Towards resolving singularities in Asymptotic Safety

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1 Introduction

Finding a theory for quantum gravity is one of the greatest challenges in theoretical physics. The perturbative quantum field theoretical techniques that highly accurately describe the other three fundamental forces, together giving the standard model, render a multitude of problems when applied to gravity. A down-to-earth strategy, then, would be to apply *nonperturbative* techniques to probe quantum gravity as an effective field theory. That is what we will do in this thesis.

All objects surrounding us in everyday life consist of countless ($10^{24} - 10^{25}$ for a glass of water) molecules containing atoms made from electrons in uncertain nontrivial orbits around a nucleus that in turn consists of protons and neutrons made from gluons and quarks. Then, how can we state anything about, say, the temperature of a glass of water without requiring a deep understanding of the fundamental theories describing these fundamental particles?

We can, using an *effective* description, as on different scales, different degrees of freedom play a role of importance. To describe a water balloon you are throwing towards your friend you do not need to consider all degrees of freedom of all water atoms. And to describe how the earth revolves around the sun, one does not need to take into account all degrees of freedom of its water balloon throwing inhabitants. To describe physics, in short, one has to take the scale at which it takes place into consideration. There exists a natural direction in considering degrees of freedom: from small to big scales. When you know everything about the atoms, you can describe the molecules. When knowing everything about the molecules, one can describe the liquid, and so on and so on.

As for degrees of freedom, likewise the description of interactions depends on the length scale at which they are probed. Or, as length is inversely proportional to energy, on the energy scale. A way to understand this is that for interactions at higher energies, virtual pairs of particles may briefly appear, thereby changing the coupling. We say the coupling runs, or gets renormalised. The standard model is built on this notion. However, when applied to general relativity, an infinite number of problems seems to arise.

Gravity is generally regarded as one of the four fundamental forces, but really stands out from the others. As described by Einsteins theory of general relativity, gravity manifests itself as a dynamical description of spacetime. Massive objects curve their surrounding spacetime, which in turn dictates how objects move. Therefore, a theory of quantum gravity is also a theory of quantum spacetime. Finding a theory of quantum gravity thus amounts to giving a fundamental description of the building blocks of our universe, see Fig. 1. It might answer questions about some of the most exotic and fascinating phenomena of our universe, like the big bang or the inside of a black hole. For both phenomena, our current theories are not sufficient. General relativity predicts singularities to reside inside black holes and at the beginning of our universe, prompting the remark that it predicts its own breakdown. Resolving these singularities is a key incentive for the search for a theory of quantum gravity.

But finding this theory is hard. When applying perturbative quantum field theory to general relativity, *i.e.* naively quantising gravity, the problem arises that new infinities arise at every (loop) expansion order [1], meaning that an infinite number of free parameters should be introduced to cancel these, rendering the theory useless. General relativity is therefore called perturbatively non-renormalisable. This is due to the fact that gravitational particles also have a gravitational field themselves, due to their energy. Thus "gravity gravitates", or "curved spacetime curves spacetime". The reason is the negative mass-dimension of Newtons constant. More precisely, the strength of the gravitational coupling is $G_N\,k^2$, where G_N is Newtons constant and k is some energy or momentum scale corresponding to the process under consideration. This energy scale appears because gravity couples to mass, which manifests itself in the fact that all the couplings of gravitons contain derivatives [2]. Thus, the higher the energy of a particle, the stronger its gravitational coupling. This leads to an ultraviolet (UV) divergence, where the gravitational coupling becomes infinite for high energy scales. This divergence is problematic as some observables like decay rates and cross sections depend on these couplings and we do not expect to observe infinite decay rates or cross sections.

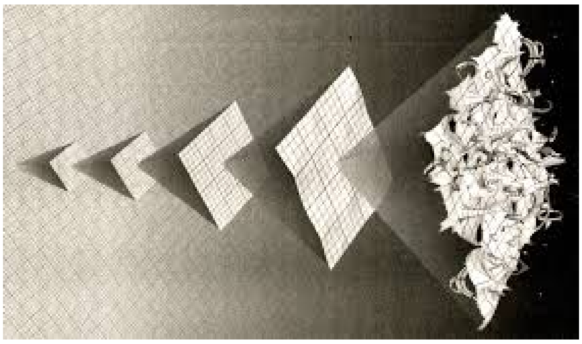

Figure 1: Artists impression of how a two-dimensional spacetime may look different on different length scales. Taken from [3].

It is possible, however, that the divergence is due to wrongly using perturbative techniques. In that case, the nonrenormalisable theory is actually a low-energy approximation of a renormalisable theory. The infinities appear due to using the approximation outside its regime of validity. The low-energy approximation then serves as an effective theory, containing degrees of freedom that may not be fundamental, and breaks down when reaching energies that are too high. Finding a theory that also works at these higher energies is called the "UV completion" of the effective theory. Technically this may work as follows. As we learned above that couplings themselves also change with the energy scale, one might hope that Newtons coupling[1] will change exactly like k^{-2} (for large k,

[1] We will write G_k from now on and call it Newtons coupling as it changes with the scale k.

at least) making the strength of the gravitational coupling $G_k k^2$ finite for all k. This idea is central to our approach.

The theoretical framework we use to search for this completion is Weinberg's idea of Asymptotic Safety [4, 5]. This program revolves around the conjecture outlined above that all couplings become finite as the energy scale at which they are probed goes to infinity. This would be ensured by an (attractive) fixed point in theory space where the dimensionless couplings are scale-independent. For a theory to be asymptotically safe, in addition to having a UV-fixed point, the surface of all trajectories ending up in the fixed point should be finite dimensional. In this way there are finitely many relevant parameters that need to be measured, such that the theory is predictive, see Fig. 2. For gravity specifically, we also require the low-energy limit to approach general relativity.

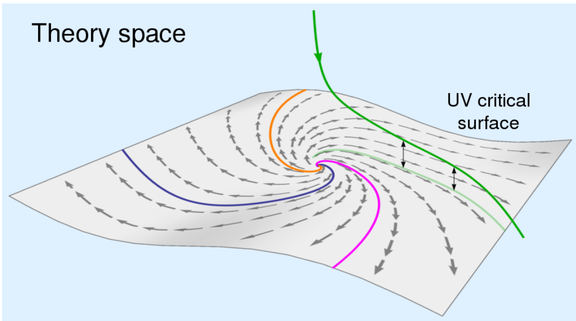

Figure 2: Renormalisation group flow trajectories in theory space, with arrows pointing from the UV to the IR. All actions that flow into the fixed point, like the orange, blue and magenta lines, are asymptotically safe and constitute the UV critical surface. The dimension of this surface is the number of parameters needed to measure to find the trajectory realised in nature. Trajectories like the green one are deemed irrelevant. Taken from [6].

The standard model revolves around a noninteracting (trivial) fixed point, where all couplings are zero, making it possible to use perturbative expansions around the uncoupled theory. For gravity, we are looking for an interacting (nontrivial) fixed point and need to use nonperturbative methods. We will now move on to see how to find the scale-dependence and fixed point(s) of a theory.

2 The renormalisation group

Our investigative tool is the (functional) renormalisation group. The idea, due to Wilson [7], is to integrate out degrees of freedom from small to big scales (from high to low energies), giving an effective description. The concept of the renormalisation group comes from statistical physics, where for example spins in a lattice are step-by-step combined in blocks, thereby changing the scale. To see how it works, consider the Wick-rotated path integral or partition function

$$\mathcal{Z} = \int_{C^\infty(M)_{\leq \Lambda_0}} \mathcal{D}\varphi \, \mathrm{e}^{-S_{\Lambda_0}[\varphi]/\hbar}, \tag{2.1}$$

where $S_{\Lambda_0}[\varphi]$ is the action describing a field theory and the integral is over the space of smooth functions with energy smaller than the so-called cut-off scale Λ_0. Changing the scale, or integrating out modes, then corresponds to performing the high-energy integral of this partition function. This gives an effective action involving only low-energy modes. This procedure can then be iterated. It starts with splitting the field φ in high ($\Lambda_1 \leq |p| \leq \Lambda_0$) and low momenta ($|p| \leq \Lambda_1$) modes

$$\begin{aligned}\varphi(x) &= \int_{|p|\leq\Lambda_0} \frac{d^d p}{(2\pi)^d} \mathrm{e}^{ip\cdot x} \tilde{\varphi}(p) \\ &= \int_{|p|\leq\Lambda} \frac{d^d p}{(2\pi)^d} \mathrm{e}^{ip\cdot x} \tilde{\varphi}(p) + \int_{\Lambda_1<|p|\leq\Lambda_0} \frac{d^d p}{(2\pi)^d} \mathrm{e}^{ip\cdot x} \tilde{\varphi}(p) \\ &\equiv \phi(x) + \chi(x),\end{aligned} \tag{2.2}$$

which also means the measure is split

$$\mathcal{D}\varphi = \mathcal{D}\phi\mathcal{D}\chi. \tag{2.3}$$

Doing the integral over the high-energy modes then gives the effective action at scale Λ_1

$$\Gamma_{k,\Lambda_1} = S^{\text{eff}}_{\Lambda_1} = -\hbar \log \left[\int_{C^\infty(M)_{(\Lambda_1,\Lambda_0]}} \mathcal{D}\chi \; \mathrm{e}^{-S_{\Lambda_0}[\phi+\chi]/\hbar} \right], \tag{2.4}$$

which thus involves only the low-energy modes. We stress that the partition function for any effective action is exactly the same as the original one: $\mathcal{Z}_{\Lambda_1}(g_i(\Lambda_1)) = \mathcal{Z}_{\Lambda_0}(g_{i,0}(\Lambda_0))$, with g_i the couplings of the theory. These couplings thus do change with the scale.

Repeating this procedure infinitesimally gives an *exact* equation describing the change in the effective action as quantum fluctuations up to a momentum scale are integrated out [8, 9]. This functional renormalisation group equation, flow equation, or Wetterich equation, adapted to gravity first in [10], is

$$k\partial_k \Gamma_k = \frac{1}{2} \mathrm{Tr} \left[(\Gamma_k^{(2)} + \mathcal{R}_k)^{-1} k\partial_k \mathcal{R}_k \right], \tag{2.5}$$

where k is again the reference scale, Γ_k the effective action, $\Gamma_k^{(2)}$ its second variation with respect to the fluctuation fields and Tr denotes a functional trace which sums over momenta, frequencies, internal indices and fields. The regulator $\mathcal{R}_k$ vanishes for $p^2 \gg k^2$, while behaving like k^2 for $p^2 \ll k^2$. Apart from these requirements, one can freely choose the regulator and the outcome of a calculation should not qualitatively depend on this choice. The requirements establish that $\mathcal{R}_k$ acts as an infrared regulator, giving a mass-like term to modes with momenta $p^2 \lesssim k^2$, removing a possible pole at $p = 0$. Meanwhile $\partial_k \mathcal{R}_k$ provides an ultraviolet regularisation, ensuring that the high-energy modes with $p^2 \gtrsim k^2$ do not contribute to the trace. This establishes that the change of Γ_k is directed by integrating out quantum fluctuations with momenta $p \approx k$.

The right-hand side of the Wetterich equation includes the *full* (infinite loop) effective propagator

$$G(\Delta) = \left(\Gamma_k^{(2)}(\Delta) + \mathcal{R}_k(\Delta) \right)^{-1}. \tag{2.6}$$

The Wetterich equation describes how, by integrating out modes (*i.e.* changing the scale), the couplings in the effective action vary to account for the change in the degrees of freedom. In this way it is ensured that the partition function is independent of the scale. The running of the couplings is captured in the beta functions of the theory, defined by

$$\beta_i \equiv k\frac{\partial g_i}{\partial k} = k\partial_k g_i. \tag{2.7}$$

The beta function conveys the quantum effect of integrating out the high-energy modes. Generally, the beta-function for g_i is a function of all couplings: $\beta_i(g_j)$. Fixed points are points where all beta functions of a theory vanish simultaneously. It is this set of points that we are interested in and will search for, using the Wetterich equation.

Although the Wetterich equation (2.5) is exact, it is probably impossible to solve it exactly for the general case. Therefore, we need to choose a theory space truncation and an ansatz for the effective action. After that, we can take the second variation, inverse and trace, while projecting on the chosen theory space. The goal of this project is to be as general as possible in these steps, to find the full momentum dependence of the coupling constants by using nonperturbative methods in our computation. In the next chapter, we will investigate what is a feasible and general ansatz. This strategy shows the strength of the Asymptotic Safety program. With the flow equation as its tool, it gives a systematic research process for quantum theories in stead of quantising a known classical theory [11].

2.1 Pseudospectral methods

We will see that explicitly including the momentum dependence of couplings will result in integro-differential equations for the flow equations, as there may arise integrals over loop-momentum from the trace. Therefore also the fixed point equations we need to solve to find the fixed points and functions (encoding the momentum dependence) will be integro-differential equations. Due to their complexity, we do not expect to solve them exactly and need to resort to numerical approximation methods. The momentum-integral entails that we need to find a solution on the entire positive real axis. Thus, approximating the function by a Taylor series (possibly with finite radius of convergence) is not suitable. We will therefore (try to) solve the fixed point equations nonperturbatively using pseudospectral methods [12–14]. We will use an ansatz of the form

$$f(x) = (1+x)^{\sigma} \sum_{n=0}^{N} a_n \, R_n(x), \tag{2.8}$$

where we have introduced the scaling parameter σ, the constant in $1+x$ ensures there is no artificial root imposed at $x = 0$, and we use a basis of Chebyshev rational functions, defined as

$$R_n(x) \equiv T_n\left(\frac{x-L}{x+L}\right) \tag{2.9}$$

as they are bounded and span $[0, \infty)$, which is the domain of our momentum squared. Here $T_n(x)$ is a Chebyshev polynomial of the first kind, defined by $T_n(\cos(x)) = \cos(nx)$, and L is an arbitrary reference scale, to be determined later. The scaling is added as a finite sum of bounded functions may not give the asymptotic behaviour that might be needed.

We will use this ansatz with the collocation method, which implies solving the equation for $N+1$ collocation points that are the roots of R_{N+1}, to determine the $N+1$ coefficients a_n. After finding a solution for a certain N, we will increase the number of coefficients and solve again. If the same solution is found, the finite-sum approximation works and this solution is deemed robust.

For integro-differential equations that are nonlinear, the method outlined above will not work. To investigate these equations, we will use the Newton-Raphson method. Consider a first order Taylor series approximation (around x_0) of some function $f(x)$:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \ldots \quad . \tag{2.10}$$

For $f(x) = 0$ we can rewrite this to

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}, \tag{2.11}$$

such that x is now a (first order) approximation of the root $f(x) = 0$. Doing this iteratively as $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ is the essence of the Newton-Raphson method to find the root of a function. If the function is sufficiently nice (meaning that $f'(x) \neq 0$ on some interval around the root and that $f''(x)$ is continues on some interval around the root) and the initial guess is sufficiently close (close enough that the first order Taylor series suffices and we can ignore higher order terms), the method abides a quadratic convergence. The method can be straightforwardly generalised to a system of equations. The algorithm then becomes

$$x_{n+1} = x_n - J(x_n)^{-1}R(x_n) \tag{2.12}$$

where $J(x_n)$ is the Jacobian at x_n and $R(x_n)$ is the residu, *i.e.* the value of the functions at the previous iteration. The goal is to bring this residu to zero for the fixed point equations.

Our strategy will be to choose an N, make some initial guess for the coefficients a_n and then apply the Newton-Raphson method to the fixed point equation evaluated at N collocation points to iteratively update these coefficients. Once the residu has become (nearly) zero, N can be increased to test if the finite sum approximation is robust.

The setup for the rest of this thesis is as follows. In the next chapter we will set the stage by looking at a nonperturbative calculation of a flow using nonlocal heat kernel techniques. This will give us some insight in this technique as well as a reasonable theory space and ansatz and some understanding of a simplification we will use. In chapter 4 we will then look at the flows including the form-factor terms $\int \sqrt{g} R F(\Delta) R$ and $\int \sqrt{g} C^{\mu\nu\rho\sigma} W(\Delta) C_{\mu\nu\rho\sigma}$. Thereafter, in chapter 5 we will determine and solve the resulting fixed point equations. First, we will investigate a system where $F_k(\Delta) = 0$ and afterwards look at the full system. In chapter 6 we will investigate the physical implications of the obtained results.

3 Einstein-Hilbert and the need for form factors

To exploit the strenght of the Asymptotic Safety program, we want to investigate an effective action (*i.e.* a set of theories) as general as possible. Previous investigations for example have been performed using a polynomial expansion of $f(R)$-gravity, where the action is some undetermined function of the Ricci scalar. The function is then determined from the fixed point data. See for example [15] for a review. Another scheme is the derivative expansion, where terms including up to a certain number of spacetime derivatives are included. These types of expansions, however are not sufficient to investigate questions related to the propagation of fields and degrees of freedom supported by the fixed point [16]. Truncating a derivative expansion of terms contributing to the propagator at finite order gives problems regarding the Ostrogradski stability [17, 18]. (The kinetic term of the Hamiltonian is no longer bounded from below, giving an instable system where the vacuum can instantly decay, negative-norm states appear, or the S-matrix becomes non-unitary.) To investigate these questions we will use an effective action with momentum dependent functions and investigate their flow. These *form factors* apart from the reference scale k also depend on the momenta of the fluctuation fields. They can be regarded as a derivative expansion of infinite order.

We will see that the form factors give nontrivial contributions to the propagator and thereby the physics described. Knowing the form factors is expected to be essential to understand the spacetime structure entailed by Asymptotic Safety [16]. Previously, it has been shown that quantising the Einstein-Hilbert action in a flat background already gives nontrivial momentum dependence of the form factors [19], showing that the nonperturbative effects are important to the momentum dependence of the graviton propagator.

Let us now look at an example where we will already see the need for form factors.

3.1 Flows from a massless scalar field minimally coupled to gravity

To understand the procedure of getting flow equations from an ansatz and getting some insight in a reasonable theory space, let us start by looking at the action describing a massless scalar field minimally coupled to gravity

$$\Gamma_k = \frac{1}{2}\int d^4x\sqrt{g}\,\varphi\Delta\varphi, \tag{3.1}$$

where $\Delta \equiv -g^{\mu\nu}D_\mu D_\nu = -D^2$ is the positive definite Laplacian[2] constructed from the metric $g_{\mu\nu}$ and g is the determinant of the metric. Here and throughout this thesis we work using the Euclidean signature. This effective action describes a quantum scalar field in a curved background, *i.e.* coupled to classical gravity without graviton fluctuations. We want to solve the Wetterich equation

$$k\partial_k\Gamma_k(\Delta) = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)}(\Delta) + \mathcal{R}_k(\Delta)\right)^{-1} k\partial_k\mathcal{R}_k(\Delta)\right], \tag{3.2}$$

[2] D_μ is the covariant derivative, working on a tensor T^α_β as $D_\mu T^\alpha_\beta = \partial_\mu T^\alpha_\beta + \Gamma^\alpha_{\mu\lambda}T^\lambda_\beta - \Gamma^\lambda_{\mu\beta}T^\alpha_\lambda$

to find the flow from the scalar field. Let us introduce the notation $k\partial_k \mathcal{R}_k \equiv \dot{\mathcal{R}}_k$ and define the operator-structure

$$O(\Delta) \equiv \left(\Gamma_k^{(2)}(\Delta) + \mathcal{R}_k(\Delta)\right)^{-1} \dot{\mathcal{R}}_k(\Delta). \tag{3.3}$$

The first step in the calculation is to obtain the second variation of our action. The fluctuating field is φ, so we have

$$\Gamma_k^{(1)} \equiv \frac{\delta \Gamma_k}{\delta \varphi} = \frac{1}{2}\Delta\varphi + \frac{1}{2}\Delta\varphi \tag{3.4}$$

and

$$\Gamma_k^{(2)} \equiv \frac{\delta \Gamma_k^{(1)}}{\delta \varphi} = \Delta. \tag{3.5}$$

The operator-structure then becomes

$$O_\varphi(\Delta) = (\Delta + \mathcal{R}_k(\Delta))^{-1} \dot{\mathcal{R}}_k(\Delta). \tag{3.6}$$

As we are interested in the full momentum dependence, we will employ nonlocal heat kernel methods to evaluate the operator trace.

3.2 Nonlocal heat kernel

To evaluate the Wetterich equation we have to take a functional trace over an operator-structure, (*i.e.* some function of Laplacians). To start, we perform an inverse Laplace transform:

$$\mathrm{Tr}\left[O(\Delta)\right] = \int_0^\infty ds\ \widetilde{O}(s)\ \mathrm{Tr}\left[e^{-s\Delta}\right]. \tag{3.7}$$

Then we identify

$$\mathrm{Tr}\left[e^{-s\Delta}\right] = \int d^d x\ \sqrt{g}\ \mathrm{Tr}\left[H(s)\right], \tag{3.8}$$

where d is the dimension and $H(s)$ is the heat kernel [20]. We will use $d = 4$ throughout this thesis. At this point we can use heat-kernel methods to evaluate the trace. We will use the non-local heat kernel expansion from [21], giving

$$\begin{aligned}\mathrm{Tr}[H(s)] = \int d^4 x\ \sqrt{g}\ \frac{1}{(4\pi s)^2}\mathrm{tr}\,\{&\mathbb{1} + \mathbb{1} s g_R(sx) R + s^2\left[\mathbb{1} R f_R(sx) R + \mathbb{1} C^{\mu\nu\alpha\beta} f_C(sx) C_{\mu\nu\alpha\beta}\right]\\ &+\mathcal{O}(R^3)\},\end{aligned} \tag{3.9}$$

where R is the Ricci scalar, $C_{\mu\nu\alpha\beta}$ is the Weyl tensor (the traceless component of Riemann tensor), tr denotes the trace over spacetime indices only, and we use the notation $x = \Delta$. The Ricci scalar and Weyl tensor constitute a full basis as a possible third basis element can be eliminated when only considering terms up to second order in the curvature, using that for $n \geq 0$

$$\int d^d x\left[R_{\mu\nu\rho\sigma}\Delta^n R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}\Delta^n R^{\mu\nu} + R\Delta^n R\right] = \mathcal{O}(R^3), \tag{3.10}$$

as can be proven using Bianchi identities [22, 23]. Note that the Weyl tensor is a function of the Riemann tensor, the Ricci tensor and the Ricci scalar. The heat kernel coefficients in (3.9) are

$$g_R(x) = -\frac{1}{2x} + \frac{1}{4x}(x+2)\mathrm{f}(x), \tag{3.11}$$

$$f_C(sx) = \frac{1}{12sx} + \frac{1}{2s^2x^2}[\mathrm{f}(sx) - 1], \tag{3.12}$$

$$f_R(sx) = \frac{1}{32}\mathrm{f}(sx) + \frac{1}{8sx}\mathrm{f}(sx) - \frac{7}{48sx} - \frac{1}{8s^2x^2}[\mathrm{f}(sx) - 1] + \frac{2}{3}f_C(sx), \tag{3.13}$$

where $\mathrm{f}(sx)$ is defined as

$$\mathrm{f}(sx) = \int_0^1 d\alpha\, \mathrm{e}^{-\alpha(1-\alpha)sx}. \tag{3.14}$$

This heat kernel expansion is non-local in the sense that it is not a local, perturbative expansion in small s. The expansion is valid for *all* s. This is very relevant as s will be identified with the momentum, meaning that this technique for taking the trace includes the *full momentum dependence*. Plugging the above into the Wetterich equation gives

$$\begin{aligned} k\partial_k\Gamma_k = \frac{1}{32\pi^2}\int_0^\infty ds\, \widetilde{O}_\varphi(s) \int d^4x\, \sqrt{g}\, \mathrm{tr}\, \big\{ \mathbb{1}s^{-2} + \mathbb{1}s^{-1}g_R(sx)R \\ + \big[\mathbb{1}C_{\mu\nu\alpha\beta}f_C(sx)C^{\mu\nu\alpha\beta} + \mathbb{1}Rf_R(sx)R\big] + \mathcal{O}(R^3)\big\}\,. \end{aligned} \tag{3.15}$$

We thus see that performing a nonperturbative calculation to find the full momentum dependence induces terms of the form $\mathbf{R}\, f_{\mathbf{R}}\, \mathbf{R}$, where $\mathbf{R}$ is some term of order R and $f_{\mathbf{R}}$ is some function of the Laplacian. This function is nonlocal and encodes momentum dependence.

Performing this little exercise therefore gave us some insight in the structure of the theory space we should consider and the ansatz we should use if we want to include the full momentum dependence, as we see that the (metric in the) Laplacian appearing in the second variation of (3.1) already induces form-factor-like terms $\mathbf{R}\, f_{\mathbf{R}}\, \mathbf{R}$.

3.3 Ansatz

We are interested in the flows induced by gravity itself. Keeping in mind that we want (our ansatz) to be as general as possible and seeing the structures appearing above, we pick the effective action ansatz

$$\Gamma_k = \frac{1}{16\pi G_k}\int d^4x\sqrt{g}\, \{2\Lambda_k - R + C^{\mu\nu\rho\sigma}W_k(\Delta)C_{\mu\nu\rho\sigma} - RF_k(\Delta)R\}\,. \tag{3.16}$$

The first two terms constitute the standard Einstein-Hilbert action that gives the Einstein field equations of general relativity (by setting their first variation to zero). In total, this effective action entails the most general pure gravity ansatz up to second order in the curvature where, like above, the Ricci scalar and Weyl tensor constitute a basis. This generality is due to the usage of the form factors $W_k(\Delta)$ and $F_k(\Delta)$, which are arbitrary functions of the Laplacian and carry the physical momentum dependence of couplings in the effective action. We will assume that the form factors admit a Laplace transform, such that we can use the nonlocal heat kernel techniques outlined above.

We will also assume that we can freely integrate by parts without generating boundary terms. Why it is important that we use a general function and not just some polynomial approximation can be seen from the following example. Imagine that the true function would be $\mathrm{e}^{-\Delta}$, a positive definite function. A series expansion, however, would introduce -at every second order- new unphysical poles in the propagator G. As this is highly unwanted, we use nonlocal form factors.
Due to the conformal factor problem [24], the spin 0 sector comes with a negative kinetic term. Therefore, we write the R^2-terms with a minus sign, such that the form factor $F_k(\Delta)$ is a positive function.

Note that this really is the most general action up to second order in the curvature, as a function of Laplacians working on the Ricci scalar only leads to total derivatives or surface terms to the action [16]:

$$\int \sqrt{g}\, g(\Delta)R = \int \sqrt{g} \int_0^\infty ds\tilde{g}(s)\mathrm{e}^{-s\Delta}R = \int \sqrt{g}\, g(0)R, \tag{3.17}$$

as we can expand the exponential and integrate by parts to find only the constant piece remains. The choice of the basis terms R and $C_{\mu\nu\rho\sigma}$ is very useful as the second variation of the Weyl tensor squared is proportional to the transverse traceless projector, when evaluated in flat space [21]. In this basis the form factor W_k is thus directly related to the propagator of the spin-2 component of the graviton fluctuation $h_{\mu\nu}$ [25–27]. In other words: then the form factors W_k and F_k give the nontrivial momentum dependence of the transverse traceless and scalar propagator, respectively, when the background metric is chosen to be flat Euclidean space. We will make use of this when investigating the physical implications in chapter 6.

In writing down this ansatz we do implicitly assume one can use some kind of curvature expansion. And using at most two orders of curvature is an approximation, but -apart from the fact that including an extra order of curvature dramatically complicates all computations- one that makes sense on physical grounds. As we have seen in equation (2.6), to determine a propagator one takes the second variation. For the flat-space propagator, one thereafter sets all curvatures to zero. This means that any term in the action of third order in the curvature will not contribute to the flat-space propagator. As truncation for the theory space we choose to include terms like the one appearing in our ansatz.

3.4 Conformally reduced setting

As gravity is a theory of spacetime, any spacetime that we work with should arise from the theory. In other words, no specific spacetime can have any distinguished role. We cope with this by using the background field formalism. This entails working with an arbitrary background spacetime and in the end verifying that the results were independent of the choice of background. This arbitrary background makes it possible to distinguish large and small wavelengths, which we need for the renormalisation framework. For most of the calculations, we will work with conformally reduced gravity, where the metric becomes

$$g_{\mu\nu} = \varphi^2 \hat{g}_{\mu\nu} \equiv \left(1 + \frac{1}{4}h\right) \hat{g}_{\mu\nu}, \tag{3.18}$$

where h is the fluctuation field and has no indices and $\hat{g}_{\mu\nu}$ is the fixed, but arbitrary background metric. This is an approximation: we only consider scalar fluctuations. We will also set all fluctuations to zero after the second variation, meaning we only consider quantities in the action up to second order in h. In the following, a hat will denote that a quantity is taken with respect to the background metric $\hat{g}_{\mu\nu}$.

The conformally reduced setting has -apart from greatly simplifying calculations- the advantage that there is no gauge freedom, such that we do not need to gauge fix and therefore also do not have any Fadeev-Popov ghosts.

This split of the metric entails the following transformations

$$\begin{aligned}
R &= \frac{1}{\varphi^2}\hat{R} - \frac{6}{\varphi^4}\varphi\hat{D}^2\varphi, \\
\sqrt{g} &= \varphi^4\sqrt{\hat{g}}, \\
C_{\mu\nu\rho\sigma} &= \varphi^2\hat{C}_{\mu\nu\rho\sigma}, \\
R_{\mu\nu} &= \hat{R}_{\mu\nu} + 2\varphi\left(\hat{D}_\mu\hat{D}_\nu - \frac{1}{4}\hat{g}_{\mu\nu}\hat{D}^2\right)\frac{1}{\varphi},
\end{aligned} \tag{3.19}$$

and the Laplacian working on a scalar χ becomes

$$\Delta\chi = \frac{1}{\varphi^2}\hat{\Delta}\chi - \frac{2}{\varphi^3}(\hat{D}_\mu\varphi)\hat{D}^\mu\chi. \tag{3.20}$$

3.5 The flow of the Einstein-Hilbert terms

It is insightful to now look at the flows induced by the first two terms in our action (3.16):

$$\begin{aligned}
\Gamma_{EH} &= \frac{1}{16\pi G_k}\int d^4x\sqrt{g}\,\{\,2\Lambda_k - R\} \\
&= \frac{1}{16\pi G_k}\int d^4x\sqrt{\hat{g}}\,\{\,2\Lambda_k\varphi^4 - \hat{R}\varphi^2 + 6\varphi\hat{D}^2\varphi\Big\} \\
&= \frac{1}{16\pi G_k}\int d^4x\sqrt{\hat{g}}\,\{2\Lambda\left(1 + \frac{1}{2}h + \frac{1}{16}h^2\right) - \hat{R} - \frac{1}{4}\hat{R}h + \frac{3}{4}(\hat{D}^2h) - \frac{3}{32}(\hat{D}_\mu h)(\hat{D}^\mu h) \\
&\qquad + \mathcal{O}(h^3)\}\,,
\end{aligned} \tag{3.21}$$

where we include all terms with up to two powers of h as discussed. The first variation, without prefactor, then becomes

$$\Gamma^{(1)}_{EH} \equiv \frac{\delta\Gamma_{EH}}{\delta h} = \Lambda\left(1 + \frac{1}{4}h\right) - \frac{1}{4}\hat{R} + \frac{3}{4}\hat{D}^2 + \frac{3}{16}(\hat{D}^2h), \tag{3.22}$$

and the second variation is

$$\Gamma^{(2)}_{EH} \equiv \frac{\delta \Gamma^{(1)}_{EH}}{\delta h} = \frac{1}{4}\Lambda + \frac{3}{16}\hat{D}^2 = \frac{1}{4}\Lambda - \frac{3}{16}\hat{\Delta}. \tag{3.23}$$

So, including the prefactor we have: $\Gamma^{(2)}_{EH} = \frac{1}{64\pi G_k}\Lambda - \frac{3}{256\pi G_k}\hat{\Delta}$. These terms are of order $\mathcal{O}(R^0)$ in the curvature. We will therefore use the nonlocal heat kernel up to second order in the curvature to take the trace. Let us denote the operator structure as

$$O_{EH}(\hat{\Delta}) = \left(\frac{1}{4}\Lambda - \frac{3}{16}\hat{\Delta} + \mathcal{R}_k\right)^{-1} \dot{\mathcal{R}}_k. \tag{3.24}$$

The flow induced by these terms (*i.e.* the right hand side of the Wetterich equation) becomes

$$\begin{aligned}
\frac{1}{2}\mathrm{Tr}[O_{EH}(\Delta)] &= \frac{1}{2}\int_0^\infty ds\, \widetilde{O}_{EH}(s) \int d^4x\, \sqrt{\hat{g}}\, \frac{1}{(4\pi s)^2}\, \mathrm{tr}\Bigg\{\mathbb{1} + \mathbb{1}s\frac{\hat{R}}{6} \\
&+ s^2\left[\mathbb{1}\hat{C}_{\mu\nu\alpha\beta} f_C(s\hat{x})\hat{C}^{\mu\nu\alpha\beta} + \mathbb{1}\hat{R} f_R(s\hat{x})\hat{R}\right] + \mathcal{O}(\hat{R}^3)\Bigg\} \\
&= \frac{1}{32\pi^2}\int_0^\infty ds\, \widetilde{O}_{EH}(s)\int d^4x\, \sqrt{\hat{g}}\, \mathrm{tr}\Bigg\{\mathbb{1}s^{-2} + \mathbb{1}s^{-1}\frac{\hat{R}}{6} \\
&+ \Bigg[\mathbb{1}\hat{C}_{\mu\nu\alpha\beta}\left(\frac{1}{12 s\hat{x}} + \frac{1}{2s^2\hat{x}^2}[\mathrm{f}(s\hat{x}) - 1]\right)\hat{C}^{\mu\nu\alpha\beta} \\
&+ \mathbb{1}\hat{R}\left(\frac{1}{32}\mathrm{f}(s\hat{x}) - \frac{13}{144 s\hat{x}} + \frac{1}{8s\hat{x}}\mathrm{f}(s\hat{x}) + \frac{5}{24 s^2\hat{x}^2}[\mathrm{f}(s\hat{x}) - 1]\right)\hat{R}\Bigg] + \mathcal{O}(\hat{R}^3)\Bigg\},
\end{aligned} \tag{3.25}$$

with $\hat{x} = \hat{\Delta}$ and where we use that $g_R(0) = \frac{1}{6}$ as we are not considering terms with Laplacians working on the Ricci scalar as mentioned above. We can see that these terms induce flows in all four couplings of our ansatz. Later on, we will see that the $F_k(\Delta)$-term will induce flows in Λ_k, G_k, $F_k(\Delta)$ and $W_k(\Delta)$. The term of $W_k(\Delta)$ will only contribute to its own flow.

To compute the above expression, we observe there are a few similar integrals to be performed. There is either an expression like $\int_0^\infty ds\, s^{-n}$ for $n = 1, 2$ or like $\int_0^\infty ds\, s^{-n}\, \mathrm{f}(s\hat{x})$ for $n = 0, 1, 2$.
In the first case, we can use that for $n > 0$

$$\int_0^\infty ds\, \widetilde{O}(s) s^{-n} = \frac{1}{\Gamma(n)}\int_0^\infty dz\, O(z) z^{n-1}, \tag{3.26}$$

which can be derived from the definition of the Gamma-function [28, 29]. For the expressions including the function $\mathrm{f}(s\hat{x})$, the integrals are worked out in Appendix A.1. The result for the trace

(3.25) is

$$
\begin{aligned}
\mathrm{Tr}[O_{EH}(\Delta)] = \frac{1}{32\pi^2}\int d^4x\,\sqrt{\hat{g}}\,\Bigg\{&\int_0^\infty dz\,z\,O_{EH}(z) + \int_0^\infty dz\,O_{EH}(z)\frac{\hat{R}}{6} \\
&+ \hat{C}_{\mu\nu\alpha\beta}\left[\frac{\hat{x}^{-1}}{12}\int_0^\infty dz\,O_{EH}(z) - \frac{\hat{x}^{-2}}{2}\int_0^\infty dz\,z\,O_{EH}(z)\right. \\
&\left.+\frac{1}{6}\int_0^{\frac{1}{4}} dv\,(1-4v)^{\frac{3}{2}}\,O_{EH}(\hat{x}v) + \int_0^\infty dv\left(v-\frac{1}{6}\right)O_{EH}(\hat{x}v)\right]\hat{C}^{\mu\nu\alpha\beta} \\
&+ \hat{R}\left[\frac{1}{16}\int_0^{\frac{1}{4}} dv\frac{1}{\sqrt{1-4v}}\,O_{EH}(\hat{x}v) - \frac{13\hat{x}^{-1}}{144}\int_0^\infty dz\,O_{EH}(z) - \frac{5\hat{x}^{-2}}{24}\int_0^\infty dz\,z\,O_{EH}(z)\right. \\
&+ \frac{\hat{x}^{-1}}{8}\left(-\int_0^{\frac{1}{4}} dv\,\sqrt{1-4v}\,O_{EH}(\hat{x}v) + \int_0^\infty dv\,O_{EH}(\hat{x}v)\right) \\
&\left.+\frac{5}{24}\left(\frac{1}{6}\int_0^{\frac{1}{4}} dv\,(1-4v)^{\frac{3}{2}}\,O_{EH}(\hat{x}v) + \int_0^\infty dv\left(v-\frac{1}{6}\right)O_{EH}(\hat{x}v)\right)\right]\hat{R} + \mathcal{O}(\hat{R}^3)\Bigg\},
\end{aligned}
\tag{3.27}
$$

as $\Gamma(2) = \Gamma(1) = 1$. To complete the calculation of the flow we will have to choose a regulator and perform the integrals.

As mentioned above, there will be contributions from the other terms in the ansatz relevant for the running of the cosmological constant and Newton's coupling. For now, for the sake of argument, let us just denote the trace of the relevant contributions as $\frac{1}{32\pi^2}\int d^4x\,\sqrt{\hat{g}}\,\left(\bar{G}_0 + \bar{G}_1\hat{R}\right)$. Then the Wetterich equation gives

$$
k\partial_k\frac{1}{16\pi}\left(2\frac{\Lambda_k}{G_k} - \frac{\hat{R}}{G_k}\right) = \frac{1}{32\pi^2}\,\left(\bar{G}_0 + \bar{G}_1\hat{R}\right). \tag{3.28}
$$

Equating orders of $\hat{R}$ on both sides we get the flow for the cosmological constant

$$
k\partial_k\frac{\Lambda_k}{G_k} = k\frac{G_k\partial_k\Lambda_k - \Lambda_k\partial_k G_k}{G_k^2} = \frac{1}{4\pi}\bar{G}_0, \tag{3.29}
$$

and for Newtons coupling

$$
k\partial_k\frac{1}{G_k} = -\frac{1}{2\pi}\bar{G}_1. \tag{3.30}
$$

As mentioned in chapter 1, we are interested in the fixed points of dimensionless couplings, so let us introduce the dimensionless quantities

$$
\begin{aligned}
g &= G_k\cdot k^2, \\
\lambda &= \Lambda_k\cdot k^{-2},
\end{aligned}
\tag{3.31}
$$

in terms of which the equations become

$$
k^4\frac{4g\lambda + kg\partial_k\lambda - k\lambda\partial_k g}{g^2} = \frac{1}{4\pi}\bar{G}_0, \tag{3.32}
$$

$$
k^2\frac{2g - k\partial_k g}{g^2} = -\frac{1}{2\pi}\bar{G}_1. \tag{3.33}
$$

The fixed point equations are now obtained by requiring $\partial_k g = \partial_k \lambda = 0$, giving

$$0 = -4\lambda + \frac{g}{4\pi}\bar{G}_0, \tag{3.34}$$

$$0 = -2 - \frac{g}{2\pi}\bar{G}_1. \tag{3.35}$$

Looking at the second equation, we note that, as the contribution $\bar{G}_1$ will be positive, g must be negative to solve this equation. As this indicates a repulsive gravitational interaction, we conclude the conformally reduced setting is not sufficient here. We will come back to this after completing the rest of the calculation, in chapter 5.

4 Flows including form factors

Let us now calculate the contributions to the flow equations from the full ansatz including the form factor terms. To do this, we first perform an inverse Laplace transform of the form factor term and then split the Laplacian into the background piece and other contributions:

$$\mathbf{R}\, f_{\mathbf{R}}(\Delta)\, \mathbf{R} = \int_0^\infty ds \mathbf{R}\, \widetilde{f}_{\mathbf{R}}(s)\, \mathrm{e}^{-s\Delta}\, \mathbf{R} = \int_0^\infty ds \widetilde{f}_{\mathbf{R}}(s)\, \mathrm{e}^{-s\left(\hat{\Delta}+\mathbb{T}_1+\mathbb{T}_2\right)}\, \mathbf{R}, \tag{4.1}$$

where $\mathbb{T}_i$ denotes the tensor structure with i powers of h. As we do not know the inverse Laplace transform $\widetilde{f}_{\mathbf{R}}(s)$, we want to undo the transform, with respect to the background Laplacian $\hat{\Delta}$. In order to retain the full momentum dependence, however, we do not want to expand around it. In stead, we use multicommutators, to write (see Appendix B.1)

$$\mathrm{e}^{X+\epsilon Y} = \mathrm{e}^{X}\left[1+\epsilon\tilde{M}(0,X,Y)+\frac{\epsilon^2}{2}\left(\tilde{M}(0,X,Y)^2+\partial_\epsilon\tilde{M}(0,X,Y)\right)+\mathcal{O}(\epsilon^3)\right], \tag{4.2}$$

with $\tilde{M}(\epsilon,X,Y)$ and $\partial_\epsilon\tilde{M}(\epsilon,X,Y)$ defined in equations (B.7) and (B.9), respectively. Using this, we can undo the Laplace transform and proceed to take the second variation and trace.

4.1 Second variation

We will look at the flow equations of our full ansatz including form factors (3.16). The first step is to take the second variation. For the first two terms we know it is $\Gamma^{(2)}_{EH} = \frac{1}{64\pi G_k}\Lambda_k - \frac{3}{256\pi G_k}\hat{\Delta}$.

4.1.1 Ricci-term

Let us now look at the Ricci-term $-\frac{1}{16\pi G_k}\int\sqrt{g}RF_k(\Delta)R$. Using equation 3.20, we find $\Delta = \hat{\Delta}-\frac{1}{4}h\hat{\Delta}-\frac{1}{4}(\hat{D}_\mu h)\hat{D}^\mu+\frac{1}{16}h^2\hat{\Delta}+\frac{1}{8}h(\hat{D}_\mu h)\hat{D}^\mu$ for a Laplacian working on the Ricci scalar. Neglecting for now the prefactor and using the framework outlined above, we then get (also see Appendix B.2)

$$\begin{aligned}
&\int\sqrt{g}RF_k(\Delta)R = \int\sqrt{g}R\int_0^\infty ds\widetilde{F}_k(s)\mathrm{e}^{-s\Delta}R\\
&= \int\sqrt{g}R\int_0^\infty ds\widetilde{F}_k(s)\mathrm{e}^{-s\hat{\Delta}}\ \{1\\
&+\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^{j+k+j}}{4(j+1)!}\binom{j}{k}s^{j+1}\hat{\Delta}^{j-k}\left(\left(h\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu-\frac{1}{4}h^2\hat{\Delta}-\frac{1}{2}h(\hat{D}_\mu h)\hat{D}^\mu\right)\hat{\Delta}^k\right)\\
&+\frac{1}{2}\left(\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^k}{4(j+1)!}\binom{j}{k}s^{j+1}\hat{\Delta}^{j-k}\left(\left(h\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu+\mathcal{O}(h^2)\right)\hat{\Delta}^k\right)\right)^2\\
&+\frac{1}{2}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{n=1}^{\infty}\sum_{l=0}^{n}\frac{(-1)^{k+l+1}}{16(n+j+2)!}\binom{n}{l}\binom{j}{k}s^{n+j+1}\hat{\Delta}^{j-k}\left(\left((h\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu)\hat{\Delta}^{n-l}\right.\right.\\
&\left.\left.\left(h\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu\right)\hat{\Delta}^l-\hat{\Delta}^{n-l}(h\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu)\hat{\Delta}^l(h\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu)\right)\hat{\Delta}^k\right)\Big\}\ R,
\end{aligned} \tag{4.3}$$

where we included terms up to second order in h and used equation (B.6) to work out the multi-commutators. Then we use equations (3.19) to write

$$R = \hat{R} - \frac{1}{4}\hat{R}h - \frac{3}{4}\hat{D}^2 h + \frac{1}{16}\hat{R}h^2 + \frac{3}{8}h\hat{D}^2 h + \frac{3}{32}(\hat{D}_\mu h)\hat{D}^\mu h), \tag{4.4}$$

$$\sqrt{g}R = \sqrt{\hat{g}}\left(\hat{R} + \frac{1}{4}\hat{R}h - \frac{3}{4}\hat{D}^2 h + \frac{3}{32}(\hat{D}_\mu h)\hat{D}^\mu h)\right). \tag{4.5}$$

Filling this into equation (4.3) and keeping terms up to second order in h we can take the second variation of each individual term, see Appendix C for all the details. As $[D_\mu, D_\nu] = \mathcal{O}(R)$, for terms of order $\hat{R}^2$ the covariant derivatives commute and we can use the momentum representation $\hat{D}_\mu g(p) = ip_\mu g(p)$, making life easier. The result for the second variation of the terms where the $R-R$-combination (*i.e.* without $F_k(\Delta)$) is quadratic in h (see Appendix C.1.1) is

$$\begin{aligned}\Gamma^{(2)}_{h^2} &= \frac{1}{8}\hat{R}^2 F_k(q^2) - \frac{1}{8}\hat{R}^2 F_k((p+q)^2) + \frac{3}{8}(\hat{D}^2 F_k(\hat{\Delta})\hat{R}) \\ &+ \frac{3}{8}(\hat{D}_\mu F_k(\hat{\Delta})\hat{R})\hat{D}^\mu + \frac{3}{8}(F_k(\hat{\Delta})\hat{R})\hat{D}^2 + \frac{9}{8}F_k(\hat{\Delta})\hat{\Delta}^2.\end{aligned} \tag{4.6}$$

where p_μ denotes the loop momentum and q_μ the external momentum.

For two of the four terms where the $R-R$-combination is linear in h, we can also go to momentum-space. The second variation is (see Appendix C.1.2)

$$\Gamma^{(2)}_{h^1-mom} = \frac{1}{4}\hat{R}^2\left(F_k(p^2 + 2pqx + q^2) - F_k(q^2)\right), \tag{4.7}$$

where we used that $(p+q)^2 = (p_{\mu)} + q_\mu)^2 = (p^2 + 2pqx + q^2)$ and we defined $x = \cos\theta$ with θ the angle between p_μ and q_μ.

The other two terms linear in h can be cleverly rewritten to make taking the variations possible (see Appendix C.1.2). This results in

$$\begin{aligned}\Gamma^{(2)}_{h^1-rest} &= \frac{3}{8}\left(F_k(\Delta)\hat{R}\right)\hat{\Delta} - \frac{3}{8}\hat{R}F_k(\Delta)\hat{\Delta} + \frac{3}{8}\sum_{n=0}^{\infty}\Big\{2\left(F_k(\Delta)\hat{\Delta}^{n+1}\hat{R}\right)\hat{\Delta}^{-n} \\ &- 2\left(\hat{\Delta}^{n+1}\hat{R}\right)F_k(\Delta)\hat{\Delta}^{-n} + \hat{\Delta}^{-n+1}\left(F_k(\Delta)\hat{\Delta}^n\hat{R}\right) - F_k(\Delta)\hat{\Delta}^{-n+1}\left(\hat{\Delta}^n\hat{R}\right) \\ &+ \left(\hat{D}^\mu\hat{\Delta}^n\hat{R}\right)\hat{D}_\mu F_k(\Delta)\hat{\Delta}^{-n} - \left(\hat{D}^\mu F_k(\Delta)\hat{\Delta}^n\hat{R}\right)\hat{D}_\mu\hat{\Delta}^{-n} \\ &+ \hat{\Delta}^{-n}\left(\hat{D}_\mu F_k(\Delta)\hat{\Delta}^n\hat{R}\hat{D}^\mu\right) - F_k(\Delta)\hat{\Delta}^{-n}\left(\hat{D}_\mu\hat{\Delta}^n\hat{R}\hat{D}^\mu\right)\Big\}.\end{aligned} \tag{4.8}$$

For the term where the $R-R$-combination is independent of h, we have to expand $F_k(\hat{\Delta})$ up to quadratic order in h (see Appendix C.1.3). We can then use that this term is of order $\hat{R}^2$ and go to momentum-space again. The second variation becomes

$$\begin{aligned}\Gamma^{(2)}_{h^0} = \frac{1}{8}\hat{R}^2\Big\{&-F_k'(q^2)\left(\frac{(2p^2+3pqx+q^2)(q^2-pqx)}{(p^2+2pqx)} - q^2 + 2pqx\right) \\ &+ \left(F_k(p^2+2pqx+q^2) - F_k(q^2)\right)\frac{(2p^2+3pqx+q^2)(q^2-pqx)}{(p^2+2pqx)^2}\Big\}.\end{aligned} \tag{4.9}$$

Note that this expression is well-defined everywhere. At the potential pole $x = -\frac{p}{2q}$, the expression $F_k(p^2 + 2pqx + q^2) = F_k(q^2)$ vanishes. When expanding the full expression around $p = 0$, we see there are no poles anywhere.

Adding all contributions, including those from the Einstein-Hilbert terms and including the prefactor we have the second variation of the Einstein-Hilbert and Ricci-term as

$$\begin{aligned}
\Gamma^{(2)}_{EH+Ricci} &= \frac{1}{64\pi G_k}\Lambda_k - \frac{3}{256\pi G_k}\hat{\Delta} - \frac{1}{16\pi G_k}\left[\frac{9}{8}F_k(\hat{\Delta})\hat{\Delta}^2 - \frac{3}{8}(\hat{\Delta}F_k(\hat{\Delta})\hat{R})\right. \\
&+ \frac{3}{8}(\hat{D}_\mu F_k(\hat{\Delta})\hat{R})\hat{D}^\mu - \frac{3}{8}\hat{R}F_k(\Delta)\hat{\Delta} - \frac{1}{8}\hat{R}^2F_k(q^2) + \frac{1}{8}\hat{R}^2F_k(p^2+2pqx+q^2) \\
&+ \frac{1}{8}\hat{R}^2\left\{-F_k'(q^2)\left(\frac{(2p^2+3pqx+q^2)(q^2-pqx)}{(p^2+2pqx)} - q^2 + 2pqx\right)\right. \\
&+ \left.\left(F_k(p^2+2pqx+q^2) - F_k(q^2)\right)\frac{(2p^2+3pqx+q^2)(q^2-pqx)}{(p^2+2pqx)^2}\right\} \\
&+ \frac{3}{8}\sum_{n=0}^{\infty}\left\{2\left(F_k(\Delta)\hat{\Delta}^{n+1}\hat{R}\right)\hat{\Delta}^{-n} - 2\left(\hat{\Delta}^{n+1}\hat{R}\right)F_k(\Delta)\hat{\Delta}^{-n}\right. \\
&+ \hat{\Delta}^{-n+1}\left(F_k(\Delta)\hat{\Delta}^n\hat{R}\right) - F_k(\Delta)\hat{\Delta}^{-n+1}\left(\hat{\Delta}^n\hat{R}\right) \\
&+ \left(\hat{D}^\mu\hat{\Delta}^n\hat{R}\right)\hat{D}_\mu F_k(\Delta)\hat{\Delta}^{-n} - \left(\hat{D}^\mu F_k(\Delta)\hat{\Delta}^n\hat{R}\right)\hat{D}_\mu\hat{\Delta}^{-n} \\
&+ \left.\left.\hat{\Delta}^{-n}\left(\hat{D}_\mu F_k(\Delta)\hat{\Delta}^n\hat{R}\hat{D}^\mu\right) - F_k(\Delta)\hat{\Delta}^{-n}\left(\hat{D}_\mu\hat{\Delta}^n\hat{R}\hat{D}^\mu\right)\right\}\right] \\
&\equiv \mathcal{O}_0 - \mathcal{R}_k + \mathcal{O}_1 + \mathcal{O}_2,
\end{aligned} \tag{4.10}$$

such that $\Gamma^{(2)}_{Ricci+EH} + \mathcal{R}_k = \mathcal{O}_0 + \mathcal{O}_1 + \mathcal{O}_2$, where we defined the operators $\mathcal{O}_i$ that are of order $\hat{R}^i$:

$$\begin{aligned}
\mathcal{O}_0 &= -\frac{3}{256\pi G_k}\left(\hat{\Delta} - \frac{4}{3}\Lambda + 6F_k(\hat{\Delta})\hat{\Delta}^2 - \frac{256\pi G_k}{3}\mathcal{R}_k(\hat{\Delta})\right), \\
\mathcal{O}_1 &= -\frac{3}{128\pi G_k}\left(-(\hat{\Delta}F_k(\hat{\Delta})\hat{R}) + (\hat{D}_\mu F_k(\hat{\Delta})\hat{R})\hat{D}^\mu - \hat{R}F_k(\Delta)\hat{\Delta}\right. \\
&+ \sum_{n=0}^{\infty}\left\{2\left(F_k(\Delta)\hat{\Delta}^{n+1}\hat{R}\right)\hat{\Delta}^{-n} - 2\left(\hat{\Delta}^{n+1}\hat{R}\right)F_k(\Delta)\hat{\Delta}^{-n} + \hat{\Delta}^{-n+1}\left(F_k(\Delta)\hat{\Delta}^n\hat{R}\right)\right. \\
&- F_k(\Delta)\hat{\Delta}^{-n+1}\left(\hat{\Delta}^n\hat{R}\right) + \left(\hat{D}^\mu\hat{\Delta}^n\hat{R}\right)\hat{D}_\mu F_k(\Delta)\hat{\Delta}^{-n} - \left(\hat{D}^\mu F_k(\Delta)\hat{\Delta}^n\hat{R}\right)\hat{D}_\mu\hat{\Delta}^{-n} \\
&+ \left.\left.\hat{\Delta}^{-n}\left(\hat{D}_\mu F_k(\Delta)\hat{\Delta}^n\hat{R}\hat{D}^\mu\right) - F_k(\Delta)\hat{\Delta}^{-n}\left(\hat{D}_\mu\hat{\Delta}^n\hat{R}\hat{D}^\mu\right)\right\}\right), \\
\mathcal{O}_2 &= -\frac{1}{128\pi G_k}\left(-\hat{R}^2F_k(q^2) + \hat{R}^2F_k(p^2+2pqx+q^2)\right. \\
&+ \hat{R}^2\left\{-F_k'(q^2)\left(\frac{(2p^2+3pqx+q^2)(q^2-pqx)}{(p^2+2pqx)} - q^2 + 2pqx\right)\right. \\
&+ \left.\left.\left(F_k(p^2+2pqx+q^2) - F_k(q^2)\right)\frac{(2p^2+3pqx+q^2)(q^2-pqx)}{(p^2+2pqx)^2}\right\}\right) \\
&\equiv -\frac{1}{128\pi G_k}\hat{R}^2\bar{K}(p,q,x).
\end{aligned} \tag{4.11}$$

4.1.2 Weyl-term

Next, we look at the contribution from the Weyl-term $\frac{1}{16\pi G_k}\int\sqrt{g}C^{\mu\nu\rho\sigma}W(\Delta)C_{\mu\nu\rho\sigma}$. In the conformally reduced framework we have

$$\begin{aligned} C_{\mu\nu\rho\sigma} &= \varphi^2\hat{C}_{\mu\nu\rho\sigma} = \left(1+\frac{1}{4}h\right)\hat{C}_{\mu\nu\rho\sigma}, \\ \sqrt{g}C^{\mu\nu\rho\sigma} &= \varphi^{-2}\sqrt{\hat{g}}\hat{C}^{\mu\nu\rho\sigma} = \left(1-\frac{1}{4}h+\frac{1}{16}h^2\right)\sqrt{\hat{g}}\hat{C}^{\mu\nu\rho\sigma}. \end{aligned} \tag{4.12}$$

The transformation of the Laplacian Δ working on a rank-four tensor is quite involved. For now, let us just informally write

$$\int\sqrt{g}\,C^{\mu\nu\rho\sigma}W(\Delta)C_{\mu\nu\rho\sigma} = \int\sqrt{g}\,C^{\mu\nu\rho\sigma}\int_0^\infty ds\tilde{W}(s)\mathrm{e}^{-s\left(\hat{\Delta}\mathbb{1}_C+\mathbb{T}_1+\mathbb{T}_2\right)^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}}C_{\alpha\beta\gamma\delta}, \tag{4.13}$$

where $\mathbb{1}_C$ is the unity operator with the same symmetries as the Weyl tensor. Using the same strategy as for the Ricci-term, we write

$$\begin{aligned} &\mathrm{e}^{-s\left(\hat{\Delta}\mathbb{1}_C+\mathbb{T}_1+\mathbb{T}_2\right)^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}} = \mathrm{e}^{-s\hat{\Delta}}\Bigg\{\mathbb{1}_C+\epsilon\sum_{j=0}^{\infty}\frac{(-1)^j}{(j+1)!}\left[-s\hat{\Delta},-s\left(\mathbb{T}_1+\mathbb{T}_2\right)\right]_j \\ &+\frac{\epsilon^2}{2}\left(\sum_{j=0}^{\infty}\frac{(-1)^j}{(j+1)!}\left[-s\hat{\Delta},-s\left(\mathbb{T}_1+\mathbb{T}_2\right)\right]_j\right)^2 \\ &+\frac{\epsilon^2}{2}\left(\sum_{j=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{j+n+1}}{(n+j+2)!}\left[-s\hat{\Delta},\left[\frac{s}{4}-s\left(\mathbb{T}_1+\mathbb{T}_2\right),\left[-s\hat{\Delta},-s\left(\mathbb{T}_1+\mathbb{T}_2\right)\right]_n\right]\right]_j\right) \\ &+\mathcal{O}(\epsilon^3)\Bigg\}^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}. \end{aligned} \tag{4.14}$$

Let us now combine this with the transformations for the Weyl tensor written above and, as we are interested in the second variation, only keep terms quadratic in h. The result is shown in equations (D.1)-(D.3) in Appendix D. As $\hat{C}_{\alpha\beta\gamma\delta} = \mathcal{O}(\hat{R})$, we can go to momentum representation for all terms. To find $\mathbb{T}_1$ and $\mathbb{T}_2$, we evaluate

$$\Delta\hat{C}_{\mu\nu\rho\sigma} = \hat{\Delta}\hat{C}_{\mu\nu\rho\sigma} + \mathbb{T}_{1\,\mu\nu\rho\sigma}^{\,\alpha\beta\gamma\delta}\hat{C}_{\alpha\beta\gamma\delta} + \mathbb{T}_{2\,\mu\nu\rho\sigma}^{\,\alpha\beta\gamma\delta}\hat{C}_{\alpha\beta\gamma\delta} = (\hat{\Delta}\mathbb{1}_C+\mathbb{T}_1+\mathbb{T}_2)\circ\hat{C}, \tag{4.15}$$

using the *xAct* [30] package in Mathematica, resulting in 24 terms for $\mathbb{T}_1+\mathbb{T}_2$. Their effect on the relevant expressions in momentum space is

$$\begin{aligned} \hat{C}^{\mu\nu\rho\sigma}(-q)h(-p)\mathbb{T}_1(p)^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}\hat{C}_{\alpha\beta\gamma\delta}(q) &= -\frac{1}{4}\left(2p^2+3pqx+q^2\right)h^2\hat{C}^2, \\ \hat{C}^{\mu\nu\rho\sigma}(-q)\mathbb{T}_1(-p)^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}h(p)\hat{C}_{\alpha\beta\gamma\delta}(q) &= -\frac{1}{4}\left(q^2-pqx\right)h^2\hat{C}^2, \\ \hat{C}^{\mu\nu\rho\sigma}(-q)\mathbb{T}_2(p,-p)^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}\hat{C}_{\alpha\beta\gamma\delta}(q) &= \frac{1}{32}\left(3p^2-12pqx+2q^2\right)h^2\hat{C}^2, \\ \hat{C}^{\mu\nu\rho\sigma}(-q)\mathbb{T}_1(-p)^{\kappa\lambda\pi\omega}_{\mu\nu\rho\sigma}\mathbb{T}_1(p)^{\alpha\beta\gamma\delta}_{\kappa\lambda\pi\omega}\hat{C}_{\alpha\beta\gamma\delta}(q) &= \frac{1}{32}\left(2q^4-4p^3qx+4pq^3x+p^2q^2(7-6x^2)\right)h^2\hat{C}^2, \end{aligned} \tag{4.16}$$

such that, including the prefactor, the total second variation for the Weyl-term becomes (see Appendix D for all the details)

$$\begin{aligned}\Gamma^{(2)}_{Weyl} &= \frac{1}{16\pi G_k}\frac{1}{8}\hat{C}^2\left\{W_k'(q^2)\frac{3p^4-2p^3qx-p^2q^2(5+18x^2)-2q^4}{2(p^2+2pqx)}\right.\\ &\left.+\left(W_k(p^2+2pqx+q^2)-W_k(q^2)\right)\left(\frac{2q^4+4pqx(q^2-p^2)+p^2q^2(7-6x^2)}{2(p^2+2pqx)^2}+1\right)\right\} \\ &\equiv \frac{1}{128\pi G_k}\hat{C}^2 K(p,q,x).\end{aligned} \tag{4.17}$$

Similar to the Ricci-term, this expression is well-defined everywhere[3]. Combining the contributions from equations (4.10) and (4.17), we have the total second variation for our ansatz (3.16) as

$$\Gamma^{(2)}_k = \mathcal{O}_0 - \mathcal{R}_k + \mathcal{O}_1 - \frac{1}{128\pi G_k}\hat{R}^2\bar{K}(p,q,x) + \frac{1}{128\pi G_k}\hat{C}^2 K(p,q,x), \tag{4.18}$$

with

$$\begin{aligned}\bar{K}(p,q,x) &= -F_k'(q^2)\left(\frac{(2p^2+3pqx+q^2)(q^2-pqx)}{(p^2+2pqx)}-q^2+2pqx\right)\\ &+\left(F_k(p^2+2pqx+q^2)-F_k(q^2)\right)\left(\frac{(2p^2+3pqx+q^2)(q^2-pqx)}{(p^2+2pqx)^2}+1\right),\\ K(p,q,x) &= W_k'(q^2)\frac{3p^4-2p^3qx-p^2q^2(5+18x^2)-2q^4}{2(p^2+2pqx)}\\ &+\left(W_k(p^2+2pqx+q^2)-W_k(q^2)\right)\left(\frac{2q^4+4pqx(q^2-p^2)+p^2q^2(7-6x^2)}{2(p^2+2pqx)^2}+1\right).\end{aligned} \tag{4.19}$$

The next steps are adding the regulator (which was already implied in the definition of $\mathcal{O}_0$), taking the inverse, multiplying by the regulator-derivative and taking the trace.

We take the inverse by expanding around flat spacetime

$$\left(\Gamma^{(2)}_k+\mathcal{R}_k\right)^{-1} = (\mathcal{O}_0+\delta\mathcal{O})^{-1} = \sum_{n\geq 0}\left(-\mathcal{O}_0^{-1}\delta\mathcal{O}\right)^n\mathcal{O}_0^{-1} \equiv \sum_{n\geq 0}\left(-G_0\delta\mathcal{O}\right)^n G_0, \tag{4.20}$$

where $G_0=\mathcal{O}_0^{-1}$ is the flat space propagator as $\mathcal{O}_0$ is the combination of all terms of order $\mathcal{O}(R^0)$ as defined above and $\delta\mathcal{O}$ are all higher order terms. Keeping terms up to second order in the curvature we obtain

$$G = G_0(\hat{\Delta}) - G_0(\hat{\Delta})\delta\mathcal{O}G_0(\hat{\Delta}) + G_0(\hat{\Delta})\delta\mathcal{O}G_0(\hat{\Delta})\delta\mathcal{O}G_0(\hat{\Delta}) + \mathcal{O}(R^3). \tag{4.21}$$

The trace can then be taken part-by-part and becomes

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}\left[G\dot{\mathcal{R}}_k(\hat{\Delta})\right] &= \frac{1}{2}\mathrm{Tr}\left[G_0(\hat{\Delta})\dot{\mathcal{R}}_k(\hat{\Delta})\right] - \frac{1}{2}\mathrm{Tr}\left[\mathcal{O}_{1,d}\,G_0(\hat{\Delta})\,\dot{\mathcal{R}}_k(\hat{\Delta})\,G_0(\hat{\Delta})\right]\\ &- \frac{1}{2}\mathrm{Tr}\left[\mathcal{O}_{1,od}\,G_0(\hat{\Delta})\,\dot{\mathcal{R}}_k(\hat{\Delta})\,G_0(\hat{\Delta})\right]\\ &+ \frac{1}{256\pi G_k}\mathrm{Tr}\left[\hat{R}^2\bar{K}(p,q,x)\,G_0(\hat{\Delta})\,\dot{\mathcal{R}}_k(\hat{\Delta})\,G_0(\hat{\Delta})\right]\\ &- \frac{1}{256\pi G_k}\mathrm{Tr}\left[\hat{C}^2 K(p,q,x)G_0(\hat{\Delta})\dot{\mathcal{R}}_k(\hat{\Delta})G_0(\hat{\Delta})\right]\\ &+ \frac{1}{2}\mathrm{Tr}\left[\mathcal{O}_1\,G_0(\hat{\Delta})\,\mathcal{O}_1\,G_0(\hat{\Delta})\,\dot{\mathcal{R}}_k(\hat{\Delta})\,G_0(\hat{\Delta})\right] + \mathcal{O}(\hat{R}^3),\end{aligned} \tag{4.22}$$

[3]The vanishing of the poles when expanding the full term can be seen in equation (5.21)

where $\mathcal{O}_{1,d}$ denotes the diagonal part, *i.e.* without uncontracted derivatives $\hat{D}_\mu$, of the terms with one order of curvature and $\mathcal{O}_{1,od}$ denotes the part containing these uncontracted derivatives, where the off-diagonal heat kernel (see Appendix A.2) will be used. The first operator to be traced over looks like

$$G_0(z)\dot{\mathcal{R}}_k(z) = \frac{\dot{\mathcal{R}}_k(z)}{\frac{-3}{256\pi G_k}\left(z - \frac{4}{3}\Lambda + 6F_k(z)z^2 - \frac{256\pi G_k}{3}\mathcal{R}_k(z)\right)} = \frac{\dot{\bar{\mathcal{R}}}_k(z) - \eta\bar{\mathcal{R}}_k(z)}{z - \frac{4}{3}\Lambda + 6F_k(z)z^2 + \bar{\mathcal{R}}_k(z)}, \tag{4.23}$$

where we defined the rescaled regulator

$$\bar{\mathcal{R}}_k \equiv -\frac{256\pi G_k}{3}\mathcal{R}_k, \tag{4.24}$$

such that

$$\dot{\mathcal{R}}_k = \frac{3}{256\pi}k\partial_k\frac{\bar{\mathcal{R}}_k}{G_k} = \frac{3}{256\pi}\frac{G_k\dot{\bar{\mathcal{R}}}_k - \bar{\mathcal{R}}_k\dot{G}_k}{G_k^2} = \frac{3}{256\pi G_k}\left(\dot{\bar{R}}_k - \eta\bar{R}_k\right), \tag{4.25}$$

where we evaluated

$$\frac{\dot{G}_k}{G_k^2} = \frac{k\partial_k\left(k^{-2}g\right)}{k^{-4}g^2} = \frac{k^2k\partial_k g - gk2k}{g^2} = k^2\frac{(2+\eta)g - 2g}{g^2} = \frac{\eta}{k^{-2}g} = \frac{\eta}{G_k}, \tag{4.26}$$

using that by definition

$$k\partial_k g = \dot{g} = (2+\eta)g, \tag{4.27}$$

with η the anomalous dimension. Let us now also define

$$V(z) \equiv G_0(z)\dot{\mathcal{R}}_k(z)G_0(z) = \frac{-\frac{256\pi G_k}{3}\left(\dot{\bar{\mathcal{R}}}_k(z) - \eta\bar{\mathcal{R}}_k(z)\right)}{\left(z - \frac{4}{3}\Lambda + 6F_k(z)z^2 + \bar{\mathcal{R}}_k(z)\right)^2} \equiv -\frac{256\pi G_k}{3}V'(z), \tag{4.28}$$

such that the total trace becomes

$$\begin{aligned}
\frac{1}{2}\mathrm{Tr}\left[G\dot{\mathcal{R}}_k(\hat{\Delta})\right] &= \frac{1}{2}\mathrm{Tr}\left[G_0(\hat{\Delta})\dot{\mathcal{R}}_k(\hat{\Delta})\right] - \frac{1}{2}\mathrm{Tr}\left[\mathcal{O}_{1,d}\,V(\hat{\Delta})\right] - \frac{1}{2}\mathrm{Tr}\left[\mathcal{O}_{1,od}\,V(\hat{\Delta})\right] \\
&+ \frac{1}{256\pi G_k}\mathrm{Tr}\left[\hat{R}^2\bar{K}(p,q,x)\,V(\hat{\Delta})\right] - \frac{1}{256\pi G_k}\mathrm{Tr}\left[\hat{C}^2K(p,q,x)V(\hat{\Delta})\right] \\
&+ \frac{1}{2}\mathrm{Tr}\left[\mathcal{O}_1(\hat{\Delta})\,G_0(\hat{\Delta})\,\mathcal{O}_1(\hat{\Delta})\;V(\hat{\Delta})\right] + \mathcal{O}(\hat{R}^3) \\
&\equiv \frac{1}{2}\mathrm{Tr}\left[O_1\right] + \frac{1}{2}\mathrm{Tr}\left[O_2\right] + \frac{1}{2}\mathrm{Tr}\left[O_3\right] + \frac{1}{2}\mathrm{Tr}\left[O_4\right] + \frac{1}{2}\mathrm{Tr}\left[O_5\right] + \frac{1}{2}\mathrm{Tr}\left[O_6\right] \\
&+ \mathcal{O}(R^3).
\end{aligned} \tag{4.29}$$

These traces will be taken using nonlocal heat kernel techniques in the next subsection

4.2 Traces

We will start with the last three traces, as these are all over operators of second order in the curvature and therefore only consist of an integral over momentum space. The trace over $O_6(\hat{\Delta})$ becomes

$$\begin{aligned}
\frac{1}{2}\mathrm{Tr}\left[O_6(\Delta)\right] &= \frac{1}{2}\mathrm{Tr}\left[\mathcal{O}_1(\hat{\Delta})\,G_0(\hat{\Delta})\,\mathcal{O}_1(\hat{\Delta})\,V(\hat{\Delta})\right] = \frac{1}{2}\mathrm{Tr}\left[\mathcal{O}_1(-q)\,G_0(p+q)\,\mathcal{O}_1(q)\,V(p^2)\right] \\
&\equiv 2\mathrm{Tr}\left[\hat{R}^2\widetilde{K}(p,q,x)V'(p^2)\right] = \frac{1}{4\pi^3}\int_0^\infty dp\,p^3\int_{-1}^1 dx\,\sqrt{1-x^2}\widetilde{K}(p,q,x)V'(p^2)\int\sqrt{\bar{g}}\hat{R}^2,
\end{aligned} \tag{4.30}$$

where we used that

$$\int \frac{d^4p}{(2\pi)^4} = \frac{1}{8\pi^3}\int dp\, p^3 \int_{-1}^{1} dx\sqrt{1-x^2}, \tag{4.31}$$

as

$$\int_0^\pi d\theta\, \sin^2(\theta) = -\int_\pi^0 \sin(\theta)d\theta \sin(\theta) = \int_{-1}^{1} dx\sqrt{1-x^2}, \tag{4.32}$$

with $x = \cos(\theta)$ as before. We also defined (see Appendix C.2.4)

$$\begin{aligned}
&\widetilde{K}(p,q,x) = \big(F_k(q^2)pqx - F_k((p+q)^2)(p+q)^2 \\
&+ \frac{(q^2-pqx)(p+q)^2}{p^2+2pqx}\left(F_k(q^2) - F_k((p+q)^2)\right) + \frac{p^4+p^2(pqx+q^2)}{p^2-q^2}\left(F_k(q^2)-F_k(p^2)\right)\Big) \\
&\frac{1}{(p+q)^2 + 6F_k((p+q)^2)(p+q)^4 + \bar{\mathcal{R}}_k((p+q)^2) - \frac{4}{3}\Lambda} \\
&\Big(-(q^2+pqx)F_k(q^2) - F_k(p^2)p^2 + \frac{p^2(2q^2+pqx)}{p^2-q^2}\left(F_k(q^2)-F_k(p^2)\right) \\
&+ \frac{(p+q)^2\left(p^2+pqx+q^2\right)}{p^2+2pqx}\left(F_k(q^2)-F_k((p+q)^2)\right)\Big).
\end{aligned} \tag{4.33}$$

The fifth trace is over $\mathcal{O}(\hat{C}^2)$, so we use the momentum representation and integral outlined above to find

$$\begin{aligned}
&-\frac{1}{256\pi G_k}\mathrm{Tr}\left[\hat{C}^2 K(p,q,x)V(\hat{\Delta})\right] = \frac{1}{3}\int \frac{d^4p}{(2\pi)^4}\, K(p,q,x)V'(p^2)\int \sqrt{\hat{g}}\hat{C}^2 \\
=&\frac{1}{24\pi^3}\int dp\, p^3 \int_{-1}^{1} dx\sqrt{1-x^2}\, K(p,q,x)V'(p^2)\int \sqrt{\hat{g}}\hat{C}^2.
\end{aligned} \tag{4.34}$$

The procedure is the same for the fourth trace:

$$\begin{aligned}
&\frac{1}{256\pi G_k}\mathrm{Tr}\left[\hat{R}^2\bar{K}(p,q,x)\, V(\hat{\Delta})\right] = -\frac{1}{3}\int \frac{d^4p}{(2\pi)^4}\, \bar{K}(p,q,x)V'(p^2)\int \sqrt{\hat{g}}\hat{R}^2 \\
=&-\frac{1}{24\pi^3}\int dp\, p^3 \int_{-1}^{1} dx\sqrt{1-x^2}\, \bar{K}(p,q,x)V'(p^2)\int \sqrt{\hat{g}}\hat{R}^2.
\end{aligned} \tag{4.35}$$

Next, we look at the second and third trace. They are both over an expression with one order of curvature, such that we will use the nonlocal heat kernel up to first order in the curvature. For the diagonal piece, the part that will project on the coupling G_k, *i.e.* the part proportional to $\hat{R}$, we can neglect total derivatives (Laplacians or uncontracted derivatives) working just on $\hat{R}$. Therefore, we trace the operator $\mathcal{O}_{rel} = \frac{3}{128\pi G_k}\left(\hat{R}F_k(\hat{\Delta})\hat{\Delta} - \hat{\Delta}\left(F_k(\hat{\Delta})\hat{R}\right) + F_k(\hat{\Delta})\hat{\Delta}\left(\hat{R}\right)\right)$ using only the zeroth order of the heat-kernel:

$$
\begin{aligned}
-\frac{1}{2}\mathrm{Tr}\left[\mathcal{O}_{rel}V(\hat{\Delta})\right] &= -\frac{3}{256\pi G_k}\mathrm{Tr}\left[\left(\hat{R}F_k(\hat{\Delta})\hat{\Delta}-\hat{\Delta}\left(F_k(0)\hat{R}\right)+F_k(\hat{\Delta})\hat{\Delta}\left(\hat{R}\right)\right)V(\hat{\Delta})\right] \\
&= \mathrm{Tr}\left[\hat{R}F_k(\hat{\Delta})\hat{\Delta}V'(\hat{\Delta})-F_k(0)\hat{R}V'(\hat{\Delta})\hat{\Delta}+\hat{R}V'(\hat{\Delta})F_k(\hat{\Delta})\hat{\Delta}\right] \\
&\equiv \mathrm{Tr}\left[\hat{R}V_1(\hat{\Delta})-F_k(0)\hat{R}V_2(\hat{\Delta})+\hat{R}V_3(\hat{\Delta})\right] \\
&= \frac{1}{16\pi^2}\int_0^\infty ds\, s^{-2}\left(\hat{R}\tilde{V}_1(s)-F_k(0)\hat{R}\tilde{V}_4(0,s)+\hat{R}\tilde{V}_5(0,s)\right) \\
&= \frac{1}{16\pi^2}\int_0^\infty dz\, z\left(\hat{R}V_1(z)-F_k(0)\hat{R}V_2(z)+\hat{R}V_3(z)\right) \\
&= \frac{1}{16\pi^2}\int_0^\infty dz\, z\left(\hat{R}F_k(z)zV'(z)-F_k(0)\hat{R}V'(z)z+\hat{R}V'(z)F_k(z)z\right) \\
&= \frac{1}{16\pi^2}\int_0^\infty dz\, z^2\, V'(z)\left(2F_k(z)-F_k(0)\right)\hat{R},
\end{aligned}
\tag{4.36}
$$

where we used equation (3.26). The diagonal piece will also project on the Ricci form factor. To achieve this structure where operators can work on the Ricci scalar we use the $g_R(sx)$ coefficient of the heat kernel and the full diagonal operator (see Appendix C.2.2):

$$
\begin{aligned}
&-\frac{1}{2}\mathrm{Tr}\left[\mathcal{O}_{1,d}V(\hat{\Delta})\right] = \frac{3}{256\pi G_k}\mathrm{Tr}\Big[-\left(\hat{\Delta}F_k(\hat{\Delta})\hat{R}\right)V(\hat{\Delta})-\hat{R}F_k(\Delta)\hat{\Delta}V(\hat{\Delta}) \\
&+\sum_{n=0}^{\infty}\Big\{2\left(F_k(\Delta)\hat{\Delta}^{n+1}\hat{R}\right)\hat{\Delta}^{-n}V(\hat{\Delta})-2\left(\hat{\Delta}^{n+1}\hat{R}\right)F_k(\Delta)\hat{\Delta}^{-n}V(\hat{\Delta}) \\
&+\left(F_k(\Delta)\hat{\Delta}^{n}\hat{R}\right)V(\hat{\Delta})\hat{\Delta}^{-n+1}-\left(\hat{\Delta}^{n}\hat{R}\right)V(\hat{\Delta})F_k(\Delta)\hat{\Delta}^{-n+1}\Big\}\Big] \\
&= \frac{\hat{x}^2\hat{R}^2}{96\pi^2}\left\{\int_0^\infty dv-\int_0^{\frac{1}{4}}dv\,\sqrt{1-4v}(1+2v)\right\}\frac{F_k(\hat{x}v)(2v^2+v)-F_k(\hat{x})(v^2+v+1)}{v-1}\,V'(\hat{x}v).
\end{aligned}
\tag{4.37}
$$

The third trace is also over one order of curvature, but includes uncontracted derivatives as well. Therefore, we will use the off-diagonal heat kernel coefficient with one power of curvature (see Appendix C.2.3):

$$
\begin{aligned}
&-\frac{1}{2}\mathrm{Tr}\left[\mathcal{O}_{1,od}\,V(\hat{\Delta})\right] = \frac{3}{256\pi G_N}\mathrm{Tr}\Big[\left(\hat{D}_\mu F_k(\hat{\Delta})\hat{R}\right)\hat{D}^\mu V(\hat{\Delta}) \\
&+\sum_{n=0}^{\infty}\Big\{\left(\hat{D}^\mu\hat{\Delta}^n\hat{R}\right)\hat{D}_\mu F_k(\hat{\Delta})\hat{\Delta}^{-n}V(\hat{\Delta})-\left(\hat{D}^\mu F_k(\Delta)\hat{\Delta}^n\hat{R}\right)\hat{D}_\mu\hat{\Delta}^{-n}V(\hat{\Delta}) \\
&+\left(\hat{D}_\mu F_k(\Delta)\hat{\Delta}^n\hat{R}\right)\hat{D}^\mu V(\hat{\Delta})\hat{\Delta}^{-n}-\left(\hat{D}_\mu\hat{\Delta}^n\hat{R}\right)\hat{D}^\mu V(\hat{\Delta})F_k(\hat{\Delta})\hat{\Delta}^{-n}\Big\}\Big] \\
&= -\frac{\hat{x}^2\hat{R}^2}{192\pi^2}\left\{\int_0^\infty dv-\int_0^{\frac{1}{4}}dv\,\sqrt{1-4v}(1+2v)\right\}F_k(\hat{x})V'(\hat{x}v),
\end{aligned}
\tag{4.38}
$$

where we used that the off-diagonal heat kernel coefficient is equal to $\frac{1}{2}g_R(s\hat{x})$, see Appendix A.2. Finally, we take the first trace. It has no curvatures, so we use the nonlocal heat kernel up to second order in the curvature, with $g_R(0)=\frac{1}{6}$ as this coefficient will now only project on the G_k-term. It becomes

$$
\begin{aligned}
\frac{1}{2}\mathrm{Tr}\left[O_1(\hat{\Delta})\right] &= \frac{1}{32\pi^2}\int_0^\infty ds\, \widetilde{O}_1(s)\mathrm{tr}\left\{\mathbb{1}s^{-2} + \mathbb{1}\frac{\hat{R}}{6s} + \mathbb{1}\hat{R}f_R(s\hat{\Delta})\hat{R} + \hat{C}_{\mu\nu\rho\sigma}f_C(s\hat{\Delta})\hat{C}^{\mu\nu\rho\sigma}\right\} \\
&= \frac{1}{32\pi^2}\int_0^\infty dz\, O_1(z)\left\{z + \frac{\hat{R}}{6}\right\} + \frac{1}{1152\pi^2}\hat{R}^2\int_0^{\frac{1}{4}} dv\, \frac{20v^2+8v-1}{\sqrt{1-4v}}\, O_1(\hat{x}v) \\
&\quad + \frac{\hat{C}^2}{384\pi^2}\int_0^{\frac{1}{4}} dv\, O_1(\hat{\Delta}v)\ (1-4v)^{3/2}\,,
\end{aligned} \tag{4.39}
$$

see Appendix C.2.1 for the second part and Appendix D.2 for the third part. Putting everything together, we have the total trace

$$
\begin{aligned}
\frac{1}{2}\mathrm{Tr}\left[G\dot{\mathcal{R}}_k(\hat{\Delta})\right] &= \frac{1}{32\pi^2}\int_0^\infty dz\, O_1(z)\left\{z + \frac{\hat{R}}{6}\right\} + \frac{1}{1152\pi^2}\hat{R}^2\int_0^{\frac{1}{4}} dv\, \frac{20v^2+8v-1}{\sqrt{1-4v}}\, O_1(\hat{x}v) \\
&\quad + \frac{\hat{C}^2}{384\pi^2}\int_0^{\frac{1}{4}} dv\, O_1(\hat{x}v)\,(1-4v)^{3/2} + \frac{\hat{R}}{16\pi^2}\int_0^\infty dz\, z^2 V'(z)\,(2F_k(z) - F_k(0)) \\
&\quad + \frac{\hat{x}^2\hat{R}^2}{192\pi^2}\left\{\int_0^\infty dv - \int_0^{\frac{1}{4}} dv\,\sqrt{1-4v}(1+2v)\right\} \\
&\quad \frac{F_k(\hat{x}v)(4v^2+2v) - F_k(\hat{x})(2v^2+3v+2)}{v-1} \\
&\quad + \frac{1}{24\pi^3}\int_0^\infty dp\, p^3\int_{-1}^1 dx\sqrt{1-x^2}\left(6\widetilde{K}(p,q,x) - \bar{K}(p,q,x)\right)V'(p^2)\int\sqrt{\hat{g}}\hat{R}^2 \\
&\quad + \frac{1}{24\pi^3}\int dp\, p^3\int_{-1}^1 dx\sqrt{1-x^2}\, K(p,q,x)V'(p^2)\int\sqrt{\hat{g}}\hat{C}^2 + \mathcal{O}(\hat{R}^3).
\end{aligned} \tag{4.40}
$$

Filling this into the Wetterich equation (2.5), we obtain the flow equations, from which we can find the fixed point equations.

5 Form factors at the renormalisation group fixed point

5.1 Fixed point equations

Now that we have the complete trace, we can determine the flow equations for our ansatz (3.16) by filling the ansatz and the trace in the Wetterich equation (3.2). Equating coefficients gives the flow equations

$$
\begin{aligned}
k\partial_k \frac{\Lambda_k}{G_k} &= k\frac{G_k\partial_k\Lambda_k - \Lambda_k\partial_k G_k}{G_k^2} = \frac{1}{4\pi}\int_0^\infty dz\, O_1(z)\, z, \\
k\partial_k \frac{1}{G_k} &= -\frac{1}{12\pi}\int_0^\infty dz\, O_1(z) - \frac{1}{\pi}\int_0^\infty dz\, z^2\, V'(z)\left(2F_k(z) - F_k(0)\right), \\
k\partial_k \frac{W_k(\Delta)}{G_k} &= k\frac{G_k\partial_k W_k(\hat{x}) - W_k(\hat{x})\partial_k G_k}{G_k^2} = \frac{1}{24\pi}\int_0^{\frac{1}{4}} dv\, O_1(\hat{\Delta} v)\; (1-4v)^{3/2} \\
&\quad + \frac{2}{3\pi^2}\int dp\, p^3 \int_{-1}^{1} dx \sqrt{1-x^2}\, K(p,q,x) V'(p^2), \\
k\partial_k \frac{F_k(\Delta)}{G_k} &= k\frac{G_k\partial_k F_k(\hat{x}) - F_k(\hat{x})\partial_k G_k}{G_k^2} = -\frac{1}{72\pi}\int_0^{\frac{1}{4}} dv\, \frac{20v^2+8v-1}{\sqrt{1-4v}}\, O_1(\hat{x}v) \\
&\quad - \frac{\hat{x}^2}{12\pi}\left\{\int_0^\infty dv - \int_0^{\frac{1}{4}} dv\, \sqrt{1-4v}(1+2v)\right\} \\
&\quad \frac{F_k(\hat{x}v)(4v^2+2v) - F_k(\hat{x})(2v^2+3v+2)}{v-1}\, V'(\hat{x}v) \\
&\quad - \frac{2}{3\pi^2}\int_0^\infty dp\, p^3 \int_{-1}^{1} dx\, \sqrt{1-x^2}\, V'(p^2)\left(6\widetilde{K}(p,q,x) - \bar{K}(p,q,x)\right).
\end{aligned}
\tag{5.1}
$$

From these equations we can again observe that the form factors are partly induced by the Einstein-Hilbert terms: even if we set $F_k = W_k = 0$ in all right-hand sides, there are still flow equations for W_k and F_k.

As mentioned in the introduction, we are interested in fixed points of the dimensionless couplings. Let us thus use the dimensionless quantities from equation (3.31) and introduce

$$
\begin{aligned}
\rho &= \hat{R}\cdot k^{-2}, \\
c &= \hat{C}\cdot k^{-2}, \\
w\left(\hat{\Delta}k^{-2}\right) &= W_k(\hat{\Delta})\cdot k^2, \\
f\left(\hat{\Delta}k^{-2}\right) &= F_k(\hat{\Delta})\cdot k^2,
\end{aligned}
\tag{5.2}
$$

such that the left-hand sides of the equations above become

$$k\frac{G_k\partial_k\Lambda_k-\Lambda_k\partial_kG_k}{G_k^2}=k^4\frac{4g\lambda+kg\partial_k\lambda_k-k\lambda\partial_kg}{g^2}, \tag{5.3}$$

$$k\partial_k\frac{1}{G_k}=k^2\frac{2g-k\partial_kg}{g^2}=k^2\frac{\eta}{g}, \tag{5.4}$$

$$k\frac{G_k\partial_kW_k(\hat{x})-W_k(\hat{x})\partial_kG_k}{G_k^2}=k\frac{g\partial_kw(\hat{x})+w(\hat{x})\partial_kg-2g\hat{x}w'(\hat{x})}{g^2}, \tag{5.5}$$

$$k\frac{G_k\partial_kF_k(\hat{x})-F_k(\hat{x})\partial_kG_k}{G_k^2}=k\frac{g\partial_kf(\hat{x}+f(\hat{x})\partial_kg-2g\hat{x}f'(\hat{x})}{g^2}. \tag{5.6}$$

The fixed point equations are obtained by setting the running of all couplings to zero: $k\partial_kc_i=0$ for $c_i=\lambda,g,w,f$. First of all, this means that $0=\dot{g}=(2+\eta)g$, so $\eta=-2$. We get:

$$0=-\lambda+\frac{g}{16\pi}\int_0^\infty dz\,z\;\frac{\dot{\bar{\mathcal{R}}}_k(z)+2\bar{\mathcal{R}}_k(z)}{z-\frac{4}{3}\Lambda+6f(z)z^2+\bar{\mathcal{R}}_k(z)}, \tag{5.7}$$

$$\begin{aligned}0=1&+\frac{g}{24\pi}\int_0^\infty dz\;\frac{\dot{\bar{\mathcal{R}}}_k(z)+2\bar{\mathcal{R}}_k(z)}{z-\frac{4}{3}\Lambda+6f(z)z^2+\bar{\mathcal{R}}_k(z)}\\&+\frac{g}{2\pi}\int_0^\infty dz\,z^2\;(2f(z)-f(0))\,\frac{\dot{\bar{\mathcal{R}}}_k(z)+2\bar{\mathcal{R}}_k(z)}{\left(z-\frac{4}{3}\Lambda+6f(z)z^2+\bar{\mathcal{R}}_k(z)\right)^2},\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}0=q^2w'(q^2)&+\frac{g}{48\pi}\int_0^{\frac{1}{4}}dv\;\frac{\dot{\bar{\mathcal{R}}}_k(q^2v)+2\bar{\mathcal{R}}_k(q^2v)}{q^2v-\frac{4}{3}\lambda+6f(q^2v)q^4v^2+\bar{\mathcal{R}}_k(q^2v)}\;(1-4v)^{3/2}\\&+\frac{g}{3\pi^2}\int_0^\infty dp\,p^3\int_{-1}^1dx\sqrt{1-x^2}\;K(p,q,x)\frac{\dot{\bar{\mathcal{R}}}_k(p^2)+2\bar{\mathcal{R}}_k(p^2)}{\left(p^2-\frac{4}{3}\lambda+6f(p^2)p^4+\bar{\mathcal{R}}_k(p^2)\right)^2},\end{aligned} \tag{5.9}$$

$$\begin{aligned}0=q^2f'(q^2)&-\frac{g}{144\pi}\int_0^{\frac{1}{4}}dv\;\frac{20v^2+8v-1}{\sqrt{1-4v}}\;\frac{\dot{\bar{\mathcal{R}}}_k(q^2v)+2\bar{\mathcal{R}}_k(q^2v)}{q^2v-\frac{4}{3}\lambda+6f(q^2v)q^4v^2+\bar{\mathcal{R}}_k(q^2v)}\\&-\frac{gq^4}{24\pi}\left\{\int_0^\infty dv-\int_0^{\frac{1}{4}}dv\;\sqrt{1-4v}(1+2v)\right\}\\&\frac{f(\hat{x}v)(4v^2+2v)-f(\hat{x})(2v^2+3v+2)}{v-1}\;\frac{\dot{\bar{\mathcal{R}}}_k(q^2v)+2\bar{\mathcal{R}}_k(q^2v)}{(q^2v-\frac{4}{3}\lambda+6f(q^2v)q^4v^2+\bar{\mathcal{R}}_k(q^2v))^2}\\&-\frac{g}{3\pi^2}\int_0^\infty dp\,p^3\int_{-1}^1dx\;\sqrt{1-x^2}\left(6\widetilde{K}(p,q,x)-\bar{K}(p,q,x)\right)\;\frac{\dot{\bar{\mathcal{R}}}_k(p^2)+2\bar{\mathcal{R}}_k(p^2)}{\left(p^2-\frac{4}{3}\Lambda+6f(p^2)p^4+\bar{\mathcal{R}}_k(p^2)\right)^2}.\end{aligned} \tag{5.10}$$

When looking at equation (5.8), we note that, as the integrals are positive, we do indeed see that g must be smaller than 0 to solve this equation, as foreshadowed in chapter 3. Comparing the equations with the ones obtained in *e.g.* [28], it becomes clear one should include gauge fixing and ghost contributions to the beta functions for λ and g. Therefore, we will use the beta functions found in [31] to improve the flow equations for the Einstein-Hilbert section. Here an harmonic gauge and a linear split of the metric

$$g_{\mu\nu}=\bar{g}_{\mu\nu}+h^t_{\mu\nu}+\frac{1}{4}\bar{g}_{\mu\nu}h, \tag{5.11}$$

are used, where h^t is the traceless part of the fluctuations and h is the trace. For $\bar{g}\to\hat{g}$ and $h^t=0$ this reduces to the conformally reduced setting we used (3.18). Accordingly, the structure of the fixed point equations for w and f should not change if we included the full fluctuation spectrum there as well. The beta functions including ghost and gauge fixing contributions are:

$$k\partial_k\lambda = (\eta_N - 2)\lambda + \frac{g}{2\pi}\left(10\Phi^1_2(-2\lambda) - 8\Phi^1_2(0) - 5\eta_N\widetilde{\Phi}^1_2(-2\lambda)\right), \tag{5.12}$$

$$k\partial_k g = (2+\eta_N)g, \tag{5.13}$$

with the new anomalous dimension

$$\eta_N(g,\lambda) = \frac{\frac{g}{3\pi}\left(5\Phi^1_1(-2\lambda) - 18\Phi^2_2(-2\lambda) - 4\Phi^1_1(0) - 6\Phi^2_2(0)\right)}{1+\frac{g}{6\pi}\left(5\widetilde{\Phi}^1_1(-2\lambda) - 18\widetilde{\Phi}^2_2(-2\lambda)\right)}, \tag{5.14}$$

and the regulator-dependent threshold functions

$$\Phi^p_n(v) = \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}\frac{\bar{\mathcal{R}}_k(z) - z\bar{\mathcal{R}}'_k(z)}{\left(z+\bar{\mathcal{R}}_k(z)+v\right)^p}, \tag{5.15}$$

$$\widetilde{\Phi}^p_n(v) = \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}\frac{\bar{\mathcal{R}}_k(z)}{\left(z+\bar{\mathcal{R}}_k(z)+v\right)^p}, \tag{5.16}$$

where $\bar{\mathcal{R}}'_k(z)$ denotes the derivative with respect to the argument. In the beta functions $\Phi(0)$ denotes a contribution with vanishing gap-parameter v, *i.e.* a ghost contribution. Let us now investigate the system for $F_k(\Delta) = 0$.

5.2 Solving the system for $F_k(\Delta) = 0$

In this section we will derive and solve the fixed-point equations for $F_k(\Delta) = 0$. When looking at the fixed point equations it becomes clear why this greatly simplifies matters. By setting $F_k(\Delta) = 0$ the fixed point equation for w becomes a *linear* integro-differential equation

$$\begin{aligned}0 &= q^2 w'(q^2) + \frac{g}{48\pi}\int_0^{\frac{1}{4}} dv\,\frac{\dot{\bar{\mathcal{R}}}_k(q^2v) + 2\bar{\mathcal{R}}_k(q^2v)}{q^2v - \frac{4}{3}\lambda + \bar{\mathcal{R}}_k(q^2v)}\,(1-4v)^{3/2}\\ &+ \frac{g}{3\pi^2}\int_0^\infty dp\, p^3\int_{-1}^1 dx\sqrt{1-x^2}\,K(p,q,x)\frac{\dot{\bar{\mathcal{R}}}_k(p^2) + 2\bar{\mathcal{R}}_k(p^2)}{\left(p^2 - \frac{4}{3}\lambda + \bar{\mathcal{R}}_k(p^2)\right)^2}\\ &\equiv F(q^2) - \mathcal{O}\circ w.\end{aligned} \tag{5.17}$$

where we have introduced the integro-differential operator $\mathcal{O}$. As mentioned, the integral appears as we integrate over loop momentum. We also see from (5.7) and (5.8) that the equations for λ and g are decoupled for $F_k = 0$. Therefore, we use the fixed point equations from [31] written above for the Einstein-Hilbert sector.

Now, not including the Ricci form factor clearly is an approximation. It means we only consider transverse traceless (spin 2) modes.

As the fixed point equations for the cosmological constant (equation (5.12) with $k\partial_k\lambda = 0$) and Newtons coupling ($\eta_N = -2$) are independent of the Weyl form factor, we solve this system first. We will use the regulator

$$\bar{\mathcal{R}}_k(z) = \mathrm{e}^{-\alpha z} \tag{5.18}$$

which is smooth and leads to rapid convergence when numerically evaluating the integrals. The results below are obtained using $\alpha = 1$ and do not change qualitatively if α is altered. The fixed point (denoted with an asterisk-subscript) is situated at

$$\lambda_* = 0.2855, \tag{5.19}$$

$$g_* = 0.3740. \tag{5.20}$$

5.2.1 Asymptotic behaviour

As the fixed point equation for w is still quite involved, let us start by getting some insight into the leading order behaviour of the solution of equation (5.17) by looking at the asymptotic behaviour. First, we can see from the definition of $K(p, q, x)$ (equation (4.19)) that the fixed point equation (5.9) is unchanged under a constant shift of $w(x)$. We thus have a free constant parameter w_∞, which is due to the conformally reduced approximation.

We now expand the integrand of the operator $\mathcal{O}$ working on w for large $\frac{q}{p}$ to find

$$\begin{aligned}
K(p,q,x) = \frac{1}{2}\Bigg(&\frac{w'(q^2)\left(3p^4 - 2p^3qx - p^2q^2\left(18x^2+5\right) - 2q^4\right)}{p^2+2pqx} + 2w(p^2+2pqx+q^2) \\
&+ \frac{\left(-4p^3qx + p^2q^2\left(7-6x^2\right) + 4pq^3x + 2q^4\right)\left(w(p^2+2pqx+q^2) - w(q^2)\right)}{\left(p^2+2pqx\right)^2} - 2w(q^2)\Bigg) \\
&\overset{q^2\to\infty}{\sim} \frac{1}{2}\left(q^4 w''(q^2) + 2q^2 w'(q^2)\right) + \frac{1}{3}pqx\left(q^4 w^{(3)}(q^2) + 3q^2 w''(q^2) - 12w'(q^2)\right) \\
&+ \frac{1}{12}p^2\left(30w'(q^2) + q^2\left(2q^4x^2w^{(4)}\left(q^2\right) + 2q^2\left(4x^2+1\right)w^{(3)}(q^2) + 3\left(2x^2+7\right)w''(q^2)\right)\right) + \mathcal{O}\left(p^3\right).
\end{aligned} \tag{5.21}$$

Thus, the leading order terms are $w_\infty + \frac{1}{2}\hat{x}^2 w''(\hat{x}) + \hat{x}w'(\hat{x})$ for $\hat{x} = q^2$. Other leading order contributions can arise from the inhomogeneous part $F(q^2)$ and from $q^2w'(q^2) = \hat{x}w'(\hat{x})$. For $F(q^2) = F(\hat{x})$:

$$\begin{aligned}
F(\hat{x}) &= \frac{g}{48\pi}\int_0^{\frac{1}{4}} dv\, O_1(v\hat{x})(1-4v)^{\frac{3}{2}} = \frac{g}{48\pi\hat{x}}\int_0^{\frac{\hat{x}}{4}} du\, O_1(u)\left(1-4\frac{u}{\hat{x}}\right)^{\frac{3}{2}} \\
&= \frac{g}{48\pi\hat{x}}\int_0^{\frac{\hat{x}}{4}} du\, O_1(u)\sum_{n=0}^{\infty}\left(-\frac{4u}{\hat{x}}\right)^n\binom{\frac{3}{2}}{n} = \frac{g}{48\pi\hat{x}}\sum_{n=0}^{\infty}\left(-\frac{4}{\hat{x}}\right)^n\binom{\frac{3}{2}}{n}\int_0^{\frac{\hat{x}}{4}} du\, O_1(u)\, u^n \\
&\overset{\hat{x}\to\infty}{\sim} \frac{g}{48\pi\hat{x}}\sum_{n=0}^{\infty}\left(-\frac{4}{\hat{x}}\right)^n\binom{\frac{3}{2}}{n}\int_0^{\infty} du\, O_1(u)\, u^n \equiv \frac{g}{48\pi\hat{x}}\sum_{n=0}^{\infty}\left(-\frac{4}{\hat{x}}\right)^n\binom{\frac{3}{2}}{n}\beta_n
\end{aligned} \tag{5.22}$$

where we use O_1 with $F_k = 0$, β_n is a regulator dependent number, and we assumed the integral and sum can be interchanged. The leading order contribution from this part will thus be $\frac{g}{48\pi\hat{x}}\beta_0 \equiv \frac{\bar{\beta}_0}{\hat{x}}$

So, for $w_\infty = 0$, to leading order, the fixed point equation (5.17) becomes

$$\frac{\alpha_0\hat{x}^3w''(\hat{x}) + 2(\alpha_0+8)\hat{x}^2w'(\hat{x}) + 16\bar{\beta}_0}{16\hat{x}} = 0, \tag{5.23}$$

where we have defined $\alpha_n \equiv \frac{8g_*}{3\pi^2}\int_0^\infty dp\int_{-1}^1 dx p^{3+2n}\sqrt{1-x^2}\frac{2\bar{\mathcal{R}}_k(p^2)+\dot{\bar{\mathcal{R}}}_k(p^2)}{\left(p^2+\bar{\mathcal{R}}_k(p^2)-\frac{4}{3}\lambda_*\right)^2}$. This gives the solution

$$w_*(\hat{x}) \overset{\hat{x}\to\infty}{\sim} \frac{\bar{\beta}_0}{\hat{x}} - \frac{\alpha_0\bar{c}_1\hat{x}^{-\frac{16}{\alpha_0}-1}}{\alpha_0+16} + \bar{c}_2 + ... \tag{5.24}$$

As $\alpha_0 > 0$ by definition, the leading order is at least $\hat{x}^{-1}$. We get the asymptotic expansion

$$w_*(x) \overset{x\to\infty}{\sim} w_\infty + \frac{\mathrm{c1}}{x} + \frac{\mathrm{c2}}{x^2} + \frac{\mathrm{c3}}{x^3} + \frac{\mathrm{c4}}{x^4} + ... \tag{5.25}$$

with

$$c1 = 0.0149493, \tag{5.26}$$

$$c2 = \frac{-0.015625\alpha_1 c1 - 0.0746288}{0.0625\alpha_0 + 2}, \tag{5.27}$$

$$c3 = \frac{-0.015625\alpha_2 c1 - 0.203125\alpha_1 c2 + 0.124924}{0.1875\alpha_0 + 3}, \tag{5.28}$$

$$c4 = \frac{-0.1875\alpha_2 c2 - 0.625\alpha_1 c3 + 0.216664}{0.375\alpha_0 + 4}. \tag{5.29}$$

5.2.2 Numerical solution

Now that we know about the asymptotic behaviour of w_*, we can go on to solve equation (5.17) numerically using pseudospectral methods. Because of the decreasing leading order term, we use the ansatz (2.8) with $\sigma = 0$. The collocation method then implies solving

$$F(q^2) = \mathcal{O} \circ w(q^2) = \mathcal{O} \circ \sum_{n=0}^{N} j_n R_n(q^2). \tag{5.30}$$

Now we take great advantage of the fact that equation (5.17) is a linear equation. This allows us to express the integro-differential operator $\mathcal{O}$ in the basis of Chebyshev rational functions, such that we get $\mathcal{O}_{in} = \mathcal{O} \circ R_n(x_i)$. The fixed point equation can then be written as

$$F(x_i) = \mathcal{O}_{in} j_n, \tag{5.31}$$

where $(q^2)_i = x_i$ are the collocation points. Finding the coefficients j_i now amounts to (simply) solving a matrix equation. As noted above, a constant will not contribute to $w(x)$. We choose $w_\infty = 0$ for now and therefore set $F(x_1) = 0$ and $\mathcal{O}_{1n} = (-1)^n$, which are the values of $R_n(0)$.

We again use the regulator defined in (5.18) for $\alpha = 1$. The obtained results are plotted in Fig. 3. We have set the reference scale to $L = 1$, granting as many collocation points (and thereby precision) to the constant as the asymptotic domain of the solution[4] The approximations for different N's agree very well, which is also clear from Fig. 4, where the difference between two solutions for different values for N is plotted. In Fig. 5 the evolution of the coefficients for $N = 30$ can be seen. We conclude that the solution is very robust.

Furthermore, the solution is globally well-defined and unique up to a constant, as explained above. Importantly, for $w_\infty > 0$, the solution is positive-definite, meaning there are no additional poles in

[4]These two parts can best be identified in the logarithmic plot used in Fig. 6.

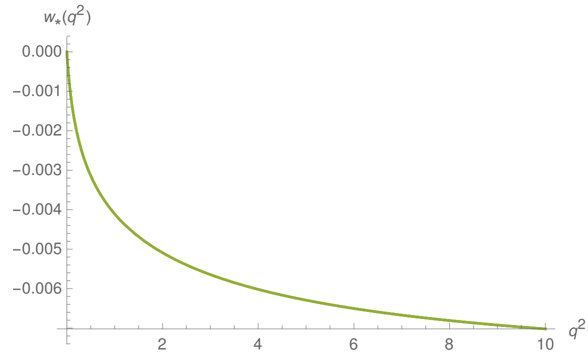

Figure 3: Plot of w_* for $N = 20, 25, 30$ and $w(0) = 0$.

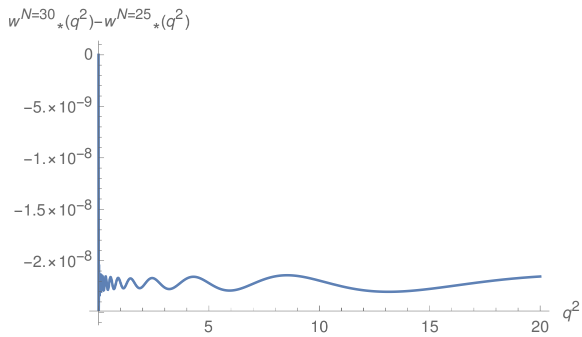

Figure 4: Plot of the difference between w_* for $N = 30$ and for $N = 25$.

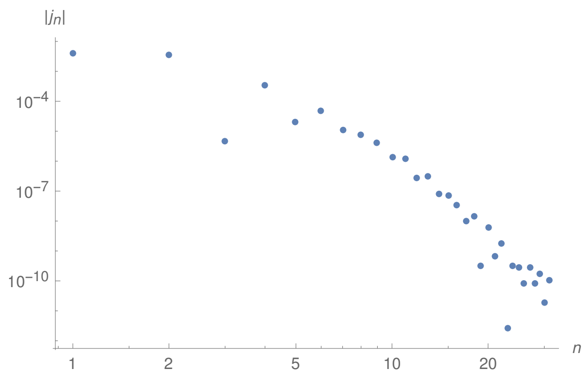

Figure 5: Plot of the absolute value of the coefficients j_n for w_* for $N = 30$.

the propagator for $q^4 > 0$.

The rather simple shape of the solution $w_*(x)$ invites to try and parametrise it. The solution and its parametrisation,

$$w_*^{fit} \approx w_\infty + \frac{\rho}{\frac{\rho}{\kappa} + q^2}, \;\; \rho \approx 0.0149, \;\; \kappa \approx 0.00817, \tag{5.32}$$

are shown in Fig. 6. (Note that ρ is equal to c1 used in the asymptotic analysis). The simple form of this approximation is expected to prove useful in the future. The parametrisation (5.32) is a non-

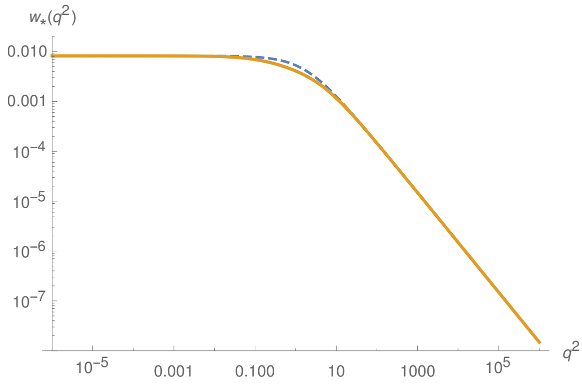

Figure 6: Plot of w_* (for $n = 30$) and w_*^{fit} for $w(\infty) = 0$.

local function (it only admits an infinite power series expansion), such that we avoid Ostrogradski instabilities.

At this point a remark is in order. We do not perform a stability analysis on W_k as one should interpret the form factor as a momentum-dependent wave function renormalisation, meaning that it is not a 'real' coupling. Its properties are related to a momentum-dependent generalisation of the anomalous dimension rather than to the critical exponents [19].

5.3 Solving the full system

Let us now look at the full system, *i.e.* with $F_k(\Delta) \neq 0$, and focus on solving the coupled system of fixed point equations for Λ_k, G_k and F_k. We *propose* the following set of fixed point equations. For the Einstein-Hilbert sector we take the equations from [31] again, but now add the second integral from equation (5.8). We thus neglect the contribution of $6f(z)z^2$ in the other gravity-propagators. This can be justified, as for small z this term is subleading, for large z the integral is suppressed by the regulator term, and for $z \sim 1$, the analysis of the previous chapter suggests the form factor might be quite small. ($w_*^{fit}(1) \simeq 0.077$). The other two fixed point equations we keep as derived above, such that we have the set

$$\begin{aligned} 0 &= -(2-\eta_N)\lambda + \frac{1}{2\pi} g \left(6\Phi^1_2(-2\lambda) - 8\Phi^1_2(0) - 5\eta_N \widetilde{\Phi}^1_2(-2\lambda)\right), \\ 0 &= (2+\bar{\eta}_N)g, \end{aligned} \tag{5.33}$$

with

$$\bar{\eta}_N = \eta_N + \frac{1}{2\pi}\int_0^\infty dz\, z^2\ \left(2f(z) - f(0)\right) \frac{\dot{\bar{\mathcal{R}}}_k(z) + 2\bar{\mathcal{R}}_k(z)}{\left(z - \frac{4}{3}\Lambda + 6f(z)z^2 + \bar{\mathcal{R}}_k(z)\right)^2}, \tag{5.34}$$

and equations (5.9) and (5.10). We thus have a system of coupled *nonlinear* integro-differential equations to solve.

5.3.1 Asymptotic behaviour

It is clearly extremely hard to solve the equations above. Let us first have a look at the asymptotic behaviour. We will investigate equation (5.10) piece by piece for $\frac{q}{p} \to \infty$. The first piece is just $xf'(x)$ for $x = q^2$. The inhomogeneous part (the second term in the equation) becomes

$$
\begin{aligned}
&- \frac{g}{144\pi} \int_0^{\frac{1}{4}} dv\, \frac{20v^2 + 8v - 1}{\sqrt{1-4v}} O_1(xv) \\
&= -\frac{g}{144\pi}\frac{1}{x} \int_0^{\frac{x}{4}} du \left(20\left(\frac{u}{x}\right)^2 + 8\left(\frac{u}{x}\right) - 1\right) O_1(u) \sum_{n=0}^{\infty} \left(\frac{-4u}{x}\right)^n \binom{-\frac{1}{2}}{n} \\
&\overset{x\to\infty}{\sim} -\frac{g}{144\pi}\frac{1}{x} \sum_{n=0}^{\infty} \left(\frac{-4}{x}\right)^n \binom{-\frac{1}{2}}{n} \int_0^{\infty} du\, O_1(u) \left(20\left(\frac{u}{x}\right)^2 + 8\left(\frac{u}{x}\right) - 1\right) u^n \\
&\equiv -\frac{g}{144\pi}\frac{1}{x} \sum_{n=0}^{\infty} \left(\frac{-4}{x}\right)^n \binom{-\frac{1}{2}}{n} \beta_n,
\end{aligned} \tag{5.35}
$$

where β_n is again a regulator-dependent number and we expanded $(1-4v)^{-\frac{1}{2}}$, wrote $u = vx$ and took the limit $x \to \infty$ in the second step.

The integral over v from 0 to $\frac{1}{4}$ on the second line of equation (5.10) can be written as

$$
\begin{aligned}
&\frac{gq^4}{24\pi} \int_0^{\frac{1}{4}} dv\, \sqrt{1-4v}(1+2v)\, I_1\, V'(q^2 v) \\
&= \frac{gx^2}{24\pi}\frac{1}{x} \int_0^{\frac{x}{4}} du \sum_{n=0}^{\infty} \left((-1)^{n+1} 2^{2n-2} \binom{\frac{1}{2}}{n-1} + (-4)^n \binom{\frac{1}{2}}{n}\right) \left(\frac{u}{x}\right)^n I_1\, V'(u) \\
&\overset{x\to\infty}{\sim} \frac{g}{24\pi} \sum_{n=0}^{\infty} x^{1-n} \left((-1)^{n+1} 2^{2n-2} \binom{\frac{1}{2}}{n-1} + (-4)^n \binom{\frac{1}{2}}{n}\right) \int_0^{\infty} du\, u^n\, I_1\, V'(u),
\end{aligned} \tag{5.36}
$$

for $n = 0$ this becomes $\frac{g}{24\pi} x \int_0^{\infty} du\, I_1\, V'(u) = \frac{gq^4}{24\pi} \int_0^{\infty} dv\, I_1\, V'(vq^2)$ which is exactly the integral over v from 0 to ∞ on the second line of equation (5.10), but with opposite sign. These terms thus cancel in the large momentum limit. For $n = 1$ we have $(-1)^{n+1} 2^{2n-2} \binom{\frac{1}{2}}{n-1} + (-4)^n \binom{\frac{1}{2}}{n} = 0$, such that the leading order contribution from the second and third integral is $\frac{g}{24\pi} x^{-1} \left(-4\binom{\frac{1}{2}}{1} + 16\binom{\frac{1}{2}}{2}\right) \int_0^{\infty} du u^2 I_1 V'(u) \sim \frac{f(x)}{x}$.

Expanding the second part of the integrand of the last term, $\bar{K}(p,q,x)$, for large $\frac{q}{p}$, or small p (which we can do because the regulators are suppressing large p), we get

$$
\begin{aligned}
\bar{K}(p,q,x) = &\frac{1}{2}\left(2q^2 f'(q^2) + q^4 f''(q^2)\right) + \frac{1}{3} px \left(q^5 f^{(3)}(q^2) + 3q^3 f''(q^2)\right) \\
&+ \frac{1}{6} p^2 \big(3q^2 x^2 f''(q^2) + 6q^2 f''(q^2) + 6f'(q^2) \\
&+ q^6 x^2 f^{(4)}(q^2) + 4q^4 x^2 f^{(3)}(q^2) + q^4 f^{(3)}(q^2)\big) + \mathcal{O}\left(p^3\right).
\end{aligned} \tag{5.37}
$$

The contribution linear in x will vanish as we integrate x from -1 to 1. The leading order will thus be $\bar{K}(p,q,x) = q^2 f'(q^2) + \frac{1}{2} q^4 f''(q^2)$.

Expanding the first part of the integrand of the last term, $\widetilde{K}(p,q,x)$ without the propagator part, for small p, we get

$$
\begin{aligned}
&\widetilde{K'}(p,q,x) = \left(q^8 f'(q^2)^2 + 2q^6 f(q^2)f'(q^2) + q^4 f(q^2)^2\right)\\
&+ p\left(2q^9 x f'(q^2)f''(q^2) + 2q^7 x f(q^2)f''(q^2) + 6q^7 x f'(q^2)^2 + 8q^5 x f(q^2)f'(q^2) + 2q^3 x f(q^2)^2\right)\\
&+ p^2\Bigg(q^2x^2 f(q^2)^2 + 4q^2 f(q^2)^2 - 2q^2 f(0)f(q^2) + q^{10}x^2 f''(q^2)^2 + \frac{4}{3}q^{10}x^2 f^{(3)}(q^2)f'(q^2)\\
&+ \frac{4}{3}q^8x^2 f(q^2)f^{(3)}(q^2) + 12q^8x^2 f'(q^2)f''(q^2) + q^8 f'(q^2)f''(q^2) + 8q^6x^2 f(q^2)f''(q^2)\\
&+ 13q^6x^2 f'(q^2)^2 + q^6 f(q^2)f''(q^2) + 4q^6 f'(q^2)^2 + 10q^4x^2 f(q^2)f'(q^2) - 2q^4 f(0)f'(q^2)\\
&+ 8q^4 f(q^2)f'(q^2)\Bigg) + \mathcal{O}\left(p^3\right),
\end{aligned}
\tag{5.38}
$$

where again the contributions linear in x will vanish. The propagator part of $\widetilde{K}(p,q,x)$ becomes

$$
\begin{aligned}
&\frac{1}{p^2 + 2pqx + q^2 + 6f(p^2+2pqx+q^2)(p^2+2pqx+q^2)^2 + \bar{\mathcal{R}}_k(p^2+2pqx+q^2) - \frac{4}{3}\lambda}\\
&\to \frac{1}{q^2 + 6f(q^2)q^4},
\end{aligned}
\tag{5.39}
$$

such that the whole leading order term for $\widetilde{K}(p,q,x)$ will be

$$
\frac{q^8 f'(q^2)^2 + 2q^6 f(q^2)f'(q^2) + q^4 f(q^2)^2}{q^2 + 6f(q^2)q^4}.
\tag{5.40}
$$

For $f(x) = \gamma x^n$ for $n > -1$, the f-term will dominate the propagator. Then the leading order will be $\widetilde{K} = \frac{1}{6}f(x) + \frac{x}{3}f'(x) + \frac{x^2}{3}\frac{f'(x)f'(x)}{f(x)}$ where now $x = q^2$.

Collecting the leading orders of all terms found above we arrive at the asymptotic fixed point equation

$$
0 = xf'(x) - \alpha x^{-1} - x^{-1}\beta - \gamma\left(f(x) + xf'(x) + x^2\frac{f'(x)f'(x)}{f(x)} - \frac{1}{2}x^2 f''(x)\right)
\tag{5.41}
$$

with again $x = q^2$ and

$$
\alpha = \frac{g}{144\pi}\int_0^\infty du\, O_1(u)\left(20\left(\frac{u}{x}\right)^2 + 8\left(\frac{u}{x}\right) - 1\right),
\tag{5.42}
$$

$$
\beta = \frac{g}{6\pi}\int_0^\infty du\, u^2 V'(u)\left[\left(\frac{1}{2} + \frac{2u}{x-u}\right)f(x) + \left(1 - \frac{2u}{x-u}\right)f(u)\right],
\tag{5.43}
$$

$$
\gamma = \frac{g}{6\pi}\int_0^\infty dp\, p^3\, \frac{\dot{\bar{\mathcal{R}}}_k(p^2) + 2\bar{\mathcal{R}}_k(p^2)}{\left(p^2 - \frac{4}{3}\lambda + 6f(p^2)p^4 + \bar{\mathcal{R}}_k(p^2)\right)^2}.
\tag{5.44}
$$

Assuming that $f(x) = cx^n, n > 0$ to leading order equation (5.10) becomes

$$
0 = cx^n\left(n - \gamma\left(1 + n + n^2 - \frac{1}{2}n(n-1)\right)\right),
\tag{5.45}
$$

$$
0 = n - \gamma\left(1 + \frac{3}{2}n + \frac{1}{2}n^2\right),
\tag{5.46}
$$

giving

$$
n_\pm = \frac{1}{2\gamma}\left(2 - 3\gamma \pm \sqrt{4 - 12\gamma + \gamma^2}\right).
\tag{5.47}
$$

The relation (5.46) is plotted in Fig. 7. From the plot, we observe that a solution $f(x) = cx^n, n > 0$ exists if $\gamma \leq 0.343$. Moreover, two solutions for $n > 0$ exist for most values of γ, one with $n < 1.35$

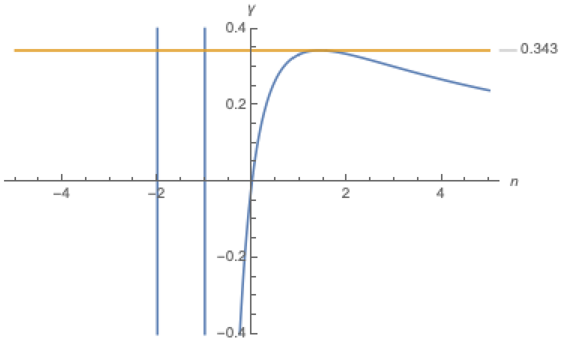

Figure 7: The solutions (5.47) plotted along a line at $\gamma = 0.343$.

and one where n might be very large for $\gamma \to 0$. The integral γ depends on $f(x)$ and is shown in Fig. 8 for positive c and n. It seems safe to say that γ will be smaller than 0.343, but we have to hope that the small-n solution will be found and dominant, as it seems $\gamma \ll 0.343$.

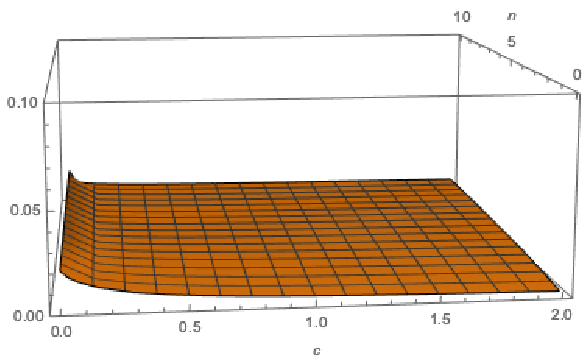

Figure 8: The value of γ for different values of c and n.

5.3.2 Newton-Raphson method

Knowing the asymptotic behaviour, let us try to (approximately) solve the system of fixed point equations using the Newton-Raphson method outlined in the introduction. Keeping the asymptotic behaviour $f(x) \sim x^n, 0 < n < 1.35$ in mind, we will use the ansatz (2.8) with $\sigma = 1$, giving

$$f(x) = (1+x) \sum_{n=0}^{N} a_n \, R_n(x). \tag{5.48}$$

In principle, one could treat σ as an additional parameter to be established by the Newton-Raphson method, but for the sake of simplifying we fix it. Our strategy will now be to choose an N, make some initial guess for $N-2$ coefficients a_n and the fixed point values λ_* and g_* and then apply the generalised Newton-Raphson method (2.12) to the fixed point equations for λ (5.7), g (5.8) and f (5.10), evaluated at N collocation points to iteratively update these coefficients and values. We will compute the Jacobian analytically and use Mathematica to compute the inverse and residu. Once the residu has become (nearly) zero, N can be increased to test if the finite sum approximation is robust.

It turns out that the success of the method is highly dependent on the initial choice of the coefficients (as was to be expected). The pattern that might work is $a_0 \sim -2a_1 \sim 2a_2 \sim ......$ At this point this system is still under investigation.

6 The nonperturbative quantum Newtonian potential

The spacetime singularities predicted by general relativity have been one of the biggest problems in theoretical physics for a long time and removing these singularities is a key motivation for a theory of quantum gravity. In the previous chapters, we have computed the fixed point values λ_* and g_* and the fixed point function $w_*(q^2)$. Let us now use these as the high-energy limits of Λ_k, G_k and $W_k(\Delta)$ (it is exactly in this limit that quantum gravity is dominant) and see what the physical implications of an effective action provided with these fixed point data are. As we only know the form factor corresponding to the Weyl-squared entry in the action, we will consider the transverse traceless spin 2 mode contributions only. We thus only quantum improve the spin 2 propagator and keep the bare scalar propagator.

To find the transverse traceless propagator we have to invert the transverse traceless contribution to the second variation of the action [32]. We expand in fluctuations around flat space (*i.e.* split the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$). Using *xAct* we find the transverse traceless part of the second variation for the Einstein-Hilbert part to be $64\pi G_k (\Delta - 2\Lambda_k)$. For the Weyl-part we find $128\pi G_k \Delta^2 W_k(\Delta)$. So neglecting the prefactor, we have the full effective transverse traceless graviton propagator

$$\mathcal{G}^{TT}(q^2) = \left(q^2 + 2(q^2)^2\, W_k(q^2)\right)^{-1} . \tag{6.1}$$

Following Donoghue's idea of gravity as an effective field theory [33, 34], we consider the process of a graviton exchange between two scalar fields with masses m_1 and m_2 minimally coupled to gravity, see the Feynman diagram in Fig. 9. We use the static limit where the scalars have infinite mass, such that we can perform the Fourier integral. Furthermore, we use the full propagator introduced above, provided with the form factor as calculated in this thesis, and work with the bare vertices.

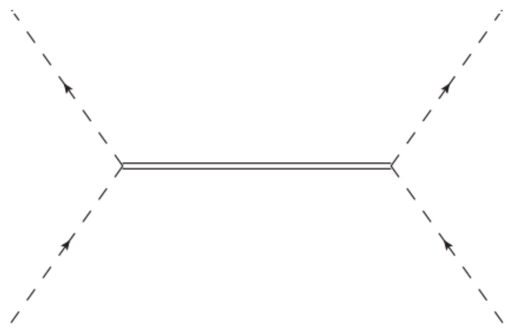

Figure 9: Feynman diagram for the interaction of two massive scalar fields (dashed lines) due to the exchange of a graviton (solid double line).

From the calculations in Appendix E, we find that the transverse traceless contribution to the scattering amplitude associated to the diagram in Fig. 9, including the prefactor, is

$$\mathcal{M}^{TT} = \frac{64\pi G_k}{3} m_1^2 m_2^2 \, \mathcal{G}^{TT}(q^2). \tag{6.2}$$

Exploiting the static limit, the Newtonian potential for the process can be computed as (see again Appendix E)

$$V(r) = -\frac{1}{2m_1}\frac{1}{2m_2}\int \frac{d^3q}{(2\pi)^3} \mathrm{e}^{iq\cdot r}\mathcal{M}. \tag{6.3}$$

Filling in the above equations and using the UV fixed point value w_* for the form factor W_k, we find the potential

$$\begin{aligned} V(r) &= -m_1 m_2 \frac{16\pi G_k}{3}\int \frac{d^3q}{(2\pi)^3}\mathrm{e}^{iq\cdot r}\,\left(q^2 + 2q^4 w_*(q^2)\right)^{-1} \\ &= -m_1 m_2 \frac{16\pi G_k}{3}\int_0^\infty \frac{dq}{(2\pi)^3} q^2 \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\varphi \mathrm{e}^{iq\cdot r\cos\theta}\,\left(q^2 + 2q^4 w_*(q^2)\right)^{-1} \\ &= -m_1 m_2 \frac{16\pi G_k}{3(2\pi)^2}\int_0^\infty dq\, q^2 \frac{2\sin qr}{qr}\,\left(q^2 + 2q^4 w_*(q^2)\right)^{-1} \\ &= -\frac{8G_k m_1 m_2}{3\pi r}\int_0^\infty dq \frac{q\sin(qr)}{q^2 + 2q^4 w_*(q^2)}. \end{aligned} \tag{6.4}$$

We see that any form factor with asymptotic behaviour $w(q^2) \overset{q^2\to\infty}{\sim} q^{-1+\xi}$ for $\xi > 0$ will change the fate of this integral.

For $w_* = 0$, corresponding to the Einstein-Hilbert action of general relativity, this becomes

$$\begin{aligned} V_c(r) &= -\frac{8G_k m_1 m_2}{3\pi r}\int_0^\infty dq \frac{\sin(qr)}{q} \\ &= -\frac{8G_k m_1 m_2}{3\pi r}\int_0^\infty d(qr) \frac{\sin(qr)}{qr} \\ &= -\frac{4}{3}\frac{G_k m_1 m_2}{r}, \end{aligned} \tag{6.5}$$

which is the classical Newtonian potential including the prefactor $\frac{4}{3}$ for the transverse traceless approximation. It is clear that this potential diverges for $r \to 0$, such that the nonrelativistic limit already shows the singularity.

Now incorporating quantum corrections by including the form factor, using $w_* = w_*^{fit}$, we get

$$V_q(r) = -\frac{8G_k m_1 m_2}{3\pi r}\int_0^\infty dq \frac{q\sin(qr)}{q^2 + w_\infty + \frac{\rho}{\frac{\rho}{\kappa}+q^2}}, \tag{6.6}$$

Performing this integral numerically gives the *quantum improved Newtonian potential* shown, together with the classical potential from above, in Fig. 10. For $w_\infty > 0$ the new quantum Newtonian potential importantly tends to a *finite* negative value as $r \to 0$, thereby removing the classical singularity. This finding has resulted in a publication in Physical Review Letters [35].

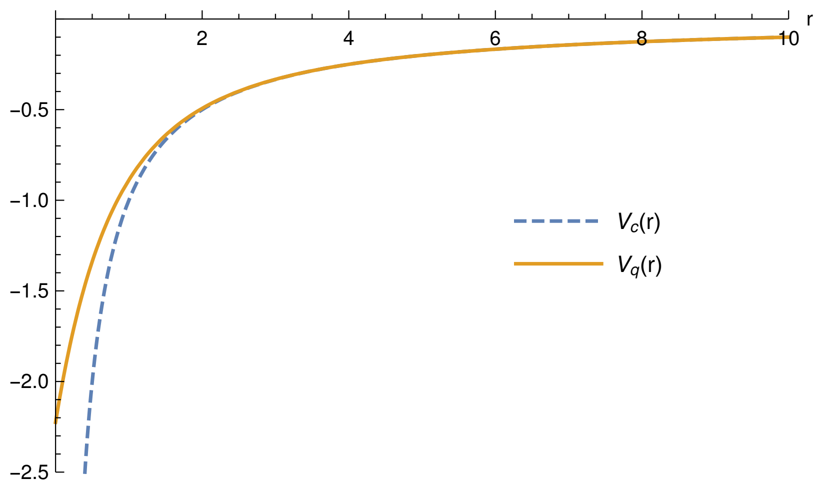

Figure 10: The classical Newtonian potential (dashed) and quantum Newtonian potential (thick). The two start to deviate around the Planck scale.

7 Discussion and outlook

In this thesis, starting from the most general gravitational action up to second order in the curvature -thereby including all relevant contributions to the flat-space propagator- we have nonperturbatively investigated the explicit and full momentum dependence of the gravitational propagator in the Asymptotic Safety framework. Using the renormalisation group formalism supplied with nonlocal heat kernel techniques, we were able to determine the fixed point equations of the system.
To find the second variations needed for the renormalisation group, analytical methods were used for the Ricci-term, while the Mathematica package *xAct* was used to find the variations of the Weyl-term.

Focusing on the transverse traceless sector, we applied psuedospectral (and collocation) numerical methods to solve the resulting integro-differential equations for the form factor corresponding to the Weyl-tensor-squared term.
Regarding these fixed point data as the high-energy limit, allows to calculate the short-distance behaviour of the quantum corrected potential. Strikingly, we found the corrections for the high-energy behaviour to resolve the classical singularity for $r \to 0$.

The results in this thesis show there are many interesting future research opportunities regarding form factors in Asymptotically Safety. Investigating the fixed point solutions for the full system including spin-0 modes (*i.e.* $F_k \neq 0$) might be considered a first sensible step.
All calculations regarding the form factors were performed in the conformally reduced setting. The fixed point equations for the cosmological constant and Newtons coupling in this setting show that this approximation might miss some important contributions. Including these contributions, on the other hand, shows that the qualitative behaviour is already captured in the conformally reduced approximation. Nonetheless, including the tensor fluctuations in the full computation is needed for a definitive statement.
Including these fluctuations will furthermore remove the shift-invariance of the form factor and establish the value of the parameter w_∞, which needs to be positive to resolve the singularity.

Let us now enter a highly speculative regime and consider the most (in)famous spacetime singularities: black holes. To really understand what the obtained results mean in this context one should find a (spherically symmetric) solution to the equations of motion for the effective action used. We can, however, recur to some speculation without this solution. Taking the Schwarzschild solution describing a black-hole

$$ds^2 = -\left(1 - 2V_c(r)\right) dt^2 + \left(1 - 2V_c(r)\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2(\theta) d\varphi^2\right), \tag{7.1}$$

and substituting in the quantum improvement $V_c(r) \to V_q(r) \simeq \alpha_1 + \alpha_2 r + ...$ modifies the spacetime singularity at $r = 0$! The singularity is still present, but it turns out to be integrable, meaning that geodesics can be extended through $r = 0$.
Thus, including form factors might be very relevant for singularity resolution in this area as well.

Acknowledgements

I am incredibly thankful to my supervisors Benjamin and Frank. It was amazing to experience and even be part of real and cutting edge research. It was a privilege to work with (future) giants of the field, something I will be forever proud of. Thanks to their supervision, I was able to learn and achieve more then I could have ever dreamed of. What an astonishing ride it was, to the (collapsed) stars, infinity, and beyond.

Benjamin established himself as a true lifesaver. From cake to connectors and from missing minus signs to a full day of explaining tensor computer algebra. Whatever saves your life he would provide. I also still have to come up with an (answerable) physics question that he will not explain extensively and gladly. It is not for a lack of trying.
I consider myself very lucky to have been able to experience Frank's expertise not only on Asymptotic Safety, but on the whole spectrum of doing research, as well as writing this up and everything that follows after that. There are some invaluable lessons I have learned here, as well as the phrase "there is a conservation of misery"[5].

All in all, this project made me realise how wonderful research can be and convinced me to take on a PhD as a next adventure.
My gratitude is highly irrelevant, as it diverges off to infinity and becomes asymptotically unsafe.

I am also thankful for many meaningful, insightful or just amusing discussions during the course of the last year with many other members of the high-energy department, specifically Chris, Joren, Amir and of course Derryk.

Last but not least, I am extremely grateful for all the love received at the Prof. Bellefroidstraat.

[5]This turned out to be a local law.

A Trace-systematics

A.1 Laplace-transform techniques

We assume we can express any operator function in its inverse Laplace transform

$$W(\Delta) = \int_0^\infty ds\, \widetilde{W}(s)\, \mathrm{e}^{-s\Delta}. \tag{A.1}$$

Using the inverse of this transform we can evaluate an integrand containing $s^0\hat{x}^0\mathrm{f}(s\hat{x})$:

$$\begin{aligned}\int_0^\infty ds\mathrm{f}(s\hat{x})\, \widetilde{W}(s) &= \int_0^\infty ds \int_0^1 d\alpha\, \mathrm{e}^{-s\hat{x}\alpha(1-\alpha)}\, \widetilde{W}(s) = \int_0^1 d\alpha\, W(\hat{x}\alpha(1-\alpha)) \\ &= 2\int_0^{\frac{1}{2}} d\alpha\, W(\hat{x}\alpha(1-\alpha)) = 2\int_0^{\frac{1}{4}} dv \frac{d\alpha}{dv}\, W(\hat{x}v) = \int_0^{\frac{1}{4}} dv \frac{2}{\sqrt{1-4v}}\, W(\hat{x}v),\end{aligned} \tag{A.2}$$

where we used that that $\alpha(1-\alpha)$ is maximal at and symmetric around $\alpha = \frac{1}{2}$, substituted $v = \alpha(1-\alpha)$ and then chose its lower root to determine $\frac{d\alpha}{dv}$.

Similarly, we can evaluate an integrand containing $s^{-1}\hat{x}^{-1}\mathrm{f}(s\hat{x})$

$$\begin{aligned}\int_0^\infty ds\, \mathrm{f}(s\hat{x})(s\hat{x})^{-1}\widetilde{W}(s) &= \hat{x}^{-1}\int_0^\infty ds\, s^{-1}\int_0^1 d\alpha\, \mathrm{e}^{-s\hat{x}\alpha(1-\alpha)}\, \widetilde{W}(s) \\ &= \int_0^\infty ds \int_0^1 d\alpha \int_1^\infty dt\alpha(1-\alpha)\mathrm{e}^{-s\hat{x}t\alpha(1-\alpha)}\, \widetilde{W}(s) \\ &= 2\int_0^{\frac{1}{2}} d\alpha \int_1^\infty dt\, \alpha(1-\alpha)\, W(\hat{x}t\alpha(1-\alpha)) \\ &= 2\int_0^{\frac{1}{2}} d\alpha \int_{\alpha(1-\alpha)}^\infty dv\, W(\hat{x}v) \\ &= 2\int_0^{\frac{1}{2}} d\alpha \left\{\int_{\alpha(1-\alpha)}^{\frac{1}{4}} dv + \int_{\frac{1}{4}}^\infty dv\right\} W(\hat{x}v) \\ &= 2\left\{\int_0^{\frac{1}{4}} dv \int_0^{\frac{1}{2}(1-\sqrt{1-4v})} d\alpha + \int_{\frac{1}{4}}^\infty dv \int_0^{\frac{1}{2}} d\alpha\right\} W(\hat{x}v) \\ &= \int_0^{\frac{1}{4}} dv\, (1-\sqrt{1-4v})\, W(\hat{x}v) + \int_{\frac{1}{4}}^\infty dv\, W(\hat{x}v) \\ &= -\int_0^{\frac{1}{4}} dv\, \sqrt{1-4v}\, W(\hat{x}v) + \int_0^\infty dv\, W(\hat{x}v)\end{aligned} \tag{A.3}$$

where we used the integral representation

$$\int_x^\infty dy\mathrm{e}^{-\alpha(1-\alpha)sy} = \frac{1}{\alpha(1-\alpha)s}\mathrm{e}^{-\alpha(1-\alpha)sx} \tag{A.4}$$

to undo the Laplace transform in the first line and substituted $v = t\alpha(1-\alpha)$ in the second line. In the third line, we split one integral to change the order of the integrals, using that $\frac{1}{2}(1-\sqrt{1-4v})$ is the lower root of $v = \alpha(1-\alpha)$. Note that in both cases the integration domains are $\alpha \in [0, \frac{1}{2}]$ and $v \in [0, \frac{1}{4}]$).

Using the same tools, we can evaluate an integrand containing $s^{-2}\hat{x}^{-2}\mathrm{f}(s\hat{x})$ as

$$\begin{aligned}
&\int_0^\infty ds\, \mathrm{f}(s\hat{x})(s\hat{x})^{-2}\widetilde{W}(s) = \hat{x}^{-2}\int_0^\infty ds\, s^{-2}\int_0^1 d\alpha\, \mathrm{e}^{-s\hat{x}\alpha(1-\alpha)}\,\widetilde{W}(s)\\
&= \int_0^\infty ds \int_0^1 d\alpha \int_1^\infty dt \int_t^\infty du\, \alpha^2(1-\alpha)^2 \mathrm{e}^{-s\hat{x}u\alpha(1-\alpha)}\,\widetilde{W}(s)\\
&= 2\int_0^{\frac{1}{2}} d\alpha \int_1^\infty dt \int_t^\infty du\, \alpha^2(1-\alpha)^2\, W(\hat{x}u\alpha(1-\alpha))\\
&= 2\int_0^{\frac{1}{2}} d\alpha \int_{\alpha(1-\alpha)}^\infty dw \int_w^\infty dv\, W(\hat{x}v)\\
&= 2\int_0^{\frac{1}{2}} d\alpha \int_{\alpha(1-\alpha)}^\infty dv \int_{\alpha(1-\alpha)}^v dw\, W(\hat{x}v)\\
&= 2\int_0^{\frac{1}{2}} d\alpha \int_{\alpha(1-\alpha)}^\infty dv\, \left(v-\alpha(1-\alpha)\right)\, W(\hat{x}v) \qquad (A.5)\\
&= 2\int_0^{\frac{1}{2}} d\alpha \left\{\int_{\alpha(1-\alpha)}^{\frac{1}{4}} dv + \int_{\frac{1}{4}}^\infty dv\right\}\left(v-\alpha(1-\alpha)\right)\, W(\hat{x}v)\\
&= 2\left\{\int_0^{\frac{1}{4}} dv \int_0^{\frac{1}{2}(1-\sqrt{1-4v})} d\alpha + \int_{\frac{1}{4}}^\infty dv \int_0^{\frac{1}{2}} d\alpha\right\}\, \left(v-\alpha(1-\alpha)\right)\, W(\hat{x}v)\\
&= \int_0^{\frac{1}{4}} dv\, \left(v(1-\sqrt{1-4v}) - \frac{1}{4}(1-\sqrt{1-4v})^2 + \frac{1}{12}(1-\sqrt{1-4v})^3\right) W(\hat{x}v)\\
&+ \int_{\frac{1}{4}}^\infty dv\, \left(v-\frac{1}{6}\right)\, W(\hat{x}v)\\
&= \frac{1}{6}\int_0^{\frac{1}{4}} dv\, (1-4v)^{\frac{3}{2}}\; W(\hat{x}v) + \int_0^\infty dv\, \left(v-\frac{1}{6}\right)\, W(\hat{x}v)
\end{aligned}$$

where we substituted $v = u\alpha(1-\alpha)$ and $w = t\alpha(1-\alpha)$. In the fifth line we change the order of the v and w integrals and subsequently perform the w integral. Thereafter, the v-integral is split and exchanged with the α-integral in the same way as above. In the last three lines the α-integrals are performed.

Collecting the results above and combining with equation (3.26), we have the full toolbox to calculate traces:

$$\int_0^\infty ds\, (s\hat{x})^{-n}\,\widetilde{W}(s) = \frac{\hat{x}^{-n}}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}W(z) = \frac{1}{\Gamma(n)}\int_0^\infty dv\, v^{n-1}W(\hat{x}v), \qquad (A.6)$$

$$\int_0^\infty ds\, \mathrm{f}(s\hat{x})\,\widetilde{W}(s) = \int_0^{\frac{1}{4}} dv \frac{2}{\sqrt{1-4v}}\, W(\hat{x}v), \qquad (A.7)$$

$$\int_0^\infty ds\, (s\hat{x})^{-1}\,\mathrm{f}(s\hat{x})\,\widetilde{W}(s) = \left\{\int_0^\infty dv - \int_0^{\frac{1}{4}} dv\, \sqrt{1-4v}\right\} W(\hat{x}v), \qquad (A.8)$$

$$\int_0^\infty ds\, (s\hat{x})^{-2}\,\mathrm{f}(s\hat{x})\,\widetilde{W}(s) = \left\{\int_0^\infty dv\, \left(v-\frac{1}{6}\right) + \frac{1}{6}\int_0^{\frac{1}{4}} dv\, (1-4v)^{\frac{3}{2}}\right\} W(\hat{x}v). \qquad (A.9)$$

A.2 Off-diagonal heat kernel

From the definition of the heat kernel [36]

$$H(s;x,y) = \frac{1}{(4\pi s)^{d/2}} \mathrm{e}^{-\frac{\sigma(x,y)}{2s}} \Omega(s;x,y) \tag{A.10}$$

with the world function $\sigma(x,y)$, we see that, as $D_\mu \sigma(x,y) = 0$, in the coincidence limit where $x = y$

$$\overline{D^x_\mu H(s;x,y)} = \frac{1}{(4\pi s)^{d/2}} \overline{D_\mu \Omega(s;x,y)}. \tag{A.11}$$

And as

$$\Omega(s;x,y) = \sum_{n\geq 0} s^n A_n(x,y) \tag{A.12}$$

and

$$\overline{D_\mu A_n(x,y)} = \frac{n\,\Gamma(n+2)}{\Gamma(2n+3)} D_\mu D^{2n-2} R \tag{A.13}$$

we arrive at

$$\overline{D_\mu \Omega(s;x,y)} = s\left(-\frac{1}{4s\Delta} + \left(\frac{1}{4s\Delta} + \frac{1}{8}\right) f(s\Delta)\right) D_\mu R \tag{A.14}$$

such that the trace of the off-diagonal heat kernel is

$$\mathrm{Tr}\left[D_\mu H(s)\right] = \frac{s}{(4\pi s)^{d/2}} \mathrm{tr}\left\{-\frac{1}{4s\Delta} + \left(\frac{1}{4s\Delta} + \frac{1}{8}\right) f(s\Delta)\right\} D_\mu R. \tag{A.15}$$

B Conformally reduced setting and retaining exact momentum dependence

To compute the flow equations, we need to expand functions of Laplacians. In doing this, we want to keep the exact momentum dependence, so we have to keep track of all the covariant derivatives. The strategy to derive the flow equations is as follows: first we perform an inverse Laplace transform of the ansatz, then we expand it in the conformally reduced setting after which we (un)do the Laplace transform with respect to the background Laplacian. Before we can perform this next step, the exponential has to be rewritten. We do this using multicommutators, introduced in the next subsection.

B.1 Multicommutators

The multicommutator of some operator structures X, Y is defined as

$$[X,Y]_l = [X,[X,Y]_{l-1}], \tag{B.1}$$

$$[X,Y]_0 = Y, \tag{B.2}$$

for any natural number l. This reduces to the standard commutator for $l = 1$. It obeys the scaling relation

$$[sX,tY]_l = s^l t[X,Y]_l. \tag{B.3}$$

Using the multicommutators we can write down the exact commutation rules for (differential) operators. Using induction, one can prove that

$$X^m Y = \sum_{l=0}^{m} \binom{m}{l} [X,Y]_l X^{m-l}, \tag{B.4}$$

$$Y X^m = \sum_{l=0}^{m} \binom{m}{l} (-1)^l X^{m-l} [X,Y]_l. \tag{B.5}$$

Another useful identity found by induction can be used when integrating over spacetime:

$$\int d^d x \sqrt{g} Y [\Delta, Z]_l X = \int d^d x \sqrt{g} \sum_{k=0}^{l} \binom{l}{k} (-1)^k (\Delta^{m-l} Y) Z (\Delta^l X). \tag{B.6}$$

Using (B.4) and (B.5), one can show that

$$\begin{aligned} \frac{d}{d\epsilon} \mathrm{e}^{X+\epsilon Y} &= \sum_{l=0}^{\infty} \frac{1}{(l+1)!} [X+\epsilon Y, Y]_l \mathrm{e}^{X+\epsilon Y} \equiv M(\epsilon, X, Y) \mathrm{e}^{X+\epsilon Y}, \\ &= \mathrm{e}^{X+\epsilon Y} \sum_{l=0}^{\infty} \frac{(-1)^l}{(l+1)!} [X+\epsilon Y, Y]_l \equiv \mathrm{e}^{X+\epsilon Y} \tilde{M}(\epsilon, X, Y). \end{aligned} \tag{B.7}$$

Expanding in ϵ, we can then use this to write the transformation rules for the exponential

$$\begin{aligned} \mathrm{e}^{X+\epsilon Y} &= \left[1 + \epsilon M(0,X,Y) + \frac{\epsilon^2}{2} \big(M(0,X,Y)^2 + \partial_\epsilon M(0,X,Y)\big) + \mathcal{O}(\epsilon^3)\right] \mathrm{e}^X \\ &= \mathrm{e}^X \left[1 + \epsilon \tilde{M}(0,X,Y) + \frac{\epsilon^2}{2} \Big(\tilde{M}(0,X,Y)^2 + \partial_\epsilon \tilde{M}(0,X,Y)\Big) + \mathcal{O}(\epsilon^3)\right], \end{aligned} \tag{B.8}$$

where we have

$$\partial_\epsilon M(0,X,Y) = \sum_{l=0}^{\infty}\sum_{k=1}^{\infty} \frac{1}{(l+k+2)!}[X,[Y,[X,Y]_k]]_l,$$
$$\partial_\epsilon \tilde{M}(0,X,Y) = \sum_{l=0}^{\infty}\sum_{k=1}^{\infty} \frac{(-1)^{k+l+1}}{(l+k+2)!}[X,[Y,[X,Y]_k]]_l. \tag{B.9}$$

B.2 Expanding the form factors

The see how the Ricci form factor in our effective action transforms we start with the Laplace transform

$$\begin{aligned}
&\int \sqrt{g}RF_k(\Delta)R = \int \sqrt{g}R\int_0^\infty ds\widetilde{F}_k(s)\mathrm{e}^{-s\Delta}R \\
&= \int \sqrt{g}R\left[\int_0^\infty ds\widetilde{F}_k(s)\mathrm{e}^{-s\left(\frac{1}{\varphi^2}\hat{\Delta}-\frac{2}{\varphi^3}(\hat{D}_\mu\varphi)\hat{D}^\mu\right)}\right]R \\
&= \int \sqrt{g}R\left[\int_0^\infty ds\widetilde{F}_k(s)\mathrm{e}^{-s\left(\hat{\Delta}+\left(-\frac{1}{4}h\hat{\Delta}-\frac{1}{4}(\hat{D}_\mu h)\hat{D}^\mu\right)+\left(\frac{1}{16}h^2\hat{\Delta}+\frac{1}{8}h(\hat{D}_\mu h)\hat{D}^\mu\right)\right)}\right]\cdot R \\
&= \int \sqrt{g}R\int_0^\infty ds\widetilde{F}_k(s)\mathrm{e}^{-s\hat{\Delta}}\ \{1 \\
&+\epsilon\sum_{j=0}^{\infty}\frac{(-1)^j}{(j+1)!}\left[-s\hat{\Delta},\frac{s}{4}\left(h\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu-\frac{1}{4}h^2\hat{\Delta}-\frac{1}{2}h(\hat{D}_\mu h)\hat{D}^\mu\right)\right]_j \\
&+\frac{\epsilon^2}{2}\left(\sum_{j=0}^{\infty}\frac{(-1)^j}{(j+1)!}\left[-s\hat{\Delta},\frac{s}{4}\left(h\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu-\frac{1}{4}h^2\hat{\Delta}-\frac{1}{2}h(\hat{D}_\mu h)\hat{D}^\mu\right)\right]_j\right)^2 \\
&+\frac{\epsilon^2}{2}\left(\sum_{j=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{j+n+1}}{(n+j+2)!}\left[-s\hat{\Delta},\left[\frac{s}{4}\left(h\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu-\frac{1}{4}h^2\hat{\Delta}-\frac{1}{2}h(\hat{D}_\mu h)\hat{D}^\mu\right),\right.\right.\right. \\
&\left.\left.\left.\left[-s\hat{\Delta},\frac{s}{4}\left(h\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu-\frac{1}{4}h^2\hat{\Delta}-\frac{1}{2}h(\hat{D}_\mu h)\hat{D}^\mu\right)\right]_n\right]\right]_j\right)+\mathcal{O}(\epsilon^3)\Bigg\}\cdot R,
\end{aligned} \tag{B.10}$$

where we rescaled $h \to \epsilon h$. We can now set $\epsilon \to 1$. In the end we want to undo the Laplace transform to write $F_k(\hat{\Delta})$ again. Working out the multicommutators using equation (B.6) gives

$$\begin{aligned}
&\int \sqrt{g}RF_k(\Delta)R = \int \sqrt{g}R\int_0^\infty ds\widetilde{F}_k(s)\mathrm{e}^{-s\hat{\Delta}}\ \{1 \\
&+\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^{j+k+j}}{4(j+1)!}\binom{j}{k}s^{j+1}\hat{\Delta}^{j-k}\left(\left(h\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu-\frac{1}{4}h^2\hat{\Delta}-\frac{1}{2}h(\hat{D}_\mu h)\hat{D}^\mu\right)\hat{\Delta}^k\right) \\
&+\frac{1}{2}\left(\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^k}{4(j+1)!}\binom{j}{k}s^{j+1}\hat{\Delta}^{j-k}\left(\left(h\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu+\mathcal{O}(h^2)\right)\hat{\Delta}^k\right)\right)^2 \\
&+\frac{1}{2}\left(\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{n=1}^{\infty}\sum_{l=0}^{n}\frac{(-1)^{k+l+1}}{16(n+j+2)!}\binom{n}{l}\binom{j}{k}s^{n+j+1}\hat{\Delta}^{j-k}\left(\left((h\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu)\hat{\Delta}^{n-l}\right.\right.\right. \\
&\left.\left.\left.\left(h\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu\right)\hat{\Delta}^l-\hat{\Delta}^{n-l}(h\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu)\hat{\Delta}^l(h\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu)\right)\hat{\Delta}^k\right)\right)\Bigg\}\cdot R
\end{aligned} \tag{B.11}$$

where we keep terms up to order $\mathcal{O}(h^1)$ in the commutator and square.

For the Weyl-term we get

$$
\begin{aligned}
&\int \sqrt{g}\, C^{\mu\nu\rho\sigma} W_k(\Delta) C_{\mu\nu\rho\sigma} = \int \sqrt{g}\, C^{\mu\nu\rho\sigma} \int_0^\infty ds \widetilde{W}_k(s) \mathrm{e}^{-s\left(\hat{\Delta}\mathbb{1}_C + \mathbb{T}_1 + \mathbb{T}_2\right)_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta}} C_{\alpha\beta\gamma\delta} \\
&= \int \sqrt{g}\, C^{\mu\nu\rho\sigma} \int_0^\infty ds \widetilde{W}_k(s) \mathrm{e}^{-s\hat{\Delta}} \left\{ \mathbb{1}_C + \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)!} \left[-s\hat{\Delta}, -s\left(\mathbb{T}_1 + \mathbb{T}_2\right)\right]_j \right. \\
&+ \frac{1}{2} \left(\sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)!} \left[-s\hat{\Delta}, -s\left(\mathbb{T}_1 + \mathbb{T}_2\right)\right]_j \right)^2 \\
&\left. + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{j+n+1}}{(n+j+2)!} \left[-s\hat{\Delta}, \left[-s\left(\mathbb{T}_1 + \mathbb{T}_2\right), \left[-s\hat{\Delta}, -s\left(\mathbb{T}_1 + \mathbb{T}_2\right)\right]_n\right]\right]_j \right\}_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta},
\end{aligned}
\tag{B.12}
$$

where we have already set $\epsilon \to 1$ and $\mathbb{T}_i$ now denotes the expansion of the Laplacian working on a rank-four symmetric tensor with i powers of h.

C Ricci-variation and -traces

C.1 Variations

From Appendix B.2 we have the expansion of $F_k(\Delta)$ in powers of h

$$\begin{aligned}
\int \sqrt{g} R F_k(\Delta) R = \int \sqrt{g} R \int_0^\infty ds \widetilde{F}_k(s) \mathrm{e}^{-s\hat{\Delta}} \ \{1 \\
+ \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(-1)^{j+k+j}}{4(j+1)!} \binom{j}{k} s^{j+1} \hat{\Delta}^{j-k} \left(\left(h\hat{\Delta} + (\hat{D}_\mu h)\hat{D}^\mu - \frac{1}{4} h^2 \hat{\Delta} - \frac{1}{2} h (\hat{D}_\mu h) \hat{D}^\mu \right) \hat{\Delta}^k \right) \\
+ \frac{1}{2} \left(\sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(-1)^{k}}{4(j+1)!} \binom{j}{k} s^{j+1} \hat{\Delta}^{j-k} \left(\left(h\hat{\Delta} + (\hat{D}_\mu h)\hat{D}^\mu + \mathcal{O}(h^2) \right) \hat{\Delta}^k \right) \right)^2 \\
+ \frac{1}{2} \left(\sum_{j=0}^{\infty} \sum_{k=0}^{j} \sum_{n=1}^{\infty} \sum_{l=0}^{n} \frac{(-1)^{k+l+1}}{16(n+j+2)!} \binom{n}{l} \binom{j}{k} s^{n+j+1} \hat{\Delta}^{j-k} \left(\left((h\hat{\Delta} + (\hat{D}_\mu h)\hat{D}^\mu) \hat{\Delta}^{n-l} \right.\right.\right. \\
\left.\left.\left. \left(h\hat{\Delta} + (\hat{D}_\mu h)\hat{D}^\mu \right) \hat{\Delta}^l - \hat{\Delta}^{n-l} (h\hat{\Delta} + (\hat{D}_\mu h)\hat{D}^\mu) \hat{\Delta}^l (h\hat{\Delta} + (\hat{D}_\mu h)\hat{D}^\mu) \right) \hat{\Delta}^k \right) \right) \Big\} R.
\end{aligned} \tag{C.1}$$

Now expanding $\sqrt{g}R$ and R, we find there is one term in the R-R expansion independent of h:

$$\hat{R}\, F_k(\Delta)\, \hat{R} \tag{C.2}$$

The terms in the R-R expansion for linear order in h:

$$\begin{array}{ll}
\hat{R}\, F_k(\Delta)\, \frac{-1}{4} \hat{R} h & \hat{R}\, F_k(\Delta)\, \frac{-3}{4} (\hat{D}^2 h) \\
\frac{1}{4} \hat{R} h\, F_k(\Delta)\, \hat{R} & \frac{-3}{4} (\hat{D}^2 h)\, F_k(\Delta)\, \hat{R}
\end{array} \tag{C.3}$$

and for quadratic order in h:

$$\begin{array}{llll}
\hat{R}\, F_k(\Delta)\, \frac{1}{16} \hat{R} h^2 & \hat{R}\, F_k(\Delta)\, \frac{3}{8} h \hat{D}^2 h & \hat{R}\, F_k(\Delta)\, \frac{3}{32} (\hat{D}_\mu h)(\hat{D}^\mu h) & \frac{-3}{4} \hat{D}^2 h\, F_k(\Delta)\, \frac{-3}{4} \hat{D}^2 h, \\
\frac{1}{4} \hat{R} h\, F_k(\Delta)\, \frac{-3}{4} \hat{D}^2 h & \frac{1}{4} \hat{R} h\, F_k(\Delta)\, \frac{-1}{4} \hat{R} h & \frac{-3}{4} \hat{D}^2 h\, F_k(\Delta)\, \frac{-1}{4} \hat{R} h & \frac{3}{32} (\hat{D}_\mu h)(\hat{D}^\mu h)\, F_k(\Delta)\, \hat{R}.
\end{array} \tag{C.4}$$

In the next subsections we will investigate these three sets of terms one by one.

C.1.1 Terms quadratic in h

The action for all terms quadratic in h admits the form

$$\Gamma_{h^2} = \int \sqrt{\hat{g}} \hat{X} \int_0^\infty ds\, \widetilde{F}_k(s)\, \mathrm{e}^{-s\hat{\Delta}}\ \{1 + \mathcal{O}(h)\}\ \hat{Y} = \int \sqrt{\hat{g}}\, \hat{X}\, F_k(\hat{\Delta})\, \hat{Y}, \tag{C.5}$$

giving

$$\begin{aligned}
\Gamma_{h^2} = \int \sqrt{\hat{g}} \Big(& \frac{1}{16} \hat{R}\, F_k(\Delta)\, \hat{R} h^2 + \frac{3}{8} \hat{R}\, F_k(\hat{\Delta})\, h \hat{D}^2 h + \frac{3}{32} \hat{R}\, F_k(\hat{\Delta})\, (\hat{D}_\mu h)(\hat{D}^\mu h) \\
& + \frac{9}{16} \hat{D}^2 h\, F_k(\hat{\Delta})\, \hat{D}^2 h - \frac{3}{16} \hat{R} h\, F_k(\hat{\Delta})\, \hat{D}^2 h - \frac{1}{16} \hat{R} h\, F_k(\hat{\Delta})\, \hat{R} h \\
& + \frac{3}{16} \hat{D}^2 h\, F_k(\hat{\Delta})\, \hat{R} h + \frac{3}{32} (\hat{D}_\mu h)(\hat{D}^\mu h)\, F_k(\hat{\Delta})\, \hat{R} \Big).
\end{aligned} \tag{C.6}$$

Let us for now neglect the numerical factors. For the first term the first and second variation are

$$\Gamma^{(1)} = 2(F_k(\hat{\Delta})\hat{R})\,\hat{R}h \tag{C.7}$$

$$\Gamma^{(2)} = 2(F_k(\hat{\Delta})\hat{R})\,\hat{R} = 2\hat{R}^2F_k(q^2), \tag{C.8}$$

as this terms is at order $\hat{R}^2$, giving us the possibility to use the momentum representation where $\hat{D}_\mu F_k(p) = ip_\mu f(p)$ and thus $\hat{\Delta}f(p) = p^2f(p)$. For the second term we find

$$\Gamma^{(1)} = (F_k(\hat{\Delta})\hat{R})\,\hat{D}^2h + \hat{D}^2(h(F_k(\hat{\Delta})\hat{R})) \tag{C.9}$$

$$\Gamma^{(2)} = \hat{D}^2(F_k(\hat{\Delta})\hat{R}) + (F_k(\hat{\Delta})\hat{R})\hat{D}^2 \tag{C.10}$$

$$= (\hat{D}^2F_k(\hat{\Delta})\hat{R}) + 2(\hat{D}_\mu F_k(\hat{\Delta})\hat{R})\hat{D}^\mu + 2(F_k(\hat{\Delta})\hat{R})\hat{D}^2 \tag{C.11}$$

For the third term the variations are

$$\Gamma^{(1)} = -2\hat{D}_\mu((\hat{D}^\mu h)F_k(\hat{\Delta})\hat{R}) \tag{C.12}$$

$$\Gamma^{(2)} = -2(\hat{D}_\mu F_k(\Delta)\hat{R})\hat{D}^\mu - 2(F_k(\Delta)\hat{R})\hat{D}^2, \tag{C.13}$$

which will turn out to be the same as the second variation resulting from the eighth term. For the fourth term we find

$$\Gamma^{(1)} = 2\hat{D}^2F_k(\hat{\Delta}\hat{D}^2h) \tag{C.14}$$

$$\Gamma^{(2)} = 2F_k(\hat{\Delta})\hat{D}^4. \tag{C.15}$$

The variations for the fifth term are

$$\Gamma^{(1)} = \hat{D}^2(F_k(\hat{\Delta})\hat{R}h) + \hat{R}(\hat{D}^2F_k(\hat{\Delta})h) \tag{C.16}$$

$$\Gamma^{(2)} = \hat{R}F_k(\hat{\Delta})\hat{D}^2 + F_k(\hat{\Delta})\hat{D}^2\hat{R}. \tag{C.17}$$

where in the second part $F_k(\hat{\Delta})\hat{D}^2$ works on everything to the right (i.e. the $\hat{R}$ and any X thereafter). For the sixth term we find

$$\Gamma^{(1)} = 2(F_k(\hat{\Delta})\hat{R}h)\hat{R} \tag{C.18}$$

$$\Gamma^{(2)} = 2\hat{R}F_k(\hat{\Delta})\hat{R} = 2\hat{R}^2F_k((p+q)^2) \tag{C.19}$$

For the seventh term, the variations are

$$\Gamma^{(1)} = (F_k(\hat{\Delta})\hat{D}^2h)\hat{R} + F_k(\hat{\Delta})\hat{D}^2(\hat{R}h) \tag{C.20}$$

$$\Gamma^{(2)} = F_k(\hat{\Delta})\hat{D}^2\hat{R} + \hat{R}F_k(\hat{\Delta})\hat{D}^2. \tag{C.21}$$

where now in the first part $F_k(\hat{\Delta})\hat{D}^2$ works on everything to its right. This second variation is the same as the fourth. As their prefactors are opposite, these contributions will cancel. For the last term we have

$$\Gamma^{(1)} = -2\hat{D}_\mu((\hat{D}^\mu h)F_k(\hat{\Delta})\hat{R}) \tag{C.22}$$

$$\Gamma^{(2)} = -2(\hat{D}_\mu F_k(\Delta)\hat{R})\hat{D}^\mu - 2(F_k(\Delta)\hat{R})\hat{D}^2. \tag{C.23}$$

So, putting it together we have

$$\begin{aligned}
\Gamma^{(2)}_{h^2} &= \frac{1}{8}\hat{R}^2 F_k(q^2) + \frac{3}{8}\left((\hat{D}^2 F_k(\hat{\Delta})\hat{R}) + 2(\hat{D}_\mu F_k(\hat{\Delta})\hat{R})\hat{D}^\mu + 2(F_k(\hat{\Delta})\hat{R})\hat{D}^2\right) \\
&- \frac{3}{16}\left(\hat{R}F_k(\hat{\Delta})\hat{D}^2 + F_k(\hat{\Delta})\hat{D}^2\hat{R}\right) + \frac{3}{16}\left(F_k(\hat{\Delta})\hat{D}^2\hat{R} + \hat{R}F_k(\hat{\Delta})\hat{D}^2\right) \\
&- \frac{1}{8}\hat{R}^2 F_k((p+q)^2) + \frac{9}{16}\left(2F_k(\hat{\Delta})\hat{D}^4\right) - \frac{3}{8}((\hat{D}_\mu F_k(\Delta)\hat{R})\hat{D}^\mu + (F_k(\Delta)\hat{R})\hat{D}^2) \\
&= \frac{1}{8}\hat{R}^2\left(F_k(q^2) - F_k((p+q)^2)\right) - \frac{3}{8}(\hat{\Delta}F_k(\hat{\Delta})\hat{R}) \\
&- \frac{3}{8}(\hat{D}_\mu F_k(\hat{\Delta})\hat{R})\hat{\Delta} + \frac{3}{8}(F_k(\hat{\Delta})\hat{R})\hat{D}^2 + \frac{9}{8}F_k(\hat{\Delta})\hat{\Delta}^2
\end{aligned} \tag{C.24}$$

C.1.2 Terms linear in h

For the terms linear in h, the action becomes

$$\begin{aligned}
\Gamma_{h^1} = &-\frac{1}{4}\int \sqrt{\hat{g}}\,\hat{R}\int_0^\infty ds\,\widetilde{F}_k(s)\,\mathrm{e}^{-s\hat{\Delta}}\Bigg\{\hat{R}h \\
&+\sum_{j=0}^\infty\sum_{k=0}^j \frac{(-1)^k}{4(j+1)!}\binom{j}{k}s^{j+1}\,\hat{\Delta}^{j-k}\left((h\,\hat{\Delta} + (\hat{D}_\mu h)\hat{D}^\mu)\hat{\Delta}^k\,\hat{R}h\right)\Bigg\} \\
&+\frac{1}{4}\int \sqrt{\hat{g}}\,\hat{R}h\int_0^\infty ds\,\widetilde{F}_k(s)\,\mathrm{e}^{-s\hat{\Delta}}\Bigg\{\hat{R} \\
&+\sum_{j=0}^\infty\sum_{k=0}^j \frac{(-1)^k}{4(j+1)!}\binom{j}{k}s^{j+1}\,\hat{\Delta}^{j-k}\left((h\,\hat{\Delta} + (\hat{D}_\mu h)\hat{D}^\mu)\hat{\Delta}^k\,\hat{R}\right)\Bigg\} \\
&-\frac{3}{4}\int \sqrt{\hat{g}}\,(\hat{D}^2 h)\int_0^\infty ds\,\widetilde{F}_k(s)\,\mathrm{e}^{-s\hat{\Delta}}\Bigg\{\,\hat{R} \\
&+\sum_{j=0}^\infty\sum_{k=0}^j \frac{(-1)^k}{4(j+1)!}\binom{j}{k}s^{j+1}\,\hat{\Delta}^{j-k}\left((h\,\hat{\Delta} + (\hat{D}_\mu h)\hat{D}^\mu)\hat{\Delta}^k\,\hat{R}\right)\Bigg\} \\
&-\frac{3}{4}\int \sqrt{\hat{g}}\,\hat{R}\int_0^\infty ds\,\widetilde{F}_k(s)\,\mathrm{e}^{-s\hat{\Delta}}\Bigg\{\hat{D}^2 h \\
&+\sum_{j=0}^\infty\sum_{k=0}^j \frac{(-1)^k}{4(j+1)!}\binom{j}{k}s^{j+1}\,\hat{\Delta}^{j-k}\left((h\,\hat{\Delta} + (\hat{D}_\mu h)\hat{D}^\mu)\hat{\Delta}^k\,\hat{D}^2 h\right)\Bigg\}.
\end{aligned} \tag{C.25}$$

As we are interested in the second variation, we only look at terms quadratic in h

$$\Gamma'_{h^1} = -\frac{1}{4}\int \sqrt{\hat{g}}\,\hat{R}\int_0^\infty ds\,\widetilde{F}_k(s)\,\mathrm{e}^{-s\hat{\Delta}}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^k}{4(j+1)!}\binom{j}{k}s^{j+1}\,\hat{\Delta}^{j-k}\Big((h\,\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu)\hat{\Delta}^k\,\hat{R}h\Big)$$

$$+\frac{1}{4}\int \sqrt{\hat{g}}\,\hat{R}h\int_0^\infty ds\,\widetilde{F}_k(s)\,\mathrm{e}^{-s\hat{\Delta}}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^k}{4(j+1)!}\binom{j}{k}s^{j+1}\,\hat{\Delta}^{j-k}\Big((h\,\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu)\hat{\Delta}^k\,\hat{R}\Big)$$

$$-\frac{3}{4}\int \sqrt{\hat{g}}\,(\hat{D}^2 h)\int_0^\infty ds\,\widetilde{F}_k(s)\,\mathrm{e}^{-s\hat{\Delta}}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^k}{4(j+1)!}\binom{j}{k}s^{j+1}\,\hat{\Delta}^{j-k}\Big((h\,\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu)\hat{\Delta}^k\,\hat{R}\Big)$$

$$-\frac{3}{4}\int \sqrt{\hat{g}}\,\hat{R}\int_0^\infty ds\,\widetilde{F}_k(s)\,\mathrm{e}^{-s\hat{\Delta}}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^k}{4(j+1)!}\binom{j}{k}s^{j+1}\,\hat{\Delta}^{j-k}\Big((h\,\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu)\hat{\Delta}^k\,\hat{D}^2 h\Big)\,.$$

As the first two terms are at $\hat{R}^2$-level, we can go to the momentum representation and write

$$\begin{aligned}
\Gamma_{R-Rh} &= -\frac{1}{16}\int\sqrt{\hat{g}}\,\hat{R}(-q)\int_0^\infty ds\,\widetilde{F}_k(s)\,\mathrm{e}^{-s\hat{\Delta}}\\
&\quad\sum_{j=0}^{\infty}\frac{s^{j+1}}{(j+1)!}\Big[\hat{\Delta}, h(-p)\hat{\Delta}+(\hat{D}_\mu h(-p))\hat{D}^\mu\Big)\Big]_j\,\hat{R}(q)h(p)\\
&= -\frac{1}{16}\int\sqrt{\hat{g}}\,\hat{R}(-q)\int_0^\infty ds\,\widetilde{F}_k(s)\,\mathrm{e}^{-sq^2}\\
&\quad\sum_{j=0}^{\infty}\frac{s^{j+1}}{(j+1)!}(-p^2-2pqx)^j\left((p^2+2pqx+q^2)+p_\mu(p+q)^\mu\right)\hat{R}(q)h(-p)h(p)\\
&= \frac{1}{16}\int\sqrt{\hat{g}}\,\hat{R}^2\int_0^\infty ds\,\widetilde{F}_k(s)\,\mathrm{e}^{-sq^2}\frac{\mathrm{e}^{-s(p^2+2pqx)}-1}{p^2+2pqx}\left(2p^2+3pqx+q^2\right)h^2,\\
\Gamma_{Rh-R} &= \frac{1}{16}\int\sqrt{\hat{g}}\,\hat{R}(-q)h(-p)\int_0^\infty ds\,\widetilde{F}_k(s)\,\mathrm{e}^{-s\hat{\Delta}}\\
&\quad\sum_{j=0}^{\infty}\frac{s^{j+1}}{(j+1)!}\Big[\hat{\Delta}, h(p)\hat{\Delta}+(\hat{D}_\mu h(p))\hat{D}^\mu\Big]_j\,\hat{R}(q)\\
&= \frac{1}{16}\int\sqrt{\hat{g}}\,\hat{R}(-q)h(-p)\int_0^\infty ds\,\widetilde{F}_k(s)\,\mathrm{e}^{-s(p^2+2pqx+q^2)}\\
&\quad\sum_{j=0}^{\infty}\frac{s^{j+1}}{(j+1)!}(p^2+2pqx)^j\left(q^2-pqx\right)h(p)\hat{R}(q)\\
&= \frac{1}{16}\int\sqrt{\hat{g}}\,\hat{R}^2h^2\int_0^\infty ds\,\widetilde{F}_k(s)\,\mathrm{e}^{-s(p^2+2pqx+q^2)}\frac{\mathrm{e}^{s(p^2+2pqx)}-1}{p^2+2pqx}\left(q^2-pqx\right),
\end{aligned} \tag{C.26}$$

where $(p+q)^2 = (p_\mu+q_\mu)^2 = p^2+q^2+2pqx$ where $x=\cos(\theta)$ and θ is the angle between p_μ and q_μ.

The second variations are then found to be

$$\begin{aligned}
\Gamma^{(2)}_{R-Rh} &= \frac{1}{8}\hat{R}^2\int_0^\infty ds\,\widetilde{F}_k(s)\,\frac{\mathrm{e}^{-s(p^2+2pqx+q^2)}-\mathrm{e}^{-sq^2}}{p^2+2pqx}\left(2p^2+3pqx+q^2\right)\\
&= \frac{1}{8}\hat{R}^2\frac{F_k(p^2+2pqx+q^2)-F_k(q^2)}{p^2+2pqx}\left(2p^2+3pqx+q^2\right),\\
\Gamma^{(2)}_{Rh-R} &= \frac{1}{8}\hat{R}^2\int_0^\infty ds\,\widetilde{F}_k(s)\,\frac{\mathrm{e}^{-sq^2}-\mathrm{e}^{-s(p^2+2pqx+q^2)}}{p^2+2pqx}\left(q^2-pqx\right)\\
&= \frac{1}{8}\hat{R}^2\frac{F_k(q^2)-F_k(p^2+2pqx+q^2)}{p^2+2pqx}\left(q^2-pqx\right).
\end{aligned} \tag{C.27}$$

Combining these terms gives $\frac{1}{4}\hat{R}^2\left(F_k(p^2+2pqx+q^2)-F_k(q^2)\right)$. The other two terms can be rewritten

$$
\begin{aligned}
\Gamma_{\hat{D}^2h-\hat{R}} &= \frac{3}{16}\int\sqrt{\hat{g}}\int_0^\infty ds\, \widetilde{F}_k(s)\, \mathrm{e}^{-s\hat{\Delta}}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^k}{(j+1)!}\binom{j}{k}s^{j+1}\,\hat{\Delta}^{j-k+1}h \\
&\left((h\,\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu)\hat{\Delta}^k\,\hat{R}\right) \\
&= \frac{3}{16}\int dy\int dz\int\sqrt{\hat{g}}\int_0^\infty ds\,\widetilde{F}_k(s)\,\mathrm{e}^{-s\hat{\Delta}_x}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^k}{(j+1)!}\binom{j}{k}s^{j+1}\,\hat{\Delta}_x^{j-k+1}h_x \\
&\left((h_y\,\hat{\Delta}_z^{k+1}\hat{R}_z+(\hat{D}_{\mu,y}h_y)\hat{D}_z^\mu\hat{\Delta}_z^k\,\hat{R}_z\right)\delta(x-y)\delta(x-z) \\
&= \frac{3}{16}\int dy\int dz\int\sqrt{\hat{g}}\,\int_0^\infty ds\,\widetilde{F}_k(s)\,\mathrm{e}^{-s\hat{\Delta}_x}(h_y\,\hat{\Delta}_z+(\hat{D}_{\mu,y}h_y)\hat{D}_z^\mu) \\
&\frac{\mathrm{e}^{s(\hat{\Delta}_x-\hat{\Delta}_z)}-1}{\hat{\Delta}_x-\hat{\Delta}_z}\hat{\Delta}_x h_x\hat{R}_z\delta(x-y)\delta(x-z) \\
&= \frac{3}{16}\int dy\int dz\int\sqrt{\hat{g}}\,\int_0^\infty ds\,\widetilde{F}_k(s)\,(h_y\,\hat{\Delta}_z+(\hat{D}_{\mu,y}h_y)\hat{D}_z^\mu) \\
&\left(\mathrm{e}^{-s\hat{\Delta}_z}-\mathrm{e}^{-s\hat{\Delta}_x}\right)\sum_{n=0}^{\infty}\hat{\Delta}_z^n\hat{\Delta}_x^{-n}h_x\hat{R}_z\delta(x-y)\delta(x-z) \\
&= \frac{3}{16}\int\sqrt{\hat{g}}\,\sum_{n=0}^{\infty}\left((h\,\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu)F_k(\Delta)\hat{\Delta}^n\hat{R}\right)\left(\hat{\Delta}^{-n}h\right) \\
&-\frac{3}{16}\int\sqrt{\hat{g}}\,\sum_{n=0}^{\infty}\left((h\,\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu)\hat{\Delta}^n\hat{R}\right)\left(F_k(\Delta)\hat{\Delta}^{-n}h\right)
\end{aligned}
\tag{C.28}
$$

and

$$
\begin{aligned}
\Gamma_{\hat{R}-\hat{D}^2h} &= \frac{3}{16}\int\sqrt{\hat{g}}\int_0^\infty ds\,\widetilde{F}_k(s)\,\mathrm{e}^{-s\hat{\Delta}}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^k}{(j+1)!}\binom{j}{k}s^{j+1}\,\hat{\Delta}^{j-k}\hat{R} \\
&\left((h\,\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu)\hat{\Delta}^{k+1}h\right) \\
&= \frac{3}{16}\int dy\int dz\int\sqrt{\hat{g}}\int_0^\infty ds\,\widetilde{F}_k(s)\,\mathrm{e}^{-s\hat{\Delta}_x}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^k}{(j+1)!}\binom{j}{k}s^{j+1}\,\hat{\Delta}_x^{j-k}\hat{R}_x \\
&\left(h_y\,\hat{\Delta}_z^{k+2}h_z+(\hat{D}_{\mu,y}h_y)\hat{D}_z^\mu\hat{\Delta}_z^{k+1}h_z\right)\delta(x-y)\delta(x-z) \\
&= \frac{3}{16}\int dy\int dz\int\sqrt{\hat{g}}\,\int_0^\infty ds\,\widetilde{F}_k(s)\,\mathrm{e}^{-s\hat{\Delta}_x}(h_y\,\hat{\Delta}_z+(\hat{D}_{\mu,y}h_y)\hat{D}_z^\mu) \\
&\frac{\mathrm{e}^{s(\hat{\Delta}_x-\hat{\Delta}_z)}-1}{\hat{\Delta}_x-\hat{\Delta}_z}\hat{\Delta}_z\hat{R}_x h_z\delta(x-y)\delta(x-z) \\
&= -\frac{3}{16}\int dy\int dz\int\sqrt{\hat{g}}\,\int_0^\infty ds\,\widetilde{F}_k(s)\,(h_y\,\hat{\Delta}_z+(\hat{D}_{\mu,y}h_y)\hat{D}_z^\mu) \\
&\left(\mathrm{e}^{-s\hat{\Delta}_z}-\mathrm{e}^{-s\hat{\Delta}_x}\right)\sum_{n=0}^{\infty}\hat{\Delta}_x^n\hat{\Delta}_z^{-n}\hat{R}_x h_z\delta(x-y)\delta(x-z) \\
&= \frac{3}{16}\int\sqrt{\hat{g}}\,\sum_{n=0}^{\infty}\left((h\,\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu)\hat{\Delta}^{-n}h\right)\left(F_k(\Delta)\hat{\Delta}^n\hat{R}\right) \\
&-\frac{3}{16}\int\sqrt{\hat{g}}\,\sum_{n=0}^{\infty}\left((h\,\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu)F_k(\Delta)\hat{\Delta}^{-n}h\right)\left(\hat{\Delta}^n\hat{R}\right),
\end{aligned}
\tag{C.29}
$$

where the subscript of a Laplacian denotes on which expression it works. We find the first variations

$$
\begin{aligned}
\Gamma^{(1)}_{\hat{D}^2h-\hat{R}} &= \frac{3}{16}\sum_{n=0}^{\infty}\hat{\Delta}^{-n}\left((h\hat{\Delta}+(\hat{D}_\mu h)\hat{D}^\mu)F_k(\Delta)\hat{\Delta}^n\hat{R}\right)+\left(F_k(\Delta)\hat{\Delta}^{n+1}\hat{R}\right)\left(\hat{\Delta}^{-n}h\right)\\
&-\hat{D}_\mu\left(\left(\hat{D}^\mu F_k(\Delta)\hat{\Delta}^n\hat{R}\right)\left(\hat{\Delta}^{-n}h\right)\right)\\
&+\frac{3}{16}\sum_{n=0}^{\infty}-F_k(\Delta)\hat{\Delta}^{-n}\left((h\hat{\Delta}-(\hat{D}_\mu h)\hat{D}^\mu)\hat{\Delta}^n\hat{R}\right)-\left(\hat{\Delta}^{n+1}\hat{R}\right)\left(F_k(\Delta)\hat{\Delta}^{-n}h\right)\\
&+\hat{D}_\mu\left(\left(\hat{D}^\mu\hat{\Delta}^n\hat{R}\right)\left(F_k(\Delta)\hat{\Delta}^{-n}h\right)\right)\\
\Gamma^{(1)}_{\hat{R}-\hat{D}^2h} &= \frac{3}{16}\sum_{n=0}^{\infty}\left(\hat{\Delta}^{-n+1}h\right)\left(F_k(\Delta)\hat{\Delta}^n\hat{R}\right)-\hat{D}_\mu\left(\hat{D}^\mu\hat{\Delta}^{-n}h\right)\left(F_k(\Delta)\hat{\Delta}^n\hat{R}\right)\\
&+\frac{3}{16}\sum_{n=0}^{\infty}\hat{\Delta}^{-n+1}\left(hF_k(\Delta)\hat{\Delta}^n\hat{R}\right)-\hat{D}^\mu\hat{\Delta}^{-n}\left((\hat{D}_\mu h)\left(F_k(\Delta)\hat{\Delta}^n\hat{R}\right)\right)\\
&+\frac{3}{16}\sum_{n=0}^{\infty}-\left(F_k(\Delta)\hat{\Delta}^{-n+1}h\right)\left(\hat{\Delta}^n\hat{R}\right)+\hat{D}_\mu\left(\hat{D}^\mu F_k(\Delta)\hat{\Delta}^{-n}h\left(\hat{\Delta}^n\hat{R}\right)\right)\\
&+\frac{3}{16}\sum_{n=0}^{\infty}-F_k(\Delta)\hat{\Delta}^{-n+1}\left(h\hat{\Delta}^n\hat{R}\right)+\hat{D}^\mu F_k(\Delta)\hat{\Delta}^{-n}\left((\hat{D}_\mu h)\left(\hat{\Delta}^n\hat{R}\right)\right).
\end{aligned}
\tag{C.30}
$$

and the second variations

$$
\begin{aligned}
\Gamma^{(2)}_{\hat{D}^2h-\hat{R}} &= \frac{3}{16}\sum_{n=0}^{\infty}\left(F_k(\Delta)\hat{\Delta}^{n+1}\hat{R}\right)\hat{\Delta}^{-n}-\hat{D}_\mu\left(\hat{D}^\mu F_k(\Delta)\hat{\Delta}^n\hat{R}\hat{\Delta}^{-n}\right)\\
&+\frac{3}{16}\sum_{n=0}^{\infty}\hat{\Delta}^{-n}\left(F_k(\Delta)\hat{\Delta}^{n+1}\hat{R}\right)+\hat{\Delta}^{-n}\left(\hat{D}^\mu F_k(\Delta)\hat{\Delta}^n\hat{R}\hat{D}_\mu\right)\\
&+\frac{3}{16}\sum_{n=0}^{\infty}-\hat{\Delta}^{n+1}\hat{R}F_k(\Delta)\hat{\Delta}^{-n}+\hat{D}_\mu\left(\hat{D}^\mu\hat{\Delta}^n\hat{R}F_k(\Delta)\hat{\Delta}^{-n}\right)\\
&+\frac{3}{16}\sum_{n=0}^{\infty}-F_k(\Delta)\hat{\Delta}^{-n}\left(\hat{\Delta}^{n+1}\hat{R}\right)-F_k(\Delta)\hat{\Delta}^{-n}\left(\hat{D}^\mu\hat{\Delta}^n\hat{R}\hat{D}_\mu\right)\\
\Gamma^{(2)}_{\hat{R}-\hat{D}^2h} &= \frac{3}{16}\sum_{n=0}^{\infty}\hat{\Delta}^{-n+1}\left(F_k(\Delta)\hat{\Delta}^n\hat{R}\right)-\hat{D}^\mu\hat{\Delta}^{-n}\left(F_k(\Delta)\hat{\Delta}^n\hat{R}\hat{D}_\mu\right)\\
&+\frac{3}{16}\sum_{n=0}^{\infty}\left(F_k(\Delta)\hat{\Delta}^n\hat{R}\right)\hat{\Delta}^{-n+1}-\hat{D}_\mu\left(F_k(\Delta)\hat{\Delta}^n\hat{R}\hat{D}^\mu\hat{\Delta}^{-n}\right)\\
&+\frac{3}{16}\sum_{n=0}^{\infty}-F_k(\Delta)\hat{\Delta}^{-n+1}\left(\hat{\Delta}^n\hat{R}\right)+\hat{D}^\mu F_k(\Delta)\hat{\Delta}^{-n}\left(\hat{\Delta}^n\hat{R}\hat{D}_\mu\right)\\
&+\frac{3}{16}\sum_{n=0}^{\infty}-\left(\hat{\Delta}^n\hat{R}\right)F_k(\Delta)\hat{\Delta}^{-n+1}+\hat{D}_\mu\left(\hat{\Delta}^n\hat{R}\hat{D}^\mu F_k(\Delta)\hat{\Delta}^{-n}\right).
\end{aligned}
\tag{C.31}
$$

Combining these terms and working out the covariant derivatives gives

$$
\begin{aligned}
\Gamma^{(2)}_{\hat{D}^2h-\hat{R},\hat{R}-\hat{D}^2h} = \frac{3}{16}\sum_{n=0}^{\infty}\Big\{&\left(F_k(\Delta)\hat{\Delta}^{n+1}\hat{R}\right)\hat{\Delta}^{-n} + F_k(\Delta)\hat{\Delta}^{n+1}\hat{R}\hat{\Delta}^{-n} \\
&- \hat{D}^\mu F_k(\Delta)\hat{\Delta}^n\hat{R}\hat{D}_\mu\hat{\Delta}^{-n} + \hat{\Delta}^{-n}\left(F_k(\Delta)\hat{\Delta}^{n+1}\hat{R}\right) \\
&+ \hat{\Delta}^{-n}\left(\hat{D}^\mu F_k(\Delta)\hat{\Delta}^n\hat{R}\hat{D}_\mu\right) - \hat{\Delta}^{n+1}\hat{R}F_k(\Delta)\hat{\Delta}^{-n} - \hat{\Delta}^{n+1}\hat{R}F_k(\Delta)\hat{\Delta}^{-n} \\
&+ \hat{D}^\mu\hat{\Delta}^n\hat{R}\hat{D}_\mu F_k(\Delta)\hat{\Delta}^{-n} - F_k(\Delta)\hat{\Delta}^{-n}\left(\hat{\Delta}^{n+1}\hat{R}\right) \\
&- F_k(\Delta)\hat{\Delta}^{-n}\left(\hat{D}^\mu\hat{\Delta}^n\hat{R}\hat{D}_\mu\right) + \hat{\Delta}^{-n+1}\left(F_k(\Delta)\hat{\Delta}^n\hat{R}\right) \\
&- \hat{\Delta}^{-n}\left(\hat{D}^\mu F_k(\Delta)\hat{\Delta}^n\hat{R}\hat{D}_\mu\right) + \hat{\Delta}^{-n}\left(F_k(\Delta)\hat{\Delta}^n\hat{R}\hat{\Delta}\right) \\
&+ \left(F_k(\Delta)\hat{\Delta}^n\hat{R}\right)\hat{\Delta}^{-n+1} - \hat{D}_\mu F_k(\Delta)\hat{\Delta}^n\hat{R}\hat{D}^\mu\hat{\Delta}^{-n} + \left(F_k(\Delta)\hat{\Delta}^n\hat{R}\hat{\Delta}^{-n+1}\right) \\
&- F_k(\Delta)\hat{\Delta}^{-n+1}\left(\hat{\Delta}^n\hat{R}\right) + F_k(\Delta)\hat{\Delta}^{-n}\left(\hat{D}^\mu\hat{\Delta}^n\hat{R}\hat{D}_\mu\right) - F_k(\Delta)\hat{\Delta}^{-n}\left(\hat{\Delta}^n\hat{R}\hat{\Delta}\right) \\
&- \left(\hat{\Delta}^n\hat{R}\right)F_k(\Delta)\hat{\Delta}^{-n+1} + \hat{D}_\mu\hat{\Delta}^n\hat{R}\hat{D}^\mu F_k(\Delta)\hat{\Delta}^{-n} - \left(\hat{\Delta}^n\hat{R}F_k(\Delta)\hat{\Delta}^{-n+1}\right)\Big\}.
\end{aligned} \tag{C.32}
$$

We can now use the relations

$$
\begin{aligned}
\sum_{n=0}^{\infty}\hat{\Delta}^{-n}\left(F_k(\Delta)\hat{\Delta}^n\hat{R}\hat{\Delta}\right) = \sum_{n=0}^{\infty}\Big\{&\hat{\Delta}^{-n+1}\left(F_k(\Delta)\hat{\Delta}^n\hat{R}\right) \\
&-\hat{\Delta}^{-n}\left(F_k(\Delta)\hat{\Delta}^{n+1}\hat{R}\right) + 2\hat{\Delta}^{-n}\left(\hat{D}_\mu F_k(\Delta)\hat{\Delta}^n\hat{R}\hat{D}^\mu\right)\Big\},
\end{aligned} \tag{C.33}
$$

and

$$
\begin{aligned}
\sum_{n=0}^{\infty}-F_k(\Delta)\hat{\Delta}^{-n}\left(\hat{\Delta}^n\hat{R}\hat{\Delta}\right) = \sum_{n=0}^{\infty}\Big\{&-F_k(\Delta)\hat{\Delta}^{-n+1}\left(\hat{\Delta}^n\hat{R}\right) \\
&+F_k(\Delta)\hat{\Delta}^{-n}\left(\hat{\Delta}^{n+1}\hat{R}\right) - 2F_k(\Delta)\hat{\Delta}^{-n}\left(\hat{D}_\mu\hat{\Delta}^n\hat{R}\hat{D}^\mu\right)\Big\},
\end{aligned} \tag{C.34}
$$

to get rid of uncontracted derivatives such that with collecting alike terms we find the second variation to be

$$
\begin{aligned}
\Gamma^{(2)}_{\hat{D}^2h-\hat{R},\hat{R}-\hat{D}^2h} &= \frac{3}{8}\sum_{n=0}^{\infty}\left(F_k(\Delta)\hat{\Delta}^{n+1}\hat{R}\right)\hat{\Delta}^{-n} + \left(F_k(\Delta)\hat{\Delta}^{n}\hat{R}\right)\hat{\Delta}^{-n+1} \\
&+ \frac{3}{8}\sum_{n=0}^{\infty} - \left(\hat{\Delta}^{n+1}\hat{R}\right)F_k(\Delta)\hat{\Delta}^{-n} - \left(\hat{\Delta}^{n}\hat{R}\right)F_k(\Delta)\hat{\Delta}^{-n+1} \\
&+ \frac{3}{8}\sum_{n=0}^{\infty}\hat{\Delta}^{-n+1}\left(F_k(\Delta)\hat{\Delta}^{n}\hat{R}\right) - F_k(\Delta)\hat{\Delta}^{-n+1}\left(\hat{\Delta}^{n}\hat{R}\right) \\
&+ \frac{3}{8}\sum_{n=0}^{\infty}\left(\hat{D}^{\mu}\hat{\Delta}^{n}\hat{R}\right)\hat{D}_{\mu}F_k(\Delta)\hat{\Delta}^{-n} - \left(\hat{D}^{\mu}F_k(\Delta)\hat{\Delta}^{n}\hat{R}\right)\hat{D}_{\mu}\hat{\Delta}^{-n} \\
&+ \frac{3}{8}\sum_{n=0}^{\infty}\hat{\Delta}^{-n}\left(\hat{D}_{\mu}F_k(\Delta)\hat{\Delta}^{n}\hat{R}\hat{D}^{\mu}\right) - F_k(\Delta)\hat{\Delta}^{-n}\left(\hat{D}_{\mu}\hat{\Delta}^{n}\hat{R}\hat{D}^{\mu}\right) \\
&= \frac{3}{8}\left(F_k(\Delta)\hat{R}\right)\hat{\Delta} + \frac{3}{8}\sum_{n=0}^{\infty}2\left(F_k(\Delta)\hat{\Delta}^{n+1}\hat{R}\right)\hat{\Delta}^{-n} \\
&- \frac{3}{8}\hat{R}F_k(\Delta)\hat{\Delta} - \frac{3}{8}\sum_{n=0}^{\infty}2\left(\hat{\Delta}^{n+1}\hat{R}\right)F_k(\Delta)\hat{\Delta}^{-n} \\
&+ \frac{3}{8}\sum_{n=0}^{\infty}\hat{\Delta}^{-n+1}\left(F_k(\Delta)\hat{\Delta}^{n}\hat{R}\right) - F_k(\Delta)\hat{\Delta}^{-n+1}\left(\hat{\Delta}^{n}\hat{R}\right) \\
&+ \frac{3}{8}\sum_{n=0}^{\infty}\left(\hat{D}^{\mu}\hat{\Delta}^{n}\hat{R}\right)\hat{D}_{\mu}F_k(\Delta)\hat{\Delta}^{-n} - \left(\hat{D}^{\mu}F_k(\Delta)\hat{\Delta}^{n}\hat{R}\right)\hat{D}_{\mu}\hat{\Delta}^{-n} \\
&+ \frac{3}{8}\sum_{n=0}^{\infty}\hat{\Delta}^{-n}\left(\hat{D}_{\mu}F_k(\Delta)\hat{\Delta}^{n}\hat{R}\hat{D}^{\mu}\right) - F_k(\Delta)\hat{\Delta}^{-n}\left(\hat{D}_{\mu}\hat{\Delta}^{n}\hat{R}\hat{D}^{\mu}\right)
\end{aligned}
\tag{C.35}
$$

C.1.3 Terms independent of h

Most involved is the expression for the R-R-term independent of h, as we have to use the expansion of $F_k(\hat{\Delta})$ up to quadratic order in h, as written in equation (B.11). As we are interested in taking the second variation with respect to h, terms linear in or independent of h can be discarded. Terms of order h^3 and higher can also be neglected, such that we get

$$
\begin{aligned}
\Gamma'_{h^0} = \int \sqrt{\hat{g}}\hat{R}\int_0^{\infty} ds\widetilde{F}_k(s)\mathrm{e}^{-s\hat{\Delta}} & \left\{\sum_{j=0}^{\infty}\frac{s^{j+1}}{4(j+1)!}\left[\hat{\Delta}, -\frac{1}{4}h^2\hat{\Delta} - \frac{1}{2}h(\hat{D}_{\mu}h)\hat{D}^{\mu}\right]_j\right. \\
+ \frac{1}{2}\left(\sum_{j=0}^{\infty}\frac{s^{j+1}}{4(j+1)!}\left[\hat{\Delta}, h\hat{\Delta} + (\hat{D}_{\mu}h)\hat{D}^{\mu}\right]_j\right) & \left(\sum_{k=0}^{\infty}\frac{s^{k+1}}{4(k+1)!}\left[\hat{\Delta}, h\hat{\Delta} + (\hat{D}_{\mu}h)\hat{D}^{\mu}\right]_k\right) \\
+ \frac{1}{2}\left(\sum_{j=0}^{\infty}\sum_{n=1}^{\infty}\frac{-s^{n+j+2}}{16(n+j+2)!}\right. & \left.\left.\left[\hat{\Delta}, \left[h\hat{\Delta} + (\hat{D}_{\mu}h)\hat{D}^{\mu}, \left[\hat{\Delta}, h\hat{\Delta} + (\hat{D}_{\mu}h)\hat{D}^{\mu}\right]_n\right]\right]_j\right)\right\}\hat{R}
\end{aligned}
\tag{C.36}
$$

We can make use of the fact that this whole term is of level $\hat{R}^2$ and go to the momentum representation to find

$$\begin{aligned}
&\bullet \left[\hat{\Delta}, -\frac{1}{4}h(-p)h(p)\hat{\Delta} - \frac{1}{2}h(-p)(\hat{D}_\mu h(p))\hat{D}^\mu\right]_j \hat{R}(q) \\
&= \left(-\frac{1}{4}h(-p)h(p)\hat{\Delta} - \frac{1}{2}h(-p)(\hat{D}_\mu h(p))\hat{D}^\mu\right)\delta_{j,0}\hat{R}(q) \\
&= \left(-\frac{1}{4}q^2 + \frac{1}{2}pqx\right)\delta_{j,0}h(-p)h(p)\hat{R}(q); \\
&\bullet \left[\hat{\Delta}, h(-p)\hat{\Delta} + (\hat{D}_\mu h(-p))\hat{D}^\mu\right]_j \left[\hat{\Delta}, h(p)\hat{\Delta} + (\hat{D}_\mu h(p))\hat{D}^\mu\right]_k \hat{R}(q) \\
&= (-p^2 - 2pqx)^j(2p^2 + 3pqx + q^2)(p^2 + 2pqx)^k(q^2 - pqx)h(-p)h(p)\hat{R}(q) \\
&\bullet \left[\hat{\Delta}, \left[h(-p)\hat{\Delta} + (\hat{D}_\mu h(-p))\hat{D}^\mu, \left[\hat{\Delta}, h(p)\hat{\Delta} + (\hat{D}_\mu h(p))\hat{D}^\mu\right]_n\right]\right]_j \hat{R}(q) \\
&= \left[h(-p)\hat{\Delta} + (\hat{D}_\mu h(-p))\hat{D}^\mu, \left[\hat{\Delta}, h(p)\hat{\Delta} + (\hat{D}_\mu h(p))\hat{D}^\mu\right]_n\right]\delta_{j,0}\hat{R}(q) \\
&= \Big((h(-p)\hat{\Delta} + (\hat{D}_\mu h(-p))\hat{D}^\mu)\left[\hat{\Delta}, h(p)\hat{\Delta} + (\hat{D}_\mu h(p))\hat{D}^\mu\right]_n \delta_{j,0}\hat{R}(q) \\
&- \left[\hat{\Delta}, h(-p)\hat{\Delta} + (\hat{D}_\mu h(-p))\hat{D}^\mu\right]_n (h(p)\hat{\Delta} + (\hat{D}_\mu h(p))\hat{D}^\mu)\Big)\delta_{j,0}\hat{R}(q) \\
&= (2p^2 + 3pqx + q^2)(p^2 + 2pqx)^n(q^2 - pqx)\delta_{j,0}h(-p)h(p)\hat{R}(q) \\
&- (-p^2 - 2pqx)^n(2p^2 + 3pqx + q^2)(q^2 - pqx)\delta_{j,0}h(-p)h(p)\hat{R}(q)
\end{aligned} \tag{C.37}$$

where we used in the first and last case that

$$\begin{aligned}
\left[\hat{\Delta}, X(-p,p)\right]_j \hat{R}(q) &= \left(\hat{\Delta}\left[\hat{\Delta}, X(-p,p)\right]_{j-1} - \left[\hat{\Delta}, X(-p,p)\right]_{j-1}\hat{\Delta}\right)\hat{R}(q) \\
&= \left((p - p + q)^2\left[\hat{\Delta}, X(-p,p)\right]_{j-1} - \left[\hat{\Delta}, X(-p,p)\right]_{j-1} q^2\right)\hat{R}(q)
\end{aligned} \tag{C.38}$$

such that the $\hat{\Delta}$ by induction gives a total contribution of $(0)^j = 0$. This means the terms are only nonzero for $j = 0$, as in that case $\left[\hat{\Delta}, X\right]_0 = X$. Using this, we can write the action as

$$\begin{aligned}
\Gamma_{h^0} &= \int \sqrt{\hat{g}}\hat{R}(-q)h(-p)h(p)\hat{R}(q)\int_0^\infty ds\widetilde{F}_k(s)\mathrm{e}^{-sq^2}\left\{\frac{s}{4}\left(-\frac{1}{4}q^2 + \frac{1}{2}pqx\right)\right. \\
&+ \frac{1}{32}\sum_{j=0}^\infty \frac{s^{j+1}}{(j+1)!}\sum_{k=0}^\infty \frac{s^{k+1}}{(k+1)!}(-1)^j(p^2 + 2pqx)^{j+k}(2p^2 + 3pqx + q^2)(q^2 - pqx) \\
&\left. - \frac{1}{32}\sum_{n=1}^\infty \frac{s^{n+2}}{(n+2)!}(1 - (-1)^n)\left(2p^2 + 3pqx + q^2)(p^2 + 2pqx)^n(q^2 - pqx)\right)\right\} \\
&= \frac{1}{8}\int \sqrt{\hat{g}}\hat{R}^2h^2\int_0^\infty ds\widetilde{F}_k(s)\mathrm{e}^{-sq^2}\left\{s\left(-\frac{1}{2}q^2 + pqx\right)\right. \\
&+ \frac{-2 + \mathrm{e}^{-s(p^2+2pqx)} + \mathrm{e}^{s(p^2+2pqx)}}{4(p^2 + 2pqx)^2}(2p^2 + 3pqx + q^2)(q^2 - pqx) \\
&\left. - \frac{-2s(p^2 + 2pqx) - \mathrm{e}^{-s(p^2+2pqx)} + \mathrm{e}^{s(p^2+2pqx)}}{4(p^2 + 2pqx)^2}(2p^2 + 3pqx + q^2)(q^2 - pqx)\right\} \\
&= \frac{1}{16}\int \sqrt{\hat{g}}\hat{R}^2h^2\int_0^\infty ds\widetilde{F}_k(s)\mathrm{e}^{-sq^2}\left\{s\left(-q^2 + 2pqx\right)\right. \\
&\left. + \left(-1 + \mathrm{e}^{-s(p^2+2pqx)} + s(p^2 + 2pqx)\right)\frac{(2p^2 + 3pqx + q^2)(q^2 - pqx)}{(p^2 + 2pqx)^2}\right\},
\end{aligned} \tag{C.39}$$

making it straightforward to obtain the second variation

$$
\begin{aligned}
\Gamma^{(2)}_{h^0} &= \frac{1}{8}\hat{R}^2 \int_0^\infty ds \widetilde{F}_k(s) \mathrm{e}^{-sq^2} \left\{ s\left(-q^2+2pqx\right) \right. \\
&+ \left. \left(-1+\mathrm{e}^{-s(p^2+2pqx)}+s(p^2+2pqx)\right) \frac{(2p^2+3pqx+q^2)(q^2-pqx)}{2(p^2+2pqx)^2} \right\} \\
&= \frac{1}{8}\hat{R}^2 \int_0^\infty ds \widetilde{F}_k(s) \left\{ s\mathrm{e}^{-sq^2}\left(-q^2+2pqx\right) + s\mathrm{e}^{-sq^2} \frac{(2p^2+3pqx+q^2)(q^2-pqx)}{(p^2+2pqx)} \right. \\
&+ \left. \left(-\mathrm{e}^{-sq^2}+\mathrm{e}^{-s(p^2+2pqx+q^2)}\right) \frac{(2p^2+3pqx+q^2)(q^2-pqx)}{(p^2+2pqx)^2} \right\} \\
&= \frac{1}{8}\hat{R}^2 \left\{ -F_k'(q^2)\left(\frac{(2p^2+3pqx+q^2)(q^2-pqx)}{(p^2+2pqx)} - q^2 + 2pqx \right) \right. \\
&+ \left. \left(F_k(p^2+2pqx+q^2)-F_k(q^2)\right) \frac{(2p^2+3pqx+q^2)(q^2-pqx)}{(p^2+2pqx)^2} \right\}.
\end{aligned}
\tag{C.40}
$$

C.2 Traces

C.2.1 First trace

Using equations (A.6), (A.7), (A.8) and (A.9), we find

$$
\begin{aligned}
&\frac{1}{2}\int_0^\infty ds\, \widetilde{O}(s) \frac{1}{16\pi^2 s^2} \mathrm{tr}\left\{ s^2 \mathbb{1}\hat{R} f_R(s\hat{x})\hat{R} \right\} \\
&= \frac{1}{32\pi^2}\hat{R}^2 \int_0^\infty ds\, \widetilde{O}(s) \left(-\frac{13}{144 s\hat{x}} - \frac{5}{24 s^2\hat{x}^2} + \frac{1}{32}\mathrm{f}(s\hat{x}) + \frac{1}{8s\hat{x}}\mathrm{f}(s\hat{x}) + \frac{5}{24 s^2\hat{x}^2}\mathrm{f}(s\hat{x}) \right) \\
&= \frac{1}{32\pi^2}\hat{R}^2 \left\{ -\frac{13}{144}\int_0^\infty dv - \frac{5}{24}\int_0^\infty dv\, v + \frac{1}{16}\int_0^{\frac{1}{4}} dv \frac{1}{\sqrt{1-4v}} \right. \\
&- \left. \frac{1}{8}\int_0^{\frac{1}{4}} dv\, \sqrt{1-4v} + \frac{1}{8}\int_0^\infty dv + \frac{5}{24}\int_0^\infty dv \left(v-\frac{1}{6}\right) + \frac{5}{144}\int_0^{\frac{1}{4}} dv\, (1-4v)^{\frac{3}{2}} \right\} O(\hat{x}v) \\
&= \frac{1}{1152\pi^2}\hat{R}^2 \int_0^{\frac{1}{4}} dv\, \frac{20v^2+8v-1}{\sqrt{1-4v}}\, O(\hat{x}v)
\end{aligned}
\tag{C.41}
$$

C.2.2 Second trace

The second trace becomes

$$
\begin{aligned}
&-\frac{1}{2}\mathrm{Tr}\left[\mathcal{O}_{1,d}G_0(\hat{\Delta})\dot{\mathcal{R}}_k(\hat{\Delta})G_0(\hat{\Delta})\right] \equiv \frac{3}{256\pi G_k}\mathrm{Tr}\Big[-\left(\hat{\Delta}F_k(\hat{\Delta})\hat{R}\right)V(\hat{\Delta}) - \hat{R}F_k(\Delta)\hat{\Delta}V(\hat{\Delta})\\
&+\sum_{n=0}^{\infty}\Big\{2\left(F_k(\Delta)\hat{\Delta}^{n+1}\hat{R}\right)\hat{\Delta}^{-n}V(\hat{\Delta}) - 2\left(\hat{\Delta}^{n+1}\hat{R}\right)F_k(\Delta)\hat{\Delta}^{-n}V(\hat{\Delta})\\
&+\left(F_k(\Delta)\hat{\Delta}^{n}\hat{R}\right)V(\hat{\Delta})\hat{\Delta}^{-n+1} - \left(\hat{\Delta}^{n}\hat{R}\right)V(\hat{\Delta})F_k(\Delta)\hat{\Delta}^{-n+1}\Big\}\Big]\\
&\equiv \frac{3}{256\pi G_k}\mathrm{Tr}\Bigg[-\left(\hat{\Delta}F_k(\hat{\Delta})\hat{R}\right)V(\hat{\Delta}) - \hat{R}V_1(\hat{\Delta}) + \sum_{n=0}^{\infty}\Big\{2\left(F_k(\Delta)\hat{\Delta}^{n+1}\hat{R}\right)V_2(n,\hat{\Delta})\\
&-\ 2\left(\hat{\Delta}^{n+1}\hat{R}\right)V_3(n,\hat{\Delta}) + \left(F_k(\Delta)\hat{\Delta}^{n}\hat{R}\right)V_4(n,\hat{\Delta}) - \left(\hat{\Delta}^{n}\hat{R}\right)V_5(n,\hat{\Delta})\Big\}\Bigg]\\
&= \frac{3}{256\pi G_k}\frac{1}{16\pi^2}\int_0^\infty ds\ \Bigg(-\left(\hat{\Delta}F_k(\Delta)\hat{R}\right)\widetilde{V}(s) - \hat{R}\widetilde{V}_1(s) + \sum_{n=0}^{\infty}\Big\{2\left(F_k(\Delta)\hat{\Delta}^{n+1}\hat{R}\right)\widetilde{V}_2(n,s)\\
&-\ 2\left(\hat{\Delta}^{n+1}\hat{R}\right)\widetilde{V}_3(n,s) + \left(F_k(\Delta)\hat{\Delta}^{n}\hat{R}\right)\widetilde{V}_4(n,s) - \left(\hat{\Delta}^{n}\hat{R}\right)\widetilde{V}_5(n,s)\Big\}\Bigg)\ s^{-1}\ g_R(s\hat{\Delta})\ \hat{R},
\end{aligned}
\tag{C.42}
$$

where we used the cyclicity of the trace in the first step and defined the operators that we trace over as

$$
\begin{aligned}
V(\hat{\Delta}) &= G_0(\hat{\Delta})\dot{\mathcal{R}}_k(\hat{\Delta})G_0(\hat{\Delta}),\\
V_1(\hat{\Delta}) &= F_k(\hat{\Delta})\hat{\Delta}V(\hat{\Delta}),\\
V_2(n,\hat{\Delta}) &= \hat{\Delta}^{-n}V(\hat{\Delta}),\\
V_3(n,\hat{\Delta}) &= F_k(\Delta)\hat{\Delta}^{-n}V(\hat{\Delta}),\\
V_4(n,\hat{\Delta}) &= V(\hat{\Delta})\hat{\Delta}^{-n+1},\\
V_5(n,\hat{\Delta}) &= V(\hat{\Delta})F_k(\Delta)\hat{\Delta}^{-n+1}.
\end{aligned}
\tag{C.43}
$$

To perform the integral over $s^{-1}g_R(s\hat{x}) = -\frac{1}{2s^2\hat{x}} + \left(\frac{1}{2s^2\hat{x}} + \frac{1}{4s}\right)\mathrm{f}(s\hat{x})$, we use equations (A.6), (A.9) and (A.8) to find

$$
\begin{aligned}
&\int_0^\infty ds\left\{-\frac{1}{2s^2\hat{x}} + \left(\frac{1}{2s^2\hat{x}} + \frac{1}{4s}\right)\mathrm{f}(s\hat{x})\right\}\bar{W}(\hat{x}v)\\
&= \int_0^\infty ds\Bigg\{-\frac{\hat{x}}{2}\int_0^\infty dv\ v + \frac{\hat{x}}{2}\int_0^\infty dv\ \left(v + \frac{1}{6}\right)\\
&+\frac{\hat{x}}{12}\int_0^{\frac{1}{4}} dv(1+4v)^{\frac{3}{2}} + \frac{\hat{x}}{4}\int_0^\infty dv - \frac{\hat{x}}{4}\int_0^{\frac{1}{4}} dv\sqrt{1-4v}\Bigg\}\bar{W}(\hat{x}v)\\
&= \frac{\hat{x}}{6}\left\{\int_0^\infty dv - \int_0^{\frac{1}{4}} dv\ \sqrt{1-4v}(1+2v)\right\}\bar{W}(\hat{x}v),
\end{aligned}
\tag{C.44}
$$

such that we find for the trace

$$
\begin{aligned}
&-\frac{1}{2}\mathrm{Tr}\left[\mathcal{O}_{1,d}G_0(\hat{x})\dot{\mathcal{R}}_k(\hat{x})G_0(\hat{x})\right]\\
&=-\frac{3}{256\pi G_k}\frac{\hat{x}}{96\pi^2}\left\{\int_0^\infty dv-\int_0^{\frac{1}{4}}dv\,\sqrt{1-4v}(1+2v)\right\}\\
&\left(\left(\hat{x}F_k(\hat{x})\hat{R}\right)V(\hat{x}v)+\hat{R}V_1(\hat{x}v)+\sum_{n=0}^{\infty}\left\{-2\left(F_k(x)\hat{x}^{n+1}\hat{R}\right)V_2(n,\hat{x}v)\right.\right.\\
&\left.\left.+2\left(\hat{x}^{n+1}\hat{R}\right)V_3(n,\hat{x}v)-\left(F_k(x)\hat{x}^n\hat{R}\right)V_4(n,\hat{x}v)+\left(\hat{x}^n\hat{R}\right)V_5(n,\hat{x}v)\right\}\right)\hat{R}\\
&=-\frac{3}{256\pi G_k}\frac{\hat{x}\hat{R}^2}{96\pi^2}\left\{\int_0^\infty dv-\int_0^{\frac{1}{4}}dv\,\sqrt{1-4v}(1+2v)\right\}\\
&\left(\hat{x}F_k(\hat{x})V(\hat{x}v)+F_k(\hat{x}v)\hat{x}vV(\hat{x}v)+\sum_{n=0}^{\infty}\left\{-2F_k(\hat{x})\hat{x}^{n+1}(\hat{x}v)^{-n}V(\hat{x}v)\right.\right.\\
&\left.\left.+\;2\hat{x}^{n+1}F_k(\hat{x}v)(\hat{x}v)^{-n}V(\hat{x}v)-F_k(x)\hat{x}^nV(\hat{x}v)(\hat{x}v)^{-n+1}+\hat{x}^nV(\hat{x}v)F_k(\hat{x}v)(\hat{x}v)^{-n+1}\right\}\right)\\
&=-\frac{3}{256\pi G_k}\frac{\hat{x}^2\hat{R}^2}{96\pi^2}\left\{\int_0^\infty dv-\int_0^{\frac{1}{4}}dv\,\sqrt{1-4v}(1+2v)\right\}\\
&\left(F_k(\hat{x})+F_k(\hat{x}v)v-v\frac{2\left(F_k(\hat{x})-F_k(\hat{x}v)\right)+v\left(F_k(\hat{x})-F_k(\hat{x}v)\right)}{v-1}\right)V(\hat{x}v)\\
&=\frac{\hat{x}^2\hat{R}^2}{96\pi^2}\left\{\int_0^\infty dv-\int_0^{\frac{1}{4}}dv\,\sqrt{1-4v}(1+2v)\right\}\frac{F_k(\hat{x}v)(2v^2+v)-F_k(\hat{x})(v^2+v+1)}{v-1}V'(\hat{x}v).
\end{aligned}
\tag{C.45}
$$

C.2.3 Third trace

The third trace becomes

$$
\begin{aligned}
&-\frac{1}{2}\mathrm{Tr}\left[\mathcal{O}_{1,od}\, G_0(\hat\Delta)\, \dot{\mathcal{R}}_k(\hat\Delta)\, G_0(\hat\Delta)\right] = \frac{3}{256\pi G_k}\mathrm{Tr}\left[\left(\hat D_\mu F_k(\hat\Delta)\hat R\right)\hat D^\mu V(\hat\Delta)\right.\\
&+\sum_{n=0}^{\infty}\left\{\left(\hat D^\mu\hat\Delta^n\hat R\right)\hat D_\mu V_3(n,\hat\Delta)-\left(\hat D^\mu F_k(\Delta)\hat\Delta^n\hat R\right)\hat D_\mu V_2(n,\hat\Delta)\right.\\
&\left.\left.+\left(\hat D_\mu F_k(\Delta)\hat\Delta^n\hat R\right)\hat D^\mu V_6(n,\hat\Delta)-\left(\hat D_\mu\hat\Delta^n\hat R\right)\hat D^\mu V_7(n,\hat\Delta)\right\}\right]\\
&=\frac{3}{256\pi G_k}\frac{1}{16\pi^2}\int_0^\infty ds\left(\left(\hat D^\mu F_k(\hat\Delta)\hat R\right)\widetilde V(s)+\sum_{n=0}^{\infty}\left\{\left(\hat D^\mu\hat\Delta^n\hat R\right)\widetilde V_3(n,s)\right.\right.\\
&-\left(\hat D^\mu F_k(\Delta)\hat\Delta^n\hat R\right)\widetilde V_2(n,s)+\left(\hat D^\mu F_k(\Delta)\hat\Delta^n\hat R\right)\widetilde V_6(n,s)\\
&\left.\left.-\left(\hat D^\mu\hat\Delta^n\hat R\right)\widetilde V_7(n,s)\right\}\right)s^{-1}\left\{-\frac{1}{4s\hat\Delta}+\left(\frac{1}{4s\hat\Delta}+\frac{1}{8}\right)\mathrm{f}(s\hat\Delta)\right\}\hat D_\mu\hat R\\
&=\frac{3}{256\pi G_k}\frac{\hat x}{192\pi^2}\left\{\int_0^\infty dv-\int_0^{\frac{1}{4}}dv\,\sqrt{1-4v}(1+2v)\right\}\\
&\left(\left(\hat D^\mu F_k(\hat x)\hat R\right)V(\hat xv)+\sum_{n=0}^{\infty}\left\{\left(\hat D^\mu\hat x^n\hat R\right)F_k(\hat xv)(\hat xv)^{-n}V(\hat xv)-\left(\hat D^\mu F_k(\hat x)\hat x^n\hat R\right)(\hat xv)^{-n}V(\hat xv)\right.\right.\\
&\left.\left.+\left(\hat D^\mu F_k(\hat x)\hat x^n\hat R\right)V(\hat xv)(\hat xv)^{-n}-\left(\hat D^\mu\hat x^n\hat R\right)V(\hat xv)F_k(\hat xv)(\hat xv)^{-n}\right\}\right)\hat D_\mu\hat R\\
&=\frac{3}{256\pi G_k}\frac{\hat x^2\hat R^2}{192\pi^2}\left\{\int_0^\infty dv-\int_0^{\frac{1}{4}}dv\,\sqrt{1-4v}(1+2v)\right\}\\
&\left(F_k(\hat x)+\sum_{n=0}^{\infty}F_k(\hat xv)v^{-n}-F_k(\hat x)v^{-n}+F_k(\hat x)v^{-n}-F_k(\hat xv)v^{-n}\right)V(\hat xv)\\
&=-\frac{\hat x^2\hat R^2}{192\pi^2}\left\{\int_0^\infty dv-\int_0^{\frac{1}{4}}dv\,\sqrt{1-4v}(1+2v)\right\}F_k(\hat x)V'(\hat xv)
\end{aligned}
\tag{C.46}
$$

where we used that the off-diagonal heat kernel coefficient is equal to $\frac{1}{2}g_R(s\hat x)$:

$$
\mathrm{Tr}\left[\hat D_\mu H(s)\right]=\frac{s}{(4\pi s)^{d/2}}\left\{-\frac{1}{4s\hat\Delta}+\left(\frac{1}{4s\hat\Delta}+\frac{1}{8}\right)\mathrm{f}(s\hat\Delta)\right\}\hat D_\mu\hat R \tag{C.47}
$$

such that we can use equation (C.44) again, and defined

$$
\begin{aligned}
V_6(n,\hat\Delta)&=V(\hat\Delta)\hat\Delta^{-n},\\
V_7(n,\hat\Delta)&=V(\hat\Delta)F_k(\Delta)\hat\Delta^{-n}.
\end{aligned}
\tag{C.48}
$$

C.2.4 Fifth trace

As the operator O_6 is of order $\hat R^2$, we can go to momentum space

$$
\frac{1}{2}\mathrm{Tr}\left[O_6\right]=\frac{1}{2}\mathrm{Tr}\left[\mathcal{O}_1(\hat\Delta)\,G_0(\hat\Delta)\,\mathcal{O}_1(\hat\Delta)\,V(\hat\Delta)\right]=\frac{1}{2}\mathrm{Tr}\left[\mathcal{O}_1(-q)\,G_0(p+q)\,\mathcal{O}_1(q)\,V(p^2)\right]. \tag{C.49}
$$

Working out the individual components (while keeping the overall structure in mind) gives

$$\begin{aligned}
\mathcal{O}_1(-q) &= \left(-\frac{3}{128\pi G_k}\right) \hat{R}(-q) \left(-q^2 F_k(q^2) + F_k(q^2)(q^2+pqx) - F_k((p+q)^2)(p+q)^2\right. \\
&+ \sum_{n=0}^{\infty} \left\{2q^{2n+2}(p+q)^{-2n}\left(F_k(q^2) - F_k((p+q)^2)\right) + p^{-2n+2}q^{2n}\left(F_k(q^2) - F_k(p^2)\right)\right. \\
&+ \left.\left. q^{2n}(pqx+q^2)(p+q)^{-2n}\left(F_k((p+q)^2) - F_k(q^2)\right) + p^{-2n}q^{2n}(pqx+q^2)\left(F_k(q^2) - F_k(p^2)\right)\right\}\right) \\
&= \left(-\frac{3}{128\pi G_k}\right) \hat{R} \left(F_k(q^2)pqx - F_k((p+q)^2)(p+q)^2 + \frac{(q^2-pqx)(p+q)^2}{p^2+2pqx}\left(F_k(q^2) - F_k((p+q)^2)\right)\right. \\
&+ \left.\frac{p^4 + p^2(pqx+q^2)}{p^2-q^2}\left(F_k(q^2) - F_k(p^2)\right)\right),
\end{aligned} \tag{C.50}$$

$$G_0(p+q) = \frac{-\frac{256\pi G_k}{3}}{(p+q)^2 + 6F_k((p+q)^2)(p+q)^4 + \mathcal{R}_k((p+q)^2) - \frac{4}{3}\Lambda}, \tag{C.51}$$

$$\begin{aligned}
\mathcal{O}_1(q) &= \left(-\frac{3}{128\pi G_k}\right) \hat{R}(q) \left(-(q^2+pqx)F_k(q^2) - F_k(p^2)p^2 + \sum_{n=0}^{\infty}\left\{2q^{2n+2}p^{-2n}\left(F_k(q^2) - F_k(p^2)\right)\right.\right. \\
&+ (p+q)^{-2n+2}q^{2n}\left(F_k(q^2) - F_k((p+q)^2)\right) + pqx\, q^{2n}p^{-2n}\left(F_k(q^2) - F_k(p^2)\right) \\
&+ \left.\left. (p+q)^{-2n}pqx\, q^{2n}\left(F_k((p+q)^2) - F_k(q^2)\right)\right\}\right) \\
&= \left(\frac{3}{128\pi G_k}\right) \hat{R}(q) \left(-(q^2+pqx)F_k(q^2) - F_k(p^2)p^2\right. \\
&+ \left.\frac{p^2(2q^2+pqx)}{p^2-q^2}\left(F_k(q^2) - F_k(p^2)\right) + \frac{(p+q)^2\left(p^2+pqx+q^2\right)}{p^2+2pqx}\left(F_k(q^2) - F_k((p+q)^2)\right)\right),
\end{aligned} \tag{C.52}$$

and

$$V(p^2) = \frac{-\frac{256\pi G_k}{3}\left(\dot{\mathcal{R}}_k(p^2) + 2\mathcal{R}_k(p^2)\right)}{\left(p^2 + 6F_k(p^2)p^4 + \mathcal{R}_k(p^2) - \frac{4}{3}\Lambda\right)^2} = -\frac{256\pi G_k}{3}V'(p^2), \tag{C.53}$$

such that the trace becomes

$$\begin{aligned}
&\frac{1}{2}\mathrm{Tr}\left[O_6\right] = \frac{1}{2}\mathrm{Tr}\left[\mathcal{O}_1(\hat{\Delta})\, G_0(\hat{\Delta})\, \mathcal{O}_1(\hat{\Delta})\, V(\hat{\Delta})\right] = \frac{1}{2}\mathrm{Tr}\left[\mathcal{O}_1(-q)\, G_0(p+q)\, \mathcal{O}_1(q)\, V(p^2)\right] \\
&= \left(\frac{3}{128\pi G_k}\right)^2 \frac{1}{2}\mathrm{Tr}\left[\hat{R}^2\left(F_k(q^2)pqx - F_k((p+q)^2)(p+q)^2\right.\right. \\
&+ \left.\frac{(q^2-pqx)(p+q)^2}{p^2+2pqx}\left(F_k(q^2) - F_k((p+q)^2)\right) + \frac{p^4+p^2(pqx+q^2)}{p^2-q^2}\left(F_k(q^2) - F_k(p^2)\right)\right) \\
&\frac{-\frac{256\pi G_k}{3}}{(p+q)^2 + 6F_k((p+q)^2)(p+q)^4 + \mathcal{R}_k((p+q)^2) - \frac{4}{3}\Lambda} \\
&\left(-(q^2+pqx)F_k(q^2) - F_k(p^2)p^2 + \frac{p^2(2q^2+pqx)}{p^2-q^2}\left(F_k(q^2) - F_k(p^2)\right)\right. \\
&+ \left.\left.\frac{(p+q)^2\left(p^2+pqx+q^2\right)}{p^2+2pqx}\left(F_k(q^2) - F_k((p+q)^2)\right)\right) \frac{-\frac{256\pi G_k}{3}\left(\dot{\mathcal{R}}_k(p^2) + 2\mathcal{R}_k(p^2)\right)}{\left(p^2 + 6F_k(p^2)p^4 + \mathcal{R}_k(p^2) - \frac{4}{3}\Lambda\right)^2}\right] \\
&\equiv 2\mathrm{Tr}\left[\hat{R}^2\widetilde{K}(p,q,x)V'(p^2)\right] = \frac{1}{4\pi^3}\int_0^\infty dp\, p^3 \int_{-1}^1 dx\, \sqrt{1-x^2}\,\widetilde{K}(p,q,x)V'(p^2)\int\sqrt{\hat{g}}\hat{R}^2,
\end{aligned} \tag{C.54}$$

with

$$
\begin{aligned}
\widetilde{K}(p,q,x) = &\big(F_k(q^2)pqx - F_k((p+q)^2)(p+q)^2 \\
&+ \frac{(q^2-pqx)(p+q)^2}{p^2+2pqx}\big(F_k(q^2)-F_k((p+q)^2)\big) + \frac{p^4+p^2(pqx+q^2)}{p^2-q^2}\big(F_k(q^2)-F_k(p^2)\big)\Big) \\
&\frac{1}{(p+q)^2+6F_k((p+q)^2)(p+q)^4+\bar{\mathcal{R}}_k((p+q)^2)-\frac{4}{3}\Lambda} \\
&\Big(-(q^2+pqx)F_k(q^2)-F_k(p^2)p^2+\frac{p^2(2q^2+pqx)}{p^2-q^2}\big(F_k(q^2)-F_k(p^2)\big) \\
&+\frac{(p+q)^2\left(p^2+pqx+q^2\right)}{p^2+2pqx}\big(F_k(q^2)-F_k((p+q)^2)\big)\Big).
\end{aligned}
\tag{C.55}
$$

D Weyl-variation and -trace

D.1 Variations

Combining the transformations to the conformally reduced setting for the Weyl tensor and metric (4.12) with the expansion of the form factor (B.12) and, as we are interested in the second variation, only keeping terms quadratic in h, we find the action for the terms where the expansion $\sqrt{g}C^{\mu\nu\rho\sigma} - C_{\mu\nu\rho\sigma}$ is quadratic in h to be

$$\begin{aligned}\Gamma_{h^2} &= -\frac{1}{16}\int\sqrt{\hat{g}}\, h\hat{C}^{\mu\nu\rho\sigma}\int_0^\infty ds\widetilde{W}_k(s)\mathrm{e}^{-s\hat{\Delta}}h\hat{C}_{\mu\nu\rho\sigma} + \frac{1}{16}\int\sqrt{\hat{g}}\, h^2\hat{C}^{\mu\nu\rho\sigma}\int_0^\infty ds\widetilde{W}_k(s)\mathrm{e}^{-s\hat{\Delta}}\hat{C}_{\mu\nu\rho\sigma} \\ &= -\frac{1}{16}\int\sqrt{\hat{g}}\, h\hat{C}^{\mu\nu\rho\sigma}W_k(\Delta)h\hat{C}_{\mu\nu\rho\sigma} + \frac{1}{16}\int\sqrt{\hat{g}}\, h^2\hat{C}^{\mu\nu\rho\sigma}W_k(\Delta)\hat{C}_{\mu\nu\rho\sigma}.\end{aligned} \tag{D.1}$$

For the two terms where the expansion $\sqrt{g}C^{\mu\nu\rho\sigma} - C_{\mu\nu\rho\sigma}$ is linear in h, we find

$$\begin{aligned}\Gamma_{h^1} = &\frac{1}{4}\int\sqrt{\hat{g}}\,\hat{C}^{\mu\nu\rho\sigma}\int_0^\infty ds\widetilde{W}_k(s)\mathrm{e}^{-s\hat{\Delta}}\left\{\sum_{j=0}^\infty\frac{(-1)^j}{(j+1)!}\left[-s\hat{\Delta},-s\mathbb{T}_1\right]_j\right\}_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta}h\hat{C}_{\alpha\beta\gamma\delta} \\ &-\frac{1}{4}\int\sqrt{\hat{g}}\, h\hat{C}^{\mu\nu\rho\sigma}\int_0^\infty ds\widetilde{W}_k(s)\mathrm{e}^{-s\hat{\Delta}}\left\{\sum_{j=0}^\infty\frac{(-1)^j}{(j+1)!}\left[-s\hat{\Delta},-s\mathbb{T}_1\right]_j\right\}_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta}\hat{C}_{\alpha\beta\gamma\delta}.\end{aligned} \tag{D.2}$$

And the final term in the action, where the expansion is independent of h and we thus have to expand the form factor up to second order in h, becomes

$$\begin{aligned}\Gamma_{h^0} = &\int\sqrt{\hat{g}}\,\hat{C}^{\mu\nu\rho\sigma}\int_0^\infty ds\widetilde{W}_k(s)\mathrm{e}^{-s\hat{\Delta}}\left\{\sum_{j=0}^\infty\frac{(-1)^j}{(j+1)!}\left[-s\hat{\Delta},-s\mathbb{T}_2\right]_j\right. \\ &+\frac{1}{2}\left(\sum_{j=0}^\infty\frac{(-1)^j}{(j+1)!}\left[-s\hat{\Delta},-s\mathbb{T}_1\right]_j\right)\left(\sum_{k=0}^\infty\frac{(-1)^k}{(k+1)!}\left[-s\hat{\Delta},-s\mathbb{T}_1\right]_k\right) \\ &\left.+\frac{1}{2}\left(\sum_{j=0}^\infty\sum_{n=1}^\infty\frac{(-1)^{j+n+1}}{(n+j+2)!}\left[-s\hat{\Delta},\left[-s\mathbb{T}_1,\left[-s\hat{\Delta},-s\mathbb{T}_1\right]_n\right]\right]_j\right)\right\}_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta}\hat{C}_{\alpha\beta\gamma\delta}.\end{aligned} \tag{D.3}$$

We can go to momentum representation for all terms to find

$$\Gamma_{h^2} = -\frac{1}{16}\int\sqrt{\hat{g}}\, h\hat{C}^{\mu\nu\rho\sigma}\, W_k((p+q)^2)h\hat{C}_{\mu\nu\rho\sigma} + \frac{1}{16}\int\sqrt{\hat{g}}\, h^2\hat{C}^{\mu\nu\rho\sigma}W_k(q^2)\hat{C}_{\mu\nu\rho\sigma}, \tag{D.4}$$

$$
\begin{aligned}
\Gamma_{h^1} &= \frac{1}{4}\int\sqrt{\hat g}\,\hat C^{\mu\nu\rho\sigma}(-q)\int_0^\infty ds\widetilde{W}_k(s)\mathrm{e}^{-sq^2}\left\{\sum_{j=0}^\infty\frac{(-1)^j(-s)^{j+1}}{(j+1)!}(-p^2-2pqx)^j\mathbb{T}_1(-p)\right\}_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta}h(p)\hat C_{\alpha\beta\gamma\delta}(q)\\
&-\frac{1}{4}\int\sqrt{\hat g}\,h(-p)\hat C^{\mu\nu\rho\sigma}(-q)\int_0^\infty ds\widetilde{W}_k(s)\mathrm{e}^{-s(p+q)^2}\left\{\sum_{j=0}^\infty\frac{(-1)^j(-s)^{j+1}}{(j+1)!}(p^2+2pqx)^j\mathbb{T}_1(p)\right\}_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta}\hat C_{\alpha\beta\gamma\delta}(q)\\
&=\frac{1}{4}\int\sqrt{\hat g}\,\hat C^{\mu\nu\rho\sigma}(-q)\int_0^\infty ds\widetilde{W}_k(s)\frac{\mathrm{e}^{-s(p+q)^2}-\mathrm{e}^{-sq^2}}{p^2+2pqx}\mathbb{T}_1(-p)_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta}h(p)\hat C_{\alpha\beta\gamma\delta}(q)\\
&-\frac{1}{4}\int\sqrt{\hat g}\,h(-p)\hat C^{\mu\nu\rho\sigma}(-q)\int_0^\infty ds\widetilde{W}_k(s)\frac{\mathrm{e}^{-s(p+q)^2}-\mathrm{e}^{-sq^2}}{p^2+2pqx}\mathbb{T}_1(p)_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta}\hat C_{\alpha\beta\gamma\delta}(q)\\
&=\frac{1}{4}\int\sqrt{\hat g}\,\hat C^{\mu\nu\rho\sigma}(-q)\frac{W_k((p+q)^2)-W_k(q^2)}{p^2+2pqx}\mathbb{T}_1(-p)_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta}h(p)\hat C_{\alpha\beta\gamma\delta}(q)\\
&-\frac{1}{4}\int\sqrt{\hat g}\,h(-p)\hat C^{\mu\nu\rho\sigma}(-q)\frac{W_k((p+q)^2)-W_k(q^2)}{p^2+2pqx}\mathbb{T}_1(p)_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta}\hat C_{\alpha\beta\gamma\delta}(q),
\end{aligned}
\tag{D.5}
$$

$$
\begin{aligned}
\Gamma_{h^0} &= \int\sqrt{\hat g}\,\hat C^{\mu\nu\rho\sigma}(-q)\int_0^\infty ds\widetilde{W}_k(s)\mathrm{e}^{-sq^2}\left\{\sum_{j=0}^\infty\frac{(-1)^j(-s)^{j+1}}{(j+1)!}\mathbb{T}_2(p,-p)\delta_{j,0}\right.\\
&+\frac{1}{2}\sum_{j=0}^\infty\sum_{k=0}^\infty\frac{(-1)^j(-s)^{j+1}}{(j+1)!}\frac{(-1)^k(-s)^{k+1}}{(k+1)!}(-1)^j(p^2+2pqx)^{j+k}\mathbb{T}_1(-p)\mathbb{T}_1(p)\\
&\left.+\frac{1}{2}\sum_{j=0}^\infty\sum_{n=1}^\infty\frac{(-1)^{j+n+1}(-s)^{n+j+2}}{(n+j+2)!}\left(1-(-1)^n\right)(p^2+2pqx)^n\mathbb{T}_1(-p)\mathbb{T}_1(p)\delta_{j,0}\right\}_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta}\hat C_{\alpha\beta\gamma\delta}(q)\\
&=\int\sqrt{\hat g}\,\hat C^{\mu\nu\rho\sigma}(-q)\int_0^\infty ds\widetilde{W}_k(s)\left\{-s\mathrm{e}^{-sq^2}\mathbb{T}_2(p,-p)_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta}\right.\\
&+\frac{1}{2}\frac{-2\mathrm{e}^{-sq^2}+\mathrm{e}^{-s(p+q)^2}+\mathrm{e}^{-s(-p^2-2pqx+q^2)}}{(p^2+2pqx)^2}\mathbb{T}_1(-p)_{\mu\nu\rho\sigma}^{\kappa\lambda\pi\omega}\mathbb{T}_1(p)_{\kappa\lambda\pi\omega}^{\alpha\beta\gamma\delta}\\
&\left.+\frac{1}{2}\frac{\mathrm{e}^{-s(p+q)^2}-\mathrm{e}^{-s(-p^2-2pqx+q^2)}+2s\mathrm{e}^{-sq^2}(p^2+2pqx)}{(p^2+2pqx)^2}\mathbb{T}_1(-p)_{\mu\nu\rho\sigma}^{\kappa\lambda\pi\omega}\mathbb{T}_1(p)_{\kappa\lambda\pi\omega}^{\alpha\beta\gamma\delta}\right\}\hat C_{\alpha\beta\gamma\delta}(q)\\
&=\int\sqrt{\hat g}\,\hat C^{\mu\nu\rho\sigma}(-q)\left\{W_k'(q^2)\mathbb{T}_2(p,-p)_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta}\right.\\
&+\frac{1}{2}\frac{-2W_k(q^2)+W_k((p+q)^2)+W_k(-p^2-2pqx+q^2)}{(p^2+2pqx)^2}\mathbb{T}_1(-p)_{\mu\nu\rho\sigma}^{\kappa\lambda\pi\omega}\mathbb{T}_1(p)_{\kappa\lambda\pi\omega}^{\alpha\beta\gamma\delta}\\
&\left.+\frac{1}{2}\frac{W_k((p+q)^2)-W_k(-p^2-2pqx+q^2)-2W_k'(q^2)(p^2+2pqx)}{(p^2+2pqx)^2}\mathbb{T}_1(-p)_{\mu\nu\rho\sigma}^{\kappa\lambda\pi\omega}\mathbb{T}_1(p)_{\kappa\lambda\pi\omega}^{\alpha\beta\gamma\delta}\right\}\hat C_{\alpha\beta\gamma\delta}(q).
\end{aligned}
\tag{D.6}
$$

Evaluating the structures involving $\mathbb{T}_{1,2}$ using equation (4.16), and combining the terms above, we (trivially) find the second variation

$$\begin{aligned}
\Gamma^{(2)}_{Weyl} &= -\frac{1}{8}\hat{C}^2\, W_k((p+q)^2) + \frac{1}{8}\hat{C}^2 W_k(q^2) - \frac{1}{2}\,\hat{C}^2\frac{W_k((p+q)^2)-W_k(q^2)}{p^2+2pqx}\frac{1}{4}\left(q^2-pqx\right) \\
&+ \frac{1}{2}\hat{C}^1\frac{W_k((p+q)^2)-W_k(q^2)}{p^2+2pqx} - \frac{1}{4}\left(2p^2+3pqx+q^2\right) \\
&+ 2\int \hat{C}^2\left\{W_k'(q^2)\frac{1}{32}\left(3p^2-12pqx+2q^2\right)\right. \\
&+ \frac{W_k((p+q)^2)-2W_k(q^2)+W_k(q^2-p^2-2pqx}{(p^2+2pqx)^2}\frac{1}{32}\left(2q^4-4p^3qx+4pq^3x+p^2q^2(7-6x^2)\right) \\
&+ \frac{W_k((p+q)^2)-W_k(-p^2-2pqx+q^2)-2W_k'(q^2)(p^2+2pqx)}{(p^2+2pqx)^2} \\
&\left.\frac{1}{32}\left(2q^4-4p^3qx+4pq^3x+p^2q^2(7-6x^2)\right)\right\} \\
&= \frac{1}{16}\hat{C}^2\left\{2W_k((p+q)^2)-2W_k(q^2)+W_k'(q^2)\frac{3p^4-2p^3qx-p^2q^2(5+18x^2)-2q^4}{p^2+2pqx}\right. \\
&+ \left.\left(W_k((p+q)^2)-W_k(q^2)\right)\frac{2q^4+4pqx(q^2-p^2)+p^2q^2(7-6x^2)}{(p^2+2pqx)^2}\right\} \\
&\equiv \hat{C}^2 K(p,q,x).
\end{aligned} \tag{D.7}$$

D.2 Trace

Using equations (A.6) and (A.9), we find

$$\begin{aligned}
&\frac{1}{32\pi^2}\int_0^\infty ds\,\widetilde{W}(s)\frac{1}{s^2}\mathrm{tr}\left\{s^2\mathbb{1}C_{\mu\nu\alpha\beta}f_C(s\hat{x})C^{\mu\nu\alpha\beta}\right\} \\
&= \frac{1}{32\pi^2}\hat{C}^2\int_0^\infty ds\,\widetilde{W}(s)\left(\frac{1}{12s\hat{x}}+\frac{1}{2s^2\hat{x}^2}\mathrm{f}(s\hat{x})-\frac{1}{2s^2\hat{x}^2}\right) \\
&= \frac{1}{32\pi^2}\hat{C}^2\left\{\frac{1}{12}\int_0^\infty dv + \frac{1}{2}\int_0^\infty dv\left(v-\frac{1}{6}\right)+\frac{1}{12}\int_0^{\frac{1}{4}} dv\,(1-4v)^{\frac{3}{2}} - \frac{1}{2}\int_0^\infty dv\,v\right\}W(\hat{x}v) \\
&= \frac{1}{384\pi^2}\hat{C}^2\int_0^{\frac{1}{4}} dv\,(1-4v)^{\frac{3}{2}}\,W(\hat{x}v).
\end{aligned} \tag{D.8}$$

E Transverse traceless contribution to the potential

Consider the Feynman diagram in Fig. 9 with masses m_1 and m_2. Incoming and outgoing momenta are k_2 and k_1 for m_1 and k_4 and k_3 for m_2, respectively. The momentum of the propagator is denoted by $q = k_2 - k_1 = k_3 - k_4$. We work with units where $\hbar = c = 1$ and use the mostly-minus metric. Then the nonrelativistic potential can be found using

$$\langle k_1, k_3|S|k_2, k_4\rangle = -i\widetilde{V}(\mathbf{q})(2\pi)\delta(E_i - E_f) = (2\pi)^4\delta^{(4)}(k_2 + k_4 - k_1 - k_3)(i\mathcal{M}), \tag{E.1}$$

where $\widetilde{V}$ is the Fourier transform of the potential. In the nonrelativistic limit the potential then becomes

$$V(\mathbf{x}) = -\frac{1}{2m_1}\frac{1}{2m_2}\int \frac{d^3k}{(2\pi)^3}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\mathcal{M}, \tag{E.2}$$

where $\mathcal{M}$ is the non-analytic part. The Feynman rules are: for a vertex with propagator-indices μ, ν, incoming momentum p and outgoing momentum p' we have $\tau^{\mu\nu}(p, p', m) = -\frac{i\kappa}{2}\left[p^\mu p'^\nu + p^\nu p'^\mu - \eta^{\mu\nu}((p\cdot p') - m^2)\right]$ and for a propagator with momentum q, incoming indices α, β and outgoing indices γ, δ we have $\frac{iP^{\alpha\beta\gamma\delta}}{q^2+i\epsilon}$ with $P^{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\eta^{\alpha\gamma}\eta\beta\delta + \eta^{\beta\gamma}\eta\alpha\delta - \eta^{\alpha\beta}\eta\gamma\delta\right)$.
Then we have

$$D = \tau^{\mu\nu}P_{\mu\nu\alpha\beta}\tau^{\alpha\beta} = -\frac{\kappa^2}{8}\left(k_2^\mu k_1^\nu + k_2^\nu k_1^\mu - \eta^{\mu\nu}((k_1\cdot k_2) - m_1^2)\right)(2\eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta}) \tag{E.3}$$

$$\left(k_3^\alpha k_4^\beta + k_3^\beta k_4^\alpha - \eta^{\alpha\beta}((k_3\cdot k_4) - m_2^2)\right) \tag{E.4}$$

$$= -\frac{\kappa^2}{8}\left(2k_{2,\alpha}k_{1,\beta} + 2k_{2,\beta}k_{1,\alpha} + \delta_\alpha^\nu\eta_{\nu\beta}q^2 - 2k_{2,\nu}k_1^\nu\eta_{\alpha\beta} - 2\eta_{\alpha\beta}q^2\right)\left(k_3^\alpha k_4^\beta + k_3^\beta k_4^\alpha - \eta^{\alpha\beta}\frac{1}{2}q^2\right) \tag{E.5}$$

$$= -\frac{\kappa^2}{8}\left(2k_{2,\alpha}k_{1,\beta} + 2k_{2,\beta}k_{1,\alpha} - \eta_{\alpha\beta}q^2 - 2\eta_{\alpha\beta}(m_1^2 - \frac{1}{2}q^2)\right)\left(k_3^\alpha k_4^\beta + k_3^\beta k_4^\alpha - \frac{1}{2}\eta^{\alpha\beta}q^2\right) \tag{E.6}$$

$$= -\frac{\kappa^2}{4}\left(k_{2,\alpha}k_{1,\beta} + k_{2,\beta}k_{1,\alpha} - \eta_{\alpha\beta}m_1^2\right)\left(k_3^\alpha k_4^\beta + k_3^\beta k_4^\alpha - \frac{1}{2}\eta^{\alpha\beta}q^2\right) \tag{E.7}$$

$$= -\frac{\kappa^2}{4}\Big((k_2\cdot k_3)(k_1\cdot k_4) + (k_2\cdot k_4)(k_1\cdot k_3) + \frac{1}{2}(k_1\cdot k_2)q^2 + (k_2\cdot k_4)(k_1\cdot k_3) \tag{E.8}$$

$$+(k_2\cdot k_3)(k_1\cdot k_4) + \frac{1}{2}(k_1\cdot k_2)q^2 - m_1^2(k_3\cdot k_4 + k_4\cdot k_3 + 2q^2)\Big) \tag{E.9}$$

$$= -\frac{\kappa^2}{2}\left((k_2\cdot k_3)(k_1\cdot k_4) + (k_2\cdot k_4)(k_1\cdot k_3) + \frac{1}{2}(k_1\cdot k_2)q^2 - m_1^2(k_3\cdot k_4 + q^2)\right) \tag{E.10}$$

$$= -\frac{\kappa^2}{2}\left((k_2\cdot k_3)(k_1\cdot k_4) + (k_2\cdot k_4)(k_1\cdot k_3) - \frac{1}{4}(q^2)^2 - m_1^2 m_2^2\right), \tag{E.11}$$

with $\kappa = \sqrt{32\pi G}$. Let us now consider nonrelativistic scattering where $q = (0, \mathbf{q})$, $k_1 = (m_1, \mathbf{q})$, $k_2 = (m_1, \mathbf{0})$, $k_3 = (m_2, -\mathbf{q})$, $k_4 = (m_2, \mathbf{0})$, such that $(k_2\cdot k_3) = (k_1\cdot k_4) = (k_2\cdot k_4) = m_1 m_2$ and $(k_1\cdot k_3) = m_1 m_2 + \mathbf{q}^2$. We then have

$$D = -\frac{\kappa^2}{2}\left(m_1^2 m_2^2 + m_1^2 m_2^2 + m_1 m_2\mathbf{q}^2 - \frac{1}{4}(q^2)^2 - m_1^2 m_2^2\right) \tag{E.12}$$

$$= -\frac{\kappa^2}{2}m_1^2 m_2^2 + \mathcal{O}(q^2), \tag{E.13}$$

such that

$$S = -\frac{i}{q^2}\tau^{\mu\nu}P_{\mu\nu\alpha\beta}\tau^{\alpha\beta} = \frac{i}{|\mathbf{q}|^2}16\pi G m_1^2 m_2^2 \tag{E.14}$$

and thus

$$\mathcal{M} = \frac{1}{|\mathbf{q}|^2}16\pi G m_1^2 m_2^2. \tag{E.15}$$

Filling this in, we find the potential

$$V(r) = -\frac{1}{2m_1}\frac{1}{2m_2}\int \frac{d^3q}{(2\pi)^3}\mathrm{e}^{iq\cdot r}\frac{1}{|\mathbf{q}|^2}16\pi G m_1^2 m_2^2 = -4\pi G m_1 m_2 \frac{1}{4\pi r} = -G\frac{m_1 m_2}{r}, \tag{E.16}$$

which is the well-known classical Newtonian potential.

Now we focus on the transverse-traceless part. Introduce the spin-2 projector [2]

$$P^{(2),\rho\sigma}_{\mu\nu} = \frac{1}{2}\left(T^\rho_\mu T^\sigma_\nu + T^\sigma_\mu T^\rho_\nu\right) - \frac{1}{d-1}T_{\mu\nu}T^{\rho\sigma}, \tag{E.17}$$

with $T^\mu_\nu = \delta^\mu_\nu - \frac{p^\mu p^\nu}{p^2}$. Acting on it with $\eta_{\rho\sigma}$ from the right, for $d = 4$ gives

$$P^{(2),\rho\sigma}_{\mu\nu}\eta_{\rho\sigma} = \left(\delta^\rho_\mu - \frac{p^\rho p_\mu}{p^2}\right)\left(\delta^\sigma_\nu - \frac{p^\sigma p_\nu}{p^2}\right)\eta_{\rho\sigma} - \frac{1}{3}T_{\mu\nu}\left(\delta^{\rho\sigma} - \frac{p^\rho p^\sigma}{p^2}\right)\eta_{\rho\sigma} \tag{E.18}$$

$$= \eta_{\mu\nu} - 2\frac{p_\mu p_\nu}{p^2} + \frac{p_\mu p_\nu}{p^2} - \frac{1}{3}T_{\mu\nu}\left(4 - \frac{p^2}{p^2}\right) = T_{\mu\nu} - T_{\mu\nu}, \tag{E.19}$$

showing that only the momentum terms will survive in the effective vertices. Using this projector to consider the same process as above, we find

$$D^{TT} = \tau^{\mu\nu}P^{(2),\rho\sigma}_{\mu\nu}\tau_{\rho\sigma} = -\frac{\kappa^2}{4}\left(k_2^\mu k_1^\nu + k_2^\nu k_1^\mu + ...\right)\left(T^\rho_\mu T^\sigma_\nu - \frac{1}{3}T_{\mu\nu}T^{\rho\sigma}\right)\left(2k_{3,\rho}k_{4,\sigma} + ...\right) \equiv \boxed{1} + \boxed{2}. \tag{E.20}$$

The second part becomes

$$\boxed{2} = \frac{\kappa^2}{6}\left(k_2^\mu k_1^\nu + k_2^\nu k_1^\mu\right)\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\left(\delta^{\rho\sigma} - \frac{q^\rho q^\sigma}{q^2}\right)\left(k_{3,\rho}k_{4,\sigma}\right) \tag{E.21}$$

$$= \frac{\kappa^2}{6}\left(2(k_1\cdot k_2) - \frac{2}{q^2}(k_1\cdot q)(k_2\cdot q)\right)\left((k_3\cdot k_4) - \frac{1}{q^2}(k_3\cdot q)(k_4\cdot q)\right) \tag{E.22}$$

$$\overset{NR}{=} \frac{\kappa^2}{3}\left(m_1^2 - \frac{q^2}{2}\right)\left(m_2^2 - \frac{q^2}{2}\right) + \mathcal{O}(q^{-2}), \tag{E.23}$$

where we took the nonrelativistic limit in the last line. The first part becomes

$$\boxed{1} = -\frac{\kappa^2}{2}\left(k_2^\mu k_1^\nu + k_2^\nu k_1^\mu\right)\left(\delta^\rho_\mu - \frac{q_\mu q^\rho}{q^2}\right)\left(\delta^\sigma_\nu - \frac{q_\nu q^\sigma}{q^2}\right)\left(k_{3,\rho}k_{4,\sigma}\right) \tag{E.24}$$

$$= -\frac{\kappa^2}{2}\Bigg((k_1\cdot k_3)(k_2\cdot k_4) - \frac{1}{q^2}(k_1\cdot q)(k_2\cdot k_4)(k_3\cdot q) - \frac{1}{q^2}(k_1\cdot k_3)(k_2\cdot q)(k_4\cdot q) \tag{E.25}$$

$$+ \frac{1}{q^4}(k_1\cdot q)(k_2\cdot q)(k_3\cdot q)(k_4\cdot q) + (1\leftrightarrow 2)\Bigg) \tag{E.26}$$

$$\overset{NR}{=} -\frac{\kappa^2}{2}\left((k_1\cdot k_3)(k_2\cdot k_4) + (k_2\cdot k_3)(k_1\cdot k_4) - \frac{1}{q^2}(k_1\cdot q)(k_3\cdot q)(k_2\cdot k_4)\right), \tag{E.27}$$

as in the nonrelativistic limit $(k_2 \cdot q) = (k_4 \cdot q) = 0$. Now we use that $(k_1 \cdot q) = -(k_3 \cdot q) = |\mathbf{q}|^2$, $(k_1 \cdot k_4) = (k_2 \cdot k_3) = (k_2 \cdot k_4) = m_1 m_2$ and $(k_1 \cdot k_3) = m_1 m_2 + |\mathbf{q}|^2$ to write this first part as

$$\boxed{1} = -\frac{\kappa^2}{2}\left((m_1 m_2 + |\mathbf{q}|^2) m_1 m_2 + m_1^2 m_2^2 + \frac{1}{q^2}|\mathbf{q}|^4 m_1 m_2\right) \tag{E.28}$$

$$= -\kappa^2 m_1 m_2 \left(m_1 m_2 + |\mathbf{q}|^2\right). \tag{E.29}$$

Combining the two parts then gives

$$D^{TT} = \kappa^2 \left(\frac{1}{3}(m_1^2 - \frac{1}{2}q^2)(m_2^2 - \frac{1}{2}q^2) - m_1 m_2 (m_1 m_2 + |\mathbf{q}|^2)\right). \tag{E.30}$$

In the limit where the particles have infinite mass we then get

$$D^{TT} = \kappa^2 m_1^2 m_2^2 \left(\frac{1}{3} - 1\right) + \mathcal{O}(q^2) = -\frac{2}{3}\kappa^2 m_1^2 m_2^2 = \frac{4}{3}D, \tag{E.31}$$

giving the relation of the transverse traceless contributions to the full contribution.

To find the potentials, we have to perform the Fourier integrals. Neglecting the prefactor $\frac{-m_1 m_2 16\pi G_k}{3}$, we have

$$V(r) = \int \frac{d^3q}{(2\pi)^3} \mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}}\, \mathcal{G}(q^2) = \int_0^\infty \frac{dq}{(2\pi)^3} q^2 \int_{-1}^1 dx \int_0^{2\pi} d\varphi \mathrm{e}^{iqrx}\, \mathcal{G}(q^2) \tag{E.32}$$

$$= \int_0^\infty \frac{dq}{(2\pi)^2} q^2 \frac{1}{iqr}\left(\mathrm{e}^{iqr} - \mathrm{e}^{-iqr}\right) \mathcal{G}(q^2) = \int_0^\infty \frac{dq}{(2\pi)^2} q^2 \frac{2}{qr} \sin qr\, \mathcal{G}(q^2) \tag{E.33}$$

$$= \frac{1}{2\pi^2 r}\int_0^\infty dq\, q\, \sin qr \mathcal{G}(q^2), \tag{E.34}$$

with $x = \cos\theta$. We can easily check that for the flat-space Einstein-Hilbert propagator $\mathcal{G}_{EH}(q^2) = q^{-2}$ we get the Newtonian potential with the prefactor $\frac{4}{3}$ as calculated above

$$V(r) = \frac{-m_1 m_2 16\pi G_k}{3} \frac{1}{2\pi^2 r}\int_0^\infty dq\, q\, \sin qr\, q^{-2} \tag{E.35}$$

$$= \frac{-m_1 m_2 8 G_k}{3\pi r}\int_0^\infty d(qr) \frac{\sin qr}{qr} \tag{E.36}$$

$$= -\frac{4}{3}\frac{G_k m_1 m_2}{r}. \tag{E.37}$$

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