ic\,\bar{\theta}\,\bar{\gamma}_\mu \end{aligned} \tag{144}$$

It turns out that $\bar{\gamma}_\mu = \gamma_\mu$. This is an immediate consequence of the reality condition for the kinetic term, if one includes the i. Not including the i in front of the kinetic term would result in an additional minus sign. Thus the sign of the generalized eigenvalue equation for $\bar{\theta}$ depends on the reality conditions through the spin metric, and for our case reads

$$\nabla_\mu \bar{\theta} = -i\sqrt{\frac{R}{48}}\,\bar{\theta}\,\gamma_\mu \,. \tag{145}$$

The identity (140) is crucial for making progress when including fermions into asymptotic safety. The reason is that this identity allows to deal with connection terms whereas commutations with other gamma matrices do not always allow to use eigenvalue equations of the form $\not{\nabla}\theta = \lambda\,\theta$.

4 Fermions in Asymptotic Safety

We are now at a point to discuss the main topic of this thesis, namely the inclusion of fermions into the asymptotic safety program. The existing literature [6] employs a flat background for the matter sector though. For a good understanding of the interplay between fermions with gravity, it is crucial to extend these computations to a curved background. We will take a bottom-up approach starting with the simplest truncation ansatz, comprising the Einstein-Hilbert action supplemented by minimally coupled Dirac fields. We will then extend our truncation ansatz by including non-minimal couplings between gravity and the fermions. This section is based on the works [47, 48].

4.1 Minimally Coupled Fermions

A natural and simplified starting point for the inclusion of fermions into the asymptotic safety program is to add them minimally coupled. On the gravitational side we include the Einstein-Hilbert action supplemented by a harmonic gauge fixing condition [49] and the corresponding ghost action, as in section 2.2. We then also add the term

$$\Gamma_k^{\text{ferm}}[g,\bar{\psi},\psi] = \int d^4x\sqrt{g}\,\bar{\psi}\big(i\not{\nabla} + m\gamma^5\big)\psi, \tag{146}$$

constructed in 3.2.3, to the truncation ansatz. Here $g_{\mu\nu}$ and ψ denote the spacetime metric and the Dirac spinors respectively and we suppressed an index enumerating the ψ's. This suppressed index runs over the range $1, ..., N_f$, i.e., all fermion fields are described by the same action. Note that we do not include wave-function renormalization for the fermions at this point. This is what is meant by 'minimally coupled'. The mass term is only here to setup conventions and to find the form of the regulator. In what follows we focus on the case of massless fermions setting $m = 0$. We employ the background method by considering fluctuations $\{h_{\mu\nu}, \bar{\chi}, \chi\}$ of the fields around a fixed background configuration $\{\bar{g}_{\mu\nu}, \bar{\theta}, \theta\}$. This is done by imposing the linear split

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,, \quad \bar{\psi} = \bar{\theta} + \bar{\chi}\,, \quad \psi = \theta + \chi\,. \tag{147}$$

We then substitute this expansion into the Wetterich equation and read off the scale-dependence of G_k and Λ_k at zeroth order in the fluctuation fields. In order to ease the computation we choose $\bar{g}_{\mu\nu}$ to be the metric of the four-sphere, just as in the Einstein-Hilbert truncation worked out in section 2.2. The background of the fermionic fields need not be specified, but one can think of these background fields being set to zero.

The final ingredient in the construction is the regulator function $\mathcal{R}_k$. this is again a matrix valued in field space. In the gravitational and ghost sector the harmonic gauge choice entails that all derivatives contained in the gravitational sector of $\Gamma_k^{(2)}$ organize themselves into Laplacians $\Delta := -\bar{g}^{\mu\nu}\bar{D}_\mu\bar{D}_\nu$ constructed from the background metric. We then follow [12, 49] and construct the entries from $\mathcal{R}_k$ via the substitution rule (Type I regulator)

$$\Delta \mapsto P_k(\Delta) = \Delta + R_k(\Delta)\,. \tag{148}$$

The shape function of $\mathcal{R}_k^{\rm grav}$ is taken to be the Litim cutoff from (16). Constructing the regulator in the fermionic sector is slightly more involved. Motivated by the mass term appearing in (146), we replace m by a k-dependent regulator of dimension one,

$$\mathcal{R}_k^{\psi} = k\,\gamma^5\,\left(1-\sqrt{\Box}/k\right)\,\Theta(1-\sqrt{\Box}/k)\,, \tag{149}$$

where $\Box$ denotes a suitable coarse-graining operator. Following [9], two canonical choices for the coarse-graining operator are $\Box = -\nabla\!\!\!/^2$ and $\Box = \Delta_\psi$. These choices are related by the Lichnerowicz formula (126). In order to treat both cases simultaneously, we then write

$$\Box = -\nabla\!\!\!/^2 + \beta\,\bar{R} \tag{150}$$

where $\beta = 0$ and $\beta = -1/4$ correspond to $\Box$ being the (squared) Dirac-operator and the Laplacian, respectively.

At this stage, we have all the ingredients to determine the scale-dependence of G_k and Λ_k. The explicit computation combines the early-time expansion of the heat-kernel, discussed in section 2.2, with standard γ-matrices manipulations. This amounts to evaluating the functional trace

$$T^{\rm ferm} = -N_f\,{\rm Tr}\big[\big(i\nabla\!\!\!/ + \mathcal{R}_k^{\psi}(\Box)\big)^{-1}\,\partial_t\mathcal{R}_k^{\psi}(\Box)\big], \tag{151}$$

projected onto the Einstein-Hilbert action monomials. The minus sign, owed to the Grassmann nature of the fermion fields, appears because of the fermion loop and an additional factor of two appears due to the matrix structure of the Hessian. For more details on this see appendix B. To evaluate the Q-integrals from the heat-kernel expansion one needs to be careful when evaluating the terms proportional to β given the argument of the step function in (16). The result of this computation is conveniently expressed in terms of the beta functions for the dimensionless couplings g_k and λ_k. Their explicit form reads

$$\begin{aligned}\beta_g &= (2+\eta_N)g \\ \beta_\lambda &= (\eta_N-2)\lambda + \frac{g}{8\pi}\left[\left(20-\frac{10}{3}\eta_N\right)\frac{1}{1-2\lambda}-16\right] - \frac{N_f}{3}\,.\end{aligned} \tag{152}$$

The anomalous dimension of Newton's coupling η_N is parameterized by

$$\eta_N = \frac{g\left(B_1^{\rm grav}(\lambda) + N_f\,B^{\rm f,minimal}\right)}{1-g\,B_2^{\rm grav}(\lambda)}\,, \tag{153}$$

with

$$\begin{aligned}B_1^{\rm grav} &= \frac{1}{3\pi}\left[-\frac{9}{(1-2\lambda)^2}+\frac{5}{1-2\lambda}-7\right]\,, \\ B_2^{\rm grav} &= \frac{1}{12\pi}\left[\frac{6}{(1-2\lambda)^2}-\frac{5}{1-2\lambda}\right]\,, \\ B^{\rm f,minimal} &= -\frac{(\pi-2)}{12\pi}\left[1-12\,(\beta+\frac{1}{4})\right]\,.\end{aligned} \tag{154}$$

The contributions from the fermionic action (146) are all proportional to N_f and vanish for $N_f = 0$. The parameter β appears in $B^{\rm f,minimal}$ only.

When investigating the beta functions (152) for $N_f = 0$ one indeed recovers the results of the Einstein-Hilbert computation. That is, the pure-gravity system admits a Gaussian fixed point (GFP) at the origin $\{\lambda_\star^{\rm GFP}, g_\star^{\rm GFP}\} = \{0,0\}$. In addition there is a NGFP situated at $\{\lambda_\star^{\rm NGFP}, g_\star^{\rm NGFP}\} = \{0.193, 0.707\}$. The critical exponents of this fixed point are $\theta_{1,2} = 1.48 \pm 3.04i$, indicating that the fixed point is UV attractive in both g_k and λ_k.

The NGFPs appearing in the minimally coupled gravity-fermion setting are given by the dashed lines in the left panels of figure 2. Notably, one obtains an infinite family of NGFPs, extending to arbitrary values N_f, for both $\beta = 0$ and $\beta = -1/4$. For $\beta = -1/4$, all NGFPs are located at

$g_\star > 0$. For $\beta = 0$ there is a critical number of fermions $N_f^{\rm crit} = 12.26$ where the NGFP transitions from $g_\star > 0$ ($N_f < N_f^{\rm crit}$) to $g_\star < 0$ ($N_f > N_f^{\rm crit}$). This is problematic because of the condition that a phenomenologically viable high-energy completion requires that the NGFP must be situated at $g_\star > 0$. The reason for this is that g_k cannot change sign along an RG trajectory [33].[9] The observational information of gravity to be attractive then entails that we must have $g_\star > 0$. Thus one concludes that not all NGFPs may be viable candidates for rendering the gravity-matter system asymptotically safe. The finite value of $N_f^{\rm crit}$, appearing for one choice of coarse-graining operator while absent for another, indicates that the validity of asymptotic safety seemingly depends on the choice of regularization procedure. This was the conclusion reached in [9]. It should be emphasized, however, that the existence of the NGFP is actually independent of the choice of regulator. Merely its position may or may not be suitable for building a viable phenomenology.

4.2 Extending Truncation Ansatz

The β-dependence discussed in the previous subsection suggests that the ansatz for Γ_k could be too simple for capturing the essential properties of the NGFPs. In order to improve our approximation systematically, it is useful to understand the mechanism underlying the presence (or absence) of $N_f^{\rm crit}$. In order to facilitate the clarity of the discussion, we will set $\lambda = 0$ and focus on the fixed point equation for Newton's coupling $\eta_N(g_\star, \lambda = 0) = -2$. In this case $g_\star$ is determined by the linear equation

$$g_\star (B_1^{\rm grav} - B_2^{\rm grav} + N_f \, B^{\rm ferm})|_{\lambda=0} = -2 \tag{155}$$

where $(B_1^{\rm grav} - B_2^{\rm grav})|_{\lambda=0} = -15/(4\pi) < 0$. This entails that for $N_f = 0$ one has $g_\star > 0$, as could be concluded by the Einstein-Hilbert truncation results. The sign of $B^{\rm ferm}$ depends on the choice of β though:[10]

$$B^{\rm ferm}|_{\beta=-1/4} = \frac{2-\pi}{12\pi} < 0 \, , \quad B^{\rm ferm}|_{\beta=0} = \frac{\pi - 2}{6\pi} > 0 \, . \tag{156}$$

If $B^{\rm ferm} < 0$ the bracket on the LHS of (155) is negative for all values of N_f resulting in $g_\star > 0$. If $B^{\rm ferm} > 0$ there is a critical value $N_f^{\rm crit}$ where the bracket changes sign and the NGFP transitions from $g_\star > 0$ to $g_\star < 0$. It turns out that restoring the λ-dependence does not alter this effect. This is the behavior exhibited by the dashed lines in figure 2.

This analysis suggests searching for additional terms in Γ_k which contribute to $B^{\rm ferm}$. If we look for an interaction term (excluding higher order kinetic terms) then there is only one possible term that contributes. This interaction term of course must contain spinors in order to contribute to $B^{\rm ferm}$ but cannot contain more than two, since for the projection onto $\int d^4x \sqrt{g}\, R$ no fermion fields can be present. The Hessian only removes two of these. This leaves us with a pure mass term, which we have already set to zero. To obtain an interaction without introducing new fields we then most resort to curvature invariants. The only curvature invariant that will contribute to the desired projection is the Ricci scalar R. For convenience (to be able to use Dirac's trick to square the propagator) a γ^5 is also be included between the bilinears. So we identified a unique canonical interaction term coupling the fermion-bilinears to the spacetime curvature:

$$\Delta\Gamma_k^{\rm ferm}[g, \bar\psi, \psi] = \tilde\alpha_k \int d^4x \sqrt{g}\, R \, \bar\psi \gamma^5 \psi \, . \tag{157}$$

This term has the structure of a mass term where the mass is set by the spacetime curvature (this also provides a piece of motivation to include the γ^5). Upon including $\Delta\Gamma_k^{\rm ferm}$ the flow of G_k and Λ_k again takes the form (152) with the new coefficient for $B^{\rm ferm}$ given by

$$B^{\rm ferm} = -\frac{1}{12\pi} \left[24\alpha + \Big(\pi - 2\Big)\Big(1 - 12(\beta + \frac{1}{4})\Big) \right] \, , \tag{158}$$

[9] The reason for this is that at $g_k = 0$ the trajectory necessarily hits the Gaussian fixed point, making it unable for the flow to cross this region. Since the trajectory is continuous this then means the sign of $g_\star$ is the same as the sign of Newton's constant at low energies.

[10] The numerical values of these coefficients differ from the ones reported in [9]. This can be traced back to the different shape of the regulator function (149) employed by the two works. Notably, this has no effect on the qualitative behavior encoded by the signs of the coefficients, showing robustness of this feature under a change of regularization procedure.

where $\alpha_k \equiv \tilde{\alpha}_k\, k$ is the dimensionless counterpart of $\tilde{\alpha}_k$. equation (158) indicates that α can play a crucial role in understanding the fixed point structure of the system. In particular, it has the potential to shift all NGFPs to $g_\star > 0$ provided that $\alpha_\star$ is sufficiently positive to compensate for the regulator contributions.

4.3 Analyzing Extended System

In the previous section it was argued that the fermion-curvature coupling (157) could be essential for understanding the fixed-point structure of gravity-fermion systems. In this section we complete the analysis by computing the beta function for the coupling α_k (section 4.3.1) and the analysis of the resulting fixed-point structure in sections 4.3.2 and 4.3.3, respectively. Explicitly the effective average action will be approximated by

$$\Gamma_k[g, \bar{\psi}, \psi] = \Gamma^{\text{grav}}[g] + \Gamma_k^{\text{ferm}}[g, \bar{\psi}, \psi] + \Delta\Gamma_k^{\text{ferm}}[g, \bar{\psi}, \psi], \tag{159}$$

where the mass of all the fermions will be set to zero and the kinetic term is still minimally coupled (for a non-minimally coupled kinetic term see 4.4).

4.3.1 Beta Functions

When computing the beta function for α, we again make use of the background field method, also retaining the structure

$$\int d^4x \sqrt{\bar{g}}\, \bar{R}\, \bar{\theta}\, \gamma^5\, \theta \tag{160}$$

on both sides of the projected Wetterich equation. Again we select the background metric to be the one of the four-sphere. Tracking the fermionic terms then also requires adopting a non-zero value of the background fermions θ, c.f. equation (147). In principle, equation (160) suggests using covariantly constant spinor fields. On the background sphere this is inconsistent, however, since the Dirac operator does not possess zero modes as mentioned in 3.2.4. We then take θ as the lowest eigenmode of the Dirac operator, which satisfies the (generalized) eigenvalue equation $\nabla_\mu\, \theta = i\sqrt{\frac{\bar{R}}{48}}\, \gamma_\mu\, \theta$.

The beta function

$$k\partial_k \alpha_k = \beta_\alpha(\lambda_k, g_k, \alpha_k) \tag{161}$$

is then found by expanding the RHS of the Wetterich equation to second order in the background fermions and identifying the term proportional to $\bar{R}$. Pictorially, the contributions to β_α are given by the Feynman diagrams shown in figure 1, encoding the self-energy corrections to the background fermions. The explicit expressions represented by these diagrams are obtained by first splitting $\Gamma_k^{(2)} + \mathcal{R}_k = A + B$ where A is independent of the background fermions and B consists of all terms containing either $\bar{\theta}, \theta$ or both. The entries of A^{-1} are the propagators of the fluctuation fields while B encodes the vertices coupling the fluctuations to the background fields. The inverse $(\Gamma_k^{(2)} + \mathcal{R}_k)^{-1}$ is then constructed as an expansion in B:

$$\big(A + B\big)^{-1} = A^{-1} - A^{-1}\, B\, A^{-1} + A^{-1}\, B\, A^{-1}\, B\, A^{-1} + \mathcal{O}(B^3). \tag{162}$$

The term A^{-1} is independent of the background fermion field while the terms of order B^3 and higher contain at least three powers of the fermion background fields. Thus these terms do not contribute to β_α. The tad-pole diagram shown in figure 1 is then generated by the term of order B while the diagrams containing the three-point vertices arise at order B^2. Thus figure 1 then includes all diagrams that contribute to the self-energy of the background fermions. For more details on this as well as the explicit expressions of the diagrams see appendix B.

The Feynman diagrams imply that β_α will be a polynomial of degree three in α. The presence of the cubic term is inferred from the observation that each three-point vertex contains a term that is proportional to α_k. In addition the fermion propagator contains a term proportional to $\alpha_k \bar{R}$. The projection onto (160) then entails an expansion of the fermion propagator in the background curvature, so that the first two diagrams in figure 1 contain contributions up to order α^3. Hence

$$\beta_\alpha = A_0 + (A_1 + 1)\, \alpha + A_2\, \alpha^2 + A_3\, \alpha^3\, . \tag{163}$$

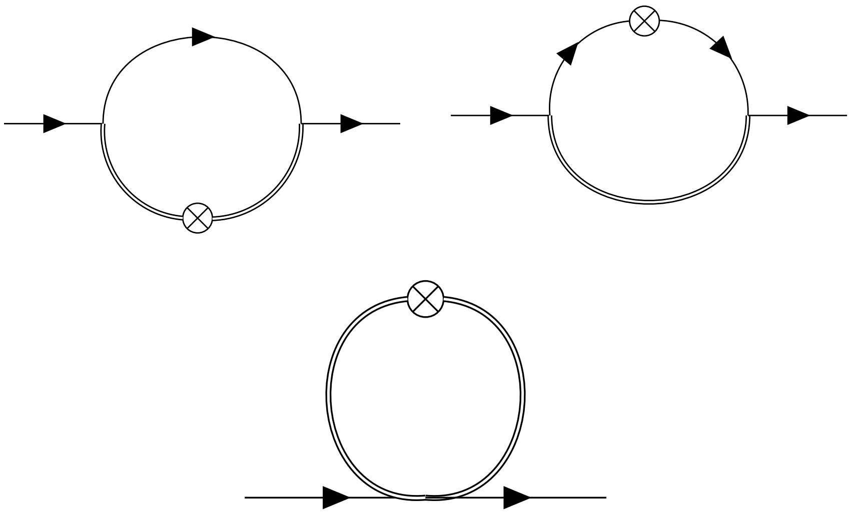

Figure 1: Feynman diagrams contributing to β_α. The bold, external lines correspond to background fermions while the solid single and double lines in the loop encode the propagators of the fermionic fluctuations and graviton fluctuations, respectively. The crossed circle marks the insertion of the regulator $\mathcal{R}_k$. The three- and four-point vertices are obtained by expanding Γ_k about the background field configuration. The bottom diagram is referred to as the tadpole diagram.

The coefficients A_i, $i = 0, 1, 2, 3$, depend on the couplings λ, g as well as the coarse-graining parameter β,

$$A_i(\lambda, g) = \frac{g}{\pi}\left(\frac{A_i^1}{(1-2\lambda)} + \frac{A_i^2}{(1-2\lambda)^2} + \frac{A_i^3}{(1-2\lambda)^3}\right), \tag{164}$$

with the non-zero numerical coefficients A_i^j, $j = 1, 2, 3$, being

$$\begin{aligned}
A_0^1 &= -\frac{3}{32} \\
A_0^2 &= \frac{3}{8} - \frac{15\pi}{128} + \frac{1}{32}\eta_N \\
A_0^3 &= \frac{7}{20} - \frac{3\pi}{32} - (\frac{179}{1120} - \frac{3\pi}{64})\eta_N \\
A_1^1 &= -\frac{7}{6} + \frac{7\pi}{16} - \frac{\beta}{2} \\
A_1^2 &= \frac{107}{30} + \frac{\pi}{32} + (\frac{1}{30} - \frac{13\pi}{64})\eta_N - \frac{\beta}{4} - (\frac{39}{40} - \frac{21\pi}{64})\beta\eta_N \\
A_1^3 &= -\frac{67}{30} - \frac{\pi}{8} + (\frac{47}{210} + \frac{\pi}{32})\eta_N \\
A_2^1 &= \frac{47}{12} - \frac{5\pi}{4} + (\frac{45}{4} - \frac{45\pi}{16})\beta \\
A_2^2 &= \frac{169}{120} - \frac{\pi}{2} + (\frac{101}{280} - \frac{\pi}{8})\eta_N \\
&\quad + (\frac{9}{2} - \frac{9\pi}{8})\beta - (\frac{61}{20} - \frac{15\pi}{16})\beta\eta_N \\
A_2^3 &= -\frac{17}{105} + (\frac{143}{630} - \frac{\pi}{16})\eta_N \\
A_3^1 &= \frac{9}{10} \\
A_3^2 &= -\frac{17}{10} + (\frac{79}{28} - \frac{27\pi}{32})\eta_N
\end{aligned} \tag{165}$$

The explicit form of the beta function (163) constitutes the main computational result of this thesis.

Obtaining these was the key challenge of this project. An outline of the derivation is given in appendix D.

4.3.2 Fixed-Point Structure

When investigating the fixed point structure for the λ-g-α system, we start with the following observations:

a) Including the fermionic sector $\Gamma_k^{\rm ferm} + \Delta\Gamma_k^{\rm ferm}$ supplements the beta functions (152) with the beta function (163). Notably, β_α is cubic in α. This guarantees that, for fixed λ, g, the equation $\beta_\alpha = 0$ has at least one real solution. Thus the NGFPs seen in the approximation $\Delta\Gamma_k^{\rm ferm} = 0$ will persist once the fermion-curvature coupling is added.

b) The coefficient A_0 in β_α is non-zero. As a consequence, the NGFPs found at minimal coupling do not generalize to fixed points with $\alpha_\star = 0$. Quantum gravity fluctuations shift the position to a non-zero value $\alpha_\star$. This shift is generated by the fermion-kinetic term and becomes visible once $\Delta\Gamma_k^{\rm ferm}$ is included. This mechanism is identical to the one generating the gravity-induced non-vanishing scalar couplings [51].

c) When investigating the transition from $N_f = 0$ to a small value, $N_f = 10^{-3}$ say, one finds that the fixed point seen at minimal coupling branches into 3 families of NGFPs discriminated by their value for $\alpha_\star$. In addition there is one family of NGFPs coming in from $\alpha_\star \to -\infty$.

d) Increasing N_f the NGFPs coming from $\alpha_\star \to -\infty$ annihilate with one branch of NGFPs emanating from the gravitational fixed point. This occurs at $N_f \approx 3$.

e) Performing a large-N_f expansion of the beta functions (152) and (163), one establishes that the two other branches extend to infinite families of NGFPs existing for all values of N_f. These solutions will be labeled $\text{NGFP}^{\rm A}$ and $\text{NGFP}^{\rm B}$.

Upon exhibiting the general structure of the fixed point system, we now investigate the properties of the solutions $\text{NGFP}^{\rm A}$ and $\text{NGFP}^{\rm B}$ numerically. Their position $\{g_\star, \lambda_\star, \alpha_\star\}$ as a function of N_f is shown in figure 2. The analysis establishes the value $\alpha_\star$ as the feature distinguishing the two branches: family A is characterized by $\alpha_\star^{\rm A} \ll 1$ with $\alpha_\star^{\rm A}$ decreasing for increasing N_f. For family B, $\alpha_\star^{\rm B} \propto N_f$ increases linearly with the number of fermions. The comparison between the solid and dashed lines shows that $\text{NGFP}^{\rm A}$ actually shares all the essential properties of the NGFPs found in the minimally coupled case. In particular, the position $g_\star^{\rm A}$ is again sensitive to the choice of β: for $\beta = 0$ there is a critical number of fermions $N_f^{\rm crit}$ at which the solution transits from $g_\star^{\rm A} > 0$ ($N_f < N_f^{\rm crit}$) to $g_\star^{\rm A} < 0$ ($N_f > N_f^{\rm crit}$). The similarity of $\text{NGFP}^{\rm A}$ to the minimally coupled case can be understood easily from the fact that $\alpha_\star^{\rm A} \ll 1$ so that $\alpha_\star = 0$ constitutes a good approximation.

The family $\text{NGFP}^{\rm B}$ is situated at $g_\star^{\rm B} > 0$ for all values of N_f, i.e, there is no critical number of fermions independent of the choice of coarse-graining operator. This feature can be understood by revisiting the fixed point condition $\eta_N^\star = -2$: since the value $\alpha_\star$ increases with an increasing number of fermions, the contribution of the coupling α always dominates over the contribution of the regulator. With $\alpha_\star > 0$ one then finds that the solution of $\eta_N^\star = -2$ is always situated at $g_\star > 0$. Moreover, $\alpha_\star \gg \beta$ also ensures that the position of the fixed point is largely independent of β.

Finally, one can study the stability of the RG flow in the vicinity of the two families of fixed points. The computation of the stability matrix $\mathbf{B}$ shows that the $\text{NGFP}^{\rm A}$ and $\text{NGFP}^{\rm B}$ come with two and three relevant directions, respectively. This result is independent of N_f and the choice of β. As a consequence a high-energy completion based on a NGFP from the family A may predict the low-energy value of α while for the fixed points in the family B this value corresponds to a free parameter which must be taken from experiment.

4.3.3 Flow Diagram and Predictivity

We close this section by illustrating the RG flow created by the beta functions (152) and (163). For this purpose, we project the full system onto the α-g–plane by setting $\lambda = 0$. For concreteness, we choose

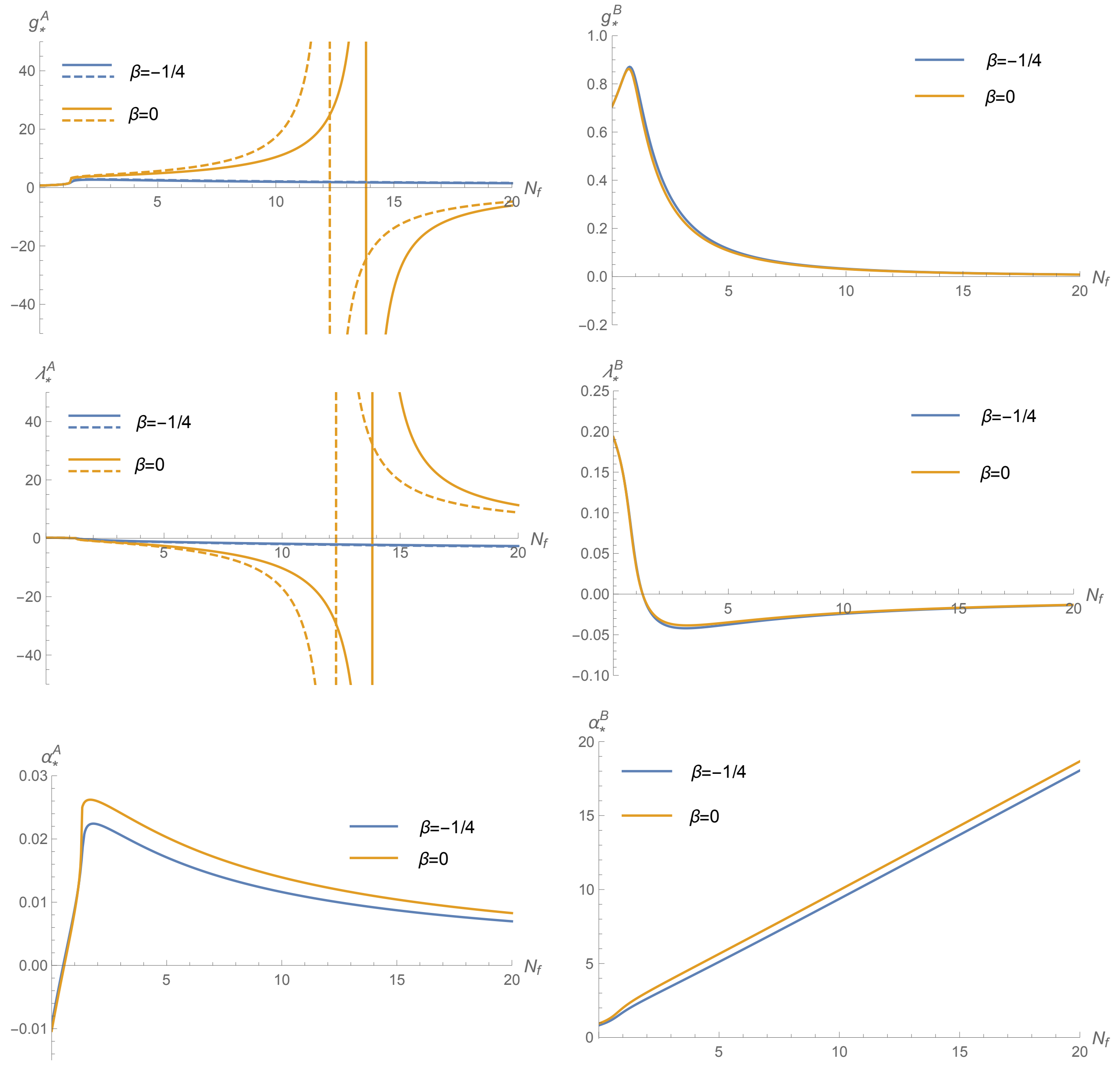

Figure 2: Position of the fixed point solutions NGFP$^{\rm A}$ and NGFP$^{\rm B}$ arising from the extended beta functions (152) and (163) as a function of N_f. The blue and orange lines correspond to the coarse-graining operator being the Laplacian ($\beta = -1/4$) and the squared Dirac operator ($\beta = 0$), respectively. The dashed lines shown in the diagrams for $g_\star^{\rm A}$ and $\lambda_\star^{\rm A}$ correspond to the position of the NGFP found at minimal coupling, $\alpha = 0$. For $\beta = 0$, NGFP$^{\rm A}$ is shifted into the unphysical region $g_\star < 0$ when $N_f > N_f^{\rm crit}$.

$N_f = 3$ and $\beta = -1/4$ which serves as an illustrative example of the general situation where one has two NGFPs situated at $g_\star > 0$. The flow is then controlled by the projection of the fixed points found for the full λ-g-α-system: $\{\alpha_\star^{\rm GFP}, g_\star^{\rm GFP}\} = \{0, 0\}$, $\{\alpha_\star^{\rm A}, g_\star^{\rm A}\} = \{0.02, 1.49\}$, and $\{\alpha_\star^{\rm B}, g_\star^{\rm B}\} = \{2.90, 0.29\}$. The GFP serves as an infrared attractor, capturing the RG flow in its vicinity as k is lowered. NGFP$^{\rm A}$ is a saddle point with the UV-attractive direction almost aligned with the $\alpha = 0$-axis. The NGFP$^{\rm B}$ is UV-attractive in both α and g. The stability properties of NGFP$^{\rm B}$ are remarkable in the sense that the two right-eigenvectors of the projected stability matrix are almost parallel, enclosing an opening angle $\theta \simeq 13°$.

The flow generated by the projected beta functions is shown in figure 3. The black lines originate from integrating the beta functions with initial conditions set along the eigenvectors of the stability matrices associated with NGFP$^{\rm A}$ and NGFP$^{\rm B}$. The bold black line connecting the NGFPs acts as a

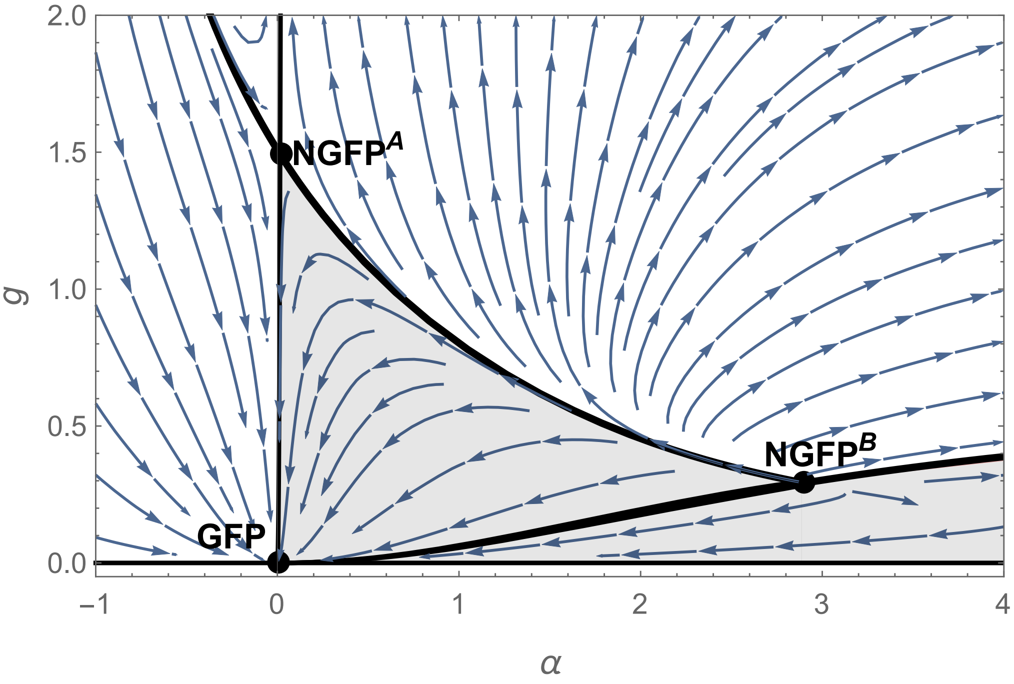

Figure 3: Illustration of the RG flow projected onto the α-g–plane obtained for $N_f = 3$ and $\beta = -1/4$. The arrows point towards the infrared, i.e., smaller values of k. The positions of the projected fixed points are marked by black dots. The thick black line marks the boundary of the region where the flow is attracted to the GFP at low energy. The figure provides a prototypical example of the interplay between the fixed points.

boundary: RG trajectories below this line are attracted to the GFP as $k \to 0$ while trajectories above this line typically terminate at a finite value of k. The trajectories in the shaded region are complete in the sense that their flow interpolates between the NGFP$^{\text{B}}$ for $k \to \infty$ and the GFP as $k \to 0$. The approach to the GFP then guarantees the existence of a classical low-energy regime where general relativity constitutes a good approximation of the gravitational physics.

4.3.4 Conclusions

Motivated by the asymptotic safety scenario [3,4] for gravity-matter systems, we studied the fixed point structure of gravity coupled to N_f Dirac fermions on a spherically symmetric background. Driven by the significant regulator dependence found at minimal coupling [9], we extended the minimal case by adding a distinguished coupling between the fermion bilinears to the Ricci scalar constructed from the spacetime metric. This coupling is induced dynamically by quantum gravity fluctuations. As a main result, we identified the two infinite families of non-Gaussian renormalization group fixed points (NGFPs) shown in figure 2 and existing for all values N_f. The first family shows the behavior reported in [9], possibly exhibiting an upper critical number of fermions for which the fixed points could be used in asymptotic safety. The second one could provide a phenomenologically viable high-energy completion for all values of N_f.

We specifically analyzed the effect of implementing different coarse-graining schemes (150) based on the squared Dirac operator ($\beta = 0$) and Laplacian ($\beta = -1/4$), respectively. In combination with the fermion-curvature coupling α introduced in equation (157) this resulted in a rather clear picture: quantum gravity fluctuations dictate that α must be non-zero at any NGFP. For the first family of NGFPs, $\alpha_\star$ is very small and well-approximated by $\alpha_\star \simeq 0$. The regularization procedure based on the squared Dirac operator then induces contributions to the flow which dominate over this coupling. As a result, the position of the NGFPs exhibits a "strong" dependence on the choice of coarse-graining operator. Notably, neither the existence nor the critical exponents are sensitive to the choice of $\Box$. It is merely the shift in position which may render the NGFPs unsuitable for a phenomenologically valid high-energy completion of the gravity-fermion system. Conversely, the second family of fixed points comes with much larger values $\alpha_\star$. As a consequence, the contribution from the regulator is subdominant and the fixed point properties show only a minor dependence on $\Box$. This suggests that it is actually the Laplacian which is the canonical choice for the coarse-graining operator, since other values of β induce additional interaction terms originating from the regularization procedure.

4.4 Including the Fermion Anomalous Dimension

We end this section by briefly discussing the inclusion of the fermion wave function renormalization. The corresponding fermionic part of the truncation is then given by

$$\Gamma_k^{\text{ferm}} = Z_k \int d^4x\sqrt{g}\,\bar{\psi}\big(i\nabla\!\!\!/ + \bar{\alpha}_k R\,\gamma^5\,\psi\big). \tag{166}$$

The presence of the extra coupling Z_k leads to an additional equation for the RG time derivative of Z_k. This coupling is different than the others discussed here in the sense that this one is inessential. What that means is that we do now have to require that this coupling runs into a fixed point value. The reason is that Z_k can always be absorbed into field redefinitions and thus never enters any observables. This means that we do not require its RG time derivative to vanish as $k \to \infty$. However the combination

$$\eta_\psi := -\frac{1}{Z_k}\partial_t Z_k, \tag{167}$$

known as the fermion anomalous dimension does enter the other beta functions. This is where we substitute in the additional equation for $\partial_t Z_k$. Thus the inclusion of the wave function renormalization does not include give rise to an extra equation at the fixed point but does *alter* the beta functions for all the other couplings. There are three reasons why we wish to include this in our ansatz:

1. According to the setup in section 2, each action monomial should come with a coupling. Minimally coupling the kinetic term is thus an approximation and it is expected that the inclusion of the wave-function renormalization will improve the quality of the approximation. Moreover the anomalous dimensions will also feature in the coefficient B^{ferm} from equation (158), and thus might have an impact on previously obtained results.

2. The anomalous dimension η_ψ allows to test the validity of a NGFP. As explained in [6] when including a non-minimal kinetic term the regulator would vanish in the UV if $\eta_\psi > 1$. This violates one of the important requirements the regulator has to satisfy and if this condition is not met the computation is no longer reliable.

3. It enables us to make contact with works employing a vertex expansion on a flat background expansion [7, 50]. There the wave function renormalization was also included and one can only compare results properly if this would be done for both methods.

For this computation we stick to the same method, in particular we still use the fermion background field (140). One can then easily determine the background action monomial which carries the information about the running of Z_k. It is given by

$$\int d^4x\sqrt{g}\,\bar{\theta}\Big(-\sqrt{\frac{R}{3}}\Big)\,\theta. \tag{168}$$

The strategy for the beta functions is the exact same as determining the beta function for α, except that now one has to project the diagrams in figure 1 onto (168) . This actually simplifies calculations significantly since now all operators commute and $\mathcal{O}(R)$ terms can be ignored. One additional replacement that needs to be made however is that now the regulator is given by

$$\mathcal{R}_k^\psi = Z_k\,k\,\gamma^5\,\Big(1-\sqrt{\Box}/k\Big)\;\Theta(1-\sqrt{\Box}/k), \tag{169}$$

thus generating an extra term in the expression for $\partial_t \mathcal{R}_k^\psi$. Including this the full set of (implicit) beta functions for this complete truncation are given by

$$
\begin{aligned}
\beta_g &= (2+\eta_N)g \\
\beta_\lambda &= -(2-\eta_N)\lambda + \frac{g}{8\pi}\Big(\frac{20}{1-2\lambda} - \frac{10}{3}\eta_N\frac{1}{1-2\lambda} - 16\Big) - N_f\,\big(\frac{1}{3} + (\frac{2}{3} - \frac{\pi}{4})\eta_\psi\big) \\
\beta_\alpha &= A + (A_1 + 1 + \eta_\psi)\,\alpha + A_2\,\alpha^2 + A_3\,\alpha^3 \\
\eta_\psi &= C_0 + C_1\,\alpha + C_2\,\alpha^2
\end{aligned} \tag{170}
$$

Here we have added the implicit expression for the fermion anomalous dimension. Newtons anomalous dimension again takes the form of (153), where the gravitational coefficients are not modified and again given by (154). The coefficient B^{ferm} recieves extra contributions and now reads

$$
\begin{aligned}
B^{\text{ferm}} = -\frac{N_f}{12\pi}\Big[&\big(24 + (72-24\pi)\eta_\psi\big)\alpha_k + 4 - 2\pi + (-2+\pi)\eta_\psi \\
&+ \big(24 - 12\pi + (-48+18\pi)\eta_\psi\big)\beta\Big].
\end{aligned} \tag{171}
$$

The coefficients in the beta function for α_k are split in the form

$$
A_i = \frac{g_k}{\pi}\Big(\frac{A_i^1 + \tilde{A}_i^1\eta_\psi}{(1-2\lambda_k)} + \frac{A_i^2 + \tilde{A}_i^2\eta_\psi}{(1-2\lambda_k)^2} + \frac{A_i^3}{(1-2\lambda_k)^3}\Big), \tag{172}
$$

with the coefficients of the minimally coupled case unaltered, and listed in (165). The new coefficients multiplying the anomalous dimension are given by

$$
\begin{aligned}
\tilde{A}_0^1 &= -\frac{9}{32} + \frac{3\pi}{32} & \tilde{A}_1^1 &= \frac{21}{16} - \frac{7\pi}{16} + (-\frac{17}{4} + \frac{45\pi}{32})\beta \\
\tilde{A}_0^2 &= -\frac{5}{32} + \frac{3\pi}{64} & \tilde{A}_1^2 &= \frac{1}{12} - \frac{\pi}{32} \\
\tilde{A}_2^1 &= \frac{83}{15} - \frac{7\pi}{4} + (-\frac{219}{20} + \frac{27\pi}{8})\beta & \tilde{A}_3^1 &= \frac{163}{10} - \frac{21\pi}{4} \\
\tilde{A}_2^2 &= \frac{49}{60} - \frac{\pi}{4} & \tilde{A}_3^2 &= 0.
\end{aligned} \tag{173}
$$

Finally, for the coefficients C appearing in the expression for the fermion anomalous dimension we have

$$
C_i = \frac{g_k}{\pi}\Big(\frac{C_i^1 + \tilde{C}_i^1\eta_\psi}{1-2\lambda_k} + \frac{C_i^2}{(1-2\lambda_k)^2}\Big), \tag{174}
$$

where

$$
\begin{aligned}
C_0^1 &= \frac{3}{4} - \frac{9\pi}{32} & C_1^1 &= \frac{21}{4} - \frac{45\pi}{32} \\
C_0^2 &= -\frac{75}{16} - \frac{9\pi}{64} + (\frac{23}{40} + \frac{3\pi}{32})\eta_N & C_1^2 &= \frac{21}{10} - \frac{9\pi}{16} + (-\frac{277}{280} + \frac{9\pi}{32})\eta_N \\
C_2^1 &= -\frac{39}{10} + \frac{9\pi}{8} & C_2^2 &= -\frac{23}{10} + \frac{3\pi}{8} + (\frac{41}{140} - \frac{3\pi}{64})\eta_N,
\end{aligned} \tag{175}
$$

and

$$
\tilde{C}_0^1 = -\frac{27}{32} + \frac{9\pi}{32} \qquad \tilde{C}_1^1 = -\frac{15}{8} + \frac{9\pi}{16} \qquad \tilde{C}_2^1 = -\frac{1}{2} + \frac{3\pi}{16}\,. \tag{176}
$$

A preliminary study of the fixed point structure entailed by (170) reveals that for the fixed point structure it appears tat $\eta_\psi < 1$ for all values of N_f, implying that the results obtained are indeed reliable. A complete analysis will be presented in [48].

5 Asymptotic Safety Beyond Flow Equations

The main topic of this thesis is the inclusion of matter into the asymptotic safety program - focusing on fermions in particular. Although this section is heavily oriented around matter, it is very much a stand alone topic. The goal of this section is to put forward possible new research ideas for the asymptotic safety program. Here the focus will not be on the flow of couplings but on the structure of fixed points. The inclusion of matter will play a central and fundamental role. Technical details will mainly be omitted as the emphasis is on outlining ideas.

As pointed out in the introduction, it seems highly plausible that theories of quantum gravity could constrain the admissible matter sectors. Here this argument will be turned upside down: A proper matter theory could constrain the dynamics on the gravitational side as well. Looking back, this happened before when motivated by the Maxwell equations, Einstein in his theory of special relativity concluded that 'spacetime' carries a geometric structure and must come with a Lorentzian metric. (Although at the time he did not call it this.) Later on, dynamics for this Lorentzian metric was provided by his theory of general relativity. From a modern perspective one could certainly conclude that the observational theory of electromagnetism dictated that spacetime should be thought of in this way. One might then wonder, how matter theories in general precisely constrain gravitational dynamics. It turns out, that under very conservative assumptions, ensuring the classical matter theory is well-behaved, the gravitational dynamics are severely constrained, as worked out by what is called 'constructive gravity' [52]. In fact, as constructive gravity claims, if one takes any form of standard model matter, it can be shown that the unique gravitational theory underlying the geometry of spacetime, is given by the Einstein-Hilbert action. Thus, the gravitational dynamics can be *derived* by the matter content of the standard model (the assumptions going into this will be discussed below). This is important for two reasons. First, in our current understanding the matter content of the standard model as well as the gravitational action are postulated separately. Of course one does not want to postulate several mutually contradicting statements. Also one wants to avoid making unnecessary postulates. Secondly, understanding such a relation between matter and the gravitational dynamics can massively impact our understanding of gravity. Especially when confronted with the current understanding of the matter-energy content of the universe. Given that only about five percent seems to be of standard model origin, it hardly seems far-fetched to propose the existence of other matter types, that might alter the gravitational theory to render the matter theory well-behaved. We will come back to this point later. In the next section we will outline the ideas and assumptions behind constructive gravity. In section 5.2 a possible connection with asymptotic safety will be discussed.

5.1 Gravitational Dynamics from Matter

A natural starting point for is to specify what is meant by 'well-behaved' classical matter theories. Here we follow the lines of [53,54]. In general, there are many requirements one might want to impose on the matter theory at hand. This approach however is very conservative and only makes requirements that are widely accepted. These requirements go under the name of *predictivity* and *quantizibility*. Both of these are requirements on the classical matter theory. A matter model needs to predictive in the sense that initial conditions determine the values of the field after the system evolves. Physics, after all, is about predicting the future. Amongst the physics community it is also believed that matter fundamentally is of quantum nature. This is where the second condition comes in. It is by no means implied that the matter theory needs to be quantized already, it merely should be quantizable. That is, there should be an observer-independent notion of positive energy. Again for a more detailed definition see [53]. Thus one starts with a matter theory in an arbitrary geometry, whose dynamics is governed by an as of yet undetermined gravitational Lagrangian and demands that this matter theory is both predictive and quantizable.

After the requirements have been made, one can then use these to set up a system of linear partial differential equations. The solution to the equations is the gravitational Lagrangian. For example, if one starts with Maxwell matter, using a constructive algorithm one can then find the system of partial differential equations. In this case these equations can be solved uniquely and the solution is the Einstein-Hilbert action with undetermined Newton and cosmological constant. Both of these appear as integration constants. The only physical input being the predictivity and quantizibility of the matter theory - the rest is pure mathematics.

Showing how these assumptions fix the kinematics requires advanced mathematical machinery including real algebraic geometry, convex analysis, and partial differential equation theory. What makes this approach practical, is that this machinery is only needed once. Furthermore in [55], a recipe is developed to go from the matter theory to the gravitational Lagrangian. The application of this recipe only requires the mathematics taught in a typical course on general relativity. We will very briefly outline this step by step recipe, and now turn our attention to the direct consequences on the two assumptions described above.

As explained above the starting point is a matter action $S_{\text{matter}}[\Phi, G]$. Here G encodes the unspecified geometry and Φ denotes the matter fields. From this matter action one can obtain the *principal polynomial*. It is precisely this object that encodes all of the causal structure. Taking the variation of $S_{\text{matter}}[\Phi, G]$ with respect to Φ yields something of the form[11]

$$\frac{\delta S_{\text{matter}}}{\delta \Phi} = \sum_{n=0}^{N} Q[G, \Phi]^{\mu_1, \dots \mu_n}(x)\partial_{\mu_1} \dots \partial_{\mu_n}\Phi \tag{177}$$

The indices μ_i run over the dimension of the manifold. It is worth to emphasize that the ∂_{μ_i} really are partial derivatives, not covariant ones. The reason for this is that in more complicated geometries it might be necessary to generalize the concept of a connection. This is left completely open. Also, the covariant derivatives of course split into partial derivatives and connection pieces. The additional connection terms can then be absorbed into the lower order Q coefficients. Note also that these Q coefficients are allowed to depend on the matter fields. That is, the matter theory need not be linear. Causality is encoded into the highest order coefficient, $Q[G, \Phi]^{\mu_1, \dots, \mu_N}$. It therefore should be no surprise that this is the only coefficient that enters into the principal polynomial. To construct this polynomial $P(x, k)$ one should fully contract the indices with a covector k_μ:

$$P(x, k) \coloneqq \det \left[Q[G, \Phi]^{\mu_1, \dots, \mu_N}(x) k_{\mu_1} \dots k_{\mu_N} \right], \tag{178}$$

here the determinant is taken over the open indices originating from the field Φ. For example if the field Φ is a spinor, the coefficients Q are matrices in spinor space and one is instructed to take the determinant in spinor space. This leads to a polynomial in k. From this one can also construct the dual polynomial (we will specify how later). It turns out, that imposing that the matter theory be predictive, implies that the principal polynomial should be hyperbolic.[12] Next to this, the condition that the matter theory be quantizable implies that the dual polynomial should also be hyperbolic. The former result is a familiar result from standard PDE theory [56]. The second result is rather new and is nontrivial in the sense that the hyperbolicity of the dual polynomial by no means follows from the hyperbolicity of the original one. This bi-hyperbolicity condition is sufficient to severely, if not fully, constrain the gravitational dynamics. We will now turn our attention to the step by step recipe, only outlining the computations needed to make everything precise. For this we work along the lines of [55].

1. **Step 1. Specify test matter dynamics:** Provide a *classical* description of a matter field Φ on a smooth manifold M. Irrespective of the chosen matter dynamics, type of matter, or geometry, coordinate invariance is guaranteed if the matter field equations are derived from a scalar functional $S_{\text{matter}}[G, \Phi]$.

   An example would be the Maxwell action

$$S_{\text{Maxwell}}[A, g] = -\frac{1}{4} \int d^4x \det(g)^{1/2} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta}, \tag{179}$$

   where $F_{\mu\nu}$ is the familiar field strength tensor constructed from the covector field A_μ. Here g is a non-degenerate $(2, 0)$ tensor field. It is referred to as the 'geometry' but as of yet plays no role other than that it specifies the coefficients in the equation of motion for the matter field.

[11] Here we restrict ourselves to local matter theories only.
[12] This property is made precise in equation (187)

2. **Step 2. Calculate the principal polynomial:** Determine the principal polynomial P defined above. Repeated factors in the polynomial should be ignored. If the theory contains a gauge symmetry one should fix this symmetry by imposing an explicit gauge condition or work with gauge independent variables.

   Taking the variation with respect to the matter field A_μ in above example after having gauge fixed, and furthermore taking the determinant over the open indices yields the principal polynomial

$$P(k) = g^{\mu\nu} k_\mu k_\nu. \tag{180}$$

3. **Step 3. Construct the dual polynomial:** Let $P_1, ..., P_f$ be the distinct irreducible factors with $P = P_1 \dots P_f$ and consider for each P_i the map DP_i that maps a covector k, with $P_i(k) = 0$ to a vector field with components

$$\big(DP_i(k)\big)^\nu := (\deg P_i)\, P_i^{\nu\,\mu_2 \dots \mu_{\deg P_i}} k_{\mu_2} \dots k_{\mu_{\deg P_i}}. \tag{181}$$

   The dual polynomial to P_i is then given by $P_i^\#$, which is the map that takes a vector field to a real number satisfying

$$P_i^\#\big(DP_i(k)\big) = 0 \quad \forall k: \;\; P_i(k) = 0. \tag{182}$$

   The dual polynomial $P^\#$ of P is then defined as the product of the irreducible ones

$$P^\#(X) := P_1^\#(X) \dots P_f^\#(X), \tag{183}$$

   for any vector field X. This dual polynomial always exists.

   For Maxwell matter the principal polynomial already is irreducible, so the only map of interest is

$$\big(DP(k)\big)^\nu = 2g^{\nu\mu} k_\mu. \tag{184}$$

   This means that the dual polynomial is given by

$$P^\#(X) = g_{\alpha\beta} X^\alpha X^\beta, \tag{185}$$

   since this satisfies the duality requirement for any k with $P(k) = 0$,

$$P^\#\big(DP(k)\big) = 4 g_{\alpha\beta} g^{\alpha\mu} g^{\beta\nu} k_\mu k_\nu = 4 g^{\mu\nu} k_\mu k_\nu = 4P(k) = 0. \tag{186}$$

4. **Step 4. Bi-hyperbolicity:** A necessary condition for the matter theory to be predictive and quantizable is the bi-hyperbolicity property. That the principal polynomial should be hyperbolic amounts to the algebraic condition that there exists a covector field h such that (i) the polynomial $P(h)$ is an everywhere non-vanishing function and (ii) for every covector field q the equation

$$P(h + \lambda q) = 0 \tag{187}$$

   admits only real-valued functions λ as solutions. Similarly, for the dual polynomial $P^\#$ with the covectors h, q replaced by vectors H, Q. These conditions impose algebraic conditions on the geometry G. This is not the only place where the bi-hyperbolicity condition comes into play although in the recipe it will only feature here.

   For the principal polynomial of Maxwell matter one can conclude that the signature should be Lorentzian (in these conventions mostly minus). Suppose that the polynomial $P(k) = g^{\mu\nu} k_\mu k_\nu$ indeed is hyperbolic, then by condition (i) there exists a covector h with $P(h) > 0$. For a covector q we then work out

$$\begin{aligned} P(h + \lambda q) &= g^{\mu\nu}(h_\mu + \lambda q_\mu)(h_\nu + \lambda q_\nu) \\ &= \lambda^2 g^{\mu\nu} q_\mu q_\nu + 2\lambda g^{\mu\nu} h_\mu q_\nu + g^{\mu\nu} h_\mu h_\nu \end{aligned} \tag{188}$$

Condition (ii) then dictates that this can only have real roots λ. This implies that the discriminant be positive,

$$\left(2g^{\mu\nu}h_\mu q_\nu\right)^2 - 4\left(g^{\mu\nu}h_\mu h_\nu\right)\left(g^{\alpha\beta}q_\alpha q_\beta\right) > 0. \tag{189}$$

One can choose a cotangent basis ϵ^0, ϵ^a with $\epsilon^0 := h$, and ϵ^a chosen such that for any $a = 1, 2, 3$,

$$g^{\mu\nu}\epsilon^0_\mu \epsilon^a_\nu = 0. \tag{190}$$

Thus, writing $q = A\epsilon^0 + B_a\epsilon^a$, the discriminant condition (189) reduces to

$$0 < 4A^2 g^{\mu\nu}h_\mu h_\nu - 4A^2 g^{\mu\nu}h_\mu h_\nu - 4B_a B_b g^{\mu\nu}\epsilon^a_\mu \epsilon^b_\nu = -4B_a B_b g^{\mu\nu}\epsilon^a_\mu \epsilon^b_\nu. \tag{191}$$

From the condition $B_a B_b g^{\mu\nu}\epsilon^a_\mu \epsilon^b_\nu > 0$ for any set of coefficients B_a one can conclude that the signature of $g^{\mu\nu}$ should be (mostly minus) Lorentzian. Conversely, one can show that a Lorentzian signature also satisfies condition (i). Hyperbolicity of the dual polynomial is automatically satisfied, since the metric has the same signature as its inverse.

The remaining steps are only quickly mentioned, avoiding technical complexity.

5. **Step 5. Determine geometric degrees of freedom.** Here one makes use of an in-explicit initial data surface in order to make use of the Hamiltonian formalism, deformation operators and Poisson brackets. Although all of these are not needed explicitly, one needs to abstractly provide some input on the geometry on the initial data surface. This will feature in the coefficients that need to be determined.

6. **Step 6. Calculate coefficients of the master equations.** Here one uses the geometry of the data surfaces from the previous step and together with the form of the principal polynomial computes the coefficients featuring in the master equations.

7. **Step 7. Write down the master equations.** The coefficients from step six completely determine the master equations. These equations are for scalar and tensorial coefficients that appear in the gravitational Lagrangian. The master equations are six partial differential equations together with five infinite sequences of equations. The solution yields the scalar and tensorial coefficients needed to construct the gravitational Lagrangian. For their explicit form see [54].

8. **Step 8. Supplement master equations by covariance equations.** Although no additional (physical nor mathematical) input is needed, one can derive so-called covariance equations to simplify the master equations. For an explicit example see appendix A of [54].

9. **Step 9. Solve the master equations.** Although solving the equations can be rather complicated, given that there are infinitely many - it can be done, at least for some matter models. Indeed for Maxwell matter the equations can be solved, where the unique solution is indeed given by the Einstein-Hilbert action. Also a second case study was considered for illustration purposes and here something different than general relativity came out. Both examples show that finding solutions to the master equations is possible.

The method described above is termed 'constructive gravity'. Indeed here it is the matter that is more fundamental. Requiring that this matter theory is predictive and quantizable then yields the classical gravitational dynamics. For any type of standard model matter general relativity comes out. This can be seen straightforwardly since for any standard model matter the principal polynomial is given by the same one as that of Maxwell matter. It is through the principal polynomial, and only the principal polynomial that the gravitational dynamics are determined. Other gravitational theories thus must carry matter theories with a different principal polynomial than that of Maxwell theory. For instance the generalization (157) discussed in this thesis would not alter anything since it is only the highest order coefficient in (177) that contributes to the principal polynomial. Notably our setup was in a Euclidean setting, which should have been left undetermined according to constructive gravity. However if some terms would give rise to a different polynomial the causal structure, presently known in the form of familiar lightcones, might radically change.

Other than a good consistency test of the theories obtained so far, this line of research has various other applications. For instance if, for phenomenological or theoretical reasons, new matter with a different principal polynomial is found, it is precisely this method that can constructively determine whether or not general relativity should be modified, and if so, in what way. A possible candidate for to be discovered matter is of course dark matter. If this matter were to have a different principal polynomial than that of standard model matter, it would generally give rise to something different than GR. Also constructive gravity turns various relevant physical questions and translates them into mathematical questions. For example the question 'is there a gravitational theory underlying spacetime?' is reduced to the question of whether a solution to the master equations exists. 'Is there one or are there many?' translates into the question of uniqueness on the space of solutions to the master equations. Finally, and most interestingly 'What is the gravitational theory?' is a matter of actually solving the master equations. It is thus through matter that these big questions should be investigated. An advantage of this is that generally matter is much closer to observational physics than gravity. Lastly, one could imagine some physical processes might be observed that would require modifications to the underlying matter theory. Indeed, these modifications could transpire to general relativity. A possible example is the vacuum split of lightrays described by birefringent quantum electrodynamics, that if observed would immediately mean general relativity could not be the correct classical description. Of course it must still serve as a good approximation. This possibility is worked out in [57], and directly relates to observational physics.

5.2 Asymptotic Safety meets Constructive Gravity

This section explores possible connections between asymptotic safety and constructive gravity. The above discussion sketched the main idea. As a consequence this section is only intended to ask some questions and inspire some potentially interesting research proposals.

Given the fairly new insights in the relation between matter and gravity offered by constructive gravity it is natural to ask if this has any consequences for quantum theories of gravity. For this it is important to keep in mind that constructive gravity makes demands on the classical matter theory only. Any quantum gravity theory, through some sort of limit, must give rise to a classical theory viable in situations where quantum fluctuations can be ignored. Although the classical theories of nature we presently have must arise (or at least be compatible with) the underlying quantum gravity theory, it is conceivable that a quantum theory of gravity will modify them. The results of these modifications must then be compatible with observational constraints. Possible examples of this are modified gravity (in this context gravitational theories with a different Lagrangian than Einstein-Hilbert) as well as a description of dark matter from quantum gravity [58]. In general it is a difficult theoretical task to show what classical theory in the end emerges.

For quantum field theories this task on the other hand is straightforward. It is for this reason that the relation between standard model matter and general relativity could be made in the first place. Since the asymptotic safety program adheres to the quantum field theoretic description that a relation to constructive gravity could be most fruitful. The first question that comes to mind is whether or not the asymptotic safety program is compatible with the assumptions made in constructive gravity. As explained in section 2.1.3, asymptotic safety takes into account all possible action monomials, parametrized by cutoff scale dependent couplings. Through the asymptotic safety mechanism some of these couplings have to be measured whereas others are predicted by the theory. This is true on both the matter as well as the gravitational side. From the resulting action one could then apply the constructive gravity algorithm and see what terms are allowed in the gravitational Lagrangian. If the asymptotic safety program predicts a term with a nonzero coupling not allowed by constructive gravity that would mean that the resulting theory is either non-predictive - thus not even classically acceptable - or not quantizable (for asymptotic safety the quantization method proceeds through fixed point and RG trajectory in a hidden way). This could thus provide a theoretical consistency test. Conversely, it might be possible to constrain theory space further as to guarantee the outcome be predictive and quantizable. This perspective could offer some crucial insights into more properly defining theory space, and as a result truncation schemes. Possibly one could then use constructive gravity to systematically build truncations with a small number of running couplings and gravitational couplings taking values given by the 'constructive gravity map'.

On the other hand, if the couplings in front of the forbidden action monomials are relevant the assumptions of predictivity and quantizibility would fix this coupling - thus enhancing the predictive power even further. Also if there exist multiple fixed points but only a subset of them give rise to forbidden terms with relevant couplings this approach would limit the viable fixed points to precisely this subset. A crucial assumption going into this line of reasoning is that the action is local. For this reason it might be that this would feature most prominently within truncation schemes only. This principle could therefore be a possible guide for constructing truncation schemes. Suppose one wants to investigate a possible extension of the standard model, where the extensions give rise to a modification of the Einstein-Hilbert action. It might be insightful to include these modifications in the truncation ansatz. This could lead to a "smart truncation building principle" similar to the one employed with the curvature-fermion coupling identified in section 4.

A current potential drawback of this, is that presently most of the asymptotic safety research is conducted in Euclidean signature. As became apparent in step four of the recipe outlined in the previous section, the hyperbolicity condition dictated the signature of the metric, in the case of standard model matter. This suggests that the asymptotic safety program first needs to become operative on Lorentzian spacetimes. Still, one could already now gain more insights into methods of constructive gravity. It might also be possible to relax certain conditions to be able to leave the signature arbitrary. Questions such as 'is there any information from the asymptotic safety perspective one could get out of knowing my truncation satisfied the master equations?' and 'What is the role of higher-order kinetic terms present in theory space in this regard' need answering, and one might already be able to make progress with this without specifying the signature. For the latter question, keep in mind that only the highest order coefficient in (177) enters into the construction of the principal polynomial. It only this object that goes into the derivation of the gravitational Lagrangian. For a finite derivative expansion, there is no problem identifying this term. In an infinite derivative expansion this is not the case[13]. With this in mind, understanding higher order kinetic terms seems a crucial ingredient.

In summary, taking over the general philosophy 'Matter first, gravity second' might provide novel insights in the asymptotic safety program. Combining asymptotic safety with constructive gravity might have the following advantages: It may

- identify some properties of high-energy part of the fixed point action independently of RG arguments ;
- lead to a more constrained and better understood theory space;
- provide a guiding principle for truncation schemes;
- give rise to a theoretical consistency test.

A better understanding of constructive gravity could shed light on these questions as well as provide other insights into building a consistent UV complete quantum theory of the fundamental interactions. It could provide a crucial piece to understand the relation between matter and gravity - a theory of nature unifying the standard model of particle physics and general relativity in one conceptual framework.

6 Summary and Outlook

We started out this thesis by giving a brief introduction to asymptotic safety, focusing on the key elements of the program including the effective average action, renormalization group trajectories and interacting renormalization group fixed points (NGFPs). At the core of the asymptotic safety mechanism is a fixed point rendering the theory safe from divergencies in the UV. This fixed point needs to be viable in the sense that it has to give rise to a low-energy phenomenology compatible with

[13] it is conceivable that the higher-derivative terms found within the derivative expansion actually conspire to combine into an entire function. This idea is at the heart of the form factor program [27]. Finding the principal polynomial for scenarios encountered in [62, 63] would then require a significant extension of the constructive gravity program.

observations. This entails in particular that it should be situated at a positive value for Newton's coupling to accommodate the attractive nature of gravity.

This thesis stressed the importance of including matter into the program. From a phenomenological viewpoint the inclusion of matter gives rise to new classes of observables which could be compared to experimental data. We argued that the smallest truncation possibly capturing the essential physics would be the standard model of particle physics supplemented by general relativity. Working out the fixed point structure in this setting constitutes a major milestone in including matter into the asymptotic safety program.

At the heart of the standard model are Dirac fermions. We briefly discussed fermions in Minkowski space before generalizing the formalism to curved spacetimes. Motivated by the structure of the Wetterich equation, we adopted Euclidean signature as well. We found that it was crucial to work a consistent set of conventions, before starting the actual calculations. This turned out to be a highly nontrivial task since many auxiliary identities had to be checked and incorporated into the framework.

Implementing a systematic refinement of the approximation made for the effective average action, we started out the inclusion of fermions into asymptotic safety by minimally coupling them. The resulting fixed point structure comprises a one-parameter family of NGFPs emanating from the gravitational fixed point when N_f is taken small. It turned out that the validity of fixed points, depends on the unphysical choice of coarse-graining operator [9]. The goal of this thesis was to investigate this problem and identify possible solutions.

Since in the exact computation such regulator dependence should not occur, we adopted the strategy of systematically improving the approximation for the effective average action, seeking out new terms that contribute to the beta functions of Newton's coupling. Using this principle we identified a canonical interaction term coupling the fermion bilinears to the Ricci curvature. Indeed we showed that this term contributed to the flow of Newtons coupling. Verifying whether this term indeed stabilizes the fixed point structure against a change of regulator required the computation of the beta function for the new interaction coupling. This required the inclusion of fermions on a curved background spacetime. We then for the first time, projected the flow onto the fermion action in a curved background, explaining our methods and strategies along the way. This calculation serves as a benchmark for other calculations involving fermions.

We found that the new coupling is crucial for discriminating two different families of non-Gaussian fixed points. Both of them exist for arbitrary N_f. the first family exhibits a behavior very similar to the one found in the minimally coupled case. The other family however was almost completely independent on the choice for coarse graining operator. In particular, the fixed point value of Newton's coupling is positive for all choices of the coarse graining procedure for any number of fermions. It is this fixed point that is a candidate to UV complete the standard model of particle physics plus gravity. We also included the fermion wavefunction renormalization. Preliminary results show that this refinement has only minor influence on the fixed point structure of the system.

At this stage the following structural remarks are in order. Throughout this work we considered massless fermions only. This is well justified when considering physics at trans-Planckian scales. Secondly, our research identified a second term which contributes to the flow of Newton's coupling. This term is given by a fermion-kinetic term involving two powers of the Dirac operator

$$\int d^4x\sqrt{g}\bar{\psi}\,\not{\nabla}^2\,\psi. \tag{192}$$

Besides the interaction term (157) included in this thesis this is the *only* other term contributing on a spherical background. With our current methods, the scale-dependence associated with this term can not be distinguished from the running of the $R\bar{\psi}\psi$-coupling, since for our background spinor θ we have $\not{\nabla}^2\theta \propto R\theta$, meaning the background action monomials are identical. Still, one could add this term to the truncation ansatz instead and see what its consequences are. If this term on its own also fixes the problem, then it is highly likely that the combination of the two terms would also do so. The same is true if the contribution of (192) gives a negligible effect to the running of Newton's coupling.

Finally, if indeed the interaction term remains a crucial piece when introducing additional fields, it is interesting to look at some phenomenological consequences. The first one that comes to mind are spectral lines in the early universe. The interaction term resembles a mass term, with the mass set by the Ricci curvature. This is particularly helpful given that the frequencies for spectral lines depend on the mass of, in this case, the electron. With the replacement

$$m \mapsto m + \bar{\alpha} R, \tag{193}$$

the interaction term would give rise to an observational effect, at least in principle, if one compares spectral lines in different backgrounds, distinguished by their value of the Ricci scalar. In the early universe there are various stages where the Ricci curvature differs from its present-day value. A good example would be the matter dominated stage. Although indeed the value of the Ricci scalar then is very small, it would still be interesting to study the resulting experimental bounds on the value of the $R\bar{\psi}\psi$ coupling and compare them to the results obtained from solving the approximate renormalization group equations.

Finally, we closed off the thesis by going into a slightly different route. We provided some suggestions to incorporate the line of reasoning of constructive gravity into asymptotic safety. In constructive gravity the matter comes first. Then, through some basic requirements heavy constraints on the gravitational dynamics are derived from physical principles. These basic requirements are at the level of the classical theory. This possible line of future research offered a further perspective, and additional motivation on the inclusion of matter into the asymptotic safety program. Notably, it could provide new guidelines for devising efficient approximation schemes for solving Wetterich's equation.

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Appendix A Variations

Similarly to step two of the computation in the Einstein-Hilbert truncation (section 2.2), the computation of the extended system (159), will also start by expanding the EAA. Here, we will focus on the fermionic part and refer to the Einstein-Hilbert truncation for details on the gravitational sector. To start with, we need the first and second order variations of both γ^μ as well as ∇_μ. Since the Hessian only instructs to take two functional variations we will ignore all terms of third order or higher in the fluctuation fields.

For a general object A, depending on fields f_i (where the i could refer to any number and any type of indices) we have

$$\delta^2 A = \frac{\partial^2 A}{\partial f_i \partial f_j}\delta f_i \delta f_j. \tag{194}$$

Thus, if $f_i = \bar{f}_i + \delta f_i$ then we have

$$A = \bar{A} + \delta A + \frac{1}{2}\delta^2 A. \tag{195}$$

Returning to the case at hand we write the background and fluctuation fields by means of

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad \bar{\psi} = \bar{\theta} + \bar{\chi}, \quad \psi = \theta + \chi, \tag{196}$$

and split the $h_{\mu\nu}$ field into a trace and trace-less part

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \frac{1}{4}\bar{g}_{\mu\nu}\, h, \quad h = \bar{g}^{\mu\nu} h_{\mu\nu}. \tag{197}$$

For the inverse metric we have

$$\bar{g}^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + h^{\mu\sigma} h_{\sigma\rho}\bar{g}^{\rho\nu}, \tag{198}$$

where indices are raised and lowered by $\bar{g}$. For the gamma matrices we then have to make sure they satisfy the anti-commutation relations

$$\left\{\bar{\gamma}^\mu + \delta\gamma^\mu + \frac{1}{2}\delta^2\gamma^\mu\, ,\, \bar{\gamma}^\nu + \delta\gamma^\nu + \frac{1}{2}\delta^2\gamma^\nu\right\} = 2\, g^{\mu\nu}\mathbb{1}_\gamma, \tag{199}$$

Working this out for all orders of h then yields

$$\delta\gamma^\mu = -\frac{1}{2}h^{\mu\nu}\gamma_\nu, \quad \delta^2\gamma^\mu = \frac{3}{8}\gamma^\alpha h_{\beta\alpha}h^{\mu\beta}. \tag{200}$$

To obtain the variations of the spin covariant derivative a similar procedure can be used, but then using the commutation relations for ∇_μ. This is a much more extensive calculation, with the result being

$$\begin{aligned} \delta\nabla_\mu &= \frac{1}{8}\left[\gamma^\alpha, \gamma^\beta\right] D_\beta h_{\mu\alpha}, \\ \delta^2\nabla_\mu &= \frac{1}{8}\left[\gamma^\alpha, \gamma^\beta\right]\left(h^\lambda_{\ \alpha} D_\beta h_{\mu\lambda} + h^\lambda_{\ \beta} D_\lambda h_{\mu\alpha} + \frac{1}{2}h^\lambda_{\ \alpha} D_\mu h_{\beta\lambda}\right). \end{aligned} \tag{201}$$

For completeness we also reproduce the variations

$$\begin{aligned} \delta\sqrt{g} &= \frac{1}{2}\sqrt{\bar{g}}\bar{g}^{\mu\nu}h_{\mu\nu} \\ \delta R &= -\bar{R}^{\mu\nu}h_{\mu\nu} + \bar{D}_\beta\bar{D}_\alpha h^{\alpha\beta} - \bar{D}^2 h^\alpha_{\ \alpha} \end{aligned} \tag{202}$$

and

$$\begin{aligned} \delta^2\sqrt{g} &= \frac{1}{2}\sqrt{\bar{g}}\big(\frac{1}{2}\bar{g}^{\mu\nu}\bar{g}^{\rho\sigma}h_{\mu\nu}h_{\rho\sigma} - h^{\mu\nu}h_{\mu\nu}\big) \\ \delta^2 R \ &= \bar{R}_{\beta\mu}h^{\beta\gamma}h_\gamma^{\ \mu} - \bar{R}_{\alpha\beta\gamma\rho}h^{\beta\gamma}h^{\alpha\rho} - 3h^{\beta\gamma}\bar{D}_\gamma\bar{D}_\alpha h^\alpha_{\ \beta} + 2h^{\beta\gamma}\bar{D}_\beta\bar{D}_\gamma h^\alpha_{\ \alpha} \\ &\quad + 2h_{\beta\gamma}\bar{D}^2 h^{\beta\gamma} - h^{\beta\gamma}\bar{D}_\alpha\bar{D}_\beta h^\alpha_{\ \gamma} - \big(\bar{D}_\alpha h^{\beta\gamma}\big)\big(\bar{D}_\beta h^\alpha_{\ \gamma}\big) \\ &\quad + \frac{3}{2}\big(\bar{D}_\lambda h_{\beta\gamma}\big)\big(\bar{D}^\lambda h^{\beta\gamma} - 2\big(\bar{D}_\gamma h^{\beta\gamma}\big)\big(\bar{D}_\alpha h^\alpha_{\ \beta}\big) + 2\big(\bar{D}_\beta h^{\beta\gamma}\big)\big(\bar{D}_\gamma h^\alpha_{\ \alpha}\big) \\ &\quad - \frac{1}{2}\big(D_\lambda h^\gamma_{\ \gamma}\big)\big(\bar{D}^\lambda h^\alpha_{\ \alpha}\big). \end{aligned} \tag{203}$$

These expressions can be simplified by specifying the background $\bar{g}$ to be that of a maximally symmetric space, particularly a four-sphere (see (66)). For θ we take the background field discussed in section 3.2.4.

We also have to take variations of the EAA with respect to the fermion fluctuation fields. This is simpler in the sense that only the field ψ depends on the fluctuations. For this we use

$$\frac{\delta \psi^i(x)}{\delta \chi^j(y)} = \delta^i_j\, \delta(x-y), \quad \frac{\delta \bar{\psi}_i(x)}{\delta \bar{\chi}_j(y)} = \delta_i^j\, \delta(x-y), \tag{204}$$

and all the other variations are zero, meaning we treat ψ and $\bar{\psi}$ as independent fields. Introducing a multiplet Υ as

$$\Upsilon = \{\hat{h}_{\mu\nu}, h, \chi^a, \bar{\chi}_a\}, \tag{205}$$

the Hessian $\Gamma_k^{(2)}$, is defined as

$$\Gamma_k^{(2)} = \frac{\overrightarrow{\delta}}{\delta \Upsilon}\, \Gamma_k\, \frac{\overleftarrow{\delta}}{\delta \Upsilon^T}. \tag{206}$$

The variations from the right are taken with respect to the transpose in spinor space. We will denote the entries of this matrix in field space by

$$\Gamma_{f_i f_j} \coloneqq \frac{\overrightarrow{\delta}}{\delta f_i}\, \Gamma_k\, \frac{\overleftarrow{\delta}}{\delta f_j^T}. \tag{207}$$

Carefully keeping track of transposes and the Grassmann nature of the fermions, by employing the identity $\left(\bar{\psi} A \psi\right)^T = -\psi^T A^T \psi^\dagger$, the off-diagonal (in the sense of boson-fermion and fermion-boson variation) entries of the Hessian for $\Gamma_k^{\text{ferm full}}$ are given below

$$\begin{aligned}
\Gamma_{\hat{h}\chi} &= \frac{i}{4}\, \overleftarrow{D}^\mu\, \bar{\theta}\gamma^\nu + \bar{\alpha}_k \overleftarrow{D}^\mu \overleftarrow{D}^\nu\, \bar{\theta}\gamma^5 \\
\Gamma_{h\chi} &= -\frac{3i}{16}\, \overleftarrow{D}^\mu\, \bar{\theta}\gamma_\mu + \frac{\bar{\alpha}_k}{4}\, (3\Delta + R)\, \bar{\theta}\gamma^5 - \frac{3}{2}\, ic\, \bar{\theta} \\
\Gamma_{\hat{h}\bar{\chi}} &= \frac{i}{4}\, \overleftarrow{D}^\mu\, \gamma^\nu\theta - \bar{\alpha}_k \overleftarrow{D}^\mu \overleftarrow{D}^\nu\, \gamma^5\theta \\
\Gamma_{h\bar{\chi}} &= -\frac{3i}{16}\, \overleftarrow{D}^\mu\, \gamma_\mu\theta - \frac{\bar{\alpha}_k}{4}\, (3\Delta + R)\, \gamma^5\theta + \frac{3}{2}\, ic\, \theta \\
\\
\Gamma_{\chi\hat{h}} &= -\frac{i}{4}\, \overleftarrow{D}^\mu\, \bar{\theta}\gamma^\nu - \bar{\alpha}_k \overleftarrow{D}^\mu \overleftarrow{D}^\nu\, \bar{\theta}\gamma^5 \\
\Gamma_{\chi h} &= \frac{3i}{16}\, \bar{\theta}\gamma^\mu D_\mu - \frac{\bar{\alpha}_k}{4}\, \bar{\theta}\gamma^5\, (3\Delta + R) + \frac{3}{2}\, ic\, \bar{\theta} \\
\Gamma_{\bar{\chi}\hat{h}} &= -\frac{i}{4}\, \gamma^\nu\theta D^\mu + \bar{\alpha}_k\, \gamma^5\theta D^\mu D^\nu \\
\Gamma_{\bar{\chi}h} &= \frac{3i}{16}\, \gamma^\mu\theta D_\mu + \frac{\bar{\alpha}_k}{4}\, \gamma^5\theta\, (3\Delta + R) - \frac{3}{2}\, ic\, \theta.
\end{aligned} \tag{208}$$

Here we have $c = \sqrt{\frac{R}{48}}$ and all objects are constructed from the background metric only. The left arrow over the derivatives arrives from partial integration and is there to signal the derivative works on the gravitational degrees of freedom. The boson-boson variations, ignoring the contributions of the kinetic term, are given by

$$\begin{aligned}
\Gamma_{\hat{h}\hat{h}} &= -\bar{\alpha}_k\, \bar{\theta}\gamma^5\theta\, (\Delta + \frac{2}{3} R) \\
\Gamma_{hh} &= \frac{3}{8} \bar{\alpha}_k\, \bar{\theta}\gamma^5\theta\, \Delta
\end{aligned} \tag{209}$$

The reason we ignore the kinetic term is because they do now contribute to the projection of the $\bar{\alpha}_k$ action monomial. For including the kinetic term this variation is important. Finally, the fermion-fermion variations result in

$$\begin{aligned} \Gamma_{\bar{\psi}\psi} &= i\nabla\!\!\!/ + \bar{\alpha}_k R\gamma^5 \\ \Gamma_{\psi\bar{\psi}} &= -\big(i\nabla\!\!\!/ + \bar{\alpha}_k R\gamma^5\big)^T. \end{aligned} \tag{210}$$

These are all the variations that enter into the calculation of the beta function for the α coupling.

Appendix B The Matrix Structure

This appendix focuses on the matrix structure in field space of the objects appearing within the flow equation. Specifically we do this in the context of the truncation ansatz (159). The goal is to derive and give an explicit expression of the diagrams appearing in figure 1.

The full Hessian in matrix form reads

$$\Gamma_k^{(2)} = \begin{pmatrix} \Gamma_{\hat{h}\hat{h}} & 0 & \Gamma_{\hat{h}\bar{\psi}} & \Gamma_{\hat{h}\psi} \\ 0 & \Gamma_{hh} & \Gamma_{h\bar{\psi}} & \Gamma_{h\psi} \\ \Gamma_{\bar{\psi}\hat{h}} & \Gamma_{\bar{\psi}h} & 0 & \Gamma_{\bar{\psi}\psi} \\ \Gamma_{\psi\hat{h}} & \Gamma_{\psi h} & \Gamma_{\psi\bar{\psi}} & 0 \end{pmatrix} \tag{211}$$

where we now no longer restrict to only the fermion part. Setting the fermion background fields to zero (temporarily) allows us to define

$$G_k^{-1} := \big(\Gamma_k^{(2)}\big|_{\theta,\bar{\theta}=0} + \mathcal{R}_k\big)^{-1} = \begin{pmatrix} G_{\hat{h}}^{-1} & & & \\ & G_h^{-1} & & \\ & & 0 & -\big(G_\psi^{-1}\big)^T \\ & & G_{\bar{\psi}}^{-1} & 0 \end{pmatrix} \tag{212}$$

where we used the following notation

$$\begin{aligned} G_\psi^{-1} &= (\Gamma_{\bar{\psi}\psi} + \mathcal{R}_k^\psi)^{-1} \\ G_{\hat{h}}^{-1} &= (\Gamma^{\text{grav}}_{\hat{h}\hat{h}} + \mathcal{R}_k^{\hat{h}})^{-1} \\ G_h^{-1} &= (\Gamma^{\text{grav}}_{hh} + \mathcal{R}_k^h)^{-1}. \end{aligned} \tag{213}$$

The gravitational propagators are given in (78) and of course remain unaltered by the inclusion of the fermion action. For the fermion propagator we use Dirac's trick

$$\begin{aligned} G_\psi^{-1} &= \Big(i\nabla\!\!\!/ + \gamma^5 R_k^\psi + \bar{\alpha}R\gamma^5\Big)^{-1} \\ &= \Big(i\nabla\!\!\!/ + \gamma^5 R_k^\psi + \bar{\alpha}R\gamma^5\Big)^{-1}\Big(i\nabla\!\!\!/ + \gamma^5 R_k^\psi + \bar{\alpha}R\gamma^5\Big)^{-1}\Big(i\nabla\!\!\!/ + \gamma^5 R_k^\psi + \bar{\alpha}R\gamma^5\Big) \\ &= \Big(-\nabla\!\!\!/^2 + (R_k^\psi)^2 + 2\bar{\alpha}R\,R_k^\psi\Big)^{-1}\Big(i\nabla\!\!\!/ + \gamma^5 R_k^\psi + \bar{\alpha}R\gamma^5\Big), \end{aligned} \tag{214}$$

where we threw away terms of order R^2 or higher. Constructed from the replacement rule (76), the regulator takes the following form in field space

$$\mathcal{R}_k = \begin{pmatrix} \mathcal{R}_k^{\hat{h}} & & & \\ & \mathcal{R}_k^h & & \\ & & 0 & R_k^\psi \\ & & -\big(R_k^\psi\big)^T & 0 \end{pmatrix} \tag{215}$$

All entries of the regulator are independent of the fermion background fields. We can then write

$$\Gamma_k^{(2)} + \mathcal{R}_k = G_k + \bar{\Gamma}_k^{(2)}, \tag{216}$$

where $\bar{\Gamma}_k^{(2)}$ contains all the terms, and only the terms that depend on either $\bar{\theta}, \theta$ or both. Explicitly it is given by

$$\bar{\Gamma}_k^{(2)} = \begin{pmatrix} \Gamma^{\text{ferm}}_{\hat{h}\hat{h}} & 0 & \Gamma_{\hat{h}\bar{\psi}} & \Gamma_{\hat{h}\psi} \\ 0 & \Gamma^{\text{ferm}}_{hh} & \Gamma_{h\bar{\psi}} & \Gamma_{h\psi} \\ \Gamma_{\bar{\psi}\hat{h}} & \Gamma_{\bar{\psi}h} & 0 & 0 \\ \Gamma_{\psi\hat{h}} & \Gamma_{\psi h} & 0 & 0 \end{pmatrix} \tag{217}$$

We then use the expansion

$$\big(\Gamma_k^{(2)} + \mathcal{R}_k\big)^{-1} = G_k^{-1} - G_k^{-1}\bar{\Gamma}_k^{(2)} G_k^{-1} + G_k^{-1}\bar{\Gamma}_k^{(2)} G_k^{-1}\bar{\Gamma}_k^{(2)} G_k^{-1} + \mathcal{O}\big[(\bar{\Gamma}_k^{(2)})^3\big]. \tag{218}$$

The G_k^{-1} are the propagators and the $\bar{\Gamma}_k^{(2)}$ encodes the vertices. This is helpful when projecting onto $\bar{\theta}\theta$ because at most diagrams with two vertices contribute (higher order ones contain too many background spinors). The zeroth order term also does not contribute.

The first order term then gives rise to the tadpole diagram in figure 1. Multiplying the matrices and then taking the trace over field space then gives

$$\begin{aligned} D^{\hat{h}}_{\text{Tad}} &= -\frac{1}{2}\text{Tr}_{TT}\Big[G_{\hat{h}}^{-1}\,\Gamma^{\text{ferm}}_{\hat{h}\hat{h}}\,G_{\hat{h}}^{-1}\,\partial_t\mathcal{R}_k^{\hat{h}}\Big] \\ D^{h}_{\text{Tad}} &= -\frac{1}{2}\text{Tr}_0\Big[G_h^{-1}\,\Gamma^{\text{ferm}}_{hh}\,G_h^{-1}\,\partial_t\mathcal{R}_k^{h}\Big]. \end{aligned} \tag{219}$$

The three point vertex diagrams ($D_3^{hh\chi}, D_3^{\chi\chi h}$ on the left and right of figure 1 respectively) are given by

$$\begin{aligned} D_3^{hh\chi} &= \text{Tr}_0\big[G_h^{-1}\,\Gamma_{h\chi}\,G_\psi^{-1}\,\Gamma_{\bar{\chi}h}\,G_h^{-1}\partial_t\mathcal{R}_k^h \\ D_3^{\chi\chi h} &= -\text{Tr}_\psi\Big[G_\psi^{-1}\,\Gamma_{\bar{\chi}h}\,G_h^{-1}\,\Gamma_{h\chi}\,G_\psi^{-1}\partial_t\mathcal{R}_k^\psi\Big]. \end{aligned} \tag{220}$$

Note that the factor of one half is canceled. This is due to the additional transposed term, that with careful treatment can be shown to always exactly be equal to the terms between brackets in (220), thus giving a factor of two. This is also the reason why the one half is canceled in (151). The minus sign for $D_3^{\chi\chi h}$ appears because of the presence of a fermion loop. Finally the diagrams with a propagating $\hat{h}-$field are obtained by making the replacement $h \mapsto \hat{h}$ and $\text{Tr}_0 \mapsto \text{Tr}_{TT}$. The traces $\text{Tr}_0, \text{Tr}_{TT}$, and Tr_ψ refer to an insertion of a full set of scalars, traceless tensors and spinors respectively. Specifically on a sphere the full set of scalar functions are the spherical harmonics.

Alternatively, one could have obtained these results by varying the flow equation with respect to the background spinors $\bar{\theta}, \theta$ and setting them to zero.

$$\frac{\delta}{\delta\bar{\theta}}\frac{\delta}{\delta\theta}\partial_t\Gamma_k\Big|_{\bar{\theta},\theta=0} = \frac{1}{2}\frac{\delta}{\delta\bar{\theta}}\frac{\delta}{\delta\theta}\text{Tr}\big[(\Gamma_k^{(2)} + \mathcal{R}_k)^{-1}\partial_t\mathcal{R}_k\big]\Big|_{\bar{\theta},\theta=0}, \tag{221}$$

In this way one would also obtain all the terms contributing to the desired projection. This avoids having to do the explicit multiplications of the matrices. However, one has to be very careful with minus signs due to the Grassmann valued spinors, also appearing in the derivatives.

Appendix C Various Identities

In this appendix we will derive some general relations needed to compute the diagrams. The first identity shows how to commute a function of the squared Dirac operator $\nabla\!\!\!/^2$ with a gamma matrix. Secondly, we derive how to pull a function of the Dirac operator through the background spinor θ satisfying $\nabla_\mu\theta = ic\gamma_\mu\theta$, discussed in section 3.2.4. Finally we show how covariant derivatives acting on the scalar structure $\bar{\theta}\,\Gamma\,\theta$, for an arbitrary matrix Γ in spinor space can be ignored when taking a functional trace. The combination of these identities allows projecting the diagrams onto the structure (160).

C.1 Identity I

Although the gamma matrices are covariantly constant, meaning $[\nabla_\mu, \gamma^\nu] = 0$, they do not commute with the Dirac operator. Here we will derive an identity of how to commute a function $f(\nabla\!\!\!/^2)$ through a gamma matrix. We limit ourselves to the four sphere simplifying commutation relations to the ones in (255). This will be done for an arbitrary spinor carrying an arbitrary spacetime rank $\psi_{\alpha..}$ where the dots stand for possible other open indices. We will limit however to the case where the index of the gamma matrix is contracted with the spinor. We start by working out

$$\nabla\!\!\!/^2 \gamma^\alpha \psi_{\alpha..} = \gamma^\mu\gamma^\nu\gamma^\alpha\, \nabla_\mu\nabla_\nu\psi_{\alpha..}. \tag{222}$$

Commuting through the gamma matrices using the Clifford algebra gives

$$\nabla\!\!\!/^2 \gamma^\alpha \psi_{\alpha..} = \gamma^\alpha \nabla\!\!\!/^2 \psi_{\alpha..} + 2\gamma^\nu \big[D_\nu, D_\alpha\big]\psi^\alpha_{..} + \frac{R}{4}\gamma^\alpha\psi_{\alpha..}. \tag{223}$$

Here we used the commutator relation (123), keeping in mind that the spinor ψ also carries spacetime indices. The commutator is always of order R and we denote the coefficient in front of R, adding the other term by b. This coefficient depends solely on the index structure of the spinor ψ. Thus we have

$$\nabla\!\!\!/^2 \gamma^\alpha \psi_{\alpha..} = \gamma^\alpha \nabla\!\!\!/^2 \psi_{\alpha..} + bR\,\gamma^\alpha\psi_{\alpha..}. \tag{224}$$

Using this we compute

$$\big(\nabla\!\!\!/^2\big)^n \gamma^\alpha \psi_{\alpha..} = \big(\nabla\!\!\!/^2\big)^{n-1}\big(\gamma^\alpha \nabla\!\!\!/^2 \psi_{\alpha..} + bR\,\gamma^\alpha\psi_{\alpha..}\big). \tag{225}$$

Since (224) works for any spinor $\psi_{\alpha..}$, it also holds for the spinor $\nabla\!\!\!/^2\psi_{\alpha..}$ meaning we can use this relation repeatedly obtaining

$$\big(\nabla\!\!\!/^2\big)^n \gamma^\alpha \psi_{\alpha..} = \gamma^\alpha \big(\nabla\!\!\!/^2\big)^n \psi_{\alpha..} + n\,bR\gamma^\alpha\big(\nabla\!\!\!/^2\big)^{n-1}\psi_{\alpha..} + \mathcal{O}(R^2). \tag{226}$$

Here we have thrown away all terms of order R^2 and higher. Thus for an arbitrary operator valued function, always represented by its tailor series, satisfies the relation

$$f(\nabla\!\!\!/^2)\,\gamma^\alpha\,\psi_{\alpha..} = \gamma^\alpha\, f(\nabla\!\!\!/^2)\,\psi_{\alpha..} + bR\,\gamma^\alpha\, f'(\nabla\!\!\!/^2)\,\psi_{\alpha..} + \mathcal{O}(R^2). \tag{227}$$

This is the relation we wanted to derive. Specifically for the case where the index α is the only index ψ carries the value for b is given by

$$b = -\frac{1}{4}. \tag{228}$$

C.2 Identity II

The identity we will derive now is crucial for dealing with the background mode θ satisfying $\nabla_\mu\theta = ic\,\gamma_\mu\,\theta$, discussed in section 3.2.4. Since this mode is not covariantly constant this is highly non trivial. The goal is to write an expression of the form

$$f(\nabla\!\!\!/^2)\,\theta\, T \tag{229}$$

in a way where no derivatives act on the background spinor any longer. Here T is a spacetime tensor of arbitrary rank. We start by noting

$$\nabla_\mu\theta\, T = \big(\nabla_\mu\theta\big)\,T + \theta\, D_\mu T \tag{230}$$

We first work out

$$\begin{aligned}
\nabla\!\!\!/^2\,\theta\, T &= \gamma^\mu\nabla_\mu\gamma^\nu\nabla_\nu\,\theta\, T \\
&= \gamma^\mu\nabla_\mu\gamma^\nu\, ic\,\gamma_\nu\,\theta\, T + \gamma^\mu\gamma^\nu\nabla_\mu\,\theta\, D_\nu T \\
&= 4ic\,\gamma^\mu(\nabla_\mu\theta)\,T + 4ic\,\gamma^\mu\theta\, D_\mu T + \gamma^\mu\gamma^\nu(\nabla_\mu\theta)\,D_\nu T + \gamma^\mu\gamma^\nu\,\theta\, D_\mu D_\nu T.
\end{aligned} \tag{231}$$

Then we write the last term as

$$\begin{aligned}\gamma^\mu\gamma^\nu\,\theta\,D_\mu D_\nu T &= \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\,\theta\,D_\mu D_\nu T + \frac{1}{2}\big[\gamma^\mu,\gamma^\nu\big]\,\theta\,D_\mu D_\nu T\\ &= -\theta\,\Delta\,T + \frac{1}{2}\gamma^\mu\gamma^\nu\,\theta\,\big[D_\mu, D_\nu\big]\,T.\end{aligned} \tag{232}$$

Thus in the end we then obtain

$$\nabla\!\!\!/^2\,\theta\,T = -16c^2\,\theta\,T + 2ic\,\gamma^\mu\theta\,D_\mu T - \theta\,\Delta\,T + \frac{1}{2}\gamma^\mu\gamma^\nu\,\theta\,\big[D_\mu, D_\nu\big]\,T. \tag{233}$$

We need to eventually work out all possible powers of $\nabla\!\!\!/^2$. The simplification we can make however is that we can ignore all terms of $\mathcal{O}(c^3)$ and higher. This is because we are only interested in the projection onto R. Note that a commutator is always of order R as well. Using this we get for the next order term

$$\nabla\!\!\!/^4\,\theta\,T = \nabla\!\!\!/^2\Big[-16c^2\,\theta\,T + 2ic\,\gamma^\mu\theta\,D_\mu T - \theta\,\Delta\,T + \frac{1}{2}\gamma^\mu\gamma^\nu\,\theta\,\big[D_\mu, D_\nu\big]\,T\Big] \tag{234}$$

After a tedious calculation, also using (227), we find

$$\begin{aligned}\nabla\!\!\!/^4\,\theta\,T =& -2*16c^2\,\theta\,(-\Delta)\,T + 2*2ic\,\gamma^\mu\,\theta\,D_\mu(-\Delta)\,T + \theta\,(-\Delta)^2\,T\\ &- 2c^2\,\gamma^\mu\gamma^\nu\,\theta\,D_\mu D_\nu\,T + 2*\frac{1}{2}\gamma^\mu\gamma^\nu\,\theta\,\big[D_\mu, D_\nu\big]\Delta\,T.\end{aligned} \tag{235}$$

Carefully analyzing the transition from $\nabla\!\!\!/^2$ to $\nabla\!\!\!/^4$ inspires the proposal

$$\begin{aligned}\big(\nabla\!\!\!/^2\big)^n =& -16c^2\,\theta\big(n(-\Delta)^{n-1}\big)\,T + \theta(-\Delta)^n\,T + \frac{1}{2}\gamma^\mu\gamma^\nu\theta\,\big[D_\mu, D_\nu\big]\,n(-\Delta)^{n-1}\,T\\ &+ 2ic\,\gamma^\mu\theta\,D_\mu\big(n(-\Delta)^{n-1}\big)\,T - 2c^2\,\gamma^\mu\gamma^\nu\theta\,D_\mu D_\nu\frac{n(n-1)}{2}(-\Delta)^{n-2}\,T.\end{aligned} \tag{236}$$

Explicitly one can check this already for the cases $n = 1, 2$. Also if this formula is true for one specific n, then one can look at what the $n+1$ case would be. For instance for the term $2ic\,\gamma^\mu\theta\,D_\mu n(-\Delta)^{n-1}\,T$ term, gains contributions, of the former evaluation, from itself and from the $\theta(-\Delta)^n\,T$ term, always leading to a factor $(n-1)+1 = n$. The number in front of the term $-2c^2\,\gamma^\mu\gamma^\nu\theta\,D_\mu D_\nu(-\Delta)^{n-2}$ gains contributions from itself and from the $n\,ic\,\gamma^\mu\theta\,D_\mu\,n(-\Delta)^{n-1}\,T$ term. If we call the coefficient a_n, we note that for $n = 2$ we have $a_2 = 1$. For $n = 1$ this term is absent. We also have the relation

$$a_n = a_{n-1} + n - 1, \tag{237}$$

Thus

$$\begin{aligned}a_n = 1 + \sum_{i=1}^{n}(i-1) = 1 + \sum_{i=1}^{n} -1\\ &= \sum_{i=1}^{n-1} i\\ &= \frac{(n-1)(n-1+1)}{2}\\ &= \frac{n(n-1)}{2},\end{aligned} \tag{238}$$

as is written in the last term of formula (236). Hence for the final form we have derived the identity

$$\begin{aligned}f(\nabla\!\!\!/^2)\,\theta\,T =& \,\theta\,f(-\Delta)\,T - 16c^2\,\theta\,f'(-\Delta)\,T + \frac{1}{2}\gamma^\mu\gamma^\nu\theta\,\big[D_\mu, D_\nu\big]\,f'(-\Delta)\,T\\ &+ 2c^2\,\theta\,\Delta f''(-\Delta)\,T + 2ic\,\gamma^\mu\theta\,D_\mu\,f'(-\Delta)\,T.\end{aligned} \tag{239}$$

The prime denotes a derivative of the operator valued function with respect to its argument.

C.3 Identity III

This third identity gives an argument that allows to pull out the spinor structure $\bar{\theta}\,\Gamma\,\theta$ from the functional traces, while various derivatives still act on it. Note that this structure is a scalar. We will use the spherical harmonics as well off-diagonal heat kernel methods.

Let $\phi(x)$ be a scalar function on the sphere. We can then expand in spherical harmonics

$$\phi(x) = \sum_{l\geq 0,\, m} c_{lm} T^{lm}(x). \tag{240}$$

The spherical harmonics satisfy

$$\Delta T^{lm} = \Lambda_l\, T^{lm}, \qquad \int d^4x \sqrt{g}\, T^{lm}\, T^{kn} = \delta^{lk}\delta^{mn} \tag{241}$$

(here the repeated index l is not summed over). The lowest eigenvalue $\Lambda_0 = 0$, and has degeneracy one. This implies the constant mode on the sphere

$$T^{00} = \mathrm{Vol}(\mathrm{S}^4)^{-1/2}. \tag{242}$$

Thus,

$$\int d^4x \sqrt{g} T^{lm}(x) = 0 \quad l \neq 0. \tag{243}$$

We note

$$\Delta\, \phi(x) = \sum_{l\geq 1,\, m} \Lambda_l\, C_{lm}\, T^{lm}(x), \tag{244}$$

i.e. the Laplacian projects out the zero mode. Focusing on the trace of interest

$$\mathrm{Tr}\Big[(\Delta\phi)\, g(\Delta)\Big] = \mathrm{Tr}\Big[\big(\sum_{l\geq 1,\, m} \Lambda_l\, C_{lm}\, T^{lm}(x)\big)\, g(\Delta)\Big]. \tag{245}$$

The off-diagonal heat kernel reduces this expression to [59]

$$\frac{1}{16\pi^2}\sum_{n=0}^{\infty} Q_{d/2-n}[g] \int d^4x \sqrt{g} \big(\sum_{l\geq 1} \Lambda_l c_{lm} T^{lm}(x)\big)\, a_{2n}\, R^n, \tag{246}$$

where the a_{2n} are some numbers known as the heat-kernel coefficients. Then by the conclusion reached in (243), the integral in (246) vanishes. Thus we have derived

$$\mathrm{Tr}\Big[(\Delta\phi)\, g(\Delta)\Big] = 0. \tag{247}$$

Appendix D Diagrams: The Computation

The computation of the diagrams displayed in figure 1 will be presented here. The emphasis will be on the overall outline and strategy. We discuss the scalar bosonic loop diagram $D_3^{hh\chi}$ first. For $D_3^{\hat{h}\hat{h}\chi}$ some additional techniques will be required, and we will only focus on those. The other three-point vertex diagrams, containing fermion loops, will be dealt with by using a cyclicity of the trace trick. This allows to relate them to the bosonic loop diagrams. The tadpole diagrams will not be discussed as all the techniques discussed for $D_3^{hh\chi}$ are more than enough to compute them.

D.1 The Scalar Bosonic Loop Diagram

By the above equation of (220), and the explicit form of the vertices 208 the object we want to calculate is

$$\begin{aligned} D_3^{hh\chi} = \mathrm{Tr}\Big[& W_h(\Delta)\,\big(\frac{3}{16}\, i\, D_\mu\bar{\theta}\gamma^\mu + \frac{\bar{\alpha}}{4}\,(3\Delta + R)\bar{\theta}\gamma^5 - \frac{3}{2} i c\,\bar{\theta}\big) \\ & \frac{i\nabla\!\!\!/ + \gamma^5 R_k^\psi + \gamma^5\,\bar{\alpha} R}{-\nabla\!\!\!/^2 + (R_k^\psi)^2 + 2\bar{\alpha} R\, R_k^\psi}\big(\frac{3}{16}\, i\,\gamma^\nu\theta\, D_\nu + \gamma^5\,\frac{\bar{\alpha}}{4}\theta\,(3\Delta + R) - \frac{3}{2} i c\,\theta\big)\Big]. \end{aligned} \tag{248}$$

Here we have for the function W_h,

$$W_h(\Delta) \coloneqq G_h^{-2}(\Delta)\, \partial_t \mathcal{R}_k^h(\Delta). \tag{249}$$

Note that using the cyclicity of the trace we have taken one propagator and the regulator to the front of the expression. The form of the fermion propagator motivates the definitions

$$\begin{aligned} W_\chi(\nabla\!\!\!/^2) &\coloneqq \frac{R_k^\psi + \bar{\alpha} R}{-\nabla\!\!\!/^2 + (R_k^\psi)^2 + 2\bar{\alpha} R\, R_k^\psi} \\ \widetilde{W}_\chi(\nabla\!\!\!/^2) &\coloneqq \frac{1}{-\nabla\!\!\!/^2 + (R_k^\psi)^2 + 2\bar{\alpha} R\, R_k^\psi}. \end{aligned} \tag{250}$$

In this way we have

$$G_\psi^{-1} = i\nabla\!\!\!/\, \widetilde{W}_\chi(\nabla\!\!\!/^2) + \gamma^5\, W_\chi(\nabla\!\!\!/^2). \tag{251}$$

The goal is to project this functional trace onto the structure

$$R\, \bar{\theta}\, \gamma^5\, \theta. \tag{252}$$

For a general matrix A we have that the projection of $\bar{\theta}\, A\, \theta$ onto $\bar{\theta}\, \gamma^5\, \theta$ is given by the projection rule (109)

$$\bar{\theta}\, A\, \theta \risingdotseq \frac{1}{4} \mathrm{tr}(A\gamma^5)\, \bar{\theta}\, \gamma^5\, \theta. \tag{253}$$

Thus, we now have the following criterion for terms that we can throw away

- Terms of $\mathcal{O}(R^2)$ or higher;
- Terms containing an odd number of derivatives acting on the complete set of scalar states;
- Terms containing an odd number of gamma matrices between the spinor structures.

The reason terms with odd number of derivatives can be ignored is because the spectrum of the derivative operator is symmetric around zero. It is important to only throw away the terms of order $\sqrt{R}$ at the very end. The strategy for computing these diagrams is then the following

1. Split the diagram up into only single vertex terms, and deal with them one by one.
2. For each of these make a choice of which part of the fermion propagator to use based on the number of derivatives (keep in mind of order $\sqrt{R}$ terms).
3. Commute the gamma matrix with the function of $\nabla\!\!\!/^2$ with Identity I (227).
4. Use identity II (239) to deal with all the derivatives acting on the background spinor.
5. Project onto the spinor structure using (253) and take this spinor structure out of the functional trace, allowed by Identity III (247).
6. Compute all the commutators and contract the indices.
7. Add all the terms together

For the commutation relations, see [59], specifically on a sphere, ignoring $\mathcal{O}(R^2)$ terms, we have

$$\left[D_\mu, W(\Delta)\right] \phi_{\alpha_1 \ldots \alpha_n} = W'(\Delta) \left[D_\mu, \Delta\right] \phi_{\alpha_1 \ldots \alpha_n}, \tag{254}$$

where the derivation most commonly uses the commutators

$$\begin{aligned} \left[D_\mu, \Delta\right] \phi^\mu &= -\frac{1}{4}\, R\, D_\mu\, \phi^\mu \\ \left[D_\mu, \Delta\right] \hat{\phi}^{\mu\nu} &= -\frac{5}{12}\, R\, D_\mu\, \hat{\phi}^{\mu\nu}, \end{aligned} \tag{255}$$

where $\hat{\phi}^{\mu\nu}$ refers to an arbitrary symmetric, traceless tensor. In the end one should be left with a trace representation of the diagram. We will now demonstrate this strategy for one vertex combination term of the diagram $D_3^{hh\chi}$. For this we take the curvature and kinetic term of the first and second vertex insertion respectively. Thus we want to compute

$$T_1 := \mathrm{Tr}\Big[W_h(\Delta)\,\frac{\bar{\alpha}}{4}(3\Delta + R)\,\bar{\theta}\gamma^5\left(i\not{\nabla}\,\widetilde{W}_\chi(\not{\nabla}^2) + \gamma^5 W_\chi(\not{\nabla}^2)\right)\frac{3i}{16}\,\gamma^\nu\theta D_\nu\Big] \tag{256}$$

According to step two we note that the terms involving $\widetilde{W}_\chi$, give some non-vanishing contributions. Although the W_χ term can give rise to some single derivative contributions, see e.g. identity (239), all these terms will be of $\mathcal{O}(\sqrt{R})$ and thus not contribute to the projection. Proceeding with step three and collecting prefactors we then arrive at

$$T_1 = -\frac{3\bar{\alpha}}{64}\,\mathrm{Tr}\Big[W_h(\Delta)\,(3\Delta + R)\,\bar{\theta}\gamma^5\,\not{\nabla}\big(\gamma^\nu\widetilde{W}_\chi(\not{\nabla}^2) - \frac{R}{4}\gamma^\nu\,\widetilde{W}'_\chi(\not{\nabla}^2)\big)\,\theta D_\nu\Big]. \tag{257}$$

Moving the $\not{\nabla}^2$-dependent functions through the theta in step four, ignoring squared order curvature terms we find

$$\begin{aligned} T_1 = -\frac{3\bar{\alpha}}{64}\,\mathrm{Tr}\Big[&W_h(\Delta)\,(3\Delta + R)\,\bar{\theta}\gamma^5\,\not{\nabla}\,\gamma^\nu\Big(\theta\,\widetilde{W}_\chi(-\Delta) - 16c^2\,\theta\,\widetilde{W}'_\chi(-\Delta) + \\ &\frac{1}{2}\gamma^\sigma\gamma^\rho\,\theta\,[D_\sigma, D_\rho]\widetilde{W}'_\chi(-\Delta) + 2c^2\,\theta\,\Delta\widetilde{W}''_\chi(-\Delta) + 2ic\,\gamma^\alpha\theta\,D_\alpha\widetilde{W}_\chi(-\Delta) \\ &-\frac{R}{4}\,\widetilde{W}'_\chi(-\Delta)\Big)\,D_\nu\Big]. \end{aligned} \tag{258}$$

The $\mathcal{O}(c)$ term does not contribute since the only way to gain an extra factor of c is through the single $\not{\nabla}$ derivative, but then the trace contains an odd number of derivatives and would thus vanish. For all other terms the single Dirac operator will act on the 'hidden' trace functionals only and can thus be moved trivially through the background spinor θ, combining with the left over single derivative D_ν to form the Laplacian. Proceeding with step five by projecting onto the right spinor structure using (253), and evaluating the commutators, i.e. performing step six, as well as plugging in the value $c^2 = \frac{R}{48}$ we arrive at

$$\begin{aligned} T_1 = &+\frac{9}{64}\bar{\alpha}\,\bar{\theta}\gamma^5\theta\,\mathrm{Tr}\big[W_h(\Delta)\,\Delta^2\widetilde{W}_\chi(-\Delta)\big] \\ &+\frac{3}{64}\bar{\alpha}\,\bar{\theta}\gamma^5\theta\,\mathrm{Tr}\big[W_h(\Delta)\,R\,\widetilde{W}_\chi(-\Delta)\big] \\ &-\frac{9}{256}\bar{\alpha}\,\bar{\theta}\gamma^5\theta\,\mathrm{Tr}\big[W_h(\Delta)\,\Delta^2\,R\,\widetilde{W}'_\chi(-\Delta)\big] \\ &+\frac{9}{32}\bar{\alpha}\,\bar{\theta}\gamma^5\theta\,\mathrm{Tr}\big[W_h(\Delta)\,\Delta^3\,\widetilde{W}''_\chi(-\Delta)\big]. \end{aligned} \tag{259}$$

Repeating the same procedure for the other vertex combinations and doing step seven then results in the trace representation for the scalar bosonic diagram

$$\begin{aligned} D_3^{hh\chi}\Big|_{\bar\theta\gamma^5\theta} = &-\frac{9}{256}\mathrm{Tr}_0\big[W_h(\Delta)\,\Delta\,W_\chi(-\Delta)\big] \\ &+\frac{3}{64}\mathrm{Tr}_0\big[W_h(\Delta)\,R\,W_\chi(-\Delta)\big] \\ &+\frac{3}{1024}\mathrm{Tr}_0\big[W_h(\Delta)\,\Delta\,R\,W_\chi'(-\Delta)\big] \\ &-\frac{3}{2048}\mathrm{Tr}_0\big[W_h(\Delta)\,\Delta^2\,R\,W_\chi''(-\Delta)\big] \\ &+\frac{9}{32}\bar\alpha\,\mathrm{Tr}_0\big[W_h(\Delta)\,\Delta^2\,\widetilde{W}_\chi(-\Delta)\big] \\ &+\frac{3}{32}\bar\alpha\,\mathrm{Tr}_0\big[W_h(\Delta)\,\Delta\,R\,\widetilde{W}_\chi(-\Delta)\big] \\ &-\frac{9}{128}\bar\alpha\,\mathrm{Tr}_0\big[W_h(\Delta)\,\Delta^2\,R\,\widetilde{W}_\chi'(-\Delta)\big] \\ &+\frac{3}{256}\bar\alpha\,\mathrm{Tr}_0\big[W_h(\Delta)\,\Delta^3\,R\,\widetilde{W}_\chi''(-\Delta)\big] \\ &+\frac{9}{16}\bar\alpha^2\,\mathrm{Tr}_0\big[W_h(\Delta)\,\Delta^2\,W_\chi(-\Delta)\big] \\ &+\frac{3}{8}\bar\alpha^2\,\mathrm{Tr}_0\big[W_h(\Delta)\,\Delta\,R\,W_\chi(-\Delta)\big] \\ &-\frac{3}{16}\bar\alpha^2\,\mathrm{Tr}_0\big[W_h(\Delta)\,\Delta^2\,R\,W_\chi'(-\Delta)\big] \\ &+\frac{3}{128}\bar\alpha^2\,\mathrm{Tr}_0\big[W_h(\Delta)\,\Delta^3\,R\,W_\chi''(-\Delta)\big]. \end{aligned} \tag{260}$$

These functional traces can be calculated using the heat kernel formula (82). Also the terms containing the double derivative W_χ'' are advised to be computed using integration by parts. This is to avoid dealing with the product of step functions with derivatives of delta functions.

D.2 The Tensorial Bosonic Loop Diagram

The computation of the diagram of $D_3^{\hat h\hat h\chi}$ is very similar to that of $D_3^{hh\chi}$. In this case however the vertices are somewhat different as well as the function $W_{\hat h}(\Delta)$, that is now constructed from the $\hat h$-propagators and regulator function. Note that in particular the vertices carry open indices. These are contracted with the full set of symmetric traceless tensors from the functional trace. One important subtlety to remember is to deal with the zero modes. These have to be taken properly into account before one can use heat kernel techniques. To deal with them we employ the York-decomposition [60]

$$\hat h_{\mu\nu} = h^{TT}_{\mu\nu} + D_\mu\xi_\nu + D_\nu\xi_\mu + \big(D_\mu D_\nu + \frac{1}{4}g_{\mu\nu}\,\Delta\big)\sigma \tag{261}$$

Here the fluctuation field $\hat h_{\mu\nu}$ is further decomposed into a transverse-traceless part $h^{TT}_{\mu\nu}$, a transverse vector ξ_μ and a scalar part σ, satisfying

$$g^{\mu\nu}h^{TT}_{\mu\nu} = 0, \quad D^\mu h^{TT}_{\mu\nu} = 0, \quad D^\mu\xi_\mu = 0. \tag{262}$$

The Jacobian associated with this decomposition is equal to one if

$$\langle \hat h_{\mu\nu}\,|\,\hat h^{\mu\nu}\rangle = 1. \tag{263}$$

This is guaranteed if one redefines the fields according to

$$\xi_\mu \mapsto \frac{1}{\sqrt{2}}\big[\Delta - \frac{1}{4}R\big]^{-1/2}\xi_\mu, \quad \sigma \mapsto \big[\frac{3}{4}\Delta^2 - \frac{1}{4}R\Delta\big]^{-1/2}\sigma. \tag{264}$$

These redefinitions can be performed at the end of the step plan for calculating the diagrams. The traceless-tensor part of the calculation drops out (this was precisely the reason we further decomposed the fluctuation field). This leaves us a transverse vector part as well as a scalar part for the diagram, both are automatically diagonalized. The step plan still remains very much the same accept that between step one and two, there is an additional step that says to take care of the index contraction. Also all covariant derivatives should be made to act on the right, by using integration by parts. This will give rise to $W'_{\hat{h}}(\Delta)$ terms. Also between step six and seven there is a step taking care of the field redefinitions, and performing a Taylor expansion of the denominators in R. This will lead to the result

$$
\begin{aligned}
D_3^{\hat{h}\hat{h}\chi}\Big|_{\bar{\theta}\gamma^5\theta} = &- \frac{1}{32}\,\mathrm{Tr}_{TV}\big[W_{\hat{h}}\,\Delta\,W_\chi(-\Delta)\big] \\
&+ \frac{1}{128}\,\mathrm{Tr}_{TV}\big[W_{\hat{h}}\,R\,W_\chi(-\Delta)\big] \\
&+ \frac{1}{96}\,\mathrm{Tr}_{TV}\big[W_{\hat{h}}\,\Delta\,R\,W'_\chi(-\Delta)\big] \\
&- \frac{1}{768}\,\mathrm{Tr}_{TV}\big[W_{\hat{h}}\,\Delta^2\,R\,W''_\chi(-\Delta)\big] \\
&+ \frac{5}{384}\,\mathrm{Tr}_{TV}\big[W'_{\hat{h}}\,\Delta\,R\,W_\chi(-\Delta)\big] \\
&- \frac{3}{64}\,\mathrm{Tr}_0\big[W_{\hat{h}}\,\Delta\,W_\chi(-\Delta)\big] \\
&+ \frac{1}{64}\,\mathrm{Tr}_0\big[W_{\hat{h}}\,R\,W_\chi(-\Delta)\big] \\
&+ \frac{1}{256}\,\mathrm{Tr}_0\big[W_{\hat{h}}\,\Delta\,R\,W'_\chi(-\Delta)\big] \\
&- \frac{1}{512}\,\mathrm{Tr}_0\big[W_{\hat{h}}\,\Delta^2\,R\,W''_\chi(-\Delta)\big] \\
&+ \frac{1}{32}\,\mathrm{Tr}_0\big[W'_{\hat{h}}\,\Delta\,R\,W_\chi(-\Delta)\big] \\
&+ \frac{3}{8}\bar{\alpha}\,\mathrm{Tr}_0\big[W_{\hat{h}}\,\Delta^2\,\widetilde{W}_\chi(-\Delta)\big] \\
&- \frac{1}{8}\bar{\alpha}\,\mathrm{Tr}_0\big[W_{\hat{h}}\,\Delta\,R\,\widetilde{W}_\chi(-\Delta)\big] \\
&- \frac{3}{32}\bar{\alpha}\,\mathrm{Tr}_0\big[W_{\hat{h}}\,\Delta^2\,R\widetilde{W}'_\chi(-\Delta)\big] \\
&+ \frac{1}{64}\bar{\alpha}\,\mathrm{Tr}_0\big[W_{\hat{h}}\,\Delta^3\,R\,\widetilde{W}''_\chi(-\Delta)\big] \\
&- \frac{1}{4}\bar{\alpha}\,\mathrm{Tr}_0\big[W'_{\hat{h}}\,\Delta^2\,R\,\widetilde{W}_\chi(-\Delta)\big] \\
&+ \frac{3}{4}\bar{\alpha}^2\,\mathrm{Tr}_0\big[W_{\hat{h}}\,\Delta^2\,W_\chi(-\Delta)\big] \\
&- \frac{1}{4}\bar{\alpha}^2\,\mathrm{Tr}_0\big[W_{\hat{h}}\,\Delta\,R\,W_\chi(-\Delta)\big] \\
&- \frac{1}{4}\bar{\alpha}^2\,\mathrm{Tr}_0\big[W_{\hat{h}}\,\Delta^2\,R\,W'_\chi(-\Delta)\big] \\
&+ \frac{1}{32}\bar{\alpha}^2\,\mathrm{Tr}_0\big[W_{\hat{h}}\,\Delta^3\,R\,W''_\chi(-\Delta)\big] \\
&- \frac{1}{2}\bar{\alpha}^2\,\mathrm{Tr}_0\big[W'_{\hat{h}}\,\Delta^2\,R\,W_\chi(-\Delta)\big].
\end{aligned}
\tag{265}
$$

Be aware that the heat kernel coefficients take on different values for a scalar or traceless vector trace.

D.3 Fermion Loop Diagrams

In principle these diagrams (see equation (220)) can be done using the Fierz reordering formula (108) since this also holds for operator valued matrices. There is however an easier way. Using a cyclicity of the trace argument we can write relate the trace evaluation of this diagram with that of the bosonic loop diagrams. For the argument we insert the unit operator $\mathbb{1} = \sum_\phi |\phi\rangle\langle\phi|$ in the following way [14]

$$
\begin{aligned}
D_3^{\chi\chi h} &= -\sum_\psi \langle\bar\psi|\Gamma_{\bar\chi h}\, G_h^{-1}\Gamma_{h\chi}G_\psi^{-1}\partial_t\mathcal{R}_k^\psi\, G_\psi^{-1}|\psi\rangle \\
&= -\sum_{\psi,\phi}\langle\bar\psi|\Gamma_{\bar\chi h}\, G_h^{-1}\,|\phi\rangle\langle\phi|\,\Gamma_{h\chi}G_\psi^{-1}\partial_t\mathcal{R}_k^\psi\, G_\psi^{-1}|\psi\rangle \\
&= -\sum_{\psi,\phi}\langle\phi|\,\Gamma_{h\chi}G_\psi^{-1}\partial_t\mathcal{R}_k^\psi\, G_\psi^{-1}|\bar\psi\rangle\langle\psi|\Gamma_{\bar\chi h}\, G_h^{-1}\,|\phi\rangle.
\end{aligned}
\tag{266}
$$

The reason we could insert a full set of scalar fields is because on both the left and right hand side the structure resulted in a scalar operator. In the second step we changed the order. Using now the relation $\sum_\psi|\psi\rangle\langle\bar\psi|$, as well as pulling the gravitational propagator to the front we arrive at

$$
D_3^{\chi\chi h} = -\sum_\phi\langle\phi|\,G_h^{-1}\,\Gamma_{h\chi}G_\psi^{-1}\partial_t\mathcal{R}_k^\psi\, G_\psi^{-1}\Gamma_{\bar\chi h}\, G_h^{-1}\,|\phi\rangle. \tag{267}
$$

This is of very similar structure of the scalar diagram, apart from the fact that the functions have changed. The changes in the objects compared to the evaluation of $D_3^{hh\chi}$ are

$$
W_h(\Delta)\mapsto G_h^{-1}(\Delta), \quad G_\psi^{-1}(\not\nabla)\mapsto G_\psi^{-2}(\not\nabla)\partial_t\mathcal{R}_k^\psi(\not\nabla). \tag{268}
$$

For the corresponding functions of $W_\chi(\not\nabla^2)$ and $\widetilde{W}_\chi(\not\nabla^2)$ one would then need to find the part of $G_\psi^{-2}\partial_t\mathcal{R}_k^\psi$ proportional to a single derivative $\not\nabla$ as well as the other part proportional to γ^5. This means that making the replacement

$$
W_\chi(\not\nabla^2)\mapsto V_\chi(\not\nabla^2), \quad \widetilde{W}_\chi(\not\nabla^2)\mapsto \widetilde{V}_\chi(\not\nabla^2), \tag{269}
$$

results in the expressions

$$
\begin{aligned}
V_\chi(\not\nabla^2) &= \frac{\not\nabla^2 + \left(R_k^\psi\right)^2 + 2\bar\alpha R\, R_k^\psi}{\left[-\not\nabla^2 + \left(R_k^\psi\right)^2 + 2\bar\alpha R\, R_k^\psi\right]^2}\partial_t R_k^\psi, \\
\widetilde{V}_\chi(\not\nabla^2) &= \frac{2(\bar\alpha R + R_k^\psi)}{\left[-\not\nabla^2 + \left(R_k^\psi\right)^2 + 2\bar\alpha R\, R_k^\psi\right]^2}\partial_t R_k^\psi.
\end{aligned}
\tag{270}
$$

Replacing these functions in the trace representation for the bosonic loop diagrams, as well as including an *overall* minus sign then yields the fermion diagrams. Various explicit checks to verify the arguments presented here have been made and all provided affirmative confirmation. The evaluation of all the functional traces gives rise to the coefficients (165).

[14]The same argument can be extended for the tensorial diagram. Instead one has to include a full set of traceless, symmetric tensors.

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