# Helicity calculation of charmonium production matrix elements with longitudinally polarized gluons and photons

*masters thesis on theoretical high energy physics*

Melvin Meijer
IMAPP, Radboud University Nijmegen

*supervisors:*
prof. dr. R. Kleiss
IMAPP, Radboud University Nijmegen

prof. dr. J. Smith
YITP, Stony Brook University

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This report is a summary of the research professor Jack Smith and I did during my internship at the State University of New York at Stony Brook. We were helped greatly by professor Willy van Neerven from the Leiden University, who unfortunately died February this year.

# 1 Introduction

The production of heavy quarkonium can be described using nonrelativistic quantum chromodynamics (NRQCD), an effective field theory where one not only expands in the coupling constant $\alpha_S$, but also in the typical velocity $v$ of the heavy quark. See Bodwin, Braaten and Lepage [1] for an indepth discussion of NRQCD. This expansion is especially suitable for heavy quarkonia, since bound states made out of two equal mass heavy quarks are automatically nonrelativistic.

A typical way for quarkonium production in particle accelerator experiments is a proton-proton collision. The partons inside a proton cannot be described perturbatively, so one uses the so-called factorization ansatz. This ansatz holds for high transverse momenta and it assumes a separation between the perturbative short-distance matrix elements, and the nonperturbative long-distance matrix elements. In this way, the production of the partons is summarized in one number, the parton density, which depends on the type of parton, the momentum, polarization etcetera. The next step, where the heavy quark pair is produced, can be calculated with NRQCD and after that the quark and antiquark form a quarkonium state, which happens in a nonperturbative way again. Because of the factorization ansatz the last part also reduces to a number, which will depend on the spin, angular momentum and colour configurations. The numerical values of both this number and the parton densities can be determined by experiment or by using lattice QCD.

One of the possible channels to make quarkonium is the fusion of two gluons where the quark pair and a third gluon are created. The third gluon makes it possible for the quark and antiquark to have a similar momentum (without it the quarks would have opposite velocity) and therefore enables the production of quarkonium. An other channel is similar, but with one photon instead of one of the gluons. We will examine these both channels, where the fusing gluons or photons are either polarized, or unpolarized.

In particular we will look at charmonium production, while the same will hold for bottomonium.

In symbolic notation we have the reactions

$$\begin{aligned} 2g &\rightarrow c\bar{c}\left({}^{2S+1}L_J\right) + g \quad \text{and} \\ \gamma + g &\rightarrow c\bar{c}\left({}^{2S+1}L_J\right) + g\,, \end{aligned}$$

where $S$, $L$ and $J$ give the absolute value of the spin, the angular momentum and the sum of both, respectively.

We use the helicity method described in the book by Gastmans and Wu [2] to calculate both longitudinally polarized and unpolarized differential cross sections. To do this, we'll first calculate

$$\begin{aligned} 3g &\to c\bar{c}\left({}^{2S+1}L_J\right) \quad \text{and} \\ \gamma + 2g &\to c\bar{c}\left({}^{2S+1}L_J\right) , \end{aligned}$$

because the gluon symmetries in these reaction are easier to see. Then we will project the charmonium onto the right colour, spin and angular momentum state. At the end of the calculations we'll switch the third gluon from incoming to outgoing again.

A schematic overview of the above can be seen in figure 1.

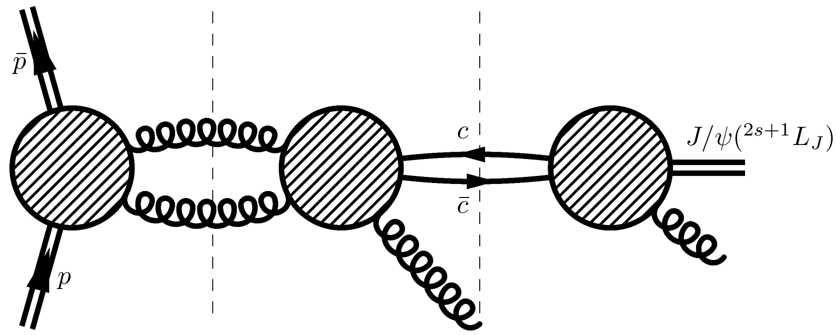



Figure 1: the total process of charmonium production, separated in three sections. The middle part can be calculated using NRQCD. The gluon emitted on the right is a mechanism to make the charmonium colour neutral while the quark pair at the end of the perturbative part may have a net colour.

# 2 Spin projections

This section largely follows a method described in Guberina et al [3].

The amplitudes for the production of charmonium with momentum $p$ in a non-relativistic approximation are given by

$$A({}^{2S+1}L_J) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \sum_{M,S_z} \psi_{LM}(\vec{q}) \mathrm{Tr}[\mathcal{O} P_{SS_z}(p,q)] \langle LM; SS_z | JJ_z \rangle ,$$

where $q$ is the relative momentum of the quarks and $P_{SS_z}(p,q)$ represent the spin projections operators, which are given by

$$\begin{aligned} P_{SS_z}(p,q) &= \frac{1}{\sqrt{m}} \sum_{s,s'} v\left(\frac{1}{2}p - q; s\right) \langle \frac{1}{2}, s; \frac{1}{2}, s' | SS_z \rangle \bar{u}\left(\frac{1}{2}p + q; s'\right) \\ &= \frac{1}{\sqrt{m(E_c+m)(E_{\bar{c}}+m)}} \left(\frac{1}{2}\not{p} - \not{q} - m\right) \frac{\Pi_{SS_z}}{\sqrt{2}} \left(\frac{1}{2}\not{p} + \not{q} + m\right) , \\ \Pi_{SS_z} &= \begin{cases} \gamma_5 & S = 0 \\ \not{\epsilon}(S_z) & S = 1 \end{cases} . \end{aligned}$$

Essentially, $P_{SS_z}(p,q)$ is given by a boost to the $c\bar{c}$ frame, applying the appropriate spin content of the pair and boosting back again.

Simplifying $P_{SS_z}(p,q)$ gives

$$\begin{aligned} P_{SS_z}(p,q) &= \frac{-1}{\sqrt{8m^3}}\left(-\frac{1}{2}\not{p}+\not{q}+m\right)\Pi_{SS_z}\left(\frac{1}{2}\not{p}+\not{q}+m\right) \\ &= \frac{-1}{\sqrt{8m^3}}\bigg[\left(-\frac{1}{2}\not{p}+\not{q}+m\right)\left(-\frac{1}{2}\not{p}-\not{q}+m\right)\Pi_{SS_z} \\ &\quad +\left(-\frac{1}{2}\not{p}+\not{q}+m\right)2\left\{q,\Pi_{SS_z}\right\}\bigg] \\ &= \frac{-1}{\sqrt{8m^3}}\left[\left(2m^2-\not{p}m-\not{q}\,\not{p}\right)\Pi_{SS_z}+(-\not{p}+2m)\left\{q,\Pi_{SS_z}\right\}\right] , \end{aligned}$$

where $\{\dots\}$ is the anti-commutator and where we used that in the non-relativistic approach $E \approx m$, and we only need terms linear in $q$, so $p^2 = M^2 \approx 4m^2$ and $(p \cdot \epsilon(S_z)) = (q \cdot p) = 0$.

We only need terms linear in $q$ so we expand

$$\begin{aligned} \mathcal{O}(q) &= \mathcal{O}(0)+\mathcal{O}^{\alpha}q_{\alpha} \\ P_{SS_z}(p,q) &= P_{SS_z}(p,0)+P^{\alpha}_{SS_z}(p)q_{\alpha} , \end{aligned}$$

which gives us

$$\begin{aligned} A(^{2S+1}S_S) &= \sum_{M,S_z}\mathrm{Tr}\left[\mathcal{O}(0)P_{SS_z}(p,0)\right]\int\frac{\mathrm{d}^3q}{(2\pi)^3}\psi_{00}(0) \\ &\equiv \frac{R_0}{\sqrt{4\pi}}\sum_{M,S_z}\mathrm{Tr}\left[\mathcal{O}(0)P_{SS_z}(p,0)\right] \\ A(^{2S+1}P_J) &= \sum_{M,S_z}\int\frac{\mathrm{d}^3q}{(2\pi)^3}q_{\alpha}\psi_{1M}(\vec{q})\langle 1M;SS_z|JJ_z\rangle \\ &\quad\times\mathrm{Tr}\left[\mathcal{O}^{\alpha}(0)P_{SS_z}(p,0)+\mathcal{O}(0)P^{\alpha}_{SS_z}(p,0)\right] \\ &= -i\epsilon^{\alpha}(M)\sqrt{\frac{3}{4\pi}}R_1'(0)\sum_{M,S_z}\langle 1M;SS_z|JJ_z\rangle \\ &\quad\times\mathrm{Tr}\left[\mathcal{O}^{\alpha}(0)P_{SS_z}(p,0)+\mathcal{O}(0)P^{\alpha}_{SS_z}(p,0)\right] . \end{aligned}$$

The two parameters $R_0$ and $R_1'$ are respectively the S-state wave function and the derivative of the P-state wave function evaluated at the origin. The former is defined in terms of the leptonic decay width

$$R_0^2 = M^2\Gamma(^3S_1 \to e^+e^-)/4\alpha^2Q_f^2 , \tag{1}$$

with $\alpha \approx 1/137$ the fine structure constant and $Q_f$ is the fractional charge of the quarks. $R_1'$ is determined from a fit to the charmonium potential and has the value

$${R_1'}^2/M_\chi^2 \approx 0.006 \quad (\mathrm{GEV})^3 , \tag{2}$$

For $S = 0, 1$ we get

$$\begin{aligned} P_{00}(p,0) &= \frac{1}{\sqrt{4M}}\,(\not{p} - M)\,\gamma_5 \\ P_{00}^{\alpha}(p) &= \frac{1}{\sqrt{M^3}}\gamma^{\alpha}\not{p}\gamma_5 \\ P_{1S_z}(p,0) &= \frac{1}{\sqrt{4M}}\,(\not{p} - M)\,\not{\epsilon}(S_z) \\ P_{1S_z}^{\alpha}(p) &= \frac{1}{\sqrt{M^3}}\left(\gamma^{\alpha}\not{p}\,\not{\epsilon}(S_z) + (\not{p} - M)\,\epsilon^{\alpha}(S_z)\right)\,, \end{aligned}$$

so

$$\begin{aligned} A(^1S_0) &= \frac{R_0}{\sqrt{16\pi M}}\mathrm{Tr}\left[\mathcal{O}(0)\,(\not{p} - M)\,\gamma_5\right] \\ A(^3S_1) &= \frac{R_0}{\sqrt{16\pi M}}\mathrm{Tr}\left[\mathcal{O}(0)\,(\not{p} - M)\,\not{\epsilon}(J_z)\right] \\ A(^1P_1) &= -i\sqrt{\frac{3}{16\pi M}}R_1'(0) \\ &\quad \mathrm{Tr}\left[\epsilon_{\alpha}(J_z)\mathcal{O}^{\alpha}\,(\not{p} - M)\,\gamma_5 + \frac{2}{M}\mathcal{O}(0)\not{\epsilon}(J_z)\not{p}\gamma_5\right] \\ A(^{2S+1}P_J) &= -i\sqrt{\frac{3}{16\pi M}}R_1'(0)\sum_{M,S_z}\epsilon_{\alpha}(M)\langle 1M;SS_z|JJ_z\rangle \\ &\quad \times\mathrm{Tr}\Big[\mathcal{O}^{\alpha}(0)\,(\not{p} - M)\,\not{\epsilon}(S_z) \\ &\qquad +\mathcal{O}(0)\frac{2}{M}\left(\gamma^{\alpha}\not{p}\,\not{\epsilon}(S_z) + (\not{p} - M)\,\epsilon^{\alpha}(S_z)\right)\Big] \end{aligned}$$

and when we use the formulae

$$\begin{aligned} \sum_{M,S_z}\langle 1M;1S_z|00\rangle\epsilon^{\alpha}(M)\epsilon^{\beta}(S_z) &= \frac{1}{\sqrt{3}}\left(g^{\alpha\beta} - \frac{p^{\alpha}p^{\beta}}{M^2}\right)\,, \\ \sum_{M,S_z}\langle 1M;1S_z|1J_z\rangle\epsilon^{\alpha}(M)\epsilon^{\beta}(S_z) &= -i\frac{1}{\sqrt{2}}\epsilon^{\alpha\beta\gamma\delta}\frac{p_{\gamma}}{M}\epsilon_{\delta}(J_z)\;\text{and} \\ \sum_{M,S_z}\langle 1M;1S_z|2J_z\rangle\epsilon^{\alpha}(M)\epsilon^{\beta}(S_z) &= \epsilon^{\alpha\beta}(J_z)\,, \end{aligned}$$

where $\epsilon^{\alpha\beta}$ denotes the polarization tensor for a spin-2 system and $\epsilon^{\alpha\beta\gamma\delta}$ is the Levi-Cevita tensor, we get

$$\begin{aligned} A(^3P_0) &= -\frac{iR_1'}{\sqrt{16\pi M}}\mathrm{Tr}[\mathcal{O}_{\alpha}(\not{p} - M)(\gamma^{\alpha} - \frac{p^{\alpha}\not{p}}{M^2}) \\ &\quad +\frac{2}{M}\mathcal{O}(\gamma^{\alpha}\not{p}\gamma_{\alpha} - \not{p} + 3(\not{p} - M))] \\ &= \frac{iR_1'}{\sqrt{16\pi M}}\mathrm{Tr}[\mathcal{O}_{\alpha}(\gamma^{\alpha} - p^{\alpha}/M)(\not{p} + M) + 6\mathcal{O}] \end{aligned}$$

$$\begin{aligned}
A(^3P_1) &= R_1'\sqrt{\frac{3}{32\pi M^3}}\epsilon^{\alpha\beta\gamma\delta}p_\gamma\epsilon_\delta(J_z) \\
&\quad\times\mathrm{Tr}[\mathcal{O}_\alpha(-\not{p}+M)\gamma_\beta - 2\mathcal{O}\gamma_\alpha\not{p}\gamma_\beta/M] \\
&= R_1'\sqrt{\frac{3}{32\pi M^3}}\epsilon^{\alpha\beta\gamma\delta}p_\gamma\epsilon_\delta(J_z) \\
&\quad\times\mathrm{Tr}[\mathcal{O}_\alpha\gamma_\beta(\not{p}+M) - 2\mathcal{O}\gamma_\alpha\not{p}\gamma_\beta/M] \\
A(^3P_2) &= iR_1'\sqrt{\frac{3}{16\pi M^3}}\epsilon^{\alpha\beta}(J_z)\mathrm{Tr}[\mathcal{O}_\alpha(-\not{p}+M)\gamma_\beta \\
&\quad-\frac{2}{M}\mathcal{O}(\gamma_\alpha\not{p}\gamma_\beta + (\not{p}-M)g_{\alpha\beta})] \\
&= iR_1'\sqrt{\frac{3}{16\pi M^3}}\epsilon^{\alpha\beta}(J_z)\mathrm{Tr}[\mathcal{O}_\alpha\gamma_\beta(\not{p}+M)] \quad ,
\end{aligned}$$

where we dropped the (0) in $R_0(0)$, $R_1'(0)$ and $\mathcal{O}(0)$ for convenience. We used $\epsilon^{\alpha\beta}(J_z) = \epsilon^{\beta\alpha}(J_z)$ and $\epsilon^{\alpha}_{\alpha}(J_z) = p_\alpha\epsilon^{\alpha\beta}(J_z) = 0$ in the above. Notice that there is an overall minus sign in $A(^3P_1)$ compared to Gastmans and Wu [2]. This difference doesn't matter for the end results, since the amplitudes will be squared.

Another property of the polarization tensor for a spin-2 system, which we will use during the calculations, is

$$\sum_{J_z}\epsilon^{\alpha\beta}(J_z)\epsilon^{\gamma\delta}(J_z) = \frac{1}{2}\left(P^{\alpha\gamma}P^{\beta\delta} + P^{\beta\gamma}P^{\alpha\delta}\right) - \frac{1}{3}P^{\alpha\beta}P^{\gamma\delta}$$

with

$$P^{\alpha\beta} = \left(-g^{\alpha\beta} + \frac{p^\alpha p^\beta}{M^2}\right) = \sum_{J_z}\epsilon^\alpha(J_z)\epsilon^\beta(J_z)\,.$$

# 3 The diagrams

Although it usually isn't very hard to see which diagrams contribute, it is of course nice to have a more scientific way.

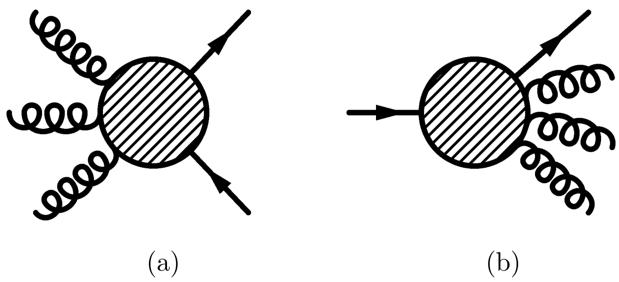
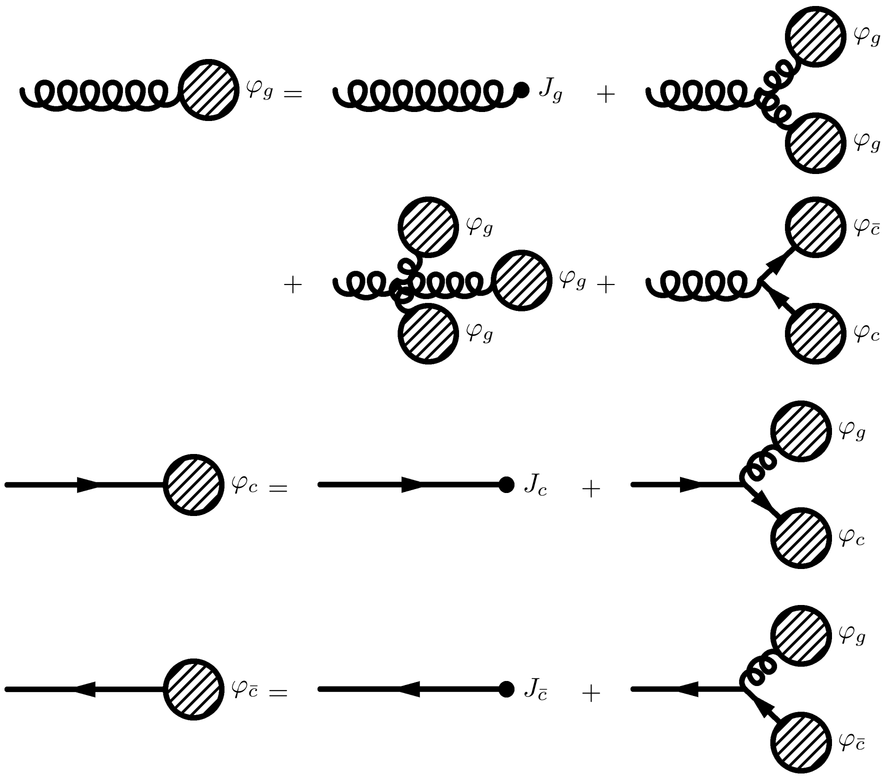



Figure 2: a diagramatic representation of the processes (a) $3g \to c + \bar{c}$ and (b) $c \to c + 3g$.

We would like to know how many and which diagrams contribute to the tree-diagram production of a charm-anticharm pair, starting with three gluons, see figure 2a. This is of course the same as the number and types of diagrams for one charm quark radiating three gluons, see figure 2b.

The diagrammatic equations




are equivalent to the formulae

$$
\begin{aligned}
\varphi_g &= J_g + \frac{g_{3g}}{2!}\varphi_g^2 + \frac{g_{4g}}{3!}\varphi_g^3 + g_{gcc}\varphi_c\varphi_{\bar{c}} \quad , \\
\varphi_c &= J_c + g_{gcc}\varphi_c\varphi_g \quad , \\
\varphi_{\bar{c}} &= J_{\bar{c}} + g_{gcc}\varphi_{\bar{c}}\varphi_g \, ,
\end{aligned}
$$

where $\varphi$ represents the field, $J$ the source and $g$ the coupling.

Because we'd like to know the total number of diagrams, we temporarily set the coupling constants all equal to one: $g_{4g} = g_{3g} = g_{gcc} = 1$.

We want to acquire expressions for the various fields in terms of the sources. We do this iteratively by

$$
\begin{aligned}
\varphi_g^{(i+1)} &= J_g + \frac{1}{2}\left(\varphi_g^{(i)}\right)^2 + \frac{1}{6}\left(\varphi_g^{(i)}\right)^3 + \varphi_c^{(i)}\varphi_{\bar{c}}^{(i)} && (3) \\
\varphi_c^{(i+1)} &= J_c + \varphi_c^{(i)}\varphi_g^{(i)} && (4) \\
\varphi_{\bar{c}}^{(i+1)} &= J_{\bar{c}} + \varphi_{\bar{c}}^{(i)}\varphi_g^{(i)} \, , && (5)
\end{aligned}
$$

where the $^{(i)}$ means the $i^{\text{th}}$ iterative step.

We use as many iterations as necessary, which means $n-1$ for a $n$-particle diagram. When we apply equations (3), (4) and (5) and drop terms higher than order $J^i$, it results in the following steps for the various fields:

$$\begin{aligned}
\varphi_g^{(1)} &= J_g \\
\varphi_c^{(1)} &= J_c \\
\varphi_{\bar{c}}^{(1)} &= J_{\bar{c}} \\
\varphi_g^{(2)} &= J_g + \frac{1}{2}\left(J_g\right)^2 + J_c J_{\bar{c}} \\
\varphi_c^{(2)} &= J_c\left(1 + J_g\right) \\
\varphi_{\bar{c}}^{(2)} &= J_{\bar{c}}\left(1 + J_g\right) \\
\varphi_g^{(3)} &= J_g + \frac{1}{2}\left(J_g\right)^2 + J_c J_{\bar{c}} + \frac{2}{3}\left(J_g\right)^3 + 3 J_g J_c J_{\bar{c}} \\
\varphi_c^{(3)} &= J_c\left(1 + J_g + \frac{3}{2}\left(J_g\right)^2 + J_c J_{\bar{c}}\right) \\
\varphi_c^{(4)} &= J_c\left(1 + J_g + \frac{3}{2}\left(J_g\right)^2 + J_c J_{\bar{c}} + \frac{8}{3}\left(J_g\right)^3 + 5 J_g J_c J_{\bar{c}}\right) .
\end{aligned}$$

So, the number of diagrams with one charm quark radiating three gluons is

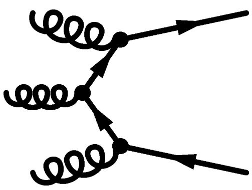
$$\#_{c\to c+3g} = \left(\frac{\partial}{\partial J_g}\right)^3 \frac{\partial}{\partial J_c}\varphi_c^{(4)} = \frac{8}{3}3! = 16 ,$$

which, as mentioned, is the same as the number of diagrams for $3g \to c + \bar{c}$.

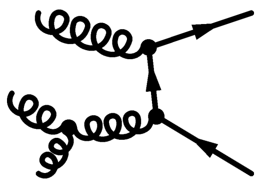
A more detailed analysis[1] gives the number of diagrams for each specific combination of vertices:

$$\begin{aligned}
\#_{3g_{gcc}} &= 6 \\
\#_{2g_{gcc},1g_{3g}} &= 6 \\
\#_{1g_{gcc},2g_{3g}} &= 3 \\
\#_{1g_{gcc},1g_{4g}} &= 1 ,
\end{aligned}$$

so we now know that we need the six permutations of both



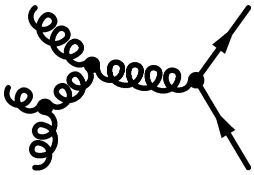
and



,

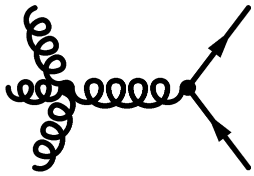
as well as the three permutations of

[1]where $g_{3g}$, $g_{4g}$ and $g_{gcc}$ are not set equal to 1



,

and the single diagram



for the three-gluon creation of the charm-anticharm pair.

# 4 Calculating matrix elements

The outgoing momenta of the charm quark and antiquark are $\frac{1}{2}p+q$ and $\frac{1}{2}p-q$ respectively, so the pair has momentum $p$. This means for the quark mass $m$:

$$\begin{aligned} m^2 &= \left(\frac{1}{2}p+q\right)^2 \\ &= \frac{1}{4}p^2+(p\cdot q)+q^2 \qquad (6) \\ m^2 &= \left(\frac{1}{2}p-q\right)^2 \\ &= \frac{1}{4}p^2-(p\cdot q)+q^2\,. \qquad (7) \end{aligned}$$

When we combine equations (6)+(7) and (6)−(7) we get

$$\begin{aligned} 2m^2 &= \frac{1}{2}p^2+2q^2 \\ &\approx \frac{1}{2}p^2 \text{ and} \\ 0 &= 2(p\cdot q)\,. \end{aligned}$$

where we used the fact that we do a non-relativistic approximation, so we can drop terms of order $q^2$ or higher. This results in the mass of the charmonium pair being approximately two times the quark mass: $M^2 \equiv p^2 \approx 4m^2$, so $M \approx 2m$.

We define the momenta of the incoming gluons to be $k_1$, $k_2$ and $k_3$. Because of overall momentum conservation, they add up to $k_1+k_2+k_3=p$.

We use the following variables in our calculations:

$$s \equiv (k_1+k_2)^2 = 2(k_1\cdot k_2)$$

$$
\begin{aligned}
t &\equiv (k_2+k_3)^2 = 2(k_2\cdot k_3)\\
u &\equiv (k_3+k_1)^2 = 2(k_3\cdot k_1)\\
N_1 &\equiv -2(p\cdot k_1) = -s-u = t-M^2\\
N_2 &\equiv -2(p\cdot k_2) = -t-s = u-M^2\\
N_3 &\equiv -2(p\cdot k_3) = -u-t = s-M^2\,,
\end{aligned}
$$

where $s$, $t$ and $u$ are the normal Mandelstam variables for a 4-particle reaction when we treat the charmonium as one particle. $N_1$, $N_2$ and $N_3$ are chosen such that our denominators will be simple.

## 4.1 Three quark gluon couplings

We call the channels without gluon gluon couplings $d$ diagrams.

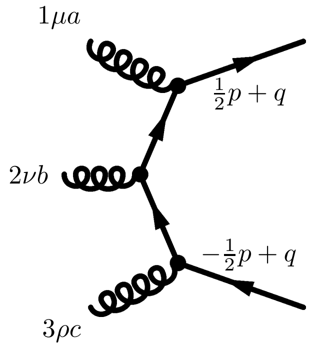



Figure 3: the diagram $d123$, which also shows the number, vector index and colour of the gluons and the momenta of the $c$ quark and antiquark.

The one in figure 3, called $d123$ because the order in which the gluons couple to the quark, has the following matrix element:

$$
\begin{aligned}
M_{d123} &= g^3 T^a_{ij}T^b_{jk}T^c_{kl}\bar{u}\left(\frac{p}{2}+q\right)\not{\epsilon}_1\frac{\frac{1}{2}\not{p}+\not{q}-\not{k}_1+m}{(\frac{1}{2}p+q-k_1)^2-m^2}\not{\epsilon}_2\\
&\quad\times\frac{-\frac{1}{2}\not{p}+\not{q}+\not{k}_3+m}{(-\frac{1}{2}p+q+k_3)^2-m^2}\not{\epsilon}_3 v\left(\frac{p}{2}-q\right)\\
&= g^3 T^a_{ij}T^b_{jk}T^c_{kl}\bar{u}\left(\frac{p}{2}+q\right)\frac{p\cdot\epsilon_1+2q\cdot\epsilon_1+(-\frac{1}{2}\not{p}-\not{q}+m)\not{\epsilon}_1+\not{\epsilon}_1\not{k}_1}{-p\cdot k_1-2q\cdot k_1}\not{\epsilon}_2\\
&\quad\times\frac{-p\cdot\epsilon_3+2q\cdot\epsilon_3+\not{\epsilon}_3(\frac{1}{2}\not{p}-\not{q}+m)-\not{k}_3\not{\epsilon}_3}{-p\cdot k_3+2q\cdot k_3}v\left(\frac{p}{2}-q\right)\\
&= -4g^3 T^a_{ij}T^b_{jk}T^c_{kl}\bar{u}\left(\frac{p}{2}+q\right)\frac{p\cdot\epsilon_1+2q\cdot\epsilon_1+\not{\epsilon}_1\not{k}_1}{N_1-4q\cdot k_1}\not{\epsilon}_2\\
&\quad\times\frac{p\cdot\epsilon_3-2q\cdot\epsilon_3+\not{k}_3\not{\epsilon}_3}{N_3+4q\cdot k_3}v\left(\frac{p}{2}-q\right)\,,
\end{aligned}
$$

where we used $\bar{u}(k)(\not{k}-m)=(\not{k}+m)v(k)=0$.
This results in

$$\begin{aligned}
O_{d123} &= \left(-4g^3T^a_{ij}T^b_{jk}T^c_{kl}\frac{p\cdot\epsilon_1+2q\cdot\epsilon_1+\not{\epsilon}_1\not{k}_1}{N_1-4q\cdot k_1}\not{\epsilon}_2\frac{p\cdot\epsilon_3-2q\cdot\epsilon_3+\not{k}_3\not{\epsilon}_3}{N_3+4q\cdot k_3}\right)_{q=0}\\
&= -4g^3T^a_{ij}T^b_{jk}T^c_{kl}\frac{p\cdot\epsilon_1+\not{\epsilon}_1\not{k}_1}{N_1}\not{\epsilon}_2\frac{p\cdot\epsilon_3+\not{k}_3\not{\epsilon}_3}{N_3} \quad \text{and}\\
O^\beta_{d123q} &= -\left(\frac{\partial}{\partial q_\beta}g^3T^a_{ij}T^b_{jk}T^c_{kl}\frac{p\cdot\epsilon_1+2q\cdot\epsilon_1+\not{\epsilon}_1\not{k}_1}{-p\cdot k_1-2q\cdot k_1}\not{\epsilon}_2\right.\\
&\quad\left.\times\frac{p\cdot\epsilon_3-2q\cdot\epsilon_3+\not{k}_3\not{\epsilon}_3}{-p\cdot k_3+2q\cdot k_3}\right)_{q=0}\\
&= -4g^3T^a_{ij}T^b_{jk}T^c_{kl}\left(2\frac{\epsilon_1^\beta}{N_1}+4k_1^\beta\frac{p\cdot\epsilon_1+\not{\epsilon}_1\not{k}_1}{N_1^2}\right)\not{\epsilon}_2\frac{p\cdot\epsilon_3+\not{k}_3\not{\epsilon}_3}{N_3}\\
&\quad +4g^3T^a_{ij}T^b_{jk}T^c_{kl}\frac{p\cdot\epsilon_1+\not{\epsilon}_1\not{k}_1}{N_1}\not{\epsilon}_2\left(2\frac{\epsilon_3^\beta}{N_3}+4k_3^\beta\frac{p\cdot\epsilon_3+\not{k}_3\not{\epsilon}_3}{N_3^2}\right).
\end{aligned}$$

Similarly, the other five permutations of the gluons give

$$\begin{aligned}
O_{d213} &= -4g^3T^b_{ij}T^a_{jk}T^c_{kl}\frac{p\cdot\epsilon_2+\not{\epsilon}_2\not{k}_2}{N_2}\not{\epsilon}_1\frac{p\cdot\epsilon_3+\not{k}_3\not{\epsilon}_3}{N_3}\\
O^\beta_{d213q} &= -4g^3T^b_{ij}T^a_{jk}T^c_{kl}\left(2\frac{\epsilon_2^\beta}{N_2}+4k_2^\beta\frac{p\cdot\epsilon_2+\not{\epsilon}_2\not{k}_2}{N_2^2}\right)\not{\epsilon}_1\frac{p\cdot\epsilon_3+\not{k}_3\not{\epsilon}_3}{N_3}\\
&\quad +4g^3T^b_{ij}T^a_{jk}T^c_{kl}\frac{p\cdot\epsilon_2+\not{\epsilon}_2\not{k}_2}{N_2}\not{\epsilon}_1\left(2\frac{\epsilon_3^\beta}{N_3}+4k_3^\beta\frac{p\cdot\epsilon_3+\not{k}_3\not{\epsilon}_3}{N_3^2}\right)\\
O_{d132} &= -4g^3T^a_{ij}T^c_{jk}T^b_{kl}\frac{p\cdot\epsilon_1+\not{\epsilon}_1\not{k}_1}{N_1}\not{\epsilon}_3\frac{p\cdot\epsilon_2+\not{k}_2\not{\epsilon}_2}{N_2}\\
O^\beta_{d132q} &= -4g^3T^a_{ij}T^c_{jk}T^b_{kl}\left(2\frac{\epsilon_1^\beta}{N_1}+4k_1^\beta\frac{p\cdot\epsilon_1+\not{\epsilon}_1\not{k}_1}{N_1^2}\right)\not{\epsilon}_3\frac{p\cdot\epsilon_2+\not{k}_2\not{\epsilon}_2}{N_2}\\
&\quad +4g^3T^a_{ij}T^c_{jk}T^b_{kl}\frac{p\cdot\epsilon_1+\not{\epsilon}_1\not{k}_1}{N_1}\not{\epsilon}_3\left(2\frac{\epsilon_2^\beta}{N_2}+4k_2^\beta\frac{p\cdot\epsilon_2+\not{k}_2\not{\epsilon}_2}{N_2^2}\right)\\
O_{d321} &= -4g^3T^c_{ij}T^b_{jk}T^a_{kl}\frac{p\cdot\epsilon_3+\not{\epsilon}_3\not{k}_3}{N_3}\not{\epsilon}_2\frac{p\cdot\epsilon_1+\not{k}_1\not{\epsilon}_1}{N_1}\\
O^\beta_{d321q} &= -4g^3T^c_{ij}T^b_{jk}T^a_{kl}\left(2\frac{\epsilon_3^\beta}{N_3}+4k_3^\beta\frac{p\cdot\epsilon_3+\not{\epsilon}_3\not{k}_3}{N_3^2}\right)\not{\epsilon}_2\frac{p\cdot\epsilon_1+\not{k}_1\not{\epsilon}_1}{N_1}\\
&\quad +4g^3T^c_{ij}T^b_{jk}T^a_{kl}\frac{p\cdot\epsilon_3+\not{\epsilon}_3\not{k}_3}{N_3}\not{\epsilon}_2\left(2\frac{\epsilon_1^\beta}{N_1}+4k_1^\beta\frac{p\cdot\epsilon_1+\not{k}_1\not{\epsilon}_1}{N_1^2}\right)\\
O_{d231} &= -4g^3T^b_{ij}T^c_{jk}T^a_{kl}\frac{p\cdot\epsilon_2+\not{\epsilon}_2\not{k}_2}{N_2}\not{\epsilon}_3\frac{p\cdot\epsilon_1+\not{k}_1\not{\epsilon}_1}{N_1}\\
O^\beta_{d231q} &= -4g^3T^b_{ij}T^c_{jk}T^a_{kl}\left(2\frac{\epsilon_2^\beta}{N_2}+4k_2^\beta\frac{p\cdot\epsilon_2+\not{\epsilon}_2\not{k}_2}{N_2^2}\right)\not{\epsilon}_3\frac{p\cdot\epsilon_1+\not{k}_1\not{\epsilon}_1}{N_1}
\end{aligned}$$

$$
\begin{aligned}
& +4g^3 T^b_{ij}T^c_{jk}T^a_{kl}\frac{p\cdot\epsilon_2+\not{\epsilon}_2\not{k}_2}{N_2}\not{\epsilon}_3\left(2\frac{\epsilon_1^\beta}{N_1}+4k_1^\beta\frac{p\cdot\epsilon_1+\not{k}_1\not{\epsilon}_1}{N_1^2}\right)\\
O_{d312} &= -4g^3 T^c_{ij}T^a_{jk}T^b_{kl}\frac{p\cdot\epsilon_3+\not{\epsilon}_3\not{k}_3}{N_3}\not{\epsilon}_1\frac{p\cdot\epsilon_2+\not{k}_2\not{\epsilon}_2}{N_2}\\
O^\beta_{d312q} &= -4g^3 T^c_{ij}T^a_{jk}T^b_{kl}\left(2\frac{\epsilon_3^\beta}{N_3}+4k_3^\beta\frac{p\cdot\epsilon_3+\not{\epsilon}_3\not{k}_3}{N_3^2}\right)\not{\epsilon}_1\frac{p\cdot\epsilon_2+\not{k}_2\not{\epsilon}_2}{N_2}\\
& +4g^3 T^c_{ij}T^a_{jk}T^b_{kl}\frac{p\cdot\epsilon_3+\not{\epsilon}_3\not{k}_3}{N_3}\not{\epsilon}_1\left(2\frac{\epsilon_2^\beta}{N_2}+4k_2^\beta\frac{p\cdot\epsilon_2+\not{k}_2\not{\epsilon}_2}{N_2^2}\right) .
\end{aligned}
$$

## 4.2 Two quark gluon couplings

The channels with two quark gluon couplings can be grouped in $s, t$ and $u$ diagrams, named after the Mandelstam variables in the denominater of the gluon propagator.

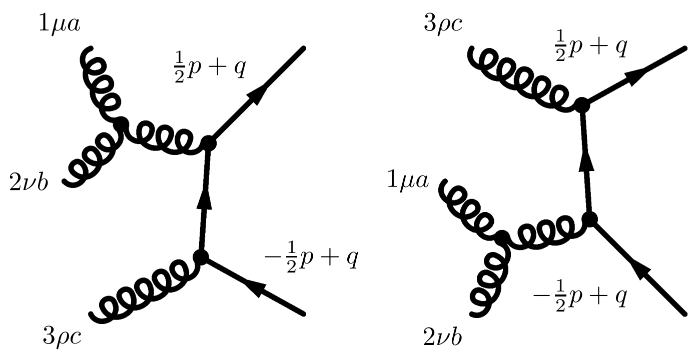



Figure 4: the $s$ diagrams. The number, vector index and colour of the gluons and the momenta of the $c$ quark and antiquark are also shown.

The diagrams in figure 4 together form the $s$ diagrams. This gives

$$
\begin{aligned}
M_s &= ig^3\frac{f^{abe}\epsilon_1^\mu\epsilon_2^\nu}{(k_1+k_2)^2}\left[(k_1-k_2)^\alpha g_{\mu\nu}+(2k_2+k_1)_\mu g^\alpha{}_\nu+(-2k_1-k_2)_\nu g^\alpha{}_\mu\right]\\
&\times\bar{u}\left(\frac{p}{2}+q\right)\left(T^e_{ij}T^c_{jl}\gamma_\alpha\frac{-\frac{1}{2}\not{p}+\not{q}+\not{k}_3+m}{(-\frac{1}{2}p+q+k_3)^2-m^2}\not{\epsilon}_3\right.\\
&\left.+T^c_{ij}T^e_{jl}\not{\epsilon}_3\frac{\frac{1}{2}\not{p}+\not{q}-\not{k}_3+m}{(\frac{1}{2}p+q-k_3)^2-m^2}\gamma_\alpha\right)v\left(\frac{p}{2}-q\right)\\
&= ig^3 f^{abe}\left[(k_1-k_2)^\alpha(\epsilon_1\cdot\epsilon_2)+2(k_2\cdot\epsilon_1)\epsilon_2^\alpha-2(k_1\cdot\epsilon_2)\epsilon_1^\alpha\right]\frac{1}{s}\\
&\times\bar{u}\left(\frac{p}{2}+q\right)\left(T^e_{ij}T^c_{jl}\gamma_\alpha\frac{-\not{p}+2\not{q}+2\not{k}_3+2m}{N_3+4q\cdot k_3}\not{\epsilon}_3\right.\\
&\left.+T^c_{ij}T^e_{jl}\not{\epsilon}_3\frac{\not{p}+2\not{q}-2\not{k}_3+2m}{N_3-4q\cdot k_3}\gamma_\alpha\right)v\left(\frac{p}{2}-q\right) ,
\end{aligned}
$$

which results in

$$
\begin{aligned}
O_s &= ig^3 f^{abe}\left[(k_1-k_2)^\alpha(\epsilon_1\cdot\epsilon_2)+2(k_2\cdot\epsilon_1)\epsilon_2^\alpha-2(k_1\cdot\epsilon_2)\epsilon_1^\alpha\right]\frac{1}{sN_3}\\
&\quad\times\left(T^e_{ij}T^c_{jl}\gamma_\alpha(-\not{p}+2\not{k}_3+2m)\not{\epsilon}_3+T^c_{ij}T^e_{jl}\not{\epsilon}_3(\not{p}-2\not{k}_3+2m)\gamma_\alpha\right)\\
O^\beta_{sq} &= 2ig^3 f^{abe}\left[(k_1-k_2)^\alpha(\epsilon_1\cdot\epsilon_2)+2(k_2\cdot\epsilon_1)\epsilon_2^\alpha-2(k_1\cdot\epsilon_2)\epsilon_1^\alpha\right]\frac{1}{sN_3}\\
&\quad\times\left(T^e_{ij}T^c_{jl}\gamma_\alpha\gamma^\beta\not{\epsilon}_3+2T^e_{ij}T^c_{jl}k_3^\beta\gamma_\alpha\frac{\not{p}-2\not{k}_3-2m}{N_3}\not{\epsilon}_3\right.\\
&\quad\left.+T^c_{ij}T^e_{jl}\not{\epsilon}_3\gamma^\beta\gamma_\alpha+2T^c_{ij}T^e_{jl}k_3^\beta\not{\epsilon}_3\frac{\not{p}-2\not{k}_3+2m}{N_3}\gamma_\alpha\right).
\end{aligned}
$$

The results for the $t$ and $u$ channels are

$$
\begin{aligned}
O_t &= ig^3 f^{bce}\left[(k_2-k_3)^\alpha(\epsilon_2\cdot\epsilon_3)+2(k_3\cdot\epsilon_2)\epsilon_3^\alpha-2(k_2\cdot\epsilon_3)\epsilon_2^\alpha\right]\frac{1}{tN_1}\\
&\quad\times\left(T^e_{ij}T^a_{jl}\gamma_\alpha(-\not{p}+2\not{k}_1+2m)\not{\epsilon}_1+T^a_{ij}T^e_{jl}\not{\epsilon}_1(\not{p}-2\not{k}_1+2m)\gamma_\alpha\right)\\
O^\beta_{tq} &= 2ig^3 f^{bce}\left[(k_2-k_3)^\alpha(\epsilon_2\cdot\epsilon_3)+2(k_3\cdot\epsilon_2)\epsilon_3^\alpha-2(k_2\cdot\epsilon_3)\epsilon_2^\alpha\right]\frac{1}{tN_1}\\
&\quad\times\left(T^e_{ij}T^a_{jl}\gamma_\alpha\gamma^\beta\not{\epsilon}_1+2T^e_{ij}T^a_{jl}k_1^\beta\gamma_\alpha\frac{\not{p}-2\not{k}_1-2m}{N_1}\not{\epsilon}_1\right.\\
&\quad\left.+T^a_{ij}T^e_{jl}\not{\epsilon}_1\gamma^\beta\gamma_\alpha+2T^a_{ij}T^e_{jl}k_1^\beta\not{\epsilon}_1\frac{\not{p}-2\not{k}_1+2m}{N_1}\gamma_\alpha\right)\\
O_u &= ig^3 f^{cae}\left[(k_3-k_1)^\alpha(\epsilon_3\cdot\epsilon_1)+2(k_1\cdot\epsilon_3)\epsilon_1^\alpha-2(k_3\cdot\epsilon_1)\epsilon_3^\alpha\right]\frac{1}{uN_2}\\
&\quad\times\left(T^e_{ij}T^b_{jl}\gamma_\alpha(-\not{p}+2\not{k}_2+2m)\not{\epsilon}_2+T^b_{ij}T^e_{jl}\not{\epsilon}_2(\not{p}-2\not{k}_2+2m)\gamma_\alpha\right)\\
O^\beta_{uq} &= 2ig^3 f^{cae}\left[(k_3-k_1)^\alpha(\epsilon_3\cdot\epsilon_1)+2(k_1\cdot\epsilon_3)\epsilon_1^\alpha-2(k_3\cdot\epsilon_1)\epsilon_3^\alpha\right]\frac{1}{uN_2}\\
&\quad\times\left(T^e_{ij}T^b_{jl}\gamma_\alpha\gamma^\beta\not{\epsilon}_2+2T^e_{ij}T^b_{jl}k_2^\beta\gamma_\alpha\frac{\not{p}-2\not{k}_2-2m}{N_2}\not{\epsilon}_2\right.\\
&\quad\left.+T^b_{ij}T^e_{jl}\not{\epsilon}_2\gamma^\beta\gamma_\alpha+2T^b_{ij}T^e_{jl}k_2^\beta\not{\epsilon}_2\frac{\not{p}-2\not{k}_2+2m}{N_2}\gamma_\alpha\right).
\end{aligned}
$$

## 4.3 Two triple gluon vertices

We call the three diagrams with two triple gluon diagrams 1, 2 and 3. They are named after the special gluon of the three.

The diagram in figure 5, called diagram 1, gives

$$
M_1 = -g^3 f^{bcf} f^{fae} T^e_{il}\frac{\epsilon_1^\mu\epsilon_2^\nu\epsilon_3^\rho}{(k_2+k_3)^2p^2}\bar{u}\left(\frac{p}{2}+q\right)\gamma_\sigma v\left(\frac{p}{2}-q\right)
$$

$$
\begin{aligned}
& \times \left[(k_2-k_3)^\alpha g_{\nu\rho} + (2k_3+k_2)_\nu g^\alpha_{\ \rho} + (-2k_2-k_3)_\rho g^\alpha_{\ \nu}\right] \\
& \times \left[(k_2+k_3-k_1)^\sigma g_{\alpha\mu} + (k_1+p)_\alpha g^\sigma_{\ \mu} + (-p-k_2-k_3)_\mu g^\sigma_{\ \alpha}\right] \\
= \;& \frac{-g^3}{tM^2} f^{bcf} f^{fae} T^e_{il} \gamma_\sigma \left[(k_2-k_3)^\alpha (\epsilon_2\cdot\epsilon_3) + 2(k_3\cdot\epsilon_2)\epsilon_3^\alpha - 2(k_2\cdot\epsilon_3)\epsilon_2^\alpha\right] \\
& \times \bar{u}\left(\frac{p}{2}+q\right)\left[(\not{p}-2\not{k}_1)\epsilon_{1\alpha} + (k_1+p)_\alpha \not{\epsilon}_1 - 2(p\cdot\epsilon_1)\gamma_\alpha\right] v\left(\frac{p}{2}-q\right) ,
\end{aligned}
$$

which results in

$$
\begin{aligned}
O_1 &= -g^3 f^{bcf} f^{fae} T^e_{il} \gamma_\sigma \left[(k_2-k_3)^\alpha (\epsilon_2\cdot\epsilon_3) + 2(k_3\cdot\epsilon_2)\epsilon_3^\alpha - 2(k_2\cdot\epsilon_3)\epsilon_2^\alpha\right] \\
&\quad \times \frac{1}{tM^2}\left[(\not{p}-2\not{k}_1)\epsilon_{1\alpha} + (k_1+p)_\alpha \not{\epsilon}_1 - 2(p\cdot\epsilon_1)\gamma_\alpha\right] \\
O^\beta_{1q} &= 0 .
\end{aligned}
$$

For diagrams 2 and 3 we get

$$
\begin{aligned}
O_2 &= -g^3 f^{caf} f^{fbe} T^e_{il} \gamma_\sigma \left[(k_3-k_1)^\alpha (\epsilon_3\cdot\epsilon_1) + 2(k_1\cdot\epsilon_3)\epsilon_1^\alpha - 2(k_3\cdot\epsilon_1)\epsilon_3^\alpha\right] \\
&\quad \times \frac{1}{uM^2}\left[(\not{p}-2\not{k}_2)\epsilon_{2\alpha} + (k_2+p)_\alpha \not{\epsilon}_2 - 2(p\cdot\epsilon_2)\gamma_\alpha\right] \\
O_3 &= -g^3 f^{abf} f^{fce} T^e_{il} \gamma_\sigma \left[(k_1-k_2)^\alpha (\epsilon_1\cdot\epsilon_2) + 2(k_2\cdot\epsilon_1)\epsilon_2^\alpha - 2(k_1\cdot\epsilon_2)\epsilon_1^\alpha\right] \\
&\quad \times \frac{1}{sM^2}\left[(\not{p}-2\not{k}_3)\epsilon_{3\alpha} + (k_3+p)_\alpha \not{\epsilon}_3 - 2(p\cdot\epsilon_3)\gamma_\alpha\right] \\
O^\beta_{2q} &= O^\beta_{3q} = 0 .
\end{aligned}
$$

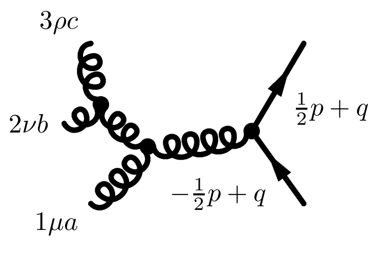



Figure 5: diagram 1. The number, vector index and colour of the gluons and the momenta of the $c$ quark and antiquark are also shown.

## 4.4 Four gluon vertex

The diagram in figure 6 is called diagram 4. It gives

$$
M_4 = -g^3 \epsilon_1^\mu \epsilon_2^\nu \epsilon_3^\rho \frac{T^e_{il}}{p^2} \bar{u}\left(\frac{p}{2}+q\right)\gamma^\sigma v\left(\frac{p}{2}-q\right)
$$

$$
\begin{aligned}
& \times\left[f^{abf}f^{fce}(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho})\right. \\
& +f^{acf}f^{fbe}(g_{\mu\nu}g_{\rho\sigma}-g_{\mu\sigma}g_{\nu\rho}) \\
& \left.+f^{cbf}f^{fae}(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\nu}g_{\rho\sigma})\right] \\
= \quad & -g^3\frac{T^e_{il}}{M^2}\bar{u}\left(\frac{p}{2}+q\right)\left[f^{abf}f^{fce}(\epsilon_1\cdot\epsilon_3\not{\epsilon}_2-\not{\epsilon}_1\epsilon_2\cdot\epsilon_3)\right. \\
& +f^{acf}f^{fbe}(\epsilon_1\cdot\epsilon_2\not{\epsilon}_3-\not{\epsilon}_1\epsilon_2\cdot\epsilon_3) \\
& \left.+f^{cbf}f^{fae}(\epsilon_1\cdot\epsilon_3\not{\epsilon}_2-\epsilon_1\cdot\epsilon_2\not{\epsilon}_3)\right]v\left(\frac{p}{2}-q\right)\,,
\end{aligned}
$$

so

$$
\begin{aligned}
O_4 \quad = \quad & -g^3\frac{T^e_{il}}{M^2}\left[f^{abf}f^{fce}(\epsilon_1\cdot\epsilon_3\not{\epsilon}_2-\not{\epsilon}_1\epsilon_2\cdot\epsilon_3)\right. \\
& +f^{acf}f^{fbe}(\epsilon_1\cdot\epsilon_2\not{\epsilon}_3-\not{\epsilon}_1\epsilon_2\cdot\epsilon_3) \\
& \left.+f^{cbf}f^{fae}(\epsilon_1\cdot\epsilon_3\not{\epsilon}_2-\epsilon_1\cdot\epsilon_2\not{\epsilon}_3)\right] \\
O^{\beta}_{4q} \quad = \quad & 0\,.
\end{aligned}
$$

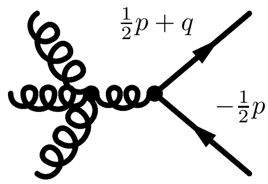



Figure 6: diagram 4. The number, vector index and colour of the gluons and the momenta of the $c$ quark and antiquark are also shown.

### 4.5 One photon, two gluons

We also want to calculate the process where two gluons and one photon produce the charmonium. Once the appropriate equations for the $3g \to c\bar{c}\left({}^{2S+1}L_J\right)$ are known, turning one of the gluons into a photon just means setting the colour matrix of that gluon equal to unity. We turn the first gluon $g(k_1,\mu,a)$ into a photon, which means $T^a_{ij}=\delta_{ij}$ and $f^{ade}=0\ \forall d,e$.

## 5 Helicities

This section is based on chapter 3 in Gastmans and Wu, [2].

We should now choose the helicities of the gluons explicitly. For one, because we'd also like to know the polarized cross section, but taking physical polarizations also ensures us we don't need to include ghost terms in our calculation.

Gluons and photons are massless and have therefore $D-2=2$ polarization states. A convenient choice for the polarization vectors will be

$$\begin{aligned}\epsilon^{\parallel}_{\mu}(k) &= \frac{(q\cdot k)p_\mu-(p\cdot k)q_\mu}{\sqrt{2(q\cdot k)(q\cdot p)(p\cdot k)}}\\ \epsilon^{\perp}_{\mu}(k) &= \frac{\epsilon_{\mu\alpha\beta\gamma}q^\alpha p^\beta k^\gamma}{\sqrt{2(q\cdot k)(q\cdot p)(p\cdot k)}}\,,\end{aligned}$$

for a gluon or photon with momentum $k$. Here $q$ and $p$ are two arbitrary vectors, which we can take to obey $p^2=q^2=0$, so

$$\begin{aligned}(\epsilon^{\parallel})^2 = {}& (\epsilon^{\perp})^2 &&= -1\,,\\ (k\cdot\epsilon^{\parallel}) = {}& (k\cdot\epsilon^{\perp}) &&= (\epsilon^{\parallel}\cdot\epsilon^{\perp}) = 0\,.\end{aligned}$$

These states can be combined into two circularly polarized states

$$\begin{aligned}\not\epsilon^{\pm} &= \gamma^\mu\frac{1}{\sqrt{2}}(\epsilon^{\parallel}_\mu \pm i\epsilon^{\perp}_\mu)\\ &= \frac{(q\cdot k)\not p-(p\cdot k)\not q \pm i\gamma^\mu\epsilon_{\mu\alpha\beta\gamma}q^\alpha p^\beta k^\gamma}{2\sqrt{(q\cdot k)(q\cdot p)(p\cdot k)}}\\ &= \frac{(q\cdot k)\not p-(p\cdot k)\not q \pm (\gamma_\alpha\gamma_\beta\gamma_\gamma-\gamma_\alpha g_{\beta\gamma}+\gamma_\beta g_{\alpha\gamma}-\gamma_\gamma g_{\alpha\beta})\gamma_5 q^\alpha p^\beta k^\gamma}{2\sqrt{(q\cdot k)(q\cdot p)(p\cdot k)}}\\ &= \frac{(q\cdot k)\not p-(p\cdot k)\not q \pm (\not q\not p\not k-\not q(p\cdot k)+\not p(q\cdot k)-\not k(q\cdot p))\gamma_5}{2\sqrt{(q\cdot k)(q\cdot p)(p\cdot k)}}\\ &= -\frac{\not k\not p\not q\,(1\pm\gamma_5)+\not q\not p\not k\,(1\mp\gamma_5)-2(p\cdot q)\not k}{4\sqrt{(q\cdot k)(q\cdot p)(p\cdot k)}}\\ &= -N\left(\not k\not p\not q(1\pm\gamma_5)+\not q\not p\not k(1\mp\gamma_5)-2(p\cdot q)\not k\right)\,,\end{aligned}$$

with $N=\frac{1}{4}[(q\cdot k)(q\cdot p)(p\cdot k)]^{-\frac{1}{2}}$.

For photons it's often convenient to write

$$\not\epsilon^{\pm} = -N\left(\not k\not p\not q(1\pm\gamma_5)-\not p\not q\not k(1\mp\gamma_5)\mp 2(p\cdot q)\not k\gamma_5\right)\,.$$

For the process $3g\to c\bar{c}\left({}^{2S+1}L_J\right)$ we choose the helicities

$$\begin{aligned}\not\epsilon^{+}_1 &= N[\not k_1\not k_2\not k_3(1-\gamma_5)-\not k_3\not k_2\not k_1(1+\gamma_5)]\\ \not\epsilon^{+}_2 &= N[\not k_2\not k_3\not k_1(1-\gamma_5)+\not k_1\not k_3\not k_2(1+\gamma_5)]\\ \not\epsilon^{+}_3 &= N[\not k_3\not k_1\not k_2(1-\gamma_5)+\not k_2\not k_1\not k_3(1+\gamma_5)]\end{aligned}$$

when all the helicities are equal and

$$\begin{aligned}\not\epsilon^{+}_1 &= N[\not k_1\not k_2\not k_3(1-\gamma_5)-\not k_3\not k_2\not k_1(1+\gamma_5)]\\ \not\epsilon^{+}_2 &= N[\not k_2\not k_1\not k_3(1-\gamma_5)+\not k_3\not k_1\not k_2(1+\gamma_5)]\\ \not\epsilon^{-}_3 &= N[\not k_3\not k_1\not k_2(1+\gamma_5)+\not k_2\not k_1\not k_3(1-\gamma_5)]\end{aligned}$$

when one of the three has an opposite sign. Because of the symmetry of interchanging the gluons and because of CP symmetry, we can make all eight helicity combinations out of these two. We dropped the term proportional to $\not{k}$ because QCD satisfies vector current conservation.

For the process $2g + \gamma \to c\bar{c}\left({}^{2S+1}L_J\right)$ we choose the helicities

$$
\begin{aligned}
\not{\epsilon}_1^{\pm} &= N[\not{k}_1\not{k}_2\not{k}_3(1 \mp \gamma_5) - \not{k}_2\not{k}_3\not{k}_1(1 \pm \gamma_5) \pm 2(k_2 \cdot k_3)\not{k}_1\gamma_5] \\
\not{\epsilon}_2^{\pm} &= N[\not{k}_3\not{k}_1\not{k}_2(1 \pm \gamma_5) + \not{k}_2\not{k}_1\not{k}_3(1 \mp \gamma_5) - 2(k_1 \cdot k_3\not{k}_2)] \\
\not{\epsilon}_3^{\pm} &= N[\not{k}_1\not{k}_2\not{k}_3(1 \pm \gamma_5) + \not{k}_3\not{k}_2\not{k}_1(1 \mp \gamma_5) - 2(k_1 \cdot k_2\not{k}_3)] \,,
\end{aligned}
$$

which will provide us with all eight helicity combinations. Because symmetry under CP conjugation still holds, we're only required to use four of them: $(+,+,+)$, $(+,+,-)$, $(+,-,+)$ and $(-,+,+)$. Actually either the second or third is redundant because one could be obtained from the other by interchanging gluons 2 and 3. However, we choose to ignore this and list all four helicity combinations.

# 6 Colour projection

The amplitudes depend on the colour configuration of the charmonium pair. When we choose a colour singlet configuration, denoted by the symbol $^{(1)}$, we should add the projection factor $\delta_{ij}/\sqrt{3}$, where $i$ and $j$ are the remaining colour indices of the matrix element. The appropriate normalization $1/\sqrt{3}$ makes sure squaring the projection and summing over the colours gives unity.

The colour octet projection is proportional to $T^a_{ij}$. Because $\mathrm{Tr}(T^aT^a) = 1/2$, the normalizing factor $\sqrt{2}$ should be added.

An other way to see that this holds, is by realizing the colour projections can not mix. This means that taking no colour projection at all and taking the absolute value squared of the amplitudes, is the same as taking the absolute value squared of the singlet and octet configuration separately and adding them. In a formula this means that for two matrix elements, $M_1$ and $M_2$, the relation

$$
\begin{aligned}
\mathrm{Tr}(M_1M_2) &= M_{1ij}\frac{\delta_{ji}}{\sqrt{3}}M_{2kl}\frac{\delta_{lk}}{\sqrt{3}} + \sum_a M_{1ij}\sqrt{2}T^a_{ji}M_{2kl}\sqrt{2}T^a_{lk} \\
&= \frac{1}{3}\mathrm{Tr}(M_1)\mathrm{Tr}(M_2) + 2\sum_a \mathrm{Tr}(M_1T^a)\mathrm{Tr}(M_2T^a)
\end{aligned}
$$

should hold. This is exactly 2 times equation (11) from appendix A, so it is indeed correct. Colour octet configuations are denoted by the symbol $^{(8)}$.

Note that naively, one would require the colour of the charm and anticharm to be the same (colour singlet), so they can form a colour neutral charmonium. Because the part where the quark and antiquark form quarkonium happens nonperturbatively, however, soft gluons (i.e. gluons with very low energy in the centre of mass frame) can be emitted very easily. In this way the pair can loose it's colour content which makes the colour octet configurations, where the colour of the quark and the antiquark are different, very important.

# 7 Matrix elements squared

We squared the matrix elements using the algebraic computer program Form. The codes we wrote can be found in appendix B.

We'll list the matrix elements squared after they have been summed over the colour states of the gluons and the polarization of the charmonium. We also mention possible differences between our results and those of Gastmans and Wu [2], which we refer to as GW.

The results are devided into 4 sections:

$$\begin{array}{lrcl} 7.1: & 3g & \to & c\bar{c}\left({}^{2S+1}L_J^{(1)}\right) \\ 7.2: & 3g & \to & c\bar{c}\left({}^{2S+1}L_J^{(8)}\right) \\ 7.3: & \gamma + 2g & \to & c\bar{c}\left({}^{2S+1}L_J^{(1)}\right) \\ 7.4: & \gamma + 2g & \to & c\bar{c}\left({}^{2S+1}L_J^{(8)}\right) . \end{array}$$

As has been said in section 5, $|M(+,+,+)|^2$ and $|M(+,+,-)|^2$ contain all the information. The other elements can be found by interchanging the gluons, so

$$\begin{aligned} |M(+,-,+)|^2 &= |M(+,+,-)|^2 \big|_{s\leftrightarrow u} \\ |M(-,+,+)|^2 &= |M(+,+,-)|^2 \big|_{s\leftrightarrow t} , \end{aligned}$$

and by CP conjugation, switching $+ \leftrightarrow -$.

To make the expressions simpler, we used the variables $M = \sqrt{s+t+u}$, $P = st + tu + us$ and $Q = stu$. See appendix C for more details how this change was done.

## 7.1 Three gluons colour singlet

${}^1S_0^{(1)}$ Here we find

$$\begin{aligned} |M(+,+,+)|^2 &= \frac{16g^6R_0^2}{\pi M}\frac{M^8P^2}{Q(Q-M^2P)^2} \\ |M(+,+,-)|^2 &= \frac{16g^6R_0^2}{\pi M}\frac{s^4P^2}{Q(Q-M^2P)^2} . \end{aligned}$$

These results agree with the squares of (8.29) and (8.40) in GW.

${}^3S_1^{(1)}$ Here we find

$$\begin{aligned} |M(+,+,+)|^2 &= 0 \\ |M(+,+,-)|^2 &= \frac{160g^6R_0^2}{9\pi M}\frac{M^2s^2(s-M^2)^2}{(Q-M^2P)^2} . \end{aligned}$$

The second result agrees with (8.50) in GW after correcting an obvious typo that the $(t-M^2)$ should read $(t-M^2)^2$.

${}^1P_1^{(1)}$ Here we agree with the results (8.55) and (8.57) in GW.

$$|M(+,+,+)|^2 = \frac{640g^6R'^2}{3\pi M^3}\frac{M^{10}(-M^2P+5Q)}{(Q-M^2P)^3}$$

$$|M(+,+,-)|^2 = \frac{640g^6R_1'^2}{3\pi M^3}\frac{M^2(M^4s^2(Q-M^2P)+2Qs^2(s^2+M^4))}{(Q-M^2P)^3}.$$

${}^3P_0^{(1)}$ Here we agree with the results in (8.59) in GW so that

$$\begin{aligned}|M(+,+,+)|^2 &= \frac{64gR_1'^2}{\pi M^3}\frac{9M^8P^2(Q-M^2P)^2}{Q(Q-M^2P)^4}\\ |M(+,+,-)|^2 &= \frac{64gR_1'^2}{\pi M^3}\frac{(s-M^2)^2}{Q(Q-M^2P)^4}\\ &\quad\times\Big[Q^2-s^2Q(s-3M^2)+3PM^2s^3\Big]^2.\end{aligned}$$

${}^3P_1^{(1)}$ We find

$$\begin{aligned}|M(+,+,+)|^2 &= 0\\ |M(+,+,-)|^2 &= \frac{96g^6R_1'^2}{\pi M^3}\frac{(s-M^2)^2s^2}{(Q-M^2P)^4}\\ &\quad\times\Big[Q(20M^4P-4M^8+4P^2+4(s-M^2)^4\\ &\qquad-16(P-sM^2)(s-M^2)^2-20s^2(s-M^2)^2)\\ &\quad-Q^2(30M^2-16s)-8M^2P^3+2M^6P^2\Big],\end{aligned}$$

which agrees with (8.63) in GW.

${}^3P_2^{(1)}$ We find

$$\begin{aligned}|M(+,+,+)|^2 &= 0\\ |M(+,+,-)|^2 &= \frac{64g^6R_1'^2}{\pi M^3}\frac{1}{Q(Q-M^2P)^4}\\ &\quad\times\Big[12M^8P^4(3s-M^2)(s-M^2)-12M^4P^5s(s-3M^2)\\ &\quad-3M^6P^3Q(s-M^2)(25s-8M^2)\\ &\quad+12M^2P^4Q(s^2-4M^2s-3M^4)\\ &\quad+M^4P^2Q^2(8s^2+9M^2s-15M^4)\\ &\quad-2P^3Q^2(s^2-5M^2s-30M^4)\\ &\quad+M^2PQ^3(29s^2-51M^2s+18M^4)\\ &\quad+2P^2Q^3(s-11M^2)-M^2Q^4(9s-11M^2)\Big],\end{aligned}$$

which agrees with (8.70) in GW.

## 7.2 Three gluons colour octet

Now we present the corresponding results for the colour octet projections. These results do not seem to be available in the literature. We have only found expressions for the differential cross sections with which we will compare later on. We give these results since we need the differences between the helicity combinations to check the longitudinally polarized differential cross sections. The constants from the wave functions are now simply renamed as $R \to \langle R\,[^1S_0^{(8)}]\,\rangle$ etcetera, since there are other definitions in the literature. We will present the relations between the definitions later on.

$^1S_0^{(8)}$ Here we find

$$
\begin{aligned}
|M(+,+,+)|^2 &= \frac{40g^6\langle R\,[^1S_0^{(8)}]\,\rangle}{\pi M}\frac{M^8(P^2-M^2Q)}{Q(Q-M^2P)^2}\\
|M(+,+,-)|^2 &= \frac{40g^6\langle R\,[^1S_0^{(8)}]\,\rangle}{\pi M}\frac{s^3}{Q(Q-M^2P)^2}\Big[PQ+s^3(s-M^2)^2-s^2Q\Big]\,.
\end{aligned}
$$

$^3S_1^{(8)}$ Here we find

$$
\begin{aligned}
|M(+,+,+)|^2 &= 0\\
|M(+,+,-)|^2 &= \frac{16g^6\langle R\,[^3S_1^{(8)}]\,\rangle}{3\pi M}\frac{(19M^4-27P)s^2(s-M^2)^2}{M^2(Q-M^2P)^2}\,.
\end{aligned}
$$

$^1P_1^{(8)}$ Here we find

$$
\begin{aligned}
|M(+,+,+)|^2 &= \frac{32g^6\langle R\,[^1P_1^{(8)}]\,\rangle}{\pi M^3}\frac{M^6}{Q(Q-M^2P)^3}\\
&\quad\times\Big[-54PQ^2+217M^4Q^2+43M^6PQ\\
&\quad-27M^2P^2Q-27M^4P^3\Big]\\
|M(+,+,-)|^2 &= \frac{32g^6\langle R\,[^1P_1^{(8)}]\,\rangle}{\pi M^3}\frac{s^3}{Q(Q-M^2P)^3}\\
&\quad\Big[+Q(t+u)(-38u^4-82tu^3-169t^2u^2-82t^3u-38t^4)\\
&\quad+Qs(-98u^4-278tu^3-468t^2u^2-278t^3u-98t^4)\\
&\quad+Qs^2(t+u)(-174u^2+26tu-174t^2)\\
&\quad+s^4(-152tu^3+10t^2u^2-152t^3u-27t^4-27u^4)\\
&\quad+s^5(t+u)(-27u^2-11tu-27t^2)\\
&\quad+t^2u^2(t+u)(-38u^3-60tu^2-60t^2u-38t^3)\Big]\,.
\end{aligned}
$$

$^3P_0^{(8)}$ Here we find

$$
\begin{aligned}
|M(+,+,+)|^2 &= \frac{160g^6\langle R\,[^3P_0^{(8)}]\,\rangle}{\pi M^3}\frac{9M^8(P^2-M^2Q)(Q-M^2P)^2}{Q(Q-M^2P)^4} \\
|M(+,+,-)|^2 &= \frac{160g^6\langle R\,[^3P_0^{(8)}]\,\rangle}{\pi M^3}\frac{(s-M^2)^2}{Q(Q-M^2P)^4}\Big[9s^8M^4(s-M^2)^2 \\
&\quad +Qs^5(-6M^8+33sM^6-42s^2M^4+6s^3M^2) \\
&\quad +Q^2s^2(44s^2M^4+4M^8-18s^3M^2+s^4) \\
&\quad +Q^3s(-2(s-M^2)^2+9sM^2)+Q^4\Big]\,.
\end{aligned}
$$

$^3P_1^{(8)}$ Here we find

$$
\begin{aligned}
|M(+,+,+)|^2 &= 0 \\
|M(+,+,-)|^2 &= \frac{960g^6\langle R\,[^3P_1^{(8)}]\,\rangle}{\pi M^3}\frac{(s-M^2)^2(P+s^2-sM^2)}{Q(Q-M^2P)^4} \\
&\quad \times\Big[Q^3(s-2M^2)+Q^2s(M^6+7s^2M^2-2s^3) \\
&\quad +Qs^3(M^8-4sM^6+11s^2M^4-10s^3M^2+s^4) \\
&\quad +s^5M^2(M^8-4sM^6+7s^2M^4-6s^3M^2+2s^4)\Big]\,.
\end{aligned}
$$

$^3P_2^{(8)}$

$$
\begin{aligned}
|M(+,+,+)|^2 &= 0 \\
|M(+,+,-)|^2 &= \frac{320g^6\langle R\,[^3P_2^{(8)}]\,\rangle}{\pi M^3}\frac{(s-M^2)^2}{Qs^4(Q-M^2P)^4} \\
&\quad \times\Big[6s^8M^4(s-M^2)^6 \\
&\quad +Qs^6M^2(18M^{12}-114sM^{10}+285s^2M^8-354s^3M^6 \\
&\qquad +225s^4M^4-66s^5M^2+6s^6) \\
&\quad +Q^2s^4(24M^{12}-132sM^{10}+313s^2M^8-336s^3M^6 \\
&\qquad +161s^4M^4-30s^5M^2+s^6) \\
&\quad +Q^3s^2(18M^{10}-78sM^8+141s^2M^6-110s^3M^4 \\
&\qquad +25s^4M^2-2s^5) \\
&\quad +Q^4(s^4-6M^2(s-M^2)^3+6sM^4(s-M^2))\Big]\,.
\end{aligned}
$$

### 7.3 One photon two gluons colour singlet

Now we present the corresponding results for the colour singlet matrix elements squared for the reaction $\gamma + 2g \to c\bar{c}\left({}^{2S+1}L_J^{(1)}\right)$. These results do not seem to be available in the literature. We have only found expressions for the differential cross sections which we will not list because the expressions don't really simplify after adding or subtracting the matrix elements for the helicity combinations. We did compare the differential cross sections with results in Klasen, Kniehl, Mihaila and Steinhauser [4], which we refer to as KKMS, as well as those in Yuan, Dong, Hao and Chao [5], which we refer to as YDHC, and in Ko, Lee and Soy [6], which we refer to as KLS. We calculated the (un)polarized differential cross sections for this reaction using a method similar to the one described in section 8.

${}^1S_0^{(1)}$ Here we find

$$|M(+,+,+)|^2 = |M(+,+,-)|^2 = |M(+,-,+)|^2 = |M(-,+,+)|^2 = 0$$

${}^3S_1^{(1)}$ Here we find

$$\begin{aligned}
|M(+,+,+)|^2 &= 0 \\
|M(+,+,-)|^2 &= \frac{128 g^4 e^2 \langle R\,[{}^3S_1^{(1)}]\,\rangle}{3\pi M}\frac{M^2 s^2 (t+u)^2}{(Q-M^2P)^2} \\
|M(+,-,+)|^2 &= \frac{128 g^4 e^2 \langle R\,[{}^3S_1^{(1)}]\,\rangle}{3\pi M}\frac{M^2 u^2 (s+t)^2}{(Q-M^2P)^2} \\
|M(-,+,+)|^2 &= \frac{128 g^4 e^2 \langle R\,[{}^3S_1^{(1)}]\,\rangle}{3\pi M}\frac{M^2 t^2 (u+s)^2}{(Q-M^2P)^2}
\end{aligned}$$

When we calculate the differential cross section for

$$\gamma(k_1) + g(k_2, b) \to q(p/2+q) + \bar{q}(p/2-q) + g(k_3, b)\,, \tag{8}$$

it agrees with (A.3) in KKMS if we make the replacement $\langle R\,[{}^3S_1^{(0)}]\,\rangle = 16\pi\langle O\,[{}^3S_1^{(0)}]\,\rangle/3$.

${}^1P_1^{(1)}$ Here we find

$$\begin{aligned}
|M(+,+,+)|^2 &= \frac{1024 g_1^4 e^2 \langle R\,[{}^1P_1^{(1)}]\,\rangle}{\pi M^3}\frac{M^{10}(5Q-M^2P)}{(Q-M^2P)^3} \\
|M(+,+,-)|^2 &= \frac{1024 g_1^4 e^2 \langle R\,[{}^1P_1^{(1)}]\,\rangle}{\pi M^3}\frac{M^2 s^2}{(Q-M^2P)^3} \\
&\quad\times[5M^4Q - 4PQ - M^6P - 2Q(t^2+u^2)] \\
|M(+,-,+)|^2 &= \frac{1024 g_1^4 e^2 \langle R\,[{}^1P_1^{(1)}]\,\rangle}{\pi M^3}\frac{M^2 u^2}{(Q-M^2P)^3} \\
&\quad\times[5M^4Q - 4PQ - M^6P - 2Q(s^2+t^2)]
\end{aligned}$$

$$|M(-,+,+)|^2 = \frac{1024 g_1^4 e^2 \langle R\,[^1P_1^{(1)}]\,\rangle}{\pi M^3} \frac{M^2 t^2}{(Q-M^2P)^3} \times [5M^4Q - 4PQ - M^6P - 2Q(s^2+u^2)]\,.$$

When we calculate the differential cross sections they agree with the unpolarized and polarized results in (A.4) in KKMS if we make the replacement $\langle R\,[^1P_1^{(0)}]\,\rangle = 8\pi \langle O\,[^1P_1^{(0)}]\,\rangle/9$.

$^3P_0^{(1)}$, $^3P_1^{(1)}$ and $^3P_2^{(1)}$ Here we find

$$|M(+,+,+)|^2 = |M(+,+,-)|^2 = |M(+,-,+)|^2 = |M(-,+,+)|^2 = 0\,.$$

## 7.4 One photon two gluons colour octet

$^1S_0^{(8)}$ Here we find

$$|M(+,+,+)|^2 = \frac{96 g^4 e^2 \langle R\,[^1S_0^{(8)}]\,\rangle}{\pi M} \frac{usM^8}{t(Q-M^2P)^2}$$

$$|M(+,+,-)|^2 = \frac{96 g^4 e^2 \langle R\,[^1S_0^{(8)}]\,\rangle}{\pi M} \frac{uss^4}{t(Q-M^2P)^2}$$

$$|M(+,-,+)|^2 = \frac{96 g^4 e^2 \langle R\,[^1S_0^{(8)}]\,\rangle}{\pi M} \frac{usu^4}{t(Q-M^2P)^2}$$

$$|M(-,+,+)|^2 = \frac{96 g^4 e^2 \langle R\,[^1S_0^{(8)}]\,\rangle}{\pi M} \frac{ust^4}{t(Q-M^2P)^2}\,.$$

When we calculate the unpolarized differential cross section, which contains the sum of these expressions, it does not agree with the first terms in (A.5) in KKMS. However the polarized differential cross section involves the difference, which agrees with the second terms in (A.5) in KKMS, if we make the replacement $\langle R\,[^1S_0^{(8)}]\,\rangle = 2\pi \langle O\,[^1S_0^{(8)}]\,\rangle$. Both the sum and the difference agrees with (A1) and (A2) in YDHC. The sum agrees with (A1) in KLS.

$^3S_1^{(8)}$ Here we find

$$|M(+,+,+)|^2 = 0$$

$$|M(+,+,-)|^2 = \frac{320 g^4 e^2 \langle R\,[^3S_1^{(8)}]\,\rangle}{3\pi M} \frac{s^2(t+u)^3 + s^3(t+u)^2}{(Q-M^2P)^2}$$

$$|M(+,-,+)|^2 = \frac{320 g^4 e^2 \langle R\,[^3S_1^{(8)}]\,\rangle}{3\pi M} \frac{u^2(t+s)^3 + u^3(t+s)^2}{(Q-M^2P)^2}$$

$$|M(-,+,+)|^2 = \frac{320 g^4 e^2 \langle R\,[^3S_1^{(8)}]\,\rangle}{3\pi M} \frac{t^2(s+u)^3 + t^3(s+u)^2}{(Q-M^2P)^2}\,.$$

The sum and the difference both agree with (A.6) in KKMS and with (A4) and (A5) in YDHC. The sum agrees with (A2) in KLS if we make the replacement $\langle R\,[^3S_1^{(8)}]\,\rangle = 2\pi \langle O\,[^3S_1^{(8)}]\,\rangle/3$.

${}^1P_1^{(8)}$ Here we find

$$
\begin{aligned}
|M(+,+,+)|^2 &= \frac{2560 g^4 e^2 \langle R\,[{}^1P_1^{(8)}]\,\rangle}{\pi M^3} \frac{M^{10}(5Q - M^2 P)}{(Q-M^2P)^3} \\
|M(+,+,-)|^2 &= \frac{2560 g^4 e^2 \langle R\,[{}^1P_1^{(8)}]\,\rangle}{\pi M^3} \\
&\quad \times \frac{M^2 s^2 (5M^4 Q - 4PQ - M^6 P - 2Q(t^2+u^2))}{(Q-M^2P)^3} \\
|M(+,-,+)|^2 &= \frac{2560 g^4 e^2 \langle R\,[{}^1P_1^{(8)}]\,\rangle}{\pi M^3} \frac{M^2 u^2 (3M^4 Q - M^6 P + 2Qu^2)}{(Q-M^2P)^3} \\
|M(-,+,+)|^2 &= \frac{2560 g^4 e^2 \langle R\,[{}^1P_1^{(8)}]\,\rangle}{\pi M^3} \frac{M^2 t^2 (3M^4 Q - M^6 P + 2Qt^2)}{(Q-M^2P)^3} \,.
\end{aligned}
$$

The sum and the difference both agree with (A.7) in KKMS if we make the replacement $\langle R\,[{}^1P_1^{(8)}]\,\rangle = \pi\langle O\,[{}^1P_1^{(8)}]\,\rangle/9$.

${}^3P_0^{(8)}$ Here we find

$$
\begin{aligned}
|M(+,+,+)|^2 &= \frac{128 g^4 e^2 \langle R\,[{}^3P_0^{(8)}]\,\rangle}{\pi M^3} \frac{9usM^8(M^2-s)^2}{t(Q-M^2P)^4} \\
&\quad \times \Big[ t^2u^2 + 2Q(M^2-s) \\
&\quad + s^2(M^2-s)^2 + 2sQ + 2s^3(M^2-s) + s^4 \Big] \\
|M(+,+,-)|^2 &= \frac{128 g^4 e^2 \langle R\,[{}^3P_0^{(8)}]\,\rangle}{\pi M^3} \frac{uss^2(M^2-s)^2}{t(Q-M^2P)^4} \Big[ 4t^2u^2(M^2-s)^2 \\
&\quad + 4Qtu(M^2-s) - 12sQ(M^2-s)^2 - 18s^2Q(M^2-s) \\
&\quad + Q^2 + 9s^4(M^2-s)^2 - 6s^3Q + 18s^5(M^2-s) + 9s^6 \Big] \\
|M(+,-,+)|^2 &= \frac{128 g^4 e^2 \langle R\,[{}^3P_0^{(8)}]\,\rangle}{\pi M^3} \frac{usu^2(M^2-u)^2}{t(Q-M^2P)^4} \Big[ 4t^2s^2(M^2-u)^2 \\
&\quad + 4Qts(M^2-u) - 12uQ(M^2-u)^2 - 18u^2Q(M^2-u) \\
&\quad + Q^2 + 9u^4(M^2-u)^2 - 6u^3Q + 18u^5(M^2-u) + 9u^6 \Big] \\
|M(-,+,+)|^2 &= \frac{128 g^4 e^2 \langle R\,[{}^3P_0^{(8)}]\,\rangle}{\pi M^3} \frac{ust^4(M^2-t)^2}{t(Q-M^2P)^4} \Big[ 4(M^2-t)^4 \\
&\quad + 4su(M^2-t)^2 + s^2u^2 + 28t(M^2-t)^3 + 14Q(M^2-t) \\
&\quad + 69t^2(M^2-t)^2 + 10tQ + 70t^3(M^2-t) + 25t^4 \Big] \,.
\end{aligned}
$$

The difference agrees with (A.8) in KKMS, if we make the replacement $\langle R\,[{}^3P_0^{(8)}]\,\rangle = 2\pi\langle O\,[{}^3P_0^{(8)}]\,\rangle$. The sum and the difference agree with (A6) and (A7) in YDHC once a typo is corrected; the last term in these equations should have been $(t+u)^{-2}$ instead of $(t+s)^{-2}$. The sum agrees with (A2) in KLS.

${}^3P_1^{(8)}$ Here we find that the answers are not shorter when expressed in terms of the variables $M^2$, $P$ and $Q$, so we give them in terms of $s$, $t$ and $u$.

$$
\begin{aligned}
|M(+,+,+)|^2 &= 0 \\
|M(+,+,-)|^2 &= \frac{1152 g^4 e^2 \langle R\,[{}^3P_1^{(8)}]\,\rangle}{\pi M^3} \frac{s^2(M^2-s)^2}{(Q-M^2P)^4} \\
&\quad \times \Big[ s(9t^2u^4 + 20t^3u^3 + 13t^4u^2 + 2t^5u) \\
&\quad + s^2(tu^4 + 26t^2u^3 + 38t^3u^2 + 13t^4u + t^5 + u^5) \\
&\quad + s^3(6tu^3 + 34t^2u^2 + 22t^3u + 3t^4 - u^4) \\
&\quad + s^4(9tu^2 + 13t^2u + 3t^3 - u^3) + s^5(t+u)^2 + t^2u^2(t+u)^3 \Big] \\
|M(+,-,+)|^2 &= \frac{1152 g1^4 e^2 \langle R\,[{}^3P_1^{(8)}]\,\rangle}{\pi M^3} \frac{u^2(M^2-u)^2}{(Q-M^2P)^4} \\
&\quad \times \Big[ u(2st^5 + 13s^2t^4 + 20s^3t^3 + 9s^4t^2) \\
&\quad + u^2(13st^4 + 38s^2t^3 + 26s^3t^2 + s^4t + s^5 + t^5) \\
&\quad + u^3(22st^3 + 34s^2t^2 + 6s^3t - s^4 + 3t^4) \\
&\quad + u^4(13st^2 + 9s^2t - s^3 + 3t^3) + u^5(s+t)^2 + s^2t^2(t+s)^3 \Big] \\
|M(-,+,+)|^2 &= \frac{1152 g^4 e^2 \langle R\,[{}^3P_1^{(8)}]\,\rangle}{\pi M^3} \frac{t^2(M^2-t)^2}{(Q-M^2P)^4} \\
&\quad \times \Big[ ts^2u^2(5(u+s)^2 - 2su) \\
&\quad + t^2((u+s)^5 + 8s^2u^2(u+s)) \\
&\quad + t^3(3(u+s)^4 - 2su(s^2+u^2)) \\
&\quad + t^4(3(u+s)^3 - 4su(u+s)) \\
&\quad + t^5(s^2+u^2) + s^2u^2(u+s)^3 - 4s^3u^3(u+s) \Big] ,
\end{aligned}
$$

and only the difference agrees with (A.9) in KKMS, if we make the replacement $\langle R\,[{}^3P_1^{(8)}]\,\rangle = \pi \langle O\,[{}^3P_1^{(8)}]\,\rangle/4$. The sum and the difference agree with (A8) and (A9) in YDHC, respectively. The sum also agrees with (A3) in KLS once the factor $(s^2-u^2)^2$ is replaced by $(s^2-u^2)^4$.

${}^3P_2^{(8)}$ Here we find

$$
\begin{aligned}
|M(+,+,+)|^2 &= 0 \\
|M(+,+,-)|^2 &= \frac{48 g^4 e^2 \langle R\,[{}^3P_2^{(8)}]\,\rangle}{\pi M^3} \frac{s^2u(M^2-s)^2}{Q(Q-M^2P)^4} \\
&\quad \times \Big[ +(12t^2u^7 + 48t^3u^6 + 72t^4u^5 + 48t^5u^4 + 12t^6u^3) \\
&\quad + s(24tu^7 + 96t^2u^6 + 171t^3u^5 + 177t^4u^4 \\
&\qquad + 105t^5u^3 + 27t^6u^2) \\
&\quad + s^2(72tu^6 + 140t^2u^5 + 187t^3u^4 + 200t^4u^3
\end{aligned}
$$

$$
\begin{aligned}
& +111t^5u^2+18t^6u+12u^7) \\
& +s^3(51tu^5+59t^2u^4+134t^3u^3+162t^4u^2 \\
& \quad +63t^5u+3t^6+24u^6) \\
& +s^4(-3tu^4+26t^2u^3+102t^3u^2+78t^4u \\
& \quad +9t^5+12u^5) \\
& +s^5(-3tu^3+27t^2u^2+39t^3u+9t^4) \\
& +s^6(3tu^2+6t^2u+3t^3))\Big]
\end{aligned}
$$

$$
\begin{aligned}
|M(+,-,+)|^2 \quad = \quad & \frac{48g^4e^2\langle R\,[^3P_2^{(8)}]\,\rangle}{\pi M^3}\frac{su^2(M^2-u)^2}{Q(Q-M^2P)^4} \\
& \times\Big[+(12s^3t^6+48s^4t^5+72s^5t^4+48s^6t^3+12s^7t^2) \\
& +u(27*s^2t^6+105s^3t^5+177s^4t^4+171s^5t^3 \\
& \quad +96s^6t^2+24s^7t) \\
& +u^2(18st^6+111s^2t^5+200s^3t^4+187s^4t^3 \\
& \quad +140s^5t^2+72s^6t+12s^7) \\
& +u^3(3t^6+63st^5+162s^2t^4+134s^3t^3 \\
& \quad +59s^4t^2+51s^5t+24s^6) \\
& +u^4(9t^5+78st^4+102s^2t^3+26s^3t^2 \\
& \quad -3s^4t+12s^5) \\
& +u^5(9t^4+39st^3+27s^2t^2-3s^3t) \\
& +u^6(3t^3+6st^2+3s^2t))\Big]
\end{aligned}
$$

$$
\begin{aligned}
|M(-,+,+)|^2 \quad = \quad & \frac{48g^4e^2\langle R\,[^3P_2^{(8)}]\,\rangle}{\pi M^3}\frac{st(M^2-t)^2}{Q(Q-M^2P)^4} \\
& \times\Big[+t(24s^2u^7+120s^3u^6+240s^4u^5+240s^5u^4 \\
& \quad +120s^6u^3+24s^7u^2) \\
& +t^2(12su^7+120s^2u^6+348s^3u^5+480s^4u^4 \\
& \quad +348s^5u^3+120s^6u^2+12s^7u) \\
& +t^3(48su^6+267s^2u^5+549s^3u^4+549s^4u^3 \\
& \quad +267s^5u^2+48s^6u) \\
& +t^4(92su^5+367s^2u^4+552s^3u^3+367s^4u^2 \\
& \quad +92s^5u) \\
& +t^5(3u^5+119su^4+334s^2u^3+334s^3u^2 \\
& \quad +119s^4u+3s^5) \\
& +t^6(9u^4+102su^3+182s^2u^2+102s^3u+9s^4) \\
& +t^7(9u^3+47su^2+47s^2u+9s^3) \\
& +t^8(3u^2+8su+3s^2) \\
& +12s^3u^7+48s^4u^6+72s^5u^5+48s^6u^4+12s^7u^3)\Big],
\end{aligned}
$$

and only the difference of the above results agrees with (A.10) in KKMS if we make the replacement $\langle R\,[{}^3P_2^{(8)}]\,\rangle = 16\pi\langle O\,[{}^3P_2^{(8)}]\,\rangle/15$. The difference also agrees with (A10) and (A11) in YDHC. The sum from our results does not agree agree with the answer in (A4) in KLS as well as with (A10) in YDHC.

# 8 Differential cross sections

First we'll change the third gluon from incoming to outgoing. This only involves changing $k_3$ into $-k_3$ and flipping the helicity of the third gluon. When we change the Mandelstam variables into $s = (k_1 + k_2)^2$, $t = (k_2 - k_3)^2$ and $u = (k_1 - k_3)^2$ we simply get

$$\begin{aligned}
|M(+,+;+)|^2 &= |M(+,+,-)|^2 \\
|M(+,+;-)|^2 &= |M(+,+,+)|^2 \\
|M(+,-;-)|^2 &= |M(+,-,+)|^2 = |M(+,+,-)|^2\Big|_{s\leftrightarrow u} \\
|M(-,+;-)|^2 &= |M(-,+,+)|^2 = |M(+,+,-)|^2\Big|_{s\leftrightarrow t} \quad ,
\end{aligned}$$

where the semicolon indicates that the first two helicities are for the incoming gluons and the last sign gives the helicity of the outgoing gluon.

The unpolarized differential cross sections, $\mathrm{d}\sigma/\mathrm{d}t$, now follow from the sum of the squares of the helicity matrix elements

$$|M(+,+;+)|^2 + |M(+,+;-)|^2 + |M(+,-;-)|^2 + |M(-,+;-)|^2\,, \tag{9}$$

while the polarized differential cross sections, $\mathrm{d}\Delta\sigma/\mathrm{d}t$ follow from

$$|M(+,+;+)|^2 + |M(+,+;-)|^2 + |M(+,-;-)|^2 + |M(-,+;-)|^2\,. \tag{10}$$

Then one should add the average over colours and spins and multiply by an overall factor of $1/(16\pi^2 s^2)$. We also substituted $g^2 = 4\pi\alpha_S$.

We devide our results in 4 sections, with either colour singlet or colour octet and either unpolarized or polarized differential cross sections.

The results can be compared with the results in GW and KKMS. The latter authors give differential cross sections as functions of polarization factors $\xi_a\xi_b$. The unpolarized cross sections are obtained by setting $\xi_a\xi_b = 0$. We call these the first terms and the coefficients of $\xi_a\xi_b$, which yield the longitudinally polarized differential cross sections, the the second terms. Note that due to the differences in the definitions of the wave functions, our comments concern the polynomial dependence of $a(s,t,u)$ and $b(s,t,u)$ on the invariants. We will also identify the prefactors.

## 8.1 Unpolarized three gluons colour singlet

${}^1S_0^{(1)}$ Here we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\pi\alpha_s^3 R_0^2}{Ms^2}\frac{P^2}{Q(Q-M^2P)^2}\Big[(P-M^4)^2 + 2M^2Q\Big]\,,$$

which agrees with (8.46) in GW. However, it does not agree with the first term in (A.16) in KKMS who use the notation where $R_0^2 = 4\pi\langle O[^1S_0^{(1)}]\rangle$.

$^3S_1^{(1)}$ Here we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{10\pi\alpha_s^3 R_0^2}{9Ms^2}\frac{M^2(P^2 - M^2Q)}{(Q - M^2P)^2},$$

which agrees with (8.52) in GW and also agrees with the first terms in (A.17) in KKMS who use the notation where $R_0^2 = 4\pi\langle O[^3S_1^{(1)}]\rangle/3$.

$^1P_1^{(1)}$ Here we find

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}t} &= \frac{40\pi\alpha_s^3 R_1'^2}{3M^3s^2}\frac{M^2}{(Q - M^2P)^3} \\ &\quad \times\Big[ - M^{10}P + M^6P^2 + Q(5M^8 - 7M^4P + 2P^2) + 4M^2Q^2\Big],\end{aligned}$$

which agrees with (8.58) in GW. It also agrees with the first terms in (A.18) in KKMS who use the notation where $R_1'^2 = 4\pi\langle O[^1P_1^{(1)}]\rangle/9$.

$^3P_0^{(1)}$ Here we find

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}t} &= \frac{4\pi\alpha_s^3 R_1'^2}{M^3s^2}\frac{1}{Q(Q - M^2P)^4} \\ &\quad \times\Big[P^2(3PM^2 - Q)^2(P - M^4)^2 + 6M^6P^3Q(3P - M^4) \\ &\quad -2M^8P^2Q^2 - 2M^2PQ^3(P - M^4) + 6M^4Q^4\Big],\end{aligned}$$

which agrees with (8.60) in GW. It does not agree with the first terms in (A.19) in KKMS who use the notation where $R_1'^2 = 4\pi\langle O[^3P_0^{(1)}]\rangle/3$.

$^3P_1^{(1)}$ Here we find

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}t} &= \frac{12\pi\alpha_s^3 R_1'^2}{M^3s^2}\frac{P^2}{(Q - M^2P)^4} \\ &\quad \times\Big[M^2P^2(M^4 - 4P) - 2Q(M^8 - 5M^4P - P^2) - 15M^2Q^2\Big],\end{aligned}$$

which agrees with (8.64) in GW. It does not agree with the first terms in (A.20) in KKMS who use the notation where $R_1'^2 = 4\pi\langle O[^3P_1^{(1)}]\rangle/9$.

${}^3P_2^{(1)}$ Here we find

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}t} &= \frac{4\pi\alpha_s^3 R_1'^2}{M^3 s^2}\frac{1}{Q(Q-M^2P)^4} \\ &\times\Big[12M^4P^4(P-M^4)^2 - 3M^2P^3Q(8M^8 - M^4P + 4P^2) \\ &-2P^2Q^2(7M^8 - 43M^4P - P^2) + M^2PQ^3(16M^4 - 61P) + 12M^4Q^4\Big],\end{aligned}$$

which agrees with (8.71) in GW after correcting a typo. They have $(8M^8 - M^4P + P^2)$ which should read $(8M^8 - M^4P + 4P^2)$. The expression is given correctly in their published paper [7]. Also it does not agree with the first terms in (A.21) in KKMSS who use the notation where $R_1'^2 = 4\pi\langle O[{}^3P_2^{(1)}]\rangle/15$.

## 8.2 Unpolarized three gluons colour octet

These can be compared with the results for the squares of the matrix elements in the appendix of Cho and Leibovich [8], which we refer to as CL and with the first parts of the expressions in Appendix A of KKMS. The relations between the squares of the wave functions are identical to those given in the previous section. We begin with

${}^1S_0^{(8)}$ Here we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{5\pi\alpha_s^3\langle R\,[{}^1S_0^{(8)}]\,\rangle}{2Ms^2}\frac{P^2 - M^2Q}{Q(Q-M^2P)^2}\Big[(P-M^4)^2 + 2M^2Q\Big],$$

which agrees with (A5a) in CL. It does not agree with the first part of (A.22) in KKMS who use the notation where $R_0^2 = 4\pi\langle O[{}^1S_0^{(1)}]\rangle$.

${}^3S_1^{(8)}$ Next we give

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\pi\alpha_s^3\langle R\,[{}^3S_1^{(8)}]\,\rangle}{3Ms^2}\frac{(P^2-M^2Q)(19M^4 - 27P)}{M^2(Q-M^2P)^2},$$

which agrees with the sum of (A5b) plus (A5c) in CL. It does not agree with the first terms in (A.23) in KKMS who use the notation where $R_0^2 = 4\pi\langle O[{}^1S_0^{(1)}]\rangle$.

${}^1P_1^{(8)}$ The expression

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}t} &= \frac{2\pi\alpha_s^3\langle R\,[{}^1P_1^{(8)}]\,\rangle}{M^3s^2}\frac{1}{Q(Q-M^2P)^3}\Big[179M^4Q^3 + 217M^{10}Q^2 - 27M^2P^5 \\ &+54M^6P^4 - 27M^{10}P^3 + 135PQ^3 + 103M^2P^2Q^2 + 27P^4Q \\ &-212M^6PQ^2 - 124M^8P^2Q + 43M^{12}PQ\Big]\end{aligned}$$

is not given in CL. It does not agree with the first terms in (A.24) in KKMS who use the notation where $R_0^2 = 4\pi\langle O[^1S_0^{(1)}]\rangle$.

${}^3P_0^{(8)}$ Now we turn to

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{10\pi\alpha_s^3\langle R\,[^3P_0^{(8)}]\,\rangle}{M^3s^2}\frac{1}{Q(Q-M^2P)^4}\Big[9M^4P^4(P-M^4)^2 - 9M^{14}P^2Q$$
$$+3M^{10}P^3Q + 27M^6P^4Q - 6M^2P^5Q + 18M^{12}PQ^2 - 32M^8P^2Q^2$$
$$-4M^4P^3Q^2 + P^4Q^2 - 13M^{10}Q^3 + 11M^6PQ^3 - M^2P^2Q^3 + 5M^4Q^4\Big]\,,$$

which agrees with (A5d) in CL. It does not agree with the first terms in (A.25) in KKMS who use the notation where $R_0^2 = 4\pi\langle O[^1S_0^{(1)}]\rangle$.

${}^3P_1^{(8)}$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{60\pi\alpha_s^3\langle R\,[^3P_1^{(8)}]\,\rangle}{M^3s^2}\frac{1}{(Q-M^2P)^4}$$
$$\times\Big[M^6P^4 - 2M^2P^5 - 2M^8P^2Q + 7M^4P^3Q$$
$$+P^4Q + M^{10}Q^2 - 3M^6PQ^2 - 9M^2P^2Q^2 + 6M^4Q^3\Big]\,,$$

which agrees with the sum of (A5e) and (A5f) in CL. It does not agree with the first terms in (A.26) in KKMS who use the notation where $R_0^2 = 4\pi\langle O[^1S_0^{(1)}]\rangle$.

${}^3P_2^{(8)}$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{20\pi\alpha_s^3\langle R\,[^3P_2^{(8)}]\,\rangle}{M^3s^2}\frac{1}{Q(Q-M^2P)^4}\Big[6M^{12}P^4 - 12M^8P^5 + 6M^4P^6$$
$$+Q(-6M^{14}P^2 - 3M^{10}P^3 + 3M^6P^4 - 6M^2P^5)$$
$$+Q^2(24M^{12}P - 29M^8P^2 + 41M^4P^3 + P^4)$$
$$+Q^3(-19M^{10} + 14M^6P - 31M^2P^2) + 11M^4Q^4\Big]\,,$$

which agrees with the sum of (A5g) plus (A5h) plus (A5i) in CL. It does not agree with the first terms in (A.26) in KKMS who use the notation where $R_0^2 = 4\pi\langle O[^1S_0^{(1)}]\rangle$.

In view of the differences in the above results we recalculated the differential cross sections by summing over the physical polarizations of the external gluons using the covariant expression

$$\sum_{\alpha=+,-}\epsilon^\mu(k,\alpha)\epsilon^\nu(k,\alpha) = P^{\mu\nu}(n,k)\,,$$

with

$$P^{\mu\nu}(n,k) = -g_{\mu\nu} + (n_\mu k_\nu + k_\mu n_\nu)/n \cdot k\,,$$

where $n_\mu$ satisfies $n_\mu P^{\mu\nu} = P^{\mu\nu} n_\nu = 0$ and $n^2 = 0$. One uses this sum for each external gluon and the answer for the square of the matrix elements should be independent of $n_\mu$. This was the case and this method yielded the same answers for the differential cross sections as those from the helicity method.

## 8.3 Polarized three gluons colour singlet

${}^1S_0^{(1)}$ Here we find

$$\begin{aligned}\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} &= \frac{\pi\alpha_s^3 \langle R\,[{}^1S_0^{(1)}]\,\rangle}{s^2 M}\frac{P^2}{Qs^2(Q-M^2P)^2} \\ &\times\Big[-2Q^2 + 4Qs(s-M^2)^2 + s^6 + s^2M^8 - s^2(s-M^2)^4\Big]\,,\end{aligned}$$

which agrees with the second terms in (A.16) in KKMS.

${}^3S_1^{(1)}$ Here we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{10\pi\alpha_s^3 \langle R\,[{}^3S_1^{(1)}]\,\rangle}{9Ms^2}\frac{M^2Q(s^2-P)}{s(Q-M^2P)^2}\,,$$

which agrees with the second terms in (A.17) in KKMS.

${}^1P_1^{(1)}$ Here we find

$$\begin{aligned}\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} &= \frac{40\pi\alpha_s^3 \langle R\,[{}^1P_1^{(1)}]\,\rangle}{3M^3 s^2}\frac{M^2}{(Q-M^2P)^3} \\ &\times\Big[stu(2u^4 + 4tu^3 + 6t^2u^2 + 4t^3u + 2t^4) \\ &\quad + s^2(5tu^4 + 7t^2u^3 + 7t^3u^2 + 5t^4u + t^5 + u^5) \\ &\quad + s^3(10tu^3 + 10t^2u^2 + 10t^3u + 4t^4 + 4u^4) \\ &\quad + s^4(10tu^2 + 10t^2u + 6t^3 + 6u^3) \\ &\quad + s^5(4tu + 4t^2 + 4u^2) + s^6(t+u) \\ &\quad + t^2u^5 + 3t^3u^4 + 3t^4u^3 + t^5u^2)\Big]\,,\end{aligned}$$

which agrees with the second terms in (A.18) in KKMS.

${}^3P_0^{(1)}$ Here we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{4\pi\alpha_s^3 \langle R\,[{}^3P_0^{(1)}]\,\rangle}{M^3 s^2}\frac{Q+s^2M^2}{Qs^6(Q-M^2P)^4}$$

$$
\begin{aligned}
&\times\Big[18s^9M^4(s-M^2)^6+9s^{10}M^6(s-M^2)^4)\\
&+Qs^7M^2(66M^{12}-327sM^{10}+684s^2M^8-762s^3M^6+462s^4M^4\\
&\quad-135s^5M^2+12s^6)\\
&+Q^2s^5(96M^{12}-422sM^{10}+750s^2M^8-663s^3M^6+286s^4M^4\\
&\quad-49s^5M^2+2s^6)\\
&+Q^3s^3(66M^{10}-260sM^8+374s^2M^6-237s^3M^4+62s^4M^2-5s^5)\\
&+Q^4s(18M^8-75sM^6+80s^2M^4-31s^3M^2+4s^4)\\
&+Q^5(6sM^2-s^2-9M^4)\Big]\,,
\end{aligned}
$$

which agrees with the second terms in (A.19) in KKMS.

${}^3P_1^{(1)}$ Here we find

$$
\begin{aligned}
\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} &= \frac{12\pi\alpha_s^3\langle R\,[{}^3P_1^{(1)}]\,\rangle}{M^3s^2}\,\frac{Q(Q+s^2M^2)}{s^5(Q-M^2P)^4}\\
&\times\Big[2s^6(-M^8+5sM^6-9s^2M^4+7s^3M^2-2s^4)\\
&+Qs^3(M^8-4sM^6+11s^2M^4-18s^3M^2+10s^4)\\
&+Q^2s(M^6-6sM^4+11s^2M^2-8s^3)\\
&+2Q^3(s-2M^2)\Big]\,,
\end{aligned}
$$

which does not agree with the second terms in (A.20) in KKMS.

${}^3P_2^{(1)}$ Here we find

$$
\begin{aligned}
\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} &= \frac{4\pi\alpha_s^3\langle R\,[{}^3P_2^{(1)}]\,\rangle}{M^3s^2}\,\frac{Q+s^2M^2}{Qs^6(Q-M^2P)^4}\\
&\times\Big[-24s^9M^4(s-M^2)^6-12s^{10}M^6(s-M^2)^4\\
&+Qs^7M^4(-48M^{10}+276sM^8-648s^2M^6+768s^3M^4-456s^4M^2\\
&\quad+108s^5)\\
&+Q^2s^6(-330sM^8+306s^2M^6-82s^3M^4-14s^4M^2+4s^5\\
&\quad+116M^{10})\\
&+Q^3s^3(48M^{10}-79sM^8+28s^2M^6-2s^4M^2-10s^5+15s^3M^4)\\
&+Q^4s(24M^8-63sM^6+34s^2M^4-5s^3M^2+8s^4)\\
&+Q^5(-12M^4+12sM^2-2s^2)\Big]\,,
\end{aligned}
$$

which agrees with the second terms in (A.21) in KKMS.

## 8.4 Polarized three gluons colour octet

These are only available in the Appendix of KKMS. We begin with

${}^1S_0^{(8)}$

$$\begin{aligned}\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} &= \frac{5\pi\alpha_s^3\langle R\,[{}^1S_0^{(8)}]\,\rangle}{2s^2M}\frac{1}{s^4Q(Q-M^2P)^2}\\ &\times\Big[2s^7M^2(M^2-s)^4+s^8M^4(M^2-s)^2\\ &+Qs^5(4M^8-15sM^6+20s^2M^4-12s^3M^2+2s^4)\\ &+Q^2s^3(4M^6-12sM^4+14s^2M^2-5s^3)\\ &+Q^3s(2M^4-5sM^2+4s^2)-Q^4\Big]\,,\end{aligned}$$

which agrees with the second terms in (A.22) in KKMS, if we make the replacement $\langle R\,[{}^1S_0^{(8)}]\,\rangle = \pi\langle O\,[{}^1S_0^{(8)}]\,\rangle/2$.

${}^3S_1^{(8)}$ Here we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{\pi\alpha_s^3\langle R\,[{}^3S_1^{(8)}]\,\rangle}{3Ms^2}\frac{Q(19M^4-27P)(s^2-P)}{M^2s(Q-M^2P)^2}\,,$$

which agrees with the second terms in (A.23) in KKMS, if we make the replacement $\langle R\,[{}^3S_1^{(8)}]\,\rangle = \pi\langle O\,[{}^3S_1^{(8)}]\,\rangle/6$.

${}^1P_1^{(8)}$ Here we find

$$\begin{aligned}\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} &= \frac{2\pi\alpha_s^3\langle R\,[{}^1P_1^{(8)}]\,\rangle}{M^3s^2}\frac{M^2-s}{Qs^6(Q-M^2P)^4}\\ &\times\Big[(27(M^2-s)^3M^6s^{11}(2M^4-3sM^2+2s^2)\\ &+Qs^9(M^2-s)M^4(-621sM^6+864s^2M^4\\ &-567s^3M^2+108s^4+173M^8)\\ &+Q^2s^7M^2(-1395sM^8+2307s^2M^6\\ &-1988s^3M^4+621s^4M^2-54s^5+249M^{10})\\ &+Q^3s^5(-1488sM^8+2314s^2M^6\\ &-1492s^3M^4+189s^4M^2+249M^{10})\\ &+Q^4s^3(-779sM^6+1379s^2M^4-449s^3M^2+173M^8)\\ &+Q^5s(-162sM^4+373s^2M^2+27s^3+54M^6)\\ &-Q^627(M^2-s))\Big]\,,\end{aligned}$$

which agrees with the second terms in (A.24) in KKMS, if we make the replacement $\langle R\,[{}^1P_1^{(8)}]\,\rangle = \pi\langle O\,[{}^1P_1^{(8)}]\,\rangle/18$.

${}^3P_0^{(8)}$ Here we find

$$\begin{aligned}
\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} &= \frac{10\pi\alpha_s^3\langle R\,[{}^3P_0^{(8)}]\,\rangle}{M^3s^2}\frac{1}{Qs^6(Q-M^2P)^4} \\
&\times\Big[9s^{11}M^6(2(M^2-s)^6+sM^2(M^2-s)^4) \\
&+3Qs^9M^4(23M^{12}-126sM^{10}+285s^2M^8-342s^3M^6+228s^4M^4 \\
&\quad -78s^5M^2+10s^6) \\
&+Q^2s^7M^2(117M^{12}-612sM^{10}+1285s^2M^8-1358s^3M^6 \\
&\quad +738s^4M^4-184s^5M^2+14s^6) \\
&+Q^3s^5(117M^{12}-553sM^{10}+1021s^2M^8-881s^3M^6+352s^4M^4 \\
&\quad -54s^5M^2+2s^6) \\
&+Q^4s^3(69M^{10}-292sM^8+439s^2M^6-275s^3M^4+68s^4M^2-5s^5) \\
&+Q^5s(18M^8-81sM^6+88s^2M^4-33s^3M^2+4s^4) \\
&-Q^6(s-3M^2)^2\Big]\,,
\end{aligned}$$

which agrees with the second terms in (A.25) in KKMS, if we make the replacement $\langle R\,[{}^3P_0^{(8)}]\,\rangle = \pi\langle O\,[{}^3P_0^{(8)}]\,\rangle/6$.

${}^3P_1^{(8)}$ Here we find

$$\begin{aligned}
\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} &= \frac{60\pi\alpha_s^3\langle R\,[{}^3P_1^{(8)}]\,\rangle}{M^3s^2}\frac{Q}{s^5(Q-M^2P)^4} \\
&\times\Big[s^7M^2(-M^{10}+5sM^8-10s^2M^6+11s^3M^4-7s^4M^2+2s^5) \\
&+Qs^5(-2M^{10}+8sM^8-14s^2M^6+17s^3M^4-12s^4M^2+2s^5) \\
&+Q^2s^3(-2M^8+4sM^6-8s^2M^4+10s^3M^2-5s^4) \\
&+Q^3s(-M^6+3sM^4-5s^2M^2+4s^3)+Q^4(-s+2M^2)\Big]\,,
\end{aligned}$$

which agrees with the second terms in (A.26) in KKMS, if we make the replacement $\langle R\,[{}^3P_1^{(8)}]\,\rangle = \pi\langle O\,[{}^3P_1^{(8)}]\,\rangle/6$.

${}^3P_2^{(8)}$ Here we find

$$\begin{aligned}
\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} &= \frac{20\pi\alpha_s^3\langle R\,[{}^3P_2^{(8)}]\,\rangle}{M^3s^2}\frac{1}{Qs^6(Q-M^2P)^4} \\
&\times\Big[-12s^{11}M^6(M^2-s)^6-6s^{12}M^8(M^2-s)^4 \\
&+Qs^9M^4(-27M^{12}+177sM^{10}-447s^2M^8+567s^3M^6 \\
&\quad -378s^4M^4+120s^5M^2-12s^6) \\
&+Q^2s^7M^2(-15M^{12}+138sM^{10}-398s^2M^8+481s^3M^6-255s^4M^4 \\
&\quad +47s^5M^2+2s^6)
\end{aligned}$$

$$+Q^3s^5(15M^{12}+5sM^{10}-89s^2M^8+115s^3M^6-35s^4M^4$$
$$-12s^5M^2+2s^6)$$
$$+Q^4s^3(27M^{10}-55sM^8+37s^2M^6-5s^3M^4+2s^4M^2-5s^5)$$
$$+Q^5s(12M^8-39sM^6+25s^2M^4-3s^3M^2+4s^4)$$
$$+Q^6(6sM^2-s^2-6M^4)\Big],$$

which agrees with the second terms in (A.27) in KKMS, if we make the replacement $\langle R\,[^3P_2^{(8)}]\,\rangle = \pi\langle O\,[^3P_2^{(8)}]\,\rangle/30$.

# 9 Towards predictions

Unfortunately the time for my master project is limited and my part of the research ends at this point. In order to give an overview, it is nice to list a very short summary of what needs to be done further.

Now we have the numerical expressions for the differential cross sections, we can calculate the distribution in transverse momentum and rapidity. For that we need to plug in the known particle density functions as well as the correct coupling constant for the specific energy scales.

Because $J/\psi$ production is slow at experiments where measurements with polarized particles are being done, like the Relativistic Heavy Ion Collider (RHIC), one has to integrate out the rapidity. Detectors can not measure to high or low rapidities, so an appropriate rapidity cut has to be applied.

Finally we can check whether our predictions agree with experimental results or not.

# References

[1] G.T. Bodwin, E. Braaten and G.P. Lepage, Phys. Rev. D **51**, 1125 (1995).

[2] R. Gastmans and T.T. Wu, The Ubiquitious Photon, Helicity Methods for QED and QCD, Oxford Science Publishers 1990.

[3] B. Gubernina, J.H. Kuhn, R.D. Peccei and R. Ruckl, Nuc. Phys. B174, 317 (1980).

[4] M. Klasen, B.A. Kniehl, L.N. Mihaila and M. Steinhauser, Phys. Rev D68, 034017 (2003), arXiv:hep-ph/0306080.

[5] F. Yuan, H-S. Dong, L-K. Hao and K.T. Chao Phys. Rev. D61, 114013 (2000), arXiv:hep-ph/9909221.

[6] P. Ko, J. Lee and H.S. Song, Phys. Rev. D54, 4315 (1996), arXiv:hep-ph/9602223,

[7] R.Gastmans, W. Troost and T.T. Wu, Nucl. Phys. B291, 731 (1987).

[8] P. Cho and A.K. Leibovich, Phys. Rev. D 53 6203 (1996). arXiv:hep-ph/9511315.

# A Colour traces

Any $3 \times 3$ matrix $M$, can be decomposed into a trace part, $\frac{1}{3}\mathrm{Tr}(M)$, and eight parts proportional to the colour matrices, $NT^a\mathrm{Tr}(MT^a)$, where $T^b = NT^a\mathrm{Tr}[T^bT^a] = NT^a\delta^{ab}/2 = NT^b/2$ tells us $N = 2$.

For a special $M$, with only one 1 on place $(m, n)$ and 0 elsewhere, the decomposition becomes

$$\begin{aligned}
\delta_{mi}\delta_{nj} &= M_{ij} \\
&= \left(2\sum_a T^a \delta_{mk}\delta_{nl}T^a_{lk} + \frac{1}{3}\delta_{mk}\delta_{nl}\delta_{lk}\right)_{ij} \\
&= 2\sum_a T^a_{ij}T^a_{nm} + \frac{1}{3}\delta_{mn}\delta_{ij}\,,
\end{aligned}$$

or in other words

$$\sum_a T^a_{ij}T^a_{nm} = \frac{1}{2}\delta_{mi}\delta_{nj} - \frac{1}{6}\delta_{mn}\delta_{ij}\,.$$

This is very usefull for the calculation of colour traces, because we get combinations with

$$\begin{aligned}
\sum_a \mathrm{Tr}(T^aM_1)\mathrm{Tr}(T^aM_2) &= \sum_a T^a_{ij}M_{1ji}T^a_{nm}M_{2mn} \\
&= \left(\frac{1}{2}\delta_{mi}\delta_{nj} - \frac{1}{6}\delta_{mn}\delta_{ij}\right)M_{1ji}M_{2mn} \\
&= \frac{1}{2}M_{1nm}M_{2mn} - \frac{1}{6}M_{1ii}M_{2mm} \\
&= \frac{1}{2}\mathrm{Tr}(M_1M_2) - \frac{1}{6}\mathrm{Tr}(M_1)\mathrm{Tr}(M_2)\,, \qquad (11)
\end{aligned}$$

where $M_1$ and $M_2$ are arbitrary matrix elements, possibly containing combinations of colour matrices. Similarly

$$\begin{aligned}
\sum_a \mathrm{Tr}(T^aM_1T^aM_2) &= \sum_a T^a_{ij}M_{1jn}T^a_{nm}M_{2mi} \\
&= \left(\frac{1}{2}\delta_{mi}\delta_{nj} - \frac{1}{6}\delta_{mn}\delta_{ij}\right)M_{1jn}M_{2mi} \\
&= \frac{1}{2}M_{1jj}M_{2ii} - \frac{1}{6}M_{1im}M_{2mi} \\
&= \frac{1}{2}\mathrm{Tr}(M_1)\mathrm{Tr}(M_2) - \frac{1}{6}\mathrm{Tr}(M_1M_2)\,. \qquad (12)
\end{aligned}$$

Formula (11) and (12) provide us the tool to calculate the colour traces relevant for use. Summation over repeated colour indices is implied in the rest of this appendix.

For $\gamma + g \longrightarrow c\bar{c}\left({}^{2S+1}L_J^{(1)}\right) + g$ we have terms with two colour matrices:

$$\begin{aligned}
\mathrm{Tr}(T^aT^b)\mathrm{Tr}(T^aT^b) &= \frac{1}{2}\mathrm{Tr}(T^bT^b) - \frac{1}{6}\mathrm{Tr}(T^b)\mathrm{Tr}(T^b) \\
&= \frac{1}{4}\delta^{bb} = 2\,.
\end{aligned}$$

For $\gamma + g \longrightarrow c\bar{c}\left({}^{2S+1}L_J^{(8)}\right) + g$ and $2g \longrightarrow c\bar{c}\left({}^{2S+1}L_J^{(1)}\right) + g$ we get terms with three matrices:

$$\begin{aligned}
\mathrm{Tr}(T^aT^bT^c)\mathrm{Tr}(T^aT^bT^c) &= \frac{1}{2}\mathrm{Tr}(T^bT^cT^bT^c) - \frac{1}{6}\mathrm{Tr}(T^bT^c)\mathrm{Tr}(T^bT^c) \\
&= \frac{1}{2}\left(\frac{1}{2}\mathrm{Tr}(T^c)\mathrm{Tr}(T^c) - \frac{1}{6}\mathrm{Tr}(T^cT^c)\right) - \frac{1}{3} \\
&= -\frac{2}{3} \\
\mathrm{Tr}(T^aT^bT^c)\mathrm{Tr}(T^cT^bT^a) &= \frac{1}{2}\mathrm{Tr}(T^bT^cT^cT^b) - \frac{1}{6}\mathrm{Tr}(T^bT^c)\mathrm{Tr}(T^cT^b) \\
&= \frac{1}{2}\left(\frac{1}{2}\mathrm{Tr}(1)\mathrm{Tr}(T^cT^c) - \frac{1}{6}\mathrm{Tr}(T^cT^c)\right) - \frac{1}{3} \\
&= \frac{7}{3}\,.
\end{aligned}$$

For $2g \longrightarrow c\bar{c}\left({}^{2S+1}L_J^{(8)}\right) + g$ we get terms with four colour matrices:

$$\begin{aligned}
\mathrm{Tr}(T^aT^bT^cT^d)\mathrm{Tr}(T^aT^bT^cT^d) &= \frac{1}{2}\mathrm{Tr}(T^bT^cT^dT^bT^cT^d) \\
&\quad -\frac{1}{6}\mathrm{Tr}(T^bT^cT^d)\mathrm{Tr}(T^bT^cT^d) \\
&= \frac{1}{2}\left(\frac{1}{2}\mathrm{Tr}(T^cT^d)\mathrm{Tr}(T^cT^d)\right. \\
&\qquad \left. -\frac{1}{6}\mathrm{Tr}(T^cT^dT^cT^d)\right) + \frac{1}{9} \\
&= \frac{2}{3} \\
\mathrm{Tr}(T^aT^bT^cT^d)\mathrm{Tr}(T^aT^cT^dT^b) &= \mathrm{Tr}(T^aT^bT^cT^d)\mathrm{Tr}(T^aT^dT^bT^c) \\
&= \mathrm{Tr}(T^aT^bT^cT^d)\mathrm{Tr}(T^aT^cT^bT^d) \\
&= \mathrm{Tr}(T^aT^bT^cT^d)\mathrm{Tr}(T^aT^bT^dT^c) \\
&= \frac{1}{2}\mathrm{Tr}(T^bT^cT^dT^bT^dT^c) \\
&\quad -\frac{1}{6}\mathrm{Tr}(T^bT^cT^d)\mathrm{Tr}(T^bT^dT^c) \\
&= \frac{1}{2}\left(\frac{1}{2}\mathrm{Tr}(T^cT^d)\mathrm{Tr}(T^dT^c)\right. \\
&\qquad \left. -\frac{1}{6}\mathrm{Tr}(T^cT^dT^dT^c)\right) - \frac{7}{18} \\
&= -\frac{1}{3} \\
\mathrm{Tr}(T^aT^bT^cT^d)\mathrm{Tr}(T^aT^dT^cT^b) &= \frac{1}{2}\mathrm{Tr}(T^bT^cT^dT^dT^cT^b) \\
&\quad -\frac{1}{6}\mathrm{Tr}(T^bT^cT^d)\mathrm{Tr}(T^dT^cT^b) \\
&= \frac{1}{2}\left(\frac{1}{2}\mathrm{Tr}(T^cT^dT^dT^c)\mathrm{Tr}(1)\right.
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{6}\mathrm{Tr}(T^cT^dT^dT^c)\Big) - \frac{7}{18} \\
= \quad & \frac{19}{6}\,.
\end{aligned}
$$

# B Form code

We wrote three types of programs:
• 1 to store the amplitudes for the different ${}^{2S+1}L_J$ states;
• 24 to calculate the matrix elements squared for different helicity configurations: one for each combination of colour, spin projection state and reaction, and
• 48 to calculate the (un)polarized differential cross sections: one for each combination of colour, spin projection state and reaction.

The programs of the same type are similar. We give an example of each type.

## B.1 basis.frm

```
* program basis.frm
* Last change: MM April 13th 2007
* This program calculates the basic diagrams and stores the values for
* different spin orbit configurations in order to process them further later on.
off statistics;
V k1,k2,k3,e1,e2,e3,e,p1,p2,p,q;
Indices mu,nu,la,si,al,be,ro;
nt [T(a)],[T(b)],[T(c)],[T(d)],[T(e)];
nt S1, S2, S3, S4, S5, S6;
S T1, T2, T3, T4, T5, T6;
S m,me,s,t,u,M,R0,R1,g1,i,N,N1,N2,N3,N1p,N1m,N2p,N2m,N3p,N3m,pl;
S x,y,z,ss,s1,t1,u1,P,Q,[(Q-M^2*P)],alphas,pi,M1,sqpim,sq3,sq2;
S [f(a,b,c)],[d(a,b,c)],[f(abc)],[d(abc)];
S [e(al,be)];

* notation k1 + k2 + k3 = P
*         N1 = t-M^2 = -2*p.k1  N1m = -2p.k1+4q.k1  N1p = -2p.k1-4q.k1
*         N2 = u-M^2 = -2*p.k2  N2m = -2p.k2+4q.k2  N2p = -2p.k2-4q.k2
*         N3 = s-M^2 = -2*p.k3  N3m = -2p.k3+4q.k3  N3p = -2p.k3-4q.k3
* remember M=2m
*
* amplitudes with triple gluon terms

Local Os=sq2*g1^3
   *((k2(al)-k1(al))*(e1.e2)+2*(k1.e2)*e1(al) -2*(k2.e1)*e2(al))
   *( ([T(a)]*[T(b)]-[T(b)]*[T(a)])*[T(c)]*[T(d)]
   *(g_(1,al)*(g_(1,p)-2*g_(1,k3)-2*m)*g_(1,e3))/N3
   -([T(c)]*([T(a)]*[T(b)]-[T(b)]*[T(a)])*[T(d)]
    *g_(1,e3)*(g_(1,p)-2*g_(1,k3)+2*m)*g_(1,al)/N3))*s^-1;

Local Osq=sq2*g1^3
  *((k2(al)-k1(al))*(e1.e2)+2*(k1.e2)*e1(al) -2*(k2.e1)*e2(al))
  *(
   -2
   *([T(a)]*[T(b)]-[T(b)]*[T(a)])*[T(c)]*[T(d)]
   *g_(1,al)*g_(1,be)*g_(1,e3)/N3
  -2
   *[T(c)]*([T(a)]*[T(b)]-[T(b)]*[T(a)])*[T(d)]
  *g_(1,e3)*g_(1,be)*g_(1,al)/N3
  -4
   *([T(a)]*[T(b)]-[T(b)]*[T(a)])*[T(c)]*[T(d)]
   *k3(be)*g_(1,al)*(g_(1,p)-2*g_(1,k3)-2*m)*g_(1,e3)/N3/N3
  -4
   *[T(c)]*([T(a)]*[T(b)]-[T(b)]*[T(a)])*[T(d)]
   *k3(be)*g_(1,e3)*(g_(1,p)-2*g_(1,k3)+2*m)*g_(1,al)/N3/N3)*s^-1;

Local Ot=sq2*g1^3
 *((k3(al)-k2(al))*(e2.e3)+2*(k2.e3)*e2(al) -2*(k3.e2)*e3(al))
   *(([T(b)]*[T(c)]-[T(c)]*[T(b)])*[T(a)]*[T(d)]
 *(g_(1,al)*(g_(1,p)-2*g_(1,k1)-2*m)*g_(1,e1))/N1
   -([T(a)]*([T(b)]*[T(c)]-[T(c)]*[T(b)])*[T(d)]
  *g_(1,e1)*(g_(1,p)-2*g_(1,k1)+2*m)*g_(1,al)/N1))*t^-1;
```

```
Local Otq=sq2*g1^3
  *((k3(al)-k2(al))*(e2.e3)+2*(k2.e3)*e2(al) -2*(k3.e2)*e3(al))
  *(-2
   *([T(b)]*[T(c)]-[T(c)]*[T(b)])*[T(a)]*[T(d)]
   *g_(1,al)*g_(1,be)*g_(1,e1)/N1
   -2
   *[T(a)]*([T(b)]*[T(c)]-[T(c)]*[T(b)])*[T(d)]
   *g_(1,e1)*g_(1,be)*g_(1,al)/N1
  -4
   *([T(b)]*[T(c)]-[T(c)]*[T(b)])*[T(a)]*[T(d)]
   *k1(be)*g_(1,al)*(g_(1,p)-2*g_(1,k1)-2*m)*g_(1,e1)/N1/N1
  -4
   *[T(a)]*([T(b)]*[T(c)]-[T(c)]*[T(b)])*[T(d)]
   *k1(be)*g_(1,e1)*(g_(1,p)-2*g_(1,k1)+2*m)*g_(1,al)/N1/N1)*t^-1;

Local Ou=sq2*g1^3
  *((k1(al)-k3(al))*(e3.e1)+2*(k3.e1)*e3(al) -2*(k1.e3)*e1(al))
   *(([T(c)]*[T(a)]-[T(a)]*[T(c)])*[T(b)]*[T(d)]
  *(g_(1,al)*(g_(1,p)-2*g_(1,k2)-2*m)*g_(1,e2))/N2
   -([T(b)]*([T(c)]*[T(a)]-[T(a)]*[T(c)])*[T(d)]
   *g_(1,e2)*(g_(1,p)-2*g_(1,k2)+2*m)*g_(1,al)/N2))*u^-1;

Local Ouq=sq2*g1^3
  *((k1(al)-k3(al))*(e3.e1)+2*(k3.e1)*e3(al) -2*(k1.e3)*e1(al))
  *(-2
   *([T(c)]*[T(a)]-[T(a)]*[T(c)])*[T(b)]*[T(d)]
   *g_(1,al)*g_(1,be)*g_(1,e2)/N2
   -2
   *[T(b)]*([T(c)]*[T(a)]-[T(a)]*[T(c)])*[T(d)]
   *g_(1,e2)*g_(1,be)*g_(1,al)/N2
  -4
   *([T(c)]*[T(a)]-[T(a)]*[T(c)])*[T(b)]*[T(d)]
  *k2(be)*g_(1,al)*(g_(1,p)-2*g_(1,k2)-2*m)*g_(1,e2)/N2/N2
  -4
   *[T(b)]*([T(c)]*[T(a)]-[T(a)]*[T(c)])*[T(d)]
   *k2(be)*g_(1,e2)*(g_(1,p)-2*g_(1,k2)+2*m)*g_(1,al)/N2/N2)*u^-1;

* amplitudes with no triple gluon couplings

* this is the diagram with 132 along the quark line, so the
* propagators contain -2p.k1 = N1 = t-M^2 and -2p.k2 = N2 = u-M^2
* so add N3 = s - M^2.
* in my notes this is diagram II

Local Od1=i*g1^3*sq2
   *(4*i*[T(a)]*[T(c)]*[T(b)])*[T(d)]
  *((p.e1)+g_(1,k1)*g_(1,e1))*g_(1,e3)*((p.e2)+g_(1,e2)*g_(1,k2))
             *(s-M^2)/N1/N2/N3;

Local Od1q=i*g1^3*sq2
   *(4*i*[T(a)]*[T(c)]*[T(b)])*[T(d)]
  *(2*e1(be)*g_(1,e3)*((p.e2)+g_(1,e2)*g_(1,k2))/N1/N2
   -2*((p.e1)+g_(1,k1)*g_(1,e1))*e2(be)*g_(1,e3)/N1/N2
   +4*k1(be)*((p.e1)+g_(1,k1)*g_(1,e1))*g_(1,e3)*((p.e2)+g_(1,e2)
             *g_(1,k2))/N1/N1/N2
   -4*k2(be)*((p.e1)+g_(1,k1)*g_(1,e1))*g_(1,e3)*((p.e2)+g_(1,e2)
             *g_(1,k2))/N1/N2/N2);

* this is the diagram with 213 along the quark line, so the
* propagators contain -2p.k2 = N2 = u - M^2 and - 2p.k3 = N3 = s - M^2
* so add N1 = t - M^2.
* in my notes this is diagram III

Local Od2=i*g1^3*sq2
   *(4*i*[T(b)]*[T(a)]*[T(c)])*[T(d)]
  *((p.e2)+g_(1,k2)*g_(1,e2))*g_(1,e1)*((p.e3)+g_(1,e3)*g_(1,k3))
            *(t-M^2)/N1/N2/N3;

Local Od2q=i*g1^3*sq2
   *(4*i*[T(b)]*[T(a)]*[T(c)])*[T(d)]
  *(2*e2(be)*g_(1,e1)*((p.e3)+g_(1,e3)*g_(1,k3))/N2/N3
   -2*((p.e2)+g_(1,k2)*g_(1,e2))*g_(1,e1)*e3(be)/N2/N3
   +4*k2(be)*((p.e2)+g_(1,k2)*g_(1,e2))*g_(1,e1)*((p.e3)
            +g_(1,e3)*g_(1,k3))
```

```
            /N2/N2/N3
   -4*k3(be)*((p.e2)+g_(1,k2)*g_(1,e2))*g_(1,e1)*((p.e3)
           +g_(1,e3)*g_(1,k3))
            /N2/N3/N3);
* this is the diagram with 321 along the quark line, so the
* propagators contain -2p.k3 = N3 = s - M^2 and -2p.k1 = N1 = t - M^2
* so add N2 = u - M^2.
* in my notes this is diagram VI

Local Od3=i*g1^3*sq2
   *(4*i*[T(c)]*[T(b)]*[T(a)])*[T(d)]
  *((p.e3)+g_(1,k3)*g_(1,e3))*g_(1,e2)*((p.e1)+g_(1,e1)*g_(1,k1))
           *(u-M^2)/N1/N2/N3;


Local Od3q=i*g1^3*sq2
   *(4*i*[T(c)]*[T(b)]*[T(a)])*[T(d)]
  *(2*e3(be)*g_(1,e2)*((p.e1)+g_(1,e1)*g_(1,k1))/N1/N3
   -2*((p.e3)+g_(1,k3)*g_(1,e3))*g_(1,e2)*e1(be)/N1/N3
   +4*k3(be)*((p.e3)+g_(1,k3)*g_(1,e3))*g_(1,e2)*((p.e1)
           +g_(1,e1)*g_(1,k1))
                /N3/N3/N1
   -4*k1(be)*((p.e3)+g_(1,k3)*g_(1,e3))*g_(1,e2)*((p.e1)
           +g_(1,e1)*g_(1,k1))
                /N3/N1/N1);

* this is the diagram with 312 along the quark line, so the
* propagators contain -2p.k3 = N3 = s - M^2 and - 2p.k2 = N2 = u - M^2
* so add N1 = t - M^2.
* in my notes this is diagram V

Local Od4=i*g1^3*sq2
   *(4*i*[T(c)]*[T(a)]*[T(b)])*[T(d)]
  *((p.e3)+g_(1,k3)*g_(1,e3))*g_(1,e1)*((p.e2)+g_(1,e2)*g_(1,k2))
           *(t-M^2)/N1/N2/N3;


Local Od4q=i*g1^3*sq2
   *(4*i*[T(c)]*[T(a)]*[T(b)])*[T(d)]
  *(2*e3(be)*g_(1,e1)*((p.e2)+g_(1,e2)*g_(1,k2))/N2/N3
   -2*((p.e3)+g_(1,k3)*g_(1,e3))*g_(1,e1)*e2(be)/N2/N3
   +4*k3(be)*((p.e3)+g_(1,k3)*g_(1,e3))*g_(1,e1)*((p.e2)
           +g_(1,e2)*g_(1,k2))
            /N2/N3/N3
   -4*k2(be)*((p.e3)+g_(1,k3)*g_(1,e3))*g_(1,e1)*((p.e2)
           +g_(1,e2)*g_(1,k2))
           /N2/N2/N3);

* this is the diagram with 123 along the quark line, so the
* propagators contain -2p.k1 = N1 = t - M^2 and - 2p.k3 = N3 = s- M^2
* so add N2 = u - M^2.
* in my notes this is diagram I

Local Od5=i*g1^3*sq2
   *(4*i*[T(a)]*[T(b)]*[T(c)])*[T(d)]
  *((p.e1)+g_(1,k1)*g_(1,e1))*g_(1,e2)*((p.e3)+g_(1,e3)*g_(1,k3))
           *(u-M^2)/N1/N2/N3;


Local Od5q=i*g1^3*sq2
   *(4*i*[T(a)]*[T(b)]*[T(c)])*[T(d)]
  *(2*e1(be)*g_(1,e2)*((p.e3)+g_(1,e3)*g_(1,k3))/N3/N1
   -2*((p.e1)+g_(1,k1)*g_(1,e1))*g_(1,e2)*e3(be)/N3/N1
   +4*k1(be)*((p.e1)+g_(1,k1)*g_(1,e1))*g_(1,e2)*((p.e3)
        +g_(1,e3)*g_(1,k3))
           /N3/N1/N1
   -4*k3(be)*((p.e1)+g_(1,k1)*g_(1,e1))*g_(1,e2)*((p.e3)
           +g_(1,e3)*g_(1,k3))
          /N3/N3/N1);
* this is the diagram with 231 along the quark line, so the
* propagators contain -2p.k2 = N2 = u - M^2 and 2p.k1 = N1 = t - M^2
* so add N3 =  s - M^2.
* in my notes this is diagram IV
```

```
Local Od6=i*g1^3*sq2
   *(4*i*[T(b)]*[T(c)]*[T(a)])*[T(d)]
  *((p.e2)+g_(1,k2)*g_(1,e2))*g_(1,e3)*((p.e1)+g_(1,e1)*g_(1,k1))
          *(s-M^2)/N1/N2/N3;


Local Od6q=i*g1^3*sq2
   *(4*i*[T(b)]*[T(c)]*[T(a)])*[T(d)]
  *(2*e2(be)*g_(1,e3)*((p.e1)+g_(1,e1)*g_(1,k1))/N1/N2
   -2*((p.e2)+g_(1,k2)*g_(1,e2))*g_(1,e3)*e1(be)/N1/N2
   +4*k2(be)*((p.e2)+g_(1,k2)*g_(1,e2))*g_(1,e3)*((p.e1)
            +g_(1,e1)*g_(1,k1))
             /N1/N2/N2
   -4*k1(be)*((p.e2)+g_(1,k2)*g_(1,e2))*g_(1,e3)*((p.e1)
            +g_(1,e1)*g_(1,k1))
             /N1/N1/N2);

*This is the graph with the 4gluon vertex.
Local O4 = sq2*e1(mu)*e2(nu)*e3(ro)*g1^3*g_(1,si)*(
   (([T(a)]*[T(b)]-[T(b)]*[T(a)])*[T(c)]-[T(c)]*([T(a)]*[T(b)]-[T(b)]*[T(a)]))*
        [T(d)]*(d_(mu,ro)*d_(nu,si)-d_(mu,si)*d_(nu,ro))
  +(([T(a)]*[T(c)]-[T(c)]*[T(a)])*[T(b)]-[T(b)]*([T(a)]*[T(c)]-[T(c)]*[T(a)]))*
        [T(d)]*(d_(mu,nu)*d_(ro,si)-d_(mu,si)*d_(nu,ro))
  +(([T(c)]*[T(b)]-[T(b)]*[T(c)])*[T(a)]-[T(a)]*([T(c)]*[T(b)]-[T(b)]*[T(c)]))*
        [T(d)]*(d_(mu,ro)*d_(nu,si)-d_(mu,nu)*d_(ro,si)))/M/M;

*This is the diagram where gluon 2 and gluon 3 join. The result fuses in turn with gluon 1;
Local O1 = sq2*e1(mu)*e2(nu)*e3(ro)*g1^3*g_(1,si)*
   (([T(b)]*[T(c)]-[T(c)]*[T(b)])*[T(a)]-[T(a)]*([T(b)]*[T(c)]-[T(c)]*[T(b)]))*[T(d)]*
   ((k2(al)-k3(al))*d_(nu,ro)+2*k3(nu)*d_(ro,al)-2*k2(ro)*d_(al,nu))*
   (-2*k1(si)*d_(mu,al)-2*p(mu)*d_(al,si)+(p(al)+k1(al))*d_(si,mu))/t/M/M;

*This is the diagram where gluon 3 and gluon 1 join. The result fuses in turn with gluon 2;
Local O2 = sq2*e1(mu)*e2(nu)*e3(ro)*g1^3*g_(1,si)*
   (([T(c)]*[T(a)]-[T(a)]*[T(c)])*[T(b)]-[T(b)]*([T(c)]*[T(a)]-[T(a)]*[T(c)]))*[T(d)]*
   ((k3(al)-k1(al))*d_(ro,mu)+2*k1(ro)*d_(mu,al)-2*k3(mu)*d_(al,ro))*
   (-2*k2(si)*d_(nu,al)-2*p(nu)*d_(al,si)+(p(al)+k2(al))*d_(si,nu))/u/M/M;

*This is the diagram where gluon 1 and gluon 2 join. The result fuses in turn with gluon 3;
Local O3 = sq2*e1(mu)*e2(nu)*e3(ro)*g1^3*g_(1,si)*
   (([T(a)]*[T(b)]-[T(b)]*[T(a)])*[T(c)]-[T(c)]*([T(a)]*[T(b)]-[T(b)]*[T(a)]))*[T(d)]*
   ((k1(al)-k2(al))*d_(mu,nu)+2*k2(mu)*d_(nu,al)-2*k1(nu)*d_(al,mu))*
   (-2*k3(si)*d_(ro,al)-2*p(ro)*d_(al,si)+(p(al)+k3(al))*d_(si,ro))/s/M/M;

*The following substitutions make the colour factors in the diagrams somewhat more synoptic.
.sort
id [T(a)]*[T(b)]*[T(c)]*[T(d)] = T1;
id [T(a)]*[T(c)]*[T(b)]*[T(d)] = T2;
id [T(b)]*[T(a)]*[T(c)]*[T(d)] = T3;
id [T(b)]*[T(c)]*[T(a)]*[T(d)] = T4;
id [T(c)]*[T(a)]*[T(b)]*[T(d)] = T5;
id [T(c)]*[T(b)]*[T(a)]*[T(d)] = T6;
.sort
Local Od=Od1+Od2+Od3+Od4+Od5+Od6;
Local O=Os+Ot+Ou+Od+O1+O2+O3+O4;

Local Odq=Od1q+Od2q+Od3q+Od4q+Od5q+Od6q;
Local Oq=Osq+Otq+Ouq+Odq;
.sort

*Here we calculate amplitudes for every spin projection. In the A3P1 case,
*we use a different notation for the epsilon, since this will make the
*substitution for the helicities easier.
Local A1S0=1/4/sqpim*R0*(O*(g_(1,p)-M)*g5_(1));
Local A3S1=1/4/sqpim*R0*(O*(g_(1,p)-M)*g_(1,e));
Local A1P1=-i*R1*sq3/4/sqpim*(e(be)*Oq*(g_(1,p)-M)*g5_(1)+2*O*g_(1,e)*g_(1,p)
            *g5_(1)/M);
Local A3P0=i*R1/4/sqpim*(Oq*(g_(1,be)-p(be)/M)*(g_(1,p)+M)+6*O);
Local A3P1=-R1*sq3/4/sq2/sqpim/M*(-i/4*g_(2,be,al,mu,nu,5_))*
            p(mu)*e(nu)*(Oq*g_(1,al)*(g_(1,p)+M)-2*O*g_(1,be)*g_(1,p)*g_(1,al)/M);
Local A3P2=1/4*i*sq3*sqpim^-1*R1*[e(al,be)]*Oq*g_(1,al)*(g_(1,p)+M);
.sort
Global A1 = A1S0;
Global A2 = A3S1;
```

```
Global A3 = A1P1;
Global A4 = A3P0;
Global A5 = A3P1;
Global A6 = A3P2;
.store
save basis.sav A1,A2,A3,A4,A5,A6;
.store
.end
```

## B.2 singlet1P1.frm

```
* program singlet1P1.frm
* Last change: JS June 4 2007
* independent of the gauge OK
*#
* This program uses stored amplitudes for different spin orbit
* configurations from basis.frm, squares them, plugs in different
* helicities and adds everything to get the final answers. It puts the
* answers in resultsinglet1P1.sav to load them in dsigsinglet1P1.frm.
off statistics;
V k1,k2,k3,e1,e2,e3,e,p1,p2,p,q;
Indices mu,nu,la,si,al,be,mu1,nu1,la1,si1,mu2,nu2,la2,si2;
nt [T(a)],[T(b)],[T(c)],[T(d)],[T(e)];
nt S1, S2, S3, S4, S5, S6;
S T1, T2, T3, T4, T5, T6;
S a1, a2, a3, b1, b2, b3;
S m,me,s,t,u,M,R0,R1,g1,i,N,N1,N2,N3,N1p,N1m,N2p,N2m,N3p,N3m,pl;
S x,y,z,ss,s1,t1,u1,P,Q,[(Q-M^2*P)],alphas,pi,M1,sqpim,sq3,sq2;
S [f(a,b,c)],[d(a,b,c)],[f(abc)],[d(abc)];
S A,B,C,D,xx,yy,zz,aa,bb,cc;

load basis.sav;

Local [M+++] = A3;
Local [M++-] = A3;
.sort
id i^2=-1;
*b i,g_(1,5_,e1,e2,e3);
*print [M+++];
*.end
* taking the color trace for color singlet production. Comment this
* part out for color octet production.
id T1=([d(abc)]+i*[f(abc)])/4/sq3/sq2;
id T2=([d(abc)]-i*[f(abc)])/4/sq3/sq2;
id T3=([d(abc)]-i*[f(abc)])/4/sq3/sq2;
id T4=([d(abc)]+i*[f(abc)])/4/sq3/sq2;
id T5=([d(abc)]+i*[f(abc)])/4/sq3/sq2;
id T6=([d(abc)]-i*[f(abc)])/4/sq3/sq2;
.sort

*In the next part the helicities are specified. This is helicity dependent.

skip [M++-];
id g_(1,e1)=
     N*(g_(1,k1)*g_(1,k2)*g_(1,k3)*(gi_(1)-g5_(1))
       +g_(1,k3)*g_(1,k2)*g_(1,k1)*(gi_(1)+g5_(1))
        -2*aa*k2.k3*g_(1,k1) );
*         );
id g_(1,e2)=
     N*(g_(1,k2)*g_(1,k3)*g_(1,k1)*(gi_(1)-g5_(1))
       +g_(1,k1)*g_(1,k3)*g_(1,k2)*(gi_(1)+g5_(1))
        -2*bb*k3.k1*g_(1,k2) );
*      );
id g_(1,e3)=
     N*(g_(1,k3)*g_(1,k1)*g_(1,k2)*(gi_(1)-g5_(1))
       +g_(1,k2)*g_(1,k1)*g_(1,k3)*(gi_(1)+g5_(1))
        -2*cc*k1.k2*g_(1,k3) );
*      );
.sort
skip [M+++];
id g_(1,e1)=
```

```
    N*(g_(1,k1)*g_(1,k2)*g_(1,k3)*(gi_(1)-g5_(1))
      +g_(1,k3)*g_(1,k2)*g_(1,k1)*(gi_(1)+g5_(1))
        -2*aa*k2.k3*g_(1,k1) );
*       );
id g_(1,e2)=
   -N*(g_(1,k2)*g_(1,k1)*g_(1,k3)*(gi_(1)-g5_(1))
      +g_(1,k3)*g_(1,k1)*g_(1,k2)*(gi_(1)+g5_(1))
        +2*bb*k1.k3*g_(1,k2) );
*      );
id g_(1,e3)=
    N*(g_(1,k3)*g_(1,k1)*g_(1,k2)*(gi_(1)+g5_(1))
      +g_(1,k2)*g_(1,k1)*g_(1,k3)*(gi_(1)-g5_(1))
        +2*cc*k1.k2*g_(1,k3) );
*      );
.sort
* The next part is general. We take the trace and apply general
* (helicity independent) equations.
id i^2=-1;
id k1.k1=0;
id k2.k2=0;
id k3.k3=0;
id k1.k2=s/2;
id k2.k3=t/2;
id k1.k3=u/2;
id e1.e1=0;
id e2.e2=0;
id e3.e3=0;
id e1.k1=0;
id e2.k2=0;
id e3.k3=0;
.sort
*b i,g_(1,5_,e1,e2,e3);
*print [M+++];
*.end
Trace4,1;
contract 0;
.sort
id e.p=0;
id i^2=-1;
id m=M/2;
id p.p=M^2;
id p = k1+k2+k3;
id k1.k1=0;
id k2.k2=0;
id k3.k3=0;
id k1.k2=s/2;
id k2.k3=t/2;
id k1.k3=u/2;
id e1.e1=0;
id e2.e2=0;
id e3.e3=0;
id e1.k1=0;
id e2.k2=0;
id e3.k3=0;
*The next part is helicity dependent again.
.sort
*b i,g_(1,5_,e1,e2,e3);
*print [M+++];
*.end
skip [M++-];
id k2.e1 = -1/2*s*t*N*aa + N*s*t;
id k3.e1 = -1/2*t*u*N*aa;
id k1.e2 = -1/2*s*u*N*bb;
id k3.e2 = -1/2*t*u*N*bb + N*t*u;
id k1.e3 = -1/2*s*u*N*cc + N*u*s;
id k2.e3 = -1/2*s*t*N*cc;
id e1.e2 = 1+1/4*aa*bb-1/2*bb;
id e2.e3 = 1+1/4*bb*cc-1/2*cc;
id e3.e1 = 1+1/4*aa*cc-1/2*aa;
.sort
skip [M+++];
id k2.e1 = -1/2*s*t*N*aa + N*s*t;
id k3.e1 = -1/2*t*u*N*aa;
id k1.e2 = -1/2*s*u*N*bb -N*s*u;
```

```
id k3.e2 = -1/2*t*u*N*bb;
id k1.e3 = 1/2*s*u*N*cc + N*s*u;
id k2.e3 = 1/2*s*t*N*cc;
id e1.e2 = 1/4*aa*bb+1/2*aa-1/2*bb;
id e2.e3 = -1/4*bb*cc;
id e3.e1 = -1/4*aa*cc-1/2*aa;
.sort
*id aa=0;
*id bb=0;
*id cc=0;
.sort
id N^2=(s^-1*t^-1*u^-1)/2;
.sort
*b i,g_(1,5_,e1,e2,e3);
*print [M+++],[M++-];
*.end
* now there is no epsilon symbol and a factor of i
* This part can give an intermediate result, if desired.
* In this particular case we find the results of Gastmans and Wu to be
* not entirely accurate. There is a overall minus sign (which will not
* matter for the cross section. Furthermore we think there should be an
* additional minus sign for a2 in the ++- case.
*Local answer = [M+++]/i*g1^-3/[d(abc)]/R1*sqpim;
*Local answer = -[M++-]/i*g1^-3/[d(abc)]/R1*sqpim;
*Local GW = a1*(e.e1)+a2*(e.e2)+a3*(e.e3)+b1*(k1.e)+b2*(k2.e)+b3*(k3.e);
*Local diff3 = answer - GW;
*Local diff4 = diff3*N1*N2*N3;
*Local diff4 = diff3*N1*N2*N3;
*.sort
*id a1 = -2*(2*t+u)/N3/N1;
*id a2 = -2*(2*u+s)/N1/N2;
*id a3 = -2*(2*s+t)/N2/N3;
*id b1 = 4*N*(-s*(t^2+u*t-u*M^2)/N3+t*(u^2+s*u-s*M^2)/N2)/N1/N1;
*id b2 = 4*N*(-t*(u^2+s*u-s*M^2)/N1+u*(s^2+t*s-t*M^2)/N3)/N2/N2;
*id b3 = 4*N*(-u*(s^2+t*s-t*M^2)/N2+s*(t^2+u*t-u*M^2)/N1)/N3/N3;
*id a1 = 0;
*id a2 = -2*s/N1/N2;
*id a3 =  2*s*(t+u-2*M^2)/N1/N2/N3;
*id b1 = -4*N*s^2*u*(1/N3+1/N2)/N1/N1;
*id b2 =  4*N*s^2*u*(1/N1-1/N3)/N2/N2;
*id b3 =  4*N*s^2*u*(1/N2+1/N1)/N3/N3;
*id N1 = -u-s;
*id N2 = -s-t;
*id N3 = -t-u;
*id N^2=(s^-1*t^-1*u^-1)/2;
*id N^-2=2*(s*t*u);
*id M^2 = s+t+u;
*id aa=0;
*id bb=0;
*id cc=0;
*.sort
*print GW;
*print answer;
*b k1.e,k2.e,k3.e,e1.e,e2.e,e3.e;
*print [M+++],[M++-];
*.end
*.sort

* Below, we multiply the amplitudes with their hermitian conjugate to
* get the absolute value squared. Because form uses a imaginary
* Levi-Cevita tensor for 4 dimensions, we have to be careful. The
* following approach is based on the fact that the amplitudes are
* imaginary (except for the i hidden in the epsilon tensor).
* first we have to remove the gluon polarization vactors entirely

.sort
skip [M++-];
* this gives e1.ev k1.ev
id e1.e = g_(2,e1,e)/4;
id e2.e = g_(2,e2,e)/4;
id e3.e = g_(2,e3,e)/4;
*+++:
id g_(2,e1)=
     N*(g_(2,k1)*g_(2,k2)*g_(2,k3)*(1-g5_(2))
```

```
       +g_(2,k3)*g_(2,k2)*g_(2,k1)*(1+g5_(2))
     -2*aa*k2.k3*g_(2,k1));
id g_(2,e2)=
     N*(g_(2,k2)*g_(2,k3)*g_(2,k1)*(1-g5_(2))
       +g_(2,k1)*g_(2,k3)*g_(2,k2)*(1+g5_(2))
     -2*bb*k1.k3*g_(2,k2));
id g_(2,e3)=
     N*(g_(2,k3)*g_(2,k1)*g_(2,k2)*(1-g5_(2))
       +g_(2,k2)*g_(2,k1)*g_(2,k3)*(1+g5_(2))
     -2*cc*k1.k2*g_(2,k3));
*
Trace4,2;
id aa=0;
id bb=0;
id cc=0;
id k1.k1 = 0;
id k2.k2 = 0;
id k3.k3 = 0;
id k1.k2 = s/2;
id k2.k3 = t/2;
id k3.k1 = u/2;
id N^2 = 1/s/t/u/2;
id e_(k1,k2,k3,e) = i*e_(k1,k2,k3,e);
id i^2=-1;
.sort
*b e_(k1,k2,k3,e),k1.e,k2.e.k3.e;
*print [M+++];
*.end
skip [M+++];
*
id e1.e = g_(2,e1,e)/4;
id e2.e = g_(2,e2,e)/4;
id e3.e = g_(2,e3,e)/4;
*+++:
id g_(2,e1)=
     N*(g_(2,k1)*g_(2,k2)*g_(2,k3)*(1-g5_(2))
       +g_(2,k3)*g_(2,k2)*g_(2,k1)*(1+g5_(2))
     -2*aa*k2.k3*g_(2,k1));
id g_(2,e2)=
    -N*(g_(2,k2)*g_(2,k1)*g_(2,k3)*(1-g5_(2))
       +g_(2,k3)*g_(2,k1)*g_(2,k2)*(1+g5_(2))
     -2*bb*k1.k3*g_(2,k2));
id g_(2,e3)=
     N*(g_(2,k3)*g_(2,k1)*g_(2,k2)*(1+g5_(2))
       +g_(2,k2)*g_(2,k1)*g_(2,k3)*(1-g5_(2))
     -2*cc*k1.k2*g_(2,k3));
*
Trace4,2;
id aa=0;
id bb=0;
id cc=0;
id k1.k1 = 0;
id k2.k2 = 0;
id k3.k3 = 0;
id k1.k2 = s/2;
id k2.k3 = t/2;
id k3.k1 = u/2;
id N^2 = 1/s/t/u/2;
id e_(k1,k2,k3,e) = i*e_(k1,k2,k3,e);
id i^2=-1;
.sort
*b e_(k1,k2,k3,e),k1.e,k2.e,k3.e;
*print [M++-];
*.end

Local [M+++cc]=[M+++];
Local [M++-cc]=[M++-];
.sort
skip [M+++],[M++-];
*id e_(k1,k2,k3,e)=-e_(k1,k2,k3,e);
id i=-i;
.sort
Local [M+++sq] = [M+++]*[M+++cc];
Local [M++-sq] = [M++-]*[M++-cc];
```

```
.sort
id i^2=-1;
.sort
*b i,g_(1,5_,e1,e2,e3);
*print [M+++sq],[M++-sq];
*.end
* producing the right color factors for the color octet case (keep in
* mind that one of the two Ts in the product should be reversed because
* of the hermitian conjugate).
id T1*T1 = 19/6;
id T1*T2 = -1/3;
id T1*T3 = -1/3;
id T1*T4 = -1/3;
id T1*T5 = -1/3;
id T1*T6 =  2/3;

id T2*T2 = 19/6;
id T2*T3 = -1/3;
id T2*T4 = 2/3;
id T2*T5 = -1/3;
id T2*T6 = -1/3;

id T3*T3 = 19/6;
id T3*T4 = -1/3;
id T3*T5 = 2/3;
id T3*T6 = -1/3;

id T4*T4 = 19/6;
id T4*T5 = -1/3;
id T4*T6 = -1/3;

id T5*T5 = 19/6;
id T5*T6 = -1/3;

id T6*T6 = 19/6;

*Giving the color factors for the singlet case.
id [d(abc)]^2=40/3;
id [f(abc)]^2=24;
id [f(abc)]*[d(abc)]=0;

id sq2^2 = 2;
id sq3^-2 = 1/3;

.sort
id e_(mu1?,nu1?,la1?,si1?)*e_(mu2?,nu2?,la2?,si2?)= +(
         -d_(mu1,mu2)*(d_(nu1,nu2)*d_(la1,la2)*d_(si1,si2)
                     + d_(si1,nu2)*d_(nu1,la2)*d_(la1,si2)
                     + d_(la1,nu2)*d_(si1,la2)*d_(nu1,si2)
                     - d_(si1,nu2)*d_(la1,la2)*d_(nu1,si2)
                     - d_(la1,nu2)*d_(nu1,la2)*d_(si1,si2)
                     - d_(nu1,nu2)*d_(si1,la2)*d_(la1,si2))
         +d_(nu1,mu2)*(d_(mu1,nu2)*d_(la1,la2)*d_(si1,si2)
                     + d_(si1,nu2)*d_(mu1,la2)*d_(la1,si2)
                     + d_(la1,nu2)*d_(si1,la2)*d_(mu1,si2)
                     - d_(si1,nu2)*d_(la1,la2)*d_(mu1,si2)
                     - d_(la1,nu2)*d_(mu1,la2)*d_(si1,si2)
                     - d_(mu1,nu2)*d_(si1,la2)*d_(la1,si2))
         -d_(la1,mu2)*(d_(mu1,nu2)*d_(nu1,la2)*d_(si1,si2)
                     + d_(si1,nu2)*d_(mu1,la2)*d_(nu1,si2)
                     + d_(nu1,nu2)*d_(si1,la2)*d_(mu1,si2)
                     - d_(si1,nu2)*d_(nu1,la2)*d_(mu1,si2)
                     - d_(nu1,nu2)*d_(mu1,la2)*d_(si1,si2)
                     - d_(mu1,nu2)*d_(si1,la2)*d_(nu1,si2))
         +d_(si1,mu2)*(d_(mu1,nu2)*d_(nu1,la2)*d_(la1,si2)
                     + d_(la1,nu2)*d_(mu1,la2)*d_(nu1,si2)
                     + d_(nu1,nu2)*d_(la1,la2)*d_(mu1,si2)
                     - d_(la1,nu2)*d_(nu1,la2)*d_(mu1,si2)
                     - d_(nu1,nu2)*d_(mu1,la2)*d_(la1,si2)
                     - d_(mu1,nu2)*d_(la1,la2)*d_(nu1,si2)));
contract 0;
.sort
*Print MM;
*.end
```

```
*A1,A2,A3,A4,A5,A6,A7,A8,A9;
id aa=0;
id bb=0;
id cc=0;
.sort
id e_(k1,k2,k3,e)*k1.e=xx;
id e_(k1,k2,k3,e)*k2.e=yy;
id e_(k1,k2,k3,e)*k3.e=zz;
.sort
id k1.k1 = 0;
id k2.k2 = 0;
id k3.k3 = 0;
id k1.k2 = s/2;
id k2.k3 = t/2;
id k3.k1 = u/2;
.sort
*print [e1.e1], [e2.e2], [e3.e3], [e1.e2], [e2.e3], [e3.e1], [k1.e1], [k1.e2],
*      [k1.e3], [k2.e1], [k2.e2], [k3.e2], [k3.e1], [k3.e2], [k3.e3],
*      [e1.ev], [e2.ev], [e3.ev], [e1b.ev], [e2b.ev], [e3b.ev],
*      [e1.evb], [e2.evb], [e3.evb], [e1b.evb], [e2b.evb], [e3b.evb],
*b e_(k1,k2,k3,ev);
*print      MM2,MM2b,MM;
*      ans1, ans2, ans3, bns1, bns2, bns3, xx1,
*      xx11,xx22,xx33,yy11,yy22,yy33,zz12,zz13,zz21,zz23,zz31,zz32;
*.end

* work out other products with
id k1.e*k2.e=-k1.k2+k1.p*k2.p/M^2;
id k1.e*k2.e=-k1.k2+k1.p*k2.p/M^2;
id k1.e*k3.e=-k1.k3+k1.p*k3.p/M^2;
id k1.e*k3.e=-k1.k3+k1.p*k3.p/M^2;
id k2.e*k3.e=-k2.k3+k2.p*k3.p/M^2;
id k2.e*k3.e=-k2.k3+k2.p*k3.p/M^2;
.sort
id k1.e*k1.e=-k1.k1+k1.p^2/M^2;
id k2.e*k2.e=-k2.k2+k2.p^2/M^2;
id k3.e*k3.e=-k3.k3+k3.p^2/M^2;
.sort
id p = k1+k2+k3;
.sort
id k1.k1 = 0;
id k2.k2 = 0;
id k3.k3 = 0;
id k1.k2 = s/2;
id k2.k3 = t/2;
id k3.k1 = u/2;
id N^2 = 1/s/t/u/2;
.sort
id aa=0;
id bb=0;
id cc=0;
.sort
id k1.e*k2.e=-k1.k2+k1.p*k2.p/M^2;
id k1.e*k2.e=-k1.k2+k1.p*k2.p/M^2;
id k1.e*k3.e=-k1.k3+k1.p*k3.p/M^2;
id k1.e*k3.e=-k1.k3+k1.p*k3.p/M^2;
id k2.e*k3.e=-k2.k3+k2.p*k3.p/M^2;
id k2.e*k3.e=-k2.k3+k2.p*k3.p/M^2;
.sort
id k1.e*k1.e=-k1.k1+k1.p^2/M^2;
id k2.e*k2.e=-k2.k2+k2.p^2/M^2;
id k3.e*k3.e=-k3.k3+k3.p^2/M^2;
.sort
id p = k1+k2+k3;
.sort
id k1.k1 = 0;
id k2.k2 = 0;
id k3.k3 = 0;
id k1.k2 = s/2;
id k2.k3 = t/2;
id k3.k1 = u/2;
id N^2 = 1/s/t/u/2;
.sort
id e.e =-3;
```

```
id i^2=-1;
.sort
id e_(k1,k2,k3,e)*k1.e=xx;
id e_(k1,k2,k3,e)*k2.e=xx;
id e_(k1,k2,k3,e)*k3.e=yy;
id e_(k1,k2,k3,e)*k1.e=yy;
id e_(k1,k2,k3,e)*k2.e=zz;
id e_(k1,k2,k3,e)*k3.e=zz;
id M^2=s+t+u;
.sort
id a1=-2*(2*t+u)/N3/N1;
id a2=-2*(2*u+s)/N1/N2;
id a3=-2*(2*s+t)/N2/N3;
id b1=2*N*N1^-1*(-s*N3^-1*(t^2+u*t-u*M^2)+N2^-1*t*(u^2+s*u-s*M^2));
id b2=2*N*N2^-1*(-t*N1^-1*(u^2+s*u-s*M^2)+u*N3^-1*(s^2+s*t-t*M^2));
id b3=2*N*N3^-1*(-u*N2^-1*(s^2+t*s-t*M^2)+s*N1^-1*(t^2+t*u-u*M^2));
id M^2=s+t+u;
id N^2 = 1/s/t/u/2;
.sort
id N1=-s-u;
id N2=-s-t;
id N3=-t-u;
.sort
* Below we apply the formula e(al)*e(be) =
*  -d_(al,be)+p(al)*p(be)/M/M explicitly.
id e.p=0;
id k1.e*k2.e=-k1.k2+k1.p*k2.p/M^2;
id k1.e*k3.e=-k1.k3+k1.p*k3.p/M^2;
id k2.e*k3.e=-k2.k3+k2.p*k3.p/M^2;
id k1.e^2=-k1.k1+k1.p^2/M^2;
id k2.e^2=-k2.k2+k2.p^2/M^2;
id k3.e^2=-k3.k3+k3.p^2/M^2;
* these are not used
id e_(k1,k2,k3,e)*k3.e = xx;
id e_(k1,k2,k3,e)*k2.e = yy;
id e_(k1,k2,k3,e)*k1.e = zz;
id e.e = -3;
.sort
id i^2=-1;
id p.p=M^2;
id p = k1+k2+k3;
id k1.k1=0;
id k2.k2=0;
id k3.k3=0;
id k1.k2=s/2;
id k2.k3=t/2;
id k1.k3=u/2;
*The next part is helicity dependent again.
.sort
skip [M++-sq];
id k2.e1 = -1/2*s*t*N*aa + N*s*t;
id k3.e1 = -1/2*t*u*N*aa;
id k1.e2 = -1/2*s*u*N*bb;
id k3.e2 = -1/2*t*u*N*bb + N*t*u;
id k1.e3 = -1/2*s*u*N*cc + N*u*s;
id k2.e3 = -1/2*s*t*N*cc;
id e1.e2 = 1+1/4*aa*bb-1/2*bb;
id e2.e3 = 1+1/4*bb*cc-1/2*cc;
id e3.e1 = 1+1/4*aa*cc-1/2*aa;
.sort
skip [M+++sq];
id k2.e1 = -1/2*s*t*N*aa + N*s*t;
id k3.e1 = -1/2*t*u*N*aa;
id k1.e2 = -1/2*s*u*N*bb -N*s*u;
id k3.e2 = -1/2*t*u*N*bb;
id k1.e3 = 1/2*s*u*N*cc + N*s*u;
id k2.e3 = 1/2*s*t*N*cc;
id e1.e2 = 1/4*aa*bb+1/2*aa-1/2*bb;
id e2.e3 = -1/4*bb*cc;
id e3.e1 = -1/4*aa*cc-1/2*aa;
.sort
*id aa=0;
*id bb=0;
*id cc=0;
```

```
.sort

* Here, we can compare our answers with results in Gastmans and Wu
* (for the color singlet case) or other sources.
Local answer = 3*[M+++sq]/640*g1^-6/R1/R1*sqpim^2*N1^4*N2^4*N3^4*M^2;
Local GW = M^8*(-M^2*P +5*Q)*M^2*N1*N2*N3;
Local answer2 = 3*[M++-sq]/640*g1^-6/R1/R1*sqpim^2*N1^4*N2^4*N3^4*M^2;
Local GW2 = (x*M^4*s^2*N1*N2*N3+2*Q*(s^4+s^2*M^4))*N1*N2*N3*M^2;
Local diff = answer - GW;
Local diff2 = answer2 - GW2;
id Q = s*t*u;
id P = s*u + u*t + t*s;
id N1 = -u-s;
id N2 = -s-t;
id N3 = -t-u;
id N^2=(s^-1*t^-1*u^-1)/2;
id N^-2=2*(s*t*u);
id M^2 = s+t+u;
id x=1;
.sort

print answer,GW,diff,answer2,GW2,diff2;
.sort
Global [A1P1+++sq] = [M+++sq];
Global [A1P1++-sq] = [M++-sq];
.store
save resultsinglet1P1.sav [A1P1+++sq],[A1P1++-sq];
.store
.end
```

## B.3 dsigoctet1S0.frm

```
* program dsigoctet1S0.frm
* Last change: MM April 17th 2007
*#
* This program uses stored squared amplitudes for different spin
* orbit configurations from octet1S0.frm. It calculates the other
* helicity configurations by interchanging gluon momenta, flips
* the third gluon from incoming to outgoing and calculates the
* final differential cross section.
off statistics;
V k1,k2,k3,e1,e2,e3,e,p1,p2,p,q;
Indices mu,nu,la,si,al,be;
nt [T(a)],[T(b)],[T(c)],[T(d)],[T(e)];
nt S1, S2, S3, S4, S5, S6;
S T1, T2, T3, T4, T5, T6;
S m,me,s,t,u,M,R0,R1,g1,i,N,N1,N2,N3,N1p,N1m,N2p,N2m,N3p,N3m,pl;
S x,y,z,ss,s1,t1,u1,P,Q,[(Q-M^2*P)],alphas,pi,M1,sqpim,sq3,sq2;
S [f(a,b,c)],[d(a,b,c)],[f(abc)],[d(abc)];
S pi, alphas;

load resultoctet1S0.sav;

Local [dsig+++] = [A1S0+++sq];
Local [dsig++-] = [A1S0++-sq];
Local [dsig+-+] = [dsig++-];
Local [dsig-++] = [dsig++-];

* Here the different helicity configurations are calculated by
* interchanging the appropriate gluon momenta.
.sort
skip [dsig+++],[dsig++-],[dsig+-+];
id t = x;
id 1/t = 1/x;
id s = t;
id 1/s = 1/t;
id x = s;
id 1/x = 1/s;
id 1/N1 = x;
id 1/N3 = 1/N1;
id x = 1/N3;
.sort
```

```
skip [dsig+++],[dsig++-],[dsig-++];
id u = x;
id 1/u = 1/x;
id s = u;
id 1/s = 1/u;
id x = s;
id 1/x = 1/s;
id 1/N2 = x;
id 1/N3 = 1/N2;
id x = 1/N3;
.sort

* Here we flip the third gluon from incoming to outgoing, by changing
* k3 into -k3 and flipping its helicity. So from now on, t = -2*k2.k3,
* u = -2*k3.k1.
Local [dsig(+,+,+)] = [dsig++-];
Local [dsig(+,+,-)] = [dsig+++];
Local [dsig(+,-,-)] = [dsig+-+];
Local [dsig(-,+,-)] = [dsig-++];
.sort

*Keep in mind that [dsig(+,+,+)] = [dsig(-,-,-)].
* This gives a factor of 2 for all dsigs.
Local DSIG=(2^-12)*pi^-1*s^-2
 *2*([dsig(+,+,+)]+[dsig(+,+,-)]+[dsig(+,-,-)]+[dsig(-,+,-)]);
.sort
id sqpim^-2 = 1/pi/M;
id g1^2 = alphas*4*pi;
.sort

*Here, we can compute try nice solutions for our answers.
Local answer = DSIG*M*s^2/pi*alphas^-3/R0/R0*(N1*N2*N3)^2*Q*2/5;
*Local RESULT = M^2*P^2*Q - 2*M^4*P^3 - 2*M^4*Q^2 + 2*M^6*P*Q
* + M^8*P^2 - M^10*Q + P^4;
Local RESULT = (P^2-M^2*Q)*((P-M^4)^2 + 2*M^2*Q);
.sort
id N1 = -u-s;
id N2 = -s-t;
id N3 = -t-u;
id N^2=(s^-1*t^-1*u^-1)/2;
id N^-2=2*(s*t*u);
id M^2 = s+t+u;
id Q = s*t*u;
id P = s*t+t*u+u*s;
.sort
*We used the expressions below to get 'answer' in a terms of P, Q and M^2.
*id s*t*u^6 = Q*(M^10-5*M^6*P+5*M^4*Q+5*M^2*P^2-5*P*Q)
*  -(s*t^6*u+s^6*t*u);
*id s*t^2*u^5 = Q*(M^6*P-3*M^2*P^2+5*P*Q-M^4*Q)
* -(s*u^2*t^5+t*s^2*u^5+t*u^2*s^5+u*t^2*s^5+u*s^2*t^5);
*id s*t^3*u^4 = Q*(M^2*P^2-P*Q-2*M^4*Q)
* -(s*u^3*t^4+t*s^3*u^4+t*u^3*s^4+u*t^3*s^4+u*s^3*t^4);
*id s^2*t^2*u^4 = Q^2*(M^4-2*P)-(s^4*t^2*u^2+s^2*t^4*u^2);
*id s^3*t^3*u^2 = Q^2*(P)-(s^2*t^3*u^3+s^3*t^2*u^3);
*id t^2*u^6 = (M^8*P^2-4*M^4*P^3+2*P^4+8*M^6*P*Q+2*P*Q^2-9*M^4*Q^2
* -2*M^10*Q)-(t^6*u^2+s^2*u^6+s^2*t^6+s^6*u^2+s^6*t^2);
*id t^3*u^5 = (M^4*P^3-2*P^4+6*M^2*P^2*Q-7*P*Q^2-3*M^6*P*Q+3*M^4*Q^2)
* -(t^5*u^3+s^3*u^5+s^3*t^5+s^5*u^3+s^5*t^3);
*id t^4*u^4 = (P^4-4*M^2*P^2*Q+4*P*Q^2+2*M^4*Q^2)-(s^4*u^4+s^4*t^4);
.sort
Local diff = answer-RESULT;
.sort
B R0,N1,N2,N3,pi,alphas;
Format Fortran;
print answer;
print RESULT;
print diff;
.end
```

# C Changing variables

We have the differential cross sections as functions of $s$, $t$ and $u$ and we would like to express some of them in variables $M = \sqrt{s+t+u}$, $P = st + tu + us$ and $Q = stu$. This should be possible for the the unpolarized cross sections, because they should not depend on interchanging the names of the gluons, which means that interchanging $s$, $t$ and $u$ for the $2g \to c\bar{c}\left({}^{2S+1}L_J\right) + g$ cross section shouldn't give a different answer.

The only problem is that it isn't easy to find what a particular combination of $s$, $t$ and $u$ will give in terms of $M$, $P$ and $Q$. Therefore, we will discuss here how such a transition is made.

An expression which is completely symmetric in $s$, $t$ and $u$ is made out of combinations like $s^a t^b u^c + s^a t^c u^b + s^b t^a u^c + s^b t^c u^a + s^c t^a u^b + s^c t^b u^a \equiv A(a,b,c)$, where we are free to choose $a \geq b \geq c$.

$$A(a,b,c) = Q^c A(a',b',0) \equiv Q^c B(a',b') \quad , \quad a' = a - c \quad , \quad b' = b - c \tag{13}$$

This function $B(a,b)$ has the following properties:

$$\begin{aligned}
B(a,b) &= B(b,a) \\
PB(a,b) &= (st+tu+us)(s^a t^b + s^a u^b + s^b t^a + s^b u^a + t^a u^b + t^b u^a) \\
&= B(a+1,b+1) + Q\big(B(a-1,b) + B(a,b-1)\big) \\
M^2 B(a,b) &= (s+t+u)(s^a t^b + s^a u^b + s^b t^a + s^b u^a + t^a u^b + t^b u^a) \\
&= B(a+1,b) + B(a,b+1) + QB(a-1,b-1)\,.
\end{aligned}$$

We can use these properties to calculate $B(a,b)$ recursively.

$$\begin{aligned}
B(a,b) &= PB(a-1,b-1) - Q\big(B(a-2,b-1) + B(a-1,b-2)\big) \\
B(a,1) &= PB(a-1,0)/2 - QB(a-2,0)/2 \\
B(a,0) &= M^2 B(a-1,0) - 2B(a-1,1)\,.
\end{aligned}$$

With initial values

$$\begin{aligned}
B(0,0) &= 6 \\
B(1,0) &= 2M^2 \\
B(1,1) &= 2P\,,
\end{aligned}$$

we get

$$\begin{aligned}
B(2,0) &= 2M^4 - 4P \\
B(3,0) &= 2M^6 - 6M^2 P + 6Q \\
B(2,1) &= M^2 P - 3Q \\
B(4,0) &= 2M^8 - 8M^4 P + 8QM^2 + 4P^2 \\
B(3,1) &= M^4 P - 2P^2 - QM^2
\end{aligned}$$

$$
\begin{aligned}
B(2,2) &= 2P^2 - 4QM^2 \\
B(5,0) &= 2M^{10} - 10M^6P + 10M^4Q + 10M^2P^2 - 10PQ \\
B(4,1) &= M^6P - 3M^2P^2 + 5PQ - M^4Q \\
B(3,2) &= M^2P^2 - PQ - 2M^4Q
\end{aligned}
$$

$$
\begin{aligned}
B(6,0) &= 2M^{12} - 12M^8P + 12M^6Q + 18M^4P^2 - 24M^2PQ - 4P^3 + 6Q^2 \\
B(5,1) &= M^8P - 4M^4P^2 + 7M^2PQ + 2P^3 - M^6Q - 3Q^2 \\
B(4,2) &= M^4P^2 - 2P^3 + 4M^2PQ - 3Q^2 - 2M^6Q \\
B(3,3) &= 2P^3 - 6M^2PQ + 6Q^2 \\
B(7,0) &= 2M^{14} - 14M^{10}P + 14M^8Q + 28M^6P^2 - 42M^4PQ - 14M^2P^3 \\
&\quad +14M^2Q^2 + 14P^2Q \\
B(6,1) &= M^{10}P - 5M^6P^2 + 9M^4PQ + 5M^2P^3 - 7P^2Q - M^8Q - 4M^2Q^2 \\
B(5,2) &= M^6P^2 - 3M^2P^3 + 3P^2Q + 6M^4PQ - 7M^2Q^2 - 2M^8Q \\
B(4,3) &= M^2P^3 - P^2Q - 3M^4PQ + 5M^2Q^2 \\
B(8,0) &= 2M^{16} - 16M^{12}P + 16M^{10}Q + 40M^8P^2 - 64M^6PQ - 32M^4P^3 \\
&\quad +24M^4Q^2 + 48M^2P^2Q + 4P^4 - 16PQ^2 \\
B(7,1) &= M^{12}P - 6M^8P^2 + 11M^6PQ + 9M^4P^3 - 17M^2P^2Q - 2P^4 \\
&\quad +8PQ^2 - M^{10}Q - 5M^4Q^2 \\
B(6,2) &= M^8P^2 - 4M^4P^3 + 2P^4 + 8M^6PQ + 2PQ^2 - 9M^4Q^2 - 2M^{10}Q \\
B(5,3) &= M^4P^3 - 2P^4 + 6M^2P^2Q - 7PQ^2 - 3M^6PQ + 3M^4Q^2 \\
B(4,4) &= 2P^4 - 8M^2P^2Q + 8PQ^2 + 4M^4Q^2
\end{aligned}
$$

$$
\begin{aligned}
B(9,0) &= 2M^{18} - 90M^8PQ + 108M^4P^2Q - 54M^2PQ^2 - 18M^{14}P + 6Q^3 \\
&\quad +18M^{12}Q + 54M^{10}P^2 - 60M^6P^3 + 36M^6Q^2 + 18M^2P^4 \\
&\quad -18P^3Q \\
B(8,1) &= M^{14}P - 7M^{10}P^2 + 13M^8PQ + 14M^6P^3 - 30M^4P^2Q - 7M^2P^4 \\
&\quad +19M^2PQ^2 + 9P^3Q - M^{12}Q - 6M^6Q^2 - 3Q^3 \\
B(7,2) &= M^{10}P^2 - 5M^6P^3 - 5M^4P^2Q + 5M^2P^4 - 5P^3Q + 10M^8PQ \\
&\quad +13M^2PQ^2 - 11M^6Q^2 - 3Q^3 - 2M^{12}Q \\
B(6,3) &= M^6P^3 - 3M^2P^4 + 3P^3Q + 9M^4P^2Q - 18M^2PQ^2 - 3M^8PQ \\
&\quad +6Q^3 + 3M^6Q^2 \\
B(5,4) &= M^2P^4 - P^3Q - 4M^4P^2Q + 7M^2PQ^2 - 3Q^3 + 2M^6Q^2
\end{aligned}
$$

$$
\begin{aligned}
B(10,0) &= 2M^{20} - 4P^5 - 120M^{10}PQ + 200M^6P^2Q - 120M^4PQ^2 \\
&\quad -80M^2P^3Q - 20M^{16}P + 20M^{14}Q + 70M^{12}P^2 - 100M^8P^3 \\
&\quad +50M^8Q^2 + 50M^4P^4 + 20M^2Q^3 + 30P^2Q^2 \\
B(9,1) &= M^{16}P - 8M^{12}P^2 + 15M^{10}PQ + 20M^8P^3 - 46M^6P^2Q
\end{aligned}
$$

$$
\begin{aligned}
& -16M^4P^4+33M^4PQ^2+31M^2P^3Q+2P^5-15P^2Q^2-M^{14}Q \\
& -7M^8Q^2-7M^2Q^3 \\
B(8,2) &= M^{12}P^2-6M^8P^3-12M^6P^2Q+9M^4P^4-8M^2P^3Q-2P^5 \\
& +P^2Q^2+12M^{10}PQ+28M^4PQ^2-13M^8Q^2-10M^2Q^3 \\
& -2M^{14}Q \\
B(7,3) &= M^8P^3-4M^4P^4+2P^5+12M^6P^2Q+6P^2Q^2-24M^4PQ^2 \\
& -3M^{10}PQ-2M^2P^3Q+11M^2Q^3+3M^8Q^2 \\
B(6,4) &= M^4P^4-2P^5+8M^2P^3Q-9P^2Q^2-4M^6P^2Q+2M^2Q^3 \\
& +2M^8Q^2 \\
B(5,5) &= 2P^5-10M^2P^3Q+10P^2Q^2+10M^4PQ^2-10M^2Q^3
\end{aligned}
$$

$$
\begin{aligned}
B(11,0) &= 2M^{22}-220M^4P^3Q-154M^{12}PQ+330M^8P^2Q-220M^6PQ^2 \\
& +132M^2P^2Q^2-22M^2P^5-22M^{18}P+22M^{16}Q+88M^{14}P^2 \\
& -154M^{10}P^3+66M^{10}Q^2+110M^6P^4+44M^4Q^3+22P^4Q \\
& -22PQ^3 \\
B(10,1) &= M^{18}P-9M^{14}P^2+17M^{12}PQ+27M^{10}P^3-30M^6P^4 \\
& +50M^6PQ^2+9M^2P^5-11P^4Q-65M^8P^2Q+70M^4P^3Q \\
& -51M^2P^2Q^2+11PQ^3-M^{16}Q-8M^{10}Q^2-12M^4Q^3 \\
B(9,2) &= M^{14}P^2-7M^{10}P^3-21M^8P^2Q+14M^6P^4-7M^4P^3Q \\
& -7M^2P^5-12M^2P^2Q^2+7P^4Q+14M^{12}PQ+47M^6PQ^2 \\
& +5PQ^3-15M^{10}Q^2-19M^4Q^3-2M^{16}Q \\
B(8,3) &= M^{10}P^3-5M^6P^4-10M^4P^3Q+5M^2P^5-5P^4Q+15M^8P^2Q \\
& +30M^2P^2Q^2-30M^6PQ^2-13PQ^3-3M^{12}PQ+14M^4Q^3 \\
& +3M^{10}Q^2 \\
B(7,4) &= M^6P^4-3M^2P^5+3P^4Q+12M^4P^3Q-24M^2P^2Q^2 \\
& -4M^8P^2Q+11PQ^3-2M^6PQ^2+6M^4Q^3+2M^{10}Q^2 \\
B(6,5) &= M^2P^5-P^4Q-5M^4P^3Q+9M^2P^2Q^2-4PQ^3+5M^6PQ^2 \\
& -7M^4Q^3
\end{aligned}
$$

$$
\begin{aligned}
B(12,0) &= 2M^{24}+4P^6+6Q^4-480M^6P^3Q-192M^{14}PQ+504M^{10}P^2Q \\
& -360M^8PQ^2+360M^4P^2Q^2+120M^2P^4Q-96M^2PQ^3 \\
& -24M^{20}P+24M^{18}Q+108M^{16}P^2-224M^{12}P^3+84M^{12}Q^2 \\
& +210M^8P^4+80M^6Q^3-48P^3Q^2-72M^4P^5 \\
B(11,1) &= -2P^6-3Q^4+130M^6P^3Q+19M^{14}PQ-87M^{10}P^2Q \\
& +70M^8PQ^2-114M^4P^2Q^2-49M^2P^4Q+37M^2PQ^3+M^{20}P \\
& -M^{18}Q-10M^{16}P^2+35M^{12}P^3-9M^{12}Q^2-50M^8P^4 \\
& -18M^6Q^3+24P^3Q^2+25M^4P^5 \\
B(10,2) &= 2P^6-3Q^4+16M^{14}PQ-32M^{10}P^2Q+70M^8PQ^2-16M^4P^5
\end{aligned}
$$

$$
\begin{aligned}
& -45M^4P^2Q^2+20M^2P^4Q+28M^2PQ^3-2M^{18}Q+M^{16}P^2 \\
& -8M^{12}P^3-17M^{12}Q^2+20M^8P^4-30M^6Q^3-6P^3Q^2 \\
B(9,3) &= M^{12}P^3-6M^8P^4-21M^6P^3Q+9M^4P^5-6M^2P^4Q-2P^6 \\
& -3P^3Q^2+18M^{10}P^2Q+63M^4P^2Q^2-36M^8PQ^2-42M^2PQ^3 \\
& -3M^{14}PQ+6Q^4+3M^{12}Q^2+17M^6Q^3 \\
B(8,4) &= M^8P^4-4M^4P^5+2P^6+16M^6P^3Q+8P^3Q^2-28M^4P^2Q^2 \\
& -4M^{10}P^2Q-4M^2P^4Q+16M^2PQ^3-4M^8PQ^2-3Q^4 \\
& +2M^{12}Q^2+8M^6Q^3 \\
B(7,5) &= M^4P^5-2P^6+10M^2P^4Q-11P^3Q^2-5M^6P^3Q+13M^2PQ^3 \\
& +5M^8PQ^2-5M^4P^2Q^2-3Q^4-5M^6Q^3 \\
B(6,6) &= 2P^6-12M^2P^4Q+12P^3Q^2+18M^4P^2Q^2-24M^2PQ^3+6Q^4 \\
& -4M^6Q^3
\end{aligned}
$$

$$
\begin{aligned}
B(13,0) &= 26M^{20}Q+26M^2P^6-182M^6P^5+130M^{18}P^2-312M^{14}P^3 \\
& +104M^{14}Q^2+364M^{10}P^4+130M^8Q^3+26M^2Q^4-26P^5Q \\
& +52P^2Q^3+2M^{26}-260M^4PQ^3-234M^{16}PQ \\
& +728M^{12}P^2Q-546M^{10}PQ^2-910M^8P^3Q+780M^6P^2Q^2 \\
& -260M^2P^3Q^2+390M^4P^4Q-26M^{22}P \\
B(12,1) &= -M^{20}Q-11M^2P^6+55M^6P^5-11M^{18}P^2+44M^{14}P^3 \\
& -10M^{14}Q^2-77M^{10}P^4-25M^8Q^3-10M^2Q^4+13P^5Q \\
& -26P^2Q^3+82M^4PQ^3+21M^{16}PQ-112M^{12}P^2Q \\
& +93M^{10}PQ^2+215M^8P^3Q-210M^6P^2Q^2+106M^2P^3Q^2 \\
& -135M^4P^4Q+M^{22}P \\
B(11,2) &= -2M^{20}Q+9M^2P^6-30M^6P^5+M^{18}P^2-9M^{14}P^3-19M^{14}Q^2 \\
& +27M^{10}P^4-43M^8Q^3-13M^2Q^4-9P^5Q-4P^2Q^3 \\
& +75M^4PQ^3+18M^{16}PQ-45M^{12}P^2Q+97M^{10}PQ^2 \\
& +15M^8P^3Q-104M^6P^2Q^2-2M^2P^3Q^2+36M^4P^4Q \\
B(10,3) &= -7M^2P^6+14M^6P^5+M^{14}P^3+3M^{14}Q^2-7M^{10}P^4+20M^8Q^3 \\
& +17M^2Q^4+7P^5Q+19P^2Q^3-80M^4PQ^3-3M^{16}PQ \\
& +21M^{12}P^2Q-42M^{10}PQ^2-35M^8P^3Q+105M^6P^2Q^2 \\
& -35M^2P^3Q^2 \\
B(9,4) &= M^{10}P^4-5M^6P^5-15M^4P^4Q+5M^2P^6-5P^5Q+20M^8P^3Q \\
& +40M^2P^3Q^2-30M^6P^2Q^2-20P^2Q^3-4M^{12}P^2Q+10M^4PQ^3 \\
& -6M^{10}PQ^2+10M^8Q^3-M^2Q^4+2M^{14}Q^2 \\
B(8,5) &= M^6P^5-3M^2P^6+3P^5Q+15M^4P^4Q-30M^2P^3Q^2 \\
& -5M^8P^3Q+14P^2Q^3-10M^6P^2Q^2+30M^4PQ^3 \\
& +5M^{10}PQ^2-5M^8Q^3-13M^2Q^4 \\
B(7,6) &= M^2P^6-P^5Q-6M^4P^4Q+11M^2P^3Q^2-5P^2Q^3+9M^6P^2Q^2 \\
& -17M^4PQ^3+8M^2Q^4-2M^8Q^3
\end{aligned}
$$

$$
\begin{aligned}
B(14,0) &= 980M^6P^4Q+2M^{28}-4P^7+70M^4Q^4-28M^{24}P+28M^{22}Q+\\
&\quad 154M^{20}P^2-420M^{16}P^3+126M^{16}Q^2+588M^{12}P^4+196M^{10}Q^3\\
&\quad -392M^8P^5+98M^4P^6-28PQ^4-560M^6PQ^3+1008M^{14}P^2Q\\
&\quad -784M^{12}PQ^2-280M^{18}PQ-1568M^{10}P^3Q+1470M^8P^2Q^2\\
&\quad -840M^4P^3Q^2-168M^2P^5Q+280M^2P^2Q^3+70P^4Q^2\\
B(13,1) &= -295M^6P^4Q+2P^7-22M^4Q^4+M^{24}P-M^{22}Q-12M^{20}P^2\\
&\quad +54M^{16}P^3-11M^{16}Q^2-112M^{12}P^4-33M^{10}Q^3+105M^8P^5\\
&\quad -36M^4P^6+14PQ^4+150M^6PQ^3-140M^{14}P^2Q\\
&\quad +119M^{12}PQ^2+23M^{18}PQ+329M^{10}P^3Q-345M^8P^2Q^2\\
&\quad +290M^4P^3Q^2+71M^2P^5Q-114M^2P^2Q^3-35P^4Q^2\\
B(12,2) &= 50M^6P^4Q-2P^7-32M^4Q^4-2M^{22}Q+M^{20}P^2-10M^{16}P^3\\
&\quad -21M^{16}Q^2+35M^{12}P^4-58M^{10}Q^3-50M^8P^5+25M^4P^6\\
&\quad +8PQ^4+152M^6PQ^3-60M^{14}P^2Q+128M^{12}PQ^2\\
&\quad +20M^{18}PQ+40M^{10}P^3Q-195M^8P^2Q^2+36M^4P^3Q^2\\
&\quad -36M^2P^5Q-44M^2P^2Q^3+13P^4Q^2\\
B(11,3) &= 16M^6P^4Q+2P^7+31M^4Q^4+M^{16}P^3+3M^{16}Q^2-8M^{12}P^4\\
&\quad +23M^{10}Q^3+20M^8P^5-16M^4P^6-19PQ^4-127M^6PQ^3\\
&\quad +24M^{14}P^2Q-48M^{12}PQ^2-3M^{18}PQ-52M^{10}P^3Q\\
&\quad +156M^8P^2Q^2-108M^4P^3Q^2+18M^2P^5Q+91M^2P^2Q^3\\
&\quad -2P^4Q^2\\
B(10,4) &= -30M^6P^4Q-2P^7+5M^4Q^4+2M^{16}Q^2+M^{12}P^4+12M^{10}Q^3\\
&\quad -6M^8P^5+9M^4P^6+14PQ^4-4M^{14}P^2Q-8M^{12}PQ^2\\
&\quad +24M^{10}P^3Q-30M^8P^2Q^2+80M^4P^3Q^2-4M^2P^5Q\\
&\quad -60M^2P^2Q^3-5P^4Q^2\\
B(9,5) &= 20M^6P^4Q+2P^7-20M^4Q^4-5M^{10}Q^3+M^8P^5-4M^4P^6\\
&\quad -PQ^4+40M^6PQ^3+5M^{12}PQ^2-5M^{10}P^3Q-15M^8P^2Q^2\\
&\quad -30M^4P^3Q^2-6M^2P^5Q+10M^2P^2Q^3+10P^4Q^2\\
B(8,6) &= M^4P^6-2P^7+12M^2P^5Q-13P^4Q^2-6M^6P^4Q+28M^2P^2Q^3\\
&\quad +9M^8P^2Q^2-12M^4P^3Q^2-10PQ^4-8M^6PQ^3-2M^{10}Q^3\\
&\quad +M^4Q^4\\
B(7,7) &= 2P^7-14M^2P^5Q+14P^4Q^2+28M^4P^3Q^2-42M^2P^2Q^3\\
&\quad +14PQ^4-14M^6PQ^3+14M^4Q^4\,.
\end{aligned}
$$