RADBOUD UNIVERSITY NIJMEGEN

FACULTY OF SCIENCE (FNWI)

THEORETICAL HIGH ENERGY PHYSICS

Conformal symmetry versus quantization of geodesic motion

Author:
Mila J. KEIJER

Supervisor:
Prof. Dr. W.J.P. BEENAKKER

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1 Introduction

The Standard Model of particle physics was developed in flat spacetime. In this process, global gauge invariances were transformed into local ones in accordance with the local characteristics of special relativity. The resulting symmetries brought with them certain interactions and conserved quantities, so one can say they are the foundation on which the Standard Model was built. Of course, since we are now aware of the fact that we live in a curved spacetime, rather than a flat one, it is important to bring the Standard Model into a curved spacetime setting. In this thesis, we implement this in the form of a background metric. The starting point is a Lagrangian with gauge symmetry before symmetry breaking. At this point, there is no need for mass and our theory is globally scale invariant. Similarly to gauge invariances, we would like to implement this global phenomenon locally and investigate the consequences. To that end, conformal symmetry is introduced. The conformal transformation transforms both the metric and the fields. Of course, electroweak symmetry breaking is eventually needed to introduce mass. This means that conformal symmetry would also need to be broken at this point, perhaps leaving interesting remnants for us to observe.

Many people have worked on conformal symmetries, including Dr. E. A. Tagirov. He investigated conformal symmetry for a spinless Quantum Field Theory (QFT) in general spacetime in [1]. He found that a coupling between the scalar field and the scalar curvature is needed for a conformal symmetry to exist. He then proceeded to compare this to an entirely different approach in which a potential term proportional to the scalar curvature arises. This second approach, described in [2], is the quantization of the non-relativistic motion of a spinless particle moving along a geodesic line. This approach was inspired by DeWitt, who did similar calculations resulting in a potential term proportional to the scalar curvature in [3]. When Tagirov compares the two coupling coefficients of the different approaches, he concludes that they are the same in 4 dimensions and goes on to say that this cannot be a coincidence. This is because the coefficient in the geodesic approach is a fingerprint of the underlying conformal symmetry of the massless theory, according to Tagirov.

We were intrigued by these findings and set out to further understand them. However, this proved challenging since the reasoning in these papers is limited and often times more complex than can be determined at first glance, which brings us to our present goal. In this thesis, the reasoning behind the comparison of the coupling with curvature in these two approaches is reviewed and documented in a clear, unambiguous, and complete manner.

To this end, we will first investigate conformal symmetry in a QFT setting in line with the first approach of Tagirov, which can be found in chapter 2. Next, we will investigate the second approach of Tagirov, the quantization of the non-relativistic motion of a spinless particle that is moving along a geodesic line, which can be found in chapter 3. Our conclusions and outlook can then be found in chapter 4.

2 Conformal transformations

We first want to investigate conformal transformations in a QFT setting. Although we want them to work similarly to the gauge transformations in the SM, they are more complex. A large part of this complexity is due to the fact that we are in a general spacetime setting, which affects our Lagrangian and equations of motion. Therefore, we first focus on QFT in a general spacetime. After this, we can focus on the conformal transformations and the resulting symmetries.
Tagirov focuses on spin-0 particles and our main focus will reflect this. However, we grew curious about the effects on spin-$\frac{1}{2}$ particles as well. Since they show some differences with respect to the spin-0 particles, we decided to include them in this chapter.

2.1 Notation and conventions

In this chapter, we use natural units. Additionally, a general n-dimensional metric $g^{\mu\nu}$ is used, where of course $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. We use the 'mostly minus' metric signature, which means the Riemann tensor is

$$R^{\alpha}_{\beta\gamma\delta} = \partial_\delta \Gamma^{\alpha}_{\beta\gamma} - \partial_\gamma \Gamma^{\alpha}_{\beta\delta} + \Gamma^{\alpha}_{\delta\lambda}\Gamma^{\lambda}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\lambda}\Gamma^{\lambda}_{\beta\delta}, \tag{1}$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\delta\alpha}\left(\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\delta\beta} - \partial_\delta g_{\beta\gamma}\right). \tag{2}$$

The scalar curvature is defined as

$$R = g^{\beta\delta} R^{\alpha}_{\beta\alpha\delta}. \tag{3}$$

2.2 Quantum Field Theory in a general spacetime

QFT is often set up in flat Minkowski spacetime[i] with metric η_{ab}. However, we want to work in a general spacetime with a general metric $g_{\mu\nu}$. In this section, we discuss the necessary changes to arrive at a QFT in a general spacetime.

2.2.1 Spin-0

In flat Minkowski spacetime, we have the scalar Lagrangian[ii] $\mathscr{L} = \frac{1}{2}\left(\eta^{ab}\partial_a\phi\partial_b\phi - m^2\phi^2\right)$, where ϕ is the scalar field and m the mass. The Euler-Lagrange equation that accompanies it is $\left(\Box + m^2\right)\phi = 0$, which is also known as the Klein-Gordon (KG) equation. Here, we have used that $\Box = \eta^{ab}\partial_a\partial_b$ in Minkowski spacetime. The switch to a general spacetime is not complicated in this case. The metric η^{ab} is replaced by the metric $g^{\mu\nu}$ and the partial derivative ∂_a is replaced by the covariant derivative ∇_μ. This results in

$$\mathscr{L} = \frac{1}{2}\sqrt{|g|}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2 - \xi R\phi^2\right), \tag{4}$$

where $g = \det(g_{\mu\nu})$ is the determinant of the metric tensor, R is the scalar curvature, and ξ is a constant. $\nabla_\mu\phi = \partial_\mu\phi$ was used since ϕ is a scalar field. We have added $\sqrt{|g|}$, which is a result of the invariant volume element $\sqrt{|g|}d^n x$. The term containing the R-ϕ^2 coupling is also added. This is allowed because in flat spacetime $R = 0$, so this term

[i] The Roman indices are used to make an explicit distinction between flat Minkowski spacetime and a general spacetime, for which Greek indices are used. This will help with clarity later on.
[ii] Note that Lagrangian density is meant whenever the term Lagrangian is used.

drops out automatically when we go back to Minkowski spacetime. The special case when $\xi = 0$ is referred to as minimal coupling. The KG-equation turns into

$$\left(\Box + m^2 + \xi R\right)\phi = 0 \tag{5}$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$.

2.2.2 Spin-$\frac{1}{2}$

Similarly to the spin-0 approach, we start with the Dirac Lagrangian in Minkowski spacetime, $\mathscr{L}_D = \frac{i}{2}\left(\bar{\psi}\gamma^a\partial_a\psi - \bar{\psi}\gamma^a\overleftarrow{\partial}_a\psi\right) - m\bar{\psi}\psi$, where ψ is the spin-$\frac{1}{2}$ field and γ^a are the Dirac gamma matrices. The Euler-Lagrange equation that accompanies it is $\left(i\gamma^a\partial_a - m\right)\psi = 0$, which is also known as the Dirac equation. In a general spacetime these become

$$\mathscr{L}_D = \sqrt{|g|}\left(\frac{i}{2}\left(\bar{\psi}\gamma^\mu\nabla_\mu\psi - \bar{\psi}\gamma^\mu\overleftarrow{\nabla}_\mu\psi\right) - m\bar{\psi}\psi\right) \tag{6}$$

and

$$\left(i\gamma^\mu\nabla_\mu - m\right)\psi = 0. \tag{7}$$

Please note that $\bar{\psi} \neq \psi^\dagger\gamma^0$ because we work in a different spacetime than Minkowski spacetime. Also note that the transformation into a general spacetime is not as straight forward as it seems. The field ψ is not scalar, which means $\nabla_\mu\psi \neq \partial_\mu\psi$ but rather somewhat more complicated. This is easily explained using the n-bein formalism as described by Parker and Toms in [4]. We will not explain the entire procedure in the same depth as they do, instead we quote the necessary definitions.

The n-bein formalism relies on the introduction of a tensor, the n-bein $e^a_\mu(x)$. With this n-bein[iii] one can relate the general spacetime to a local orthonormal frame (Minkowski spacetime) in the following way:

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}. \tag{8}$$

This can then also be used for other quantities, for example $A^a = e^a_\mu A^\mu$. This raises the question of what $\nabla_\mu A^a$ represents. A^a is not a scalar and therefore ∇_μ does not only involve a partial derivative but also a connection. However, this connection is not the Christoffel symbol $\Gamma^\lambda_{\mu\nu}$ but rather something involving the Minkowski indices. We introduce the connection $\omega^a_{\mu b}$ in the following way,

$$\nabla_\mu A^a = \partial_\mu A^a + \omega^a_{\mu b}A^b. \tag{9}$$

In order to fully define this connection we use that $\nabla_\mu e^a_\nu = 0$ and arrive at

$$\omega^a_{\mu b} = -e^\nu_b(\partial_\mu e^a_\nu - \Gamma^\lambda_{\mu\nu}e^a_\lambda). \tag{10}$$

The exact steps taken to arrive at this definition can be found on page 223 of [4]. Note that, with this definition, we can rewrite the Riemann tensor, and thus the scalar curvature R, in terms of the connection $\omega^a_{\mu b}$ and the n-bein e^a_μ.

Of course, what we are actually interested in is $\nabla_\mu\psi$. If we follow the reasoning in [4], we arrive at

$$\nabla_\mu\psi = \partial_\mu\psi + \frac{1}{8}\omega^{ab}_\mu[\gamma_a,\gamma_b]\psi \tag{11}$$

[iii]Note that the n-bein is not unique, nothing stops us from using a Lorentz-transformed n-bein. This means we need to make sure that the choice of n-bein has no impact on the end result.

where γ_a are the Dirac gamma matrices in Minkowski spacetime satisfying $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$. One can bring those into our general spacetime as well, $\gamma^\mu = e^\mu_a \gamma^a$, where they satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

In Minkowski spacetime, the Dirac equation squared equals the Klein-Gordon equation. However, in our general spacetime with the new covariant derivative this becomes

$$\left(\gamma^\mu \nabla_\mu\right)^2 \psi = \Box\psi + \frac{1}{4} R\psi. \tag{12}$$

Here, we see that a coupling with the scalar curvature R arises naturally and with a fixed coefficient.

2.3 Conformal transformations

Now we have QFT in a general spacetime, we can investigate the conformal transformations. As mentioned in the introduction, we want to implement local scale invariance, which is why conformal transformations are introduced. As part of the conformal transformations, not only the fields transform but the metric itself transforms as well. The transformations are defined as follows,

$$\phi'(x) = \Omega(x)^{\frac{2-n}{2}} \phi(x) \tag{13}$$

$$\psi'(x) = \Omega(x)^{\frac{1-n}{2}} \psi(x) \tag{14}$$

$$g'_{\mu\nu}(x) = \Omega(x)^2 g_{\mu\nu}(x). \tag{15}$$

Here, Ω is the factor describing the local scale transformation and is itself dependent on the spacetime coordinates x. As one can see, the power of Ω is determined by the mass-dimension of the transformed field. These are the basic transformed quantities, but note that the transformation of the metric means that $\Gamma^\lambda_{\mu\nu}$ and R transform as well. In order to not interrupt the flow of this thesis too much, all step-by-step derivations of relevant transformations can be found in appendix A. In this section, we will discuss the results of those transformations.

Before we perform the conformal transformations of the Lagrangians and Euler-Lagrange equations mentioned above, we remove the mass. This is done because the conformal symmetry is based on the scale invariance of these equations, which is only applicable when they do not contain mass.

2.3.1 Spin-0

For spin-0, we are interested in the transformation of the Lagrangian and the Klein-Gordon equation, since their invariance under the transformation will signal a conformal symmetry. We begin with the scalar Lagrangian, which behaves in the following way under conformal transformation,

$$\mathscr{L}' = \mathscr{L} + \nabla_\mu \left(-\sqrt{-g} \left(\frac{n}{2} - 1 \right) \frac{1}{2} \Omega^{-1} g^{\mu\nu} \phi^2 \partial_\nu \Omega \right). \tag{16}$$

It is invariant up to a total four-divergence, which is quite common in symmetries. The transformation of the Klein-Gordon equation yields the following result

$$\Box'\phi' + \xi R'\phi' = \Omega^{-\frac{n}{2}-1} \left(\Box\phi + \xi R\phi \right) = 0, \tag{17}$$

which is also invariant. However, both of these invariances only hold for a coupling coefficient of $\xi = \xi_c = \frac{n-2}{4(n-1)}$. This is the same value Tagirov finds in [1] and Birrell and Davies have found in [5]. This also corresponds to the findings of Penrose [6], who did similar calculations in four dimensions which resulted in a coupling coefficient of $\frac{1}{6}$. Since the Klein-Gordon equation and the Lagrangian are invariant, we are indeed dealing with a conformal symmetry.

Note that it was necessary to introduce the coupling $\xi R\phi^2$ between the scalar curvature and the field, with a specific value for the coupling coefficient, in order to arrive at a conformal symmetry. Now, we have followed Tagirov's first approach and have come to similar conclusions. This means that we have everything we need to continue with the second approach, the quantization of geodesic motion. However, before we do this, we have added the transformations for spin-$\frac{1}{2}$ particles below for completeness.

2.3.2 Spin-$\frac{1}{2}$

For spin-$\frac{1}{2}$ we are interested in the transformation of the Lagrangian and the Dirac equation, since their invariance under the transformation will again signal a conformal symmetry. The Dirac Lagrangian behaves as follows under conformal transformation,

$$\mathscr{L}'_D = \mathscr{L}_D, \tag{18}$$

so it is invariant. The transformed Dirac equation becomes,

$$i\gamma'^{\mu}\nabla'_{\mu}\psi' = \Omega^{\frac{1-n}{2}-1} i\gamma^{\mu}\nabla_{\mu}\psi = 0, \tag{19}$$

which is invariant. In this case, it was not necessary to introduce any extra couplings in order to arrive at a conformal symmetry due to the nature of $\nabla_{\mu}\psi$. Even though we do not need to add any extra terms, we have seen in equation 12 that a coupling between the Dirac field and the scalar curvature arises naturally. It is interesting to note that it is impossible to make the coefficients $\frac{1}{4}$ and $\xi_c = \frac{n-2}{4(n-1)}$ for the different spin cases match for any number of dimensions. With this we have completed our investigation of the conformal transformations and will continue with the quantization of geodesic motion.

3 Quantization of geodesic motion

Now we have completed our investigation of conformal transformations, we can continue with the second approach of Tagirov. In this approach, the non-relativistic motion of a spinless particle, moving along a geodesic line in a general spacetime, is quantized. In this quantization, we pay special attention to some rules of ordering, which become more complex in a general spacetime. We then hope to find that the quantized Hamiltonian has a potential term proportional to the scalar curvature R. In this section, we will follow the outline of sections 2 and 3 of Tagirov's paper [2] and add our own calculations and observations.

3.1 Notation and conventions

In this chapter, we focus on the quantization process and we will therefore not be working in natural units. For the sake of simplicity, we will be working with a slightly different metric than before. We take time foliations of the spacetime to separate a single component, $t = x^0/c$, from the rest of the metric in the following way,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = (c\,dt)^2 - \omega_{ij}(t, \mathrm{x}) d\mathrm{x}^i d\mathrm{x}^j. \tag{20}$$

Since $g_{\mu\nu}$ is n-dimensional, it follows that ω_{ij} is $(n-1)$-dimensional. The slightly different notation between dx and $d\mathrm{x}$ is used to highlight the difference between coordinates before and after separating the time component.[iv] This coordinate system is chosen without loss of generality.

3.2 Classical Hamiltonian along a geodesic line

Before we start the quantization process, we must first find a classical Hamiltonian for us to quantize. To this end, we start with an on-shell spinless particle moving along a geodesic line, adhering to the following constraint,

$$g^{\mu\nu} p_\mu p_\nu = m^2 c^2. \tag{21}$$

Similarly to the metric, we can separate the time component and arrive at

$$\left(p_0 + mc\sqrt{1 + \frac{2H_0}{mc^2}}\right)\left(p_0 - mc\sqrt{1 + \frac{2H_0}{mc^2}}\right) = 0, \tag{22}$$

where H_0 is

$$H_0 = \frac{1}{2m}\omega^{ij}(\mathrm{x}) p_i p_j. \tag{23}$$

We then choose the solution for which $H = cp_0 > 0$ and arrive at the classical Hamiltonian

$$H = mc^2\sqrt{1 + \frac{2H_0}{mc^2}}. \tag{24}$$

Since we are interested in a non-relativistic particle, we can expand H into

$$H \approx mc^2 + H_0 \tag{25}$$

plus higher order terms.

[iv]Note that in this section the difference between the Greek and Roman indices is not for Minkowski spacetime but for the separation of the time component.

3.3 Quantization and rules of ordering

In canonical quantization the primary observables x and p become operators, $\hat{q}$ and $\hat{p}$ respectively, satisfying the canonical commutation relations

$$[\hat{q}^i, \hat{q}^j] = 0 \qquad [\hat{p}_i, \hat{p}_j] = 0 \qquad [\hat{q}^i, \hat{p}_j] = i\hbar\delta^i_j. \tag{26}$$

Note that upper indices are used for the position operators and lower indices for the momentum operators from now on.[v] We will mostly be working in coordinate representation, where the operators become

$$\hat{q}^i \xrightarrow{\text{coord. repr.}} \mathrm{x}^i \tag{27a}$$

$$\hat{p}_j \xrightarrow{\text{coord. repr.}} -i\hbar\left(\frac{\partial}{\partial \mathrm{x}^j} + \frac{1}{4}\left(\frac{\partial}{\partial \mathrm{x}^j}\ln\omega(\mathrm{x})\right)\right) = -i\hbar\omega^{-\frac{1}{4}}(\mathrm{x})\frac{\partial}{\partial \mathrm{x}^j}\omega^{\frac{1}{4}}(\mathrm{x}). \tag{27b}$$

Here, $\omega(\mathrm{x}) = \det(\omega_{ij}(\mathrm{x}))$ is the determinant of the metric tensor and $\frac{\partial}{\partial \mathrm{x}^j}$ acts on everything to its right.

Obviously, we want to quantize the classical Hamiltonian, equation 25. The non-trivial part of this quantization is H_0 and, as such, we will focus on the quantization of equation 23. In this quantization, the rule of hermitian operator ordering becomes rather important. Just like Tagirov, we choose a combination of two rules of ordering to see which one might be better suited to our goal.

The first rule of ordering we choose is Weyl ordering. This popular ordering takes all possible orderings of non-commuting operators and weighs them equally. A one dimensional example is

$$\mathrm{x}p^2 \xrightarrow{Q} \frac{1}{3}(\hat{q}\hat{p}^2 + \hat{p}\hat{q}\hat{p} + \hat{p}^2\hat{q}). \tag{28}$$

The second rule of ordering is Rivier ordering. In this case, all position operators are grouped on one side and all momentum operators on the other side, supplemented by the reversed expression and a weight of $\frac{1}{2}$. The above one dimensional example then becomes

$$\mathrm{x}p^2 \xrightarrow{Q} \frac{1}{2}(\hat{q}\hat{p}^2 + \hat{p}^2\hat{q}). \tag{29}$$

We then combine the two by taking ν times Weyl ordering and $(1-\nu)$ times Rivier ordering.

In our case, the position operators $\hat{q}$ only appear inside the metric $\omega^{ij}(\hat{q})$, which poses a problem. We do not know what form $\omega^{ij}(\mathrm{x})$ may take or how many position operators it contains. To tackle this problem we will use the method of implementing an ordering by means of integrals, where we make the assumption that $\omega^{ij}(\mathrm{x})$ can be written as a polynomial. The method of implementing an ordering using integrals in Euclidean space is described in detail by Berezin and Shubin in [7]. Our version in the general spacetime with metric ω^{ij} is based on that method. The goal is to write $\hat{f}\Psi(\mathrm{x})$, a random function of operators acting on a state function, as an integral of the symbol f. The symbol is the classical analogue of the operator function $\hat{f}$, the way you would intuitively write it. We use this because although we might not know $\hat{f}$, we do know the function we want to quantize, f. This naturally means that a rule of ordering must be chosen and implemented in order to switch between a symbol and a quantized function. This integral method is an important part of our reasoning and the entire process is thus described in detail in this section.

[v] If the indices were both upper/lower, the last commutation relation would contain the metric instead of the delta-function, which would complicate the calculations.

3.3.1 Weyl ordering using integrals

We will start with Weyl ordering, since it poses the biggest problem in our investigation. As mentioned before, in Weyl ordering, every ordering of non-commuting operators has an equal contribution. To write this down simply, we use the Weyl-ordered symmetric product $\left(A_1^{k_1}...A_N^{k_N}\right)_W$ of non-commuting operators $A_1^{k_1},...,A_N^{k_N}$, which is defined as follows,

$$(\alpha_1 A_1 + ... + \alpha_N A_N)^k = \sum_{k_1+...+k_N=k} \frac{k!}{k_1!...k_N!} \alpha_1^{k_1}...\alpha_N^{k_N} \left(A_1^{k_1}...A_N^{k_N}\right)_W. \tag{30}$$

In our case there are only pairs of non-commuting position and momentum operators, so $N = 2$ and the previous equation becomes

$$\left(\alpha\hat{q} + \beta\hat{p}\right)^k = \sum_{l+m=k} \frac{k!}{l!m!} \alpha^l \beta^m \left(\hat{q}^l \hat{p}^m\right)_W \tag{31}$$

for one such $(\hat{q},\hat{p})$ pair. This means, for example,

$$\left(\hat{q}\hat{p}\right)_W = \frac{1}{2}\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right) \qquad \left(\hat{q}^2\hat{p}\right)_W = \frac{1}{3}\left(\hat{q}^2\hat{p} + \hat{q}\hat{p}\hat{q} + \hat{p}\hat{q}^2\right). \tag{32}$$

We then define our polynomial Weyl-ordered function $\hat{f}_W$ as follows in $n-1$ spatial dimensions

$$\hat{f}_W = \sum_{\sigma_\alpha,\sigma_\beta \leq m} c_{\vec{\alpha}\vec{\beta}} (\hat{q}_1^{\alpha_1}\hat{p}_1^{\beta_1})_W ... (\hat{q}_{n-1}^{\alpha_{n-1}}\hat{p}_{n-1}^{\beta_{n-1}})_W, \tag{33}$$

where $\sigma_\alpha = \alpha_1 + ... + \alpha_{n-1}$, $\sigma_\beta = \beta_1 + ... + \beta_{n-1}$, m is the maximum degree of the polynomial, and $c_{\vec{\alpha}\vec{\beta}}$ is a coefficient. To clarify the notation, with $\hat{q}_k^{\alpha_k}$ we mean that every operator $\hat{q}_k$ has its own corresponding power α_k, where the latter are integers and the repeated k-indices are not summed over. This series of symmetric products is implemented in the form of an exponential function, similarly to [7],

$$\begin{aligned} e^{i\left(r_i\hat{q}^i + s^j\hat{p}_j\right)} &= e^{i\left(r_1\hat{q}^1 + s^1\hat{p}_1\right)} ... e^{i\left(r_{n-1}\hat{q}^{n-1} + s^{n-1}\hat{p}_{n-1}\right)} \\ &= \sum_{k_1=0}^{\infty} \frac{i^{k_1}}{k_1!}\left(r_1\hat{q}^1 + s^1\hat{p}_1\right)^{k_1} ... \sum_{k_{n-1}=0}^{\infty} \frac{i^{k_{n-1}}}{k_{n-1}!}\left(r_{n-1}\hat{q}^{n-1} + s^{n-1}\hat{p}_{n-1}\right)^{k_{n-1}}. \end{aligned} \tag{34}$$

Each of the terms above is of the form of equation 31, so we automatically arrive at something of the form of equation 33. To deal with the lack of knowledge about the precise functional form of the operator $\hat{f}$, we use a Fourier Transformation (FT) of position and momentum at the same time. To this end, we start by writing $\hat{f}$ as a FT of some function $\varphi(r,s)$,

$$\hat{f}_W(\hat{q},\hat{p}) = \int e^{i\left(r_i\hat{q}^i + s^j\hat{p}_j\right)} \varphi(r,s)\, dr\, ds. \tag{35}$$

Here, we have used the compact notation $dr = dr_1...dr_{n-1}$ and $ds = ds^1...ds^{n-1}$, the same holds true for dp and dq. We also write the classical symbol f in a similar way,

$$f(q,p) = \int e^{i\left(r_i q^i + s^j p_j\right)} \varphi(r,s)\, dr\, ds, \tag{36}$$

and its inverse

$$\varphi(r,s) = (2\pi)^{2(1-n)} \int e^{-i\left(r_i q^i + s^j p_j\right)} f(q,p)\, dq\, dp. \tag{37}$$

As mentioned before, we are interested in the effect of $\hat{f}_W$ on a state function $\Psi(\mathrm{x})$,

$$\hat{f}_W \Psi(\mathrm{x}) = \int e^{i\left(r_i\hat{q}^i + s^j\hat{p}_j\right)} \Psi(\mathrm{x}) \varphi(r,s)\, dr\, ds. \tag{38}$$

We want to know how the exponential acts on the state function, but this is made difficult by the appearance of both position and momentum operators in the exponent. As a short intermezzo, we will explore this.

To study the effect of $e^{i\left(r_i\hat{q}^i+s^j\hat{p}_j\right)}$ on $\Psi(\mathrm{x})$, we will first split the exponential into components with a single operator type and then investigate the effect on $\Psi(\mathrm{x})$. To split the exponential, we define

$$U(t)=e^{-its^k\hat{p}_k}e^{-itr_l\hat{q}^l}e^{it\left(r_f\hat{q}^f+s^m\hat{p}_m\right)}. \tag{39}$$

Then we take the derivative

$$\begin{aligned}\frac{\partial U(t)}{\partial t}&=-ie^{-its^k\hat{p}_k}(s^a\hat{p}_a)e^{-itr_l\hat{q}^l}e^{it\left(r_f\hat{q}^f+s^m\hat{p}_m\right)}\\&-ie^{-its^k\hat{p}_k}e^{-itr_l\hat{q}^l}(r_b\hat{q}^b)e^{it\left(r_f\hat{q}^f+s^m\hat{p}_m\right)}\\&+ie^{-its^k\hat{p}_k}e^{-itr_l\hat{q}^l}\left(r_c\hat{q}^c+s^d\hat{p}_d\right)e^{it\left(r_f\hat{q}^f+s^m\hat{p}_m\right)}\\&=-ie^{-its^k\hat{p}_k}\left(s^a\hat{p}_ae^{-itr_l\hat{q}^l}-e^{-itr_l\hat{q}^l}s^d\hat{p}_d\right)e^{it\left(r_f\hat{q}^f+s^m\hat{p}_m\right)}\\&=-ie^{-its^k\hat{p}_k}\left(e^{-itr_l\hat{q}^l}s^a\hat{p}_a-\hbar ts^br_be^{-itr_l\hat{q}^l}-e^{-itr_l\hat{q}^l}s^d\hat{p}_d\right)e^{it\left(r_f\hat{q}^f+s^m\hat{p}_m\right)}\\&=i\hbar ts^br_be^{-its^k\hat{p}_k}e^{-itr_l\hat{q}^l}e^{it\left(r_f\hat{q}^f+s^m\hat{p}_m\right)}\\&=i\hbar ts^br_bU(t).\end{aligned} \tag{40}$$

We know $U(0)=1$ so we can solve this to get

$$U(t)=e^{i\hbar t^2s^kr_k/2}=e^{-its^k\hat{p}_k}e^{-itr_l\hat{q}^l}e^{it\left(r_f\hat{q}^f+s^m\hat{p}_m\right)}. \tag{41}$$

This means

$$e^{it\left(r_f\hat{q}^f+s^m\hat{p}_m\right)}=e^{i\hbar t^2s^kr_k/2}e^{itr_l\hat{q}^l}e^{its^m\hat{p}_m} \tag{42}$$

and for $t=1$

$$e^{i\left(r_f\hat{q}^f+s^m\hat{p}_m\right)}=e^{i\hbar s^kr_k/2}e^{ir_l\hat{q}^l}e^{is^m\hat{p}_m}. \tag{43}$$

Now, we have our rule for splitting the exponential in terms of components with a single operator type, which means we can continue to the action of the exponential on the state function $\Psi(\mathrm{x})$:

$$\begin{aligned}e^{i\left(r_f\hat{q}^f+s^m\hat{p}_m\right)}\Psi(\mathrm{x})&=e^{i\hbar s^kr_k/2}e^{ir_l\hat{q}^l}e^{is^m\hat{p}_m}\Psi(\mathrm{x})\\&=\langle\mathrm{x}|e^{i\hbar s^kr_k/2}e^{ir_l\hat{q}^l}e^{is^m\hat{p}_m}|\Psi\rangle\\&=e^{i\hbar s^kr_k/2}\langle\mathrm{x}|e^{ir_l\hat{q}^l}e^{is^m\hat{p}_m}|\Psi\rangle\\&=e^{i\hbar s^kr_k/2}e^{ir_l\mathrm{x}^l}\langle\mathrm{x}|e^{is^m\hat{p}_m}|\Psi\rangle\\&=e^{ir_k(s^k\hbar/2+\mathrm{x}^k)}\omega^{-\frac{1}{4}}(\mathrm{x})\Psi'(\mathrm{x}+\hbar s).\end{aligned} \tag{44}$$

In the last step, we used that

$$
\begin{aligned}
e^{-ia^k\hat{p}_k/\hbar} &= \sum_{l=0}^{\infty}\frac{1}{l!}\left(-ia^k\hat{p}_k/\hbar\right)^l \\
&= \sum_{l=0}^{\infty}\frac{1}{l!}\left(-ia^k(-i\hbar)\omega^{-\frac{1}{4}}(\mathrm{x})\frac{\partial}{\partial \mathrm{x}^k}\omega^{\frac{1}{4}}(\mathrm{x})/\hbar\right)^l \\
&= \sum_{l=0}^{\infty}\frac{1}{l!}\omega^{-\frac{1}{4}}(\mathrm{x})\left(-a^k\frac{\partial}{\partial \mathrm{x}^k}\right)^l\omega^{\frac{1}{4}}(\mathrm{x}) \\
&= \omega^{-\frac{1}{4}}(\mathrm{x})e^{-a^k\partial/\partial \mathrm{x}^k}\omega^{\frac{1}{4}}(\mathrm{x}),
\end{aligned}
\tag{45}
$$

where we used equation 27b for the momentum operator in coordinate representation. This results in the following translation, which is implemented in the last step of equation 44,

$$
\begin{aligned}
e^{-ia^k\hat{p}_k/\hbar}\Psi(\mathrm{x}) &= \omega^{-\frac{1}{4}}(\mathrm{x})e^{-a^k\partial/\partial \mathrm{x}^k}\omega^{\frac{1}{4}}(\mathrm{x})\Psi(\mathrm{x}) \\
&= \omega^{-\frac{1}{4}}(\mathrm{x})\Psi'(\mathrm{x}-a),
\end{aligned}
\tag{46}
$$

where we defined the new state function as $\Psi'(\mathrm{x}) = \omega^{\frac{1}{4}}(\mathrm{x})\Psi(\mathrm{x})$. This concludes our intermezzo.

Now, we can go back to $\hat{f}_W\Psi(\mathrm{x})$ as defined in equation 38, where we can implement what we learned in equation 44,

$$
\begin{aligned}
\hat{f}_W\Psi(\mathrm{x}) &= \int e^{i\left(r_i\hat{q}^i+s^j\hat{p}_j\right)}\Psi(\mathrm{x})\varphi(r,s)\,dr\,ds \\
&= \int e^{ir_k(s^k\hbar/2+\mathrm{x}^k)}\omega^{-\frac{1}{4}}(\mathrm{x})\Psi'(\mathrm{x}+\hbar s)\varphi(r,s)\,dr\,ds \\
&= (2\pi)^{2(1-n)}\int e^{ir_k(s^k\hbar/2+\mathrm{x}^k)}e^{-i\left(r_iq^i+s^jp_j\right)}f(q,p)\omega^{-\frac{1}{4}}(\mathrm{x})\Psi'(\mathrm{x}+\hbar s)\,dq\,dp\,dr\,ds \\
&= (2\pi)^{2(1-n)}\int e^{i\left(r_k(s^k\hbar/2+\mathrm{x}^k-q^k)-s^jp_j\right)}f(q,p)\omega^{-\frac{1}{4}}(\mathrm{x})\Psi'(\mathrm{x}+\hbar s)\,dq\,dp\,dr\,ds \\
&= (2\pi)^{1-n}\int e^{-is^jp_j}\delta\left(s\hbar/2+\mathrm{x}-q\right)f(q,p)\omega^{-\frac{1}{4}}(\mathrm{x})\Psi'(\mathrm{x}+\hbar s)\,dq\,dp\,ds \\
&= (2\pi)^{1-n}\int e^{-is^jp_j}f(s\hbar/2+\mathrm{x},p)\omega^{-\frac{1}{4}}(\mathrm{x})\Psi'(\mathrm{x}+\hbar s)\,dp\,ds.
\end{aligned}
\tag{47}
$$

In the third step, we used the inverse FT for $\varphi(r,s)$ as described in equation 37. We then define new coordinates $\mathrm{y} = \mathrm{x}+\hbar s$ and rewrite equation 47,

$$
\begin{aligned}
\hat{f}_W\Psi(\mathrm{x}) &= (2\pi\hbar)^{1-n}\int e^{ip_j(\mathrm{x}^j-\mathrm{y}^j)/\hbar}f(\frac{\mathrm{x}+\mathrm{y}}{2},p)\omega^{-\frac{1}{4}}(\mathrm{x})\Psi'(\mathrm{y})\,dp\,d\mathrm{y} \\
&= (2\pi\hbar)^{1-n}\int e^{ip_j(\mathrm{x}^j-\mathrm{y}^j)/\hbar}f(\frac{\mathrm{x}+\mathrm{y}}{2},p)\omega^{-\frac{1}{4}}(\mathrm{x})\Psi(\mathrm{y})\omega^{\frac{1}{4}}(\mathrm{y})\,dp\,d\mathrm{y}.
\end{aligned}
\tag{48}
$$

This integral can be used to determine the Weyl ordering of a random polynomial function $\hat{f}$ of non-commuting operators $\hat{q}$ and $\hat{p}$. Our integral differs from that of Tagirov in [2] in that the phase has a different sign. We checked this by using a $\hat{f}_W$ with an odd number of momentum operators (x^2p), where our phase gives the correct expression. However, this difference between our findings and those of Tagirov is inconsequential to our current investigation since we are dealing with an even number of momentum operators in $\hat{H}_0$.

We will take a another short intermezzo to show that this integral indeed produces Weyl ordered functions. We perform a check with the one dimensional example of equation 28, where $f(\mathrm{x}, p) = \mathrm{x}p^2$:

$$
\begin{aligned}
\hat{f}_W \Psi(\mathrm{x}) &= (2\pi\hbar)^{-1} \int e^{ip(\mathrm{x}-\mathrm{y})/\hbar} \frac{\mathrm{x}+\mathrm{y}}{2} p^2 \omega^{-\frac{1}{4}}(\mathrm{x}) \Psi'(\mathrm{y})\, dp\, d\mathrm{y} \\
&= (2\pi\hbar)^{-1} \int (i\hbar)^2 \left(\frac{\partial^2}{\partial \mathrm{y}^2} e^{ip(\mathrm{x}-\mathrm{y})/\hbar} \right) \frac{\mathrm{x}+\mathrm{y}}{2} \omega^{-\frac{1}{4}}(\mathrm{x}) \Psi'(\mathrm{y})\, dp\, d\mathrm{y} \\
&= \int -\hbar^2 \left(\frac{\partial^2}{\partial \mathrm{y}^2} \delta(\mathrm{x}-\mathrm{y}) \right) \frac{\mathrm{x}+\mathrm{y}}{2} \omega^{-\frac{1}{4}}(\mathrm{x}) \Psi'(\mathrm{y})\, d\mathrm{y} \\
&= -\hbar^2 \omega^{-\frac{1}{4}}(\mathrm{x}) \int \delta(\mathrm{x}-\mathrm{y}) \frac{\partial^2}{\partial \mathrm{y}^2} \left(\frac{\mathrm{x}+\mathrm{y}}{2} \Psi'(\mathrm{y}) \right) d\mathrm{y} \\
&= -\hbar^2 \omega^{-\frac{1}{4}}(\mathrm{x}) \int \delta(\mathrm{x}-\mathrm{y}) \left(\frac{\partial}{\partial \mathrm{y}} \Psi'(\mathrm{y}) + \frac{\mathrm{x}+\mathrm{y}}{2} \frac{\partial^2}{\partial \mathrm{y}^2} \Psi'(\mathrm{y}) \right) d\mathrm{y} \\
&= -\hbar^2 \omega^{-\frac{1}{4}}(\mathrm{x}) \left(\frac{\partial}{\partial \mathrm{x}} \Psi'(\mathrm{x}) + \mathrm{x} \frac{\partial^2}{\partial \mathrm{x}^2} \Psi'(\mathrm{x}) \right) \\
&= -\hbar^2 \omega^{-\frac{1}{4}}(\mathrm{x}) \frac{1}{3} \left(\mathrm{x} \frac{\partial^2}{\partial \mathrm{x}^2} \Psi'(\mathrm{x}) + \frac{\partial}{\partial \mathrm{x}} \left(\mathrm{x} \frac{\partial}{\partial \mathrm{x}} \Psi'(\mathrm{x}) \right) + \frac{\partial^2}{\partial \mathrm{x}^2} \left(\mathrm{x} \Psi'(\mathrm{x}) \right) \right) \\
&= \frac{1}{3} \left(\hat{q}\hat{p}^2 + \hat{p}\hat{q}\hat{p} + \hat{p}^2\hat{q} \right) \Psi(\mathrm{x}).
\end{aligned}
\tag{49}
$$

This is exactly the same result as before. Note that there are more ways than one to rewrite the before last equation, resulting in seemingly different weights for each term. For example,

$$
\begin{aligned}
\hat{f}_W \Psi(\mathrm{x}) &= -\hbar^2 \omega^{-\frac{1}{4}}(\mathrm{x}) \left(\frac{\partial}{\partial \mathrm{x}} \Psi'(\mathrm{x}) + \mathrm{x} \frac{\partial^2}{\partial \mathrm{x}^2} \Psi'(\mathrm{x}) \right) \\
&= -\hbar^2 \omega^{-\frac{1}{4}}(\mathrm{x}) \frac{1}{4} \left(\mathrm{x} \frac{\partial^2}{\partial \mathrm{x}^2} \Psi'(\mathrm{x}) + 2 \frac{\partial}{\partial \mathrm{x}} \left(\mathrm{x} \frac{\partial}{\partial \mathrm{x}} \Psi'(\mathrm{x}) \right) + \frac{\partial^2}{\partial \mathrm{x}^2} \left(\mathrm{x} \Psi'(\mathrm{x}) \right) \right) \\
&= \frac{1}{4} \left(\hat{q}\hat{p}^2 + 2\hat{p}\hat{q}\hat{p} + \hat{p}^2\hat{q} \right) \Psi(\mathrm{x}).
\end{aligned}
\tag{50}
$$

This is still Weyl ordering, just rewritten in a different form. We will make explicit use of this later on, while determining $\hat{H}_0$.

3.3.2 Rivier ordering

Returning to our original problem, the function $\hat{f}$ that we are interested in contains exactly two momentum operators and an unknown number of position operators in the form of $\omega^{ij}(\hat{q})$. Equation 48 tells us how to deal with this in Weyl ordering, but we wanted to use a combination of Weyl and Rivier ordering. Rivier ordering, in this case, is rather simple since the unknown part of $\hat{f}$ only contains position operators and, in Rivier ordering, they do not mix with the momentum operators. Thus, we can simply write

$$
\hat{f}_R = \frac{1}{2} \left(\omega^{ij}(\hat{q}) \hat{p}_i \hat{p}_j + \hat{p}_i \hat{p}_j \omega^{ij}(\hat{q}) \right) \tag{51}
$$

for Rivier ordering in our specific case. As required, $\hat{f}_R$ is hermitian. The general case with an unknown number of position and momentum operators, similar to the approach above for Weyl ordering, is described in detail in appendix B.

3.4 Quantized Hamiltonian

We can now continue with the quantization of H_0, equation 23. Using the integral of equation 48 for Weyl ordering and equation 51 for Rivier ordering, and weighing them with ν and $1-\nu$ respectively, we get

$$\hat{H}_0\Psi(\mathrm{x}) = \left(\frac{2-\nu}{8m}\omega^{ij}(\hat{q})\hat{p}_i\hat{p}_j + \frac{\nu}{4m}\hat{p}_i\omega^{ij}(\hat{q})\hat{p}_j + \frac{2-\nu}{8m}\hat{p}_i\hat{p}_j\omega^{ij}(\hat{q})\right)\Psi(\mathrm{x}). \tag{52}$$

The full derivation can be found in appendix C. This is exactly the same as equation (25) in [2] of Tagirov.

This may look as though the different terms in Weyl ordering do not share the same weight, but this is not the case. The terms are simply rewritten in the same way as the example of equation 50. To prove the validity of the integral method for Weyl ordering, we perform a check by way of a Taylor expansion of the metric $\omega^{ij}(\mathrm{x})$. The full Taylor expansion can be found in appendix D and agrees with equation (52). In this appendix, one can also find that equation 52 can be written as

$$\hat{H}_0\Psi(\mathrm{x}) = \left(\frac{1}{2m}\hat{p}_i\omega^{ij}(\hat{q})\hat{p}_j - \frac{\hbar^2}{4m}(1-\frac{\nu}{2})\left(\frac{\partial}{\partial\hat{q}^i}\frac{\partial}{\partial\hat{q}^j}\omega^{ij}(\hat{q})\right)\right)\Psi(\mathrm{x}), \tag{53}$$

where the dependence on the ordering is purely in the second term.

3.4.1 Separating kinetic and potential terms

As usual, we would like to separate $\hat{H}_0$ into a purely kinetic term and a potential term. Before doing so, we introduce some definitions that will simplify our calculations. First of all, we introduce the Laplace-Beltrami operator Δ_ω,

$$\begin{aligned}\Delta_\omega\Psi(\mathrm{x}) = \nabla\cdot\nabla\Psi(\mathrm{x}) &= \frac{1}{\sqrt{\omega(\mathrm{x})}}\frac{\partial}{\partial\mathrm{x}^i}\left(\sqrt{\omega(\mathrm{x})}\omega^{ij}(\mathrm{x})\frac{\partial}{\partial\mathrm{x}^j}\Psi(\mathrm{x})\right)\\ &= \frac{1}{2}\omega^{-1}(\mathrm{x})\omega^{ij}(\mathrm{x})\left(\frac{\partial}{\partial\mathrm{x}^i}\omega(\mathrm{x})\right)\left(\frac{\partial}{\partial\mathrm{x}^j}\Psi(\mathrm{x})\right) + \left(\frac{\partial}{\partial\mathrm{x}^i}\omega^{ij}(\mathrm{x})\right)\left(\frac{\partial}{\partial\mathrm{x}^j}\Psi(\mathrm{x})\right) + \omega^{ij}(\mathrm{x})\left(\frac{\partial}{\partial\mathrm{x}^i}\frac{\partial}{\partial\mathrm{x}^j}\Psi(\mathrm{x})\right).\end{aligned} \tag{54}$$

Next, we introduce the Christoffel symbol,

$$\Gamma^k_{ij}(\mathrm{x}) = \frac{1}{2}\omega^{kl}(\mathrm{x})\left(\frac{\partial}{\partial\mathrm{x}^i}\omega_{jl}(\mathrm{x}) + \frac{\partial}{\partial\mathrm{x}^j}\omega_{il}(\mathrm{x}) - \frac{\partial}{\partial\mathrm{x}^l}\omega_{ij}(\mathrm{x})\right), \tag{55}$$

which we can use to define

$$\Gamma_i(\mathrm{x}) = \Gamma^k_{ik}(\mathrm{x}) = \frac{1}{2}\omega^{kl}(\mathrm{x})\left(\frac{\partial}{\partial\mathrm{x}^i}\omega_{kl}(\mathrm{x}) + \frac{\partial}{\partial\mathrm{x}^k}\omega_{il}(\mathrm{x}) - \frac{\partial}{\partial\mathrm{x}^l}\omega_{ik}(\mathrm{x})\right) = \frac{1}{2}\omega^{kl}(\mathrm{x})\frac{\partial}{\partial\mathrm{x}^i}\omega_{kl}(\mathrm{x}), \tag{56}$$

where the last equation holds because of the symmetry in the indices k, l. We then use the identity

$$\ln\omega(\mathrm{x}) = \mathrm{Tr}(\ln\omega_{ij}(\mathrm{x})), \tag{57}$$

to get

$$\frac{1}{\omega(\mathrm{x})}\frac{\partial}{\partial\mathrm{x}^k}\omega(\mathrm{x}) = \omega^{ij}(\mathrm{x})\frac{\partial}{\partial\mathrm{x}^k}\omega_{ij}(\mathrm{x}), \tag{58}$$

which can be proven directly using the fully diagonalised version of the metric. This is then connected to the Christoffel symbol, resulting in the following identity:

$$\frac{\partial}{\partial\mathrm{x}^k}\omega(\mathrm{x}) = 2\omega(\mathrm{x})\Gamma_k(\mathrm{x}). \tag{59}$$

Now, we can separate $\hat{H}_0$ into a kinetic term and a potential. We start with equation 53 and work this out until we have the separation we are looking for.

$$
\begin{aligned}
\hat{H}_0\Psi(\mathrm{x}) &= \left(\frac{1}{2m}\hat{p}_i\omega^{ij}(\hat{q})\hat{p}_j - \frac{\hbar^2}{4m}(1-\frac{\nu}{2})\left(\frac{\partial}{\partial\hat{q}^i}\frac{\partial}{\partial\hat{q}^j}\omega^{ij}(\hat{q})\right)\right)\Psi(\mathrm{x}) \\
&= -\frac{\hbar^2}{2m}\omega^{-\frac{1}{4}}(\mathrm{x})\frac{\partial}{\partial\mathrm{x}^i}\left(\omega^{ij}(\mathrm{x})\frac{\partial}{\partial\mathrm{x}^j}\left(\omega^{\frac{1}{4}}(\mathrm{x})\Psi(\mathrm{x})\right)\right) - \frac{\hbar^2}{4m}(1-\frac{\nu}{2})\left(\frac{\partial}{\partial\mathrm{x}^i}\frac{\partial}{\partial\mathrm{x}^j}\omega^{ij}(\mathrm{x})\right)\Psi(\mathrm{x}) \\
&= -\frac{\hbar^2}{2m}\omega^{-\frac{1}{4}}(\mathrm{x})\left[\left(\frac{\partial}{\partial\mathrm{x}^i}\omega^{ij}(\mathrm{x})\right)\left(\omega^{\frac{1}{4}}(\mathrm{x})\frac{\partial}{\partial\mathrm{x}^j}\Psi(\mathrm{x}) + \Psi(\mathrm{x})\frac{\partial}{\partial\mathrm{x}^j}\omega^{\frac{1}{4}}(\mathrm{x})\right) + \omega^{ij}(\mathrm{x})\left(\left(\frac{\partial}{\partial\mathrm{x}^i}\frac{\partial}{\partial\mathrm{x}^j}\omega^{\frac{1}{4}}(\mathrm{x})\right)\Psi(\mathrm{x})\right.\right. \\
&\qquad \left.\left. + \left(\frac{\partial}{\partial\mathrm{x}^i}\omega^{\frac{1}{4}}(\mathrm{x})\right)\left(\frac{\partial}{\partial\mathrm{x}^j}\Psi(\mathrm{x})\right) + \left(\frac{\partial}{\partial\mathrm{x}^j}\omega^{\frac{1}{4}}(\mathrm{x})\right)\left(\frac{\partial}{\partial\mathrm{x}^i}\Psi(\mathrm{x})\right) + \omega^{\frac{1}{4}}(\mathrm{x})\left(\frac{\partial}{\partial\mathrm{x}^i}\frac{\partial}{\partial\mathrm{x}^j}\Psi(\mathrm{x})\right)\right)\right] \\
&\quad - \frac{\hbar^2}{4m}(1-\frac{\nu}{2})\left(\frac{\partial}{\partial\mathrm{x}^i}\frac{\partial}{\partial\mathrm{x}^j}\omega^{ij}(\mathrm{x})\right)\Psi(\mathrm{x}) \\
&= -\frac{\hbar^2}{2m}\left[\left(\frac{\partial}{\partial\mathrm{x}^i}\omega^{ij}(\mathrm{x})\right)\left(\frac{\partial}{\partial\mathrm{x}^j}\Psi(\mathrm{x})\right) + \frac{1}{4}\omega^{-1}(\mathrm{x})\Psi(\mathrm{x})\left(\frac{\partial}{\partial\mathrm{x}^i}\omega^{ij}(\mathrm{x})\right)\left(\frac{\partial}{\partial\mathrm{x}^j}\omega(\mathrm{x})\right) + \frac{1}{4}\omega^{-1}(\mathrm{x})\omega^{ij}(\mathrm{x})\Psi(\mathrm{x})\left(\frac{\partial}{\partial\mathrm{x}^i}\frac{\partial}{\partial\mathrm{x}^j}\omega(\mathrm{x})\right)\right. \\
&\qquad - \frac{3}{16}\omega^{-2}(\mathrm{x})\omega^{ij}(\mathrm{x})\left(\frac{\partial}{\partial\mathrm{x}^i}\omega(\mathrm{x})\right)\left(\frac{\partial}{\partial\mathrm{x}^j}\omega(\mathrm{x})\right)\Psi(\mathrm{x}) + \frac{1}{2}\omega^{-1}(\mathrm{x})\omega^{ij}(\mathrm{x})\left(\frac{\partial}{\partial\mathrm{x}^i}\omega(\mathrm{x})\right)\left(\frac{\partial}{\partial\mathrm{x}^j}\Psi(\mathrm{x})\right) \\
&\qquad \left. + \omega^{ij}(\mathrm{x})\left(\frac{\partial}{\partial\mathrm{x}^i}\frac{\partial}{\partial\mathrm{x}^j}\Psi(\mathrm{x})\right) + \frac{1}{2}(1-\frac{\nu}{2})\left(\frac{\partial}{\partial\mathrm{x}^i}\frac{\partial}{\partial\mathrm{x}^j}\omega^{ij}(\mathrm{x})\right)\Psi(\mathrm{x})\right] \\
&= -\frac{\hbar^2}{2m}\Delta_\omega\Psi(\mathrm{x}) - \frac{\hbar^2}{4m}\left[(1-\frac{\nu}{2})\left(\frac{\partial}{\partial\mathrm{x}^i}\frac{\partial}{\partial\mathrm{x}^j}\omega^{ij}(\mathrm{x})\right) + \frac{1}{2}\omega^{-1}(\mathrm{x})\frac{\partial}{\partial\mathrm{x}^i}\left(\omega^{ij}(\mathrm{x})\left(\frac{\partial}{\partial\mathrm{x}^j}\omega(\mathrm{x})\right)\right)\right. \\
&\qquad \left. - \frac{3}{8}\omega^{-2}(\mathrm{x})\omega^{ij}(\mathrm{x})\left(\frac{\partial}{\partial\mathrm{x}^i}\omega(\mathrm{x})\right)\left(\frac{\partial}{\partial\mathrm{x}^j}\omega(\mathrm{x})\right)\right]\Psi(\mathrm{x}) \\
&= -\frac{\hbar^2}{2m}\Delta_\omega\Psi(\mathrm{x}) - \frac{\hbar^2}{4m}\left[(1-\frac{\nu}{2})\left(\frac{\partial}{\partial\mathrm{x}^i}\frac{\partial}{\partial\mathrm{x}^j}\omega^{ij}(\mathrm{x})\right) + \frac{\partial}{\partial\mathrm{x}^i}\left(\omega^{ij}(\mathrm{x})\Gamma_j(\mathrm{x})\right) + \frac{1}{2}\omega^{ij}(\mathrm{x})\Gamma_i(\mathrm{x})\Gamma_j(\mathrm{x})\right]\Psi(\mathrm{x}) \\
&= -\frac{\hbar^2}{2m}\Delta_\omega\Psi(\mathrm{x}) + V(\mathrm{x})\Psi(\mathrm{x})
\end{aligned} \tag{60}
$$

The first term is the kinetic term and

$$
V(\mathrm{x}) = -\frac{\hbar^2}{4m}\left[(1-\frac{\nu}{2})\left(\frac{\partial}{\partial\mathrm{x}^i}\frac{\partial}{\partial\mathrm{x}^j}\omega^{ij}(\mathrm{x})\right) + \frac{\partial}{\partial\mathrm{x}^i}\left(\omega^{ij}(\mathrm{x})\Gamma_j(\mathrm{x})\right) + \frac{1}{2}\omega^{ij}(\mathrm{x})\Gamma_i(\mathrm{x})\Gamma_j(\mathrm{x})\right] \tag{61}
$$

represents the potential term, which Tagirov calls the quantum potential in [2]. It is interesting to note that, although we are dealing with a 'free' particle, there is still a potential term because we are working in a general spacetime instead of a flat one. Also note that $V(\mathrm{x})$ still contains ν and, as such, is dependent on the choice of a rule of ordering. At this point, we have no means of preferring a specific ordering.
If we compare our potential to that of Tagirov in [2] (equation 27), we can see that the two results are distinctly different. To investigate this, we directly replicated Tagirov's calculations but still end up with our own results. Therefore, we conclude that we are in disagreement with Tagirov.

3.4.2 Coupling with R

A slightly different approach described by DeWitt in [3] investigates the path integral to see how a particle behaves that is moving along a geodesic line. This was done because, in the classical limit, a freely moving particle will follow a geodesic line. Along this investigation some assumptions were made that come down to the following: The extent of the particle's wave packet is very small in comparison to the distance scale on which the metric changes appreciably, so the particle can effectively be treated as a point particle. The other assumptions are that this particle is moving non-relativistically and that the metric is static. DeWitt uses the same Hamiltonian as we did and quantizes H_0 as $\hat{H}_0 = \frac{1}{2m}\hat{p}_i\omega^{ij}(\hat{q})\hat{p}_j$, which corresponds to our findings if we take $\nu = 2$. He then uses the path integral to arrive at

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\left(\Delta_\omega - \frac{1}{6}R_\omega\right), \tag{62}$$

where $\hat{H}_0$ is split into a kinetic term and a potential term proportional to the scalar curvature R with a coefficient of $\frac{1}{6}$.

To cast our $\hat{H}_0$ into a similar form, we use Riemannian normal coordinates y^μ. These coordinates arise from the idea that a unique geodesic line can be found between two points that are sufficiently close together. The new coordinates are defined as follows:

$$y^\mu = sa^\mu, \tag{63}$$

where s is the distance along the geodesic line between the two points and a^μ is the tangent vector along the same geodesic line. In the vicinity of the point $y^\mu = 0$, which we will call the origin (O), we can approximate the metric as

$$g_{\mu\nu} \approx g_{\mu\nu}(O) + \frac{1}{3}R_{\mu\alpha\nu\beta}(O)y^\alpha y^\beta, \tag{64}$$

plus higher order terms which we will omit in the current calculations. As a result of the Riemannian normal coordinates, $\frac{\partial}{\partial x^\alpha}g_{\mu\nu}(O) = 0$ and therefore $\Gamma^\mu_{\alpha\beta}(O) = 0$. We assume that all time derivatives of ω^{ij} are zero, so we work in a globally static spacetime, as a consequence, $R = R_\omega$. Implementing this into our potential term of equation 61, we arrive at

$$V(O) = -\frac{\hbar^2}{4m}\left[-\left(2-\frac{\nu}{2}\right)\frac{1}{3}R_\omega(O)\right] = -\frac{\hbar^2}{2m}\left[-\left(2-\frac{\nu}{2}\right)\frac{1}{6}R_\omega(O)\right]. \tag{65}$$

We have indeed found a potential term proportional to the scalar curvature and if we take $\nu = 2$ we arrive at the same result as DeWitt.

If we follow the same procedure described above for the potential term of Tagirov (equation 27 in [2]), we arrive at

$$V^{(T)}(O) = -\frac{\hbar^2}{2m}\left[-\left(1+\frac{\nu}{2}\right)\frac{1}{6}R_\omega(O)\right]. \tag{66}$$

It is the same as ours but with a different prefactor involving ν. In his paper, Tagirov describes following the same procedure but he arrives at

$$V^{(T)}(O) = -\frac{\hbar^2}{2m}\left[-\left(\frac{\nu}{2}\right)\frac{1}{6}R_\omega(O)\right], \tag{67}$$

which is the same as DeWitt for $\nu = 2$, unlike equation 66. It seems that the difference in the potential term compared to ours and the difference in the prefactor above compared to our calculations magically cancel to arrive at the same equation as DeWitt in the case of $\nu = 2$. Tagirov then goes on to say that ν must be 2 so that the result agrees with DeWitt. However, DeWitt did not take into account different rules of ordering and we still do not see any reason to prefer a certain rule of ordering.

Like we set out to do in the beginning of this chapter, we have found that the quantized Hamiltonian has a potential term proportional to the scalar curvature R in these Riemannian normal coordinates. The corresponding coefficient is $\left(2 - \frac{\nu}{2}\right) \frac{1}{6}$, which is still dependent on the choice of rule of ordering, ν. If $\nu = 2$, the coefficient becomes $\frac{1}{6}$, which is the same as the conformal coefficient found in section 2.3.1, $\xi_c = \frac{n-2}{4(n-1)}$, in four dimensions. However, the factor $\frac{1}{3}$ in equation 65 is a direct consequence of the use of normal coordinates so it is unclear whether this similarity is a fingerprint of the underlying conformal symmetry of the massless theory or this is merely a coincidence. Furthermore, as of yet we still have no clear compelling reason[vi] to prefer a certain rule of ordering. This concludes our investigation of the second approach, quantization of geodesic motion, which means we can proceed to some interesting conclusions and areas of future research.

[vi]If you require the Hamiltonian to be purely kinetic, that could be realised by taking $\nu = 4$, which coincides with minimal coupling or $\xi = 0$. However, we do not have a compelling reason for this requirement yet.

4 Conclusions and outlook

In the introduction, we mentioned that the goal of this thesis was to review and document the reasoning behind the QFT approach with conformal symmetry and the approach of quantization of geodesic motion presented by Tagirov in a clear, unambiguous and complete manner. The two approaches have been explored and described in detail and we can draw some interesting conclusions from our research.

In chapter 2, the first approach of setting up a Quantum Field Theory (QFT) in a general spacetime and searching for a conformal symmetry was investigated thoroughly. From that investigation we can conclude that, for a massless spin-0 field, there is a conformal symmetry if we introduce the coupling $\xi R\phi^2$ to the scalar Lagrangian and take the coupling coefficient to be $\xi = \xi_c = \frac{n-2}{4(n-1)}$, where n is the number of spacetime dimensions. This is exactly the same result found by Tagirov in [1].

We have also investigated the same approach for a massless spin-$\frac{1}{2}$ Dirac field. In this case, there is a conformal symmetry without the need for introducing extra terms. Furthermore, in writing the squared of the Dirac equation we arrive at a Klein-Gordon-like equation with a coupling between the scalar curvature and the Dirac field with a coupling coefficient of $\frac{1}{4}$. This coupling is a result of the action of the derivatives on the Dirac field in a general spacetime. It is interesting to note that it is impossible to make the coefficients for the different spin cases match for any number of dimensions.

After we concluded the first approach with similar results to Tagirov, we investigated the second approach in chapter 3. The second approach of quantization of the non-relativistic motion of a spinless particle moving along a geodesic line was investigated thoroughly and a few different conclusions can be drawn from this investigation. First of all, we verified the method of using integrals to implement Weyl ordering and Rivier ordering for a random function of operators working on a state function.

Secondly, we found the same integrals as Tagirov did in [2] except for a different sign for the phase. This had no impact on our calculations since $\hat{H}_0$ contains two momentum operators and we arrived at the same expression for $\hat{H}_0$ as Tagirov. However, it is important to use the correct sign in any future research, especially when working with an odd number of momentum operators.

Finally, we separated $\hat{H}_0$ into a kinetic and a potential term and arrived at a distinctly different expression compared to Tagirov in [2]. After we used Riemannian normal coordinates, the potential term was proportional to the scalar curvature. The resulting coefficient agreed with the calculations of DeWitt if we take $\nu = 2$ for the ordering parameter in our calculation, but we have not found a compelling reason to prefer this choice for the rule of ordering, nor any other choice. For $\nu = 2$, the coefficient became $\frac{1}{6}$ which corresponds to the coefficient found in the first approach for a massless spin-0 field in four dimensions. However, it is still unclear whether this similarity is a fingerprint of the underlying conformal symmetry of the massless theory, or merely a coincidence. Even though Tagirov uses the same method of using Riemannian normal coordinates, we find a different coefficient for his potential term than he does. Inexplicably, the coefficient he arrives at becomes the same as that of DeWitt for $\nu = 2$.

These conclusions, of course, give rise to some interesting areas for future research. The first is to investigate whether there is a guiding principle that tells us which rule of ordering we should choose. An argument could be made for taking $\nu = 4$, which coincides with minimal coupling, because the Hamiltonian would be purely kinetic in that case. However, we do not have a compelling reason to set this requirement yet. If a clear reason arises for choosing $\nu = 2$, resulting in the same coefficient for the two approaches in four dimensions, it would also be interesting to investigate whether an underlying reason for the correspondence can be determined or not.

Another interesting area for future research is to follow the second approach of quantization of geodesic motion but for a spin-$\frac{1}{2}$ particle. Of course, this changes $\hat{H}_0$ and the spin-$\frac{1}{2}$ character of the particle needs to be taken into account throughout the calculations. It would be interesting to see whether a similar coupling with the scalar curvature arises, and whether this corresponds to the coupling found in the QFT approach.

Regardless of the direction of future research, we hope that this thesis can serve as a solid foundation.

A Conformal transformations

In this appendix, the derivations of the conformal transformations can be found. It has been split into four parts. The first section contains metric related quantities such as the Christoffel symbol, the Ricci curvature scalar, and the spinor connection. The second section contains n-bein related quantities such as the n-bein itself, the Dirac gamma matrix and the connection. The third section contains equations containing the scalar field such as the Klein-Gordon equation and the scalar Lagrangian. The fourth section contains equations containing the Dirac field such as the Dirac equation and the Dirac Lagrangian.

A.1 Metric $g^{\mu\nu}$

In this section, the derivation of conformal transformations of metric related quantities can be found.

A.1.1 Christoffel symbol $\Gamma^{\lambda}_{\mu\nu}$

$$\begin{aligned}
\Gamma'^{\lambda}_{\mu\nu} &= \frac{1}{2} g'^{\lambda\rho}\left(\partial_\mu g'_{\nu\rho} + \partial_\nu g'_{\mu\rho} - \partial_\rho g'_{\mu\nu}\right) \\
&= \frac{1}{2}\Omega^{-2} g^{\lambda\rho}\left(\partial_\mu(\Omega^2 g_{\nu\rho}) + \partial_\nu(\Omega^2 g_{\mu\rho}) - \partial_\rho(\Omega^2 g_{\mu\nu})\right) \\
&= \frac{1}{2}\Omega^{-2} g^{\lambda\rho}\left(2\Omega\partial_\mu\Omega g_{\nu\rho} + \Omega^2\partial_\mu g_{\nu\rho} + 2\Omega\partial_\nu\Omega g_{\mu\rho} + \Omega^2\partial_\nu g_{\mu\rho} - 2\Omega\partial_\rho\Omega g_{\mu\nu} - \Omega^2\partial_\rho g_{\mu\nu}\right) \\
&= \Gamma^{\lambda}_{\mu\nu} + \Omega^{-1} g^{\lambda\rho}\left(g_{\nu\rho}\partial_\mu\Omega + g_{\mu\rho}\partial_\nu\Omega - g_{\mu\nu}\partial_\rho\Omega\right) \\
&= \Gamma^{\lambda}_{\mu\nu} + \Omega^{-1}\partial_\sigma\Omega\left(\delta^\lambda_\mu\delta^\sigma_\nu + \delta^\lambda_\nu\delta^\sigma_\mu - g_{\mu\nu}g^{\lambda\sigma}\right)
\end{aligned} \tag{68}$$

A.1.2 Ricci curvature scalar R

$$\begin{aligned}
R' = {} & g'^{\beta\delta}\left(\partial_\delta\Gamma'^{\alpha}_{\beta\alpha} - \partial_\alpha\Gamma'^{\alpha}_{\beta\delta} + \Gamma'^{\alpha}_{\delta\lambda}\Gamma'^{\lambda}_{\beta\alpha} - \Gamma'^{\alpha}_{\alpha\lambda}\Gamma'^{\lambda}_{\beta\delta}\right) \\
= {} & \Omega^{-2} g^{\beta\delta}\Big[\partial_\delta\Gamma^{\alpha}_{\beta\alpha} + \partial_\delta\left(\Omega^{-1}\partial_\lambda\Omega(\delta^\alpha_\beta\delta^\lambda_\alpha + \delta^\alpha_\alpha\delta^\lambda_\beta - g_{\beta\alpha}g^{\alpha\lambda})\right) \\
& - \partial_\alpha\Gamma^{\alpha}_{\beta\delta} - \partial_\alpha\left(\Omega^{-1}\partial_\lambda\Omega(\delta^\alpha_\beta\delta^\lambda_\delta + \delta^\alpha_\delta\delta^\lambda_\beta - g_{\beta\delta}g^{\alpha\lambda})\right) \\
& + \Gamma^{\alpha}_{\delta\lambda}\Gamma^{\lambda}_{\beta\alpha} + \Gamma^{\alpha}_{\delta\lambda}\Omega^{-1}\partial_\mu\Omega(\delta^\lambda_\beta\delta^\mu_\alpha + \delta^\lambda_\alpha\delta^\mu_\beta - g_{\beta\alpha}g^{\lambda\mu}) + \Omega^{-1}\partial_\mu\Omega(\delta^\alpha_\delta\delta^\mu_\lambda + \delta^\alpha_\lambda\delta^\mu_\delta - g_{\delta\lambda}g^{\alpha\mu})\Gamma^{\lambda}_{\beta\alpha} \\
& + \Omega^{-1}\partial_\mu\Omega(\delta^\alpha_\delta\delta^\mu_\lambda + \delta^\alpha_\lambda\delta^\mu_\delta - g_{\delta\lambda}g^{\alpha\mu})\Omega^{-1}\partial_\nu\Omega(\delta^\lambda_\beta\delta^\nu_\alpha + \delta^\lambda_\alpha\delta^\nu_\beta - g_{\beta\alpha}g^{\lambda\nu}) \\
& - \Gamma^{\alpha}_{\alpha\lambda}\Gamma^{\lambda}_{\beta\delta} - \Gamma^{\alpha}_{\alpha\lambda}\Omega^{-1}\partial_\mu\Omega(\delta^\lambda_\beta\delta^\mu_\delta + \delta^\lambda_\delta\delta^\mu_\beta - g_{\beta\delta}g^{\lambda\mu}) - \Omega^{-1}\partial_\mu\Omega(\delta^\alpha_\alpha\delta^\mu_\lambda + \delta^\alpha_\lambda\delta^\mu_\alpha - g_{\alpha\lambda}g^{\alpha\mu})\Gamma^{\lambda}_{\beta\delta} \\
& - \Omega^{-1}\partial_\mu\Omega(\delta^\alpha_\alpha\delta^\mu_\lambda + \delta^\alpha_\lambda\delta^\mu_\alpha - g_{\alpha\lambda}g^{\alpha\mu})\Omega^{-1}\partial_\nu\Omega(\delta^\lambda_\beta\delta^\nu_\delta + \delta^\lambda_\delta\delta^\nu_\beta - g_{\beta\delta}g^{\lambda\nu})\Big]
\end{aligned} \tag{69}$$

With clarity and conciseness in mind, we examine the terms in between the square brackets proportional to Ω^0, $\partial_\mu\left(\Omega^{-1}\partial_\lambda\Omega\right)$, $\Omega^{-1}\partial_\mu\Omega$, and $\Omega^{-2}\partial_\mu\Omega\partial_\nu\Omega$ separately. We start with the term proportional to Ω^0,

$$\Omega^{-2} g^{\beta\delta}\left[\partial_\delta\Gamma^{\alpha}_{\beta\alpha} - \partial_\alpha\Gamma^{\alpha}_{\beta\delta} + \Gamma^{\alpha}_{\delta\lambda}\Gamma^{\lambda}_{\beta\alpha} - \Gamma^{\alpha}_{\alpha\lambda}\Gamma^{\lambda}_{\beta\delta}\right] = \Omega^{-2}R. \tag{70}$$

Next, we examine the term proportional to $\partial_\mu\left(\Omega^{-1}\partial_\lambda\Omega\right)$,

$$\begin{aligned}
&\Omega^{-2}g^{\beta\delta}\left[\partial_\delta\left(\Omega^{-1}\partial_\lambda\Omega(\delta^\alpha_\beta\delta^\lambda_\alpha+\delta^\alpha_\alpha\delta^\lambda_\beta-g_{\beta\alpha}g^{\alpha\lambda})\right)-\partial_\alpha\left(\Omega^{-1}\partial_\lambda\Omega(\delta^\alpha_\beta\delta^\lambda_\delta+\delta^\alpha_\delta\delta^\lambda_\beta-g_{\beta\delta}g^{\alpha\lambda})\right)\right]\\
&\quad=\Omega^{-2}g^{\beta\delta}\left[\partial_\delta(n\Omega^{-1}\partial_\beta\Omega)-\partial_\beta(\Omega^{-1}\partial_\delta\Omega)-\partial_\delta(\Omega^{-1}\partial_\beta\Omega)+\partial_\alpha(\Omega^{-1}\partial_\lambda\Omega g_{\beta\delta}g^{\alpha\lambda})\right]\\
&\quad=\Omega^{-2}\left[(n-2)g^{\beta\delta}\partial_\delta(\Omega^{-1}\partial_\beta\Omega)+n\partial_\alpha(\Omega^{-1}g^{\alpha\lambda}\partial_\lambda\Omega)+\Omega^{-1}g^{\beta\delta}g^{\alpha\lambda}\partial_\lambda\Omega\partial_\alpha g_{\beta\delta}\right]\\
&\quad=\Omega^{-2}\left[(n-2)g^{\beta\delta}\partial_\delta(\Omega^{-1}\partial_\beta\Omega)+n\left(g^{\alpha\lambda}\partial_\alpha(\Omega^{-1}\partial_\lambda\Omega)+\Omega^{-1}\partial_\lambda\Omega\partial_\alpha g^{\alpha\lambda}\right)+\Omega^{-1}g^{\beta\delta}g^{\alpha\lambda}\partial_\lambda\Omega\partial_\alpha g_{\beta\delta}\right]\\
&\quad=\Omega^{-2}\left[2(n-1)g^{\beta\delta}\partial_\delta(\Omega^{-1}\partial_\beta\Omega)+n\Omega^{-1}\partial_\lambda\Omega\partial_\alpha g^{\alpha\lambda}+\Omega^{-1}g^{\beta\delta}g^{\alpha\lambda}\partial_\lambda\Omega\partial_\alpha g_{\beta\delta}\right]\\
&\quad=\Omega^{-2}\left[2(n-1)\Omega^{-1}g^{\beta\delta}\partial_\beta\partial_\delta\Omega-2(n-1)\Omega^{-2}g^{\beta\delta}\partial_\beta\Omega\partial_\delta\Omega+n\Omega^{-1}\partial_\lambda\Omega\partial_\alpha g^{\alpha\lambda}+\Omega^{-1}g^{\beta\delta}g^{\alpha\lambda}\partial_\lambda\Omega\partial_\alpha g_{\beta\delta}\right].
\end{aligned}\tag{71}$$

Then, we examine the term proportional to $\Omega^{-1}\partial_\mu\Omega$,

$$\begin{aligned}
&\Omega^{-2}g^{\beta\delta}\Big[\Gamma^\alpha_{\delta\lambda}\Omega^{-1}\partial_\mu\Omega(\delta^\lambda_\beta\delta^\mu_\alpha+\delta^\lambda_\alpha\delta^\mu_\beta-g_{\beta\alpha}g^{\lambda\mu})+\Omega^{-1}\partial_\mu\Omega(\delta^\alpha_\delta\delta^\mu_\lambda+\delta^\alpha_\lambda\delta^\mu_\delta-g_{\delta\lambda}g^{\alpha\mu})\Gamma^\lambda_{\beta\alpha}\\
&\qquad-\Gamma^\alpha_{\alpha\lambda}\Omega^{-1}\partial_\mu\Omega(\delta^\lambda_\beta\delta^\mu_\delta+\delta^\lambda_\delta\delta^\mu_\beta-g_{\beta\delta}g^{\lambda\mu})-\Omega^{-1}\partial_\mu\Omega(\delta^\alpha_\alpha\delta^\mu_\lambda+\delta^\alpha_\lambda\delta^\mu_\alpha-g_{\alpha\lambda}g^{\alpha\mu})\Gamma^\lambda_{\beta\delta}\Big]\\
&\quad=\Omega^{-3}\partial_\mu\Omega g^{\beta\delta}\left(\Gamma^\mu_{\delta\beta}+\Gamma^\alpha_{\alpha\delta}\delta^\mu_\beta-\Gamma^\alpha_{\delta\lambda}g_{\beta\alpha}g^{\lambda\mu}+\Gamma^\mu_{\beta\delta}+\Gamma^\alpha_{\alpha\beta}\delta^\mu_\delta-g_{\delta\lambda}g^{\alpha\mu}\Gamma^\lambda_{\beta\alpha}-\Gamma^\alpha_{\alpha\beta}\delta^\mu_\delta-\Gamma^\alpha_{\alpha\delta}\delta^\mu_\beta+\Gamma^\alpha_{\alpha\lambda}g_{\beta\delta}g^{\lambda\mu}-n\Gamma^\mu_{\beta\delta}\right)\\
&\quad=\Omega^{-3}\partial_\mu\Omega\left((2-n)g^{\beta\delta}\Gamma^\mu_{\beta\delta}-\Gamma^\alpha_{\alpha\lambda}g^{\lambda\mu}-\Gamma^\beta_{\beta\alpha}g^{\alpha\mu}+n\Gamma^\alpha_{\alpha\lambda}g^{\lambda\mu}\right)\\
&\quad=\Omega^{-3}\partial_\mu\Omega(2-n)\left(g^{\beta\delta}\Gamma^\mu_{\beta\delta}-\Gamma^\alpha_{\alpha\lambda}g^{\lambda\mu}\right)\\
&\quad=\Omega^{-3}\partial_\mu\Omega(2-n)\left(\frac{1}{2}g^{\beta\delta}g^{\mu\lambda}(\partial_\beta g_{\delta\lambda}+\partial_\delta g_{\beta\lambda}-\partial_\lambda g_{\beta\delta})-\frac{1}{2}g^{\lambda\mu}g^{\alpha\nu}(\partial_\alpha g_{\nu\lambda}+\partial_\lambda g_{\nu\alpha}-\partial_\nu g_{\alpha\lambda})\right)\\
&\quad=\Omega^{-3}\partial_\mu\Omega(2-n)g^{\beta\delta}g^{\mu\lambda}(\partial_\delta g_{\beta\lambda}-\partial_\lambda g_{\beta\delta})\\
&\quad=\Omega^{-3}\partial_\mu\Omega(2-n)\left(2g^{\beta\delta}\Gamma^\mu_{\beta\delta}-g^{\beta\delta}g^{\mu\lambda}\partial_\beta g_{\delta\lambda}\right).
\end{aligned}\tag{72}$$

Finally, we examine the term proportional to $\Omega^{-2}\partial_\mu\Omega\partial_\nu\Omega$,

$$\begin{aligned}
&\Omega^{-2}g^{\beta\delta}\Big[\Omega^{-1}\partial_\mu\Omega(\delta^\alpha_\delta\delta^\mu_\lambda+\delta^\alpha_\lambda\delta^\mu_\delta-g_{\delta\lambda}g^{\alpha\mu})\Omega^{-1}\partial_\nu\Omega(\delta^\lambda_\beta\delta^\nu_\alpha+\delta^\lambda_\alpha\delta^\nu_\beta-g_{\beta\alpha}g^{\lambda\nu})\\
&\qquad-\Omega^{-1}\partial_\mu\Omega(\delta^\alpha_\alpha\delta^\mu_\lambda+\delta^\alpha_\lambda\delta^\mu_\alpha-g_{\alpha\lambda}g^{\alpha\mu})\Omega^{-1}\partial_\nu\Omega(\delta^\lambda_\beta\delta^\nu_\delta+\delta^\lambda_\delta\delta^\nu_\beta-g_{\beta\delta}g^{\lambda\nu})\Big]\\
&\quad=\Omega^{-4}\partial_\mu\Omega\partial_\nu\Omega g^{\mu\nu}(n^2-3n+2).
\end{aligned}\tag{73}$$

Now, we gather all the terms again and use $\Box\Omega=g^{\mu\nu}\partial_\mu\partial_\nu\Omega-\Gamma^\lambda_{\mu\nu}\partial_\lambda\Omega g^{\mu\nu}$:

$$\begin{aligned}
R'&=\Omega^{-2}\Big[R+\Omega^{-2}\partial_\mu\Omega\partial_\nu\Omega g^{\mu\nu}(n^2-3n+2)+\Omega^{-1}\partial_\mu\Omega(2-n)\left(2g^{\beta\delta}\Gamma^\mu_{\beta\delta}-g^{\beta\delta}g^{\mu\lambda}\partial_\beta g_{\delta\lambda}\right)\\
&\qquad+2(n-1)\Omega^{-1}g^{\beta\delta}\partial_\beta\partial_\delta\Omega-2(n-1)\Omega^{-2}g^{\beta\delta}\partial_\beta\Omega\partial_\delta\Omega+n\Omega^{-1}\partial_\lambda\Omega\partial_\alpha g^{\alpha\lambda}+\Omega^{-1}g^{\beta\delta}g^{\alpha\lambda}\partial_\lambda\Omega\partial_\alpha g_{\beta\delta}\Big]\\
&=\Omega^{-2}\big[R+\Omega^{-2}\partial_\mu\Omega\partial_\nu\Omega g^{\mu\nu}(n^2-5n+4)+2(n-1)\Omega^{-1}\Box\Omega\\
&\qquad+2\Omega^{-1}\partial_\mu\Omega g^{\beta\delta}\Gamma^\mu_{\beta\delta}-(2-n)\Omega^{-1}\partial_\mu\Omega g^{\beta\delta}g^{\mu\lambda}\partial_\beta g_{\delta\lambda}+n\Omega^{-1}\partial_\lambda\Omega\partial_\alpha g^{\alpha\lambda}+\Omega^{-1}g^{\beta\delta}g^{\alpha\lambda}\partial_\lambda\Omega\partial_\alpha g_{\beta\delta}\Big]
\end{aligned}\tag{74}$$

Focussing on the last four terms, we can show that they amount to 0.

$$\begin{aligned}
&2\Omega^{-1}\partial_\mu\Omega g^{\beta\delta}\Gamma^\mu_{\beta\delta} - (2-n)\Omega^{-1}\partial_\mu\Omega g^{\beta\delta}g^{\mu\lambda}\partial_\beta g_{\delta\lambda} + n\Omega^{-1}\partial_\lambda\Omega\partial_\alpha g^{\alpha\lambda} + \Omega^{-1}g^{\beta\delta}g^{\alpha\lambda}\partial_\lambda\Omega\partial_\alpha g_{\beta\delta}\\
&\quad = \Omega^{-1}\partial_\mu\Omega g^{\beta\delta}g^{\mu\lambda}(\partial_\beta g_{\delta\lambda} + \partial_\delta g_{\beta\lambda} - \partial_\lambda g_{\beta\delta}) - (2-n)\Omega^{-1}\partial_\mu\Omega g^{\beta\delta}g^{\mu\lambda}\partial_\beta g_{\delta\lambda}\\
&\quad\quad + n\Omega^{-1}\partial_\lambda\Omega\partial_\alpha g^{\alpha\lambda} + \Omega^{-1}g^{\beta\delta}g^{\alpha\lambda}\partial_\lambda\Omega\partial_\alpha g_{\beta\delta}\\
&\quad = n\Omega^{-1}\partial_\lambda\Omega(g^{\beta\delta}g^{\mu\lambda}\partial_\beta g_{\delta\mu} + \partial_\alpha g^{\alpha\lambda})\\
&\quad = n\Omega^{-1}\partial_\lambda\Omega\left(\partial_\beta(g^{\beta\delta}g^{\lambda\mu}g_{\delta\mu}) - g^{\beta\delta}g_{\delta\mu}\partial_\beta g^{\lambda\mu} - g^{\lambda\mu}g_{\delta\mu}\partial_\beta g^{\beta\delta} + \partial_\alpha g^{\alpha\lambda}\right)\\
&\quad = n\Omega^{-1}\partial_\lambda\Omega(2\partial_\alpha g^{\alpha\lambda} - \delta^\beta_\mu\partial_\beta g^{\lambda\mu} - \delta^\lambda_\delta\partial_\beta g^{\beta\delta})\\
&\quad = n\Omega^{-1}\partial_\lambda\Omega(2\partial_\alpha g^{\alpha\lambda} - 2\partial_\alpha g^{\alpha\lambda})\\
&\quad = 0
\end{aligned} \tag{75}$$

In the before last step, we used that $\partial_\beta\delta^\lambda_\delta = 0$. The transformation of R then results in

$$R' = \Omega^{-2}\left[R + (n-1)(n-4)\Omega^{-2}g^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega + 2(n-1)\Omega^{-1}\Box\Omega\right]. \tag{76}$$

A.2 n-bein e^a_μ

In this section, the derivation of conformal transformations of n-bein related quantities can be found.

A.2.1 n-bein e^a_μ

We use the definition of e^a_μ, given in equation 8, and the transformation of the metric.

$$g'_{\mu\nu} = \Omega^2 g_{\mu\nu} = e'^a_\mu e'^b_\nu \eta_{ab} \qquad g'^{\mu\nu} = \Omega^{-2}g^{\mu\nu} = e'^\mu_a e'^\nu_b \eta^{ab} \tag{77}$$

This means that the n-bein transforms as

$$e'^a_\mu = \Omega e^a_\mu \qquad e'^\mu_a = \Omega^{-1}e^\mu_a. \tag{78}$$

A.2.2 Dirac gamma matrix γ^μ

We use that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{79}$$

which results in

$$\gamma'^\mu = \Omega^{-1}\gamma^\mu \tag{80}$$

A.2.3 Connection $\omega^a_{\mu b}$

$$\begin{aligned}
\omega'^a_{\mu b} &= -e'^{\ \nu}_b(\partial_\mu e'^a_\nu - \Gamma'^\lambda_{\mu\nu}e'^a_\lambda)\\
&= -\Omega^{-1}e^\nu_b\left(\partial_\mu(\Omega e^a_\nu) - \left(\Gamma^\lambda_{\mu\nu} + \Omega^{-1}\partial_\sigma\Omega(\delta^\lambda_\mu\delta^\sigma_\nu + \delta^\lambda_\nu\delta^\sigma_\mu - g_{\mu\nu}g^{\lambda\sigma})\right)\Omega e^a_\lambda\right)\\
&= -\Omega^{-1}e^\nu_b\left(e^a_\nu\partial_\mu\Omega + \Omega\partial_\mu e^a_\nu - \Gamma^\lambda_{\mu\nu}\Omega e^a_\lambda - \partial_\sigma\Omega(\delta^\lambda_\mu\delta^\sigma_\nu + \delta^\lambda_\nu\delta^\sigma_\mu - g_{\mu\nu}g^{\lambda\sigma})e^a_\lambda\right)\\
&= \omega^a_{\mu b} - \Omega^{-1}\left(e^\nu_b e^a_\nu\partial_\mu\Omega - e^\nu_b\partial_\sigma\Omega(\delta^\lambda_\mu\delta^\sigma_\nu + \delta^\lambda_\nu\delta^\sigma_\mu - g_{\mu\nu}g^{\lambda\sigma})e^a_\lambda\right)\\
&= \omega^a_{\mu b} - \Omega^{-1}\left(e^\nu_b e^a_\nu\partial_\mu\Omega - e^\sigma_b e^a_\mu\partial_\sigma\Omega - e^\nu_b e^a_\nu\partial_\mu\Omega + g_{\mu\rho}g^{\lambda\sigma}e^\rho_b e^a_\lambda\partial_\sigma\Omega\right)\\
&= \omega^a_{\mu b} - \Omega^{-1}\left(g_{\mu\rho}g^{\lambda\sigma}e^\rho_b e^a_\lambda\partial_\sigma\Omega - e^\sigma_b e^a_\mu\partial_\sigma\Omega\right)
\end{aligned} \tag{81}$$

Here, we used the transformation of the Christoffel symbol from section A.1.1.
We are also interested in the transformation of $\omega_\mu^{ab}[\gamma_a,\gamma_b]$, where we will use the transformation of $\omega_{\mu b}^{a}$ described above:

$$\begin{aligned}
\omega_\mu'^{\,ab}[\gamma_a,\gamma_b] &= \omega'^{a}_{\mu c}\eta^{cb}[\gamma_a,\gamma_b] \\
&= \eta^{cb}\left(\omega^{a}_{\mu c} - \Omega^{-1}\left(g_{\mu\rho}g^{\lambda\sigma}e^{\rho}_{c}e^{a}_{\lambda}\partial_\sigma\Omega - e^{\sigma}_{c}e^{a}_{\mu}\partial_\sigma\Omega\right)\right)[\gamma_a,\gamma_b] \\
&= \omega^{ab}_{\mu}[\gamma_a,\gamma_b] - \Omega^{-1}\eta^{cb}\left(g_{\mu\rho}g^{\lambda\sigma}e^{\rho}_{c}e^{a}_{\lambda}\partial_\sigma\Omega - e^{\sigma}_{c}e^{a}_{\mu}\partial_\sigma\Omega\right)[\gamma_a,\gamma_b] \\
&= \omega^{ab}_{\mu}[\gamma_a,\gamma_b] - \Omega^{-1}\left(g_{\mu\rho}g^{\lambda\sigma}\partial_\sigma\Omega[\gamma_\lambda,\gamma^\rho] - \partial_\sigma\Omega[\gamma_\mu,\gamma^\sigma]\right) \\
&= \omega^{ab}_{\mu}[\gamma_a,\gamma_b] - \Omega^{-1}\left(\partial_\sigma\Omega g^{\lambda\sigma}[\gamma_\lambda,\gamma_\mu] - \partial_\sigma\Omega g^{\lambda\sigma}[\gamma_\mu,\gamma_\lambda]\right) \\
&= \omega^{ab}_{\mu}[\gamma_a,\gamma_b] - 2\Omega^{-1}\partial_\sigma\Omega g^{\lambda\sigma}[\gamma_\lambda,\gamma_\mu].
\end{aligned} \tag{82}$$

A.3 Scalar field ϕ

In this section, the conformal transformations of the Klein-Gordon equation and the scalar Lagrangian can be found.

A.3.1 Klein-Gordon equation $\Box\phi + \xi R\phi = 0$

First, we focus on the d'Alembertian of the scalar field, $\Box\phi$:

$$\begin{aligned}
\Box'\phi' &= \frac{1}{\sqrt{-g'}}\partial_\mu\left(\sqrt{-g'}g'^{\mu\nu}\partial_\nu\phi'\right) \\
&= \frac{1}{\sqrt{-\Omega^{2n}g}}\partial_\mu\left(\sqrt{-\Omega^{2n}g}\Omega^{-2}g^{\mu\nu}\partial_\nu(\Omega^{1-\frac{n}{2}}\phi)\right) \\
&= \frac{1}{\sqrt{-g}}\Omega^{-n}\partial_\mu\left(\sqrt{-g}\Omega^{n-2}g^{\mu\nu}\left(\Omega^{1-\frac{n}{2}}\partial_\nu\phi + (1-\frac{n}{2})\phi\Omega^{-\frac{n}{2}}\partial_\nu\Omega\right)\right) \\
&= \frac{1}{\sqrt{-g}}\Omega^{-n}\partial_\mu\left(\Omega^{\frac{n}{2}-1}\sqrt{-g}g^{\mu\nu}\partial_\nu\phi + \Omega^{\frac{n}{2}-2}\sqrt{-g}g^{\mu\nu}\phi(1-\frac{n}{2})\partial_\nu\Omega\right) \\
&= \frac{1}{\sqrt{-g}}\Omega^{-n}\left(\Omega^{\frac{n}{2}-1}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) + \sqrt{-g}g^{\mu\nu}\partial_\nu\phi(\frac{n}{2}-1)\Omega^{\frac{n}{2}-2}\partial_\mu\Omega + (\frac{n}{2}-2)\Omega^{\frac{n}{2}-3}\partial_\mu\Omega\sqrt{-g}g^{\mu\nu}\phi(1-\frac{n}{2})\partial_\nu\Omega\right. \\
&\qquad\left. + \Omega^{\frac{n}{2}-2}\phi(1-\frac{n}{2})\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Omega) + \Omega^{\frac{n}{2}-2}\sqrt{-g}g^{\mu\nu}(1-\frac{n}{2})\partial_\nu\Omega\partial_\mu\phi\right) \\
&= \Omega^{-\frac{n}{2}-1}\left(\Box\phi - \Omega^{-2}\frac{(n-2)(n-4)}{4}g^{\mu\nu}\phi\partial_\mu\Omega\partial_\nu\Omega - \frac{n-2}{2}\Omega^{-1}\phi\Box\Omega\right).
\end{aligned} \tag{83}$$

Then, we implement this and the transformation of R into the full Klein-Gordon equation:

$$\begin{aligned}\Box'\phi' + \xi R'\phi' &= \Omega^{-\frac{n}{2}-1}\left(\Box\phi - \Omega^{-2}\frac{(n-2)(n-4)}{4}g^{\mu\nu}\phi\partial_\mu\Omega\partial_\nu\Omega - \frac{n-2}{2}\Omega^{-1}\phi\Box\Omega\right)\\ &\quad + \xi\Omega^{-\frac{n}{2}-1}\left(R\phi + (n-1)(n-4)\Omega^{-2}g^{\mu\nu}\phi\partial_\mu\Omega\partial_\nu\Omega + 2(n-1)\Omega^{-1}\phi\Box\Omega\right)\\ &= \Omega^{-\frac{n}{2}-1}(\Box\phi + \xi R\phi) + \Omega^{-\frac{n}{2}-1}\left[\left(\xi(n-1)(n-4) - \frac{(n-2)(n-4)}{4}\right)\Omega^{-2}g^{\mu\nu}\phi\partial_\mu\Omega\partial_\nu\Omega\right.\\ &\qquad\left. + \left(\xi 2(n-1) - \frac{n-2}{2}\right)\Omega^{-1}\phi\Box\Omega\right]\\ &= \Omega^{-\frac{n}{2}-1}\left(\Box\phi + \xi R\phi\right) = 0\end{aligned} \tag{84}$$

Here, the last equation only holds for $\xi = \xi_c = \frac{n-2}{4(n-1)}$.

A.3.2 Scalar Lagrangian $\mathscr{L}$

$$\begin{aligned}\mathscr{L}' &= \frac{1}{2}\sqrt{-g'}\left(g'^{\mu\nu}\partial_\mu\phi'\partial_\nu\phi' - \xi R'\phi'^2\right)\\ &= \frac{1}{2}\Omega^n\sqrt{-g}\left[\Omega^{-2}g^{\mu\nu}\partial_\mu(\Omega^{1-\frac{n}{2}}\phi)\partial_\nu(\Omega^{1-\frac{n}{2}}\phi) - \xi\Omega^{-2}\left(R + (n-1)(n-4)\Omega^{-2}g^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega + 2(n-1)\Omega^{-1}\Box\Omega\right)(\Omega^{1-\frac{n}{2}}\phi)^2\right]\\ &= \frac{1}{2}\Omega^n\sqrt{-g}\left[\Omega^{-2}g^{\mu\nu}\left((1-\frac{n}{2})\Omega^{-\frac{n}{2}}\phi\partial_\mu\Omega + \Omega^{1-\frac{n}{2}}\partial_\mu\phi\right)\left((1-\frac{n}{2})\Omega^{-\frac{n}{2}}\phi\partial_\nu\Omega + \Omega^{1-\frac{n}{2}}\partial_\nu\phi\right)\right.\\ &\qquad\left. -\xi\left(R + (n-1)(n-4)\Omega^{-2}g^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega + 2(n-1)\Omega^{-1}\Box\Omega\right)\Omega^{-n}\phi^2\right]\\ &= \frac{1}{2}\sqrt{-g}\left[g^{\mu\nu}\Omega^{n-2}\left((1-n+\frac{n^2}{4})\Omega^{-n}\phi^2\partial_\mu\Omega\partial_\nu\Omega + (1-\frac{n}{2})\Omega^{1-n}\phi\partial_\mu\Omega\partial_\nu\phi + (1-\frac{n}{2})\Omega^{1-n}\phi\partial_\nu\Omega\partial_\mu\phi + \Omega^{2-n}\partial_\mu\phi\partial_\nu\phi\right)\right.\\ &\qquad\left. -\xi\left(R + (n-1)(n-4)\Omega^{-2}g^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega + 2(n-1)\Omega^{-1}\Box\Omega\right)\phi^2\right]\\ &= \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \xi R\phi^2\right) + \frac{1}{2}\sqrt{-g}\left[\left((1-n+\frac{n^2}{4}) - \xi(n-1)(n-4)\right)\Omega^{-2}g^{\mu\nu}\phi^2\partial_\mu\Omega\partial_\nu\Omega\right.\\ &\qquad\left. + (2-n)\Omega^{-1}g^{\mu\nu}\phi\partial_\mu\Omega\partial_\nu\phi - 2(n-1)\xi\Omega^{-1}\phi^2\Box\Omega\right]\end{aligned} \tag{85}$$

Now, we plug in $\xi = \xi_c = \frac{n-2}{4(n-1)}$:

$$\begin{aligned}\mathscr{L}' &= \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \xi R\phi^2\right) + \frac{1}{2}\sqrt{-g}\left((\frac{n}{2}-1)\Omega^{-2}g^{\mu\nu}\phi^2\partial_\mu\Omega\partial_\nu\Omega + (2-n)\Omega^{-1}g^{\mu\nu}\phi\partial_\mu\Omega\partial_\nu\phi - (\frac{n}{2}-1)\Omega^{-1}\phi^2\Box\Omega\right)\\ &= \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \xi R\phi^2\right) + \frac{1}{2}(\frac{n}{2}-1)\left(\sqrt{-g}\Omega^{-2}g^{\mu\nu}\phi^2\partial_\mu\Omega\partial_\nu\Omega - 2\sqrt{-g}\Omega^{-1}\partial_\mu\Omega\partial_\nu\phi g^{\mu\nu}\phi - \Omega^{-1}\sqrt{-g}\phi^2\Box\Omega\right)\\ &= \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \xi R\phi^2\right) + \frac{1}{2}(\frac{n}{2}-1)\left(\sqrt{-g}\Omega^{-2}g^{\mu\nu}\phi^2\partial_\mu\Omega\partial_\nu\Omega - 2\sqrt{-g}\Omega^{-1}\partial_\mu\Omega\partial_\nu\phi g^{\mu\nu}\phi - \Omega^{-1}\phi^2\nabla_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Omega)\right)\\ &= \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \xi R\phi^2\right) + \nabla_\mu\left(-\sqrt{-g}(\frac{n}{2}-1)\frac{1}{2}\Omega^{-1}g^{\mu\nu}\phi^2\partial_\nu\Omega\right)\\ &= \mathscr{L} + \nabla_\mu\left(-\sqrt{-g}(\frac{n}{2}-1)\frac{1}{2}\Omega^{-1}g^{\mu\nu}\phi^2\partial_\nu\Omega\right),\end{aligned} \tag{86}$$

which is the same as the original scalar Lagrangian up to a total four-divergence.

A.4 Dirac field ψ

In this section, the conformal transformations of the Dirac equation and the Dirac Lagrangian can be found.

A.4.1 Dirac equation $i\gamma^\mu\nabla_\mu\psi = 0$

$$
\begin{aligned}
i\gamma'^\mu\nabla'_\mu\psi' &= i\gamma'^\mu\left(\partial_\mu + \frac{1}{8}\omega'^{\;ab}_\mu[\gamma_a,\gamma_b]\right)\psi' \\
&= \Omega^{\frac{1-n}{2}-1} i\gamma^\mu\left(\partial_\mu + \frac{1}{8}\omega^{ab}_\mu[\gamma_a,\gamma_b] - \frac{1}{4}\Omega^{-1}\partial_\sigma\Omega g^{\lambda\sigma}[\gamma_\lambda,\gamma_\mu] + \frac{1-n}{2}\Omega^{-1}\partial_\mu\Omega\right)\psi \\
&= \Omega^{\frac{1-n}{2}-1}\left(i\gamma^\mu\nabla_\mu\psi - \frac{i}{4}\Omega^{-1}\partial_\sigma\Omega g^{\lambda\sigma}\gamma^\mu[\gamma_\lambda,\gamma_\mu]\psi + i\frac{1-n}{2}\Omega^{-1}\gamma^\mu\partial_\mu\Omega\psi\right) \\
&= \Omega^{\frac{1-n}{2}-1}\left(i\gamma^\mu\nabla_\mu\psi + i\frac{n-1}{2}\Omega^{-1}\partial_\sigma\Omega g^{\lambda\sigma}\gamma_\lambda\psi + i\frac{1-n}{2}\Omega^{-1}\gamma^\mu\partial_\mu\Omega\psi\right) \\
&= \Omega^{\frac{1-n}{2}-1}\left(i\gamma^\mu\nabla_\mu\psi - i\frac{1-n}{2}\Omega^{-1}\gamma^\mu\partial_\mu\Omega\psi + i\frac{1-n}{2}\Omega^{-1}\gamma^\mu\partial_\mu\Omega\psi\right) \\
&= \Omega^{\frac{1-n}{2}-1} i\gamma^\mu\nabla_\mu\psi = 0
\end{aligned}
\tag{87}
$$

Here, we used $\gamma^\mu[\gamma_\lambda,\gamma_\mu] = -(2n-2)\gamma_\lambda$ and the transformations described in section A.2.

A.4.2 Dirac Lagrangian $\mathscr{L}_D$

$$
\begin{aligned}
\mathscr{L}'_D &= \frac{i}{2}\sqrt{|g'|}\left(\bar\psi'\gamma'^\mu\nabla'_\mu\psi' - \bar\psi'\gamma'^\mu\overleftarrow{\nabla}'_\mu\psi'\right) \\
&= \frac{i}{2}\Omega^n\sqrt{|g|}\left(\Omega^{\frac{1-n}{2}}\Omega^{\frac{1-n}{2}-1}\bar\psi\gamma^\mu\nabla_\mu\psi - \Omega^{\frac{1-n}{2}}\Omega^{\frac{1-n}{2}-1}(\bar\psi\gamma^\mu\overleftarrow{\nabla}_\mu)\psi\right) \\
&= \Omega^{(1-n)-1}\Omega^n\frac{i}{2}\sqrt{|g|}\left(\bar\psi\gamma^\mu\nabla_\mu\psi - \bar\psi\gamma^\mu\overleftarrow{\nabla}_\mu\psi\right) \\
&= \mathscr{L}_D
\end{aligned}
\tag{88}
$$

B Rivier ordering in integral form

In this section, we describe the method of implementing the Rivier ordering for a random polynomial function $\hat{f}_R(\hat{q},\hat{p})$ using integrals.

As with Weyl ordering we start with the general description of $\hat{f}_R$,

$$\hat{f}_R(\hat{q},\hat{p}) = \sum_{\sigma_\alpha,\sigma_\beta \leq m} c_{\vec{\alpha}\vec{\beta}} \frac{1}{2}\left((\hat{q}^1)^{\alpha_1}...(\hat{q}^{n-1})^{\alpha_{n-1}}\hat{p}_1^{\beta_1}...\hat{p}_{n-1}^{\beta_{n-1}} + \hat{p}_1^{\beta_1}...\hat{p}_{n-1}^{\beta_{n-1}}(\hat{q}^1)^{\alpha_1}...(\hat{q}^{n-1})^{\alpha_{n-1}}\right), \tag{89}$$

where $\sigma_\alpha = \alpha_1 + ... + \alpha_{n-1}$, $\sigma_\beta = \beta_1 + ... + \beta_{n-1}$, m is the maximum degree of the polynomial, and $c_{\vec{\alpha}\vec{\beta}}$ is a coefficient. A reminder of the notation, with $(\hat{q}^k)^{\alpha_k}$ we mean that every operator $\hat{q}^k$ has its own corresponding power α_k, where the latter are integers and the repeated k-indices are not summed over. Generally, Rivier ordering groups all operators of the same type, as mentioned in chapter 3. All position operators are grouped on one side and all momentum operators on the other side, supplemented by the reversed expression and a weight of $\frac{1}{2}$. For example, $\mathrm{x}p^2 \xrightarrow{Q} \frac{1}{2}(\hat{q}\hat{p}^2 + \hat{p}^2\hat{q})$. More generally, we can write

$$\hat{f}_R = \frac{1}{2}(\hat{f}_l + \hat{f}_r), \tag{90}$$

where $\hat{f}_l$ and $\hat{f}_r$ have all position operators to the left and right of all momentum operators respectively. Since the two parts are easily separable and very similar, the following description of the integral method is done for the first half of Rivier ordering, $\hat{f}_l$.

First, we define the kernel K in the following way

$$K(\mathrm{x},\mathrm{y}) = \langle \mathrm{x}|\hat{f}|\mathrm{y}\rangle. \tag{91}$$

In our case, we will only be working with $\hat{f}_l$ so we will use

$$K(\mathrm{x},\mathrm{y}) = \langle \mathrm{x}|\hat{f}_l|\mathrm{y}\rangle. \tag{92}$$

Then, we write the symbol f as a Fourier transformation (FT) of the kernel

$$f(q,p) = \int L(q,p|\mathrm{x},\mathrm{y})K(\mathrm{x},\mathrm{y})\, d\mathrm{x}\, d\mathrm{y}. \tag{93}$$

In order to determine $L(q,p|\mathrm{x},\mathrm{y})$, a few steps are needed as well as some ingredients. The first ingredients are the following four kernels:

$$\begin{aligned}
\langle \mathrm{x}|\hat{p}_j\hat{f}_l|\mathrm{y}\rangle &= \int \langle \mathrm{x}|\hat{p}_j|\mathrm{z}\rangle\langle \mathrm{z}|\hat{f}_l|\mathrm{y}\rangle\, dz \\
&= \int (-i\hbar)\left(\frac{\partial}{\partial \mathrm{x}^j} + \frac{1}{4}\frac{\partial}{\partial \mathrm{x}^j}\ln\omega(\mathrm{x})\right)\delta(\mathrm{x}-\mathrm{z})K(\mathrm{z},\mathrm{y})\, dz \\
&= -i\hbar\left(\frac{\partial}{\partial \mathrm{x}^j} + \frac{1}{4}\frac{\partial}{\partial \mathrm{x}^j}\ln\omega(\mathrm{x})\right)K(\mathrm{x},\mathrm{y}) \qquad (94a)\\
\langle \mathrm{x}|\hat{f}_l\hat{p}_j|\mathrm{y}\rangle &= \int \langle \mathrm{x}|\hat{f}_l|\mathrm{z}\rangle\langle \mathrm{z}|\hat{p}_j|\mathrm{y}\rangle\, dz \\
&= \int K(\mathrm{x},\mathrm{z})(-i\hbar)\left(\frac{\partial}{\partial \mathrm{z}^j} + \frac{1}{4}\frac{\partial}{\partial \mathrm{z}^j}\ln\omega(\mathrm{z})\right)\delta(\mathrm{z}-\mathrm{y})\, dz \\
&= +i\hbar\left(\frac{\partial}{\partial \mathrm{y}^j} - \frac{1}{4}\frac{\partial}{\partial \mathrm{y}^j}\ln\omega(\mathrm{y})\right)K(\mathrm{x},\mathrm{y}) \qquad (94b)\\
\langle \mathrm{x}|\hat{q}^j\hat{f}_l|\mathrm{y}\rangle &= \mathrm{x}^j\langle \mathrm{x}|\hat{f}_l|\mathrm{y}\rangle = \mathrm{x}^jK(\mathrm{x},\mathrm{y}) \qquad (94c)\\
\langle \mathrm{x}|\hat{f}_l\hat{q}^j|\mathrm{y}\rangle &= \mathrm{y}^j\langle \mathrm{x}|\hat{f}_l|\mathrm{y}\rangle = \mathrm{y}^jK(\mathrm{x},\mathrm{y}). \qquad (94d)
\end{aligned}$$

The next ingredients are the following operators and their corresponding symbols.

$$\hat{p}_j\hat{f}_l \to \left(p_j - i\hbar\frac{\partial}{\partial q^j}\right)f \qquad (95a)$$

$$\hat{f}_l\hat{p}_j \to p_jf \qquad (95b)$$

$$\hat{q}^j\hat{f}_l \to q^jf \qquad (95c)$$

$$\hat{f}_l\hat{q}^j \to \left(q^j - i\hbar\frac{\partial}{\partial p_j}\right)f. \qquad (95d)$$

Note that we can only translate the operators to their corresponding symbols once they are in the right ordering, for which we used the commutation relations of equation 26. To determine $L(q,p|\mathrm{x},\mathrm{y})$, we first use equation 93 in combination with equations 94a and 95a,

$$\begin{aligned}
\left(p_j - i\hbar\frac{\partial}{\partial q^j}\right)f(q,p) &= \int\left(p_j - i\hbar\frac{\partial}{\partial q^j}\right)L(q,p|\mathrm{x},\mathrm{y})K(\mathrm{x},\mathrm{y})\, d\mathrm{x}\, d\mathrm{y} \\
&= \int L(q,p|\mathrm{x},\mathrm{y})(-i\hbar)\left(\frac{\partial}{\partial \mathrm{x}^j} + \frac{1}{4}\frac{\partial}{\partial \mathrm{x}^j}\ln\omega(\mathrm{x})\right)K(\mathrm{x},\mathrm{y})\, d\mathrm{x}\, d\mathrm{y} \qquad (96)\\
&= \int (+i\hbar)K(\mathrm{x},\mathrm{y})\left(\frac{\partial}{\partial \mathrm{x}^j} - \frac{1}{4}\frac{\partial}{\partial \mathrm{x}^j}\ln\omega(\mathrm{x})\right)L(q,p|\mathrm{x},\mathrm{y})\, d\mathrm{x}\, d\mathrm{y},
\end{aligned}$$

to arrive at

$$\left(p_j - i\hbar\frac{\partial}{\partial q^j}\right)L(q,p|\mathrm{x},\mathrm{y}) = +i\hbar\left(\frac{\partial}{\partial \mathrm{x}^j} - \frac{1}{4}\frac{\partial}{\partial \mathrm{x}^j}\ln\omega(\mathrm{x})\right)L(q,p|\mathrm{x},\mathrm{y}). \qquad (97)$$

Then, we do the same thing, using equations 94b and 95b,

$$\begin{aligned}
f(q,p)p_j &= \int p_jL(q,p|\mathrm{x},\mathrm{y})K(\mathrm{x},\mathrm{y})\, d\mathrm{x}\, d\mathrm{y} \\
&= \int L(q,p|\mathrm{x},\mathrm{y})i\hbar\left(\frac{\partial}{\partial \mathrm{y}^j} - \frac{1}{4}\frac{\partial}{\partial \mathrm{y}^j}\ln\omega(\mathrm{y})\right)K(\mathrm{x},\mathrm{y})\, d\mathrm{x}\, d\mathrm{y} \qquad (98)\\
&= \int (-i\hbar)K(\mathrm{x},\mathrm{y})\left(\frac{\partial}{\partial \mathrm{y}^j} + \frac{1}{4}\frac{\partial}{\partial \mathrm{y}^j}\ln\omega(\mathrm{y})\right)L(q,p|\mathrm{x},\mathrm{y})\, d\mathrm{x}\, d\mathrm{y},
\end{aligned}$$

to arrive at

$$p_j L(q,p|\mathrm{x},\mathrm{y}) = -i\hbar\left(\frac{\partial}{\partial \mathrm{y}^j} + \frac{1}{4}\frac{\partial}{\partial \mathrm{y}^j}\ln\omega(\mathrm{y})\right) L(q,p|\mathrm{x},\mathrm{y}). \tag{99}$$

Similarly, we use equations 94c and 95c,

$$q^j f(p,q) = \int q^j L(q,p|\mathrm{x},\mathrm{y}) K(\mathrm{x},\mathrm{y})\, d\mathrm{x}\, d\mathrm{y} = \int L(q,p|\mathrm{x},\mathrm{y})\mathrm{x}^j K(\mathrm{x},\mathrm{y})\, d\mathrm{x}\, d\mathrm{y}, \tag{100}$$

to arrive at

$$q^j L(q,p|\mathrm{x},\mathrm{y}) = x^j L(q,p|\mathrm{x},\mathrm{y}). \tag{101}$$

And finally, we use equations 94d and 95d,

$$\begin{aligned}\left(q^j - i\hbar\frac{\partial}{\partial p_j}\right) f(p,q) &= \int \left(q^j - i\hbar\frac{\partial}{\partial p_j}\right) L(q,p|\mathrm{x},\mathrm{y}) K(\mathrm{x},\mathrm{y})\, d\mathrm{x}\, d\mathrm{y} \\ &= \int \mathrm{y}^j L(q,p|\mathrm{x},\mathrm{y}) K(\mathrm{x},\mathrm{y})\, d\mathrm{x}\, d\mathrm{y},\end{aligned} \tag{102}$$

to arrive at

$$\left(q^j - i\hbar\frac{\partial}{\partial p_j}\right) L(q,p|\mathrm{x},\mathrm{y}) = \mathrm{y}^j L(q,p|\mathrm{x},\mathrm{y}). \tag{103}$$

We have acquired a set of differential equations, formed by equations 97, 99, 101, and 103, and now we solve these to find $L(q,p|\mathrm{x},\mathrm{y})$. We begin by using equation 99

$$\left(\frac{i}{\hbar}p_j - \frac{1}{4}\frac{\partial}{\partial \mathrm{y}^j}\ln\omega(\mathrm{y})\right) L(q,p|\mathrm{x},\mathrm{y}) = \frac{\partial}{\partial \mathrm{y}^j} L(q,p|\mathrm{x},\mathrm{y}), \tag{104}$$

which results in

$$L(q,p|\mathrm{x},\mathrm{y}) = e^{\frac{i}{\hbar}p_j\mathrm{y}^j - \frac{1}{4}\ln\omega(\mathrm{y})} L_1(q,p|\mathrm{x}) = e^{ip_j\mathrm{y}^j/\hbar}\omega^{-\frac{1}{4}}(\mathrm{y}) L_1(q,p|\mathrm{x}). \tag{105}$$

Then, we use equation 97

$$\begin{aligned} \left(-\frac{i}{\hbar}p_j + \frac{1}{4}\frac{\partial}{\partial \mathrm{x}^j}\ln\omega(\mathrm{x})\right) L(q,p|\mathrm{x},\mathrm{y}) &= \left(\frac{\partial}{\partial \mathrm{x}^j} + \frac{\partial}{\partial q^j}\right) L(q,p|\mathrm{x},\mathrm{y}) \\ \Rightarrow \quad \left(-\frac{i}{\hbar}p_j + \frac{1}{4}\frac{\partial}{\partial \mathrm{x}^j}\ln\omega(\mathrm{x})\right) e^{ip_k\mathrm{y}^k/\hbar}\omega^{-\frac{1}{4}}(\mathrm{y}) L_1(q,p|\mathrm{x}) &= \left(\frac{\partial}{\partial \mathrm{x}^j} + \frac{\partial}{\partial q^j}\right) e^{ip_k\mathrm{y}^k/\hbar}\omega^{-\frac{1}{4}}(\mathrm{y}) L_1(q,p|\mathrm{x}) \\ \Rightarrow \quad \left(-\frac{i}{\hbar}p_j + \frac{1}{4}\frac{\partial}{\partial \mathrm{x}^j}\ln\omega(\mathrm{x})\right) L_1(q,p|\mathrm{x}) &= \left(\frac{\partial}{\partial \mathrm{x}^j} + \frac{\partial}{\partial q^j}\right) L_1(q,p|\mathrm{x}). \end{aligned} \tag{106}$$

This results in

$$L_1(q,p|\mathrm{x}) = e^{-ip_j\mathrm{x}^j/\hbar + \frac{1}{4}\ln\omega(\mathrm{x})} L_2(q-\mathrm{x},p) = \omega^{\frac{1}{4}}(\mathrm{x}) e^{-ip_j\mathrm{x}^j/\hbar} L_2(q-\mathrm{x},p), \tag{107}$$

which means $L(q,p|\mathrm{x},\mathrm{y})$ becomes

$$L(q,p|\mathrm{x},\mathrm{y}) = \omega^{\frac{1}{4}}(\mathrm{x})\omega^{-\frac{1}{4}}(\mathrm{y}) e^{-ip_j(\mathrm{x}^j-\mathrm{y}^j)/\hbar} L_2(q-\mathrm{x},p). \tag{108}$$

Using equation 101, we arrive at

$$\begin{aligned} q^j\omega^{\frac{1}{4}}(\mathrm{x})\omega^{-\frac{1}{4}}(\mathrm{y}) e^{-ip_j(\mathrm{x}^j-\mathrm{y}^j)/\hbar} L_2(q-\mathrm{x},p) &= \mathrm{x}^j\omega^{\frac{1}{4}}(\mathrm{x})\omega^{-\frac{1}{4}}(\mathrm{y}) e^{-ip_j(\mathrm{x}^j-\mathrm{y}^j)/\hbar} L_2(q-\mathrm{x},p) \\ \Rightarrow \quad q^j L_2(q-\mathrm{x},p) &= \mathrm{x}^j L_2(q-\mathrm{x},p). \end{aligned} \tag{109}$$

This implies

$$L_2(q-\mathrm{x},p)=\delta(\mathrm{x}-q)L_3(p), \tag{110}$$

resulting in the following for $L(q,p|\mathrm{x},\mathrm{y})$,

$$L(q,p|\mathrm{x},\mathrm{y})=\omega^{\frac{1}{4}}(\mathrm{x})\omega^{-\frac{1}{4}}(\mathrm{y})e^{-ip_j(\mathrm{x}^j-\mathrm{y}^j)/\hbar}\delta(\mathrm{x}-q)L_3(p). \tag{111}$$

Finally, we use equation 103

$$\begin{aligned}
\left(q^j-i\hbar\frac{\partial}{\partial p_j}\right)\omega^{\frac{1}{4}}(\mathrm{x})\omega^{-\frac{1}{4}}(\mathrm{y})e^{-ip_j(\mathrm{x}^j-\mathrm{y}^j)/\hbar}\delta(\mathrm{x}-q)L_3(p)&=\mathrm{y}^j\omega^{\frac{1}{4}}(\mathrm{x})\omega^{-\frac{1}{4}}(\mathrm{y})e^{-ip_j(\mathrm{x}^j-\mathrm{y}^j)/\hbar}\delta(\mathrm{x}-q)L_3(p)\\
\Rightarrow\qquad \left(\mathrm{x}^j-i\hbar\frac{\partial}{\partial p_j}\right)e^{-ip_j(\mathrm{x}^j-\mathrm{y}^j)/\hbar}L_3(p)&=\mathrm{y}^je^{-ip_j(\mathrm{x}^j-\mathrm{y}^j)/\hbar}L_3(p)\\
\Rightarrow\qquad \frac{\partial}{\partial p_j}L_3(p)&=0,
\end{aligned} \tag{112}$$

so $L_3(p)$ is a constant. We know that for $f=1$, $K(\mathrm{x},\mathrm{y})=\langle\mathrm{x}|\mathrm{y}\rangle=\delta(\mathrm{x}-\mathrm{y})$ and thus equation 93 becomes

$$1=\int\omega^{\frac{1}{4}}(\mathrm{x})\omega^{-\frac{1}{4}}(\mathrm{y})e^{-ip_j(\mathrm{x}^j-\mathrm{y}^j)/\hbar}\delta(\mathrm{x}-q)L_3(p)\delta(\mathrm{x}-\mathrm{y})\,d\mathrm{x}\,d\mathrm{y}. \tag{113}$$

This is correct if $L_3(p)=1$ and we finally arrive at

$$L(q,p|\mathrm{x},\mathrm{y})=\omega^{\frac{1}{4}}(\mathrm{x})\omega^{-\frac{1}{4}}(\mathrm{y})e^{-ip_j(\mathrm{x}^j-\mathrm{y}^j)/\hbar}\delta(\mathrm{x}-q). \tag{114}$$

Now we know $L(q,p|\mathrm{x},\mathrm{y})$, we can insert it into equation 93 to get

$$\begin{aligned}
f(q,p)&=\int\omega^{\frac{1}{4}}(\mathrm{x})\omega^{-\frac{1}{4}}(\mathrm{y})K(\mathrm{x},\mathrm{y})e^{-ip_j(\mathrm{x}^j-\mathrm{y}^j)/\hbar}\delta(\mathrm{x}-q)\,d\mathrm{x}\,d\mathrm{y}\\
&=\int\omega^{\frac{1}{4}}(q)\omega^{-\frac{1}{4}}(\mathrm{y})K(q,\mathrm{y})e^{-ip_j(q^j-\mathrm{y}^j)/\hbar}\,d\mathrm{y}.
\end{aligned} \tag{115}$$

The inverse FT is given by

$$\omega^{\frac{1}{4}}(\mathrm{x})\omega^{-\frac{1}{4}}(\mathrm{y})K(\mathrm{x},\mathrm{y})=(2\pi\hbar)^{1-n}\int e^{ip_j(\mathrm{x}^j-\mathrm{y}^j)/\hbar}f(\mathrm{x},p)\,dp, \tag{116}$$

which can be written as

$$K(\mathrm{x},\mathrm{y})=(2\pi\hbar)^{1-n}\int e^{ip_j(\mathrm{x}^j-\mathrm{y}^j)/\hbar}\omega^{-\frac{1}{4}}(\mathrm{x})\omega^{\frac{1}{4}}(\mathrm{y})f(\mathrm{x},p)\,dp. \tag{117}$$

Like before with Weyl ordering, we want to know $\hat{f}_l\Psi(\mathrm{x})$

$$\begin{aligned}
\hat{f}_l\Psi(\mathrm{x})&=\langle\mathrm{x}|\hat{f}_l|\Psi\rangle\\
&=\int\langle\mathrm{x}|\hat{f}_l|\mathrm{y}\rangle\langle\mathrm{y}|\Psi\rangle\,d\mathrm{y}\\
&=\int K(\mathrm{x},\mathrm{y})\Psi(\mathrm{y})\,d\mathrm{y}\\
&=(2\pi\hbar)^{1-n}\omega^{-\frac{1}{4}}(\mathrm{x})\int f(\mathrm{x},p)\omega^{\frac{1}{4}}(\mathrm{y})e^{ip_j(\mathrm{x}^j-\mathrm{y}^j)/\hbar}\Psi(\mathrm{y})\,d\mathrm{y}\,dp.
\end{aligned} \tag{118}$$

For $\hat{f}_r$, we can follow the same method as for $\hat{f}_l$ and arrive at

$$\hat{f}_r\Psi(\mathrm{x}) = (2\pi\hbar)^{1-n}\omega^{-\frac{1}{4}}(\mathrm{x})\int f(\mathrm{y},p)\omega^{\frac{1}{4}}(\mathrm{y})e^{ip_j(\mathrm{x}^j-\mathrm{y}^j)/\hbar}\Psi(\mathrm{y})\,d\mathrm{y}\,dp. \tag{119}$$

If we then add $\hat{f}_l$ and $\hat{f}_r$ with weight $\frac{1}{2}$, we arrive at our Rivier ordering in integral form

$$\hat{f}_R\Psi(\mathrm{x}) = (2\pi\hbar)^{1-n}\omega^{-\frac{1}{4}}(\mathrm{x})\int \frac{1}{2}\big(f(\mathrm{x},p)+f(\mathrm{y},p)\big)\omega^{\frac{1}{4}}(\mathrm{y})e^{ip_j(\mathrm{x}^j-\mathrm{y}^j)/\hbar}\Psi(\mathrm{y})\,d\mathrm{y}\,dp. \tag{120}$$

As with the Weyl ordering, this integral differs from that of Tagirov in [2] in that the phase has a different sign. Again, this was checked with an odd number of momentum operators, where our phase gives the correct expression. In the case described in chapter 3 this difference is inconsequential since we are dealing with an even number of momentum operators.

C Quantization of H_0 using the integral method

In this appendix, the derivation of equation 52, the quantization of H_0 using the integral method, can be found. Before quantization, H_0 is

$$H_0 = \frac{1}{2m} p_i \omega^{ij}(\mathrm{x}) p_j = f. \tag{121}$$

Of course, this is also our symbol f. To get the Weyl ordering, we use the integral of equation 48 with the above symbol as input and fully work it out along the lines of the example in equation 49:

$$\begin{aligned}
\hat{f}_W \Psi(\mathrm{x}) &= (2\pi\hbar)^{1-n} \omega^{-\frac{1}{4}}(\mathrm{x}) \int e^{i p_k (\mathrm{x}^k - \mathrm{y}^k)/\hbar} \frac{1}{2m} p_i \omega^{ij}(\frac{\mathrm{x}+\mathrm{y}}{2}) p_j \Psi'(\mathrm{y})\, dp\, d\mathrm{y} \\
&= (2\pi\hbar)^{1-n} \frac{1}{2m} \omega^{-\frac{1}{4}}(\mathrm{x}) \int \left[(i\hbar)^2 \frac{\partial}{\partial \mathrm{y}^i} \frac{\partial}{\partial \mathrm{y}^j} e^{i p_k (\mathrm{x}^k - \mathrm{y}^k)/\hbar} \right] \omega^{ij}(\frac{\mathrm{x}+\mathrm{y}}{2}) \Psi'(\mathrm{y})\, dp\, d\mathrm{y} \\
&= (2\pi\hbar)^{1-n} \frac{(-\hbar^2)}{2m} \omega^{-\frac{1}{4}}(\mathrm{x}) \int \left[\frac{\partial}{\partial \mathrm{y}^i} \frac{\partial}{\partial \mathrm{y}^j} (2\pi\hbar)^{n-1} \delta(\mathrm{x}-\mathrm{y}) \right] \omega^{ij}(\frac{\mathrm{x}+\mathrm{y}}{2}) \Psi'(\mathrm{y})\, d\mathrm{y} \\
&= \frac{-\hbar^2}{2m} \omega^{-\frac{1}{4}}(\mathrm{x}) \int \delta(\mathrm{x}-\mathrm{y}) \frac{\partial}{\partial \mathrm{y}^i} \left[\frac{\partial}{\partial \mathrm{y}^j} \left(\omega^{ij}(\frac{\mathrm{x}+\mathrm{y}}{2}) \Psi'(\mathrm{y}) \right) \right] d\mathrm{y} \\
&= \frac{-\hbar^2}{2m} \omega^{-\frac{1}{4}}(\mathrm{x}) \Bigg[\omega^{ij}(\mathrm{x}) \left(\frac{\partial}{\partial \mathrm{x}^i} \frac{\partial}{\partial \mathrm{x}^j} \Psi'(\mathrm{x}) \right) + \frac{1}{2} \left(\frac{\partial}{\partial \mathrm{x}^i} \omega^{ij}(\mathrm{x}) \right) \left(\frac{\partial}{\partial \mathrm{x}^j} \Psi'(\mathrm{x}) \right) + \frac{1}{2} \left(\frac{\partial}{\partial \mathrm{x}^j} \omega^{ij}(\mathrm{x}) \right) \left(\frac{\partial}{\partial \mathrm{x}^i} \Psi'(\mathrm{x}) \right) \\
&\qquad + \frac{1}{4} \left(\frac{\partial}{\partial \mathrm{x}^i} \frac{\partial}{\partial \mathrm{x}^j} \omega^{ij}(\mathrm{x}) \right) \Psi'(\mathrm{x}) \Bigg] \\
&= \frac{-\hbar^2}{2m} \omega^{-\frac{1}{4}}(\mathrm{x}) \Bigg[\frac{1}{4} \omega^{ij}(\mathrm{x}) \left(\frac{\partial}{\partial \mathrm{x}^i} \frac{\partial}{\partial \mathrm{x}^j} \Psi'(\mathrm{x}) \right) + \frac{1}{4} \frac{\partial}{\partial \mathrm{x}^i} \left(\omega^{ij}(\mathrm{x}) \frac{\partial}{\partial \mathrm{x}^j} \Psi'(\mathrm{x}) \right) + \frac{1}{4} \frac{\partial}{\partial \mathrm{x}^j} \left(\omega^{ij}(\mathrm{x}) \frac{\partial}{\partial \mathrm{x}^i} \Psi'(\mathrm{x}) \right) \\
&\qquad + \frac{1}{4} \frac{\partial}{\partial \mathrm{x}^i} \frac{\partial}{\partial \mathrm{x}^j} \left(\omega^{ij}(\mathrm{x}) \Psi'(\mathrm{x}) \right) \Bigg] \\
&= \frac{-\hbar^2}{2m} \omega^{-\frac{1}{4}}(\mathrm{x}) \left[\frac{1}{4} \omega^{ij}(\mathrm{x}) \left(\frac{\partial}{\partial \mathrm{x}^i} \frac{\partial}{\partial \mathrm{x}^j} \Psi'(\mathrm{x}) \right) + \frac{1}{2} \frac{\partial}{\partial \mathrm{x}^i} \left(\omega^{ij}(\mathrm{x}) \frac{\partial}{\partial \mathrm{x}^j} \Psi'(\mathrm{x}) \right) + \frac{1}{4} \frac{\partial}{\partial \mathrm{x}^i} \frac{\partial}{\partial \mathrm{x}^j} \left(\omega^{ij}(\mathrm{x}) \Psi'(\mathrm{x}) \right) \right] \\
&= \frac{1}{2m} \left(\frac{1}{4} \omega^{ij}(\hat{q}) \hat{p}_i \hat{p}_j + \frac{1}{2} \hat{p}_i \omega^{ij}(\hat{q}) \hat{p}_j + \frac{1}{4} \hat{p}_i \hat{p}_j \omega^{ij}(\hat{q}) \right) \Psi(\mathrm{x})
\end{aligned} \tag{122}$$

In the before last step we have used the symmetry in the indices i, j. We combine this with equation 51 for the Rivier ordering according to the rule 'ν·Weyl ordering+$(1-\nu)$·Rivier ordering' and arrive at

$$\begin{aligned}
\hat{H}_0 \Psi(\mathrm{x}) &= \frac{1}{2m} \left(\frac{\nu}{4} \omega^{ij}(\hat{q}) \hat{p}_i \hat{p}_j + \frac{\nu}{2} \hat{p}_i \omega^{ij}(\hat{q}) \hat{p}_j + \frac{\nu}{4} \hat{p}_i \hat{p}_j \omega^{ij}(\hat{q}) + \frac{1-\nu}{2} \omega^{ij}(\hat{q}) \hat{p}_i \hat{p}_j + \frac{1-\nu}{2} \hat{p}_i \hat{p}_j \omega^{ij}(\hat{q}) \right) \Psi(\mathrm{x}) \\
&= \left(\frac{2-\nu}{8m} \omega^{ij}(\hat{q}) \hat{p}_i \hat{p}_j + \frac{\nu}{4m} \hat{p}_i \omega^{ij}(\hat{q}) \hat{p}_j + \frac{2-\nu}{8m} \hat{p}_i \hat{p}_j \omega^{ij}(\hat{q}) \right) \Psi(\mathrm{x}).
\end{aligned} \tag{123}$$

D Taylor expansion of ω^{ij}

In this appendix, we use the Taylor expansion of the metric ω^{ij} to investigate the ordering of quantum observables belonging to $H_0 = \frac{1}{2m}\omega^{ij}(\mathrm{x})p_i p_j$, equation 23. This is done to verify the results of the integral method, equation 52.

First, we take the Taylor expansion of $\omega^{ij}(\mathrm{x})$ with respect to the reference point $\bar{\mathrm{x}}$,

$$\omega^{ij}(\mathrm{x}) = \sum_{s=0}^{\infty} \frac{1}{s!}\left(\partial_{k_1}...\partial_{k_s}\omega^{ij}(\bar{\mathrm{x}})\right)(\mathrm{x}^{k_1} - \bar{\mathrm{x}}^{k_1})...(\mathrm{x}^{k_s} - \bar{\mathrm{x}}^{k_s}). \tag{124}$$

Here, we used that $\partial_{k_1}...\partial_{k_s}\omega^{ij}(\bar{\mathrm{x}}) \equiv \frac{\partial}{\partial \mathrm{x}^{k_1}}\cdots\frac{\partial}{\partial \mathrm{x}^{k_s}}\omega^{ij}(\mathrm{x})|_{\mathrm{x}=\bar{\mathrm{x}}}$. We will make use of the following series

$$\begin{aligned}
\sum_{l=0}^{s} 1 &= s+1 \\
\sum_{l=0}^{s} l &= \frac{1}{2}s(s+1) \\
\sum_{l=0}^{s} l^2 &= \frac{1}{6}s(s+1)(2s+1) \\
\sum_{l=0}^{s} l^3 &= \frac{1}{4}s^2(s+1)^2.
\end{aligned} \tag{125}$$

Next, we implement Rivier ordering,

$$\frac{1}{4m}\left(\omega^{ij}(\hat{q})\hat{p}_i\hat{p}_j + \hat{p}_i\hat{p}_j\omega^{ij}(\hat{q})\right), \tag{126}$$

and Weyl ordering,

$$\begin{aligned}
&\frac{1}{2m}\sum_{s=0}^{\infty}\frac{1}{s!}\frac{2!s!}{(s+2)!}\left(\partial_{k_1}...\partial_{k_s}\omega^{ij}(\bar{\mathrm{x}})\right)\sum_{l=0}^{s}\sum_{r=0}^{s-l}(\hat{q}^{k_1} - \bar{\mathrm{x}}^{k_1})...(\hat{q}^{k_l} - \bar{\mathrm{x}}^{k_l})\hat{p}_i\cdot \\
&(\hat{q}^{k_{l+1}} - \bar{\mathrm{x}}^{k_{l+1}})...(\hat{q}^{k_{l+r}} - \bar{\mathrm{x}}^{k_{l+r}})\hat{p}_j(\hat{q}^{k_{l+r+1}} - \bar{\mathrm{x}}^{k_{l+r+1}})...(\hat{q}^{k_s} - \bar{\mathrm{x}}^{k_s}).
\end{aligned} \tag{127}$$

Here, we have separated the s position operators in 3 parts, each divided by a momentum operator. Because of symmetry in the indices $k_1,...,k_s$, the order of the operators within the different parts does not matter, resulting in a factor $\binom{s+2}{2}^{-1}$. The order of the momentum operators does not matter either because of the symmetry in the indices i, j. The resulting factor $\frac{2!s!}{(s+2)!}$ is the expected weight associated with Weyl ordering and is in agreement with the one dimensional example of equation 28 for $\mathrm{x}p^2$, where we have one position operator and for $s = 1$ the above factor becomes $\frac{1}{3}$.

We then combine the expressions using the rule 'ν·Weyl ordering+$(1-\nu)$·Rivier ordering' and bring $\hat{p}_i$ to the left and $\hat{p}_j$ to the right using the commutation relations of equation 26, the series described above, and the symmetry in the indices $k_1,...,k_s$ and i,j:

$$\begin{aligned}
\hat{H}_0 =& \frac{1}{2m}\hat{p}_i\left((1-\nu)\omega^{ij}(\hat{q}) + \sum_{s=0}^{\infty}\frac{2\nu}{(s+2)!}\left(\partial_{k_1}...\partial_{k_s}\omega^{ij}(\bar{\mathrm{x}})\right)\sum_{l=0}^{s}\sum_{r=0}^{s-l}(\hat{q}^{k_1}-\bar{\mathrm{x}}^{k_1})...(\hat{q}^{k_s}-\bar{\mathrm{x}}^{k_s})\right)\hat{p}_j \\
&+\frac{1}{2m}\sum_{s=0}^{\infty}\frac{1}{s!}\left(\partial_{k_1}...\partial_{k_s}\omega^{ij}(\bar{\mathrm{x}})\right)\left[i\hbar s\delta_i^{k_1}(\hat{q}^{k_2}-\bar{\mathrm{x}}^{k_2})...(\hat{q}^{k_s}-\bar{\mathrm{x}}^{k_s})\hat{p}_j\frac{1-\nu}{2}\right. \\
&-i\hbar s\delta_j^{k_s}\hat{p}_i(\hat{q}^{k_1}-\bar{\mathrm{x}}^{k_1})...(\hat{q}^{k_{s-1}}-\bar{\mathrm{x}}^{k_{s-1}})\frac{1-\nu}{2} \\
&+\frac{2\nu}{(s+1)(s+2)}\sum_{l=0}^{s}\sum_{r=0}^{s-l}\Big(i\hbar l\delta_i^{k_1}(\hat{q}^{k_2}-\bar{\mathrm{x}}^{k_2})...(\hat{q}^{k_{l+r}}-\bar{\mathrm{x}}^{k_{l+r}})\hat{p}_j(\hat{q}^{k_{l+r+1}}-\bar{\mathrm{x}}^{k_{l+r+1}})...(\hat{q}^{k_s}-\bar{\mathrm{x}}^{k_s}) \\
&\left.-i\hbar(s-l-r)\delta_j^{k_s}\hat{p}_i(\hat{q}^{k_1}-\bar{\mathrm{x}}^{k_1})...(\hat{q}^{k_{s-1}}-\bar{\mathrm{x}}^{k_{s-1}})\Big)\right] \\
=& \frac{1}{2m}\hat{p}_i\omega^{ij}(\hat{q})\hat{p}_j + \hat{Q}.
\end{aligned} \tag{128}$$

The term on the first line is responsible for the first term at the end, which means the rest of the terms are collected in $\hat{Q}$. The latter is worked out by bringing all $\hat{q}$ operators to the right-hand side of the remaining $\hat{p}$ operator, giving rise to additional non-zero commutators.

$$\begin{aligned}
\hat{Q} =& \frac{1-\nu}{4m}\sum_{s=0}^{\infty}\frac{1}{s!}\left(\partial_{k_1}...\partial_{k_s}\omega^{ij}(\bar{\mathrm{x}})\right)i\hbar s\left[\hat{p}_j\delta_i^{k_1}(\hat{q}^{k_2}-\bar{\mathrm{x}}^{k_2})...(\hat{q}^{k_s}-\bar{\mathrm{x}}^{k_s})\right. \\
&\left.-\hat{p}_i\delta_j^{k_s}(\hat{q}^{k_1}-\bar{\mathrm{x}}^{k_1})...(\hat{q}^{k_{s-1}}-\bar{\mathrm{x}}^{k_{s-1}}) + i\hbar(s-1)\delta_i^{k_1}\delta_j^{k_2}(\hat{q}^{k_3}-\bar{\mathrm{x}}^{k_3})...(\hat{q}^{k_s}-\bar{\mathrm{x}}^{k_s})\right] \\
&+\frac{\nu}{m}\sum_{s=0}^{\infty}\frac{1}{(s+2)!}\left(\partial_{k_1}...\partial_{k_s}\omega^{ij}(\bar{\mathrm{x}})\right)i\hbar\sum_{l=0}^{s}\sum_{r=0}^{s-l}\left[l\delta_i^{k_1}\hat{p}_j(\hat{q}^{k_2}-\bar{\mathrm{x}}^{k_2})...(\hat{q}^{k_s}-\bar{\mathrm{x}}^{k_s})\right. \\
&\left.-(s-l-r)\delta_j^{k_s}\hat{p}_i(\hat{q}^{k_1}-\bar{\mathrm{x}}^{k_1})...(\hat{q}^{k_{s-1}}-\bar{\mathrm{x}}^{k_{s-1}}) + i\hbar(l+r-1)l\delta_i^{k_1}\delta_j^{k_2}(\hat{q}^{k_3}-\bar{\mathrm{x}}^{k_3})...(\hat{q}^{k_s}-\bar{\mathrm{x}}^{k_s})\right]
\end{aligned} \tag{129}$$

We use the following series:

$$\begin{aligned}
\sum_{l=0}^{s}\sum_{r=0}^{s-l} l =& \sum_{l=0}^{s} l(s-l+1) = -\frac{1}{6}s(s+1)(2s+1)+\frac{1}{2}s(s+1)^2 = \frac{1}{6}s(s+1)(s+2) \\
\sum_{l=0}^{s}\sum_{r=0}^{s-l}(l+r-s) =& \frac{1}{2}\sum_{l=0}^{s}(l-s)(s-l+1) = -\frac{1}{12}s(s+1)(2s+1)+\frac{1}{4}s(s+1)(2s+1)-\frac{1}{2}s(s+1)^2 \\
=& -\frac{1}{6}s(s+1)(s+2) \\
\sum_{l=0}^{s}\sum_{r=0}^{s-l}(l^2+rl-l) =& \sum_{l=0}^{s}(s-l+1)(\frac{1}{2}l^2+\frac{1}{2}l(s-2)) = \sum_{l=0}^{s}(-\frac{1}{2}l^3+\frac{3}{2}l^2+\frac{1}{2}l(s+1)(s-2)) \\
=& -\frac{1}{8}s^2(s+1)^2+\frac{1}{4}s(s+1)(2s+1)+\frac{1}{4}s(s+1)^2(s-2) \\
=& \frac{1}{8}s(s+1)(-s^2-s+4s+2+2s^2-2s-4) = \frac{1}{8}(s-1)s(s+1)(s+2).
\end{aligned} \tag{130}$$

We implement these in $\hat{Q}$ and arrive at

$$\begin{aligned}\hat{Q} =& -\hbar^2 \frac{1-\nu}{4m} \sum_{s=2}^{\infty} \frac{1}{(s-2)!} \left(\partial_{k_3}...\partial_{k_s}(\partial_i\partial_j\omega^{ij}(\bar{\mathrm{x}}))\right)(\hat{q}^{k_3}-\bar{\mathrm{x}}^{k_3})...(\hat{q}^{k_s}-\bar{\mathrm{x}}^{k_s}) \\ & -\hbar^2 \frac{\nu}{8m} \sum_{s=2}^{\infty} \frac{1}{(s-2)!} \left(\partial_{k_3}...\partial_{k_s}(\partial_i\partial_j\omega^{ij}(\bar{\mathrm{x}}))\right)(\hat{q}^{k_3}-\bar{\mathrm{x}}^{k_3})...(\hat{q}^{k_s}-\bar{\mathrm{x}}^{k_s}) \\ =& -\frac{\hbar^2}{4m}(1-\frac{\nu}{2})\partial_i\partial_j\omega^{ij}(\hat{q}).\end{aligned} \tag{131}$$

At first order in $\hbar$, all orderings are equivalent. This follows from the mirror symmetry of $\hat{p}_i$ and $\hat{p}_j$ of every hermitian ordering. At highest order (2^{nd}) in $\hbar$, we see the only differences between the different orderings. Therefore, it is sufficient to use two representative orderings, and thus one parameter ν, to parametrize the dependence of $\hat{H}_0$ on the quantum ordering. If $\hat{H}_0$ would have contained more than two momentum operators, one parameter would not suffice. The ordering of quantum observables belonging to H_0 finally results in

$$\hat{H}_0 = \frac{1}{2m}\hat{p}_i\omega^{ij}(\hat{q})\hat{p}_j - \frac{\hbar^2}{4m}(1-\frac{\nu}{2})\left(\partial_i\partial_j\omega^{ij}(\hat{q})\right). \tag{132}$$

To show that this is in agreement with equation 52, we rewrite the latter:

$$\begin{aligned}\hat{H}_0\Psi(\mathrm{x}) &= \left(\frac{2-\nu}{8m}\omega^{ij}(\hat{q})\hat{p}_i\hat{p}_j + \frac{\nu}{4m}\hat{p}_i\omega^{ij}(\hat{q})\hat{p}_j + \frac{2-\nu}{8m}\hat{p}_i\hat{p}_j\omega^{ij}(\hat{q})\right)\Psi(\mathrm{x}) \\ &= -\frac{\hbar^2}{2m}\left[\frac{2-\nu}{4}\omega^{ij}(\mathrm{x})\omega^{-\frac{1}{4}}(\mathrm{x})\left(\partial_i\partial_j\Psi'(\mathrm{x})\right) + \frac{\nu}{2}\omega^{-\frac{1}{4}}(\mathrm{x})\partial_i\left(\omega^{ij}(\mathrm{x})\partial_j\Psi'(\mathrm{x})\right) + \frac{2-\nu}{4}\omega^{-\frac{1}{4}}(\mathrm{x})\partial_i\partial_j\left(\omega^{ij}(\mathrm{x})\Psi'(\mathrm{x})\right)\right] \\ &= -\frac{\hbar^2}{2m}\left[\frac{2-\nu}{4}\omega^{ij}(\mathrm{x})\omega^{-\frac{1}{4}}(\mathrm{x})\left(\partial_i\partial_j\Psi'(\mathrm{x})\right) + \frac{\nu}{2}\omega^{-\frac{1}{4}}(\mathrm{x})\partial_i\left(\omega^{ij}(\mathrm{x})\partial_j\Psi'(\mathrm{x})\right) + \frac{2-\nu}{4}\omega^{-\frac{1}{4}}(\mathrm{x})\left(\partial_i\partial_j\omega^{ij}(\mathrm{x})\right)\Psi'(\mathrm{x})\right. \\ &\left.+\frac{2-\nu}{2}\omega^{-\frac{1}{4}}(\mathrm{x})\left(\partial_i\omega^{ij}(\mathrm{x})\right)\left(\partial_j\Psi'(\mathrm{x})\right) + \frac{2-\nu}{4}\omega^{-\frac{1}{4}}(\mathrm{x})\omega^{ij}(\mathrm{x})\left(\partial_i\partial_j\Psi'(\mathrm{x})\right)\right] \\ &= -\frac{\hbar^2}{2m}\left[\frac{2-\nu}{2}\omega^{-\frac{1}{4}}(\mathrm{x})\partial_i\left(\omega^{ij}(\mathrm{x})\partial_j\Psi'(\mathrm{x})\right) - \frac{2-\nu}{2}\omega^{-\frac{1}{4}}(\mathrm{x})\left(\partial_i\omega^{ij}(\mathrm{x})\right)\left(\partial_j\Psi'(\mathrm{x})\right) + \frac{\nu}{2}\omega^{-\frac{1}{4}}(\mathrm{x})\partial_i\left(\omega^{ij}(\mathrm{x})\partial_j\Psi'(\mathrm{x})\right)\right. \\ &\left.+\frac{2-\nu}{4}\omega^{-\frac{1}{4}}(\mathrm{x})\left(\partial_i\partial_j\omega^{ij}(\mathrm{x})\right)\Psi'(\mathrm{x}) + \frac{2-\nu}{2}\omega^{-\frac{1}{4}}(\mathrm{x})\left(\partial_i\omega^{ij}(\mathrm{x})\right)\left(\partial_j\Psi'(\mathrm{x})\right)\right] \\ &= -\frac{\hbar^2}{2m}\left[\omega^{-\frac{1}{4}}(\mathrm{x})\partial_i\left(\omega^{ij}(\mathrm{x})\partial_j\Psi'(\mathrm{x})\right) + \frac{1}{2}(1-\frac{\nu}{2})\omega^{-\frac{1}{4}}(\mathrm{x})\left(\partial_i\partial_j\omega^{ij}(\mathrm{x})\right)\Psi'(\mathrm{x})\right] \\ &= \frac{1}{2m}\hat{p}_i\omega^{ij}(\hat{q})\hat{p}_j\Psi(\mathrm{x}) - \frac{\hbar^2}{4m}(1-\frac{\nu}{2})\left(\partial_i\partial_j\omega^{ij}(\hat{q})\right)\Psi(\mathrm{x})\end{aligned} \tag{133}$$

This is indeed the same as equation 132, which means both methods give the same result.

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