# Hybrid renormalization and the $\beta$-functions of the real scalar Higgs Lagrangians from the scalar spectral action

J. de Jong

Master thesis Physics & Mathematics

IMAPP, Radboud universiteit Nijmegen

Supervision:

dr. W.D. van Suijlekom (Mathematical physics)

dr. W.J.P. Beenakker (Theoretical high energy physics)

June 21, 2013

Radboud Universiteit Nijmegen

# Contents

**Overview** **2**

**1 An introduction to the rsHL** **4**
1.1 The Lagrangian . . . 4
1.2 Background field . . . 7
1.3 Power counting . . . 10
1.4 Renormalization . . . 13
1.5 Divergent diagrams and $\beta$-functions . . . 20

**2 Expansion of the spectral action** **29**
2.1 Introduction . . . 29
2.2 Operator calculus . . . 30
2.3 The spectral triple . . . 39
2.4 Asymptotic expansion . . . 43
2.5 Computation of the flat Seeley-De Witt coefficients . . . 48

**3 Physical structure of the scalar action** **55**
3.1 Gamma matrices . . . 55
3.2 The higher order asymptotic spectral action . . . 58
3.3 Higher derivative field theory of the ersHL . . . 61

**4 The UV-behaviour of the ersHL** **67**
4.1 Calculation methods . . . 67
4.2 Higher derivative ersHL-integrals . . . 71
4.3 All divergent amplitudes . . . 78
4.4 The $\beta$-functions . . . 80
4.5 Tadpole consistency condition . . . 86

**5 Conclusions** **94**

# An overview of this thesis

The spectral action from non-commutative geometry has a well-known link to quantum field theory and particle physics. Even the Standard model has a non-commutative description. In this work the spectral action will be analyzed in a scalar setting using renormalization techniques. We will try to explore the possibilities of using higher order corrections of the real scalar spectral action as regulating terms. Such terms are usually neglected, since they correspond to field operators of dimensionality higher than that of spacetime. However, the spectral action provides an eigenvalue cutoff with the dimensions of mass that could be applied to make such terms more interesting. In the end this mass parameter should be sent to infinity. This limit will turn out the be the crux of this work, because it provides us with an asymptotic expansion that allows us to expand the spectral action in real scalar Higgs Lagrangians and it acts as a natural regulator in the treatment of this action.
An interesting property of this spectral action for checking the effect of these higher order regulating terms is the UV-behaviour, described by the $\beta$-functions, which indicate the changing of the coupling constants, fields and masses as the energy scale shifts. If a higher order theory describes the same physics with an extended Lagrangian, these $\beta$-functions should be the same. Checking these is a relatively easy first step.
As a result of the aforementioned higher order field operators, higher derivative field theory on a Euclidean spacetime will be needed. Since this is not completely standard, we will introduce this and some practical renormalization concepts in Chapter 1 on the basis of the real scalar Higgs Lagrangian (rsHL). The aim in this chapter is to find the $\beta$-functions for this theory.
In Chapter 2 we will demonstrate the basics of obtaining a physical theory from the spectral action using a heat kernel expansion. Here we will calculate the first few Seeley-De Witt coefficients on a flat Riemannian compact manifold without a gauge field (curvature). Once we have done this, it will be fairly simple to construct the real scalar Higgs Lagrangian and its extension, which we do in Chapter 3. After we have described some of the basic properties of this theory, we will try to determine its $\beta$-functions in Chapter 4. Similar projects have been undertaken for Yang-Mills theories[1], [2] and with added BRS-invariant terms instead of higher spectral action terms [3].
We will see that the renormalization of higher derivative field theories is not a straightforward generalization of the standard case, but carries some surprises. For example, it may give some information on the possible extensions of the theory in the form of restrictions on the possible coupling constants.

A particular feature of the spectral action in this thesis is that it gives rise to a negative mass term. Obtaining a positive mass term requires that we expand the potential around its minimum, which is usually called spontaneous symmetry breaking in the physics literature. Due to this we name this theory the real scalar Higgs Lagrangian (rsHL) and its regulating extension the extended real scalar Higgs Lagrangian (ersHL). However, since the scalar field is real and we consider only self-interactions, this Lagrangian has nothing to do with the usual Higgs Lagrangian from high energy physics. It is a $\varphi^4$-theory with spontaneous symmetry breaking with an extension provided by the spectral action. Furthermore, this symmetry is discrete, contrary to the common broken symmetries.

# 1 An introduction to the rsHL

This chapter forms an introduction into the quantum field theory needed. For example, we will work on a Euclidean space, which differs on certain points from quantum field theory [4], [5] on a Minkowski space. We will do this on the basis of a specific example, the real scalar Higgs Lagrangian, equation (1.1).

## 1.1 The Lagrangian

Quantum field theories usually deal solely with Lagrangians that are functionals of a set of fields and their first derivatives, as is the case in the example below. However, in Chapter 3 we will encounter a Lagrangian containing higher derivatives of $\varphi$ as well, so that we are forced to deal with so called higher derivative field theory. Therefore we have to set up everything in a manner suitable for such a treatment as well.

### 1.1.1 Feynman path integrals and the propagator

In the non-relativistic limit of Minkowski space, used in quantum mechanics, time is treated differently from space and particles are considered stable. They are not created from the vacuum or annihilated by antiparticles. However, they move, so we can study their propagation.
The squared propagator $|\langle\vec{x}_2, t_2|\vec{x}_1, t_1\rangle|^2$ is the probability for a particle to propagate from $(\vec{x}_1, t_1)$ to $(\vec{x}_2, t_2)$. We use the Hamiltonian as the generator of time evolution to write $\langle\vec{x}_2, t_2|\vec{x}_1, t_1\rangle = \langle\vec{x}_2|e^{-\frac{i}{\hbar}H(t_2-t_1)}|\vec{x}_1\rangle$. Usually this is taken as a starting point to find the Feynman path integral. In Minkowski space one divides the time interval $[t_1, t_2]$ in $N$ parts and approximates this to find

$$\langle\vec{x}_2, t_2|\vec{x}_1, t_1\rangle = \int \mathcal{D}x\, e^{\frac{i}{\hbar}\int_{t_1}^{t_2} \mathrm{d}t\, L}$$

in the limit where $N \to \infty$, where $L$ is the classical Lagrangian and $\mathcal{D}x$ is a formal notation for an integration over all paths from $\vec{x}_1$ to $\vec{x}_2$ with several constants absorbed. The expression $\int \mathcal{D}x\, e^{\frac{i}{\hbar}S}$ is called the Feynman path integral [6]. A precise meaning of the Feynman path integral in quantum mechanics can be found in [7].
However, the outline of Feynman's idea is clear from this formulation: attach to every path a phase factor $e^{\frac{i}{\hbar}S}$ and add the contributions. We will do the same in the relativistic situation.
We make the transition from non-relativistic quantum mechanics to relativistic quantum field theory without interactions. This means that we go from a fixed number of particles to fields, and from trajectories to probability amplitudes. We then interpret the quanta of these fields as particles. This includes that the integration over all paths is changed to an integral over all field configurations. Essentially, all the information is encoded in a Lagrangian (density) $\mathcal{L}[\varphi]$, which we have taken to depend on one field $\varphi$ only.

*Remark* 1.1. If we change all space variables $\vec{x}$ to $\vec{x}_E = -i\vec{x}$, by a Wick antirotation, we see that the exponent $e^{\frac{i}{\hbar}\int \mathrm{d}^4x\,\mathcal{L}}$ changes to $e^{\frac{(-i)^3 i}{\hbar}\int \mathrm{d}^4x_E\,\mathcal{L}}$. This changes the Minkowski picture to a Euclidean one. Now, probabilities $e^{-\frac{S}{\hbar}}$ are attached to each path.

*Remark* 1.2. According to our conventions, the Lagrangian is given by $L = T + V$, where $T$ is the kinetic and $V$ the potential energy, where in a Minkowski metric and in classical mechanics this is $L = T - V$. The difference is the special role of time in the latter two.

*Remark* 1.3. We will use natural units in this project, which is standard in quantum field theory. This means that we set $\hbar = c = 1$, where $\hbar$ is the reduced Planck constant and $c$ the speed of light. Furthermore, we work with a 4-dimensional manifold $M$.

*Example* 1.1. We have the real scalar Higgs Lagrangian (rSHL)

$$\mathcal{L} = \frac{1}{(4\pi)^2}\left(4\Lambda^4 f_4 - 4\Lambda^2 f_2\tilde{\varphi}^2 + 2f_0\tilde{\varphi}^4 + 2f_0(\partial\tilde{\varphi})^2\right) \quad , \tag{1.1}$$

where $\tilde{\varphi}$ is a real scalar field and $(\partial\tilde{\varphi})^2 = \partial^\mu\tilde{\varphi}\partial_\mu\tilde{\varphi}$ and summation over repeated upper and lower indices is implied, according to the Einstein summation convention. The origin of the universal energy scale $\Lambda$, the factors of $4\pi$ and the constants $f_0, f_2$ and $f_4$ will become clear in Chapters 2 and 3.

*Remark* 1.4. The interaction term lacks a coupling constant. We could cure this by rescaling the field, $\tilde{\varphi} \to (2f_0)^{-1/2}\tilde{\varphi}$, which gives a coupling constant $(2f_0)^{-1}$. We will not do this, to simplify the comparison of this theory with the Lagrangian from Proposition 3.2.
However, we should keep in mind that $f_0$ is a large parameter, since otherwise the coupling constant is not small.

By interpreting the quantized field $\varphi$ at a point $x$ as the creation/annihilation operator of a particle at the point $x$, we recognize a term in a Lagrangian with three or more fields as an interaction term, where several fields 'collide'.
The quantum mechanical propagator $\langle\vec{x}_2, t_2|\vec{x}_1, t_1\rangle$ is replaced by the two point Green's function $\frac{1}{\int \mathcal{D}\varphi\, e^{-S_0[\varphi]}}\int \mathcal{D}\varphi\,\varphi(x)\varphi(y)e^{-S_0[\varphi]}$, where $x$ and $y$ are now four-vectors. We interpret it as a particle created at $x$, destroyed at $y$ and that has propagated in between. Such a term we call a (field) propagator term. For this we need the mass and kinetic terms of the Lagrangian and combined they form the free Lagrangian. In example 1.1, these are the second and fourth term of the Lagrangian (1.1), respectively. We will use the free action to determine the propagator, $S_0 = \int_M \mathcal{L}_0$ .
In the classical case the paths satisfying the variational equation $\delta S = 0$ give the stationary points of this action and hence the classical paths. The analogue equation for this is $\delta S[\varphi] = 0$. To study propagation this motivates us to formulate the following definition.

**Definition 1.1.** [Equation of motion]
For a Lagrangian $\mathcal{L}[\varphi, \ldots, \partial_\mu^n\varphi]$, depending on one real scalar field and its $n$ derivatives, the equation of motion is given by

$$0 = \sum_{k=0}^{n}(-1)^k\partial_{\mu_1}\ldots\partial_{\mu_k}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\ldots\partial_{\mu_k}\varphi)} \quad .$$

However, since this is an interacting theory, the propagation will be fundamentally different from the free field case. For propagation without interaction we need the equation of free motion, which is given by the equation of motion for the non-interacting part $\mathcal{L}_0$ of the Lagrangian. If we combine all derivatives as $\Delta_x = -\partial_\mu\partial^\mu$ we write this in a generic way as

$$q_{\Delta_x}\varphi(x) = \sum_{i=0}^{n} c_i \Delta_x^i \varphi(x) = 0 \quad .$$

All derivatives that we encounter in equations of motion can be rewritten as $\Delta_x$. This enables us to put all derivative terms and the mass term in $q_{\Delta_x}\varphi(x)$.

**Definition 1.2.** [Propagator]
The propagator is given by $D(x-y)$, where $D(x-y)$ is the Green's function such that $q_{\Delta_x} D(x-y) = \delta^4(x-y)$.

Below is a formal calculation, standard in many quantum field textbooks, demonstrating why the propagator is defined in this way.

*Remark* 1.5. The free generating functional is given by

$$Z_0[J] \equiv \frac{\int \mathcal{D}\varphi\, e^{-\int \mathrm{d}^4x\, \mathcal{L}_0(x) + J(x)\varphi(x)}}{\int \mathcal{D}\varphi\, e^{-S_0[\varphi]}} \quad . \tag{1.2}$$

where $J(x)$ is a 'source' term. It serves primarily as a calculation tool, creating particles out of nothing. This allows us to determine the behaviour of free propagation of particles, without wondering where they are coming from. The field propagator is given by

$$\frac{1}{\int \mathcal{D}\varphi\, e^{-S_0[\varphi]}} \int \mathcal{D}\varphi\, \varphi(x)\varphi(y) e^{-S_0[\varphi]} = \frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} Z_0[J]\Big|_{J=0} \tag{1.3}$$

We now apply the shift $\varphi(x) \to \varphi(x) - \int \mathrm{d}^4y\, D(x-y)J(y)$ to the free Lagrangian. Since this transformation has Jacobian 1, the 'measure' remains unaffected. This yields

$$\begin{aligned}
\int_M \mathcal{L}_0[\varphi] + J\varphi &= \int_M \mathrm{d}^4x \left(\frac{1}{2}\varphi(x) q_{\Delta_x}\varphi(x) + J(x)\varphi(x)\right) \\
&\to \int_M \mathrm{d}^4x \Big(\frac{1}{2}\varphi(x) q_{\Delta_x}\varphi(x) + J(x)\varphi(x) - \int_M \mathrm{d}^4y\, J(x)D(x-y)J(y) \\
&- \int_M \mathrm{d}^4y\, \frac{1}{2}\varphi(x) q_{\Delta_x} D(x-y)J(y) \\
&+ \iint_M \mathrm{d}^4y\, \mathrm{d}^4z\, \frac{1}{2} D(x-y)J(y) q_{\Delta_x} D(x-z)J(z) \\
&- \int_M \mathrm{d}^4y\, \frac{1}{2} D(x-y)J(y) q_{\Delta_x}\varphi(x)\Big) \qquad (1.4) \\
&= \int_M \mathrm{d}^4x \Big(\mathcal{L}_0[\varphi] + J(x)\varphi(x) - \int_M \mathrm{d}^4y\, J(x)D(x-y)J(y) \\
&- \int_M \mathrm{d}^4y\, \frac{1}{2}\varphi(x)\delta^4(x-y)J(y) - \frac{1}{2}\int_M \mathrm{d}^4y\, \left(q_{\Delta_x} D(x-y)\right) J(y)\varphi(x) \\
&+ \iint_M \mathrm{d}^4y\, \mathrm{d}^4z\, \frac{1}{2} D(x-y)J(y)\delta^4(x-z)J(z)\Big) \\
&= \int_M \mathrm{d}^4x \left(\mathcal{L}_0[\varphi] - \int_M \mathrm{d}^4y\, \frac{1}{2} J(x)D(x-y)J(y)\right) \qquad (1.5)
\end{aligned}$$

Notice that we have shifted $q_{\Delta_x}$ in equation (1.4) to let it act on $D(x-y)$ by adding a boundary term that vanishes under the integral by Stokes' theorem. This is possible

since all partial derivatives in $q_{\Delta_x}$ are contracted, $\Delta_x = -\partial_\mu \partial^\mu$, so that all terms shift with a positive sign under repeated partial integration.
Inserting this result in equation (1.2) and noticing that the second term of equation (1.5) does not depend on $\varphi_0$, we find that

$$Z_0[J] = e^{\frac{1}{2}\iint_M \mathrm{d}^4x\,\mathrm{d}^4y\, J(x)D(x-y)J(y)} \frac{\int \mathcal{D}\varphi\, e^{-\int \mathrm{d}^4x\, \mathcal{L}_0[\varphi](x)}}{\int \mathcal{D}\varphi\, e^{-\int \mathrm{d}^4x\, \mathcal{L}_0[\varphi](x)}} \quad . \tag{1.6}$$

In the denominator and the numerator the same factor appears. It consists of all so called vacuum bubbles, which in the free theory are just propagators beginning and ending at the same space-time point on the manifold. These vacuum bubbles are everywhere and of no importance. This is also the reason that we divide by $\int \mathcal{D}\varphi\, e^{-S_0[\varphi]}$ in the field propagator.
Now, combining equations (1.6) and (1.3) it follows that

$$\frac{1}{\int \mathcal{D}\varphi\, e^{-S_0[\varphi]}} \int \mathcal{D}\varphi\, \varphi(x)\varphi(y) e^{-S_0[\varphi]} = D(x-y) \tag{1.7}$$

This shows that the Green's function of the equation of free motion can be interpreted as the free field propagator.

Mostly, one works with the Fourier transformed of the propagator.

**Lemma 1.1.** The Fourier transformed propagator $\hat{D}(p)$ is given by $\dfrac{1}{q_{p^2}}$.

*Proof.* We want to solve $q_{\Delta_x} D(x) = \delta^{(4)}(x)$, where $q_{\Delta_x}\varphi(x) = 0$ is the equation of free motion. We use the Fourier transform

$$D(x) = \int \mathrm{d}^4p\, \frac{e^{ip\cdot x}}{(2\pi)^4} \hat{D}(p) \quad .$$

Acting with $q_{\Delta_x}$ on this we see that $\Delta_x$ becomes $p^2$. Requiring that

$$\delta^{(4)}(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\, e^{ip\cdot x} q_{p^2} \hat{D}(p) \quad ,$$

gives us $\hat{D}(p) = \frac{1}{q_{p^2}}$. □

But before we can actually determine the propagators, we first have to resolve the negative mass problem.

## 1.2 Background field

For a (Euclidean) real scalar field theory a mass term is of the form $+\frac{1}{2}m^2\tilde{\varphi}^2$. In equation (1.1) we see that we have a negative mass term here. We will fix this using a constant real background field $\chi_0$. This means that we split the field $\tilde{\varphi}(x) = \chi_0 + \varphi(x)$, where $\chi_0$ is chosen to minimize the constant field potential. Then $\varphi$ contains the quantum fluctuations around this minimum.

**Definition 1.3.** The potential $V_\mathcal{L}(\chi)$ for a constant field $\chi$ corresponding to a Lagrangian $\mathcal{L}$ is given by

$$V_\mathcal{L}(\chi) \equiv \mathcal{L}[\tilde{\varphi} = \chi] \quad .$$

*Remark* 1.6. Since we have chosen $\chi$ a constant field, all derivative terms in $\mathcal{L}$ will vanish and we are left with a polynomial in $\chi$. From this it is clear that the potential is a function of the real parameter $\chi$.

**Lemma 1.2.** Switching to the background field, $\tilde{\varphi}(x) = \chi_0 + \varphi(x)$, where $\chi_0$ minimizes the potential $V_{\mathcal{L}}$, gives a positive mass term.

*Proof.* We Taylor expand the Lagrangian, depending on one field $\varphi$ and its $n$ derivatives, around $\chi_0$.

$$\begin{aligned}\mathcal{L}[\tilde{\varphi}(x), \partial_\mu \tilde{\varphi}(x), \ldots, \partial_{\mu_1} \ldots \partial_{\mu_n} \tilde{\varphi}(x)])_{\tilde{\varphi}(x)=\chi_0+\varphi(x)} &= \mathcal{L}[\chi_0] + \frac{\partial \mathcal{L}}{\partial \tilde{\varphi}(x)}[\chi_0]\ \varphi(x) \\ &+ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \tilde{\varphi}(x))}[\chi_0]\ \partial_\mu \varphi(x) + \ldots + \frac{\partial \mathcal{L}}{\partial(\partial_{\mu_1} \ldots \partial_{\mu_n} \tilde{\varphi}(x))}[\chi_0]\ \partial_{\mu_1} \ldots \partial_{\mu_n} \varphi(x) \\ &+ \frac{1}{2!} \frac{\partial^2 \mathcal{L}}{\partial \tilde{\varphi}(x)^2}[\chi_0]\ \varphi(x)^2 + \frac{\partial^2 \mathcal{L}}{\partial(\tilde{\varphi}(x) \partial_\mu \tilde{\varphi}(x))}[\chi_0]\ \varphi(x) \partial_\mu \varphi(x) + \ldots \quad . \end{aligned} \tag{1.8}$$

Minimizing the potential $V_{\mathcal{L}}(\chi)$ means solving the equation

$$\frac{\mathrm{d}V_{\mathcal{L}}}{\mathrm{d}\chi}(\chi_0) = 0 \quad .$$

This is precisely the same equation as

$$\frac{\partial \mathcal{L}}{\partial \tilde{\varphi}(x)}[\tilde{\varphi} = \chi_0] = \frac{\partial \mathcal{L}}{\partial \varphi(x)}[\varphi = 0] = 0 \quad ,$$

so that the term $\frac{\partial \mathcal{L}}{\partial \tilde{\varphi}(x)}[\chi_0]\ \varphi(x)$ vanishes in equation (1.8). The other terms with a single $\varphi$ are all total derivatives of $\varphi$ and vanish therefore when integrated over in the action. Secondly, minimizing the potential means

$$\frac{\mathrm{d}^2 V_{\mathcal{L}}}{\mathrm{d}\chi^2}(\chi_0) \geq 0 \quad ,$$

which is again the same as

$$\frac{\partial^2 \mathcal{L}}{\partial \tilde{\varphi}(x)^2}[\chi_0] = \frac{\partial^2 \mathcal{L}}{\partial \varphi(x)^2}[0] \geq 0 \quad .$$

This shows that the mass term $\frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial \tilde{\varphi}(x)^2}[\chi_0]\ \varphi(x)^2$ in equation (1.8) is indeed positive. □

*Example* 1.2. The result of applying the above is that the Lagrangian from equation (1.1), example 1.1 now looks like

$$\begin{aligned}\mathcal{L} &= \frac{1}{(4\pi)^2} \Big( \left(4f_4\Lambda^4 - 4f_2\chi_0^2\Lambda^2 + 2f_0\chi_0^4\right) \\ &\quad + \frac{1}{2}\left(-8f_2\Lambda^2 + 24f_0\chi_0^2\right)\varphi^2 + \frac{1}{2}\varphi\left(4f_0\Delta\varphi\right) + \left(8f_0\chi_0\right)\varphi^3 + 2f_0\varphi^4\Big) \\ &= \frac{1}{(4\pi)^2}\left(\left(4f_4 - 2\frac{f_2^2}{f_0}\right)\Lambda^4 + \frac{1}{2}m^2\varphi^2 + \frac{1}{2}\varphi\left(\alpha\Delta\varphi\right) + \lambda\varphi^3 + g\varphi^4\right) \quad , \end{aligned} \tag{1.9}$$

where we have split the field $\tilde{\varphi} = \chi_0 + \varphi$ and minimized the potential by a constant field. This gives us the following reparametrization of the theory

$$\chi_0' \approx \sqrt{\frac{f_2}{f_0}}\ , \quad \alpha = 4f_0\ , \quad m^2 = (-8f_2 + 24f_0(\chi_0')^2)\Lambda^2\ , \quad \lambda = 8f_0\chi_0\ , \quad g = 2f_0 \quad ,$$

(1.10)

where the dimensionless minimum of the potential $\chi_0' = \Lambda^{-1}\chi_0$ is given at its classical value. We will come back to this in paragraph 1.4.5.

*Example* 1.3. Now that we have a sensible Lagrangian, we can determine the propagator, Definition 1.2.

*Corollary* 1.1. The (Fourier transformed) propagator for the Lagrangian (1.9) is given by

$$\hat{D}(p) = \frac{16\pi^2\alpha^{-1}}{p^2 + \frac{m^2}{\alpha}} \quad .$$

*Proof.* Applying Definition 1.1 to the free Lagrangian $\mathcal{L}_0 = (4\pi)^{-2}(\frac{1}{2}m^2\varphi^2 + \frac{1}{2}\varphi(\alpha\Delta\varphi))$ gives us that $q_{\Delta_x} = (4\pi)^{-2}(m^2+\alpha\Delta)$. From Lemma 1.1 the statement is now clear. □

### 1.2.1 Interactions

A free field only propagates, which we have treated now. Since we know everything about the non-interacting theory, it is now time to include interactions to the theory.

**Definition 1.4.** For a Lagrangian a term with three or more fields involved is called an interaction term or vertex. The number of fields involved is the valence of the vertex.

Graphically these interactions are commonly represented by Feynman diagrams. The number of fields in an interaction term corresponds to the number of lines attached to the vertex diagrammatically. Feynman diagrams are usually evaluated in momentum space. We already have an expression for the propagator in momentum space, but not yet for the vertices.

We want to find the vertices in momentum space. We start as we did for the propagator. The strength of, for example, the $\varphi^3$ vertex, is the coefficient accompanying $\left(\frac{1}{\int \mathcal{D}\varphi\, e^{-S[\varphi]}} \int \mathcal{D}\varphi\, \varphi^3 e^{-S[\varphi]}\right)$. This is called the coupling constant. First of all we split the Lagrangian into a free part and an interaction part, where we multiply the interaction part by $a \in \mathbb{R}$:

$$\mathcal{L} = \mathcal{L}_0 + a\mathcal{L}_I \quad .$$

We interpret $\int \mathcal{D}\varphi\, e^{-S[\varphi]}$ again as all vacuum bubbles. Then, the strength of any interaction can be read out from

$$\frac{\mathrm{d}}{\mathrm{d}a} Z[a]\Big|_{a=1} = \frac{\mathrm{d}}{\mathrm{d}a} \frac{\int \mathcal{D}\varphi e^{-\int \mathrm{d}x\, \mathcal{L}_0 + a\mathcal{L}_I}}{\int \mathcal{D}\varphi e^{-S[\varphi]}}\Big|_{a=1} \quad .$$

However, these interactions are still in the momentum representation. Therefore we Fourier transform all fields $\varphi_j(x) = \int \frac{\mathrm{d}^4 p_j}{(2\pi)^4}\, \varphi_j(p_j) e^{ip_j \cdot x}$ in the interaction terms, where we have numbered the fields to keep track of the momenta. Notice that $\partial^\mu \varphi_j(x)$ would transform to $i \int \frac{\mathrm{d}^4 p_j}{(2\pi)^4}\, p_j^\mu \varphi_j(p_j) e^{ip_j \cdot x}$.
When defining the momenta positive for outgoing particles, every interaction term is accompanied by the integral

$$\int_M \mathrm{d}^4 x\, e^{ix \cdot (\sum_{i=j}^n p_j)} = (2\pi)^4 \delta^4 (\sum_{i=j}^n p_j) \quad ,$$

a Dirac delta function. These will fix most, if not all, integrals $\int \frac{\mathrm{d}^4 p_j}{(2\pi)^4}$ and we have to integrate over undetermined momenta, occurring in loop diagrams only, since in tree diagrams all momenta are fixed by the Dirac delta functions and the external momenta.
From this it is clear that the strength of an interaction is precisely $-1$ times the constant accompanying the interaction term, possibly together with some momentum depending factors.

*Example* 1.4. For the Lagrangian (1.9) from example 1.2 the strength of the 3-vertex is given by $\frac{-\lambda}{(4\pi)^2}$ and that of the 4-vertex by $\frac{-g}{(4\pi)^2}$.

*Remark* 1.7. Now that we have the propagators and the interactions we only need the external lines to build Feynman diagrams. In momentum representation they are just given by 1. This may seem strange, but its origin lies in the fact that Feynman diagrams represent scattering amplitudes, not Green's functions. The essential point is that the propagators of the particles to the detectors are considered as a part of the external laboratory dynamics and not as a part of the properties of the specific theory one studies. Then the factor 1 is just the coupling strength of the external line in the diagram to the external propagator it has to connect to.
A further clarification on this can be found in any text book on quantum field theory [8].

## 1.3 Power counting

A loop in a Feynman diagram contains an undetermined circulating momentum, which has to be integrated. It is not surprising that such an integral may diverge, since we (falsely) assume that the theory is valid up to all energy scales. When this happens because the integral does not fall off fast enough at high momenta, we say that the diagram is UV-divergent.
These divergences are regarded unphysical, since yet unknown physics may appear at higher energies, making the Lagrangian unsuited for these energy scales.
However, at low energies such theories can make sense. To keep any predictive power there, we need to know which parts make diagrams divergent.
If we can split a diagram in two parts by cutting one propagator, the momentum of this propagator is fixed, since it cannot be part of a loop. Therefore, a UV-divergence cannot be caused by such a propagator and we do not have to consider it as a part of a divergent building block.
In the UV-range of the momentum spectrum finite momenta can be neglected. We can pretend that all momenta are the same. These observations lead us to the following definitions.

**Definition 1.5.** A Feynman diagram is called amputated when by cutting one single propagator carrying an external momentum the diagram can not be split in two non-trivial parts.

**Definition 1.6.** [Superficial degree of divergence]
The amplitude of an amputated, connected Feynman diagram goes as $p^\omega$, when all loop momenta are set equal (to $p$) and go simultaneously to infinity. The superficial degree of divergence for this diagram is $\omega$.
If $\omega \geq 0$ the diagram is superficially divergent, where $\omega = 0$ is called logarithmically divergent.

A finite theory contains no divergent diagrams, no renormalization is needed. When there are only finitely many divergent diagrams, all divergences are absorbed after renormalization at some loop order. At higher loop orders no new divergent diagrams appear and all renormalization is done. Such theories are called superrenormalizable. If divergences occur at all loop levels, but only for a finite number of amplitudes, one has to renormalize at every loop order the same parameters, making the theory renormalizable. If the number of divergent amplitudes increases with higher loop orders, an infinite number of parameters is needed to cancel all divergences. Such theories have no predictive power and are not renormalizable.

**Definition 1.7.** [Superficial renormalizability]
The physical theory obtained from a Lagrangian $\mathcal{L}$ is

- finite, if no diagrams are superficially divergent;
- superrenormalizable, if a finite number of diagrams is superficially divergent;
- renormalizable, if a finite number of amplitudes is superficially divergent;
- non-renormalizable, if the number of superficially divergent amplitudes is infinite.

*Remark* 1.8. As will become clear in a moment, the cancellation of divergences is based on the adjustment of Lagrangian parameters. Furthermore, the classification in Definition 1.7 presumes independent parameters and amplitudes. It may happen that shifting one parameter renders two divergent amplitudes finite or that a tunable parameter has no effect on the amplitudes. However, such situations will not occur here and this slightly simplified definition suffices.

*Example* 1.5. We continue with example 1.2.

**Lemma 1.3.** The superficial degree of divergence for a loop diagram in the real scalar Higgs theory is given by $\omega = 4 - N - V_3$, where $N$ is the number of external lines and $V_3$ the number of vertices with valence 3.

*Proof.* Besides the introduced numbers we need the number of propagators $P$ and the number of vertices with valence 4, $V_4$. If we count the powers of the momentum in a loop diagram, we see that every integrated loop gives 4 powers of $p$ and every propagator $-2$, as can be seen from example 1.3. The vertices in this theory do not contain any power of $p$, see example 1.4.
For a connected diagram the number of loops is given by $L = P + 1 - V_3 - V_4$. The number of loops is equal to the number of undetermined momenta, i.e. the number of propagators, minus the number of conditions on these momenta, which equals the number of vertices. One vertex is not needed, because of overall momentum conservation. Summing, we find $L = P - V + 1$. This gives $\omega = 4 + 2P - 4V_3 - 4V_4$.
Finally we use that a propagator connects to two vertices, while an external line connects to one vertex. A vertex of valence $n$ connects to $n$ propagators and external lines. Together this gives $N + 2P = 3V_3 + 4V_4$. Using this identity we find that $\omega = 4 - N - V_3$. □

Just by constructing diagrams that obey these rules we can make five connected, amputated superficially divergent 1-loop Feynman diagrams. They are given in figure 1.1.

*Example* 1.6. From the formula for the superficial degree of divergence in example 1.5, Lemma 1.3 we see that the theory defined by the rSHL is renormalizable.

*Remark* 1.9. We number the diagrams from Figure 1.1 from left to right, up to down with 1 to 5, so that the two-point diagrams are number 4 and 5. For this we need the amplitude of the diagrams, which we denote with $\mathcal{M}_n$, where $n$ indicates the diagram.

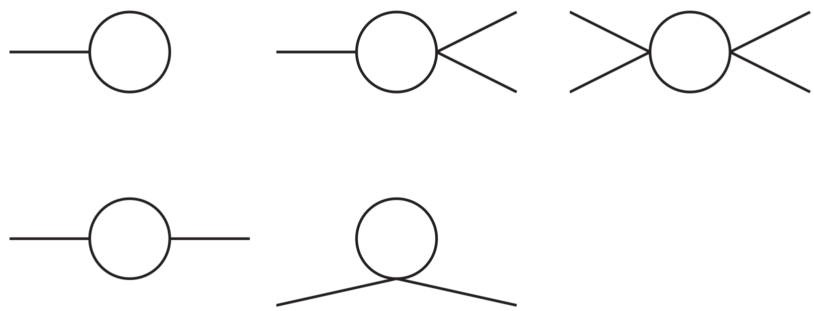

Figure 1.1: The divergent diagrams. Numbered from left ro right, up to down with 1, 2, 3, 4 and 5.

### 1.3.1 Diagram multiplicity

If we consider four external lines and one vertex with valence 4, there are 4! ways in which the external legs can be connected to the vertices. This is the first example of a multiplicity. When quantitatively analyzing the theory, it is not only of importance that we can compute single diagrams, but also that we know how often a diagram occurs.
We think of a Feynman diagram as the set of external lines, the set of vertices and the set of edges, i.e. propagators, between those. In this way the same structure guarantees the same amplitude by the rules above.
Experiments are conducted by colliding a number of particles and sorting the results on the basis of the outgoing, detected particles. Each external line thus represents a particle on its way to the detector and is therefore distinguishable from the other external lines. For the given set of external lines all possible diagrams are then written down (up to a certain loop order). For each diagram all the vertices are fixed. There are now several ways in which the vertices can be connected with each other and the external lines, such that the resulting structure is the same. This gives the multiplicity of the diagram.
We do not allow external lines to be only connected to other external lines, leading to disconnected diagrams. This would contribute to a trivial part of a transition amplitude, while we study the non-trivial part only.

### Mandelstam variables

For a four-point amplitude $(p_1, p_2) \to (p_3, p_4)$ without loops in the theory from example 1.2, there are two options to construct the corresponding diagrams. Either with one vertex or with two. In the first case there is no orientation needed, all external lines are connected to one vertex. In the last case the two vertices should be connected by a propagator. For the momentum carried by this propagator there are three options: $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$ or $u = (p_1 - p_4)^2$.
These variables $s$, $t$ and $u$ are called Mandelstam variables and the resulting diagrams are called the $s$-, $t$- and $u$-channel. In case of a loop diagram, the internal structure would carry a netto momentum from one side of the diagram to the other, so this distinction can be made for loop diagrams as well. The four possible tree diagrams are shown in Figure 1.2.

*Remark* 1.10. Contrary to common practice, we do not specify a time direction in the Feynman diagrams, because the momentum dependence will disappear almost completely from our amplitudes. In this way there are less Feynman diagrams that

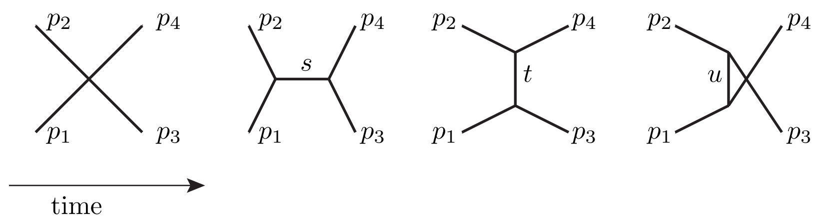


Figure 1.2: Four options for the RSHL four-point tree diagram with incoming momenta $p_1$ and $p_2$ and outgoing momenta $p_3$ and $p_4$. From left to right these are the one-vertex, $s$-, $t$- and $u$-channel diagram. Time flows from left to right.

we have to draw. However, we should remind ourselves that we have to sum over these different orientations to obtain the complete amplitude. In all cases that we will encounter, this will give a factor 3 for the $s$- $t$- and $u$-channel.

*Example* 1.7. Continuing with example 1.5 we find the following multiplicities for the diagrams of Figure 1.1.

| # | multiplicity |
|---|---|
| $n_1$ | 3 |
| $n_2$ | 72 |
| $n_3$ | 288 |
| $n_4$ | 18 |
| $n_5$ | 12 |

Table 1.1: Multiplicities of the five superficially divergent Feynman diagrams from Figure 1.1.

## 1.4 Renormalization

As we have shown in paragraph 1.3 a theory can contain diagrams that diverge when entering high energies. If we have no way to obtain information out of these divergences, this (non-renormalizable) theory can only be used in lowest order approximation, where no diagram diverges. If we are able, despite of these divergent diagrams, to retrieve predictive power at high energies, we call such a theory renormalizable. Such theories can be used perturbatively, since we then have a way of getting information out of divergent loop diagrams. For an introduction into renormalization and $\beta$-functions that is independent of context and subject, see for example [9].
We remark that we neglect IR-divergences in this discussion, because they could only occur in this context for negative or zero mass.

### 1.4.1 Dimensional regularization

As mentioned before, we have to deal with infinities to renormalize the theory. We do this by introducing regulators. In a regularization scheme we slightly alter the expressions by introducing a regulator such that the original integral is obtained in

the limit where the regulating parameter goes to its original value. One can determine the divergent and finite parts of the expressions in this way. In the limit the original theory is recovered.
In the literature there are numerous regulators available. In this chapter we will use two. The first one is dimensional regularization.

The regulator we introduce is $\varepsilon \in \mathbb{C}$ that gives the deviation from the dimension of the manifold to be 4. This means that we perform the integral over the momenta in an arbitrary number of dimensions $d = 4 - 2\varepsilon$, where the factor 2 is for convenience. We analytically continue the obtained expressions to all dimensions $d$. We then get a part that is infinite as $\varepsilon \to 0$, a finite part and vanishing terms as $\varepsilon \to 0$. We will then subtract the infinite parts from the Feynman diagrams.

*Remark* 1.11. Being introduced by hand, the regulator $\varepsilon$ is not a physical parameter. This means that any finite part that in some way depends on the way we chose $\varepsilon$ and its pole should not determine any physics. There is an ambiguity in adding a finite part to the pole, so we will have to define what we use as a pole.

### 1.4.2 The natural regulator $\Lambda$

As we will see in Chapter 2 $\Lambda$ has a natural role as regulator as well, when at the end of the calculation we take $\Lambda \to \infty$. This is our second regulator. However, we see from example 1.2 that $\Lambda$ is the only mass scale in the theory. This means that all masses will become infinite, and zero after subtraction of the infinities. From this observation only it is already clear that $\Lambda$ is a natural regulator that has a physical meaning. From the reasoning in remark 1.11, mutatis mutandis, it is clear that a small modification in the way $\Lambda \to \infty$ may have a physical meaning. So we can split $\Lambda \to \Lambda_0 + \infty$ in the limit. In this way we have somehow substituted $\Lambda$ for $\Lambda_0$. Precisely how will become clear in a moment.
Determining what is used as infinity to subtract is done by choosing a renormalization scheme, in our case an extended version of modified minimal subtraction called $\overline{\text{eMS}}$. Usual (modified) minimal subtraction only deals with one dimensional regulator $\varepsilon$.

**Definition 1.8.** [ $\overline{\text{eMS}}$ ]
The infinite part of a Feynman diagram is the term proportional to

$$\frac{\Gamma(\varepsilon)}{1-\varepsilon} \quad , \qquad \text{as } \varepsilon \to 0 \qquad \text{and}$$

$$(\Lambda - \Lambda_0) \text{ or } \log\left(\frac{\Lambda}{\Lambda_0}\right) \text{ as } \Lambda \to \infty \quad ,$$

where $\Lambda_0$ is an arbitrary parameter with dimensions of mass.
Since we have chosen to work in 4 dimensions, we prefer to take the limit $\Lambda \to \infty$ in 4 dimensions as well. Therefore we take the limit $\varepsilon \to 0$ first and then the limit $\Lambda \to \infty$. This particular order of limits will be meant when writing this as $\lim_{\substack{\varepsilon \to 0 \\ \Lambda \to \infty}} \mathcal{M}$.

For a consistent treatment, we should take the limit $\Lambda \to \infty$ in the classical approximation of the theory as well, i.e. at tree level. Contrary to usual physics, where one has to renormalize the theory as a result of its UV-behaviour, we already have to do this at the classical level.
From the mass term in the Lagrangian (1.9) it is clear that the limit $\Lambda \to \Lambda_0 + \infty$ sends the mass to infinity. Therefore we will also speak of this limit as the infinite mass limit, although this is not strictly correct. Another interesting aspect can be seen when calculating the tree level four-point diagram with two vertices. As a result

of the limit $\Lambda - \Lambda_0 \to \infty$ this diagram is just a multiple of the four-point tree diagram with one vertex.

*Remark* 1.12. Notice that introducing a dimensional regulator causes the integration measure $\mathrm{d}^{4-2\varepsilon}p$ to have mass dimension $4 - 2\varepsilon$, instead of 4. Therefore the dimensionality of the entire expression is different for different values of $\varepsilon$. To cure this we introduce an extra arbitrary mass $\mu$, so that in dimensional regularization the loop integral becomes $\mu^{2\varepsilon} \int \frac{\mathrm{d}^d p}{(2\pi)^d}$ instead of $\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$.
In the end the predictions made by the theory should not depend on $\mu$, so at some stage it has to vanish from our theory.

### 1.4.3 Multiplicative renormalization

Recapitulating, we have a Lagrangian, an arbitrary mass parameter $\mu$ and some finite and infinite loop corrections for a measurable physical process. The most standard technique to cope with these infinities is to split at every loop order the (meaningless) Lagrangian parameter into the physical parameter and a counterterm that absorbs all the infinities. This is called multiplicative renormalization, because it can be achieved by multiplying all parameters by a constant.

Here we will demonstrate one of the essential features of multiplicative renormalization. We scale all Lagrangian parameters to the corresponding physical parameter, for example $m_0^2 \to Z_m m^2$, defining that the various $Z$'s renormalize the theory.
The essential part of multiplicative renormalization is that one can include all irreducible two-point diagrams in the propagator. The diagrams that are proportional to the squared external momentum are put in an overall factor , while the diagrams that are not proportional to the external momenta renormalize the mass. The overall factor is then shifted from the propagators to the fields, since any propagator is attached to two fields.
However, for the external lines we overcompensate, leading to the equality for the $n$-point function $G_0^{(n)} = \prod_{i=1}^k Z_{\varphi_i}^{-n_i/2} G^{(n)}$, where $\sum_{i=1}^k n_i = n$ is the number of external lines and $G_0$ indicates the unrenormalized Green's functions.

**Definition 1.9.** A Feynman diagram is one-particle irreducible (1PI) if it is connected and cannot become disconnected by removing one propagator.

**Definition 1.10.** The total propagator is the sum of all two-point diagrams.

*Example* 1.8. We will renormalize the mass here, so we denote the Lagrangian parameter with $m_0$ and the physical mass with $m$. We gather all 1PI-diagrams into the self energy function $\Sigma(p^2)$. We sum the total propagator

$$\begin{aligned}
\tilde{\hat{D}}(p^2) &= \hat{D}(p^2) + \hat{D}(p^2)\Sigma(p^2)\hat{D}(p^2) + \hat{D}(p^2)\left(\Sigma(p^2)\hat{D}(p^2)\right)^2 + \ldots = \frac{1}{\frac{1}{\hat{D}} - \Sigma} \\
&= \frac{16\pi^2\alpha^{-1}}{p^2 + \frac{m^2}{\alpha} + (\frac{m_0^2}{\alpha} - \frac{m^2}{\alpha}) - \frac{16\pi^2}{\alpha}\Sigma(-\frac{m^2}{\alpha}) - \frac{16\pi^2}{\alpha}\sum_{n=1}^{\infty}\frac{1}{n!}\Sigma^{(n)}(-\frac{m^2}{\alpha})(p^2 + \frac{m^2}{\alpha})^n} \\
&= \frac{1}{1 - 16\pi^2\alpha^{-1}\sum_{n=1}^{\infty}\frac{1}{n!}\Sigma^{(n)}(-\frac{m^2}{\alpha})(p^2 + \frac{m^2}{\alpha})^{n-1}} \frac{16\pi^2\alpha^{-1}}{p^2 + \frac{m^2}{\alpha}} \\
&= Z_\varphi(p^2)\,\frac{16\pi^2\alpha^{-1}}{p^2 + \frac{m^2}{\alpha}} \quad , \qquad (1.11)
\end{aligned}$$

where we choose $m_0$ such that the pole of the propagator is shifted to the (experimentally measured) physical value:

$$\frac{m_0^2}{\alpha} \equiv \frac{m^2}{\alpha} + \frac{16\pi^2}{\alpha}\Sigma(-\frac{m^2}{\alpha}) \quad . \tag{1.12}$$

Here we have assumed that the function $\Sigma(p^2)$ has a full Taylor expansion. The factor $Z_\varphi(\frac{-m^2}{\alpha})$ is the (multiplicative) field renormalization factor.

**Definition 1.11.** [(Non-multiplicative renormalization)]
If it is not possible to write the total propagator of a renormalizable theory as a multiple of the free propagator we call the theory non-multiplicatively renormalizable. If this is possible, we call the theory multiplicatively renormalizable.

From the introduction of the $\mu$-parameter it is already clear that it has no physical significance and has to drop out of the full theory. This means that the Green's function with all orders in perturbation theory included is $\mu$-independent. However, a truncated expansion will depend on $\mu$. Any method to remove this arbitrariness is not just another magic trick, but it gives us a hint of what will happen at higher order in the perturbation series.
The contributions from several orders can sum to give Lagrangian parameters with values that are shifted with respect to the lowest order. This indicates that the parameters in the Lagrangian, possibly together with a finite number of new parameters that are just zero at lowest order, can be made $\mu$-dependent to remove the overall scale dependence.
Assuming for the moment that the theory is multiplicatively renormalizable, we can state the following lemma.

**Lemma 1.4.** [Callan-Symanzik equation]
For a Lagrangian depending on the coupling constants $\{\lambda_1, \ldots, \lambda_l\}$, the masses $\{m_1, \ldots, m_j\}$ and consisting of the fields $\{\varphi_1, \ldots, \varphi_k\}$ with $n$-point Green's function $G^{(n)}$ that does not depend on $\mu$ the following holds:

$$\begin{aligned} 0 &= \left(\mu\frac{\partial}{\partial\mu} + \sum_{i=1}^{l}\mu\frac{\partial\lambda_i}{\partial\mu}\frac{\partial}{\partial\lambda_i} + \sum_{i=1}^{j}\mu\frac{\partial m_i}{\partial\mu}\frac{\partial}{\partial m_i} + \sum_{i=1}^{k}\frac{n_i\mu}{2Z_{\varphi_i}}\frac{\partial Z_{\varphi_i}}{\partial\mu}\right)G^{(n)} \\ &= \left(\mu\frac{\partial}{\partial\mu} + \sum_{i=1}^{l}\beta(\lambda_i)\frac{\partial}{\partial\lambda_i} + \sum_{i=1}^{j}m_i\gamma_{m_i}(\{\lambda\})\frac{\partial}{\partial m_i} - \sum_{i=1}^{k}n_i\gamma(\{\lambda\})\right)G^{(n)} \ , \end{aligned} \tag{1.13}$$

where $n_i$ is the number of external lines of type $\varphi_i$, so that $\sum_{i=1}^{k} n_i = n$.

*Proof.* In Remark 1.12 we introduced an arbitrary energy parameter $\mu$. In fixing the renormalized Lagrangian parameters an actual value for $\mu$ is chosen, but the experimentally observable Green's function $G^{(n)}$ should still not depend on $\mu$. This explains the left hand side of the first line. From this it follows that

$$0 = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}G^{(n)} \quad .$$

The first line is then found by working this out.
All the renormalized coupling constants, masses and fields may depend on $\mu$ and therefore we can make the derivative explicit. The renormalized $n$-point Green's function depends on the unrenormalized Green's function by $G_0^{(n)} = \prod_{i=1}^{k} Z_{\varphi_i}^{-n_i/2}G^{(n)}$, where $Z_{\varphi_i}$ renormalizes the field $\varphi_i$.
The final result is obtained by the redefinitions $\beta(\lambda_i) \equiv \mu\frac{\partial\lambda_i}{\partial\mu}$, $m_i\gamma_{m_i}(\{\lambda\}) \equiv \mu\frac{\partial m_i}{\partial\mu}$ and $-\gamma(\{\lambda\}) \equiv \mu\frac{\partial}{\partial\mu}\log\left(\sqrt{Z_{\varphi_i}}\right)$. □

The Callan-Symanzik equation shows that the renormalized parameters depend on $\mu$, but in such a manner that the end result is still sensible. The functions introduced relate the theory on different energy scales. In a full perturbation series $\mu$ should drop out. However, a truncated series will in general depend on $\mu$, so that we have to eliminate it manually.

*Remark* 1.13. It is important that the counterterms do not depend directly on $\mu$. To derive the Callan-Symanzik equation we have used that the derivative $\frac{\mathrm{d}}{\mathrm{d}\mu}$ can be split in a series of partial derivatives. If one of the counterterms depends on $\mu$ as well, we should include that, obscuring the equation and changing essentially nothing. We would have unnecessarily added a useless parameter, instead of using the counterterm as a place to put the meaningless divergences in.

*Example* 1.9. The coupling constants referred to in Lemma 1.4 should be small. In example 1.8 this refers to $(2f_0)^{-1}$, see remark 1.4. When applying the Callan-Symanzik equation, Lemma 1.4, to the (multiplicatively) renormalized theory from example 1.2 we may consider $\lambda$ as an independent coupling constant as well. We will not do this, and keep seeing $\lambda$ as the product of the coupling $g$ and the minimum of the potential $\chi_0$ and renormalize the latter instead. This is meant by the classical value of example 1.2.

### 1.4.4 Non-multiplicative renormalization

From Definition 1.11 we infer that not every quantum field theory is multiplicatively renormalizable. We will encounter an example of this in Chapter 3. Then we have no other option than to renormalize non-multiplicatively. To get used to this we will introduce and apply it already here.
The differences between these two ways of renormalizing are only technical. When they are both applicable, they should produce the same results and physics. Therefore we are free here to chose one. This can be put the other way around as well: if two ways of renormalizing do not produce the same physics, at least one of them is not applicable. This will become clear in paragraph 1.5 and Chapter 4. First we must introduce the Rota-Baxter algebra [10], [11].

**Definition 1.12.** [Rota-Baxter algebra]
A real Rota-Baxter algebra is an algebra $\mathscr{A}$ over $\mathbb{R}$ with a linear map $\mathcal{R}$ and weight $\vartheta \in \mathbb{R}$ that satisfies the relation

$$\mathcal{R}(a)\mathcal{R}(b) + \vartheta\mathcal{R}(ab) = \mathcal{R}\left(\mathcal{R}(a)b\right) + \mathcal{R}\left(a\mathcal{R}(b)\right) \qquad \forall a, b \in \mathscr{A} \quad . \tag{1.14}$$

*Remark* 1.14. It is clear that definition 1.12 can be generalized to any field, but we will only use the real case.

**Lemma 1.5.** A map $\mathcal{R}$ is a Rota-Baxter map of weight $\vartheta$, if and only if $\vartheta - \mathcal{R}$ is a Rota-Baxter map of weight $\vartheta$.

*Proof.* Since $\vartheta - (\vartheta - \mathcal{R}) = \mathcal{R}$ it suffices to prove only one direction. The Rota-Baxter identity from Definition 1.12 for the map $\rho - X = \mathcal{R}$, $\rho \in \mathbb{R}$ on $a, b \in \mathscr{A}$ is given by

$$\begin{aligned}0 = \mathcal{R}(a)\mathcal{R}(b) + \vartheta\mathcal{R}(ab) - \mathcal{R}\left(\mathcal{R}(a)b\right) - \mathcal{R}\left(a\mathcal{R}(b)\right) = (\rho a - X(a)) \cdot (\rho b - X(b)) \\ + \vartheta\left(\rho ab - X(ab)\right) - (\rho a - X(a))\,\rho b + X\left((\rho a - X(a))\, b\right) - \rho a\left(\rho b - X(b)\right) \\ + X\left(a\left(\rho b - X(b)\right)\right) \\ = X(a)X(b) + (2\rho - \vartheta)X(ab) - X\left(X(a)b\right) - X\left(aX(b)\right) + \rho(\vartheta - \rho)ab \quad ,\end{aligned}$$

from which we see directly that only for $\rho = \vartheta$ the Rota-Baxter identity is satisfied by $X = \rho - \mathcal{R}$. The weight of $\rho - \mathcal{R}$ is then given by $2\rho - \vartheta = \vartheta$. ☐

The $\mathcal{R}$-operator from Definition 1.12 is in a physical context the projection operator $T$, which is used to construct the Bogolyubov $R$-operator that renormalizes general Feynman diagrams [12]. The linear operator $T$ projects a 1PI-amplitude $I_{\mathcal{M}}$ on the infinities, defined in Definition 1.8. Then it is clear that the operator $1-T$ subtracts the multiples of the infinities, so that $(1-T)(I_{\mathcal{M}})$ is finite. This means that we use BPHZ-subtraction using regulators instead of the external momenta to expand our diagram in. See for a comprehensive introduction [13, §16.4].

**Definition 1.13.** The subtraction operator $\overline{\mathfrak{T}} \equiv 1-T$.

From Lemma 1.5 we see that $\overline{\mathfrak{T}}$ is a Rota-Baxter map of weight 1 if and only if $T$ is. Defining a 1PI-subtraction operator $\overline{\mathfrak{T}}$ of weight 1 automatically defines a projection $T$ on the infinities of weight 1.
Since we will only work on 1-loop diagrams, the action of the subtraction operator $\overline{\mathfrak{T}}$ equals that of the Bogolyubov $R$-operator, so we will only work with $\overline{\mathfrak{T}}$. We define this renormalization operator in accordance with Definition 1.8.

**Definition 1.14.** The Rota-Baxter map of weight 1 we use to subtract the singularities from a formal Laurent series is given by

$$\overline{\mathfrak{T}}_{\varepsilon}\left(\sum_{n=-\infty}^{N} a_n z^n\right) = \sum_{n=-\infty}^{0} a_n z^n \quad , \quad N \in \mathbb{N}, \quad \sum_{n=-\infty}^{N} a_n z^n \in \mathbb{R}[[z^{-1}, z] \quad ,$$

where $z = \frac{\Gamma(\varepsilon)}{1-\varepsilon}$.

For the natural regulator $\Lambda$ there are two interesting options, both with their own advantages, which we will explore in the remaining part of this chapter.

**Definition 1.15.** The Rota-Baxter map $\overline{\mathfrak{T}}^{(1)}_{\Lambda-\Lambda_0}$ of weight 1 subtracts formal power series in $\Lambda-\Lambda_0$ and is defined by

$$\overline{\mathfrak{T}}^{(1)}_{\Lambda-\Lambda_0}\left(\sum_{n=0}^{\infty} a_n(\Lambda-\Lambda_0)^n\right) = a_0 \quad , \quad \sum_{n=0}^{\infty} a_n(\Lambda-\Lambda_0)^n \in \mathbb{R}[[(\Lambda-\Lambda_0)]] \quad .$$

**Definition 1.16.** The Rota-Baxter map $\overline{\mathfrak{T}}^{(2)}_{\Lambda-\Lambda_0}$ of weight 1 that subtracts the divergent parts from a formal Laurent series in $\Lambda-\Lambda_0$ and $\log\left(\frac{\Lambda}{\Lambda_0}\right)$ is defined by

$$\overline{\mathfrak{T}}^{(2)}_{\Lambda-\Lambda_0}\left(\sum_{n=-\infty}^{N_1}(\Lambda-\Lambda_0)^n\left(a_n+\sum_{m=1}^{N_2} b_{m,n}\left(\log\left(\frac{\Lambda}{\Lambda_0}\right)\right)^m\right)\right) = \overline{\mathfrak{T}}^{(2)}_{\Lambda-\Lambda_0} I_{(\{a\},\{b\})}$$
$$= \sum_{n=-\infty}^{0} a_n(\Lambda-\Lambda_0)^n \quad ,$$
$$N_{1,2} \in \mathbb{N}, \quad I_{(\{a_n\}^{N_1}_{-\infty},\{b_m\}^{N_2}_{-\infty})} \in \mathbb{R}[[(\Lambda-\Lambda_0)^{-1}, (\Lambda-\Lambda_0), \log(\Lambda\Lambda_0^{-1})] \quad .$$

*Remark* 1.15. Each formal Laurent series should have a $N \in \mathbb{N}$ for which $a_{n>N} = 0$, since otherwise the product of two series is ill-defined. This is the reason for the appearance of $N$ in Definitions 1.14 and 1.16.

Mass terms in quantum field theory appear almost always in specific places. It is therefore useful to see what is the difference between the renormalization operators $\overline{\mathfrak{T}}^{(1)}_{\Lambda-\Lambda_0}$ and $\overline{\mathfrak{T}}^{(2)}_{\Lambda-\Lambda_0}$.

**Lemma 1.6.** On the functions $\log(\Lambda)$, $\log(1+\Lambda)$, $\Lambda^n$ and $\Lambda^{-n}$ with $n \in \mathbb{N}$ the subtraction operators $\overline{\mathfrak{T}}^{(1,2)}_{\Lambda-\Lambda_0}$ from Definitions 1.15 and 1.16 act as:

| | $\log(\Lambda)$ | $\log(1+\Lambda)$ | $\Lambda^n$ | $\Lambda^{-n}$ |
|---|---|---|---|---|
| $\lim_{\Lambda-\Lambda_0\to\infty} \overline{\mathfrak{T}}^{(1)}_{\Lambda-\Lambda_0}$ | $\log(\Lambda_0)$ | $\log(1+\Lambda_0)$ | $\Lambda_0^n$ | $\Lambda_0^{-n}$ |
| $\lim_{\Lambda-\Lambda_0\to\infty} \overline{\mathfrak{T}}^{(2)}_{\Lambda-\Lambda_0}$ | $\log(\Lambda_0)$ | $\log(\Lambda_0)$ | $\Lambda_0^n$ | 0 |

*Proof.* By applying the binomial identity to

$$\Lambda^n = (\Lambda - \Lambda_0 + \Lambda_0)^n = \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} (\Lambda - \Lambda_0)^{n-i} \Lambda_0^n$$

it is clear that $\overline{\mathfrak{T}}^{(1,2)}_{\Lambda-\Lambda_0}(\Lambda^n) = \Lambda_0^n$. This we can use for the $\overline{\mathfrak{T}}^{(1)}_{\Lambda-\Lambda_0}$-identities:

$$\overline{\mathfrak{T}}^{(1)}_{\Lambda-\Lambda_0} : \log(\Lambda) = \log(1+(\Lambda-1)) = -\sum_{i=1}^{\infty} \frac{(1-\Lambda)^i}{i} \mapsto -\sum_{i=1}^{\infty} \frac{(1-\Lambda_0)^i}{i} = \log(\Lambda_0) \quad ;$$

$$\overline{\mathfrak{T}}^{(1)}_{\Lambda-\Lambda_0} : \log(1+\Lambda) = -\sum_{i=1}^{\infty} \frac{(-\Lambda)^i}{i} \mapsto -\sum_{i=1}^{\infty} \frac{(-\Lambda_0)^i}{i} = \log(1+\Lambda_0) \quad ;$$

$$\overline{\mathfrak{T}}^{(1)}_{\Lambda-\Lambda_0} : \frac{1}{\Lambda} = \frac{1}{\Lambda_0} \sum_{i=0}^{\infty} \left( \frac{\Lambda_0 - \Lambda}{\Lambda_0} \right)^i \mapsto \frac{1}{\Lambda_0} \quad \text{for } n = 1 \quad ,$$

which generalizes straightforwardly for $n \in \mathbb{N}$.
Now we continue for $\overline{\mathfrak{T}}^{(2)}_{\Lambda-\Lambda_0}$. Because we take $\Lambda \to \infty$ we assume that $\Lambda > 2\Lambda_0 > 0$. The series we use should converge now, so we find that for $n = 1$

$$\overline{\mathfrak{T}}^{(2)}_{\Lambda-\Lambda_0} : \frac{1}{\Lambda} = \sum_{i=0}^{\infty} \frac{(-\Lambda_0)^i}{(\Lambda-\Lambda_0)^{i+1}} \mapsto \sum_{i=0}^{\infty} \frac{(-\Lambda_0)^i}{(\Lambda-\Lambda_0)^{i+1}} \to 0, \quad (\Lambda - \Lambda_0) \to \infty \quad ,$$

which is again easily generalized to the case $n \in \mathbb{N}$. That $\overline{\mathfrak{T}}^{(2)}_{\Lambda-\Lambda_0}(\log(\Lambda)) = \log(\Lambda_0)$ follows directly from Definition 1.16. The final identity is found from

$$\begin{aligned} \lim_{\Lambda-\Lambda_0\to\infty} \overline{\mathfrak{T}}^{(2)}_{\Lambda-\Lambda_0} \log(1+\Lambda) &= \lim_{\Lambda-\Lambda_0\to\infty} \overline{\mathfrak{T}}^{(2)}_{\Lambda-\Lambda_0} \left( \log(\Lambda) + \log\left(1 + \frac{1}{\Lambda}\right)\right) \\ &= \log(\Lambda_0) - \lim_{\Lambda-\Lambda_0\to\infty} \overline{\mathfrak{T}}^{(2)}_{\Lambda-\Lambda_0} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{-1}{\Lambda}\right)^n = \log(\Lambda_0) \quad . \end{aligned}$$

□

*Remark* 1.16. The limit of the subtraction operator

$$\lim_{\substack{\varepsilon\to 0 \\ \Lambda\to\infty}} \overline{\mathfrak{T}}_{\Lambda-\Lambda_0} \overline{\mathfrak{T}}_{\varepsilon}$$

is unpractical to work with, because for every negative power of $z = \frac{\Gamma(\varepsilon)}{1-\varepsilon}$ one has to determine the series in $\Lambda - \Lambda_0$. All terms remaining when $\Lambda - \Lambda_0 \to \infty$ vanish then as $z \to \infty$. In practice one usually first subtracts the UV-divergent parts, then sends

$\varepsilon \to 0$ and then the parts that diverge as $(\Lambda - \Lambda_0) \to \infty$. So in practice we use that $\overline{\mathfrak{T}}$ is defined by

$$\lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}} I_{\mathcal{M}} = \lim_{\Lambda-\Lambda_0\to\infty} \overline{\mathfrak{T}}^{(1,2)}_{\Lambda-\Lambda_0} \lim_{\varepsilon\to 0} \overline{\mathfrak{T}}_\varepsilon \, I_{\mathcal{M}} \quad ,$$

where $\overline{\mathfrak{T}}_\varepsilon$ subtracts multiples and powers of $\frac{\Gamma(\varepsilon)}{1-\varepsilon}$ and $\overline{\mathfrak{T}}^{(1,2)}_{\Lambda-\Lambda_0}$ those of $\Lambda - \Lambda_0$.

*Remark* 1.17. Combined with Lemma 1.6 we will see in section 1.5 that the operator $\overline{\mathfrak{T}}^{(1)}_{\Lambda-\Lambda_0}$ effectively substitutes $\Lambda$ for $\Lambda_0$. It only regularizes divergences that arise from the limit $\Lambda - \Lambda_0 \to \infty$. This means that using the subtraction operator $\overline{\mathfrak{T}}_1 = \overline{\mathfrak{T}}^{(1)}_{\Lambda-\Lambda_0}\overline{\mathfrak{T}}_\varepsilon$ is the same as modified minimal subtraction with dimensional regularization only. Therefore, we will call this regularization scheme from now on just dimensional regularization, where we substitute $\Lambda$ for $\Lambda_0$.
By contrast, regularizing using $\overline{\mathfrak{T}}_2 = \overline{\mathfrak{T}}^{(2)}_{\Lambda-\Lambda_0}\overline{\mathfrak{T}}_\varepsilon$ deviates from just dimensional regularization. In this case the second regulator does more than just substituting and we call this hybrid regularization. From now on we will indicate which of these two subtraction operators we are using on an amplitude by a superscript: $\mathcal{M}^{(1,2)}$.

What we have introduced here is one possibility for non-multiplicative renormalization. There are many more, just as there are many multiplicative renormalization procedures. And probably there are several other ways to make the distinction between the two. There is one more aspect of renormalization we should cover here. More information and a good general overview of renormalization can be found in [14].

### 1.4.5 Tadpole renormalization

As we can see from Figure 1.1, the first loop order gives us a vertex that we did not have in the Lagrangian. In paragraph 1.2 we removed all one-point diagrams by expanding around the minimum of the potential and now they return at one-loop level. Such one-point diagrams are called tadpoles. We prefer to work without tadpoles.
At the classical level we minimized the potential to remove the one-point terms. To remove the tadpoles again at first loop-order we change the value of $\chi_0$. This means that it is no longer at its classical value. Therefore, this introduces a one-point term in the Lagrangian (1.9). This shift with respect to the classical minimum will cancel the tadpoles.
On this first order term combined with the tadpole, the Callan-Symanzik equation should return zero. This automatically implies that we will treat $\chi_0' = \Lambda^{-1}\chi_0$ as a new coupling constant in the Callan-Symanzik equation from Lemma 1.4.
Due to all this, the potential and its minimum may change at higher order. So, for example, the vacuum expectation value of the field $\varphi$ may not be invariant under the renormalization group. This illustrates that we should not interchange the minimum $\chi_0' \approx \sqrt{f_2/f_0}$ with its classical value. However, we can neglect these higher order corrections of $\chi_0'$ when they are of higher order than we are interested in.

*Remark* 1.18. Since the shift is only used to cancel the tadpole we expect that the renormalized $\chi_0'$ stays at the minimum of the potential, albeit at higher order level now, i.e. $\chi_{0R}' = \sqrt{f_{2R}/f_{0R}}$, where the subscript $R$ indicates renormalized values.

## 1.5 Divergent diagrams and $\beta$-functions

This chapter started with a Lagrangian (1.1) and a general introduction to the theory we needed. This final section is just the completion of example 1.1. We will calculate the divergent diagrams using both regularization methods and determine the

$\beta$-functions for both. Comparing this may give us some information on the quality and applicability of the hybrid regularization method, which we can use in Chapter 4.

*Remark* 1.19. We use the same variables as before, although we make them independent of $\Lambda$ during the calculation and denote them with an additional prime, see example 1.2. This is to make all the $\Lambda$-dependence explicit.

### 1.5.1 Calculation of the divergent diagrams using dimensional regularization

By applying standard textbook techniques [8, §7.5] we can compute the diagrams from figure 1.1, where we use dimensional regularization. As explained in remark 1.17 the subtraction operator $\overline{\mathfrak{T}}_1 = \overline{\mathfrak{T}}^{(1)}_{\Lambda-\Lambda_0}\overline{\mathfrak{T}}_\varepsilon$ will substitute $\Lambda$ for $\Lambda_0$ at the end of the calculation. To simplify the calculations a bit we first mention a helpful equation. In Lemma 4.4 we will prove a standard identity that expresses a typical loop integral in terms of gamma functions

$$\int_0^\infty \mathrm{d}z\,\frac{z^\alpha}{(z-\tau)^\beta} = \frac{\Gamma(\beta-\alpha-1)\Gamma(\alpha+1)}{\Gamma(\beta)}(-\tau)^{\alpha-\beta+1} \quad , \tag{1.15}$$

so we only have to bring the integrals in spherical form denoting the radius by $z$. The first diverging diagram yields

$$\begin{aligned}
\mathcal{M}_1^{(1)} &= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_1 \frac{-\lambda}{16\pi^2}\mu^{2\varepsilon}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\,\frac{16\pi^2\alpha^{-1}}{p^2+\frac{m^2}{\alpha}} \\
&= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_1 \frac{-\lambda\mu^{2\varepsilon}}{\alpha}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\,\frac{1}{p^2+\frac{m^2}{\alpha}} = \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_1 \frac{-\lambda}{(4\pi)^2\alpha}\,\frac{(4\pi\mu^2)^\varepsilon}{\Gamma(\frac{d}{2})}\int_0^\infty \mathrm{d}z\,\frac{z^{\frac{d}{2}-1}}{z+\frac{m^2}{\alpha}} \\
&= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_1 \frac{-\Lambda\lambda'}{(4\pi)^2\alpha}(4\pi\mu^2)^\varepsilon\Gamma(\varepsilon-1)\left(\frac{(\Lambda m')^2}{\alpha}\right)^{1-\varepsilon} \\
&= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_1 \frac{\Lambda\lambda'(\Lambda m')^2}{(4\pi\alpha)^2}\,\frac{\Gamma(\varepsilon)}{1-\varepsilon}\left(1-\varepsilon\log\left(\frac{(\Lambda m')^2}{4\pi\alpha\mu^2}\right)\right) \\
&= \frac{-\Lambda_0\lambda'(\Lambda_0 m')^2}{(4\pi\alpha)^2}\log\left(\frac{(\Lambda_0 m')^2}{4\pi\alpha\mu^2}\right) \quad .
\end{aligned} \tag{1.16}$$

Except diagram 5, the other diagrams have more than one propagator. It is then standard to use the Schwinger trick to make the denominator spherically symmetric.

**Definition 1.17.** The $n$-simplex $\Delta_n$ is given by

$$\{(s_0, s_1, \ldots, s_n) \in \mathbb{R}_+^{n+1} | \sum_{i=0}^n s_i = 1\}$$

and we write for the integral over $\Delta_n$

$$\int_0^1 \mathrm{d}s_1 \int_0^{1-s_1} \mathrm{d}s_2 \ldots \int_0^{1-\sum_{j=1}^{n-1} s_j} \mathrm{d}s_n \equiv \int_{\Delta_n} \mathrm{d}^n s \quad .$$

**Lemma 1.7.** [Schwinger trick]
For $A_i \in \mathbb{C}$ with $\mathrm{Re}(A_i) > 0$ and $1 \le i \le n$

$$\prod_{i=1}^n \frac{1}{A_i} = \int_{\Delta_n} \mathrm{d}^n s_i \frac{(n-1)!}{\left(\sum_{i=1}^n s_i A_i\right)^n} \quad .$$

*Proof.* Requiring that $\mathrm{Re}(A_i) > 0$ we start with the identity

$$\prod_{i=1}^{n} \frac{1}{A_i} = \prod_{i=1}^{n} \int_0^\infty \mathrm{d}\nu_i e^{-\sum_{i=1}^{n} A_i \nu_i} \quad ,$$

which is a straightforward generalization of this formula for $n = 1$. Now we substitute $s_i = \frac{\nu_i}{\nu}$, where $\nu = \sum_{i=1}^{n} \nu_i$,

$$\begin{aligned}\prod_{i=1}^{n} \int_0^\infty \mathrm{d}\nu_i &= \int_0^\infty \mathrm{d}\nu\, \nu^n \prod_{i=1}^{n} \int_0^1 \mathrm{d}s_i\, \delta\Big(\nu\big(1 - \sum_{i=1}^{n} s_i\big)\Big) \\ &= \int_0^\infty \mathrm{d}\nu\, \nu^{n-1} \prod_{i=1}^{n} \int_0^1 \mathrm{d}s_i\, \delta\Big(1 - \sum_{i=1}^{n} s_i\Big) \ ,\end{aligned}$$

where we have used the scaling property of the Dirac delta function

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \delta(ax) = |a|^{-1} \int_{-\infty}^{\infty} \mathrm{d}x\, \delta(x) \quad .$$

Then we find that

$$\begin{aligned}\prod_{i=1}^{n} \frac{1}{A_i} &= \left(\prod_{i=1}^{n} \int_0^1 \mathrm{d}s_i\, \delta\left(1 - \sum_{i=1}^{n} s_i\right)\right) \int_0^\infty \mathrm{d}\nu\, \nu^{n-1} e^{-\nu \sum_{i=1}^{n} s_i A_i} \\ &= \int_{\Delta_n} \mathrm{d}^n s\, \frac{\Gamma(n)}{\left(\sum_{i=1}^{n} s_i A_i\right)^n} \quad .\end{aligned}$$

Using that for $n \in \mathbb{N}$ $\Gamma(n) = (n-1)!$ we conclude the proof. □

The parameters $s_i$ are called Feynman parameters and therefore applying the Schwinger trick is also known as using Feynman parameters.
We put the total external momentum incoming from the left to $k$. However, it should be noticed that $k$ will not be the same for $s$-, $t$- and $u$-channel diagrams. It will be clear in paragraph 1.5.3 this will not make any difference for the $\beta$-functions, which are found for general $k$. With the Schwinger trick added to our list of steps, we find for the other four diagrams

$$\begin{aligned}\mathcal{M}_2^{(1)} &= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_1 \frac{-g}{(4\pi\alpha)^2} \frac{(-\lambda)}{(4\pi)^2} \mu^{2\varepsilon} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{(16\pi^2)^2}{p^2 + \frac{m^2}{\alpha}} \frac{1}{(p+k)^2 + \frac{m^2}{\alpha}} \\ &= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_1 \alpha^{-2} \int_0^1 \mathrm{d}s_1 \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{g\lambda\mu^{2\varepsilon}}{(l^2 + \frac{m^2}{\alpha} + k^2(s_1 - s_1^2))^2} &(1.17) \\ &= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_1 \frac{g\lambda}{(4\pi\alpha)^2} \frac{\Gamma(\varepsilon)}{1-\varepsilon} \left(1 - \varepsilon - \varepsilon \int_0^1 \mathrm{d}s_1 \log\left(\frac{\frac{m^2}{\alpha} + k^2(s_1 - s_1^2)}{4\pi\mu^2}\right)\right) \\ &= \frac{-g\Lambda_0\lambda'}{(4\pi\alpha)^2} \left(1 + \int_0^1 \mathrm{d}s_1 \log\left(\frac{\frac{(\Lambda_0 m')^2}{\alpha} + k^2(s_1 - s_1^2)}{4\pi\mu^2}\right)\right) \quad , &(1.18)\end{aligned}$$

where we used $l = p + s_1 k$ to obtain equation (1.17). With the same steps

$$\begin{aligned}
\mathcal{M}_3^{(1)} &= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\overline{\mathfrak{T}}}_1 \left(\frac{-g}{(4\pi)^2\alpha}\right)^2 \mu^{2\varepsilon} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{(16\pi^2)^2}{p^2+\frac{m^2}{\alpha}} \frac{1}{(p+k)^2+\frac{m^2}{\alpha}} \\
&= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\overline{\mathfrak{T}}}_1 \frac{g^2}{(4\pi\alpha)^2} \frac{\Gamma(\varepsilon)}{1-\varepsilon} \left(1-\varepsilon-\varepsilon\int_0^1 \mathrm{d}s_1 \, \log\left(\frac{\frac{m^2}{\alpha}+k^2(s_1-s_1^2)}{4\pi\mu^2}\right)\right) \\
&= \frac{-g^2}{(4\pi\alpha)^2}\left(1+\int_0^1 \mathrm{d}s_1\,\log\left(\frac{\frac{(\Lambda_0 m')^2}{\alpha}+k^2(s_1-s_1^2)}{4\pi\mu^2}\right)\right) \quad ; \qquad (1.19) \\
\mathcal{M}_4^{(1)} &= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\overline{\mathfrak{T}}}_1 \left(\frac{(-\lambda)}{(4\pi)^2\alpha}\right)^2 \mu^{2\varepsilon} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{(16\pi^2)^2}{p^2+\frac{m^2}{\alpha}} \frac{1}{(p+k)^2+\frac{m^2}{\alpha}} \\
&= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\overline{\mathfrak{T}}}_1 \frac{\lambda^2}{(4\pi\alpha)^2} \frac{\Gamma(\varepsilon)}{1-\varepsilon} \left(1-\varepsilon-\varepsilon\int_0^1 \mathrm{d}s_1 \, \log\left(\frac{\frac{m^2}{\alpha}+k^2(s_1-s_1^2)}{4\pi\mu^2}\right)\right) \\
&= \frac{-(\Lambda_0\lambda')^2}{(4\pi\alpha)^2}\left(1+\int_0^1 \mathrm{d}s_1\,\log\left(\frac{\frac{(\Lambda_0 m')^2}{\alpha}+k^2(s_1-s_1^2)}{4\pi\mu^2}\right)\right) \quad ; \qquad (1.20)
\end{aligned}$$

(1.21)

$$\begin{aligned}
\mathcal{M}_5^{(1)} &= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\overline{\mathfrak{T}}}_1 \mu^{2\varepsilon} \frac{-g}{(4\pi)^2\alpha} \int \frac{\mathrm{d}^d p}{(2\pi^d} \frac{16\pi^2}{p^2+\frac{m^2}{\alpha}} \\
&= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\overline{\mathfrak{T}}}_1 \frac{m^2 g}{(4\pi\alpha)^2} \frac{\Gamma(\varepsilon)}{1-\varepsilon} \left(1-\varepsilon\log\left(\frac{m^2}{4\pi\alpha\mu^2}\right)\right) \\
&= -\frac{(\Lambda_0 m')^2 g}{(4\pi\alpha)^2} \log\left(\frac{(\Lambda_0 m')^2}{4\pi\alpha\mu^2}\right) \quad . \qquad (1.22)
\end{aligned}$$

For completeness we will perform the remaining integral

$$\begin{aligned}
&\int_0^1 \mathrm{d}s_1\,\log\left(\frac{\frac{m^2}{\alpha}+k^2(s_1-s_1^2)}{4\pi\mu^2}\right) \\
&= \log\left(\frac{k^2}{4\pi\mu^2}\right) + \int_0^1 \mathrm{d}s_1\,\log\left(-s_1^2+s_1+\frac{\frac{m^2}{\alpha}}{k^2}\right) \\
&= \log\left(\frac{k^2}{4\pi\mu^2}\right) + \int_0^1 \mathrm{d}s_1\,\left(\log\left(x_+-s_1\right)+\log\left(s_1-x_-\right)\right) \\
&= \log\left(\frac{k^2}{4\pi\mu^2}\right) - 2 + 2\left(\frac{1}{2}-\frac{1}{2}\sqrt{1+\frac{4\frac{m^2}{\alpha}}{k^2}}\right)\log\left(-\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{4\frac{m^2}{\alpha}}{k^2}}\right) \\
&+ 2\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{4\frac{m^2}{\alpha}}{k^2}}\right)\log\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{4\frac{m^2}{\alpha}}{k^2}}\right) \\
&= \log\left(\frac{m^2}{4\pi\alpha\mu^2}\right) - 2 - \sqrt{1+\frac{4\frac{m^2}{\alpha}}{k^2}}\log\left(\frac{-1+\sqrt{1+\frac{4\frac{m^2}{\alpha}}{k^2}}}{1+\sqrt{1+\frac{4\frac{m^2}{\alpha}}{k^2}}}\right) \qquad (1.23) \\
&= \log\left(\frac{m^2}{4\pi\alpha\mu^2}\right) - 2 + 2\sqrt{1+\frac{4\frac{m^2}{\alpha}}{k^2}}\,\mathrm{arctanh}\left(\frac{1}{\sqrt{1+\frac{4\frac{m^2}{\alpha}}{k^2}}}\right) \quad . \qquad (1.24)
\end{aligned}$$

### 1.5.2 Calculation of divergent diagrams with hybrid regularization

We can now calculate the diagrams using the hybrid regularization method as well. The main difference with the previous case is that instead of the Schwinger trick, we use an expansion of the fractions. The reason to do that here is that it is quicker than using Feynman parameters, which give unpleasant integrals for three or more propagators, although we could obtain the same answers by figuring out the limits of the diagrams (1.16)-(1.22) as $\Lambda - \Lambda_0 \to \infty$. However, in paragraph 4.1.1 we will see that we cannot do without a method avoiding the Schwinger trick.
For a scaled loop momentum $\tau = p\Lambda^{-1}$, external momentum $k$ and constant $l$ it is proved in Lemma 4.2 that

$$\frac{1}{(\tau + \frac{k}{\Lambda})^2 - l} = \frac{1}{\tau^2 - l} - \frac{2\tau \cdot \frac{k}{\Lambda} + \frac{k^2}{\Lambda^2}}{(\tau^2 - l)^2} + \frac{4\left(\tau \cdot \frac{k}{\Lambda}\right)^2}{(\tau^2 - l)^3} + \mathcal{O}(\Lambda^{-3}) \quad . \tag{1.25}$$

This is sufficient to determine the five divergent diagrams from Lemma 1.3 and Figure 1.1. We will not perform these calculations completely, since in Chapter 4 a more complicated computation will be done in full detail, covering these as well. Instead we give a short description of the steps here.
We proceed loosely in the same way as in paragraph 1.5.1, meaning that we work towards equation (1.15). However, instead of applying Schwinger's trick when we encounter multiple propagators, we rescale the integration measure $p \to \tau = p\Lambda^{-1}$ and expand the fraction as in equation (1.25). Then the first term in this expansion gives rise to a UV-divergence. The other terms are finite and we can first send $\varepsilon \to 0$ and then $\Lambda - \Lambda_0 \to \infty$. The terms that are left have a spherical denominator and can be integrated.
From equation (1.25) it can already be seen that this expansion causes inner products of the integration variable with an external vector. Since these occur only in the finite part, meaning that $\varepsilon = 0$, they can simply be integrated, yielding a factor $\frac{1}{4}$ for the squared inner product and 0 for the single inner product. This is further worked out in section 4.2.2, remark 4.3 and equation (4.14). Finally, we recall the action of the subtraction operator $\overline{\mathfrak{T}}_2 = \overline{\mathfrak{T}}^{(2)}_{\Lambda-\Lambda_0}\overline{\mathfrak{T}}_\varepsilon$ from Lemma 1.6 and Definitions 1.14 and 1.15.
If we denote the total incoming external momenta with $k$, the amplitudes for the superficially divergent diagrams from Lemma 1.3 are given by

$$\begin{aligned}
\mathcal{M}_1^{(2)} &= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2 \frac{\lambda'(m')^2}{(4\pi\alpha)^2}\Lambda^3 \frac{\Gamma(\varepsilon)}{1-\varepsilon}\left(1 - \varepsilon\log\left(\frac{(m')^2\Lambda^2}{4\pi\alpha\mu^2}\right)\right) \\
&= \frac{-\lambda'(m')^2}{(4\pi\alpha)^2}\Lambda_0^3 \log\left(\frac{(m')^2\Lambda_0^2}{4\pi\alpha\mu^2}\right) && (1.26)\\
\mathcal{M}_2^{(2)} &= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2 \frac{g\lambda'}{(4\pi\alpha)^2}\Lambda \frac{\Gamma(\varepsilon)}{1-\varepsilon}\left(1 - \varepsilon - \varepsilon\log\left(\frac{(m')^2\Lambda^2}{4\pi\alpha\mu^2}\right)\right) \\
&= \frac{-g\lambda'}{(4\pi\alpha)^2}\Lambda_0\left(1 + \log\left(\frac{(m')^2\Lambda_0^2}{4\pi\alpha\mu^2}\right)\right) \quad ; && (1.27)\\
\mathcal{M}_3^{(2)} &= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2 \frac{g^2}{(4\pi\alpha)^2} \frac{\Gamma(\varepsilon)}{1-\varepsilon}\left(1 - \varepsilon - \varepsilon\log\left(\frac{(m')^2\Lambda^2}{4\pi\alpha\mu^2}\right)\right) \\
&= \frac{-g^2}{(4\pi\alpha)^2}\left(1 + \log\left(\frac{(m')^2\Lambda_0^2}{4\pi\alpha\mu^2}\right)\right) \quad ; && (1.28)\\
& && (1.29)
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_4^{(2)} &= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\overline{\mathfrak{T}}}_2 \frac{(\lambda')^2}{(4\pi\alpha)^2}\Lambda^2 \frac{\Gamma(\varepsilon)}{1-\varepsilon}\left(1-\varepsilon-\varepsilon\log\left(\frac{(m')^2\Lambda^2}{4\pi\alpha\mu^2}\right)\right) - \frac{1}{6}\frac{k^2(\lambda')^2}{16\pi^2\alpha(m')^2} \\
&= \frac{-(\lambda')^2}{(4\pi\alpha)^2}\Lambda_0^2\left(1+\log\left(\frac{(m')^2\Lambda_0^2}{4\pi\alpha\mu^2}\right)\right) - \frac{1}{6}\frac{k^2(\lambda')^2}{16\pi^2\alpha(m')^2} \quad ; \qquad (1.30) \\
\mathcal{M}_5^{(2)} &= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\overline{\mathfrak{T}}}_2 \frac{(m')^2 g}{(4\pi\alpha)^2}\Lambda^2 \frac{\Gamma(\varepsilon)}{1-\varepsilon}\left(1-\varepsilon\log\left(\frac{(m')^2\Lambda^2}{4\pi\mu^2\alpha}\right)\right) \\
&= -\frac{(m')^2 g}{(4\pi\alpha)^2}\Lambda_0^2\log\left(\frac{(m')^2\Lambda_0^2}{4\pi\mu^2\alpha}\right) \quad . \qquad (1.31)
\end{aligned}$$

### 1.5.3 The $\beta$-functions of the RSHL

Now that we have calculated all divergent amplitudes at one-loop level we can compute the corresponding $\beta$-functions. The parameter $\mu$ has not vanished from our calculations. In case of a multiplicative renormalization scheme we know that we can resolve this using the Callan-Symanzik equation. However, we still impose the condition that this equation, now named renormalization group equation, acting on the correlator returns zero. The squared matrix element $|\mathcal{M}|^2$ determines the measurable quantities, so that we solve exactly the same equation. In this way we adopt scale dependent coupling constants to cure the issue of the non-vanishing $\mu$-dependence, as explained in paragraph 1.4.3. So, for the $n$-point Green's function $G^{(n)}$ we want to solve

$$0 = \left(\mu\frac{\partial}{\partial\mu} + \beta_{\varkappa_0}\frac{\partial}{\partial\varkappa_0} + \beta_{f_2}\frac{\partial}{\partial f_2} + \beta_{\chi_0'}\frac{\partial}{\partial\chi_0'} - n\gamma_\varphi\right) G^{(n)} \quad . \qquad (1.32)$$

*Remark* 1.20. It will be most convenient to determine the $\beta$-functions for the parameters from Lagrangian (1.1), where we should change $f_0$ to $\varkappa_0 \equiv (f_0)^{-1} \propto g^{-1}$ in the amplitudes, as mentioned in remark 1.4. This means that we are looking for $\beta_{\varkappa_0}$, $\beta_{f_2}$ and $\gamma_\varphi$. The $\beta$-function for $\Lambda_0$, $\gamma_{\Lambda_0}$, is set to zero, since it is not independent.

*Remark* 1.21. At lowest order all our diagrams are at tree level. In that case we do not need any regularization and hence we have no $\mu$-dependence and need no non-trivial $\beta$-functions to solve the renormalization group equation (1.32). We expect therefore that a power series in the coupling constant of our running parameters becomes only $\mu$-dependent at higher order. This means that the lowest order contribution should be $\beta_{\varkappa_0} \propto \varkappa_0^2$, $\beta_{f_2} \propto \varkappa_0$ and $\gamma_\varphi \propto \varkappa_0$. We have no à priori reason to expect specifically these powers of $\varkappa_0$ instead of higher powers, but certainly no lower power. This makes explicit that we only need the divergent parts of the loop-diagrams. Hence, we only use the reduced one-loop Green's function $\tilde{G}^{(n)}$, given by the tree-level diagrams and the divergent one-loop diagrams.
This discussion makes clear that only $\frac{\partial}{\partial\mu}$ in equation (1.32) does not raise the order.

*Remark* 1.22. We can see from equation (1.23) that the kinetic details from the diagram can be decoupled from the $\mu$-dependence and only contribute to the renormalization group equation at higher order. Since the rest is the same for both subtraction operators, it is immediately clear that the $\beta$-functions are equal at lowest order. Below we will explicitly calculate the $\beta$-functions with the hybrid regularization from paragraph 1.5.2.

*Remark* 1.23. All multiplicities in this chapter will be denoted with $n_i$ for diagram $i$, so that it will be clear which $\beta$-functions depend on which diagrams. The multiplicities can be found in Table 1.1.

*Remark* 1.24. In paragraph 1.4.5 we explained that the finite value of the tadpole is an indication that the vacuum expectation value $\chi_0'$ of the field $\varphi$ will run under the renormalization group. Therefore, $\beta_{\chi_0'}$ is added to the renormalization group. The vanishing of the one-point function under the renormalization group equation

$$0 = (\text{ RGE })\,\tilde{G}^{(1)} = (\text{ RGE })\,\frac{\Lambda_0^3}{(4\pi)^2}\left(8f_2\chi_0' - 8\frac{(\chi_0')^3}{\varkappa_0} - n_1\frac{\lambda'(m')^2}{\alpha^2}\log\left(\frac{(\Lambda_0 m')^2}{4\pi\alpha\mu^2}\right)\right)$$

then gives us an opportunity to check that the sketched picture fits.

Before we can check this we must find the $\beta$-functions. We start with the four-point correlator. At tree level this is proportional to $\varkappa_0^{-1}$ and $\mu\frac{\partial}{\partial\mu}\varkappa_0^{-1} = -\varkappa_0^{-2}\beta_{\varkappa_0} = \mathcal{O}(1)$, as is $\mu\frac{\partial}{\partial\mu}\log(\mu^2)$ and $\gamma_\varphi\varkappa_0^{-1}$. All other terms are of higher order, as explained in remark 1.21. This yields

$$\beta_{\varkappa_0}\frac{\partial}{\partial\varkappa_0}\frac{48}{(4\pi)^2\varkappa_0} = \frac{-n_3}{32}\frac{24}{(4\pi)^2}\mu\frac{\partial}{\partial\mu}\left(1 + \log\left(\frac{\Lambda_0^2 f_2\varkappa_0}{\pi\mu^2}\right)\right) + 8\frac{\gamma_\varphi}{\varkappa_0}\frac{24}{(4\pi)^2} \quad ,$$

where on the right hand side the factor 3 comes from the $s$-, $t$- and $u$-channel diagrams, see remark 1.10, and $n_3$ is the multiplicity of the diagram. From this it is easy to see that

$$\beta_{\varkappa_0} = -\frac{n_3}{32}\varkappa_0^2 - 4\gamma_\varphi\varkappa_0 \quad . \tag{1.33}$$

If we want to know $\beta_{\varkappa_0}$ at higher order, we need to know at least the values of $\beta_{f_2}$ and $\gamma_\varphi$ at $\mathcal{O}(\varkappa_0)$, but also all four-point diagrams of $\mathcal{O}(\varkappa_0)$.
To obtain $\beta_{\chi_0'}$ we write down the condition for the three-point correlator

$$\tilde{G}^{(3)} = \frac{-\Lambda_0}{(4\pi)^2}\left(48\frac{\chi_0'}{\varkappa_0} + 3n_2\chi_0'\log\left(\frac{(\Lambda_0 m')^2}{4\pi\alpha\mu^2}\right)\right) \quad .$$

This is

$$0 = -3\gamma_\varphi\left(48\frac{\chi_0'}{\varkappa_0}\right) - 48\frac{\chi_0'}{\varkappa_0^2}\beta_{\varkappa_0} + 48\frac{1}{\varkappa_0}\beta_{\chi_0'} - 6n_2\chi_0' \quad ,$$

which gives using equation (1.33) and $\chi_0' \approx \sqrt{f_2\varkappa_0}$

$$\beta_{\chi_0'} = \left(\frac{n_2}{8} - \frac{n_3}{32}\right)\sqrt{f_2}\varkappa_0^{\frac{3}{2}} - \gamma_\varphi\sqrt{f_2\varkappa_0} \quad . \tag{1.34}$$

Now we continue with the two-point function $\tilde{G}^{(2)}$. Treating the two-point function as just another coupling with one-loop diagrams $\Sigma(k)$ we find

$$\frac{-(2!)}{(4\pi)^2}\left(\frac{1}{2}k^2 + \frac{1}{2}m^2\right) + \Sigma(k) \quad ,$$

which is precisely the denominator of all the two point diagrams, i.e. the left hand side of equation (1.11). At first loop order diagrams 4 and 5 from Figure 1.1 should be taken into account. Their multiplicities are respectively $n_4$ and $n_5$. From equations (1.30) and (1.31) we obtain

$$\begin{aligned}\tilde{G}^{(2)} = {} & \frac{4k^2}{\varkappa_0} + \left(-8f_2\Lambda_0^2 + 24\frac{(\chi_0')^2}{\varkappa_0}\Lambda_0^2\right) + n_5\varkappa_0\left(-f_2 + 3\frac{(\chi_0')^2}{\varkappa_0}\right)\Lambda_0^2\log\left(\frac{\Lambda_0^2 f_2\varkappa_0}{\pi\mu^2}\right) \\ & + 4n_4 f_2\varkappa_0\Lambda_0^2\left(1 + \log\left(\frac{\Lambda_0^2 f_2\varkappa_0}{\pi\mu^2}\right)\right) + \frac{n_4}{6}k^2 \quad .\end{aligned}$$

To be able to solve the corresponding renormalization group equations at the appropriate order, we first determine the next order contributions. In this case these consist

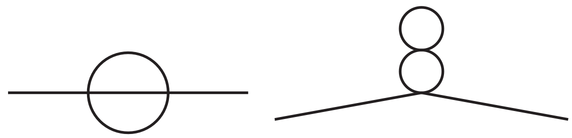

Figure 1.3: The next-to-next-to-leading order two point diagrams.

of the two diagrams from Figure 1.3. From which one can easily deduce that these are of the form $c_1 \cdot \varkappa_0 k^2 + c_2 \cdot \varkappa_0^2 m^2$. So we should make sure that lower orders vanish.

The renormalization group equation on this should be zero for all momenta $k$, so this equation actually contains two conditions. Normally one fixes the renormalization condition at the mass shell, but we have removed a part of our kinematics and we see directly from equation (1.35) that this can be done for any $k$. Since the kinetic part of diagram 4 has no $\mu$-dependence, it is only relevant at higher order. This means that we obtain for the renormalization of the two-point function $\tilde{G}^{(2)}$ the conditions

$$0 = k^2 \left(4\beta_{\varkappa_0} + 8\varkappa_0\gamma_\varphi\right) \quad ; \tag{1.35}$$

$$\begin{aligned} 0 = &-8\beta_{f_2} - 2\gamma_\varphi(-8f_2 + \frac{24}{\varkappa_0}(\chi_0')^2) - 24\frac{(\chi_0')^2}{\varkappa_0^2}\beta_{\varkappa_0} + 48\frac{\chi_0'}{\varkappa_0}\beta_{\chi_0'} \\ &- n_5\varkappa_0(-2f_2 + 6\frac{(\chi_0')^2}{\varkappa_0}) - 8n_4(\chi_0')^2 \quad , \end{aligned} \tag{1.36}$$

so that

$$\beta_{f_2} = \left(\frac{2}{\varkappa_0}\gamma_\varphi - (\frac{n_5}{2} + n_4 - \frac{3n_2}{4} + \frac{3n_3}{32})\right)\varkappa_0 f_2 \tag{1.37}$$

$$\gamma_\varphi = -\frac{n_3}{64}\varkappa_0 \quad . \tag{1.38}$$

Using the obtained value of $\gamma_\varphi$ and the multiplicities of Table 1.1 this yields the $\beta$-functions at lowest order for the rsHL

$$\beta_{\varkappa_0} = 9\varkappa_0^2 \quad ; \tag{1.39}$$

$$\beta_{\chi_0'} = \frac{9}{2}\sqrt{f_2}\varkappa_0^{\frac{3}{2}} \quad ; \tag{1.40}$$

$$\beta_{f_2} = -6f_2\varkappa_0 \quad ; \tag{1.41}$$

$$\gamma_\varphi = -\frac{9}{2}\varkappa_0 \quad . \tag{1.42}$$

We have demanded that the renormalization group equation working on the two-, three- and four-point function returns zero, but we should have required this for the one-point function as well, as mentioned in remark 1.24. Going through the same calculations we find that

$$(\text{ RGE })\,\tilde{G}^{(1)} = \frac{(2\sqrt{f_2\varkappa_0}\Lambda_0)^3}{(4\pi)^2}\left(2n_1 + \frac{n_2}{2} - \frac{n_3}{16} - n_4 - \frac{n_5}{2}\right) = 0 \tag{1.43}$$

for the multiplicities in Table 1.1. This confirms that both the calculations and the multiplicities are correct.
Although we have worked in a Euclidean setting here, it should be clear from the

construction of this higher derivative quantum field theory that the same $\beta$-functions would be obtained working in Minkowski space. The Lagrangian (1.9) was obtained by breaking the discrete symmetry. One could equally well have stated this Lagrangian from the start and would have found the same $\beta$-functions, if the same relations between the parameters were imposed. In that case it would have been more natural to study the $\beta$-functions for the three- and four-point coupling, which are related by $\chi_0'$. Then one would find $\beta_3 = \chi_0'\beta_4$, from which it is clear that the vacuum does not actually run, as already suggested in remark 1.18.

*Remark* 1.25. The minus-sign in front of the $\varkappa_0^2$-term in equation (1.33) is perhaps confusing, since this term in the $\beta$-function for ordinary $\varphi^4$-theory is positive. The difference lies in the specific combinations we have used. If we would have rescaled our fields according to $\varphi \to \frac{1}{\sqrt{2f_0}}\varphi$ we would have had real $\varphi^4$-theory with spontaneous symmetry breaking, having the same positive $\beta$-function for the four-point coupling as ordinary $\varphi^4$-theory, modulo some factors of $4\pi$ and the wave function renormalization. This can easily be checked by rescaling the fields in the renormalization group equation and solving again.

The powers of $\varkappa_0$ are just as we predicted in remark 1.21. Although we have used this prediction in a few places, the outcome is not influenced by it. Alternatively, we could have set up a system of four linear equations and solve that, which would have given precisely the same answer at the appropriate order. Finding these $\beta$-functions completes our example and this chapter.

We have determined the $\beta$-functions for this theory using a renormalization scheme that hardly uses the kinematical structure. This is what we meant in remark 1.10. Although not a full proof, the correspondence between the two sets of $\beta$-functions gives us an argument for using $\overline{\overline{\mathfrak{T}}}_2$ to obtain the $\beta$-functions of more difficult theories. This is what we will do in the remainder of this work. Furthermore, since both operators are valid subtraction operators, the renormalization theory should make sense without an experimental link to physics.

# 2 The heat kernel expansion of the spectral action

As mentioned in the last paragraph of Chapter 1, we have a way to study the $\beta$-functions of more difficult, higher order theories. Instead of guessing a more difficult Lagrangian that might give comparable $\beta$-functions, we will see that the spectral action provides us with such a theory. To be able to see this, we first need to show how to obtain a Lagrangian from the spectral action. We will do this for a compact Riemannian manifold with vanishing Christoffel symbols and without gauge connections. In this chapter we will construct an asymptotic expansion of the spectral action [15]. To expand this action for explicit analysis of the physical theory it defines, we will need to calculate the heat kernel expansion coefficients $a_k$ for it. This is the main goal in this chapter.

## 2.1 Introduction

We will first give a brief description of some of the formulae involved. Then, in paragraphs 2.2 and 2.3 we will introduce some of the needed machinery.
The spectral action is given by

$$S[\varphi] = \mathrm{Tr}\ f(\frac{\not{D} + \gamma_5\varphi}{\Lambda}) \quad , \tag{2.1}$$

where the trace is over $L^2(M, S)$, with $M$ a smooth compact manifold of dimension 4 without boundary, $S$ a spinor bundle over $M$ and $\varphi \in C^\infty(M)$ is a smooth real scalar (quantum) field acting on spinors by pointwise multiplication. Therefore $\varphi$ is uniformly bounded, for example a test function. The Dirac operator is given by $\not{D} = i\gamma^\mu\nabla_\mu$, where summation over $\mu \in \{0, 1, 2, 3\}$ is implied. The spin connection $\nabla$ reduces in our case to partial derivatives. Notice that the $\gamma$-matrices require a spinor bundle to act on. Furthermore, $f$ is given by the Laplace-Stieltjes transform

$$f(x) = \int_0^\infty \mathrm{d}t\, e^{-tx^2} g(t) \quad ,$$

which requires that $\mathrm{d}G(t) = g(t)\,\mathrm{d}t$ is of bounded variation. This choice of $f$ is made to ensure that the trace is well defined. Since $G$ is differentiable this means that $\int_0^T |g(t)|\,\mathrm{d}t < \infty$ and that $\lim_{T\to\infty} \int_0^T \mathrm{d}t\, e^{-tx^2} g(t)$ exists. Since we will be mainly interested in the moments of $f$ we could turn it around. By the Stieltjes moment problem there is a suitable measure $\mathrm{d}G(t)$ for a sequence of moments under certain conditions. Instead of using this, we will make the sufficient assumption that $g$ is of bounded variation and that the integral

$$\int_0^\infty \mathrm{d}t\, |g(t)| t^m < \infty \qquad \forall m \in \mathbb{N} \quad . \tag{2.2}$$

*Remark* 2.1. Notice that $\not{D}$ and $\Lambda$ have units of mass, to let $f(\frac{\not{D}+\gamma_5\varphi}{\Lambda})$ be defined.

The idea now is to expand the trace of this exponential and obtain a physical action. This gives a Lagrangian that defines a quantum field theory. The coefficients $a_m$ will give the fields and derivative terms, while the coefficients $f_{4-m}$ are the coupling constants and $\Lambda$ is the mass parameter.

$$S[\varphi] = \sum_m \Lambda^{4-m} f_{4-m} \int_M a_m(x, \not{D}_\varphi^2) \quad , \tag{2.3}$$

where $f_k = \int_{t>0} \mathrm{d}t\, g(t) t^{-\frac{k}{2}}$.

From equation (2.3) it is clear that we need a way to integrate over the compact manifold $M$. Below we will give a short description of what is needed to integrate the function $\psi$ over $M$.

Compactness of $M$ means that every open cover has a finite subcover. It is a sufficient condition for the existence of a partition of unity, subordinate to an open cover of $M$.

**Definition 2.1.** [Partition of unity]

A partition of unity subordinate to an open cover $\{U_\iota\}_{\iota\in\mathcal{I}}$ of a manifold M is a set of continuous functions, such that:

- $\sum_{\iota\in\mathcal{I}} \rho_\iota(x) = 1 \ \forall x \in M$;
- any $x \in M$ has a neighbourhood where only finitely many $\rho_\iota$ are non-zero ;
- the support of $\rho_\iota \subseteq U_\iota$, $\forall \iota \in \mathcal{I}$.

Globally, the coordinate charts of the manifold are glued together with a partition of unity. Each patch can be mapped injectively to Euclidean space by a coordinate function $\phi_\iota : U_\iota \hookrightarrow \mathbb{R}^4$, where we can integrate it. This means that

$$\int_M \psi = \sum_{\iota\in\mathcal{I}} \int_{U_\iota} \mathrm{d}^4 y\, \psi(\phi_\iota^{-1}(y))\rho_\iota(y) \quad .$$

In Chapter 1 we analyzed the UV-divergences of a quantum field theory. This is related to physics on a very small scale and therefore we are working on a single coordinate patch. However, it is not yet clear whether there is a open set in $M$ that maps to entire $\mathbb{R}^4$. To ensure this, we proceed as follows. Take a open ball of radius $r$ around $\phi_\iota(x) \in \mathbb{R}^4$. Scaling the size of the ball by $x \mapsto \frac{x}{r-x}$, we see that it covers all of $\mathbb{R}^4$. In this way we obtain a new atlas, that guarantees that we can locally work on entire $\mathbb{R}^4$.

## 2.2 Operator calculus

We first mention that $(\not{D} + \gamma_5\varphi)^2 = \Delta - E$, where $E$ maps $x \in M$ smoothly to an endomorphism of $S$, i.e. $E \in C^\infty(End(S))$. This statement will be proved in Lemma 3.3. From Lemma 3.3 and remark 3.2 we infer that $E$ is self-adjoint and bounded. The Laplacian on the flat manifold is denoted by $\Delta = \not{D}^2 = -\partial_\mu\partial^\mu$. From the description in the previous paragraph, having changed the notation of equation (2.1) slightly, we would find

$$\mathrm{Tr}\left(e^{-t(\Delta-E)}\right) \quad . \tag{2.4}$$

So, we have to make sense of the exponential of the self-adjoint operator $\Delta$ on the Hilbert space $L^2(M)$. We will briefly explain how this is done. For more details we refer to [16] and [17].

### 2.2.1 Unbounded operators

All operators discussed here are linear. The Hellinger-Toeplitz theorem states that an operator $A$ defined on the entire Hilbert space $\mathscr{H}$ and satisfying $\langle A\zeta, \xi\rangle = \langle \zeta, A\xi\rangle$ is bounded, i.e. $A \in \mathscr{B}(\mathscr{H})$. This makes it clear that an unbounded self-adjoint operator cannot be defined everywhere. Usually, such operators will have a dense domain $D(A)$. In the case of the Laplacian it follows from Stokes' theorem that for any $\xi, \zeta \in C^\infty(M)$

$$\int_M \mathrm{d}x\, \overline{\xi(x)}\Delta\zeta(x) = \int_M \mathrm{d}x\, \overline{\Delta\xi(x)}\zeta(x) \quad . \tag{2.5}$$

**Definition 2.2.** [Adjoint]
Let $A$ be an operator on a Hilbert space $\mathscr{H}$ with dense domain $D(A)$. Then, $D(A^*)$ is the set of $\zeta \in \mathscr{H}$, such that there is a $\xi \in \mathscr{H}$ satisfying $\langle \zeta, A\psi\rangle = \langle \xi, \psi\rangle\ \forall \psi \in D(A)$. The operator $A^* : D(A^*) \to \mathscr{H}$, given by $A^* : \zeta \mapsto \xi$ is called the adjoint.
Furthermore, if $A\psi = A^*\psi\ \forall\psi \in D(A)$, $A$ is Hermitean and $D(A) \subset D(A^*)$. If in addition also $D(A) = D(A^*)$, $A$ is called self-adjoint.

From equation 2.5 it is clear that $D(\Delta) \subset D(\Delta^*)$ and that $\Delta$ is thus Hermitean.

**Definition 2.3.** [Graph]
For a operator $A : D(A) \subset \mathscr{H} \to \mathscr{H}$ the graph

$$\Gamma(A) = \{(\psi, A\psi)|\psi \in D(A)\}$$

is a subset of $\mathscr{H} \times \mathscr{H}$ with inner product $\langle(\psi_1, \xi_1), (\psi_2, \xi_2)\rangle = \langle\psi_1, \psi_2\rangle + \langle\xi_1, \xi_2\rangle$.
The operator $A$ is closed if $\Gamma(A)$ is a closed subset. If $\Gamma(A) \subset \Gamma(A_e)$, we call $A_e$ an extension of $A$ and write $A \subset A_e$. If $A$ has a closed extension, it is closable. The smallest is called the closure $\overline{A}$.

**Lemma 2.1.** A densely defined operator $A$ on a Hilbert space $\mathscr{H}$ has closed adjoint $A^*$.

*Proof.* The operator $J : (\xi, \zeta) \mapsto (-\zeta, \xi)$ on $\mathscr{H} \times \mathscr{H}$ is unitary, since $\langle J(\xi_1, \zeta_1), J(\xi_2, \zeta_2)\rangle = \langle(\xi_1, \zeta_1), (\xi_2, \zeta_2)\rangle$. So it preserves subspaces $J(V^\perp) = J(V)^\perp$. Then,

$$\begin{aligned} (\zeta, \xi) \in J(\Gamma(A))^\perp \quad &\Leftrightarrow \quad \langle(\zeta, \xi), (-A\psi, \psi)\rangle = 0 \quad \forall\psi \in D(A) \\ &\Leftrightarrow \quad \langle\zeta, A\psi\rangle = \langle\xi, \psi\rangle \quad \forall\psi \in D(A) \quad \Leftrightarrow \quad (\zeta, \xi) \in \Gamma(A^*) \quad , \end{aligned}$$

so $J(\Gamma(A))^\perp = \Gamma(A^*)$, which is always closed. □

**Corollary 2.1.** A self-adjoint operator is closed.

**Definition 2.4.** [Spectrum]
For an operator $A$ the resolvent $\rho(A)$ set contains the numbers $\lambda \in \mathbb{C}$ such that $A - \lambda : \mathscr{H} \to \mathscr{H}$ is injective, has dense range and $(A - \lambda)^{-1} \in \mathscr{B}(\mathscr{H})$ exists.
The spectrum of $A$ is $\sigma(A) \equiv \mathbb{C} \setminus \rho(A)$.
An operator $A : \mathscr{H} \to \mathscr{H}$ has a bounded inverse $B$ if $0 \in \rho(A)$. Then, $AB = 1$ and $BA|_{D(A)} = 1|_{D(A)}$.

**Lemma 2.2.** An operator $A$ on $\mathscr{H}$ has bounded inverse if and only if $\mathrm{Ker}(A) = 0$, $\mathrm{Ran}(A) = \mathscr{H}$ and $\Gamma(A)$ is closed. Furthermore, the inverse is unique.

*Proof.* If $B$ is the bounded inverse, then $\mathrm{Ker}(A) = 0$, since $BA \subset 1$. From $AB = 1$ it follows that $\mathrm{Ran}(A) = \mathscr{H}$. Because $B$ is bounded

$$\Gamma(A) = \{(\xi, A\xi) \in \mathscr{H} \times \mathscr{H}|\xi \in\ D(A)\} = \{(B\zeta, \zeta) \in \mathscr{H} \times \mathscr{H}|\zeta \in \mathscr{H}\}$$

is closed.
Since $\mathrm{Ker}(A) = 0$ and $\mathrm{Ran}(A) = \mathscr{H}$ the inverse $A^{-1}$ is well-defined. Furthermore, that $\Gamma(A)$ is closed implies that $\Gamma(B)$ is closed. Because the projection $P : (\zeta, B\zeta) \mapsto \zeta$ is bounded and bijective, $P^{-1}$ is bounded. The composition with the projection $(\zeta, B\zeta) \mapsto B\zeta$ shows that $B$ is bounded.
Suppose there are $B_1, B_2 \in \mathscr{B}(\mathscr{H})$ such that $B_{1,2}A \subset 1$ and $AB_{1,2} = 1$. Because $A(B_1 - B_2) = 0$ and $\mathrm{Ker}(A) = 0$, $B_1 = B_2$. □

**Lemma 2.3.** Let $A$ be a closed operator. If $\lambda \in \rho(A)$, then the open disk $B(\lambda, \|R_\lambda(A)\|^{-1}) \subset \rho(A)$. In particular, $\rho(A)$ is open and $\sigma(A)$ is closed.

*Proof.* If $|\mu - \lambda| < \|R_\lambda\|^{-1}$, then $S = \sum_{i=0}^{\infty}(\mu - \lambda)^i R_\lambda^i$ converges in norm. The composed operator $R_\lambda S = (A - \mu)^{-1}$ and is bounded, so $R_\lambda S = R_\mu$. □

**Lemma 2.4.** Let $A$ be a Hermitean operator with dense domain $D(A)$. The closure is the Hermitean operator $\overline{A}$ with domain $D(\overline{A}) \supset D(A)$.

*Proof.* We have to check that $\overline{\Gamma(A)}$ is the graph of an operator $\overline{A}$.
Therefore, $(\psi, \xi), (\psi, \zeta) \in \overline{\Gamma(A)}$ must imply that $\xi = \zeta$. There exist sequences $(\psi_n, A\psi_n) \to (\psi, \xi)$ and $(\psi'_n, A\psi'_n) \to (\psi, \zeta)$. Take the inner product of $\xi$ with an element $\chi \in D(A)$:

$$\langle \chi, \xi \rangle = \lim_{n\to\infty} \langle \chi, A\psi_n \rangle = \lim_{n\to\infty} \langle A\chi, \psi_n \rangle = \langle A\chi, \psi \rangle \quad .$$

Identically, $\langle \chi, \zeta \rangle = \langle A\chi, \psi \rangle$. Since $D(A)$ lies dense, this means $\xi = \zeta$.
With $\xi_n \to \xi$ and $\zeta_n \to \zeta$ it is clear that the linear combination $\lambda\xi_n + \mu\zeta_n \to \lambda\xi + \mu\zeta$. The linearity of $\overline{A}$ then follows from

$$\|\overline{A}(\lambda\xi + \mu\zeta) - \lambda\overline{A}(\xi) - \mu\overline{A}(\zeta)\| = \|\lim_{n\to\infty} (A(\lambda\xi_n + \mu\zeta_n) - \lambda A(\xi_n) - \mu A(\zeta_n))\| = 0 \quad .$$

A similar argument proves that $\overline{A}$ is Hermitean. Two vectors $\xi, \zeta \in D(\overline{A})$ with sequences $\xi_n, \zeta_m$ in $D(A)$ satisfy

$$\langle \overline{A}\xi, \zeta \rangle - \langle \xi, \overline{A}\zeta \rangle = \lim_{m,n\to\infty} \langle A\xi_n, \zeta_m \rangle - \langle \xi_n, A\zeta_m \rangle = 0 \quad ,$$

proving that $\overline{A}$ is Hermitean.
Take $\lambda \in \rho(A)$. It is clear that $\overline{A} - \lambda$ is surjective and closed. To see that it is injective, take $\zeta \in \mathrm{Ker}(\overline{A} - \lambda)$ and a sequence $\zeta_n \to \zeta$. The sequence $\psi_n = (\overline{A} - \lambda)\zeta_n \to 0$, so $\zeta_n = R_\lambda \psi_n \to 0$. This proves $\rho(A) \subset \rho(\overline{A})$.
Conversely, for $\lambda \in \rho(\overline{A})$, $0 = \mathrm{Ker}(A - \lambda) \subset \mathrm{Ker}(\overline{A} - \lambda)$. We have to check that $\overline{(A - \lambda)D(A)} = \mathscr{H}$. Since $\lambda \in \rho(\overline{A})$ we know that $\overline{(\overline{A} - \lambda)\tilde{D}(A)} = \mathscr{H}$.
For any $\psi \in \mathscr{H}$ there is a sequence $\xi_n$ in $\tilde{D}$ such that $(\overline{A} - \lambda)\xi_n = \psi_n \to \psi$. For every $\varepsilon > 0$ all $n$ larger than some $N$ give $\|\psi - \psi_n\| < \varepsilon$. Take $\zeta_N \in B(\xi_N, \frac{\varepsilon}{\|R_\lambda(\overline{A})\|^{-1}}) \cap D(A)$.
For $\varepsilon > 0$ and $n > N$ we have that $\|\psi - (\overline{A} - \lambda)\zeta_n\| < 2\varepsilon$. This proves that $(A - \lambda)\zeta_n = (\overline{A} - \lambda)\zeta_n \to \psi$.
That the inverse is bounded is clear from the fact that $R_\lambda(\overline{A})$ is bounded, so $\lambda \in \rho(A)$.
Finally, because $R_\lambda(A)$ and $R_\lambda(\overline{A})$ are bounded, defined on the same domain and the same on a dense subset of the range, they are equal. □

**Definition 2.5.** [Essentially self-adjoint] A Hermitean operator $A$ is essentially self-adjoint, if its closure $\overline{A}$ is self-adjoint.

**Proposition 2.1.** For a Hermitean operator $A$ on a Hilbert space $\mathscr{H}$ the following are equivalent

1. $A$ is self-adjoint;
2. $A$ is closed and $\mathrm{Ker}(A^* \pm i) = 0$;
3. $\mathrm{Ran}(A \pm i) = \mathscr{H}$.

*Proof.* If $A^*\xi = \pm\xi$, then

$$\pm\langle\xi,\xi\rangle = \langle\xi, A\xi\rangle = \langle A\xi,\xi\rangle = \mp\langle\xi,\xi\rangle \quad ,$$

so that $\xi = 0$. Combined with Corollary 2.1 we see that 1 implies 2.
Next, we prove 2. ⇒ 3.. A vector $\zeta \in \mathrm{Ran}(A \mp i)^\perp$ satisfies $\langle\zeta,(A \mp i)\xi\rangle\ \forall\xi \in D(A)$, so $\zeta \in D(A^*)$ and $(T^* \pm i)\zeta = 0$. This is only possible, if $\zeta = 0$. This proves that the range is dense. To see that it is actually closed, we first observe

$$\|(A \mp i)\xi\|^2 = \|\xi\|^2 + \|A\xi\|^2 \quad , \quad \xi \in D(A) \quad .$$

There is a sequence $\psi_n \in D(A)$ such that $(A - i)\psi_n \to \xi$. This equation implies then that $\psi_n$ and $A\psi_n$ converge too. Since $A$ is closed, $\lim\psi_n = \psi \in D(A)$ and $A\psi = \lim A\psi_n\xi + i\psi$. Similarly, $\mathrm{Ran}(A + i) = \mathscr{H}$.
To show that 3 ⇒ 1, we take $\zeta \in \mathrm{Ker}(A \pm i)$, then $\zeta \in \mathrm{Ran}(A \pm i)^\perp$ and hence $\zeta = 0$, so $\mathrm{Ker}(A \pm i) = 0$.
Take any $\xi \in D(A^*)$. Since $\mathrm{Ran}(A - i) = \mathscr{H}$, there is a $\psi \in D(A)$, such that $(A-i)\psi = (A^*-i)\xi$. Because $D(A) \subset D(A^*)$, $(\xi-\psi) \in D(A^*)$ and $(A^*-i)(\xi-\psi) = 0$. So, $\xi = \psi \in D(A)$. This proves that $A$ is self-adjoint. □

**Corollary 2.2.** Let $A$ be a Hermitean operator on a Hilbert space $\mathscr{H}$. Then the following are equivalent:

1. $A$ is essentially self-adjoint;
2. $\mathrm{Ker}(A^* \pm i) = 0$;
3. $\overline{\mathrm{Ran}(A \pm i)} = \mathscr{H}$.

**Lemma 2.5.** Let $A$ be a Hermitean operator and $p, q \in \mathbb{R}$.

1. For each $\xi \in D(A)$: $\|\,(A - (p + iq))\,\xi)\|^2 = \|\,(A - p)\,\xi\|^2 + q^2\|\xi\|^2$.
2. $\mathrm{Ker}(A - p - iq) = 0$.
3. If $A$ is closed, then $\mathrm{Ran}(A - p)$ is closed.

*Proof.* The first statement follows from

$$\begin{aligned}\|\,(A - (p + iq))\,\xi)\|^2 &= \langle\xi,(A - p + iq)\,(A - p - iq)\,\xi\rangle \\ &= \|\,(A - p)\,\xi\|^2 + q^2\|\xi\|^2 \quad .\end{aligned} \tag{2.6}$$

The second is found when considering $q = 0$. Consider $\{\psi_n\} \subset D(A)$ such that $(A - p - iq)\psi_n \to \zeta$. Because $\|\xi\|^2 \le q^{-2}\|(A - p - iq)\xi\|^2$, $\psi_n \to \psi$ is a Cauchy sequence in $\mathscr{H}$. The graph $\Gamma(A)$ contains $(\psi_n,(A - p - iq)\psi_n)$ and since $A$ is closed also $(\psi,\zeta)$. This proves that $\mathrm{Ran}(A - p - iq)$ is closed. □

**Lemma 2.6.** If $S, T$ are closed subspaces of $\mathscr{H}$ and $S \cap T^\perp = 0$, then $\dim(\mathrm{S}) \le \dim(T)$.

*Proof.* Define the projections $P_1 : \mathscr{H} \twoheadrightarrow T$ and $P_2 : S \to T$, $P_2\xi = P_1\xi$. Then $P_2$ is injective. For a finite dimensional subspace $V \subset S$ we have $\dim(V) = \dim(P_2V) \le \dim(T)$. Since $V$ is arbitrary, this proves the claim. □

The following technical lemma [16, X.2.7] is essential in analyzing the spectrum of a Hermitean unbounded operator.

**Lemma 2.7.** For a closed Hermitean operator $A$ the function $\dim(\mathrm{Ker}(A-\lambda))$ is a constant for $\mathrm{Im}(\lambda) > 0$ and constant for $\mathrm{Im}(\lambda) < 0$.

*Proof.* Take $\lambda = p + iq$ and $\omega$ both complex numbers and $p$ and $q \neq 0$ both real. We first show that if $|\lambda - \omega| < |q|$, $\mathrm{Ker}(A^* - \lambda) \cap \mathrm{Ker}(A^* - \omega) = 0$.
Suppose this is not so. Take $\xi \in \mathrm{Ker}(A^* - \lambda) \cap \mathrm{Ker}(A^* - \omega)$ with $\|\xi\| = 1$. Since it is closed by Lemma 2.5, statement 3, $\mathrm{Ran}(A - \overline{\lambda}) = \mathrm{Ker}(A^* - \lambda)^\perp$ contains $\xi$. We define $\psi \in D(A)$ such that $(A - \overline{\lambda})\psi = \xi$. Since $\xi \in \mathrm{Ker}(A^* - \omega)$

$$0 = \langle \psi, (A^* - \omega)\xi \rangle = \langle (A - \overline{\lambda} + \overline{\lambda} - \overline{\omega}), \xi \rangle = \|\xi\|^2 + (\lambda - \omega)\langle \psi, \xi \rangle \quad ,$$

so that $1 = \|\xi\|^2 \leq |\lambda - \omega|\|\psi\|$. However, from the first statement from Lemma 2.5 $1 = \|(A - \overline{\lambda})\zeta\| \geq |q|\|\zeta\|$, implying that $\|\zeta\| \leq |q^{-1}|$. Combining these two estimates

$$1 \leq |\lambda - \omega|\|\zeta\| \leq |\lambda - \omega| \cdot |q^{-1}| < 1$$

yields a contradiction.
The kernels are closed subspaces. Lemma 2.6 shows that
$\dim(\mathrm{Ker}(A^* - \omega)) \leq \dim(\mathrm{Ker}(A^* - \lambda))$, if $|\lambda - \omega| < \mathrm{Im}(\lambda)$. If furthermore $|\lambda - \omega| < \frac{1}{2}|q|$, then $|\lambda - \omega| < \mathrm{Im}(\omega)$ and the converse inequality for the dimensions of the kernels holds as well. This proves that $\lambda \mapsto \dim(\mathrm{Ker}(A - \lambda))$ is locally constant on $\mathbb{C} \setminus \mathbb{R}$. □

**Proposition 2.2.** If $A$ is a closed Hermitean operator, then one of the following four is true:

1. $\sigma(A) = \mathbb{C}$;
2. $\sigma(A) = \{\lambda \in \mathbb{C} | \mathrm{Im}(\lambda) \geq 0\}$;
3. $\sigma(A) = \{\lambda \in \mathbb{C} | \mathrm{Im}(\lambda) \leq 0\}$;
4. $\sigma(A) \subset \mathbb{R}$.

*Proof.* Define $Q_\pm = \{\lambda \in \mathbb{C} | \pm\mathrm{Im}(\lambda) > 0\}$. From Lemma 2.5 $A - \lambda$ is injective and has closed range, if $\lambda \in Q_\pm$, so $\lambda \in \rho(A)$, if $A - \lambda$ is surjective. Lemma 2.7 implies then that $Q_\pm \subset \sigma(A)$ or $Q_\pm \cap \sigma(A) = \varnothing$. The proposition is proved, since the spectrum must be closed. □

**Corollary 2.3.** Let $A$ be a closed Hermitean operator. Then the following are equivalent.

1. $A$ is self-adjoint.
2. $\sigma(A) \subset \mathbb{R}$.
3. $\mathrm{Ker}(A^* \pm i) = 0$

*Proof.* $A$ is self-adjoint. Take $p, q \in \mathbb{R}$, then equation (2.6) implies for $q \neq 0$ that $\|(A - \lambda - iq)x\| \geq q\|x\|$. Lemma 2.5 implies then that $A - p - iq$ is injective and has a bounded inverse on its range. Since it is self-adjoint, $\mathrm{Ran}(A) = \mathscr{H}$ and $p + iq \in \rho(A)$, if $q \neq 0$. This proves that $\sigma(A) \subset \mathbb{R}$. So, 1 implies 2.
$A$ has a real spectrum. $\mathrm{Ker}(A^* \pm i) = \mathrm{Ran}(A \mp i)^\perp = \mathscr{H}^\perp = 0$. This implies the third statement.
That 3 implies that $A$ is self-adjoint was proved in Proposition 2.1. □

In Lemma 2.4 we proved that the closure of a Hermitean operator is Hermitean as well. Furthermore, we know from Lemma 2.4 that the resolvent set and the spectrum remain the same taking closures. This means that a Hermitean operator is essentially self-adjoint, if it's spectrum is contained in $\mathbb{R}$, which we alternatively could have taken as definition.
Using Proposition 2.2 it is enough to show that $\rho(A) \cap \mathbb{R} \neq \varnothing$ to prove that the Laplacian is essentially self-adjoint. This we will do by positivity.

**Definition 2.6.** [Positivity]
An operator $B$ on a Hilbert space $\mathscr{H}$ satisfying $\langle \psi, B\psi \rangle \geq 0 \; \forall \psi \in D(B)$ is called positive, $B \geq 0$. The operator $B$ is said to be bounded from below by $m \in \mathbb{R}$, if $B - m$ is positive.

**Lemma 2.8.** Let $A$ be a Hermitean operator that is bounded below by $m$. A real number $\lambda < m$ is contained in the resolvent, if and only if $A - \lambda$ is dense. $A$ is essentially self-adjoint, if $A - \lambda$ is dense.

*Proof.* From the inequality $\langle \xi, (A - \lambda)\xi >\geq (m - \lambda)\|\xi\|^2$ injectivity and $\|R_\lambda\| \leq (m - \lambda)^{-1}$. Then, $\lambda \in \rho(A)$ if and only if $A - \lambda$ is dense. Corollary 2.3 then implies that $A$ is essentially self-adjoint. □

**Proposition 2.3.** The Laplacian $\Delta$ with domain $C^\infty(M) \subset L^2(M)$ is essentially self-adjoint.

*Proof.* We already mentioned that equation (2.5) implies that the Laplacian is Hermitean. By partial integration

$$\int_M \mathrm{d}x \, \overline{\xi(x)} \Delta \xi(x) = \int_M \mathrm{d}x \, |\nabla \xi(x)|^2 \geq 0$$

it follows that it is positive and bounded from below by 0. To show that it is essentially self-adjoint it suffices to prove that $\Delta + 1$ has dense range.
Here, we will give an argument for the Laplacian on $\mathbb{R}^n$, where we have to take functions with compact support, because $\mathbb{R}^n$ is not bounded. More general cases can be found in [18, §5].
If $\Delta + 1$ does not have dense range, then $0 \neq \xi \in L^2(\mathbb{R}^n)$ would exist with $\langle \xi, (\Delta + 1)\psi \rangle = 0$ for any smooth function with compact support $\psi \in C_c^\infty(\mathbb{R}^n)$. Fourier transforming this would imply $\langle \hat{\xi}, (p^2+1)\hat{\psi} \rangle = 0$. Since $\{(p^2+1)\psi | \psi \in C_c^\infty(\mathbb{R}^n)\}$ lies dense in $L^2(\mathbb{R}^n)$ this implies $\hat{\xi} = 0$ and hence $\xi = 0$. □

### 2.2.2 Functional calculus

Now that we have introduced some general concepts from unbounded operator theory, we will briefly describe how the functional calculus on a separable Hilbert space can be seen.

*Remark* 2.2. Before we go into this, it may be worthwhile to recall the operator calculus of matrices. In linear algebra normality of a matrix $\mathscr{M}$ is a necessary and sufficient condition for diagonalizability, $\mathscr{M} = \sum_{j=1}^n \lambda_j \mathcal{E}_j$, where $\mathcal{E}_j$ is a projection on a subspace of dimension one and $\lambda_j$ is the corresponding eigenvalue of the matrix. Diagonal matrices can be multiplied and summed as ordinary numbers Therefore, any function $f$ that is defined on all the diagonal entries of the matrix, is defined on the matrix as well, $f(\mathscr{M}) = \sum_{j=1}^n f(\lambda_j)\mathcal{E}_j$.

A metric space is called separable if it contains a countable dense set. Since $M$ is compact, the Hilbert space $L^2(M)$ is separable.
A polynomial $f(x) = \sum_{n=1}^N a_n x^n$ can be generalized to $f(A) = \sum_{n=1}^N a_n A^n$, since

multiplication and addition are defined. A power series $f(x) = \sum_{n=1}^{\infty} a_n x^n$ with radius of convergence $R$ can be generalized for operators $\|A\| < R$. Since the operator space is complete, $f(A)$ is a convergent series.
Since for any polynomial $P$, the identity $\|P(A)\| = \sup_{\lambda \in \sigma(A)} |P(\lambda)|$ is true for self-adjoint operators, this can be generalized to functions continuous on $\sigma(A)$. This can then be generalized to Borel functions.

**Definition 2.7.** The Borel sets of $\mathbb{R}$ is the smallest family of subsets of $\mathbb{R}$ that

1. is closed under complements;
2. is closed under countable unions;
3. contains each open interval of $\mathbb{R}$.

**Definition 2.8.** A real valued function $f : \mathbb{R} \to \mathbb{R}$ is a Borel function, if for all $a < b$ $f^{-1}(a, b)$ is a Borel set.

Now, one can define a spectral measure $\mu_\psi$ for $\psi \in \mathscr{H}$, such that $\langle \psi, f(A)\psi \rangle = \int_{\sigma(A)} \mathrm{d}\mu_\psi f_\lambda$ for a Borel function $f$ and a self-adjoint operator $A$.

**Definition 2.9.** [Projection-valued measure]
For a Borel set $\Omega \subset \mathbb{R}$ the characteristic function $\chi_\Omega$ is a Borel function and by the above there is an associated projection $P_\Omega = \chi_\Omega(A)$. The family $\{P_\Omega\}$ has the properties:

1. every $P_\Omega$ is an orthogonal projections, $P^2 = P = P^*$;
2. the family contains the projections $P_\varnothing = 0$ and $P_{(-\infty,\infty)} = I$;
3. the projections satisfy $P_{\Omega_1} P_{\Omega_2} = P_{\Omega_1 \cap \Omega_2}$;
4. if $\Omega = \bigcup_{n=1}^{N} \Omega_n$ with $\Omega_n \cap \Omega_m = \varnothing$, if $n \neq m$, then

$$P_\Omega = s - \lim_{N\to\infty} \sum_{n=1}^{N} P_{\Omega_n} \quad .$$

*Remark* 2.3. It is clear that the projections are all contained in the bounded operators $\mathscr{B}(\mathscr{H})$. The strong limit, $s - \lim$, means that it converges in the strong operator topology on $\mathscr{B}(\mathscr{H})$, which can be defined as the weakest topology such that all evaluation maps

$$e_\xi : \mathscr{B}(\mathscr{H}) \to \mathscr{H}\ , \text{ given by } e_\xi : A \mapsto A\xi$$

are continuous. This topology is called strong, since it is stronger than the weak operator topology, which is the weakest such that all $e_{\zeta,\xi} : A \mapsto \langle \zeta, A\xi \rangle$ are continuous. The strong operator topology is weaker than the norm topology, which is induced by the operator norm.

As before, $\langle \psi, P\psi \rangle \equiv \mathrm{d}(P, \psi)$ is a well-defined Borel measure for every $\psi \in \mathscr{H}$ on $\mathbb{R}$. With respect to this the following proposition holds [17, VII.7 & VIII.6].

**Proposition 2.4.** [Borel functional calculus]
There is a one-to-one correspondence between self-adjoint operators $A$ and projection-valued measures $\{P_\Omega\}$ on $\mathscr{H}$. If $f$ is a Borel function on the support of $P_\Omega$, there is a unique operator $B$, denoted by $\int \mathrm{d}P_\lambda\, f(\lambda)$, so that

$$\langle \psi, B\psi \rangle = \int \mathrm{d}(P, \psi)\, f(\lambda) \quad , \forall \psi \in \mathscr{H} \quad .$$

Then the correspondence between $A$ and $P_\Omega$ is given by $A = \int_{-\infty}^{\infty} \mathrm{d}P_\lambda \, \lambda$. If $f$ is real-valued on $\mathbb{R}$, then $f(A) = \int_{-\infty}^{\infty} \mathrm{d}P_\lambda \, f(\lambda)$ is self-adjoint on

$$D_f = \{\psi \in \mathscr{H} | \int_{-\infty}^{\infty} \mathrm{d}(P, \psi) \, |f(\lambda)|^2 < \infty\} \quad .$$

If $f$ is furthermore bounded, then so is $f(A) \in \mathscr{B}(\mathscr{H})$. Finally, the spectral mapping property holds: $\sigma(f(A)) = f(\sigma(A)) = \{f(\lambda) | \lambda \in \sigma(A)\}$.

Proposition 2.4 makes clear what is meant by $e^{-\lambda(\Delta - E)}$ from equation (2.4). Furthermore, it says that it is self-adjoint and bounded as well. In fact, it is even a compact operator.

**Definition 2.10.** A set $U \subset \mathscr{H}$ is called precompact if its closure $\overline{U}$ is compact. An operator $K \in \mathscr{B}(\mathscr{H})$ is called compact if it maps any bounded set $X$ into a precompact set $U$.

A direct consequence of the Bolzano-Weierstrass theorem is the following lemma.

**Lemma 2.9.** A compact operator $K \in \mathscr{B}(\mathscr{H})$ is compact if and only if for every bounded sequence $\{\xi_n\}$ there is a convergent subsequence $\{K\xi_{n_i}\}$.

In finite dimensional vector spaces every bounded set is precompact and every bounded operator compact. So it is immediately clear that operators of finite rank are compact. A couple of propositions describing all of compact operator theory that we need follows now. The proofs can be found in [17, §VI.5].

**Lemma 2.10.** Every compact operator $B$ on a separable Hilbert space $\mathscr{H}$ is the norm limit of a sequence of operators of finite rank.

**Proposition 2.5.** [Riesz-Schauder theorem]
A compact operator $K$ on $\mathscr{H}$ has a discrete spectrum $\sigma(K)$ with only possible limit point $\lambda = 0$. Any non-zero $\lambda \in \sigma(K)$ is an eigenvalue of finite multiplicity.

**Proposition 2.6.** [Hilbert-Schmidt theorem]
For a self-adjoint compact operator $K$ on $\mathscr{H}$ there is a orthonormal basis $\{\zeta_n\}$ so that $K\zeta_n = \lambda_n \zeta_n$, where the singular values $\lambda_n \to 0$.

For any $B \in \mathscr{B}(\mathscr{H})$ the operator $B^*B$ is self-adjoint and positive and by the operator calculus $T = \sqrt{B^*B}$ is therefore sensible.

**Definition 2.11.** [Schatten norm]
The $p$-th Schatten norm of a compact operator $K$ is given by

$$\|K\|_p = \left( \sum_n \langle \zeta_n, T^p \zeta_n \rangle \right)^{\frac{1}{p}} \quad , \quad T = \sqrt{K^*K}$$

and $\{\zeta_n\}$ is the orthonormal basis relative to which $T$ is diagonalized.

The bounded operators with finite $p$-th Schatten norm form the $p$-th Schatten ideal $\mathscr{I}_p$. For $p = 2$, these are called Hilbert-Schmidt operators. For $p = 1$ they are called trace-class operators. It are precisely these operators, where the trace

$$\mathrm{Tr}\,(B) = \sum_n \langle \zeta_n, B\zeta_n \rangle < \infty$$

and is basis independent.

### Operators with integral kernels

The following two lemmas and Proposition 2.7 show that there is a close connection between trace-class and Hilbert-Schmidt operators and integral kernels. They are not necessary to understand equation (2.1), so the reader in a hurry can jump to Lemma 2.13.

**Lemma 2.11.** All trace-class operators are Hilbert-Schmidt, $\mathscr{I}_1 \subset \mathscr{I}_2$.

*Proof.* Since $\mathscr{I}_1$ is an ideal, $B^*B \in \mathscr{I}_1$ is a positive self-adjoint positive operator and

$$\mathrm{Tr}(B^*B) = \mathrm{Tr}\left(\sqrt{(B^*B)^2}\right) < \infty \quad .$$

□

**Lemma 2.12.** If $B \in \mathscr{I}_1$ or $B \in \mathscr{I}_2$, then $B$ is compact.

*Proof.* From Lemma 2.11 we know that $B \in \mathscr{I}_2$, then

$$\mathrm{Tr}\,(B^*B) = \sum_{n=1}^{\infty} \|B\zeta_n\|^2 < \infty$$

for any orthonormal basis $\{\zeta_n\}$. We approximate $B$ by finite rank operators. If $Q_N = \{\psi \in \mathscr{H} | \|\psi\| = 1 \text{ and } \langle\psi, \zeta_n\rangle = 0, 1 \leq n \leq N\}$ the operator norm of the remainder can be estimated by

$$\sup_{\psi\in Q_N} \|B\psi\|^2 \leq \mathrm{Tr}\,(B^*B) - \sum_{n=1}^{N} \|B\zeta_n\|^2 \to 0\,, \quad N \to \infty \quad .$$

So $B$ is compact. □

**Proposition 2.7.** If $\mathscr{H} = L^2(M)$, then $B \in \mathscr{B}(\mathscr{H})$ is Hilbert-Schmidt if and only if there exists a kernel $k \in L^2(M \times M)$ such that

$$B\psi(x) = \int \mathrm{d}y\, k(x,y)\psi(y) \quad .$$

Then $\mathrm{Tr}\,(B^*B) = \int \mathrm{d}x\;\mathrm{d}y\,|k(x,y)|^2$.

*Proof.* For a kernel $k$ the associated operator $B_k$ is defined by the $L^2(M)$-inner product. From

$$\|B_k\|^2 = \sup_{\|\psi\|=1} \|B_k\psi\|^2 \leq \|k\|^2_{L^2(M\times M)} < \infty$$

it is clear that $B_k$ is bounded.
If $\{\zeta_n\}$ is an orthonormal basis for $L^2(M)$, then $\{\overline{\zeta_n}\zeta_m\}$ is one for $L^2(M \times M)$. Then we can decompose

$$k = \sum_{n,m=1}^{\infty} b_{n,m}\overline{\zeta_n}\zeta_m \quad . \tag{2.7}$$

Calculating

$$\begin{aligned}\mathrm{Tr}\,(B_k^*B_k) &= \sum_{n=1}^{\infty} \|B_k\zeta_n\|^2 = \sum_{n,m=1}^{\infty} |b_{n,m}|^2 = \langle \sum_{n,m=1}^{\infty} b_{n,m}\overline{\zeta_n}\zeta_m, \sum_{n,m=1}^{\infty} b_{n,m}\overline{\zeta_n}\zeta_m\rangle \\ &= \|k\|_{L^2(M\times M)} < \infty \quad ,\end{aligned}$$

which proves that $B_k \in \mathscr{I}_2$. Defining the inner product on $\mathscr{I}_2$ by $\langle A, B\rangle_{\mathscr{I}_2} = \mathrm{Tr}\,(A^*B)$ we see that $k \mapsto B_k$ is an isometry and its range is closed. From the decomposition above it follows that the operators of finite rank come from kernels and lie dense in $\mathscr{I}_2$. This proves that $k \mapsto B_k$ covers $\mathscr{I}_2$. □

**Lemma 2.13.** Let $k_B \in L^2(M \times M)$ denote the kernel for $B \in \mathscr{I}_1$. Then

$$\mathrm{Tr}\,(B) = \int_M \mathrm{d}x\, k_B(x,x) \quad .$$

*Proof.* From Lemma 2.11 we know that $B \in \mathscr{I}_2$. For a orthonormal basis $\{\zeta_n\}$ of $L^2(M)$ we decompose as in equation (2.7)

$$k_B = \sum_{n,m=1}^{\infty} b_{n,m}\overline{\zeta_n}\zeta_m \quad .$$

Now, the statement follows immediately from

$$\mathrm{Tr}\,(B) = \sum_{n=1}^{\infty}\langle\zeta_n, B\zeta_n\rangle = \sum_{n=1}^{\infty} b_{n,n} = \int_M \mathrm{d}x\, k_B(x,x) \quad .$$

□

**Lemma 2.14.** The negative exponent of the Laplacian is trace-class, $e^{-\lambda(\Delta - E)} \in \mathscr{I}_1$ $\lambda \in \mathbb{R}_+$.

*Proof.* We first make the estimate

$$\mathrm{Tr}\left(e^{-\lambda(\Delta-E)}\right) \leq e^{\lambda\|E\|}\mathrm{Tr}\left(e^{-\lambda\Delta}\right) \quad .$$

In Lemma 2.19 we will show that $e^{-\lambda\Delta}$ has a heat kernel. Since this operator is self-adjoint and positive, calculating the trace will prove that it is trace-class. We will do this in equation (2.16). □

This provides us with a meaning of the trace in equation (2.4). For more on this subject we refer to [19]. Before we continue with the expansion of this, we first take a small detour into non-commutative geometry to see where it comes from.

## 2.3 The spectral triple

Before we start working with the spectral action (2.1), it is convenient to have an idea where it is coming from. It is derived from the spectral triple, which is a central object in non-commutative geometry [20], [21]. We will give here a minimal introduction to particle physics from non-commutative geometry [22]. Only the concepts needed for this work will be mentioned, without technicalities and generality. All details making this description possible can be found in many textbooks, like [20], [23] and [24].

A unital algebra is an algebra containing the identity. We denote the unit with $I$, where a possible subscript may indicate the algebra. By the Gelfand-Naimark theorem there is an isometry between unital commutative $C^*$-algebras and compact Hausdorff spaces. This duality is called Gelfand-duality. It can be generalized to non-unital $C^*$-algebras, giving non-compact Hausdorff spaces. This means that the space can be reconstructed from the commutative $C^*$-algebra. Supposing this stretches to non-commutative $C^*$-algebras, its dual is called a non-commutative topology. It turns out that this duality exists for Riemannian spin geometries and in that spectral triples are its generalization.

**Definition 2.12.** [Spectral triple]
A spectral triple $(\mathscr{A}, \mathscr{H}, D)$ is given by an involutive algebra $\mathscr{A}$ faithfully represented $\mathscr{A} \to \mathscr{B}(\mathscr{H})$ on a Hilbert space $\mathscr{H}$ with a self-adjoint Dirac operator $D$ such that

- the resolvent $(D - I_{\mathscr{A}}\lambda)^{-1}$, $\lambda \notin \sigma(D)$, is a compact operator on $\mathscr{H}$ and
- $[D, a] \in \mathscr{B}(\mathscr{H}) \quad \forall a \in \mathscr{A}$.

The spectral triple is even if there exists a $\mathbb{Z}_2$-grading operator $\gamma$ on $\mathscr{H}$ such that

$$\gamma = \gamma^* \, ; \quad \gamma^2 = 1 \, ; \quad [\gamma, a] = \gamma a - a\gamma = 0 \; \forall a \in \mathscr{A} ; \quad \{\gamma, D\} = \gamma D + D\gamma = 0 \, .$$

Otherwise, it is called odd.
The triple is called real when there is an antilinear isomorphism $J$ on $\mathscr{H}$ such that

$$J^2 = \varepsilon \, ; \quad JD = \varepsilon' DJ \, ; \quad J\gamma = \varepsilon'' \gamma J \, ; \quad [a, JbJ^{-1}] = 0 \, ; \quad [[D, a], JbJ^{-1}] = 0 \, ,$$

for any $a, b \in \mathscr{A}$. The third condition is only applicable for even spectral triples. The signs $\varepsilon, \varepsilon'$ and $\varepsilon''$ determine then the KO-dimension modulo 8, given in the Table 2.1.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon$ | 1 | 1 | −1 | −1 | −1 | −1 | 1 | 1 |
| $\varepsilon'$ | 1 | −1 | 1 | 1 | 1 | −1 | 1 | 1 |
| $\varepsilon''$ | 1 | | −1 | | 1 | | −1 | |

Table 2.1: The KO-dimension of a spectral triple.

*Example* 2.1. [Canonical spectral triple]
The canonical spectral triple is given by

$$\left(C^\infty(M), L^2(M, S), \not{D}\right) \quad ,$$

where $\not{D}$ is the Dirac operator and the smooth functions $C^\infty$ act by pointwise multiplication on the spinors. The spinors are a Hilbert space with inner product

$$\langle \psi, \psi' \rangle = \int_M \mathrm{d}x \, \langle \psi(x), \psi'(x) \rangle_{S_x} \quad .$$

If the dimension of the manifold is even, then so is the spectral triple. In four dimensions the grading operator is $\gamma_5$.
That $[\not{D}, a]$ is bounded follows from $[\not{D}, a]\zeta = i\gamma_\mu(\partial^\mu a)\zeta$. Checking that the Dirac operator has compact resolvent is more involved [23, §7.A]. The resolvent can be seen as a pseudo-differential operator mapping into a Sobolev space. This guarantees that it is indeed the inverse of the shifted Laplacian. Rellich's theorem states that this operation and hence the inclusion of the Sobolev space into the Hilbert space is compact.

Another special case is the finite spectral triple $(\mathscr{A}_F, \mathscr{H}_F, D_F)$. The Dirac operator is then a matrix on the Hilbert space, so the commutator with an algebra element is automatically bounded and the resolvent is finite dimensional and therefore compact. As in the more general case they can have an even and real structure as well.
They are called finite because they have a finite dimensional Hilbert space. The possibility to combine the canonical triple, on which Einstein's general relativity theory works, with a finite triple suitable for gauge theory makes this description very interesting from a physical point of view.

*Example* 2.2. A simple example is the one-point space with real structure. The continuous functions on it are given by $\mathbb{C}$, which is naturally represented on the Hilbert space $\mathbb{C}$. Since the Hilbert space is one dimensional there is no non-trivial grading possible and the trivial grading $\gamma_F = 1$ would imply $D_F = 0$. A natural choice for the real structure is given by complex conjugation. This means that $J^2 = 1$ and the triple must have KO-dimension 1 or 7. From $[D, \lambda] \in \mathbb{C}$ for any $\lambda \in \mathbb{C}$ it is clear that $D$ is itself just a complex number. In KO-dimension 7 $[D, J] = 0$ which requires that $D$ is real. As spectral triple we obtain $(\mathbb{C}, \mathbb{C}, D_F, J)$.
In KO-dimension 1 the condition $\{D, J\} = 0$ forces $D$ to be purely imaginary. Since it must self-adjoint as well, it must be zero. So the triple is then given by $(\mathbb{C}, \mathbb{C}, 0, J)$.

**Definition 2.13.** [Almost commutative manifold]
The product $M \times F$ of an even canonical and an even finite spectral triple $F$ gives an almost commutative manifold

$$\left(C^\infty(M, \mathscr{A}_F), L^2(M, S) \otimes \mathscr{H}_F, \not{D} \otimes 1 + \gamma \otimes D_F, \gamma \otimes \gamma_F)\right) \quad .$$

The operator $J$ is closely related to the charge conjugation operator in particle physics. Since we are concerned with real scalar fields, we can omit it. From Definition 2.13 we see that the product of $(C^\infty(M), L^2(M, S), \not{D}, \gamma_5)$ and $(\mathbb{C}, \mathbb{C}, D_F)$ from example 2.2 has a Dirac operator that resembles the $\not{D} + \gamma_5 \varphi$ from equation (2.1), if $D_F$ would be equal to $\varphi$. However, the finite Dirac operator is just an element of $\mathbb{C}$, while $\varphi$ is a smooth field on the manifold. So in this way, we would only obtain constant trivial fields.
In the setting of particle physics, the finite Hilbert space is usually some representation space of a symmetry group. Typically, the action would be invariant under the unitary group, or a subgroup of it. A possible choice for this is the trace of an operator. This operator should of course be trace-class. Combined with the simple formula for the trace from Lemma 2.13 this motivates our choice, see Lemma 2.14. The symmetry of the action appears as equivalence of spectral triples.

**Definition 2.14.** [Equivalence of spectral triples]
Two spectral triples $(\mathscr{A}_i, \mathscr{H}_i, D_i)$ with representations $\pi_i : \mathscr{A}_i \to \mathscr{B}(\mathscr{H}_i)$ are unitarily equivalent if an intertwining operator $U : \mathscr{H}_1 \to \mathscr{H}_2$ and an isomorphism $\alpha : \mathscr{A}_1 \to \mathscr{A}_2$ exists, such that

$$UD_1U^* = D_2\,; \quad U\pi_1(a)U^* = \pi_2(\alpha(a)) \quad \forall a \in \mathscr{A}_1 \quad .$$

Moreover, for even triples $U\gamma_1 U^* = \gamma_2$ and for real triples $UJ_1U^* = J_2$ are required.

**Lemma 2.15.** The even spectral triple $(\mathscr{A}, \mathscr{H}, D, \gamma)$ with $\pi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ is equivalent to $(\mathscr{A}, \mathscr{H}, UDU^*, \gamma)$ with representation $\pi \circ \alpha_u$, where $U = \pi(u)$ for $u \in U(\mathscr{A})$ and $\alpha_u(a) = uau^*$ for $a \in \mathscr{A}$.

*Proof.* We check that $U\pi(a)U^* = \pi(u)\pi(a)\pi(u^*) = \pi(uau^*) = \pi \circ \alpha_u(a)$. From Definition 2.12 we know that $[\gamma, u] = 0$, so that it is clear that $U\gamma U^* = \gamma$. The condition on the Dirac operator is trivially satisfied. □

So the equivalent triples are given by the intertwining operators $\pi(U(\mathscr{A}))$. The gauge group is the non-trivial subgroup of this $\mathscr{G}(\mathscr{A}) = \pi(U(\mathscr{A}))/\mathrm{Ker}(\pi)$. The origin of this equivalence is the Morita equivalence of the algebra $\mathscr{A}$ with another algebra. The definition of a spectral triple then leads to an altered Dirac operator, called an inner deformation. In the case of an even spectral triple without a real structure this is given by [21, §10.8]

$$D \mapsto D + A = D_A \quad ,$$

where the self-adjoint $A$ is of the form $\sum_j a_j[D,b_j]$ with $a_j, b_j \in \mathscr{A}$. In the context of particle physics it are these inner fluctuations that give rise to the gauge fields, although usually there is a real structure in such cases. The transformation of these inner fluctuations is given by

$$UD_AU^* = uDu^* + uAu^* = D + uAu^* + u[D,u^*] \quad ,$$

which is well-known as the transformation of the covariant derivative in gauge theory.

*Example* 2.3. An even spectral triple of the diagonal matrix algebra $\mathbb{R}^2$ represented on the finite dimensional Hilbert space $\mathbb{C}^2$ has to satisfy the following conditions. For the even structure

$$\gamma_F = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$$

the commutator $[\gamma_F, a] = 0$ , for any $a \in \mathscr{A}$, so that $g_2 = g_3 = 0$. Furthermore, it must be Hermitean, so that $g_1$ and $g_4$ are real. Since it's square must be $I_2$ we obtain $g_1, g_4 \in \{\pm 1\}$. It is a diagonal matrix of eigenvalues. Finally, it must be a grading and have eigenvalues $+1$ and $-1$ this fixes $g_1 = -g_4$. We choose $g_1 = 1$.
For the finite Dirac operator

$$D_F = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}$$

the anticommutator $\{\gamma_F, D_F\} = 0$, which requires $d_1 = d_4 = 0$. It is only self-adjoint if $d_2 = \overline{d_3}$, leaving one degree of freedom.
There are four unitary elements in $\mathscr{A}$, given by $(\pm 1, \pm 1) \in \mathbb{R}^2$. Only $\pi\left((1,1)\right) = I_2$, so that the gauge group is given by $\mathbb{Z}_2 \times \mathbb{Z}_2$.

*Example* 2.4. We consider the almost commutative manifold given by the product of the four dimensional canonical spectral triple with the finite spectral triple from example 2.3, which according to Definition 2.13 is given by

$$\left(C^\infty(M,\mathbb{R}^2), L^2(M,S)\otimes\mathbb{C}^2, \not{D}\otimes I + \gamma_5\otimes D_F, \gamma_5\otimes\gamma_F\right) \quad .$$

The algebra appears as two separate copies, $C^\infty(M,\mathbb{R}^2) \simeq C^\infty(M) \oplus C^\infty(M) \simeq C^\infty(M)\otimes\mathbb{R}^2$. This can be seen from the explicit decomposition

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & a_2 \end{pmatrix} = a_1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_2 \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad .$$

The inner fluctuations of this triple are composed by elements of the form

$$\begin{aligned} a[D,b] &= a[\not{D}\otimes I_2 + \gamma_5\otimes D_F, b] = a[\not{D}\otimes I_2, b] + a[\gamma_5\otimes D_F, b] \\ &= a\left(i\gamma_\mu\partial^\mu b + \gamma_5(b_2 - b_1)\gamma_F D_F\right) \quad , \end{aligned} \tag{2.8}$$

where we have denoted

$$b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \quad , \quad b_1, b_2 \in C^\infty(M) \quad .$$

Choosing $b$ constant the first tern of equation (2.8) vanishes. For suitable $a$ the inner fluctuations are then given by

$$A = \gamma_5\varphi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \varphi \in C^\infty(M,\mathbb{R}) \quad . \tag{2.9}$$

The unitary elements of $C^\infty(M, \mathbb{R}^2)$ are given by the functions $M \to \mathbb{R}^2$ mapping to the unitary matrices, which are $x \mapsto (\pm 1, \pm 1)$. All unitary elements are constant functions, so we obtain the same gauge group $\mathbb{Z}_2 \times \mathbb{Z}_2$ as in example 2.3.

*Remark* 2.4. The inner fluctuation (2.9) of the almost commutative manifold from example 2.4 gives a fluctuated Dirac operator

$$D_A = \not{D} \otimes I_2 + \gamma_5 \varphi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \tag{2.10}$$

which is not the same as the operator from equation (2.1). However, for the spectral action (2.4) used not the operator, but its square $(\not{D} + \gamma_5\varphi)^2 = \Delta - E$ is relevant. The square of the fluctuated Dirac operator (2.10) is

$$D_A^2 = (\Delta - E) \otimes I_2 \quad ,$$

where $E$ is the same as in Lemma 3.3. From paragraph 2.4 and in particular Proposition 2.8 it will become clear that the heat expansion of both operators is the same, except an overall factor two from the trace over $I_2$. This factor will be absorbed in the coupling constants $f_m$ from equation (2.3). For notational simplicity we will continue as if the fluctuated Dirac operator was given by

$$D_A = \not{D} + \gamma_5 \varphi \quad ,$$

instead of equation (2.10) and focus on the heat expansion.

## 2.4 Asymptotic expansion

In this paragraph we will demonstrate how the spectral action can be expanded asymptotically. More information on this and more general cases are discussed in [25]. To do so, we proceed as follows. From Lemma 3.3 we write $(\not{D} + \gamma_5\varphi)^2 = \Delta - E$, where $E$ maps $x \in M$ smoothly to an endomorphism of $S$, i.e. $E \in C^\infty(End(S))$ and $\Delta = \not{D}^2 = -\partial_\mu \partial^\mu$ the Laplacian on the manifold. Furthermore, we write $\lambda = \frac{t}{\Lambda^2}$.

**Lemma 2.16.** [Duhamel]

$$e^{-\lambda(\Delta - E)} = e^{-\lambda\Delta} + \int_0^\lambda \mathrm{d}\lambda_1 \, e^{-(\lambda-\lambda_1)(\Delta-E)} E e^{-\lambda_1 \Delta} \tag{2.11}$$

*Proof.* $e^{-\lambda(\Delta-E)}u(0)$ is the unique solution of the Cauchy problem

$$\begin{cases} (\frac{\mathrm{d}}{\mathrm{d}\lambda} + \Delta - E)u(\lambda) = 0 \\ u(0) = 1 \end{cases}$$

We check that $e^{-\lambda\Delta} + \int_0^\lambda \mathrm{d}\lambda_1 \, e^{-(\lambda-\lambda_1)(\Delta-E)} E e^{-\lambda_1\Delta}$ solves this problem as well:

$$\begin{aligned} &(\frac{d}{d\lambda} + \Delta - E)(e^{-\lambda\Delta} + \int_0^\lambda \mathrm{d}\lambda_1 \, e^{-(\lambda-\lambda_1)(\Delta-E)} E e^{-\lambda_1\Delta}) \\ &= -Ee^{-\lambda\Delta} + (\Delta - E) \int_0^\lambda \mathrm{d}\lambda_1 \, e^{-(\lambda-\lambda_1)(\Delta-E)} E e^{-\lambda_1\Delta} + Ee^{-\lambda\Delta} \\ &- (\Delta - E) \int_0^\lambda \mathrm{d}\lambda_1 \, e^{-(\lambda-\lambda_1)(\Delta-E)} E e^{-\lambda_1\Delta} = 0 \quad . \end{aligned}$$

□

Applying the expansion (2.11) to itself yields the following lemma.

**Lemma 2.17.** If Lemma 2.16 is an expansion with $n = 1$, we can expand for $n \in \mathbb{N}$

$$e^{-\lambda(\Delta-E)} = \sum_{m=0}^{n-1} \int_0^\lambda \mathrm{d}\lambda_1 \dots \int_0^{\lambda-\sum_{j=1}^{m-1}\lambda_j} \mathrm{d}\lambda_m e^{-(\lambda-\sum_{j=1}^{m}\lambda_j)\Delta} E e^{-\lambda_m\Delta} E \dots E e^{-\lambda_1\Delta}$$
$$+ \int_0^\lambda \mathrm{d}\lambda_1 \dots \int_0^{\lambda-\sum_{j=1}^{n-1}\lambda_j} \mathrm{d}\lambda_n e^{-(\lambda-\sum_{j=1}^{n}\lambda_j)(\Delta-E)} E e^{-\lambda_n\Delta} E \dots E e^{-\lambda_1\Delta} \quad . \tag{2.12}$$

*Proof.* Apply the expansion from Lemma 2.16 to $e^{-(\lambda-\lambda_1)(\Delta-E)}$ in equation (2.11):

$$\begin{aligned} e^{-\lambda(\Delta-E)} &= e^{-\lambda\Delta} + \int_0^\lambda \mathrm{d}\lambda_1 \Big( e^{-(\lambda-\lambda_1)\Delta} \\ &+ \int_0^{(\lambda-\lambda_1)} \mathrm{d}\lambda_2\, e^{-(\lambda-\lambda_1-\lambda_2)(\Delta-E)} E e^{-\lambda_2\Delta} \Big) E e^{-\lambda_1\Delta} \\ &= e^{-\lambda\Delta} + \int_0^\lambda \mathrm{d}\lambda_1\, e^{-(\lambda-\lambda_1)\Delta} E e^{-\lambda_1\Delta} \\ &+ \int_0^\lambda \mathrm{d}\lambda_1 \int_0^{(\lambda-\lambda_1)} \mathrm{d}\lambda_2\, e^{-(\lambda-\lambda_1-\lambda_2)(\Delta-E)} E e^{-\lambda_2\Delta} E e^{-\lambda_1\Delta} \quad . \end{aligned}$$

This completes the proof for $n$=2. It is now straightforward that this process can be iterated $n-2$ times, which proves our claim. ☐

We make a small change in notation now.

**Definition 2.15.** For an expansion in $n$ terms we define $\lambda_i = s_i\lambda$, where $0 \leq s_i \leq 1$ for all $1 \leq i \leq n$ and $\sum_{i=1}^n \leq 1$. We take $s_0^{[n]} = 1-\sum_{j=1}^n s_j$, so that $s_0^{[n]}+\sum_{i=1}^n s_i = 1$.

*Remark* 2.5. The in Definition 2.15 defined $s_0^{[n]}$ depends on the variables $s_1, \dots, s_n$. For notational convenience we will omit the $[n]$, since the specific $n$ will be clear in all cases.

*Remark* 2.6. The integrals we get by substituting $\lambda_i$ for $s_i\lambda$ in equation (2.12) are of the form $\lambda^m \int_0^1 \mathrm{d}s_1 \int_0^{1-s_1} \mathrm{d}s_2 \dots \int_0^{1-\sum_{j=1}^{m-1}s_j} \mathrm{d}s_m = \lambda^m \int_{\Delta_m} \mathrm{d}^m s$, as it is just the integral over the $m$-simplex, see Definition 1.17.

We are interested in the trace of the exponential $e^{-\lambda(\Delta-E)}$, so we introduce the following notation.

**Definition 2.16.** For $E_i \in C^\infty(End(S))$ we define

$$\begin{aligned} \langle E_1, E_2, \dots, E_m \rangle_\lambda &\equiv \int_{\Delta_m} \mathrm{d}^m s \,\mathrm{Tr}\left(e^{-s_0\lambda\Delta} E_1 e^{-s_m\lambda\Delta} E_2 \dots e^{-s_2\lambda\Delta} E_m e^{-s_1\lambda\Delta}\right) \\ &= \int_{\Delta_m} \mathrm{d}^m s \,\mathrm{Tr}\left(e^{-s_m\lambda\Delta} E_1 e^{-s_{m-1}\lambda\Delta} E_2 \dots e^{-s_1\lambda\Delta} E_m e^{-s_0\lambda\Delta}\right) \quad . \end{aligned}$$

*Remark* 2.7. Since the notation $\langle E, E, \dots, E\rangle_\lambda$ does not show the number of $E$'s, we can use a superscript to make clear there are $m$ $E$'s: $\langle E, E, \dots, E\rangle_\lambda^{(m)}$.

**Definition 2.17.** [Asymptotic expansion]
The sum $\sum_{m=0}^\infty a_m x^m$ as $x \to L$ is an asymptotic expansion of $f(x)$, denoted with $f(x) \sim \sum_{m=0}^\infty a_m x^m$, $x \to L$, if for any $n \in \mathbb{Z}_{\geq 0}$

$$\lim_{x\to L} x^{-n}\left(f(x) - \sum_{m=0}^n a_m x^m\right) = 0 \quad .$$

*Remark* 2.8. This definition tells us that for a fixed $n$ the difference between the asymptotic expansion and the function $f(x)$ gets smaller as $x \to L$. For higher $n$ this goes faster. However, there is no guarantee that $\lim_{n\to\infty} \sum_{m=0}^{n} a_m x^m$ approaches $f(x)$ or even converges.

By the next lemma an asymptotic expansion is obtained. More general versions of the estimates can be found in [26] and [27].

**Lemma 2.18.** There is the asymptotic expansion as $\lambda \to 0$:

$$\mathrm{Tr}\left(e^{-\lambda(\Delta - E)}\right) \sim \sum_{m=0}^{\infty} \lambda^m \langle E, E, \dots, E\rangle_\lambda^{(m)} \quad .$$

*Proof.* We want to approximate $\mathrm{Tr}\left(e^{-\lambda\Delta}\right)$ with $n+1$ terms, where $n \geq 2$. This is done by using Lemma 2.17. The difference with the claimed asymptotic expansion can be estimated by

$$\begin{aligned}
&\left|\mathrm{Tr}\left(e^{-\lambda(\Delta - E)}\right) - \sum_{m=0}^{n} \lambda^m \langle E, E, \dots, E\rangle_\lambda^{(m)}\right| \\
&\quad = \left|\lambda^{n+1} \int_{\Delta_{n+1}} \mathrm{d}^{n+1}s\, \mathrm{Tr}\left(e^{-s_0\lambda(\Delta - E)} E e^{-s_{n+1}\lambda\Delta} E \dots E e^{-s_1\lambda\Delta}\right)\right| && (2.13) \\
&\quad \leq \left|\lambda^{n+1} \frac{\|E\|^{n+1}}{(n+1)!} \mathrm{Tr}\left(e^{-\frac{\lambda}{2}\Delta}\right)\right| && (2.14) \\
&\quad \leq \left|\frac{4\lambda^{n-1}}{(4\pi)^2} \frac{\|E\|^{n+1}}{(n+1)!} \int_M \mathrm{Tr}_{S_x}(I)\right| \quad .
\end{aligned}$$

The estimate of equation (2.13) that yields (2.14) is found from Hölder's inequality $|\mathrm{Tr}\,(T_0 \dots T_j)\,| \leq \|T_0\|_{s_0^{-1}} \dots \|T_j\|_{s_j^{-1}}$ with $\sum_{i=0}^{j} s_i = 1$. These norms are Schatten norms, see Definition 2.11. Using furthermore the estimate

$$\|E e^{-s_i t\Delta}\|_{s_i^{-1}} \leq \|E\| \left(\mathrm{Tr}\left(e^{-\frac{t}{2}\Delta}\right)\right)^{s_i}$$

all terms are handled, except $\mathrm{Tr}\left(e^{-\lambda(\Delta-E)}\right)$. For two positive self-adjoint operators $A$ and $B$, $\mathrm{Tr}\left(e^{-A-B}\right) \leq \mathrm{Tr}\left(e^{-A}\right)$. Since $\Delta$ and $(\|E\| - E)$ are both positive, it is clear that

$$\mathrm{Tr}\left(e^{-\lambda(\Delta-E)}\right) = \mathrm{Tr}\left(e^{-\lambda\Delta - \lambda(\|E\| - E)} e^{\lambda\|E\|}\right) \leq e^{\lambda\|E\|}\, \mathrm{Tr}\left(e^{-\lambda\Delta}\right) \quad .$$

The exponent $e^{\lambda\|E\|} = 1 + \mathcal{O}(\lambda)$. Since the simplex coordinates $\sum_{i=0}^{n+1} s_i = 1$ in the exponent of $\mathrm{Tr}\left(e^{-\frac{1}{2}\lambda\Delta}\right)$ only the integral over these simplex coordinates remain, yielding $\int_{\Delta_{n+1}} \mathrm{d}^{n+1}s = \frac{1}{(n+1)!}$.
To obtain the last line from equation (2.14) we have used that we can actually calculate $\mathrm{Tr}\left(e^{-\frac{\lambda}{2}\Delta}\right)$, see paragraph 2.5.1. This completes the proof. □

Now we can state the key proposition.

**Proposition 2.8.** The action (2.1) can be asymptotically expanded for $\Lambda \to \infty$ as

$$S[\varphi] \sim \sum_{m=0}^{\infty} \Lambda^{-2m} \int_{t>0} \mathrm{d}t\, g(t) t^m \langle E, E, \dots, E\rangle_t^{(m)} \qquad (2.15)$$

*Proof.* We want to proof that the difference between the action and the asymptotic expansion of order $N$ vanishes as $\Lambda^{-2N+2}$ as $\Lambda \to \infty$.

$$\begin{aligned}
&\left| S[\varphi] - \sum_{m=0}^{N} \Lambda^{-2m} \int_{t>0} \mathrm{d}t\, g(t) t^m \langle E, E, \ldots, E\rangle_t^{(m)} \right| \\
&\quad = \left| \int_{t>0} \mathrm{d}t\, g(t) \left( \mathrm{Tr}\left(e^{-t\Lambda^{-2}(\Delta - E)}\right) - \sum_{m=0}^{N} \frac{t^m}{\Lambda^{2m}} \langle E, E, \ldots, E\rangle_t^{(m)} \right) \right| \\
&\quad \leq \Lambda^{-2N+2} \int_{t>0} \mathrm{d}t\, |g(t)|\, t^{N-1} \cdot \left| 4 \frac{\|E\|^{N+1}}{(N+1)!} \int_M \mathrm{Tr}_{S_x}(I) \right| \quad .
\end{aligned}$$

The integral over $M$ is finite, since the manifold is compact. From equation (2.2) in paragraph 2.1 we see that the integral over $t$ is finite. This concludes the proof. Notice that we have changed our asymptotic expansion parameter from $\lambda \to 0$ to $\Lambda \to \infty$ by substituting $\lambda = t\Lambda^{-2}$. ☐

*Remark* 2.9. This asymptotic expansion is the origin of the regulator $\Lambda - \Lambda_0 \to \infty$ as we required in paragraph 1.4.2.

*Remark* 2.10. In a different point of view we could have chosen the moments of $f$ to be zero for $n \geq N$.

*Remark* 2.11. Comparing equations (2.15) and (2.3), we see that $a_m(x, D_\varphi^2)$ are determined by computing the traces $\langle E, \ldots, E\rangle_t^{(n\leq m)}$.

**Definition 2.18.** For an asymptotic expansion of $f(x) \sim \sum_{m=0}^{\infty} a_m x^m$ in a limit $x \to L$ the $n$-th order of $f$ is given by

$$f(x) \overset{\mathcal{O}(L^{-n})}{\sim} a_n x^n \quad .$$

*Example* 2.5. For $m = 2$ the formula (2.15) has the trace

$$\Lambda^{-4} \mathrm{Tr}\left(e^{-s_2 t\Lambda^{-2}\Delta} E e^{-s_1 t\Lambda^{-2}\Delta} E e^{-s_0 t\Lambda^{-2}\Delta}\right) \quad .$$

If we would naively expand all exponentials, we would see that the lowest order in occurring is 4, when none of the derivatives acts on any $E$. Therefore, if we look for the terms of order 4, we can say that the above trace contains a term $\mathrm{Tr}\left(E^2 e^{-t\Lambda^{-2}\Delta}\right)$.

$$\Lambda^{-4} \mathrm{Tr}\left(e^{-s_2 t\Lambda^{-2}\Delta} E e^{-s_1 t\Lambda^{-2}\Delta} E e^{-s_0 t\Lambda^{-2}\Delta}\right) \overset{\mathcal{O}(\Lambda^{-4})}{\sim} \Lambda^{-4} \mathrm{Tr}\left(E^2 e^{-t\Lambda^{-2}\Delta}\right)$$

In the next section we want to calculate the coefficients $a_m$ for $m \leq 8$. Therefore, we need a way to evaluate the trace. However, already half of the work is done.

*Remark* 2.12. Comparing equations (2.15) and (2.3) we directly see that we will only get non-zero $a_m$ for $m$ even, since both $E$ and $\Delta$ are accompanied by $\Lambda^{-2}$.

## Heat equation

In the previous paragraph we have shown how the spectral action is expanded. To obtain the final action, see equation (2.3), we have to calculate the traces. We will do this by means of the heat kernel. Locally, on a Euclidean space, we know what the heat kernel looks like. In this paragraph we derive some results to be used in the calculations of the traces in the next section.

**Lemma 2.19.** On a flat 4-dimensional manifold $M$ for continuous $\psi_0(x)$ the solution of the initial value problem

$$\begin{cases} (\frac{\mathrm{d}}{\mathrm{d}t}+\Delta)\psi(x,t)=0 \\ \psi(x,0)=\psi_0(x) \end{cases} \quad \text{is given by}$$

$$\psi(x,t)=\int_M \mathrm{d}y\, k_t(x,y)\psi_0(y) \quad \text{, where } k_t(x,y)=\frac{1}{(4\pi t)^2}e^{\frac{-\|x-y\|^2}{4t}}.$$

*Proof.* The boundary condition follows from the fact that $\lim_{t\downarrow 0} k_t(x,y)=\delta^{(4)}(x-y)$ and that $\psi_0(x)$ is continuous. That it satisfies the differential equation follows from

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(x,t)=\int_M \mathrm{d}y\left(\frac{-2}{t}+\frac{\|x-y\|^2}{4t^2}\right)k_t(x,y)\psi_0(y) \quad \text{and}$$

$$\Delta\psi(x,t)=-\partial_\mu\partial^\mu\psi(x,t)=\int_M \mathrm{d}y\left(\frac{4}{2t}-\frac{\|x-y\|^2}{4t^2}\right)k_t(x,y)\psi_0(y) \quad .$$

□

Globally, the heat kernel is connected by a partition of unity, since it glues the coordinate patches of the manifold together. The local solution is translated to the entire manifold by

$$e^{-t\Delta}\psi_0(x)=\int_M \mathrm{d}y\, k_t(x,y)\psi_0(y)=\sum_{\iota\in I}\int_{\phi_\iota(U_\iota)}\mathrm{d}^4y\,\rho_\iota(y)k_t(\phi_\iota(x),y)\psi_0(y) \quad ,$$

where $x\in M$ and we assume $\psi_0(x)$ to be smooth.

**Lemma 2.20.** The heat kernel $k_t(x,y)$, as in Lemma 2.19, has the following properties.

1. $\partial_x^\mu k_t(x,y)=-\partial_y^\mu k_t(x,y)=-\frac{x^\mu-y^\mu}{2t}k_t(x,y)$
2. $\partial_x^\mu\partial_x^\nu k_t(x,y)=\left(\frac{-g^{\mu\nu}}{2t}+\frac{(x^\mu-y^\mu)(x^\nu-y^\nu)}{4t^2}\right)k_t(x,y)$,
   where the Euclidean metric $g^{\mu\nu}=\delta^{\mu\nu}=\begin{cases}1 & \text{, if } \mu=\nu\\ 0 & \text{, if } \mu\neq\nu\end{cases}$ equals the Kronecker delta.

*Proof.* Writing $k_t(x,y)=\frac{1}{(4\pi t)^2}e^{\frac{-(x-y)^\mu(x-y)_\mu}{4t}}$ and straightforwardly differentiating concludes the proof. □

The following lemma we will need several times for the calculation of the traces.

**Lemma 2.21.** For $F,G\in C^\infty(End(S))$ the trace is given by

$$\langle F,G\rangle_t=\frac{\Lambda^4}{(4\pi t)^2}\int_{\Delta_2}\mathrm{d}^2s\int_M \mathrm{d}x\,\mathrm{Tr}_{S_x}\left(F\left(e^{-s_1(1-s_1)t\Lambda^{-2}\Delta}G\right)\right) \quad ,$$

where the exponential acts solely on $G$ and $\mathrm{Tr}_{S_x}$ is the trace over the fibre $S_x$, while we denote with Tr the trace over $L^2(M,S)$.

*Proof.* We rewrite the exponentials as heat kernels. These functions can then be combined, since they have the same spatial arguments, $k_\alpha(x,y)$ and $k_\beta(y,x) = k_\beta(x,y)$. Then,

$$
\begin{aligned}
\langle F, G\rangle_t &= \int_{\Delta_2} \mathrm{d}^2 s \,\mathrm{Tr}\left(F e^{-s_1 t\Lambda^{-2}\Delta} G e^{-(1-s_1)t\Lambda^{-2}\Delta}\right) \\
&= \int_{\Delta_2} \mathrm{d}^2 s \iint_M \mathrm{d}x \,\mathrm{Tr}_{S_x}\left(\mathrm{d}y\, F(x) k_{s_1 t\Lambda^{-2}}(x,y) G(y) k_{(1-s_1)t\Lambda^{-2}}(y,x)\right) \\
&= \int_{\Delta_2} \mathrm{d}^2 s \iint_M \mathrm{d}x \\
&\mathrm{Tr}_{S_x}\left(\mathrm{d}y\, F(x)G(y) \frac{\Lambda^4}{(4\pi s_1 t)^2} e^{\frac{-\|x-y\|^2\Lambda^2}{4 s_1 t}} \frac{\Lambda^4}{(4\pi(1-s_1)t)^2} e^{\frac{-\|y-x\|^2\Lambda^2}{4(1-s_1)t}}\right) \\
&= \int_{\Delta_2} \mathrm{d}^2 s \frac{\Lambda^4}{(4\pi t)^2} \iint_M \mathrm{d}x \,\mathrm{Tr}_{S_x}\left(\mathrm{d}y\, F(x)G(y) \frac{(4\pi)^{-2}\Lambda^4}{(s_1(1-s_1)t)^2} e^{\frac{-\|x-y\|^2\Lambda^2}{4 s_1(1-s_1)t}}\right) \\
&= \int_{\Delta_2} \mathrm{d}^2 s \frac{\Lambda^4}{(4\pi t)^2} \iint_M \mathrm{d}x \,\mathrm{Tr}_{S_x}\left(\mathrm{d}y\, F(x)\left(k_{s_1(1-s_1)t\Lambda^{-2}}(x,y)\, G(y)\right)\right) \\
&= \int_{\Delta_2} \mathrm{d}^2 s \frac{\Lambda^4}{(4\pi t)^2} \int_M \mathrm{d}x \,\mathrm{Tr}_{S_x}\left(F\left(e^{-s_1(1-s_1)t\Lambda^{-2}\Delta} G\right)\right) \quad .
\end{aligned}
$$

□

**Lemma 2.22.** For $E \in C^\infty(End(S))$ we have that

$$
[e^{-\Delta}, E] + \int_0^1 \mathrm{d}u\, e^{-u\Delta}[\Delta, E] e^{-(1-u)\Delta} = 0 \quad .
$$

*Proof.* We choose $f(u) = e^{-u\Delta} E e^{-(1-u)\Delta}$ and then the formula is directly found, since $f(1) - f(0) = \int_0^1 \mathrm{d}u\, f'(u)$ . □

## 2.5 Computation of the flat Seeley-De Witt coefficients

Now we have prepared enough to actually compute the coefficients. In the end this will give us the action in a more practical form.

### 2.5.1 Calculating $a_0$

We only get a contribution to $a_0$ from the $m$=0-term from equation (2.15), since all other terms are already of higher order:

$$
\begin{aligned}
\int_{t>0} \mathrm{d}t\, g(t)\,\mathrm{Tr}\left(e^{-t\Lambda^{-2}\Delta}\right) &= \int_{t>0} \mathrm{d}t\, g(t) \int_M \mathrm{d}x\,\mathrm{Tr}_{S_x}\left(\frac{\Lambda^4}{(4\pi t)^2} e^{-\frac{\|x-x\|^2\Lambda^2}{4t}} I\right) \\
&= \frac{\Lambda^4}{(4\pi)^2} \int_{t>0} \mathrm{d}t\, g(t) t^{-2} \int_M \mathrm{Tr}_{S_x}(I) = \frac{\Lambda^4 f_4}{(4\pi)^2} \int_M \mathrm{Tr}_{S_x}(I) \quad . \qquad (2.16)
\end{aligned}
$$

Here, $I$ is the identity operator on the fibre and the trace indicated in the final line is just the trace over the fibre, in this case giving the dimensionality. Now, this is of the desired form given in equation (2.3). We have found the first coefficient:

$$
a_0(x, D_\varphi^2) = \frac{1}{(4\pi)^2} \mathrm{Tr}_{S_x}(I) \quad . \qquad (2.17)
$$

### 2.5.2 Calculating $a_2$

Since there is no $E$ in the $m$=0-term, there is nothing the Laplacian could act on, so the complete $m$=0-term is of order 0. Then, for $a_2$ the only contribution will be from $m = 1$ in equation (2.15). Furthermore, any derivative in $e^{-s_1 t\Lambda^{-2}\Delta}$ acting on $E$ would raise the order. Therefore, up to order 2

$$\begin{aligned}\Lambda^{-2}\int_{t>0} \mathrm{d}t\, g(t)t\, \langle E\rangle_t &= \Lambda^{-2}\int_{t>0} \mathrm{d}t\, g(t)t \int_0^1 \mathrm{d}s_1 \operatorname{Tr}\left(Ee^{-t\Lambda^{-2}\Delta}\right)\\ &= \Lambda^{-2}\int_{t>0} \mathrm{d}t\, g(t)t\frac{\Lambda^4}{(4\pi t)^2}\int_0^1 \mathrm{d}s_1 \int_M \mathrm{d}x\, \operatorname{Tr}_{S_x}\left(E(x)e^{-\frac{\Lambda^2\|x-x\|^2}{4t}}\right)\\ &= \Lambda^2\int_{t>0} \mathrm{d}t\, g(t)\frac{1}{t(4\pi)^2}\int_M \mathrm{d}x\, \operatorname{Tr}_{S_x}(E(x)) = \Lambda^2 f_2\frac{1}{(4\pi)^2}\int_M \operatorname{Tr}_{S_x}(E) \quad .\end{aligned}$$

So that we have found that

$$a_2(x, D_\varphi^2) = \frac{1}{(4\pi)^2}\operatorname{Tr}_{S_x}(E) \quad . \tag{2.18}$$

*Remark* 2.13. Notice that we wrote equality above. This is because, any derivative acting on $E$ would give us something of the form $\int_M \operatorname{Tr}_{S_x}(\partial^\mu \ldots \partial^\nu E)$. Because we integrate over $M$, which has no boundary, by Stokes' theorem all these terms would vanish, since they are just total differentials.

### 2.5.3 Calculating $a_4$

By the previous remark, the $m$=1-term in equation (2.15) gives no further contributions and we move on to the $m$=2-term. From equation (2.3) it is clear that the order of $a_4$ is zero. Therefore, we look for the terms of order 4 in $\Lambda^{-1}$, see example 2.5.

$$\begin{aligned}&\Lambda^{-4}\int_{t>0} \mathrm{d}t\, g(t)t^2\, \langle E, E\rangle_t\\ &\quad= \Lambda^{-4}\int_{t>0} \mathrm{d}t\, g(t)t^2\frac{\Lambda^4}{(4\pi t)^2}\int_{\Delta_2} \mathrm{d}^2 s\int_M \operatorname{Tr}_{S_x}\left(E\left(e^{-s_1(1-s_1)t\Lambda^{-2}\Delta}E\right)\right)\\ &\quad\overset{\mathcal{O}(\Lambda^{-0})}{\sim} \Lambda^{-4}\int_{t>0} \mathrm{d}t\, g(t)t^2\int_0^1 \mathrm{d}s_1\,(1-s_1)\frac{\Lambda^4}{(4\pi t)^2}\int_M \operatorname{Tr}_{S_x}\left(E^2\right)\\ &\quad= \int_{t>0} \mathrm{d}t\, g(t)\frac{1}{(4\pi)^2}\int_M \operatorname{Tr}_{S_x}\left(\frac{1}{2}E^2\right) \quad ,\end{aligned} \tag{2.19}$$

by applying Lemma 2.21 in the first line. Hence, we have found that

$$a_4(x, D_\varphi^2) = \frac{1}{(4\pi)^2}\operatorname{Tr}_{S_x}\left(\frac{1}{2}E^2\right) \quad . \tag{2.20}$$

*Remark* 2.14. Letting the Laplacians in the exponential act on $E$ in equation (2.19) would raise the order. By comparing powers of $\Lambda$ in the result with equation (2.3) we see that this gives us our first contributions to $a_6$ and $a_8$.

### 2.5.4 Calculating $a_6$

For $a_6$ we expect contributions from $m = 3$ with no Laplacians and $m = 2$ with one Laplacian. The $m$=1-term with two Laplacians is a total differential and therefore zero by Stokes' theorem.

$m = 3$**, no Laplacians**

Any derivative acting on $E$ would raise the order, so we find that

$$\begin{aligned}
&\Lambda^{-6} \int_{t>0} \mathrm{d}t\, g(t) t^3 \langle E, E, E \rangle_t \\
&\quad \overset{\mathcal{O}(\Lambda^{-2})}{\sim} \Lambda^{-6} \int_{t>0} \mathrm{d}t\, g(t) t^3 \int_{\Delta_3} \mathrm{d}^3 s\, \mathrm{Tr} \left( E^3 e^{-t\Lambda^{-2}\Delta} \right) \\
&\quad = \Lambda^{-2} f_{-2} \left( \int_0^1 \mathrm{d}s_1 \int_0^{1-s_1} \mathrm{d}s_2 \int_0^{1-s_1-s_2} \mathrm{d}s_3 \right) \int_M \mathrm{Tr}_{S_x} \left( \frac{1}{(4\pi)^2} E^3 \right) \\
&\quad = \Lambda^{-2} f_{-2} \int_M \frac{1}{(4\pi)^2} \mathrm{Tr}_{S_x} \left( \frac{1}{6} E^3 \right)
\end{aligned} \tag{2.21}$$

$m = 2$**, one Laplacian**

We apply Lemma 2.21 to

$$\begin{aligned}
&\Lambda^{-4} \int_{t>0} \mathrm{d}t\, g(t) t^2 \langle E, E \rangle_t \\
&\quad = \Lambda^{-4} \int_{t>0} \mathrm{d}t\, g(t) t^2 \int_0^1 \mathrm{d}s_1 \int_0^{1-s_1} \mathrm{d}s_2\, \mathrm{Tr} \left( E e^{-s_1 t \Lambda^{-2}\Delta} E e^{-(1-s_1)t\Lambda^{-2}\Delta} \right) \\
&\quad = \int_{t>0} \mathrm{d}t\, g(t) \frac{1}{(4\pi)^2} \int_0^1 \mathrm{d}s_1 \int_0^{1-s_1} \mathrm{d}s_2 \int_M \mathrm{Tr}_{S_x} \left( E \left( e^{-s_1(1-s_1)t\Lambda^{-2}\Delta} E \right) \right) \\
&\quad \overset{\mathcal{O}(\Lambda^{-2})}{\sim} \frac{\Lambda^{-2}}{(4\pi)^2} \int_{t>0} \mathrm{d}t\, g(t) t \int_0^1 \mathrm{d}s_1 \int_0^{1-s_1} \mathrm{d}s_2\, (s_1^2 - s_1) \int_M \mathrm{Tr}_{S_x} \left( E(\Delta E) \right) \\
&\quad = \frac{\Lambda^{-2}}{(4\pi)^2} f_{-2} \int_M \mathrm{Tr}_{S_x} \left( \frac{-1}{12} E(\Delta E) \right) \quad .
\end{aligned} \tag{2.22}$$

So we find the total expression of order 2 by adding 2.21 and 2.22, resulting in

$$\Lambda^{-2} f_{-2} \int_M \frac{1}{(4\pi)^2} \mathrm{Tr}_{S_x} \left( \frac{1}{6} E^3 - \frac{1}{12} E(\Delta E) \right) \quad .$$

From this we find

$$a_6(x, D_\varphi^2) = \frac{1}{(4\pi)^2} \mathrm{Tr}_{S_x} \left( \frac{1}{6} E^3 - \frac{1}{12} E(\Delta E) \right) \quad . \tag{2.23}$$

### 2.5.5 Calculating $a_8$

Using the same arguments as before, we expect contributions from $m = 4$ without Laplacians, $m = 3$ with one Laplacian and $m = 2$ with two Laplacians.

$m = 4$**, no Laplacians**

Using exactly the same reasoning as for (2.21)

$$\begin{aligned}
&\Lambda^{-8} \int_{t>0} \mathrm{d}t\, g(t) t^4 \langle E, E, E, E \rangle_t \\
&\quad \overset{\mathcal{O}(\Lambda^{-4})}{\sim} \Lambda^{-8} \int_{t>0} \mathrm{d}t\, g(t) t^4 \int_{\Delta_4} \mathrm{d}^4 s\, \mathrm{Tr} \left( E^4 e^{-t\Lambda^{-2}\Delta} \right) \\
&\quad = \Lambda^{-4} f_{-4} \int_M \frac{1}{(4\pi)^2} \mathrm{Tr}_{S_x} \left( \frac{1}{24} E^4 \right) \quad .
\end{aligned} \tag{2.24}$$

**$m = 2$, 2 Laplacians**

Using the $n$=2-part from equation (2.19) we find that

$$\begin{aligned}
&\Lambda^{-4}\int_{t>0} \mathrm{d}t\, g(t)t^2\ \langle E, E\rangle_t \\
&\quad = \Lambda^{-4}\int_{t>0} \mathrm{d}t\, g(t)t^2 \int_0^1 \mathrm{d}s_1\,(1-s_1)\,\mathrm{Tr}\left(Ee^{-s_1 t\Lambda^{-2}\Delta}Ee^{-(1-s_1)t\Lambda^{-2}\Delta}\right) \\
&\quad = \Lambda^{-4}\int_{t>0} \mathrm{d}t\, g(t)t^2 \frac{\Lambda^4}{(4\pi t)^2}\int_0^1 \mathrm{d}s_1\,(1-s_1)\int_M \mathrm{Tr}_{S_x}\left(E\left(e^{-s_1(1-s_1)t\Lambda^{-2}\Delta}E\right)\right) \\
&\quad \overset{\mathcal{O}(\Lambda^{-4})}{\sim} \Lambda^{-4}\int_{t>0} \mathrm{d}t\, g(t)t^2 \int_0^1 \mathrm{d}s_1\,(1-s_1) \\
&\quad \times \frac{1}{(4\pi)^2}\int_M \frac{(-s_1(1-s_1))^2}{(2!)}\mathrm{Tr}_{S_x}\left(E(\Delta^2 E)\right) \\
&\quad = \Lambda^{-4} f_{-4}\int_M \frac{1}{(4\pi)^2}\mathrm{Tr}_{S_x}\left(\frac{1}{120}E(\Delta^2 E)\right) \quad . \qquad (2.25)
\end{aligned}$$

**$m = 3$, one Laplacian**

This term requires some extra attention, since it is more difficult to find than all previous ones. We are calculating $a_8$ and therefore we are only interested in terms of order 8 in $\Lambda^{-1}$, meaning '3 $E$'s and 2 $\partial$'s'. All such terms are contained in $\langle E, E, E\rangle_t$, so we calculate this first, using Lemma 2.22 several times:

$$\begin{aligned}
\langle E, E, E\rangle_t &= \int_{\Delta_3} \mathrm{d}^3 s\,\mathrm{Tr}\left(E^2 e^{-s_1\lambda\Delta}Ee^{-(1-s_1)\lambda\Delta}\right. \\
&\quad \left. -[e^{-s_2\lambda\Delta}, E]Ee^{-s_1\lambda\Delta}Ee^{-(1-s_1-s_2)\lambda\Delta}\right) \\
&= \int_{\Delta_3} \mathrm{d}^3 s\,\frac{s_1(s_1-1)}{(4\pi)^2\lambda}\int_M \mathrm{Tr}_{S_x}\left(E^2(\Delta E)\right) \qquad (2.26) \\
&\quad - \int_{\Delta_3} \mathrm{d}^3 s\,\mathrm{Tr}\left([e^{-s_2\lambda\Delta}, E]Ee^{-s_1\lambda\Delta}Ee^{-(1-s_1-s_2)\lambda\Delta}\right) \quad , \qquad (2.27)
\end{aligned}$$

where the first term is found by applying Lemma 2.21. Since (2.26) is of the desired form, we continue with only (2.27):

$$
\begin{aligned}
&-\int_{\Delta_3} \mathrm{d}^3 s\, \mathrm{Tr}\left([e^{-s_2\lambda\Delta}, E]Ee^{-s_1\lambda\Delta}Ee^{-(1-s_1-s_2)\lambda\Delta}\right)\\
&\quad= \int_{\Delta_3} \mathrm{d}^3 s\, s_2\lambda \int_0^1 \mathrm{d}u\, \mathrm{Tr}\left(e^{-us_2\lambda\Delta}[\Delta, E]e^{-(1-u)s_2\lambda\Delta}Ee^{-s_1\lambda\Delta}Ee^{-(1-s_1-s_2)\lambda\Delta}\right)\\
&\quad= \int_{\Delta_3} \mathrm{d}^3 s\, s_2\lambda \int_0^1 \mathrm{d}u\, \mathrm{Tr}\Big(e^{-us_2\lambda\Delta}\left(\Delta E - 2E_{,\mu}\partial^\mu\right) e^{-(1-u)s_2\lambda\Delta}\\
&\quad\times Ee^{-s_1\lambda\Delta}Ee^{-(1-s_1-s_2)\lambda\Delta}\Big)\\
&\quad= \int_{\Delta_3} \mathrm{d}^3 s\, s_2\lambda \int_0^1 \mathrm{d}u\, \mathrm{Tr}\left((\Delta E)E^2e^{-\lambda\Delta}\right)\\
&\quad- \int_{\Delta_3} \mathrm{d}^3 s\, 2s_2\lambda \int_0^1 \mathrm{d}u\, \mathrm{Tr}\left(e^{-us_2\lambda\Delta}\left(E_{,\mu}\partial^\mu\right) e^{-(1-u)s_2\lambda\Delta}Ee^{-s_1\lambda\Delta}Ee^{-(1-s_1-s_2)\lambda\Delta}\right)\\
&\quad= \frac{1}{(4\pi)^2\lambda}\int_{\Delta_3} \mathrm{d}^3 s\, s_2 \int_M \mathrm{Tr}_{S_x}\left(E^2(\Delta E)\right) \qquad (2.28)\\
&\quad- \int_{\Delta_3} \mathrm{d}^3 s\, 2s_2\lambda \int_0^1 \mathrm{d}u\, \mathrm{Tr}\Big(e^{-us_2\lambda\Delta}\left(E_{,\mu}\partial^\mu\right) e^{-(1-u)s_2\lambda\Delta}\\
&\quad\times Ee^{-s_1\lambda\Delta}Ee^{-(1-s_1-s_2)\lambda\Delta}\Big) \quad . \qquad (2.29)
\end{aligned}
$$

Sum equations (2.26), (2.28) and (2.29) to summarize:

$$
\begin{aligned}
\langle E, E, E\rangle_t &= \frac{1}{(4\pi)^2\lambda}\int_{\Delta_3} \mathrm{d}^3 s\, (s_1^2 - s_1 + s_2) \int_M \mathrm{Tr}_{S_x}\left(E^2(\Delta E)\right)\\
&- \int_{\Delta_3} \mathrm{d}^3 s\, 2s_2\lambda \int_0^1 \mathrm{d}u\, \mathrm{Tr}\left(e^{-us_2\lambda\Delta}\left(E_{,\mu}\partial^\mu\right) e^{-(1-u)s_2\lambda\Delta}Ee^{-s_1\lambda\Delta}Ee^{-(1-s_1-s_2)\lambda\Delta}\right)
\end{aligned}
$$

We again apply Lemma 2.22 to equation (2.29),

$$
\begin{aligned}
&-\int_{\Delta_3} \mathrm{d}^3 s\, 2s_2\lambda \int_0^1 \mathrm{d}u\, \mathrm{Tr}\Big(e^{-us_2\lambda\Delta}\left(E_{,\mu}\partial^\mu\right) e^{-(1-u)s_2\lambda\Delta}E\left(\left[e^{-s_1\lambda\Delta}, E\right]\right.\\
&\quad\left.+Ee^{-s_1\lambda\Delta}\right) e^{-(1-s_1-s_2)\lambda\Delta}\Big)\\
&\quad= \int_{\Delta_3} \mathrm{d}^3 s\, 2s_1s_2\lambda^2 \int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}v\, \mathrm{Tr}\Big(e^{-us_2\lambda\Delta}\left(E_{,\mu}\partial^\mu\right) e^{-(1-u)s_2\lambda\Delta}\\
&\quad\times Ee^{-vs_1\lambda\Delta}\left[\Delta, E\right] e^{-(1-v)s_1\lambda\Delta}e^{-(1-s_1-s_2)\lambda\Delta}\Big)\\
&\quad- \int_{\Delta_3} \mathrm{d}^3 s\, 2s_2\lambda \int_0^1 \mathrm{d}u\, \mathrm{Tr}\left(e^{-us_2\lambda\Delta}\left(E_{,\mu}\partial^\mu\right) e^{-(1-u)s_2\lambda\Delta}E^2e^{-(1-s_2)\lambda\Delta}\right)\\
&\quad= -\int_{\Delta_3} \mathrm{d}^3 s\, 4s_1s_2\lambda^2 \int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}v\, \mathrm{Tr}\left(E_{,\mu}EE_{,\nu}\partial^\mu\partial^\nu e^{-\lambda\Delta}\right)\\
&\quad- \int_{\Delta_3} \mathrm{d}^3 s\, 2s_2\lambda \int_0^1 \mathrm{d}u\, \mathrm{Tr}\left(\left(E_{,\mu}\partial^\mu\right) e^{-(1-u)s_2\lambda\Delta}E^2e^{-(1-(1-u)s_2)\lambda\Delta}\right)\\
&\quad= \int_{\Delta_3} \mathrm{d}^3 s\, \frac{2s_1s_2}{(4\pi)^2\lambda} \int_M \mathrm{Tr}_{S_x}\left(EE_{,\mu}E^{,\mu}\right) \qquad (2.30)\\
&\quad- \int_{\Delta_3} \mathrm{d}^3 s\, 2s_2\lambda \int_0^1 \mathrm{d}u\, \mathrm{Tr}\left(\left(E_{,\mu}\partial_x^\mu\right) e^{-(1-u)s_2\lambda\Delta}E^2e^{-(1-(1-u)s_2)\lambda\Delta}\right) \quad , \qquad (2.31)
\end{aligned}
$$

using Lemma 2.20 to find equation (2.30). We only have to compute the contributions from equation (2.31). Again we want to take the trace by integrating over the diagonal of the operator acting on $\psi$. This is given by

$$\begin{aligned}
&\left(E_{,\mu}\partial^\mu e^{-(1-u)s_2\lambda\Delta}E^2 e^{-(1-(1-u)s_2)\lambda\Delta}\psi\right)(x)\\
&\quad= \iint_M \mathrm{d}z\ \mathrm{d}y\, E_{,\mu}(x)\partial_x^\mu k_{(1-u)s_2\lambda}(x,y)E^2(y)k_{(1-(1-u)s_2)\lambda}(y,z)\psi(z)\\
&\quad= \iint_M \mathrm{d}z\ \mathrm{d}y\, E_{,\mu}(x)\frac{-(x^\mu-y^\mu)}{2(1-u)s_2\lambda}k_{(1-u)s_2\lambda}(x,y)E^2(y)k_{(1-(1-u)s_2)\lambda}(y,z)\psi(z)\ ,
\end{aligned}$$

where we differentiated using Lemma 2.20. By integrating over the diagonal $x=z$ of this operator we have from contribution (2.31):

$$\begin{aligned}
&-\int_{\Delta_3}\mathrm{d}^3s\, 2s_2\lambda\int_0^1 \mathrm{d}u\, \mathrm{Tr}\left((E_{,\mu}\partial_x^\mu)\, e^{-(1-u)s_2\lambda\Delta}E^2e^{-(1-(1-u)s_2)\lambda\Delta}\right)\\
&\quad= -\int_{\Delta_3}\mathrm{d}^3s\, 2s_2\lambda\int_0^1\mathrm{d}u\iint_M \mathrm{d}x\, \mathrm{Tr}_{S_x}\Big(\mathrm{d}y\, E_{,\mu}(x)\frac{-(x^\mu-y^\mu)}{2(1-u)s_2\lambda}\\
&\quad\times k_{(1-u)s_2\lambda}(x,y)E^2(y)k_{(1-(1-u)s_2)\lambda}(y,x)\Big) \qquad (2.32)\\
&\quad= -\int_{\Delta_3}\mathrm{d}^3s\,\frac{2s_2}{(4\pi)^2\lambda}\int_0^1\mathrm{d}u\iint_M \mathrm{d}x\, \mathrm{Tr}_{S_x}\Big(\mathrm{d}y\, E_{,\mu}(x)\frac{-(x^\mu-y^\mu)}{2(1-u)s_2\lambda}\\
&\quad\times k_{(1-(1-u)s_2)((1-u)s_2)\lambda}(x,y)E^2(y)\Big)\\
&\quad= -\int_{\Delta_3}\mathrm{d}^3s\,\frac{2s_2}{(4\pi)^2\lambda}\int_0^1\mathrm{d}u\ (1-(1-u)s_2)\iint_M \mathrm{d}x\, \mathrm{Tr}_{S_x}\Big(\mathrm{d}y\, E_{,\mu}(x)\\
&\quad\times \partial_x^\mu k_{(1-(1-u)s_2)((1-u)s_2)\lambda}(x,y)E^2(y)\Big) \qquad (2.33)\\
&\quad= -\int_{\Delta_3}\mathrm{d}^3s\,\frac{2s_2}{(4\pi)^2\lambda}\int_0^1\mathrm{d}u\ (1-(1-u)s_2)\\
&\quad\int_M \mathrm{d}x\, \mathrm{Tr}_{S_x}\left(E_{,\mu}(x)\partial_x^\mu\left(e^{-(1-(1-u)s_2)((1-u)s_2)\lambda\Delta}E^2\right)(x)\right)\\
&\quad\overset{\mathcal{O}(\Lambda^{-8})}{\sim} -\int_{\Delta_3}\mathrm{d}^3s\,\frac{4s_2}{(4\pi)^2\lambda}\int_0^1\mathrm{d}u\ (1-(1-u)s_2)\int_M \mathrm{d}x\,\mathrm{Tr}_{S_x}\left(E_{,\mu}E^{,\mu}E\right)\\
&\quad= \int_{\Delta_3}\mathrm{d}^3s\,\frac{-4s_2+2s_2^2}{(4\pi)^2\lambda}\int_M \mathrm{d}x\,\mathrm{Tr}_{S_x}\left(EE_{,\mu}E^{,\mu}\right)\quad , \qquad (2.34)
\end{aligned}$$

applying Lemma 2.20 in steps (2.32) and (2.33). If we now add terms (2.26), (2.28) (2.30) and (2.34) we have a practical expression for the trace up to order 8

$$\begin{aligned}
\langle E,E,E\rangle_t &\overset{\mathcal{O}(\Lambda^{-8})}{\sim} \frac{\Lambda^2}{(4\pi)^2t}\int_{\Delta_3}\mathrm{d}^3s\int_M \mathrm{Tr}_{S_x}\left((s_1^2-s_1+s_2)E^2(\Delta E)\right.\\
&\left.+(2s_1s_2-4s_2+2s_2^2)EE_{,\mu}E^{,\mu}\right)
\end{aligned}$$

Inserting this in the $m$=3-term in equation (2.15) gives us

$$\begin{aligned}
&\Lambda^{-4}f_{-4}\int_0^1\mathrm{d}s_1\int_0^{1-s_1}\mathrm{d}s_2\,(1-s_1-s_2)\int_M\frac{1}{(4\pi)^2}\mathrm{Tr}_{S_x}\left((s_1^2-s_1+s_2)E^2(\Delta E)\right.\\
&\quad\left.+(2s_1s_2-4s_2+2s_2^2)EE_{,\mu}E^{,\mu}\right)\\
&\quad= \Lambda^{-4}f_{-4}\int_M\frac{1}{(4\pi)^2}\mathrm{Tr}_{S_x}\left(\frac{1}{60}E^2(\Delta E)+\frac{-7}{60}EE_{,\mu}E^{,\mu}\right)\quad . \qquad (2.35)
\end{aligned}$$

So we find by adding equations (2.24), (2.25), (2.35) and shifting a total differential between the terms in equation (2.35):

$$a_8(x, D_\varphi^2) = \frac{1}{(4\pi)^2}\mathrm{Tr}_{S_x}\left(\frac{1}{24}E^4 - \frac{1}{12}EE(\Delta E) + \frac{1}{12}E(E_{,\mu}E_{,}^{\ \mu}) + \frac{1}{120}E(\Delta^2 E)\right) \quad . \tag{2.36}$$

These are the same as the much more general answers found by Avramidi [28, equation 3.85] and Fujiwara [29, equation B6]. When we combine terms two and three we find

$$a_8(x, D_\varphi^2) = \frac{1}{(4\pi)^2}\mathrm{Tr}_{S_x}\left(\frac{1}{24}E^4 - \frac{1}{24}EE(\Delta E) + \frac{1}{120}E(\Delta^2 E)\right) \quad . \tag{2.37}$$

# 3 The physical structure of the scalar spectral action

As made clear at the end of Chapter 1 we are interested in $\beta$-functions of quantum field theories. We determined the $\beta$-functions for the rsHL from example 1.1 using two different subtraction operators and found that they were the same. The first subtraction operator $\overline{\mathfrak{T}}_1$ acted hardly differently from commonly used modified minimal subtraction, while the second operator $\overline{\mathfrak{T}}_2$ added a computationally convenient mass-limit to this.
In this chapter we will see that the example theory from Chapter 1 was only a lower order approximation of the asymptotically expanded spectral action from Chapter 2. Using renormalization tools from Chapter 1 and [3] we will then be able to obtain a set of $\beta$-functions for the ersHL, an extended version of the rsHL. However, this will have to wait until Chapter 4.
In this chapter we will analyze the action obtained in Chapter 2 in a way similar to Chapter 1. This means that we will work on a Euclidean space, which is one chart of the global Riemannian manifold. We will first write down the Lagrangian for this action in a usable way. From there we will interpret this higher derivative scalar field theory as a quantum field theory and finally study its divergences. In Chapter 2 we have found an explicit expansion of the action, equation (2.3), where the coefficients $a_0$, $a_2$, $a_4$, $a_6$ and $a_8$ are given respectively by equations (2.17), (2.18), (2.20), (2.23) and (2.36). These give us the full action, if the function $g$ is such that $f_k = 0$ for $k < -4$. Therefore, we need an expression for the operator $E \in C^\infty(End(S))$ in terms of the (physical) field $\varphi$. Besides that, we will need some lemmas about the gamma matrices.

## 3.1 Gamma matrices

**Definition 3.1.** [Gamma matrices]
The defining property of the gamma matrices is

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} I_4 \text{ for } \mu, \nu = 0, 1, 2, 3 \quad .$$

Next to this we define that

$$\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^0 \quad .$$

**Lemma 3.1.** The gamma matrices have the following properties:

1. $\gamma^5\gamma^5 = I_4$ , and $\{\gamma^5, \gamma^\mu\} = 0$, for $0 \leq \mu \leq 3$;
2. $\mathrm{Tr}\,(\gamma^\iota) = 0$, when $\iota = 0, 1, 2, 3, 5$;

3. $\mathrm{Tr}\left(\gamma^\iota\gamma^\lambda\right) = 4\delta^{\iota\lambda}$, when $\iota, \lambda = 0, 1, 2, 3, 5$;
4. $\mathrm{Tr}\left(\gamma^\iota\gamma^\lambda\gamma^\xi\right) = 0$, when $\iota, \lambda, \xi = 0, 1, 2, 3, 5$;
5. $\mathrm{Tr}\left(\gamma^{\mu_1}\dots\gamma^{\mu_{2n+1}}\right) = 0$, where $\mu_i = 0, 1, 2, 3$ and $n \in \mathbb{N}$.

*Proof.* To prove statement 1.

$$(\gamma^5)^2 = \gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 = (-1)^6\gamma^1\gamma^1\gamma^2\gamma^2\gamma^3\gamma^3\gamma^0\gamma^0 = I_4 \quad .$$

Then,

$$\{\gamma^5, \gamma^\mu\} = \gamma^\mu\gamma^1\gamma^2\gamma^3\gamma^0 + \gamma^1\gamma^2\gamma^3\gamma^0\gamma^\mu = (-1)^3\gamma^1\gamma^2\gamma^3\gamma^0\gamma^\mu + \gamma^1\gamma^2\gamma^3\gamma^0\gamma^\mu = 0 \quad .$$

Take $\mu \neq \iota$:

$$\mathrm{Tr}\left(\gamma^\iota\right) = \mathrm{Tr}\left(\gamma^\iota\gamma^\mu\gamma^\mu\right) = -\mathrm{Tr}\left(\gamma^\mu\gamma^\iota\gamma^\mu\right) \quad ,$$

where we have inserted $\gamma^\mu\gamma^\mu = I_4$ and anticommuted $\gamma^\iota$ and $\gamma^\mu$. Next we use the cyclicity of the trace, $((\mathrm{Tr}\,(AB) = \mathrm{Tr}\,(BA))$, and find

$$\mathrm{Tr}\left(\gamma^\iota\right) = -\mathrm{Tr}\left(\gamma^\iota\gamma^\mu\gamma^\mu\right) = -\mathrm{Tr}\left(\gamma^\iota\right) = 0 \quad .$$

This proves 2.
Again, by the cyclicity of the trace and the anticommutation relation

$$\mathrm{Tr}\left(\gamma^\iota\gamma^\lambda\right) = \frac{1}{2}(\mathrm{Tr}\left(\gamma^\iota\gamma^\lambda\right) + \mathrm{Tr}\left(\gamma^\lambda\gamma^\iota\right)) = \mathrm{Tr}\left(\delta^{\iota\lambda}I_4\right) = 4\delta^{\iota\lambda} \quad .$$

We use here $\delta$ instead of $g$, because there is no fifth component of $g$, while it matches $\delta$ for the other indices. This proves statement 3.
For the fourth notice that when two of the three matrices are equal, we can rearrange the gamma matrices to the trace of one gamma matrix, which is zero. When all three gamma matrices are different and none of them is $\gamma^5$, we see from the definition of $\gamma^5$ that we can write $\gamma^\iota\gamma^\lambda\gamma^\xi = \pm\gamma^\mu\gamma^5$, $0 \leq \mu \leq 3$, which has trace zero. If one of them is $\gamma^5$, we write $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^0$ and see that $\gamma^\iota\gamma^\lambda\gamma^\xi = \pm\gamma^\mu\gamma^\nu$, $1 \leq \mu < \nu \leq 3$, which has trace zero.
For statement 5, we can assume that $n \geq 2$, since we have already proved the other cases. There are only four different gamma matrices, so we can reduce the number of gamma matrices by anticommuting different gamma matrices and using that $\gamma^\mu\gamma^\mu = I_4$. Since these matrices vanish by two, we are left with an odd number. By repeating this we end up with one or three matrices having trace zero.
This concludes the proof. □

*Remark* 3.1. Notice that the trace in the above lemma is the trace we denoted in the previous chapter by $\mathrm{Tr}_{S_x}$.

*Remark* 3.2. The conjugation properties of the gamma matrices are not fixed by the anticommutation relation. We choose the chiral Euclidean representation. The Pauli spin matrices are then denoted by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ,$$

the chiral Euclidean gamma matrices are given in $2 \times 2$-block form by

$$\gamma^j = \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix} \text{ for } j = 1, 2, 3, \quad \gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \quad .$$

We will need one more lemma concerning gamma matrices.

**Lemma 3.2.** For some differentiable function $\varphi$:

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\partial_\mu\varphi\partial_\nu\varphi\partial_\rho\varphi\partial_\sigma\varphi\right) = 4\partial_\mu\varphi\partial^\mu\varphi\partial_\nu\varphi\partial^\nu\varphi \quad .$$

*Proof.* All indices take values 0,1,2,3. By the summation convention this is a sum with $4^4$ terms. We take any term and denote by $k$ the number of different indices of the gamma matrices. This can be 1,2,3,4. For example: $\gamma^1\gamma^1\gamma^1\gamma^1$ has $k$=1 and $\gamma^1\gamma^2\gamma^3\gamma^0$ has $k$=4.
Using Lemma 3.1 we can anticommute some of the matrices, if necessary, and bring a term to a typical form. Then we can evaluate its trace immediately using Lemma 3.1 again. We have summarized this all in the table below.
Notice that for $k$=2 there are two possible distributions, (2,2) and (3,1). For example, $\gamma^1\gamma^2\gamma^1\gamma^2$ and $\gamma^3\gamma^3\gamma^1\gamma^3$, respectively.

| $k$ | distribution | form | trace |
|---|---|---|---|
| 4 | | $\pm\gamma^5$ | 0 |
| 3 | | $\pm\gamma^\mu\gamma^5$ | 0 |
| 2 | $(3,1)$ | $\pm\gamma^\mu\gamma^{\nu\neq\mu}$ | 0 |
| 2 | $(2,2)$ | $\pm I_4$ | $\pm 4$ |
| 1 | | $I_4$ | 4 |

So, for $k$=1 we get the contribution $\sum_{j=0}^{3} 4\partial_j\varphi\partial^j\varphi\partial_j\varphi\partial^j\varphi$.
For $k$=2,(2,2) we first observe the terms where the first and the fourth gamma matrix are different, $\mu \neq \sigma$. The values $\mu$ and $\sigma$ are fixed here, so there is no summation. Then, these terms are of the forms $\gamma^\mu\gamma^\mu\gamma^\sigma\gamma^\sigma$ and $\gamma^\mu\gamma^\sigma\gamma^\mu\gamma^\sigma$. Since in the summation over such term both occur in pairs and have the same corresponding function $\partial_\mu\varphi\partial^\mu\varphi\partial_\sigma\varphi\partial^\sigma\varphi$, we can sum these two terms and this gives $\gamma^\mu\left(\gamma^\mu\gamma^\sigma + \gamma^\sigma\gamma^\mu\right)\gamma^\sigma = 0$. Hence, we are left with the terms $\mu = \sigma$ and these give terms of the form $\gamma^\mu\gamma^{\nu\neq\mu}\gamma^\nu\gamma^\mu = I_4$ with corresponding function $\partial_\mu\varphi\partial^\mu\varphi\partial_{\nu\neq\mu}\varphi\partial^\nu\varphi$. Summing these with the $k$=1-contribution proves this lemma. □

**Lemma 3.3.** For $\not{D} = i\gamma^\mu\partial_\mu$, $\gamma$ as in Definition 3.1 and $\varphi \in C^\infty(M)$:

$$(\not{D} + \gamma^5\varphi)^2 = \Delta - E \quad \text{, where}$$

$$E = i\gamma^5\gamma^\mu\partial_\mu\varphi - \varphi^2 \quad \text{, with } E \in C^\infty(End(S)).$$

*Proof.* Using Lemma 3.1 several times we see that

$$\begin{aligned}(\not{D} + \gamma^5\varphi)^2 &= (i\gamma^\mu\partial_\mu + \gamma^5\varphi)^2 = -\gamma^\mu\partial_\mu\gamma^\nu\partial_\nu + \gamma^5\varphi\gamma^5\varphi + i\gamma^\mu\partial_\mu\gamma^5\varphi + \gamma^5\varphi i\gamma^\mu\partial_\mu \\ &= -\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + \varphi^2 + i\gamma^\mu\gamma^5\partial_\mu\varphi + i\{\gamma^\mu,\gamma^5\}\varphi\partial_\mu \\ &= -\partial_\mu\partial^\mu + \varphi^2 + i\gamma^\mu\gamma^5\partial_\mu\varphi \\ &= \Delta - \left(i\gamma^5\gamma^\mu\partial_\mu\varphi - \varphi^2\right) \quad ,\end{aligned}$$

omitting the $4\times4$-identity matrix $I_4$. Since $\varphi$ is smooth, we read that

$$E = i\gamma^5\gamma^\mu\partial_\mu\varphi - \varphi^2 \quad .$$

□

## 3.2 The higher order asymptotic spectral action

These lemmas give us everything needed to write the action, equation (2.3), in a way similar to Lagrangian (1.1).

**Proposition 3.1.** If $g$ is such that $f_k = 0$ for $k < -4$, then the action is given by

$$\begin{aligned} S[\varphi] = \int_M \frac{1}{(4\pi)^2}\Big( & 4\Lambda^4 f_4 - 4\Lambda^2 f_2\varphi^2 + 2f_0\varphi^4 + 2f_0\partial_\mu\varphi\partial^\mu\varphi - 2\Lambda^{-2}f_{-2}\varphi^2\partial_\mu\varphi\partial^\mu\varphi \\ & - \frac{1}{3}\Lambda^{-2}f_{-2}\partial_\mu\varphi\Delta\partial^\mu\varphi - \frac{1}{3}\Lambda^{-2}f_{-2}\varphi^2\Delta\varphi^2 - \frac{2}{3}\Lambda^{-2}f_{-2}\varphi^6 + \frac{1}{6}\Lambda^{-4}f_{-4}\varphi^8 \\ & + \Lambda^{-4}f_{-4}\varphi^4\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{30}\Lambda^{-4}f_{-4}\partial_\mu\varphi\Delta^2\partial^\mu\varphi + \frac{1}{30}\Lambda^{-4}f_{-4}\varphi^2\Delta^2\varphi^2 \\ & + \frac{1}{6}\Lambda^{-4}f_{-4}\partial_\mu\varphi\partial^\mu\varphi\Delta\varphi^2 + \frac{1}{6}\Lambda^{-4}f_{-4}\varphi^4\Delta\varphi^2 + \frac{1}{3}\Lambda^{-4}f_{-4}\varphi^2\partial_\mu\varphi\Delta\partial^\mu\varphi \\ & + \frac{1}{6}\Lambda^{-4}f_{-4}\partial_\mu\varphi\partial^\mu\varphi\partial_\nu\varphi\partial^\nu\varphi\Big) \quad . \end{aligned} \tag{3.1}$$

*Proof.* We insert our result for $E$ from Lemma 3.3 in the coefficients $a_m$ and calculate the trace over $S_x$ to obtain our action, equation (2.3), in terms of $\varphi(x)$. To determine which terms have non-vanishing trace we will use Lemma 3.1 repeatedly.
For $a_0$, equation (2.17), $a_0 = (4\pi)^{-2}\mathrm{Tr}\,(I_4) = 4(4\pi)^{-2}$. Since the first term of $E$ is traceless, we get from $a_2$, equation (2.18) $a_2 = -(4\pi)^{-2}4\varphi^2$.
For $a_4$, equation 2.20, we have to take the trace of

$$\frac{1}{2}E^2 = \frac{1}{2}(i\gamma^5\gamma^\mu\partial_\mu\varphi - \varphi^2)^2 = \frac{1}{2}\gamma^\mu\partial_\mu\varphi\gamma^\nu\partial_\nu\varphi + \frac{1}{2}\varphi^4 - \varphi^2 i\gamma^5\gamma^\mu\partial_\mu\varphi \quad .$$

The last term has trace zero, so that $a_4 = (4\pi)^{-2}(2\varphi^4 + 2\partial_\mu\varphi\partial^\mu\varphi)$.
In the same fashion the contribution from $a_6$, equation (2.23), is obtained:

$$\begin{aligned} a_6 &= (4\pi)^{-2}\mathrm{Tr}\left(\frac{1}{6}(i\gamma^5\gamma^\mu\partial_\mu\varphi - \varphi^2)^3 - \frac{1}{12}(i\gamma^5\gamma^\mu\partial_\mu\varphi - \varphi^2)\Delta(i\gamma^5\gamma^\nu\partial_\nu\varphi - \varphi^2)\right) \\ &= (4\pi)^{-2}\mathrm{Tr}\left(-\frac{1}{2}\varphi^2\gamma^\mu\gamma^\nu\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{6}\varphi^6 - \frac{1}{12}\varphi^2\Delta\varphi^2 - \frac{1}{12}\gamma^\mu\gamma^\nu\partial_\mu\varphi\Delta\partial_\nu\varphi\right) \\ &= (4\pi)^{-2}\left(-2\varphi^2\partial_\mu\varphi\partial^\mu\varphi - \frac{2}{3}\varphi^6 - \frac{1}{3}\varphi^2\Delta\varphi^2 - \frac{1}{3}\partial_\mu\varphi\Delta\partial^\mu\varphi\right) \end{aligned}$$

Finally, from equation (2.37)

$$\begin{aligned} a_8 &= (4\pi)^{-2}\mathrm{Tr}\Big(\frac{1}{24}(i\gamma^5\gamma^\mu\partial_\mu\varphi - \varphi^2)^4 - \frac{1}{24}(i\gamma^5\gamma^\mu\partial_\mu\varphi - \varphi^2)^2\Delta(i\gamma^5\gamma^\mu\partial_\mu\varphi - \varphi^2) \\ &\quad + \frac{1}{120}(i\gamma^5\gamma^\mu\partial_\mu\varphi - \varphi^2)\Delta^2(i\gamma^5\gamma^\mu\partial_\mu\varphi - \varphi^2)\Big) \\ &= (4\pi)^{-2}\mathrm{Tr}\Big(\frac{1}{24}\gamma^5\gamma^\mu\partial_\mu\varphi\gamma^5\gamma^\nu\partial_\nu\varphi\gamma^5\gamma^\rho\partial_\rho\varphi\gamma^5\gamma^\sigma\partial_\sigma\varphi - \frac{1}{4}\varphi^4\gamma^5\gamma^\mu\partial_\mu\varphi\gamma^5\gamma^\nu\partial_\nu\varphi \\ &\quad + \frac{1}{24}\varphi^8 + \frac{1}{24}(-\gamma^5\gamma^\mu\partial_\mu\varphi\gamma^5\gamma^\nu\partial_\nu\varphi + \varphi^4)\Delta\varphi^2 + \frac{-1}{12}\varphi^2\gamma^5\gamma^\mu\partial_\mu\varphi\Delta\gamma^5\gamma^\nu\partial_\nu\varphi \\ &\quad + \frac{-1}{120}\gamma^5\gamma^\mu\partial_\mu\varphi\Delta^2\gamma^5\gamma^\nu\partial_\nu\varphi + \frac{1}{120}\varphi^2\Delta^2\varphi^2\Big) \\ &= (4\pi)^{-2}\Big(\frac{1}{6}\partial_\mu\varphi\partial^\mu\varphi\partial_\nu\varphi\partial^\nu\varphi + \varphi^4\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{6}\varphi^8 + \frac{1}{6}\partial_\mu\varphi\partial^\mu\varphi\Delta\varphi^2 + \frac{1}{6}\varphi^4\Delta\varphi^2 \\ &\quad + \frac{1}{3}\varphi^2\partial_\mu\varphi\Delta\partial^\mu\varphi + \frac{1}{30}\partial_\mu\varphi\Delta^2\partial^\mu\varphi + \frac{1}{30}\varphi^2\Delta^2\varphi^2\Big) \quad , \end{aligned}$$

where we have applied Lemma 3.2 to find the first term. Summing the coefficients $a_m$ with the factors $\Lambda^{4-m}f_{4-m}$ and integrating over $M$ gives the stated action. ∎

*Remark* 3.3. The guiding example of Chapter 1, example 1.1 is retrieved from Proposition 3.1, setting $f_{-2}$ and $f_{-4}$ to zero, see remark 2.10. Therefore we can regard the action as the extended real scalar Higgs action.

*Remark* 3.4. In the Lagrangian (1.1) from Chapter 1 $\Lambda$ appeared in the mass term and the zero-point energy, which is mainly interesting from a cosmological point of view, but not really for small scale physics. It therefore played hardly a role. In the extended case this is quite different. We see that the added terms all come with negative powers of the large cut-off parameter $\Lambda$. It acts as a regulator.

It seems natural to ask what the difference between these two theories is. To be able to answer this question we will have to explore other characteristics of this theory first.

### 3.2.1 Spontaneous symmetry breaking

In equation (3.1) we see that we have the same negative mass term here as we had in example 1.1. We will deal with this in the same way as in section 1.2. This includes that we use the same notation as before. So the field from Proposition 3.1 is called $\tilde{\varphi}$ and we will expand it according to $\tilde{\varphi} = \chi_0 + \varphi$.

*Remark* 3.5. Notice that the shape of the potential $V_{\mathcal{L}}(\chi) = \mathcal{L}[\tilde{\varphi} = \chi]$ does not depend on $\Lambda$, since all terms are of the form $\Lambda^4(\chi\Lambda^{-1})^m$. Furthermore, the minimum $\chi_0$ of the potential is determined by

$$0 = \frac{\partial V}{\partial \chi}[\chi_0] = \Lambda^3 \left( -8f_2 \left(\frac{\chi_0}{\Lambda}\right) + 8f_0 \left(\frac{\chi_0}{\Lambda}\right)^3 - 4f_{-2} \left(\frac{\chi_0}{\Lambda}\right)^5 + \frac{4}{3} f_{-4} \left(\frac{\chi_0}{\Lambda}\right)^7 \right) \quad ,$$

from which it is clear that $\chi_0 = \chi_0'\Lambda$, where $\chi_0'$ only depends on $f_2, f_0, f_{-2}$ and $f_{-4}$. Now, assuming that $V_{\mathcal{L}}(\chi)$ has a minimum we state the following lemma.

**Lemma 3.4.** The conditions for the classical minimum of the Lagrangian are:

- $0 = \frac{\partial \mathcal{L}}{\partial \tilde{\varphi}}[\chi_0] = \frac{\partial \mathcal{L}}{\partial \varphi}[0] = -8\Lambda^2 f_2 \chi_0 + 8f_0\chi_0^3 - 4\Lambda^{-2} f_{-2}\chi_0^5 + \frac{4}{3}\Lambda^{-4} f_{-4}\chi_0^7$
- $0 \leq \frac{\partial^2 \mathcal{L}}{\partial \tilde{\varphi}^2}[\chi_0] = \frac{\partial^2 \mathcal{L}}{\partial \varphi^2}[0] = -8\Lambda^2 f_2 + 24f_0\chi_0^2 - 20\Lambda^{-2} f_{-2}\chi_0^4 + \frac{28}{3}\Lambda^{-4} f_{-4}\chi_0^6$

We see that independent of which minimum we choose we will get a positive mass term.

*Proof.* This follows immediately by applying Lemma 1.2. □

*Remark* 3.6. We could ask ourselves whether it would be possible to actually determine $\chi_0$. To answer this question we notice that $V_{\mathcal{L}}(\chi)$ is symmetric in $\chi$, so we only need to determine the positive minima. The local minimum condition $\frac{\mathrm{d}V_{\mathcal{L}}}{\mathrm{d}\chi}(\chi_0) = 0$ gives us a polynomial of degree 7 with a zero at $\chi = 0$, belonging to a local maximum. Dividing by $(\chi - 0)$ and defining $\chi^2 = z$ gives us the following third order polynomial in $z$ for which we want to find the zeroes:

$$-8\Lambda^2 f_2 + 8f_0 z - 4\Lambda^{-2} f_{-2} z^2 + \frac{4}{3}\Lambda^{-4} f_{-4} z^3 = 0 \quad .$$

For this equation, the number of zeroes and their locations depend on the precise values of the coefficients $f_m$. We conclude that we can say nothing more than that there is a minimum.

Now we state a lemma that will help us to rewrite this Lagrangian.

**Lemma 3.5.** The following combinations of derivatives of a smooth function $\zeta : M \to \mathbb{R}$ can be written as:

1. $\partial_\mu\zeta\Delta^m\partial^\mu\zeta = \zeta\Delta^{m+1}\zeta + \partial_\mu(\zeta\Delta^m\partial^\mu\zeta)$, where $m \in \mathbb{Z}_{\geq 0}$;
2. $\zeta^m\Delta\zeta^2 = 2m\zeta^m\partial_\mu\zeta\partial^\mu\zeta - \partial_\mu(\zeta^m\partial^\mu\zeta^2)$
$= \frac{2m}{m+1}\zeta^{m+1}\Delta\zeta - \partial_\mu(\zeta^m\partial^\mu\zeta^2 - \frac{2m}{m+1}\zeta^{m+1}\partial^\mu\zeta)$, where $m \in \mathbb{Z}_{\geq 0}$;
3. $\zeta\Delta^2\zeta^2 = \zeta^2\Delta^2\zeta + \partial_\mu(\zeta^2\partial^\mu\Delta\zeta + \partial^\mu\zeta\Delta\zeta^2 - \Delta\zeta\partial^\mu\zeta^2 - \zeta\partial^\mu\Delta\zeta^2)$;
4. $\partial_\mu\zeta\partial^\mu\zeta\Delta\zeta = \zeta\Delta\zeta\Delta\zeta - \frac{1}{2}\zeta^2\Delta^2\zeta + \partial_\mu(\zeta\partial^\mu\zeta\Delta\zeta - \frac{1}{2}\zeta^2\partial^\mu\Delta\zeta)$;
5. $\zeta\partial_\mu\zeta\partial^\mu\Delta\zeta = \frac{1}{2}\zeta^2\Delta^2\zeta + \partial_\mu(\frac{1}{2}\zeta^2\partial^\mu\Delta\zeta)$;
6. $\zeta^2\Delta^2\zeta^2 = 4\partial_\mu\zeta\partial^\mu\zeta\partial_\nu\zeta\partial^\nu\zeta + \frac{4}{3}\zeta^3\Delta^2\zeta + \partial_\mu(\frac{4}{3}\zeta^3\partial^\mu\Delta\zeta - 4\zeta^2\partial^\mu\zeta\Delta\zeta + \partial^\mu\zeta^2\Delta\zeta^2 - \zeta^2\partial^\mu\Delta\zeta^2)$;
7. $\partial_\mu\zeta\partial^\mu\zeta\Delta\zeta^2 = -2\partial_\mu\zeta\partial^\mu\zeta\partial_\nu\zeta\partial^\nu\zeta + \zeta^2\Delta\zeta\Delta\zeta - \frac{1}{3}\zeta^3\Delta^2\zeta + \partial_\mu(\zeta^2\partial^\mu\zeta\Delta\zeta - \frac{1}{3}\zeta^3\partial^\mu\Delta\zeta)$;
8. $\zeta^2\partial_\mu\zeta\partial^\mu\Delta\zeta = \frac{1}{3}\zeta^3\Delta^2\zeta + \partial_\mu(\frac{1}{3}\zeta^3\partial^\mu\Delta\zeta)$.

*Proof.* All these identities can be proved by writing out the right hand side. □

**Proposition 3.2.** [extended real scalar Higgs Lagrangian]
Classifying the interaction vertices $\{i\}$ with $i = (n, k)$, with $n$ the valence of the vertex and $k$ the total number of derivatives, the Lagrangian $\mathcal{L}[\varphi]$ of the action $S[\tilde{\varphi}]$ from Proposition 3.1 with respect to the background field is given by

| | |
|---|---|
| $\mathcal{L}[\varphi(x)] = (4\pi)^{-2}\big($ | |
| $4\Lambda^4 f_4 - 4\Lambda^2 f_2\chi_0^2 + 2f_0\chi_0^4 - \frac{2}{3}f_{-2}\Lambda^{-2}\chi_0^6 + \frac{1}{6}\Lambda^{-4}f_{-4}\chi_0^8$ | zero-point energy |
| $+\frac{1}{2}m^2\varphi^2(x)$ | mass term |
| $+\frac{1}{2}\varphi(x)\big((4f_0 - \frac{20}{3}\Lambda^{-2}f_{-2}\chi_0^2 + \frac{14}{3}\Lambda^{-4}f_{-4}\chi_0^4)\Delta$ | |
| $+(-\frac{2}{3}\Lambda^{-2}f_{-2} + \frac{14}{15}\Lambda^{-4}f_{-4}\chi_0^2)\Delta^2 + \frac{1}{15}\Lambda^{-4}f_{-4}\Delta^3\big)\varphi(x)$ | kinetic terms |
| $+(8f_0\chi_0 - \frac{40}{3}\Lambda^{-2}f_{-2}\chi_0^3 + \frac{28}{3}\Lambda^{-4}f_{-4}\chi_0^5)\varphi^3(x)$ | $i = (3, 0)$ |
| $+(2f_0 - 10\Lambda^{-2}f_{-2}\chi_0^2 + \frac{35}{3}\Lambda^{-4}f_{-4}\chi_0^4)\varphi^4(x)$ | $i = (4, 0)$ |
| $+(-4\Lambda^{-2}f_{-2}\chi_0 + \frac{28}{3}\Lambda^{-4}f_{-4}\chi_0^3)\varphi^5(x)$ | $i = (5, 0)$ |
| $+(-\frac{2}{3}\Lambda^{-2}f_{-2} + \frac{14}{3}\Lambda^{-4}f_{-4}\chi_0^2)\varphi^6(x)$ | $i = (6, 0)$ |
| $+\frac{4}{3}\Lambda^{-4}f_{-4}\chi_0\varphi^7(x)$ | $i = (7, 0)$ |
| $+\frac{1}{6}\Lambda^{-4}f_{-4}\varphi^8(x)$ | $i = (8, 0)$ |
| $+(-\frac{10}{3}\Lambda^{-2}f_{-2}\chi_0 + \frac{14}{3}\Lambda^{-4}f_{-4}\chi_0^3)\varphi^2(x)\Delta\varphi(x)$ | $i = (3, 2)$ |
| $+(-\frac{10}{9}\Lambda^{-2}f_{-2} + \frac{14}{3}\Lambda^{-4}f_{-4}\chi_0^2)\varphi^3(x)\Delta\varphi(x)$ | $i = (4, 2)$ |
| $+\frac{7}{3}\Lambda^{-4}f_{-4}\chi_0\varphi^4(x)\Delta\varphi(x)$ | $i = (5, 2)$ |
| $+\frac{7}{15}\Lambda^{-4}f_{-4}\varphi^5(x)\Delta\varphi(x)$ | $i = (6, 2)$ |
| $+\frac{3}{10}\Lambda^{-4}f_{-4}\chi_0\varphi^2(x)\Delta^2\varphi(x) + \frac{1}{3}\Lambda^{-4}f_{-4}\chi_0\varphi(x)\Delta\varphi(x)\Delta\varphi(x)$ | $i = (3, 4)_{\mathrm{a,b}}$ |
| $+\frac{1}{10}\Lambda^{-4}f_{-4}\varphi^3(x)\Delta^2\varphi(x) + \frac{1}{6}\Lambda^{-4}f_{-4}\varphi^2(x)\Delta\varphi(x)\Delta\varphi(x)$ | |
| $-\frac{1}{30}\Lambda^{-4}f_{-4}\partial_\mu\varphi(x)\partial^\mu\varphi(x)\partial_\nu\varphi(x)\partial^\nu\varphi(x)\ \big)$ . | $i = (4, 4)_{\mathrm{a,b,c}}$ |

*Proof.* Starting from equation (3.1), we substitute $\tilde{\varphi}(x) = \chi_0 + \varphi(x)$. By Lemma 1.2 and Lemma 3.4 we see that the terms linear in $\varphi(x)$ vanish at tree level:

$$\frac{\partial\mathcal{L}}{\partial\varphi}[0]\varphi(x) = \left(-8\Lambda^2 f_2\chi_0 + 8f_0\chi_0^3 - 4\Lambda^{-2}f_{-2}\chi_0^5 + \frac{4}{3}\Lambda^{-4}f_{-4}\chi_0^7\right)\varphi(x) = 0 \quad .$$

We neglect the terms

$$-\frac{1}{3}\Lambda^{-2}f_{-2}\chi_0^2\Delta\varphi^2(x)\ , \quad -\frac{2}{3}\Lambda^{-2}f_{-2}\chi_0^3\Delta\varphi(x)\ , \quad \frac{1}{30}\Lambda^{-4}f_{-4}\chi_0^2\Delta^2\varphi^2(x) \quad ,$$
$$\frac{1}{15}\Lambda^{-4}f_{-4}\chi_0^3\Delta^2\varphi(x)\ , \quad \frac{1}{6}\Lambda^{-4}f_{-4}\chi_0^4\Delta\varphi^2(x) \text{ and } \frac{1}{3}\Lambda^{-4}f_{-4}\chi_0^5\Delta\varphi(x) \quad ,$$

since they are total derivatives and therefore will vanish under the integral.
Ordering the terms we find that

$$\frac{1}{2}m^2 = -4\Lambda^2 f_2 + 12 f_0\chi_0^2 - 10\Lambda^{-2}f_{-2}\chi_0^4 + \frac{14}{3}\Lambda^{-4}f_{-4}\chi_0^6 \quad . \tag{3.2}$$

The kinetic terms all have two $\varphi$'s and various derivatives acting on them. By applying Lemma 3.5 to these terms and neglecting the total derivatives, since they vanish under the integral, we find the stated expression.
All interaction terms are similarly found by gathering the contributing terms and rewriting them if necessary by the identities from Lemma 3.5, where the total derivatives should be neglected as well. □

**Corollary 3.1.** The Lagrangian from Proposition 3.2 can be trivially rewritten as

$$\begin{aligned}\mathcal{L}[\varphi(x)] = (4\pi)^{-2}\big(&C_0 + \frac{1}{2}m^2\varphi^2(x) + \frac{1}{2}\varphi(x)q_{\Delta_x}(\Lambda,\chi)\varphi(x) + \lambda_3\varphi^3(x) + \lambda_4\varphi^4(x)\\
&+ \lambda_5\varphi^5(x) + \lambda_6\varphi^6(x) + \lambda_7\varphi^7(x) + \lambda_8\varphi^8(x) + \xi_3\varphi^2(x)\Delta\varphi(x) + \xi_4\varphi^3(x)\Delta\varphi(x)\\
&+ \xi_5\varphi^4(x)\Delta\varphi(x) + \xi_6\varphi^5(x)\Delta\varphi(x) + \vartheta_{3a}\varphi^2(x)\Delta^2\varphi(x) + \vartheta_{3b}\varphi(x)\Delta\varphi(x)\Delta\varphi(x)\\
&+ \vartheta_{4a}\varphi^3(x)\Delta^2\varphi(x) + \vartheta_{4b}\varphi^2(x)\Delta\varphi(x)\Delta\varphi(x)\\
&+ \vartheta_{4c}\partial_\mu\varphi(x)\partial^\mu\varphi(x)\partial_\nu\varphi(x)\partial^\nu\varphi(x)\big) \quad ,\end{aligned}$$

where the order of the terms is kept the same. The mass term $\frac{1}{2}m^2$ is given in equation (3.2).

*Remark* 3.7. Notice that we could have removed the $f_2$-dependence or the $\chi_0^6$-term of the mass, using the first equation of Lemma 3.4 for the minimum of the potential. However, these equations are only true at lowest order and using them to modify the Lagrangian ruins the renormalizability of the theory.

*Remark* 3.8. In Chapter 1 we already used $\Lambda$ as a regulator, see paragraph 1.4.2. In Remark 2.9 we saw that the requirement originates from the asymptotic expansion of the spectral action that made it possible to describe the action in physically more practical terms. Continuing, we saw that the limit $\Lambda \to \Lambda_0 + \infty$ suppressed the non-trivial momentum dependence.
We expect that the same will happen here. It allows the addition of two kinetic terms, suppressing them by negative powers of $\Lambda$, which will make the propagator decrease faster in the UV-limit. Furthermore, it will effectively kill the dimensionally impossible vertices at tree level, see Lemma 1.6.

## 3.3 Higher derivative field theory of the ershl

From the Lagrangian obtained in Proposition 3.2 we want an interpretation as a physical quantum field theory. Again, we will do this parallel to Chapter 1.

**Lemma 3.6.** The equation of free motion is given by

$$\begin{aligned}0 = \frac{1}{(4\pi)^2}[&m^2 + (4f_0 - \frac{20}{3}\Lambda^{-2}f_{-2}\chi_0^2 + \frac{14}{3}\Lambda^{-4}f_{-4}\chi_0^4)\Delta\\
&+ (-\frac{2}{3}\Lambda^{-2}f_{-2} + \frac{14}{15}\Lambda^{-4}f_{-4}\chi_0^2)\Delta^2 + \frac{1}{15}\Lambda^{-4}f_{-4}\Delta^3]\varphi(x) = q_{\Delta_x}(\Lambda,\chi)\varphi(x) \quad .\end{aligned}$$

*Proof.* Apply Definition 1.1 to the free Lagrangian

$$\mathcal{L}_0 = (4\pi)^2\big(\frac{1}{2}m^2\varphi^2(x) + \frac{1}{2}\varphi(x)(4f_0 - \frac{20}{3}\Lambda^{-2}f_{-2}\chi_0^2 + \frac{14}{3}\Lambda^{-4}f_{-4}\chi_0^4)\Delta\varphi(x) + \frac{1}{2}\varphi(x)(-\frac{2}{3}\Lambda^{-2}f_{-2} + \frac{14}{15}\Lambda^{-4}f_{-4}\chi_0^2)\Delta^2\varphi(x) + \frac{1}{2}\varphi(x)\frac{1}{15}\Lambda^{-4}f_{-4}\Delta^3\varphi(x)\big) \quad .$$

□

**Lemma 3.7.** The Fourier transformed $\hat{D}(p)$ of the propagator $D$ is given by

$$\hat{D}(p) = \frac{16\pi^2\Lambda^4}{\frac{f_{-4}}{15}p^6 + (-\frac{2}{3}\Lambda^2 f_{-2} + \frac{14}{15}f_{-4}\chi_0^2)p^4 + (4\Lambda^4 f_0 - \frac{20}{3}\Lambda^2 f_{-2}\chi_0^2 + \frac{14}{3}f_{-4}\chi_0^4)p^2 + m^2\Lambda^4} \ .$$

*Proof.* From Lemma 1.1 this follows in one step. □

*Remark* 3.9. If $\hat{D}$ is well-defined, we need that the terms multiplying $p^4$ and $p^2$ are positive. Lemma 3.4 is not sufficient for this, so we have to make additional requirements about the values of $f_k$.

Since the expression for the propagator is notationally not very practical, we introduce the following notation.

**Definition 3.2.** We define the coefficients $\alpha, \beta, \gamma$ and $\delta$ such that the propagator can be written as

$$\hat{D}(p) = \frac{16\pi^2\Lambda^4\alpha^{-1}}{p^6 + \beta p^4 + \gamma p^2 + \delta} \quad , \tag{3.3}$$

where

$$\begin{aligned}
\alpha &= \frac{f_{-4}}{15} \quad ; \\
\beta &= \Lambda^2\frac{30}{f_{-4}}(-\frac{1}{3}f_{-2} + \frac{7}{15}f_{-4}(\chi_0')^2) \quad ; \\
\gamma &= \Lambda^4\frac{30}{f_{-4}}(2f_0 - \frac{10}{3}f_{-2}(\chi_0')^2 + \frac{7}{3}f_{-4}(\chi_0')^4) \quad ; \\
\delta &= \Lambda^6\frac{30}{f_{-4}}(-4f_2 + 12f_0(\chi_0')^2 - 10f_{-2}(\chi_0')^4 + \frac{14}{3}f_{-4}(\chi_0')^6) \quad .
\end{aligned}$$

*Remark* 3.10. Because $f_2, f_0, f_{-4} \geq 0$ and $f_{-2} \leq 0$ it is clear that $\alpha, \beta$ and $\gamma$ are real and positive. And since $\alpha > 0$ and we require that $0 < m^2\Lambda^2\alpha^{-1}$ we know that $\delta$ is positive as well.

*Remark* 3.11. Since the denominator of the propagator is non-zero for all values of $p$, we can substitute for the moment $p^2 \equiv z$. This denominator is the cubic function $z^3 + \beta z^2 + \gamma z + \delta$. The discriminant of this cubic function is $18\beta\gamma\delta - 4\beta^3\delta + \beta^2\gamma^2 - 4\gamma^3 - 27\delta^2$. If both the discriminant and $\beta^2 - 3\alpha\gamma$ are zero, the polynomial has one real root and therefore equals $(z - r_1)^3$, but if only the discriminant is zero, then there are 2 real roots of which one occurs twice. If the discriminant is positive, it has three distinct real roots. These three roots would all be negative. If the discriminant is negative, it has one negative real root and two complex roots, of which one is the complex conjugated of the other, so all three roots are distinct again.
Because we do not know the value of $\chi_0$, we cannot determine the sign of the discriminant. We see this theory as an extension of the one discussed in Chapter 1, see example 1.2. This means that we treat the additional parameters $f_{-2}$ and $f_{-4} \neq 0$ as free and tunable. Therefore we can assume that the discriminant is non-zero and has

one fixed sign and so three distinct roots. This makes that we can write the propagator as

$$\hat{D}(p) = \frac{16\pi^2\Lambda^4\alpha^{-1}}{(p^2 - l_a)(p^2 - l_b)(p^2 - l_c)} \quad , \tag{3.4}$$

where it is clear from Definition 3.2 and dimensional arguments that $l_\iota = \Lambda^2 l'_\iota$.

### 3.3.1 Interactions

In the same way as in Chapter 1 we determine the interaction vertices. Again their strength is given by $-1$ times the coupling constant. This we demonstrated in paragraph 1.2.1. However, we now have several more interactions.
From Proposition 3.2 we can read off what the interaction vertices are. We have drawn them in Figure 3.1. The first row consists of the $(n, 0)$-vertices, the second of the $(n, 2)$-vertices, the third of the $(4, 4)$-vertices and the last one of the $(3, 4)$-vertices. When a Laplacian is acting on a field, this is represented by a small double line around the corresponding line. Two Laplacians are then represented by a quadruple line. The only exception to this is the third diagram on the third row, corresponding to term $(4, 4)_c$ in Proposition 3.2, since it has no Laplacians, but partial derivatives acting on the fields. In interaction vertex $(4, 4)_c$ the derivative lines connected by their endpoints are contracted.

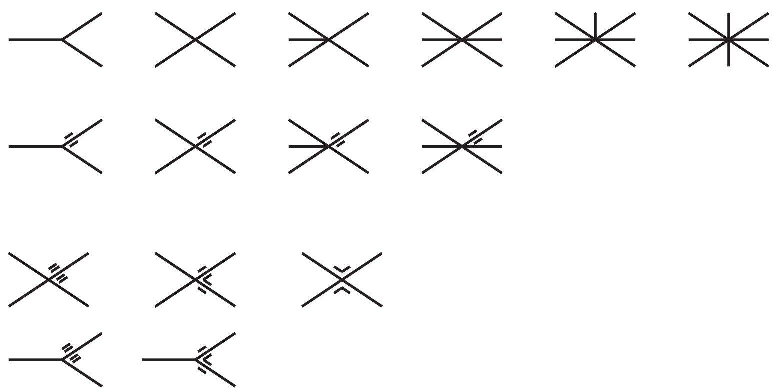

Figure 3.1: All vertices of the Lagrangian from Proposition 3.2.

### 3.3.2 Power counting

As in section 1.3 the superficial degree of divergence depends on the specific vertices one has. Therefore we perform the same analysis as in Lemma 1.3 to find all superficially divergent diagrams.

**Lemma 3.8.** For the ERSHL the superficial degree of divergence $\omega$ for a connected Feynman diagram is given by

$$\omega = 4 + N - \sum_i (n_i + 4 - k_i)V_i \quad ,$$

where $N$ is the number of external particles, $V_i$ is the number of vertices of type $i$, $k_i$ the number of derivatives on loop momenta in vertex $i$ and $n_i$ is the valence of interaction $i$.

*Proof.* The momentum dependence $p^\omega$ of a diagram in the UV-limit is given by $\omega = 4L - 6P + \sum_i k_i V_i$, since every loop comes with an integral $\int d^4p$, every derivative in a loop adds $p^1$ and every propagator adds a factor $p^{-6}$.
As explained in Lemma 1.3 the number of loops is given by $L = P + 1 - \sum_i V_i$. Furthermore, in a diagram each propagator connects to two vertices and every external line to one vertex, giving that $\sum_i n_i V_i = N + 2P$. Combining this all results in

$$\omega = 4L - 6P + \sum_i k_i V_i = -2P + 4 - 4\sum_i V_i + \sum_i k_i V_i = 4 + N - \sum_i (n_i + 4 - k_i) V_i \ .$$

□

**Proposition 3.3.** For a connected, amputated Feynman ersHL-diagram with $N \geq 1$ having $\omega \geq 0$:

1. the loop number $L = 1$;
2. no vertex has $k_i = 0$;
3. if $k_i = 2$ for some vertex, then $\sum_i V_i = 1$, there is only one vertex;
4. if $k_i = 4$ for some vertex, then $1 \leq \sum_i V_i \leq 2$;
5. $1 \leq N \leq 4$;

with the same notation as in Lemma 3.8.

*Proof.* We rewrite using $L = P + 1 - \sum_i V_i$

$$\omega = 4L - 6P + \sum_i k_i V_i = 6 - 2L + \sum_i (k_i - 6) V_i \quad .$$

Since $k_i - 6 \leq -2$, we see that for $L = 2$ only one vertex with $k_i = 4$ is allowed, but it is not possible to have two loops and an external line on one vertex with four derivatives, see Figure 3.1.
For the same reason $L \geq 3$ is no possibility and for $L = 0$ there is no UV-divergence. So, all divergent diagrams must have $L = 1$.
For $L = 1$, the condition for divergence reads

$$4 + \sum_i (k_i - 6) V_i \geq 0 \quad .$$

We see that a vertex with $k_i = 0$ will make the diagram convergent. For $k_i = 2$, only diagrams with one vertex will be divergent. And for $k_i = 4$ a diagram with either one or two vertices is divergent. This proofs 2, 3 and 4.
The condition $L = P + 1 - \sum_i V_i$ now becomes $P = \sum_i V_i$, so that we can write $\sum_i n_i V_i = N + 2P$ as $N = \sum_i (n_i - 2) V_i$. From statement 3 and 4 above in combination with the set of vertices, see Figure 3.1, we can read that one $(n, k) = (6, 2)$ vertex or two $(4, 4)$-vertices give $N = 4$. See also Figure 3.2. □

**Corollary 3.2.** According to the classification of Definition 1.7 the theory defined by the Lagrangian from Proposition 3.2 is superrenormalizable.

*Remark* 3.12. The $N \geq 1$ requirement is needed to connect the diagram to other diagrams or serve as an external line, which can be observed. Diagrams with $N = 0$ contribute to the zero-point energy only, which we neglect.
However, all $L = 2$ divergent vacuum bubbles can be found by connecting the vertices of the third row in Figure 3.1 in all five possible ways.

*Remark* 3.13. Since we will work with these diagrams extensively in Chapter 4, we will number the diagrams from left to right, up to down. So the diagram in the upper left corner is number 1, the diagram on row nine is number 25.

To conclude this chapter we make some remarks on what we can see from this. By extending the Lagrangian (1.1) to that of Proposition 3.1 the theory has become better behaved in the UV-regime. One may wonder whether such improvements are made as well when further extending the theory. From Chapter 2 we understand that the contribution from $a_{2n}$ to the Lagrangian probably consists of all terms with an even number of fields and an even number of derivatives, so that it has mass dimension $2n$, which is then corrected with powers of $\Lambda$ to bring the mass dimension back to 4. This means that the propagator will have as highest power of the momentum $k^{2(n-1)}$ and the three- and four-point vertices will have up to $2(n-2)$ derivatives. From simple power counting we can see then that diagrams like number 5 and 21 from Figure 3.2, with the maximal number of derivatives on the propagators, will remain divergent. Hence, such a Higgs Lagrangian will not become a finite theory for any $n$.

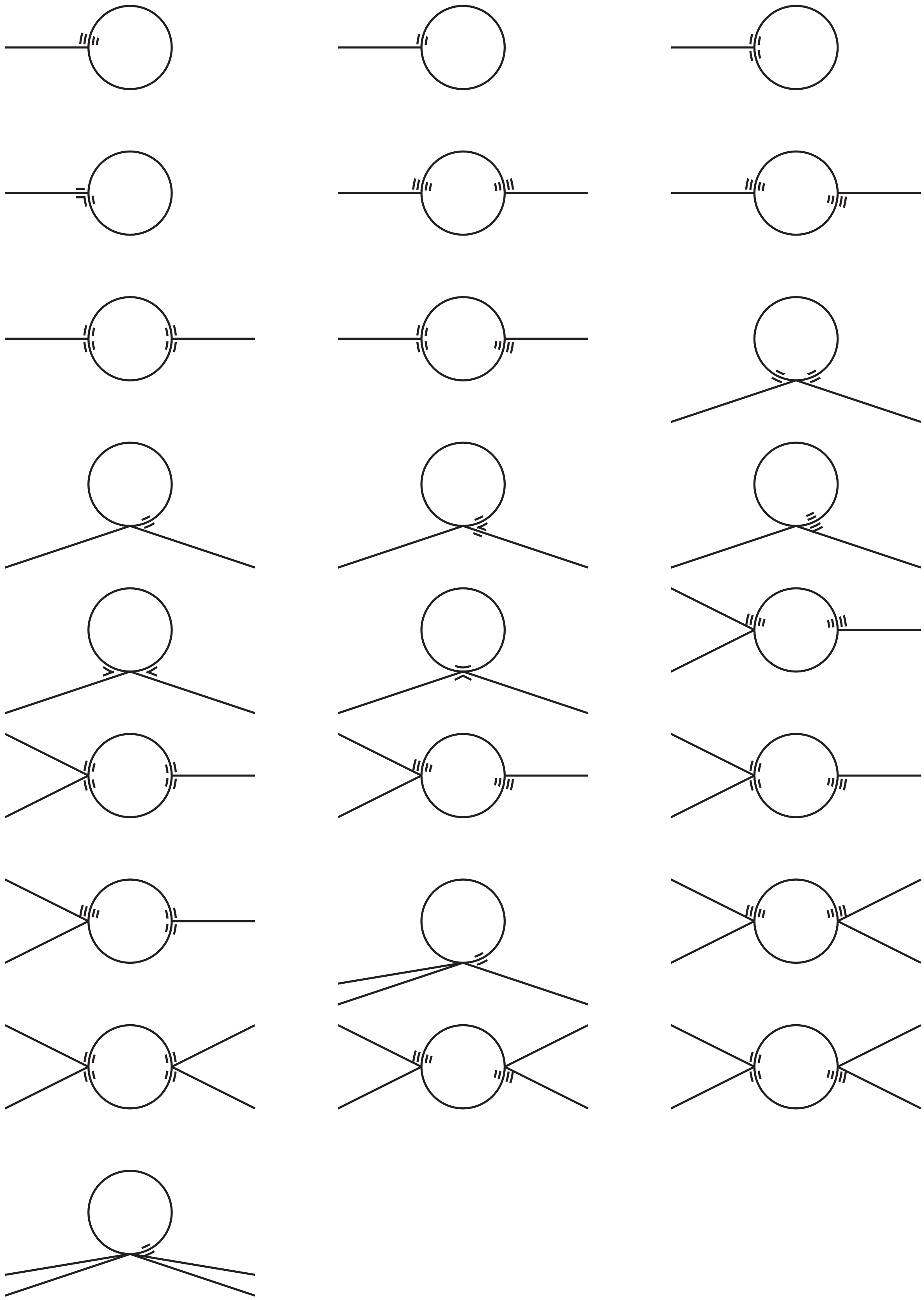

Figure 3.2: All divergent 1PI-diagrams, numbered 1-25 from left to right, top to bottom.

# 4 The UV-behaviour of the ersHL

In this chapter we will determine all parts of the superficially divergent diagrams found in paragraph 3.3.2 that are relevant for the $\beta$-functions. This we will do by the hybrid regularization procedure and subtraction operator $\overline{\overline{\mathfrak{T}}}_2$ introduced in paragraph 1.4.4, where we first dimensionally regularize the expressions and then take the cutoff parameter $\Lambda - \Lambda_0$ to infinity. We will determine the divergent amplitudes for all superficially divergent diagrams from Figure 3.2 and compare the $\beta$-functions herefore with (1.39)-(1.42) found in Chapter 1.
Recall that the scattering amplitude corresponding to diagram $n$, renormalized by the subtraction operator $\overline{\overline{\mathfrak{T}}}_2$, is called $\mathcal{M}_n^{(2)}$, where the numbering is introduced in Remark 3.13.

## 4.1 Calculation methods

**Lemma 4.1.** Suppose that $l_a, l_b$ and $l_c$ are different from each other and that furthermore $l_a^2(l_c - l_b) + l_b^2(l_a - l_c) + l_c^2(l_b - l_a) \neq 0$ . Then there exists the following partial fraction decomposition:

$$\frac{1}{(z-l_a)(z-l_b)(z-l_c)} = \frac{A}{z-l_a} + \frac{B}{z-l_b} + \frac{C}{z-l_c} \quad , \tag{4.1}$$

where $A, B$ and $C$ are given by:

$$\begin{aligned} A &= \frac{1}{(l_a-l_b)(l_a-l_c)} \\ B &= \frac{1}{(l_b-l_a)(l_b-l_c)} \\ C &= \frac{1}{(l_c-l_a)(l_c-l_b)} \quad . \end{aligned}$$

*Proof.* By rewriting the right hand side of equation (4.1) as one rational function we see that we have to solve

$$1 = A(z-l_b)(z-l_c) + B(z-l_a)(z-l_c) + C(z-l_a)(z-l_b) \quad .$$

Since this should be true for all values of $z$, this can be written in matrix form as

$$\begin{pmatrix} 1 & 1 & 1 \\ -l_b-l_c & -l_a-l_c & -l_a-l_b \\ l_bl_c & l_al_c & l_al_b \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad ,$$

where the first row of the matrix are the terms multiplying $z^2$, the second row those multiplying $z$ and the third row are the constant terms. Under the assumptions the determinant of this matrix is non-zero, so we can invert it. This gives us the expressions for $A, B$ and $C$. □

**Lemma 4.2.** For $\iota = a, b, c$ we can expand

$$\frac{1}{(\tau+\frac{k}{\Lambda})^2 - l_\iota} = \frac{1}{\tau^2 - l_\iota} - \frac{2\tau\cdot\frac{k}{\Lambda} + \frac{k^2}{\Lambda^2}}{(\tau^2-l_\iota)^2} + \frac{4\left(\tau\cdot\frac{k}{\Lambda}\right)^2}{(\tau^2-l_\iota)^3} + \mathcal{O}(\Lambda^{-3}) \quad .$$

*Proof.* We start with the identity

$$\frac{1}{a+b} = \frac{1}{a} - \frac{b}{a^2} + \frac{b^2}{a^3} - \frac{b^3}{a^3}\frac{1}{a+b} \quad . \tag{4.2}$$

Setting $b = 2\tau\cdot\frac{k}{\Lambda} + \frac{k^2}{\Lambda^2}$ and $a = \tau^2 - l_\iota$, we find that

$$\frac{1}{(\tau+\frac{k}{\Lambda})^2 - l_\iota} = \frac{1}{\tau^2 - l_\iota} - \frac{2\tau\cdot\frac{k}{\Lambda} + \frac{k^2}{\Lambda^2}}{(\tau^2-l_\iota)^2} + \frac{4\left(\tau\cdot\frac{k}{\Lambda}\right)^2}{(\tau^2-l_\iota)^3} + \mathcal{O}(\Lambda^{-3}) \quad .$$

□

Lemma 4.1 is just a special case of the following more general lemma.

**Lemma 4.3.** For $Q(z) = \prod_{i=1}^{3}(z-l_i)^{\nu_i}$, $m \in \mathbb{Z}_{\geq 0}$ and $\nu_i \in \mathbb{N}$ such that $m < \nu_1 + \nu_2 + \nu_3$ and $l_i \in \mathbb{C}$ there exists the partial fraction decomposition

$$\frac{z^m}{Q} = \sum_{j=1}^{3}\sum_{k=1}^{\nu_j} A_{(j,k)}\frac{1}{(z-l_j)^k} \quad ,$$

where $A_{(j_1,k)}$ is given by

$$A_{(j_1,k)} = (-1)^{m+\nu_{j_1}-k} \sum_{t=0}^{\min(m,\nu_{j_1}-k)} \begin{pmatrix} m \\ t \end{pmatrix} (-l_{j_1})^{m-t} \sum_{s=0}^{\nu_{j_1}-k-t} (l_{j_1}-l_{j_2})^{-\nu_{j_2}-s}$$
$$\times \begin{pmatrix} \nu_{j_2}+s-1 \\ s \end{pmatrix} \begin{pmatrix} \nu_{j_3}+\nu_{j_1}-k-t-s-1 \\ \nu_{j_1}-k-t-s \end{pmatrix} (l_{j_1}-l_{j_3})^{-\nu_{j_3}-\nu_{j_1}+k+t+s} \quad ,$$

where $\{j_1, j_2, j_3\} = \{1, 2, 3\}$ and $(f)^{(m)}(x)$ denotes the $m$-th derivative of the function $f$ evaluated in $x$.

*Proof.* We recognize the Taylor expansion of order $\nu_j - 1$ of

$$z^m\frac{(z-l_j)^{\nu_j}}{Q} = \sum_{k=1}^{\nu_j} A_{(j,k)}(z-l_j)^{\nu_j-k} + \mathcal{O}((z-l_j)^{\nu_j}) \quad ,$$

where $A_{(j,k)} = \frac{1}{(\nu_j-k)!}\left(z^m\frac{(z-l_j)^{\nu_j}}{Q}\right)^{(\nu_j-k)}(l_j)$. Both sides of

$$z^m - Q\sum_{k=1}^{\nu_j} A_{(j,k)}(z-l_j)^{-k} = \mathcal{O}((z-l_j)^{\nu_j})$$

must be divisible by $(z-l_j)^{\nu_j}$ over the polynomials. Clearly, $Q\sum_{k=1}^{\nu_j} A_{(j,k)}(z-l_j)^{-k}$ can be divided by $(z-l_i)^{\nu_i}$ for $i \neq j$. This implies that

$$z^m - Q\sum_{j=1}^{3}\sum_{k=1}^{\nu_j} A_{(j,k)}(z-l_j)^{-k}$$

can also be divided by $Q$ over the polynomials. However,

$$\deg(Q\sum_{j=1}^{3}\sum_{k=1}^{\nu_j} A_{(j,k)}(z-l_j)^{-k}) < \deg(Q) \quad ,$$

so that

$$z^m - Q\sum_{j=1}^{3}\sum_{k=1}^{\nu_j} A_{(j,k)}(z-l_j)^{-k} = 0 \quad .$$

Dividing by $Q$ gives the Taylor expansion.
The stated coefficient $A_{(j_1,k)}$ is found for $1 \leq k \leq \nu_{j_1}$ by

$$\begin{aligned}
A_{(j_1,k)} &= \frac{1}{(\nu_{j_1}-k)!}\left(\frac{\mathrm{d}^{\nu_{j_1}-k}}{\mathrm{d}z^{\nu_{j_1}-k}}\frac{z^m}{(z-l_{j_2})^{\nu_{j_2}}(z-l_{j_3})^{\nu_{j_3}}}\right)(l_{j_1}) \\
&= \sum_{t=0}^{\nu_{j_1}-k}\begin{pmatrix}\nu_{j_1}-k\\ t\end{pmatrix}(z^m)^{(t)}(l_{j_1})\frac{1}{(\nu_{j_1}-k)!}\sum_{s=0}^{\nu_{j_1}-k-t}\begin{pmatrix}\nu_{j_1}-k-t\\ s\end{pmatrix} \\
&\left(\frac{1}{(z-l_{j_2})^{\nu_{j_2}}}\right)^{(s)}(l_{j_1})\times\left(\frac{1}{(z-l_{j_3})^{\nu_{j_3}}}\right)^{(\nu_{j_1}-k-t-s)}(l_{j_1}) \\
&= \sum_{t=0}^{\nu_{j_1}-k}\begin{pmatrix}\nu_{j_1}-k\\ t\end{pmatrix}(z^m)^{(t)}(-l_{j_1})\frac{(-1)^{m+t}}{(\nu_{j_1}-k)!} \\
&\times\sum_{s=0}^{\nu_{j_1}-k-t}(-1)^{\nu_{j_1}-k-t}\begin{pmatrix}\nu_{j_1}-k-t\\ s\end{pmatrix}\frac{(\nu_{j_2}+s-1)!}{(\nu_{j_2}-1)!}(l_{j_1}-l_{j_2})^{-\nu_{j_2}-s} \\
&\times\frac{(\nu_{j_3}+\nu_{j_1}-k-t-s-1)!}{(\nu_{j_3}-1)!}(l_{j_1}-l_{j_3})^{-\nu_{j_3}-\nu_{j_1}+k+t+s} \\
&= (-1)^{m+\nu_{j_1}-k}\sum_{t=0}^{\nu_{j_1}-k}\frac{1}{t!}(z^m)^{(t)}(-l_{j_1})\sum_{s=0}^{\nu_{j_1}-k-t}\begin{pmatrix}\nu_{j_2}+s-1\\ s\end{pmatrix} \\
&\times(l_{j_1}-l_{j_2})^{-\nu_{j_2}-s}\begin{pmatrix}\nu_{j_3}+\nu_{j_1}-k-t-s-1\\ \nu_{j_1}-k-t-s\end{pmatrix}(l_{j_1}-l_{j_3})^{-\nu_{j_3}-\nu_{j_1}+k+t+s} \quad .
\end{aligned}$$

□

*Remark* 4.1. The formula for $A_{(j_1,k)}$ in the above lemma is symmetric under interchanging $j_2$ and $j_3$, as it should be.

**Lemma 4.4.** For $\tau \notin \mathbb{R}_+$:

$$\int_0^\infty \mathrm{d}z\,\frac{z^\alpha}{(z-\tau)^\beta} = \frac{\Gamma(\beta-\alpha-1)\Gamma(\alpha+1)}{\Gamma(\beta)}(-\tau)^{\alpha-\beta+1} \quad .$$

*Proof.* Apply the following transformation to the left hand side

$$y \equiv \frac{-\tau}{z-\tau} \quad ,$$

yielding

$$\int_0^\infty \mathrm{d}z \, \frac{z^\alpha}{(z-\tau)^\beta} = (-\tau)^{\alpha-\beta+1} \int_0^1 \mathrm{d}y \, y^{\beta-\alpha-2}(1-y)^\alpha = (-\tau)^{\alpha-\beta+1} B(\beta-\alpha-1, \alpha+1) \, ,$$

which equals the right hand side by the standard properties of the Beta-function. □

**Lemma 4.5.** The subtraction operator $\overline{\mathfrak{T}}_2$ completely subtracts any ersHL-diagram $I_{n\geq 5}$ with five or more external lines:

$$\overline{\mathfrak{T}}_2 I_n = 0 \quad .$$

*Proof.* In the extended real scalar Higgs Lagrangian of Proposition (3.2) only the field $\varphi$, $\Lambda$ and $\partial$ have non-zero mass dimension, if we write $\chi_0 = \chi_0'\Lambda$. Since the action must be dimensionless, any ersHL-diagram has dimensionality $4-n$, where $n$ is the number of external lines and the 4 is the dimensionality of spacetime. Since the expansion from Lemma 4.2 only gives powers of momenta in the numerator, any diagram with $n \geq 5$ must be proportional to $\Lambda^{-m}$, where $m \geq n-4$. By Lemma 1.6 it is now immediately clear that the diagram vanishes. □

### 4.1.1 Integration without Feynman parameters

Before we actually calculate some diagrams in this higher order theory we will make clear why we cannot use Feynman parameters here. We saw in remarks 3.10 and 3.11 that the coefficients $\alpha, \beta, \gamma$ and $\delta$ of the propagator are positive, but that the roots of the cubic polynomial $z^3 + \beta z^2 + \gamma z + \delta$ are distinct and not real and positive. However, it is possible that there are two complex roots that both have a positive real part. Since they would be each others conjugate, one would lie above the real axis, the other one below.
When we would combine these roots using Schwinger's trick, we would eventually have to integrate over all convex combinations of the three roots, and would not be able to avoid the positive real axis, see Figure 4.1. Said differently, we cannot guarantee that the roots satisfy the condition $\mathrm{Re}(-l_\iota) > 0$ from Lemma 1.7.

### 4.1.2 Integration method

In paragraph 1.5.2 the loop diagrams were integrated by means of an asymptotic expansion, instead of Schwinger's trick. We mentioned there that we needed a different integration prescription and in paragraph 4.1.1 we saw why. However, one may wonder whether another suitable method exists. If that is the case, $\Lambda$ will be a finite parameter again and consequently will lose its function as regulator. If we sacrifice the role of $\Lambda$ in the asymptotic expansion in Proposition 2.8, we still need something to make the higher order field operators in the action of proposition 3.1 small. Would it, for example, be possible to send $f_{-2}$ and $f_{-4}$ to zero in such a way that we can perform the integration?
It is not possible to make all three roots equal en still send $f_{-2}$ and $f_{-4}$ to zero, since it takes two free parameters to make all three roots equal as mentioned before in remark 3.11. Since we cannot use $\Lambda$ for this, it would consume all our parameter freedom. However, it is possible to make the discriminant zero by choosing $f_{-2}$ as an appropriate multiple of $f_{-4}$. Then all three roots would be real and negative, but

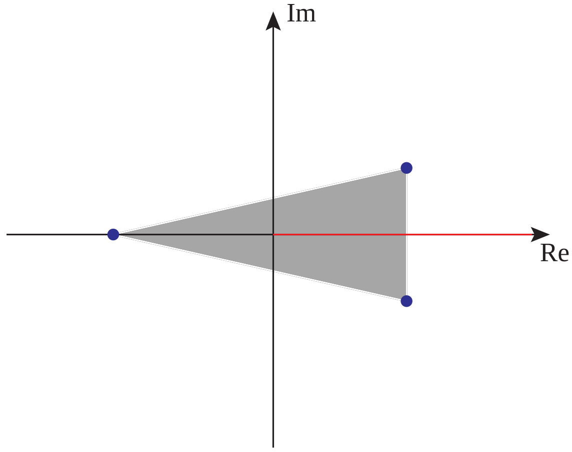


Figure 4.1: The complex plane with one negative root and two complex roots with a positive real part and the area of integration for the Feynman parameters. The blue dots are the roots, the grey triangle corresponds to the convex combinations of the three roots, partially overlapping the red positive real axis.

not equal, and we would be able to perform the integration by applying Schwinger's trick, although the theory still would not be multiplicatively renormalizable. Another nice feature of this method is that sending $f_{-4}$ and $f_{-2}$ to zero, brings the mass and vacuum expectation value $\chi_0$ of the ersHL immediately to the values for the rsHL. Since we would lose the asymptotic expansion we will not proceed along this route.

## 4.2 Higher derivative ersHL-integrals

### 4.2.1 Calculation of a tadpole diagram

Using the lemmas from the previous paragraph all divergent diagrams from section 1.3 can be calculated. In this and the next paragraph we will do two calculations in full detail. These can be generalized to all diagrams.

We will start with diagram 2, see Remark 3.13 and Figure 3.2. For the coupling constants we use the notation from Corollary 3.1, because this makes it easier generalizing to other diagrams. From paragraph 3.3 and Remark 1.12 we determine that the integral for this diagram is

$$\begin{aligned}
\mathcal{M}_2^{(2)} &= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2 \frac{-\xi_3\mu^{2\varepsilon}}{(4\pi)^2} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{16\pi^2\Lambda^4\alpha^{-1}p^2}{p^6+\beta p^4+\gamma p^2+\delta} \\
&= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2 \frac{-\Lambda^4\xi_3\mu^{2\varepsilon}}{\alpha} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{p^2}{(p^2-l_a)(p^2-l_b)(p^2-l_c)} \qquad (4.3) \\
&= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2 \frac{-\Lambda^4\xi_3\mu^{2\varepsilon}}{\alpha} \int \frac{\mathrm{d}^d p}{(2\pi)^d} p^2 \left(\frac{A}{p^2-l_a}+\frac{B}{p^2-l_b}+\frac{C}{p^2-l_c}\right) \qquad (4.4) \\
&= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2 \frac{-\Lambda^4\xi_3\mu^{2\varepsilon}}{\alpha} \int \frac{\mathrm{d}\Omega_d}{(4\pi^2)^{\frac{d}{2}}} \int_0^\infty \mathrm{d}r\, r^{d-1} r^2 \left(\frac{A}{r^2-l_a}+\frac{B}{r^2-l_b}+\frac{C}{r^2-l_c}\right) \\
&= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2 \frac{-2\Lambda^4\xi_3\mu^{2\varepsilon}}{\alpha\Gamma(\frac{d}{2})(4\pi)^{\frac{d}{2}}} \int_0^\infty \mathrm{d}r\, r^{d+1} \left(\frac{A}{r^2-l_a}+\frac{B}{r^2-l_b}+\frac{C}{r^2-l_c}\right) \qquad (4.5) \\
&= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2 \frac{-\Lambda^4\xi_3\mu^{2\varepsilon}}{\alpha\Gamma(\frac{d}{2})(4\pi)^{\frac{d}{2}}} \int_0^\infty \mathrm{d}z\, z^{\frac{d}{2}} \left(\frac{A}{z-l_a}+\frac{B}{z-l_b}+\frac{C}{z-l_c}\right) \\
&= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2 \frac{-\Lambda^4\xi_3\mu^{2\varepsilon}}{\alpha(4\pi)^{\frac{d}{2}}} \frac{\Gamma(\frac{-d}{2})\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{d}{2})\Gamma(1)} \left(A(-l_a)^{\frac{d}{2}}+B(-l_b)^{\frac{d}{2}}+C(-l_c)^{\frac{d}{2}}\right) , \qquad (4.6)
\end{aligned}$$

where equation (4.3) follows from Remark 3.11 and equation (4.4) from Lemma 4.1. Applying either Lemma 4.1 or 4.3 here makes no difference.
Since the integrand in equation (4.4) only depends on $p^2$, we have performed the integral over the angular variables in line (4.5), where $r = \sqrt{p^2}$, the radius of the sphere we are integrating over. Finally, to obtain equation (4.6) we have substituted $z = r^2$ and applied Lemma 4.4 to the three terms. Using the Γ-function properties $z\Gamma(z) = \Gamma(z+1)$, $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$ and $d = 4-2\varepsilon$ we rewrite the Γ-functions in equation (4.6) as

$$\frac{\Gamma(\frac{-d}{2})\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{d}{2})\Gamma(1)} = \frac{d}{2}\Gamma(-\frac{d}{2}) = -\Gamma(-\frac{d}{2}+1) = \frac{\Gamma(\varepsilon)}{1-\varepsilon} = \frac{1}{\varepsilon}+1-\gamma_E+\mathcal{O}(\varepsilon) \quad . \qquad (4.7)$$

Since all other parts do not contain any poles for $\varepsilon \to 0$ this is sufficient to determine all divergent and finite parts. The other parts we now have to know up to order $\varepsilon$. To do so, we use that we can rewrite

$$(4\pi\mu^2)^\varepsilon(-l_a)^{-\varepsilon} = \left(\frac{-l_a}{4\pi\mu^2}\right)^{-\varepsilon} = e^{-\varepsilon\log(\frac{-l_a}{4\pi\mu^2})} = 1-\varepsilon\log(\frac{-l_a}{4\pi\mu^2})+\mathcal{O}(\varepsilon^2).$$

Using this and the equality $Al_a^2+Bl_b^2+Cl_c^2 = 1$, implied by Lemma 4.1, we write

$$\begin{aligned}
\mathcal{M}_2^{(2)} = \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2 \frac{-\Lambda^4\xi_3}{(4\pi)^2\alpha} \frac{\Gamma(\varepsilon)}{1-\varepsilon} \Bigg(1 &- \varepsilon A l_a^2 \log\left(\frac{-l_a}{4\pi\mu^2}\right) - \varepsilon B l_b^2 \log\left(\frac{-l_b}{4\pi\mu^2}\right) \\
&-\varepsilon C l_c^2 \log\left(\frac{-l_c}{4\pi\mu^2}\right)\Bigg) \quad . \qquad (4.8)
\end{aligned}$$

Next we use that the roots scale with $l_\iota = l'_\iota\Lambda^2$, where $l'_\iota$ is a number independent of $\Lambda$. Writing $\xi_3 = \Lambda^{-1}\xi'_3$, see Proposition 3.2 and Corollary 3.1, we can now subtract all $\varepsilon$- and $\Lambda$-dependence in the non-vanishing terms, see Definitions 1.14, 1.16 and

Lemma 1.6

$$\mathcal{M}_2^{(2)} = \frac{\Lambda_0^3\xi_3'}{(4\pi)^2\alpha}\Big(\log(\frac{\Lambda_0^2}{4\pi\mu^2}) + \frac{(l_a')^2}{(l_a'-l_b')(l_a'-l_c')}\log(-l_a')$$
$$+ \frac{-(l_b')^2}{(l_a'-l_b')(l_b'-l_c')}\log(-l_b') + \frac{(l_c')^2}{(l_a'-l_c')(l_b'-l_c')}\log(-l_c')\Big) \quad (4.9)$$
$$\equiv \xi_3'\Lambda_0^3\left(T_2(\mu)+R_2\right) \quad , \quad (4.10)$$

where we have defined the functions $T_2(\mu)$ and $R_2$ for reference convenience.

*Remark* 4.2. This is the first example in this chapter of a divergent part as we defined it in Definition 1.8, see equation (4.7). In the next paragraphs we will put all such divergent parts before subtraction into functions $T_n(\mu)$ and any finite part into a constant $R_n$. The divergent parts contain before subtraction divergent logarithms and polynomials in $\Lambda$ as $\Lambda-\Lambda_0 \to \infty$ or $\frac{\Gamma(\varepsilon)}{1-\varepsilon}$ as $\varepsilon \to 0$. Terms with units (of mass) are kept explicit, so that after subtraction the parameter $\mu$ reminds us of a UV-divergence.

### 4.2.2 Calculation of a two-point diagram

We will calculate diagram 5, again with the notation from Corollary 3.1. This means that it is given by the integral

$$\mathcal{M}_5^{(2)} = \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2(\mu^2)^\varepsilon \int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{\left(\vartheta_{3a}'\Lambda^{-3}\frac{-1}{16\pi^2\alpha}\right)^2(16\pi^2\Lambda^4)^2 p^8}{p^6+\beta p^4+\gamma p^2+\delta}$$
$$\times \frac{1}{(p+k)^6+\beta(p+k)^4+\gamma(p+k)^2+\delta}$$
$$= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2(\mu^2)^\varepsilon \int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{(\Lambda\alpha^{-1}\vartheta_{3a}')^2 p^8}{(p^2-l_a)(p^2-l_b)(p^2-l_c)}$$
$$\times \frac{1}{((p+k)^2-l_a)((p+k)^2-l_b)((p+k)^2-l_c)}$$
$$= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2(\mu^2)^\varepsilon \int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{(\Lambda\alpha^{-1}\vartheta_{3a}')^2 p^2}{((p+k)^2-l_a)((p+k)^2-l_b)((p+k)^2-l_c)}$$
$$\times \frac{p^6}{(p^2-l_a)(p^2-l_b)(p^2-l_c)}$$
$$= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2(\frac{\mu^2}{\Lambda^2})^\varepsilon \int \frac{\mathrm{d}^d \tau}{(2\pi)^d}\frac{\Lambda^2\left(\frac{\vartheta_{3a}'}{\alpha}\right)^2\tau^2}{((\tau+\frac{k}{\Lambda})^2-l_a')((\tau+\frac{k}{\Lambda})^2-l_b')((\tau+\frac{k}{\Lambda})^2-l_c')}$$
$$\times \frac{\tau^6}{(\tau^2-l_a')(\tau^2-l_b')(\tau^2-l_c')} \quad , \quad (4.11)$$

where the integration variable is changed to $\tau = \Lambda^{-1}p$ and it is used that $l_\iota' = \Lambda^{-2}l_\iota$. The denominator can be simplified using Lemma 4.2 several times. Doing this for $a$,$b$

and $c$, the expansion of the first denominator in equation (4.11) is obtained

$$\frac{1}{((\tau+\frac{k}{\Lambda})^2-l'_a)((\tau+\frac{k}{\Lambda})^2-l'_b)((\tau+\frac{k}{\Lambda})^2-l'_c)} = \frac{1}{(\tau^2-l'_a)(\tau^2-l'_b)(\tau^2-l'_c)}$$
$$\times\left(1-\frac{k^2}{\Lambda^2}\left(\frac{1}{\tau^2-l'_a}+\frac{1}{\tau^2-l'_b}+\frac{1}{\tau^2-l'_c}\right)+\frac{4(\tau\cdot k)^2}{\Lambda^2}\left(\frac{1}{(\tau^2-l'_a)^2}\right.\right.$$
$$+\frac{1}{(\tau^2-l'_b)^2}+\frac{1}{(\tau^2-l'_c)^2}+\frac{1}{(\tau^2-l'_a)(\tau^2-l'_b)}+\frac{1}{(\tau^2-l'_a)(\tau^2-l'_c)}$$
$$\left.\left.+\frac{1}{(\tau^2-l'_b)(\tau^2-l'_c)}\right)\right)+\mathcal{O}(\Lambda^{-3}) \quad . \tag{4.12}$$

In this expansion only the integral over the first term in the sum is UV-divergent, since in Lemma 4.2 $\frac{b}{a}=\mathcal{O}(\tau^{-1})$. For all the finite parts, let $\varepsilon\to 0$ and bring the limit $\Lambda-\Lambda_0\to\infty$ inside the integral, whereby terms $\Lambda^{n\leq -3}$ the expansion (4.12) vanish. This yields

$$\mathcal{M}_5^{(2)} = \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2\left(\frac{\vartheta'_{3a}}{\alpha}\right)^2(\frac{\mu^2}{\Lambda^2})^{\varepsilon}\Lambda^2\int\frac{\mathrm{d}^d\tau}{(2\pi)^d}\frac{\tau^8}{(\tau^2-l'_a)^2(\tau^2-l'_b)^2(\tau^2-l'_c)^2}$$
$$+\left(\frac{\vartheta'_{3a}}{\alpha}\right)^2\int\frac{\mathrm{d}^4\tau}{(2\pi)^4}\frac{\tau^8}{(\tau^2-l'_a)^2(\tau^2-l'_b)^2(\tau^2-l'_c)^2}\left(-k^2\left(\frac{1}{\tau^2-l'_a}+\frac{1}{\tau^2-l'_b}\right.\right.$$
$$\left.+\frac{1}{\tau^2-l'_c}\right)+4(\tau\cdot k)^2\left(\frac{1}{(\tau^2-l'_a)^2}+\frac{1}{(\tau^2-l'_b)^2}+\frac{1}{(\tau^2-l'_c)^2}\right.$$
$$\left.\left.+\frac{1}{(\tau^2-l'_a)(\tau^2-l'_b)}+\frac{1}{(\tau^2-l'_a)(\tau^2-l'_c)}+\frac{1}{(\tau^2-l'_b)(\tau^2-l'_c)}\right)\right) \quad . \tag{4.13}$$

Next we have to perform the integration over the angular variables in $\int \mathrm{d}^4\tau$, because we integrate over the inner product $(\tau\cdot k)^2$, leading to a $\cos^2(\theta_1)$-dependence.

*Remark* 4.3. Integration over the $n$-sphere is done by

$$\int_{S^n}\mathrm{d}\Omega = \int_0^{\pi}\mathrm{d}\theta_1\ldots\int_0^{\pi}\mathrm{d}\theta_{n-2}\int_0^{2\pi}\mathrm{d}\theta_{n-1}\,\sin^{n-2}(\theta_1)\sin^{n-3}(\theta_2)\ldots\sin(\theta_{n-2}) \quad .$$

For $n=4$ this gives us

$$\int_0^{\pi}\mathrm{d}\theta_1\int_0^{\pi}\mathrm{d}\theta_2\int_0^{2\pi}\mathrm{d}\theta_3\,\sin^2(\theta_1)\sin(\theta_2)\cos^2(\theta_1) = \frac{2\pi^2}{4} \quad . \tag{4.14}$$

Changing variables by setting $z = \tau^2$ yields

$$\mathcal{M}_5^{(2)} = \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2 \left(\frac{\vartheta'_{3a}\Lambda}{4\pi\alpha}\right)^2 (\frac{4\pi\mu^2}{\Lambda^2})^{\varepsilon} \int_0^\infty \mathrm{d}z \, \frac{z^{\frac{d}{2}+3}}{\Gamma(\frac{d}{2})} \frac{1}{(z-l'_a)^2(z-l'_b)^2(z-l'_c)^2} \tag{4.15}$$

$$+ \left(\frac{\vartheta'_{3a}}{4\pi\alpha}\right)^2 k^2 \int_0^\infty \mathrm{d}z \, \frac{z^5}{(z-l'_a)^2(z-l'_b)^2(z-l'_c)^2} \Bigg(-\frac{1}{z-l'_a} - \frac{1}{z-l'_b} - \frac{1}{z-l'_c}$$
$$+\frac{z}{(z-l'_a)^2} + \frac{z}{(z-l'_b)^2} + \frac{z}{(z-l'_c)^2} + \frac{z}{(z-l'_a)(z-l'_b)} + \frac{z}{(z-l'_a)(z-l'_c)}$$
$$+\frac{z}{(z-l'_b)(z-l'_c)}\Bigg) \tag{4.16}$$

$$= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2 \left(\frac{\vartheta'_{3a}}{4\pi\alpha}\right)^2 \Lambda^2 \frac{2\Gamma(\varepsilon)}{(1-\varepsilon)}$$
$$\times \Bigg( (-l'_a)^5 \left(\frac{-4\pi\mu^2}{\Lambda^2 l'_a}\right)^{\varepsilon} \left(\frac{1}{(l'_a-l'_b)^2(l'_a-l'_c)^3} + \frac{1}{(l'_a-l'_b)^3(l'_a-l'_c)^2}\right)$$
$$+(-l'_b)^5 \left(\frac{-4\pi\mu^2}{\Lambda^2 l'_b}\right)^{\varepsilon} \left(\frac{1}{(l'_b-l'_a)^2(l'_b-l'_c)^3} + \frac{1}{(l'_b-l'_a)^3(l'_b-l'_c)^2}\right)$$
$$+(-l'_c)^5 \left(\frac{-4\pi\mu^2}{\Lambda^2 l'_c}\right)^{\varepsilon} \left(\frac{1}{(l'_c-l'_b)^2(l'_c-l'_a)^3} + \frac{1}{(l'_c-l'_b)^3(l'_c-l'_a)^2}\right) \Bigg)$$
$$+ \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2 \left(\frac{\vartheta'_{3a}}{4\pi\alpha}\right)^2 \Lambda^2 \frac{5\Gamma(\varepsilon)}{1-\varepsilon} \Bigg( \left(\frac{-4\pi\mu^2}{\Lambda^2 l'_a}\right)^{\varepsilon} \frac{(-l'_a)^4}{(l'_a-l'_b)^2(l'_a-l'_c)^2}$$
$$+ \left(\frac{-4\pi\mu^2}{\Lambda^2 l'_b}\right)^{\varepsilon} \frac{(-l'_b)^4}{(l'_b-l'_a)^2(l'_b-l'_c)^2} + \left(\frac{-4\pi\mu^2}{\Lambda^2 l'_c}\right)^{\varepsilon} \frac{(-l'_c)^4}{(l'_c-l'_b)^2(l'_c-l'_a)^2}\Bigg) \tag{4.17}$$

$$
\begin{aligned}
&- \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2 \left(\frac{\vartheta'_{3a}}{4\pi\alpha}\right)^2 \Lambda^2 \left(\frac{(-l'_a)^4}{(l'_a-l'_b)^2(l'_a-l'_c)^2} + \frac{(-l'_b)^4}{(l'_b-l'_a)^2(l'_b-l'_c)^2}\right.\\
&\left.+\frac{(-l'_c)^4}{(l'_c-l'_b)^2(l'_c-l'_a)^2}\right)\\
&+k^2\left(\frac{\vartheta'_{3a}}{4\pi\alpha}\right)^2\Bigg[\left(\frac{(-l'_a)^3}{(l'_a-l'_b)^2(l'_a-l'_c)^2}+a\leftrightarrow b+a\leftrightarrow c\right)\frac{\Gamma(1)\Gamma(3)}{\Gamma(4)}\\
&-\left((-l'_a)^4\left(\frac{1}{(l'_a-l'_b)^2(l'_a-l'_c)^3}+\frac{1}{(l'_a-l'_b)^3(l'_a-l'_c)^2}\right)+\frac{5(-l'_a)^3}{(l'_a-l'_b)^2(l'_a-l'_c)^2}\right.\\
&\left.+a\leftrightarrow b+a\leftrightarrow c\right)\frac{\Gamma(1)\Gamma(2)}{\Gamma(3)}\\
&+\left((-l'_a)^5\left(\frac{1}{(l'_a-l'_b)^4(l'_a-l'_c)^2}+\frac{1}{(l'_a-l'_b)^3(l'_a-l'_c)^3}+\frac{1}{(l'_a-l'_b)^2(l'_a-l'_c)^4}\right)\right.\\
&+5(-l'_a)^4\left(\frac{1}{(l'_a-l'_b)^3(l'_a-l'_c)^2}+\frac{1}{(l'_a-l'_b)^2(l'_a-l'_c)^3}\right)\\
&\left.+10(-l'_a)^3\frac{1}{(l'_a-l'_b)^2(l'_a-l'_c)^2}+a\leftrightarrow b+a\leftrightarrow c\right)\frac{\Gamma(1)\Gamma(1)}{\Gamma(2)}\\
&-\left(\left(10\frac{(-l'_a)^3}{(l'_a-l'_b)^2(l'_a-l'_c)^2}+10\frac{(-l'_a)^4}{(l'_a-l'_b)^3(l'_a-l'_c)^2}+10\frac{(-l'_a)^4}{(l'_a-l'_b)^2(l'_a-l'_c)^3}\right.\right.\\
&+6\frac{(-l'_a)^5}{(l'_a-l'_b)^3(l'_a-l'_c)^3}+6\frac{(-l'_a)^5}{(l'_a-l'_b)^4(l'_a-l'_c)^2}+6\frac{(-l'_a)^5}{(l'_a-l'_b)^2(l'_a-l'_c)^4}\\
&+2\frac{(-l'_a)^6}{(l'_a-l'_b)^5(l'_a-l'_c)^2}+2\frac{(-l'_a)^6}{(l'_a-l'_b)^2(l'_a-l'_c)^5}+2\frac{(-l'_a)^6}{(l'_a-l'_b)^3(l'_a-l'_c)^4}\\
&\left.\left.+2\frac{(-l'_a)^6}{(l'_a-l'_b)^4(l'_a-l'_c)^3}\right)(\log(-l'_a))+a\leftrightarrow b+a\leftrightarrow c\right)\Bigg] \qquad (4.18)\\
&\equiv(\vartheta'_{3a})^2\left(\Lambda_0^2T_5(\mu)+\Lambda_0^2R'_5+k^2R_5\right) \quad . \qquad (4.19)
\end{aligned}
$$

Applying Lemma 4.3 and Lemma 4.4 to equations (4.15) and (4.16) results in equations (4.17) and (4.18) respectively. All $\mu$-dependent terms we put into $T_5(\mu)$ and all finite terms into $R_5$ and $R'_5$.

*Remark* 4.4. From equation (4.17) we can find that $T_5(\mu)$ is given by

$$
T_5(\mu)=\frac{-1}{(4\pi\alpha)^2}\log\left(\frac{\Lambda_0^2}{4\pi\mu^2}\right) \quad .
$$

It is clear that the coefficient for the logarithm $\log\left(\frac{\Lambda_0^2}{4\pi\mu^2}\right)$ and the diverging part $\frac{\Gamma\varepsilon)}{1-\varepsilon}$ are the same, since

$$
\begin{aligned}
&\left(\frac{-\Lambda^2 l'_a}{4\pi\mu^2}\right)^{-\varepsilon}\Upsilon_a+\left(\frac{-\Lambda^2 l'_b}{4\pi\mu^2}\right)^{-\varepsilon}\Upsilon_b+\left(\frac{-\Lambda^2 l'_c}{4\pi\mu^2}\right)^{-\varepsilon}\Upsilon_c=(\Upsilon_a+\Upsilon_b+\Upsilon_c)\\
&\times\left(1-\varepsilon\log\left(\frac{\Lambda^2}{4\pi\mu^2}\right)\right)-\varepsilon\log(-l'_a)\Upsilon_a-\varepsilon\log(-l'_b)\Upsilon_b-\varepsilon\log(-l'_c)\Upsilon_c \quad ,
\end{aligned}
$$

where $\Upsilon_\iota$ is the rational function from equation (4.17). This can be seen by rewriting the divergent part. Substituting $z=z-l'_\iota+l'_\iota$, with $\iota=a,b,c$, in the numerator of the integral in equation (4.15) yields

$$
\begin{aligned}
&\frac{1}{\Gamma(2-\varepsilon)}\int_0^\infty \mathrm{d}z\, z^{5-\varepsilon}\frac{1}{(z-l'_a)^2(z-l'_b)^2(z-l'_c)^2} \\
&= \frac{1}{\Gamma(2-\varepsilon)}\int_0^\infty \mathrm{d}z\, z^{-\varepsilon}\frac{1}{(z-l'_a)} + \mathcal{O}(1) = \frac{\Gamma(\varepsilon)}{1-\varepsilon} + \mathcal{O}(1) \quad ,
\end{aligned}
$$

from which it is clear that $\Upsilon_a + \Upsilon_b + \Upsilon_c = 1$.
By exactly the same steps we could have seen that $Al_a^2 + Bl_b^2 + Cl_c^2 = 1$ in paragraph 4.2.1. Both identities can be obtained algebraically as well.
These simple checks confirm that we have found the right divergent parts, which we will need to determine the $\beta$-functions later on.

*Remark* 4.5. The last contribution to equation (4.18) originates from the integral

$$
\int_0^\infty \mathrm{d}z\, \frac{1}{z-l'_a} = \lim_{\varsigma\to\infty} \log(\varsigma) - \log(-l'_a) \quad ,
$$

where one can check in a way similar to remark 4.4, although more elaborate, that the diverging logarithm vanishes, because the coefficients multiplying it sum to zero:

$$
\begin{aligned}
&(c_1+c_2+c_3)\lim_{\varsigma\to\infty}\log(\varsigma) - c_1\log(-l'_a) - c_2\log(-l'_b) - c_3\log(-l'_c) \\
&= -c_1\log(-l'_a) - c_2\log(-l'_b) - c_3\log(-l'_c) \quad .
\end{aligned}
$$

Here, $c_1$ is fully written out in equation (4.18) and $c_2$ and $c_3$ are obtained from it by interchanging $a \leftrightarrow b$ and $a \leftrightarrow c$ respectively.

*Remark* 4.6. When applying Lemma 4.3 to equation (4.15) we have a freedom to choose the (integer) power of $z$, called $m$ in the lemma. However, after integrating the obtained result using Lemma 4.4 we get well-defined $\Gamma$-functions due to the non-zero $\varepsilon$. So, any choice for $m$ will give the same result.
This is not the case for the finite part, equation (4.16), where we have to take $m$ maximal, since otherwise applying Lemma 4.4 would give $\Gamma$-functions of negative integers, which are not defined. Then we would lose control over the finite part. This is the reason that we have obtained the last term of equation (4.18) by 'manual' integration. If we instead had applied Lemma 4.4, we had found $\Gamma(0)$, which would have vanished because the coefficients summed to zero, see Remark 4.5. There would not have been any finite part.

## 4.3 All divergent amplitudes

**Proposition 4.1.** The 1PI superficially divergent Feynman diagrams for the ersHL are given by

$$\mathcal{M}_1^{(2)} = -\Lambda_0^3 \vartheta'_{3a} \left(T_1(\mu) + R_1\right) \quad ;$$
$$\mathcal{M}_2^{(2)} = \xi'_3 \Lambda_0^3 \left(T_2(\mu) + R_2\right) \quad ;$$
$$\mathcal{M}_3^{(2)} = -\Lambda_0^3 \vartheta'_{3b} \left(T_1(\mu) + R_1\right) \quad ;$$
$$\mathcal{M}_4^{(2)} = 0 \quad ;$$
$$\mathcal{M}_5^{(2)} = (\vartheta'_{3a})^2 \left(\Lambda_0^2 T_5(\mu) + \Lambda_0^2 R'_5 + k^2 R_5\right) \quad ;$$
$$\mathcal{M}_6^{(2)} = (\vartheta'_{3a})^2 \left(\Lambda_0^2 T_5(\mu) + \Lambda_0^2 R'_5 + k^2 R_5 + k^2 R_6\right) \quad ;$$
$$\mathcal{M}_7^{(2)} = (\vartheta'_{3b})^2 \left(\Lambda_0^2 T_5(\mu) + \Lambda_0^2 R'_5 + k^2 R_5 + k^2 R_6\right) \quad ;$$
$$\mathcal{M}_8^{(2)} = \vartheta'_{3a} \vartheta'_{3b} \left(\Lambda_0^2 T_5(\mu) + \Lambda_0^2 R'_5 + k^2 R_5 + \frac{1}{3} k^2 R_6\right) \quad ;$$
$$\mathcal{M}_9^{(2)} = -\Lambda_0^2 \vartheta'_{4b} \left(T_1(\mu) + R_1\right) \quad ;$$
$$\mathcal{M}_{10}^{(2)} = \Lambda_0^2 \xi'_4 \left(T_2(\mu) + R_2\right) \quad ;$$
$$\mathcal{M}_{11}^{(2)} = k^2 \vartheta'_{4b} \left(T_2(\mu) + R_2\right) \quad ;$$
$$\mathcal{M}_{12}^{(2)} = -\Lambda_0^2 \vartheta'_{4a} \left(T_1(\mu) + R_1\right) \quad ;$$
$$\mathcal{M}_{13}^{(2)} = \frac{1}{4} k^2 \vartheta'_{4c} \left(T_2(\mu) + R_2\right) \quad ;$$
$$\mathcal{M}_{14}^{(2)} = k^2 \vartheta'_{4c} \left(T_2(\mu) + R_2\right) \quad ;$$
$$\mathcal{M}_{15}^{(2)} = \vartheta'_{3a} \vartheta'_{4a} \Lambda_0 \left(T_5(\mu) + R'_5\right) \quad ;$$
$$\mathcal{M}_{16}^{(2)} = \vartheta'_{3b} \vartheta'_{4b} \Lambda_0 \left(T_5(\mu) + R'_5\right) \quad ;$$
$$\mathcal{M}_{17}^{(2)} = \vartheta'_{3a} \vartheta'_{4a} \Lambda_0 \left(T_5(\mu) + R'_5\right) \quad ;$$
$$\mathcal{M}_{18}^{(2)} = \vartheta'_{3a} \vartheta'_{4b} \Lambda_0 \left(T_5(\mu) + R'_5\right) \quad ;$$
$$\mathcal{M}_{19}^{(2)} = \vartheta'_{3b} \vartheta'_{4a} \Lambda_0 \left(T_5(\mu) + R'_5\right) \quad ;$$
$$\mathcal{M}_{20}^{(2)} = \xi'_5 \Lambda_0 \left(T_2(\mu) + R_2\right) \quad ;$$
$$\mathcal{M}_{21}^{(2)} = (\vartheta'_{4a})^2 \left(T_5(\mu) + R'_5\right) \quad ;$$
$$\mathcal{M}_{22}^{(2)} = (\vartheta'_{4b})^2 \left(T_5(\mu) + R'_5\right) \quad ;$$
$$\mathcal{M}_{23}^{(2)} = (\vartheta'_{4a})^2 \left(T_5(\mu) + R'_5\right) \quad ;$$
$$\mathcal{M}_{24}^{(2)} = \vartheta'_{4a} \vartheta'_{4b} \left(T_5(\mu) + R'_5\right) \quad ;$$
$$\mathcal{M}_{25}^{(2)} = \xi'_6 \left(T_2(\mu) + R_2\right) \quad .$$

*Proof.* The loop-momentum structures of the diagrams 4, 10, 11, 14, 20 and 25 are equal to that of diagram 2. Therefore we can express those diagrams in terms of $T_2$ and $R_2$, equation (4.10), which we have defined for this purpose. Diagrams 10, 11 and 14 are proportional to the square of the external momentum $k$.

Only slightly different is diagram 1. Like in paragraph 4.2.1 we can compute $\mathcal{M}_1^{(2)}$:

$$\begin{aligned}
\mathcal{M}_1^{(2)} &= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2 \frac{-\vartheta_{3a}}{(4\pi)^2}\mu^{2\varepsilon}\int \mathrm{d}^d p \, \frac{16\pi^2\Lambda^4\alpha^{-1}p^4}{(p^2-l_a)(p^2-l_b)(p^2-l_c)} \\
&= \lim_{\substack{\varepsilon\to 0\\ \Lambda\to\infty}} \overline{\mathfrak{T}}_2 \frac{-\Lambda^3\vartheta'_{3a}}{(4\pi)^2\alpha}\left(\frac{\Gamma(\varepsilon)}{1-\varepsilon}\left(\frac{(l'_a)^3}{(l'_a-l'_b)(l'_a-l'_c)}+\frac{(l'_b)^3}{(l'_b-l'_a)(l'_b-l'_c)}\right.\right. \\
&\left. +\frac{(l'_c)^3}{(l'_c-l'_a)(l'_c-l'_b)}\right) - \frac{(l'_a)^3\log\left(\frac{-l'_a\Lambda^2}{4\pi\mu^2}\right)}{(l'_a-l'_b)(l'_a-l'_c)} - \frac{(l'_b)^3\log\left(\frac{-l'_b\Lambda^2}{4\pi\mu^2}\right)}{(l'_b-l'_a)(l'_b-l'_c)} \\
&\left. - \frac{(l'_c)^3\log\left(\frac{-l'_c\Lambda^2}{4\pi\mu^2}\right)}{(l'_c-l'_a)(l'_c-l'_b)}\right) \\
&= \frac{-\Lambda_0^3\vartheta'_{3a}}{(4\pi)^2\alpha}\left(\beta\Lambda_0^{-2}\log\left(\frac{\Lambda_0^2}{4\pi\mu^2}\right) - \frac{(l'_a)^3\log(-l'_a)}{(l'_a-l'_b)(l'_a-l'_c)} - \frac{(l'_b)^3\log(-l'_b)}{(l'_b-l'_a)(l'_b-l'_c)}\right. \\
&\left. - \frac{(l'_c)^3\log(-l'_c)}{(l'_c-l'_a)(l'_c-l'_b)}\right) \\
&\equiv -\Lambda_0^3\vartheta'_{3a}\left(T_1(\mu)+R_1\right) \quad . \qquad (4.20)
\end{aligned}$$

This diagram has superficial degree of divergence 2, instead of 0. The appearance of $\beta$, Definition 3.2, can be seen by rewriting

$$\begin{aligned}
&\frac{-(l'_a)^3}{(l'_a-l'_b)(l'_a-l'_c)} + a\leftrightarrow b + a\leftrightarrow c \\
&= -l'_a - \frac{l'_a l'_b}{l'_a-l'_b} - \frac{l'_a l'_c}{l'_a-l'_c} - \frac{l'_a l'_b l'_c(l'_b-l'_c)}{(l'_a-l'_b)(l'_a-l'_c)(l'_b-l'_c)} + a\leftrightarrow b + a\leftrightarrow c \\
&= -l'_a - l'_b - l'_c = \beta \quad .
\end{aligned}$$

Furthermore, the structure of diagrams 3, 9 and 12 is equal to that of diagram 1.
Due to momentum conservation there is no momentum on the external vertex of diagram 4, therefore $\mathcal{M}_4^{(2)} = 0$.
Diagrams 15 and 21 can be expressed in terms of $T_5$, $R_5$ and $R'_5$, equation (4.19). However, due to the extra $\Lambda^{-1}$-terms the finite part, $R_5$, vanishes under $\overline{\mathfrak{T}}_2$, see Lemma 1.6.
Diagram 6 and 8 can be seen as an extension of diagram 5. So we apply the same method as in paragraph 4.2.2. This yields

$$\begin{aligned}
\mathcal{M}_6^{(2)} &= (\vartheta'_{3a})^2\Big(\Lambda_0^2 T_5(\mu) + \Lambda_0^2 R'_5 + k^2 R_5 \\
&+ \frac{3k^2}{(4\pi\alpha)^2}\int_0^\infty \mathrm{d}z\, \frac{z^4}{(z-l'_a)^2(z-l'_b)^2(z-l'_c)^2}\Big) \quad ,
\end{aligned}$$

where we define the additional part as $R_6$. For completeness, $R_6$ is calculated in equation (4.21). Having determined $\mathcal{M}_6^{(2)}$, the amplitude

$$\begin{aligned}
\mathcal{M}_8^{(2)} &= \vartheta'_{3a}\vartheta'_{3b}\Big(\Lambda_0^2 T_5(\mu) + \Lambda_0^2 R'_5 \\
&+ k^2 R_5 + \frac{k^2}{(4\pi\alpha)^2}\int_0^\infty \mathrm{d}z\, \frac{z^4}{(z-l'_a)^2(z-l'_b)^2(z-l'_c)^2}\Big)
\end{aligned}$$

is straightforward.
The obtained answers, in terms of $T_5$, $R_5$, $R'_5$ and $R_6$, can be used as well for the similar diagrams 7, 16, 17, 18, 19 22, 23 and 24. Except diagrams 6, 7 and 8, all these

diagrams have vanishing finite parts as a result of negative powers of $\Lambda$, similar to diagrams 15 and 21.
In diagram 13 there is an angular dependence as a result of the inner products between the external momentum $k$ and the loop momentum $p$. Compared to $T_2$ this results in an extra factor 1/4 from the angular integral in four dimensions, as discussed in remark 4.3. □

*Remark* 4.7. In equation (4.20) $\beta$ can be obtained as described in remark 4.4. Since this diagram diverges quadratically also the second order is relevant. Using the same steps as in paragraphs 4.2.1 and 4.2.2 and remark 4.4

$$\begin{aligned}
\Lambda^{2\varepsilon}\int \mathrm{d}^d p\,\frac{p^4}{(p^2-l_a)(p^2-l_b)(p^2-l_c)} &= \frac{\pi^{2-\varepsilon}\Lambda^2}{\Gamma(2-\varepsilon)}\int_0^\infty \mathrm{d}z\,\frac{z^{3-\varepsilon}}{(z-l'_a)(z-l'_b)(z-l'_c)}\\
&= \frac{\pi^{2-\varepsilon}\Lambda^2}{\Gamma(2-\varepsilon)}\int_0^\infty \mathrm{d}z\,z^{-\varepsilon} + \frac{\pi^{2-\varepsilon}\Lambda^2}{\Gamma(2-\varepsilon)}\int_0^\infty \mathrm{d}z\left(\frac{l'_a z^{-\varepsilon}}{z-l'_a}+\frac{l'_b z^{-\varepsilon}}{z-l'_b}+\frac{l'_c z^{-\varepsilon}}{z-l'_c}\right)+\mathcal{O}(1)\\
&= \frac{\pi^{2-\varepsilon}\Lambda^2}{\Gamma(2-\varepsilon)}\int_0^\infty \mathrm{d}z\,z^{-\varepsilon}\frac{z+1}{z+1} + \pi^2\Lambda^2(l'_a+l'_b+l'_c)\frac{\Gamma(\varepsilon)}{1-\varepsilon}+\mathcal{O}(1)\\
&= \pi^{2-\varepsilon}\Lambda^2\left(\frac{\Gamma(\varepsilon)}{\varepsilon-1}+\frac{\Gamma(\varepsilon)}{1-\varepsilon}\right)-\pi^2\beta\frac{\Gamma(\varepsilon)}{1-\varepsilon}+\mathcal{O}(1) \quad ,
\end{aligned}$$

where the unlabeled infinity vanishes when expressed in $\Gamma$-functions.

This concludes the regularization of the superficially divergent diagrams. There is only one integral left to do, which is $R_6$. By making use of Lemmas 4.3 and 4.4 again we find that

$$\begin{aligned}
&\int_0^\infty \mathrm{d}z\,\frac{z^4}{(z-l'_a)^2(z-l'_b)^2(z-l'_c)^2} =\\
&-2\Big((-l'_a)^4\left(\frac{\log(-l'_a)}{(l'_a-l'_b)^2(l'_a-l'_c)^3}+\frac{\log(-l'_a)}{(l'_a-l'_b)^3(l'_a-l'_c)^2}\right)+2\frac{(-l'_a)^3\log(-l'_a)}{(l'_a-l'_b)^2(l'_a-l'_c)^2}\\
&+(-l'_b)^4\left(\frac{\log(-l'_b)}{(l'_b-l'_a)^2(l'_b-l'_c)^3}+\frac{\log(-l'_b)}{(l'_b-l'_a)^3(l'_b-l'_c)^2}\right)+2\frac{(-l'_b)^3\log(-l'_b)}{(l'_b-l'_a)^2(l'_b-l'_c)^2}\\
&+(-l'_c)^4\left(\frac{\log(-l'_c)}{(l'_c-l'_b)^2(l'_c-l'_a)^3}+\frac{\log(-l'_c)}{(l'_c-l'_b)^3(l'_c-l'_a)^2}\right)+2\frac{(-l'_c)^3\log(-l'_c)}{(l'_c-l'_a)^2(l'_c-l'_b)^2}\Big)\\
&+\frac{\Gamma(1)\Gamma(1)}{\Gamma(2)}\left(\frac{(-l'_a)^3}{(l'_a-l'_b)^2(l'_a-l'_c)^2}+\frac{(-l'_b)^3}{(l'_b-l'_a)^2(l'_b-l'_c)^2}+\frac{(-l'_c)^3}{(l'_c-l'_b)^2(l'_c-l'_a)^2}\right)\\
&=\frac{(4\pi\alpha)^2}{3}R_6 \quad ,
\end{aligned} \tag{4.21}$$

where the infinite parts from the logarithms vanish in the same way as in remark 4.5. Similar to the example case in Chapter 1 we have to determine the multiplicities of the diagrams from Figure 3.2, where we count as described in paragraph 1.3.1. The multiplicities of the superficially divergent Feynman diagrams from Figure 3.2 are given in Table 4.1.

## 4.4 The $\beta$-functions

In this section we will try to determine the $\beta$-functions for the Lagrangian from Proposition 3.2. If we succeed we may compare them to the $\beta$-functions (1.39), (1.40), (1.41) and (1.42) from Chapter 1. Let us mention that we are still free to choose values for $f_{-2}$ and $f_{-4}$, since they are the additional variables with respect to the Lagrangian (1.1). However, as we have pointed out in Chapter 3 choices of these variables have

| # | multiplicity | | # | multiplicity | | # | multiplicity |
|---|---|---|---|---|---|---|---|
| $n_1$ | 2 | | $n_{10}$ | 6 | | $n_{19}$ | 12 |
| $n_2$ | 2 | | $n_{11}$ | 8 | | $n_{20}$ | 24 |
| $n_3$ | 1 | | $n_{12}$ | 6 | | $n_{21}$ | 36 |
| $n_4$ | 2 | | $n_{13}$ | 8 | | $n_{22}$ | 8 |
| $n_5$ | 4 | | $n_{14}$ | 4 | | $n_{23}$ | 36 |
| $n_6$ | 4 | | $n_{15}$ | 12 | | $n_{24}$ | 48 |
| $n_7$ | 2 | | $n_{16}$ | 4 | | $n_{25}$ | 120 |
| $n_8$ | 8 | | $n_{17}$ | 12 | | | |
| $n_9$ | 2 | | $n_{18}$ | 8 | | | |

Table 4.1: Multiplicities of the superficially divergent diagrams from Figure 3.2.

influence on $\chi_0'$ as well, which we have to take into account, see paragraph 1.4.5.
From these non-multiplicatively renormalized amplitudes the $\beta$-functions can be obtained by solving the renormalization group equation [3]. As in paragraph 1.5 the $\mu$-dependence should disappear from our theory by making the parameters dependent on it. The tree-level diagrams together with the parts that came from the UV-divergences at one-loop level are sufficient to accomplish this at lowest order, as explained in paragraph 1.5.3. We will denote the reduced correlator again with $\tilde{G}$.
From Proposition 3.3 and Corollary 3.2 it is clear that these $\tilde{G}$'s contain all the UV-divergences at all loop orders. In this case this means that everything below is at one-loop level. As in paragraph 1.5 we keep the multiplicities $n_i$ of the diagrams explicit. These numbers can be found in Table 4.1.
In proposition 4.1 we found all divergent diagrams and separated the $\mu$-dependent part $T_i$ from the finite parts. Let us recall from equations (4.20), (4.10) and (4.19) with remark 4.4 what these $T_i$ are

$$T_1(\mu) = \frac{450}{(4\pi f_{-4})^2}\left(-\frac{f_{-2}}{3} + \frac{7}{15} f_{-4}(\chi_0')^2\right) \log\left(\frac{\Lambda_0^2}{4\pi\mu^2}\right) \quad ; \tag{4.22}$$

$$T_2(\mu) = \frac{15}{(4\pi)^2 f_{-4}} \log\left(\frac{\Lambda_0^2}{4\pi\mu^2}\right) \quad ; \tag{4.23}$$

$$T_5(\mu) = \frac{-225}{(4\pi f_{-4})^2} \log\left(\frac{\Lambda_0^2}{4\pi\mu^2}\right) \quad , \tag{4.24}$$

where the definitions of $\alpha$ and $\beta$ can be found in Definition 3.2. The lowest order amplitudes together with all $\mu$-dependence from the loop corrections for the two-point function are given by

$$\begin{aligned} \tilde{G}^{(2)} = \Lambda_0^{-4}\alpha\Bigg( & k^6 + \beta k^4 + \gamma k^2 + \delta - \frac{16\pi^2}{\alpha}\Lambda_0^4\Big(\Lambda_0^2 T_5(\mu)\,\big((n_5+n_6)(\vartheta_{3a}')^2 + n_7(\vartheta_{3b}')^2 \\ & + n_8\vartheta_{3a}'\vartheta_{3b}'\big) - \Lambda_0^2 T_1(\mu)(n_9\vartheta_{4b}' + n_{12}\vartheta_{4a}') + n_{10}\xi_4'\Lambda_0^2 T_2(\mu) + k^2 T_2(\mu)(n_{11}\vartheta_{4b}' \\ & + \frac{n_{13}}{4}\vartheta_{4c}' + n_{14}\vartheta_{4c}')\Big)\Bigg) \quad . \end{aligned} \tag{4.25}$$

Besides $f_0$, also $f_{-2}$ and $f_{-4}$ should be treated as inverse couplings. Therefore, we define the coupling constants $\varkappa_{-4} = 1/f_{-4}$, $\varkappa_{-2} = 1/f_{-2}$ and $\varkappa_0 = 1/f_0$, as we did in

Chapter 1. From Proposition 3.1 we know that $\beta_{\varkappa_{-2,-4}}$ are related to couplings without UV-divergences and from the multiplicative picture of Lemma 1.4 we may expect physically that these two $\beta$-functions are therefore equal to zero. However, we will not insert this and see under which conditions this is retrieved. The renormalization group equation from Lemma 1.4 we want to solve is

$$\left(\mu\frac{\partial}{\partial\mu}+\beta_{f_2}\frac{\partial}{\partial f_2}+\beta_{\varkappa_0}\frac{\partial}{\partial\varkappa_0}+\beta_{\varkappa_{-2}}\frac{\partial}{\partial\varkappa_{-2}}+\beta_{\varkappa_{-4}}\frac{\partial}{\partial\varkappa_{-4}}+\beta_{\chi_0'}\frac{\partial}{\partial\chi_0'}-n\gamma_\varphi\right)G^{(n)}=0 \quad .$$

The first term in the renormalization group equation is the only one that does not raise the order. This stems from the fact that at lowest order there is no $\mu$-dependence. Hence, $\beta_{\varkappa_i}$ is of higher order than $\varkappa_i$.

As can be seen directly from the renormalization group equation, any condition will be of the form $0=a\times\beta+b$, where we want to find $\beta$. This is solved by $\beta=-b/a$. Although the bookkeeping may become tedious, mathematically it is all straightforward.

Before we try to solve the renormalization group equations for these correlators, it is a good idea to rewrite them using Definition 3.2 and Corollary 3.1 in the variables that we actually want to use. The order of equation (4.25) is preserved in the first line of equation (4.26). This yields

$$\begin{aligned}\tilde{G}^{(2)}&=\Lambda_0^{-4}\frac{f_{-4}}{15}k^6+\Lambda_0^{-2}\left(-\frac{2}{3}f_{-2}+\frac{14}{15}f_{-4}(\chi_0')^2\right)k^4+k^2\left(4f_0-\frac{20}{3}f_{-2}(\chi_0')^2\right.\\&\left.+\frac{14}{3}f_{-4}(\chi_0')^4\right)+\Lambda_0^2\left(-8f_2+24f_0(\chi_0')^2-20f_{-2}(\chi_0')^4+\frac{28}{3}f_{-4}(\chi_0')^6\right)\\&-k^2\log\left(\frac{\Lambda_0^2}{4\pi\mu^2}\right)\left(\frac{5}{2}n_{11}-\frac{n_{13}}{8}-\frac{n_{14}}{2}\right)+\Lambda_0^2\log\left(\frac{\Lambda_0^2}{4\pi\mu^2}\right)\left((\chi_0')^2(\frac{81}{4}n_5\right.\\&+\frac{81}{4}n_6+25n_7+\frac{45}{2}n_8+35n_9+21n_{12}-70n_{10})\\&\left.-\frac{f_{-2}}{f_{-4}}(25n_9+15n_{12}-\frac{50}{3}n_{10})\right)\quad .\end{aligned} \tag{4.26}$$

Since $\lambda_3$ is the only coupling of valence three that contains no negative powers of $\Lambda$ we obtain

$$\begin{aligned}\tilde{G}^{(3)}&=\overline{\mathfrak{T}}_2\left(\frac{-6\lambda_3}{(4\pi)^2}\right)+3\Lambda_0T_5(\mu)((n_{15}+n_{17})\vartheta_{3a}'\vartheta_{4a}'+n_{16}\vartheta_{3b}'\vartheta_{4b}'+n_{18}\vartheta_{3a}'\vartheta_{4b}'\\&+n_{19}\vartheta_{3b}'\vartheta_{4a}')+\Lambda_0T_2(\mu)n_{20}\xi_5'\\&=\frac{\Lambda_0}{(4\pi)^2}\left(-6\left(\frac{8}{\varkappa_0}\chi_0'-\frac{40}{3\varkappa_{-2}}(\chi_0')^3+\frac{28}{3\varkappa_{-4}}(\chi_0')^5\right)+\chi_0'\log\left(\frac{\Lambda_0^2}{4\pi\mu^2}\right)\right.\\&\left.\times(35n_{20}-\frac{81}{4}(n_{15}+n_{17})-\frac{75}{2}n_{16}-\frac{135}{4}n_{18}-\frac{45}{2}n_{19})\right)\quad ,\end{aligned} \tag{4.27}$$

where we have summed the $s$- $t$- and $u$-channels, all having the same amplitude. Similarly, the four-point function is given by

$$\begin{aligned}\tilde{G}^{(4)} &= \frac{-(4!)\lambda_4}{(4\pi)^2} + 3T_5(\mu)((n_{21}+n_{23})(\vartheta'_{4a})^2 + n_{22}(\vartheta'_{4b})^2 + n_{24}\vartheta'_{4a}\vartheta'_{4b}) + n_{25}\xi'_6 T_2(\mu) \\ &= \frac{-(4!)}{(4\pi)^2}(2f_0 - 10f_{-2}(\chi'_0)^2 + \frac{35}{3}f_{-4}(\chi'_0)^4) \\ &+ 3\frac{-225}{(4\pi f_{-4})^2}\log\left(\frac{\Lambda_0^2}{4\pi\mu^2}\right)\left((n_{21}+n_{23})\frac{f_{-4}^2}{100} + n_{22}\frac{f_{-4}^2}{36} + n_{24}\frac{f_{-4}^2}{60}\right) \\ &+ n_{25}\frac{7}{15}f_{-4}\frac{15}{(4\pi)^2 f_{-4}}\log\left(\frac{\Lambda_0^2}{4\pi\mu^2}\right) \\ &= \frac{1}{4(4\pi)^2}\Bigg(-96\left(\frac{2}{\varkappa_0} - \frac{10}{\varkappa_{-2}}(\chi'_0)^2 + \frac{35}{3\varkappa_{-4}}(\chi'_0)^4\right) \\ &- \log\left(\frac{\Lambda_0^2}{4\pi\mu^2}\right)(27(n_{21}+n_{23}) + 75n_{22} + 45n_{24} - 28n_{25})\Bigg) \quad . \end{aligned} \tag{4.28}$$

Six independent conditions are enough to find six $\beta$-functions. The results can be checked using the tadpoles and as in equation (1.43) we expect

$$\begin{aligned} 0 = (\text{ RGE })\,\tilde{G}^{(1)} = \\ &(\text{ RGE })\,\Lambda_0^3\Big(\frac{-(1!)}{(4\pi)^2}\left(-8f_2\chi'_0 + 8f_0(\chi'_0)^3 - 4f_{-2}(\chi'_0)^5 + \frac{4}{3}f_{-4}(\chi'_0)^7\right) \\ &- (n_1\vartheta'_{3a} + n_3\vartheta'_{3b})T_1(\mu) + n_2\xi'_3 T_2(\mu)\Big) \\ &= (\text{ RGE })\,\Lambda_0^3\Bigg(\frac{1}{(4\pi)^2}\Bigg(8f_2\chi'_0 - 8\frac{(\chi'_0)^3}{\varkappa_0} + 4\frac{(\chi'_0)^5}{\varkappa_{-2}} - \frac{4}{3}\frac{(\chi'_0)^7}{\varkappa_{-4}} \\ &+ \log\left(\frac{\Lambda_0^2}{4\pi\mu^2}\right)\chi'_0\left(5\frac{\varkappa_{-4}}{\varkappa_{-2}} - 7(\chi'_0)^2\right)N_1\Bigg)\Bigg) \quad . \end{aligned} \tag{4.29}$$

**Definition 4.1.** We define the following numbers, where the $n_i$ refer to Table 4.1:

- $N_1 = 9n_1 - 10n_2 + 10n_3 = 8$ ;
- $N_{2,1} = \frac{81}{4}n_5 + \frac{81}{4}n_6 + 25n_7 + \frac{45}{2}n_8 + 35n_9 + 21n_{12} - 70n_{10} = 168$ ;
- $N_{2,2} = 25n_9 + 15n_{12} - \frac{50}{3}n_{10} = 40$ ;
- $N_{2,3} = \frac{5}{2}n_{11} - \frac{1}{8}n_{13} - \frac{1}{2}n_{14} = 17$ ;
- $N_3 = \frac{81}{2}n_{15} + 75n_{16} + \frac{81}{2}n_{17} + \frac{135}{2}n_{18} + 45n_{19} - 70n_{20} = 672$ ;
- $N_4 = 27n_{21} + 75n_{22} + 27n_{23} + 45n_{24} - 28n_{25} = 1344$ .

We try to find the $\beta$-functions in the same way as we did in paragraph 1.5.3. We solve the renormalization group equation on the independent parts of the Green's functions. Implicitly we use the already solved $\beta$-functions to obtain the following one. We start with the $k^6$-terms of $\tilde{G}^{(2)}$, equation (4.26):

$$0 = \frac{\Lambda_0^{-4}k^6}{15}\left(-2\frac{\gamma_\varphi}{\varkappa_{-4}} + \beta_{\varkappa_{-4}}\frac{-1}{\varkappa_{-4}^2}\right) \quad \Rightarrow \quad \beta_{\varkappa_{-4}} = -2\varkappa_{-4}\gamma_\varphi \quad . \tag{4.30}$$

Continuing with the $k^4$-term of $\tilde{G}^{(2)}$,

$$0 = -2\gamma_\varphi \left( \frac{14}{15} \frac{(\chi_0')^2}{\varkappa_{-4}} - \frac{2}{3\varkappa_{-2}} \right) + \frac{2\beta_{\varkappa_{-2}}}{3\varkappa_{-2}^2} - \frac{14(\chi_0')^2 \beta_{\varkappa_{-4}}}{15\varkappa_{-4}^2} + \frac{28\chi_0'\beta_{\chi_0'}}{15\varkappa_{-4}}$$
$$= \frac{2\beta_{\varkappa_{-2}}}{3\varkappa_{-2}^2} + \frac{4\gamma_\varphi}{3\varkappa_{-2}} + \frac{28\chi_0'\beta_{\chi_0'}}{15\varkappa_{-4}} \quad ,$$

so that

$$\beta_{\varkappa_{-2}} = -2\gamma_\varphi \varkappa_{-2} - \frac{14\chi_0'\varkappa_{-2}^2}{5\varkappa_{-4}} \beta_{\chi_0'} \quad . \tag{4.31}$$

From the $k^2$-term of equation (4.26) $\beta_{\varkappa_0}$ is obtained:

$$0 = -2\gamma_\varphi \left( \frac{4}{\varkappa_0} - \frac{20(\chi_0')^2}{3\varkappa_{-2}} + \frac{14(\chi_0')^4}{3\varkappa_{-4}} \right) - \frac{4\beta_{\varkappa_0}}{\varkappa_0^2} + \frac{20(\chi_0')^2\beta_{\varkappa_{-2}}}{3\varkappa_{-2}^2} - \frac{14(\chi_0')^4\beta_{\varkappa_{-4}}}{3\varkappa_{-4}^2}$$
$$+ 2N_{2,3} + \beta_{\chi_0'} \left( \frac{56(\chi_0')^3}{3\varkappa_{-4}} - \frac{40\chi_0'}{3\varkappa_{-2}} \right)$$
$$= \frac{-4\beta_{\varkappa_0}}{\varkappa_0^2} + 2N_{2,3} - \frac{8\gamma_\varphi}{\varkappa_0} - \frac{40\chi_0'\beta_{\chi_0'}}{3\varkappa_{-2}} \quad ,$$

yielding

$$\beta_{\varkappa_0} = \frac{N_{2,3}\varkappa_0^2}{2} - 2\gamma_\varphi\varkappa_0 - \frac{10\chi_0'\varkappa_0^2}{3\varkappa_{-2}}\beta_{\chi_0'} \quad . \tag{4.32}$$

The mass term of $\tilde{G}^{(2)}$ is the only one depending on $f_2$, so we use this to find $\beta_{f_2}$:

$$0 = -2\gamma_\varphi \left( -8f_2 + \frac{24(\chi_0')^2}{\varkappa_0} - \frac{20(\chi_0')^4}{\varkappa_{-2}} + \frac{28(\chi_0')^6}{3\varkappa_{-4}} \right) - 2(\chi_0')^2 N_{2,1} + 2\frac{\varkappa_{-4}}{\varkappa_{-2}} N_{2,2}$$
$$- 8\beta_{f_2} - \frac{24(\chi_0')^2\beta_{\varkappa_0}}{\varkappa_0^2} + \frac{20(\chi_0')^4\beta_{\varkappa_{-2}}}{\varkappa_{-2}^2} - \frac{28(\chi_0')^6\beta_{\varkappa_{-4}}}{3\varkappa_{-4}^2}$$
$$+ \beta_{\chi_0'} \left( \frac{48\chi_0'}{\varkappa_0} - \frac{80(\chi_0')^3}{\varkappa_{-2}} + \frac{56(\chi_0')^5}{\varkappa_{-4}} \right)$$
$$= -8\beta_{f_2} - 2(\chi_0')^2 N_{2,1} + 2\frac{\varkappa_{-4}}{\varkappa_{-2}} N_{2,2} + 16 f_2 \gamma_\varphi + \frac{48\chi_0'\beta_{\chi_0'}}{\varkappa_0} - 12(\chi_0')^2 N_{2,3} \quad .$$

This leads to the mass renormalization

$$\beta_{f_2} = \frac{\varkappa_{-4}N_{2,2}}{4\varkappa_{-2}} - \frac{(\chi_0')^2}{4}\left(N_{2,1} + 6N_{2,3}\right) + 2\gamma_\varphi f_2 + \frac{6\chi_0'}{\varkappa_0}\beta_{\chi_0'} \quad . \tag{4.33}$$

The last two are more tedious. We find the running of $\chi_0'$ from the four-point function (4.28)

$$0 = -4\gamma_\varphi \left( -96 \left( \frac{2}{\varkappa_0} - \frac{10(\chi_0')^2}{\varkappa_{-2}} + \frac{35(\chi_0')^4}{3\varkappa_{-4}} \right) \right) + 2N_4 + 96\frac{2\beta_{\varkappa_0}}{\varkappa_0^2} - 96\frac{10(\chi_0')^2\beta_{\varkappa_{-2}}}{\varkappa_{-2}^2}$$
$$+ 96\frac{35(\chi_0')^4\beta_{\varkappa_{-4}}}{3\varkappa_{-4}^2} - 96\beta_{\chi_0'} \left( \frac{-20\chi_0'}{\varkappa_{-2}} + \frac{140(\chi_0')^3}{3\varkappa_{-4}} \right)$$
$$= 2N_4 + 96N_{2,3} + 96\gamma_\varphi \left( \frac{4}{\varkappa_0} - \frac{20(\chi_0')^2}{\varkappa_{-2}} + \frac{70(\chi_0')^4}{3\varkappa_{-4}} \right)$$
$$- 96\beta_{\chi_0'} \left( \frac{-40\chi_0'}{3\varkappa_{-2}} + \frac{56(\chi_0')^3}{3\varkappa_{-4}} \right) \quad ,$$

leading to

$$\beta_{\chi_0'} = \frac{1}{4\left(\frac{-5\chi_0'}{\varkappa_{-2}} + \frac{7(\chi_0')^3}{\varkappa_{-4}}\right)} \left(\frac{N_4}{32} + \frac{3N_{2,3}}{2} + 3\gamma_\varphi \left(\frac{2}{\varkappa_0} - \frac{10(\chi_0')^2}{\varkappa_{-2}} + \frac{35(\chi_0')^4}{3\varkappa_{-4}}\right)\right) \quad . \tag{4.34}$$

Finally, the three-point correlator (4.27) yields

$$\begin{aligned} 0 = &-3\gamma_\varphi \left(-6\left(\frac{8\chi_0'}{\varkappa_0} - \frac{40(\chi_0')^3}{3\varkappa_{-2}} + \frac{28(\chi_0')^5}{3\varkappa_{-4}}\right)\right) + \chi_0' N_3 + \frac{48\chi_0'\beta_{\varkappa_0}}{\varkappa_0^2} - \frac{80(\chi_0')^3\beta_{\varkappa_{-2}}}{\varkappa_{-2}^2} \\ &+ \frac{56(\chi_0')^5\beta_{\varkappa_{-4}}}{\varkappa_{-4}^2} + \beta_{\chi_0'}\left(\frac{-48}{\varkappa_0} + \frac{240(\chi_0')^2}{\varkappa_{-2}} - \frac{280(\chi_0')^4}{\varkappa_{-4}}\right) \\ &= 8\left(\gamma_\varphi\chi_0' - \beta_{\chi_0'}\right)\left(\frac{6}{\varkappa_0} - \frac{10(\chi_0')^2}{\varkappa_{-2}} + \frac{7(\chi_0')^4}{\varkappa_{-4}}\right) + 24\chi_0' N_{2,3} + \chi_0' N_3 \quad , \end{aligned}$$

which gives

$$\gamma_\varphi - \frac{\beta_{\chi_0'}}{\chi_0'} = \frac{-3N_{2,3} - \frac{N_3}{8}}{\frac{6}{\varkappa_0} - \frac{10(\chi_0')^2}{\varkappa_{-2}} + \frac{7(\chi_0')^4}{\varkappa_{-4}}} \quad . \tag{4.35}$$

Combining equations (4.34) and (4.35) we find

$$\begin{aligned} \beta_{\chi_0'} = &\frac{-\chi_0'}{2\left(\frac{6}{\varkappa_0} - \frac{10(\chi_0')^2}{\varkappa_{-2}} + \frac{7(\chi_0')^4}{\varkappa_{-4}}\right)^2} \left(\left(\frac{N_4}{16} + 3N_{2,3}\right)\left(\frac{6}{\varkappa_0} - \frac{10(\chi_0')^2}{\varkappa_{-2}} + \frac{7(\chi_0')^4}{\varkappa_{-4}}\right)\right. \\ &\left. - \left(6N_{2,3} + \frac{N_3}{4}\right)\left(\frac{6}{\varkappa_0} - \frac{30(\chi_0')^2}{\varkappa_{-2}} + \frac{35(\chi_0')^4}{\varkappa_{-4}}\right)\right) \end{aligned} \tag{4.36}$$

and

$$\begin{aligned} \gamma_\varphi = &\frac{-1}{2\left(\frac{6}{\varkappa_0} - \frac{10(\chi_0')^2}{\varkappa_{-2}} + \frac{7(\chi_0')^4}{\varkappa_{-4}}\right)^2} \left(\left(\frac{N_4}{16} + \frac{N_3}{4} + 9N_{2,3}\right)\left(\frac{6}{\varkappa_0} - \frac{10(\chi_0')^2}{\varkappa_{-2}} + \frac{7(\chi_0')^4}{\varkappa_{-4}}\right)\right. \\ &\left. - \left(6N_{2,3} + \frac{N_3}{4}\right)\left(\frac{6}{\varkappa_0} - \frac{30(\chi_0')^2}{\varkappa_{-2}} + \frac{35(\chi_0')^4}{\varkappa_{-4}}\right)\right) \quad . \end{aligned} \tag{4.37}$$

By inserting these expressions in equations (4.30)-(4.34), we obtain all $\beta$-functions of the ERSHL.

**Proposition 4.2.** The lowest order $\beta$-functions of the $\overline{\mathfrak{T}}_2$-renormalized ersHL are given by

$$\beta_{\varkappa_0} = \frac{\mathcal{F}}{48}\varkappa_0^2\Big(1176\varkappa_{-2}^2\varkappa_0^2 N_{2,3}(\chi_0')^8 - 7\varkappa_{-4}\varkappa_{-2}\varkappa_0(\chi_0')^4\big(\varkappa_{-2}(720N_{2,3}+48N_3-3N_4) + 5\varkappa_0(528N_{2,3}+20N_3-N_4)(\chi_0')^2\big) + 2\varkappa_{-4}^2\big(9\varkappa_{-2}^2(96N_{2,3}+N_4) + 60\varkappa_{-2}\varkappa_0 N_3(\chi_0')^2 + 25\varkappa_0^2(288N_{2,3}+12N_3-N_4)(\chi_0')^4\big)\Big) \quad ; \tag{4.38}$$

$$\beta_{\varkappa_{-2}} = \frac{\mathcal{F}}{80}\varkappa_{-2}^2\varkappa_0\Big(-49\varkappa_{-2}^2\varkappa_0(432N_{2,3}+20N_3-N_4)(\chi_0')^6 + 10\varkappa_{-4}^2\big(3\varkappa_{-2}(48N_{2,3}+N_4) + 5\varkappa_0(144N_{2,3}+8N_3-N_4)(\chi_0')^2\big) + 7\varkappa_{-4}\varkappa_{-2}(\chi_0')^2\big(6\varkappa_{-2}(-48N_{2,3}-4N_3+N_4) + 5\varkappa_0(144N_{2,3}+8N_3-N_4)(\chi_0')^2\big)\Big) \quad ; \tag{4.39}$$

$$\beta_{\varkappa_{-4}} = \frac{\mathcal{F}}{16}\varkappa_{-4}^2\varkappa_{-2}\varkappa_0\Big(6\varkappa_{-4}\varkappa_{-2}(48N_{2,3}+N_4) + 10\varkappa_{-4}\varkappa_0(144N_{2,3}+8N_3-N_4)(\chi_0')^2 + 7\varkappa_{-2}\varkappa_0\big(-16(21N_{2,3}+N_3)+N_4\big)(\chi_0')^4\Big) \quad ; \tag{4.40}$$

$$\beta_{f_2} = \frac{\mathcal{F}}{16\varkappa_{-2}}\Big(-196\varkappa_{-2}^3\varkappa_0^2(N_{2,1}+6N_{2,3})(\chi_0')^{10} + 16\varkappa_{-4}^3 N_{2,2}\big(3\varkappa_{-2}-5\varkappa_0(\chi_0')^2\big)^2 + 7\varkappa_{-4}\varkappa_{-2}^2\varkappa_0(\chi_0')^4\big(\varkappa_{-2}\varkappa_0 f_2(336N_{2,3}+16N_3-N_4) - 3\varkappa_{-2}(16N_{2,1}-336N_{2,3}-20N_3+N_4)(\chi_0')^2 + 4\varkappa_0(20(N_{2,1}+6N_{2,3})+7N_{2,2})(\chi_0')^4\big) - 2\varkappa_{-4}^2\varkappa_{-2}\big(3\varkappa_{-2}^2\varkappa_0 f_2(48N_{2,3}+N_4) + \varkappa_{-2}\big[5\varkappa_0^2 f_2(144N_{2,3}+8N_3-N_4) + 9\varkappa_{-2}(8N_{2,1}-4N_3+N_4)\big](\chi_0')^2 - 3\varkappa_{-2}\varkappa_0(80N_{2,1}-720N_{2,3}-60N_3+5N_4+56N_{2,2})(\chi_0')^4 + 40\varkappa_0^2(5N_{2,1}+30N_{2,3}+7N_{2,2})(\chi_0')^6\big)\Big) \quad ; \tag{4.41}$$

$$\beta_{\chi_0'} = \frac{\mathcal{F}}{32}\varkappa_{-4}\varkappa_{-2}\varkappa_0(\chi_0')\Big(6\varkappa_{-4}\varkappa_{-2}(48N_{2,3}+4N_3-N_4) + 10\varkappa_{-4}\varkappa_0(-12(20N_{2,3}+N_3)+N_4)(\chi_0')^2 + 7\varkappa_{-2}\varkappa_0(432N_{2,3}+20N_3-N_4)(\chi_0')^4\Big) \quad ; \tag{4.42}$$

$$\gamma_\varphi = \frac{\mathcal{F}}{32}\varkappa_{-4}\varkappa_{-2}\varkappa_0\Big(-6\varkappa_{-4}\varkappa_{-2}(48N_{2,3}+N_4) + 10\varkappa_{-4}\varkappa_0\big(-8(18N_{2,3}+N_3)+N_4\big)(\chi_0')^2 + 7\varkappa_{-2}\varkappa_0(336N_{2,3}+16N_3-N_4)(\chi_0')^4\Big) \quad , \tag{4.43}$$

where the denominator is defined by

$$\mathcal{F} \equiv \frac{1}{(6\varkappa_{-2}\varkappa_{-4} - 10\varkappa_0\varkappa_{-4}(\chi_0')^2 + 7\varkappa_0\varkappa_{-2}(\chi_0')^4)^2}$$

and the numbers $N_i$ are given in Definition 4.1.

## 4.5 Tadpole consistency condition

The following computation checks whether the renormalization group equation on the one-point functions is valid. We obtain an expression for $\beta_{\chi_0'}$ from equation (4.29), pretending that it is still unknown, and compare the new obtained $\beta$-function with equation (4.42). Recalling the tadpole function from equation (4.29)

$$\begin{aligned}
0 = &-\gamma_\varphi \left( 8f_2\chi_0' - \frac{8(\chi_0')^3}{\varkappa_0} + \frac{4(\chi_0')^5}{\varkappa_{-2}} - \frac{4(\chi_0')^7}{3\varkappa_{-4}} \right) - 2\chi_0' N_1 \left( 5\frac{\varkappa_{-4}}{\varkappa_{-2}} - 7(\chi_0')^2 \right) \\
&+ 8\chi_0'\beta_{f_2} + \frac{8(\chi_0')^3}{\varkappa_0^2}\beta_{\varkappa_0} - \frac{4(\chi_0')^5}{\varkappa_{-2}^2}\beta_{\varkappa_{-2}} + \frac{4(\chi_0')^7}{3\varkappa_{-4}^2}\beta_{\varkappa_{-4}} \\
&+ \beta_{\chi_0'} \left( 8f_2 - \frac{24(\chi_0')^2}{\varkappa_0} + \frac{20(\chi_0')^4}{\varkappa_{-2}} - \frac{28(\chi_0')^6}{3\varkappa_{-4}} \right) \\
= &-\chi_0' \left( 3N_{2,3} + \frac{N_3}{8} \right) \frac{8f_2 - \frac{8(\chi_0')^2}{\varkappa_0} + \frac{4(\chi_0')^4}{\varkappa_{-2}} - \frac{4(\chi_0')^6}{3\varkappa_{-4}}}{\frac{6}{\varkappa_0} - \frac{10(\chi_0')^2}{\varkappa_{-2}} + \frac{7(\chi_0')^4}{\varkappa_{-4}}} \\
&+ \chi_0' \frac{\varkappa_{-4}}{\varkappa_{-2}} (2N_{2,2} - 10N_1) + (\chi_0')^3 (14N_1 - 8N_{2,3} - 2N_{2,1}) \\
&+ \beta_{\chi_0'} \left( 16f_2 + \frac{16(\chi_0')^2}{\varkappa_0} - \frac{8(\chi_0')^4}{3\varkappa_{-2}} + \frac{8(\chi_0')^6}{15\varkappa_{-4}} \right) \quad . \qquad (4.44)
\end{aligned}$$

By the first condition from Lemma 3.4 we see that the first line of equation (4.44) is zero at lowest order. Since $5N_1 - N_{2,2} = 0$ according to Definition 4.1, this results in

$$\begin{aligned}
\beta_{\chi_0'} &= \frac{(\chi_0')^3}{2f_2 + 2\frac{(\chi_0')^2}{\varkappa_0} - \frac{(\chi_0')^4}{3\varkappa_{-2}} + \frac{(\chi_0')^6}{15\varkappa_{-4}}} \left( \frac{1}{4}N_{2,1} + N_{2,3} - \frac{7}{4}N_1 \right) \\
&= \frac{\chi_0'}{\frac{4}{\varkappa_0} - \frac{4(\chi_0')^2}{3\varkappa_{-2}} + \frac{2(\chi_0')^4}{5\varkappa_{-4}}} \left( \frac{1}{4}N_{2,1} + N_{2,3} - \frac{7}{4}N_1 \right) \quad , \qquad (4.45)
\end{aligned}$$

where the denominator in the second line is simplified using the first condition from Lemma 3.4 again.

The renormalization of this theory is only successful if the two expressions for the renormalization of the minimum of the potential are equal. Equating (4.42) and (4.45) with the numbers from Definition 4.1 we obtain the following proposition.

**Proposition 4.3.** [Tadpole consistency condition]
The extended real scalar Higgs Lagrangian from Proposition 3.2 is only renormalizable with the $\beta$-functions as given in equations (4.38)-(4.43), if at lowest order

$$\varkappa_{-2} = 10\varkappa_{-4} \frac{\varkappa_0(\chi_0')^4 + 12\varkappa_{-4}}{186(\chi_0')^2\varkappa_{-4} - 7\varkappa_0(\chi_0')^6}$$

holds, which is the same as

$$f_{-2} = \frac{f_{-4}(\chi_0')^2}{10} \frac{186f_0 - 7f_{-4}(\chi_0')^4}{12f_0 + f_{-4}(\chi_0')^4} \quad .$$

Although it appeared that the ersHL had better renormalization properties than the rsHL, an additional condition, specifying the ratio between the two additional coupling constants $\varkappa_{-2}$ and $\varkappa_{-4}$, was needed to save the renormalization picture. When making the step from the rsHL to the ersHL we no longer required that both $f_{-2}$ and $f_{-4}$ were zero. However, renormalizability then forces the ratio between them.

Ignoring the fact that the analysis is not valid in the limit that both $f_{-2}$ and $f_{-4}$ go to zero, the additional condition is satisfied by this rsHL-solution. If this reflects the fact that the rsHL is renormalizable, then the real scalar Higgs Lagrangian with $f_{k\leq -4} = 0$ would only be renormalizable, if $f_{-2} = 0$.

With the extra constraint we can again try to find the minimum of the potential, see remark 3.6. According to Lemma 3.4 the minimum of the potential is found by solving

$$0 = -8f_2 + 8f_0(\chi_0')^2 - 4f_{-2}(\chi_0')^4 + \frac{4}{3}f_{-4}(\chi_0')^6 \quad .$$

Substituting $f_{-2}$ from Proposition 4.3 and rearranging terms, this becomes

$$0 = -24f_2f_0 + 24f_0^2(\chi_0')^2 - 2f_2f_{-4}(\chi_0')^4 - \frac{63}{5}f_0f_{-4}(\chi_0')^6 + \frac{31}{30}f_{-4}^2(\chi_0')^{10} \quad . \tag{4.46}$$

We see that the tadpole consistency condition relates renormalizability to the higher derivatives of $f$ in the spectral action (2.1). This means that the spectral action, interpreted as renormalizable field theory contains less parameter freedom than expected beforehand.
Giving up $\varkappa_{-2}$ as an independent coupling changes the status of the corresponding $\beta$-function. If the running of both sides of the tadpole consistency condition is the same, we can remove $\varkappa_{-2}$ from the lowest order theory and replace it with a combination of other parameters. Then, $\beta_{\varkappa_{-2}}$ from $\tilde{G}^{(2)}$ in equation (4.31) is the same as the renormalization group equation acting on the right hand side of the tadpole consistency condition. In that case

$$\beta_{\varkappa_{-2}} = (\text{ RGE })\, \varkappa_{-2} = (\text{ RGE }) \left(10\varkappa_{-4}\frac{\varkappa_0(\chi_0')^4 + 12\varkappa_{-4}}{186(\chi_0')^2\varkappa_{-4} - 7\varkappa_0(\chi_0')^6}\right) \tag{4.47}$$

should hold. Applying equations (4.38)-(4.43) to this, substituting $\varkappa_{-2}$ using the tadpole consistency condition and writing the result in terms of

$$\tilde{f}_{-4} = \frac{f_{-4}}{f_0}(\chi_0')^4 \tag{4.48}$$

we obtain that equation (4.47) holds, if

$$\frac{-1296 + \tilde{f}_{-4}(5148 + \tilde{f}_{-4}(-3876 + 185\tilde{f}_{-4}))}{\tilde{f}_{-4}(186 - 7\tilde{f}_{-4})^2(\tilde{f}_{-4} - 6)^2}\frac{(\chi_0')^2}{f_0^2} = 0 \quad .$$

All finite solutions for this in terms of $\tilde{f}_{-4}$ are complex valued, so this holds when $f_{-4}$ goes to infinity. Both couplings $\varkappa_{-2}$ and $\varkappa_{-4}$ vanish then.
It should not surprise us that equation (4.47) is not preserved under the renormalization group equation. This is the only term that does not raise the order, which indicates that it is violated at higher order. However, to verify that the theory is also renormalizable at the next order we need the amplitudes and $\beta$-functions also at higher order.

In all ersHL-equations encountered, we have always found a combination of $f_{-4}$ and $\chi_0'$. This means that they are not completely independent and that the physical theory depends on a combination of the two. First, we can simplify equation (4.46) and all the $\beta$-functions using $\tilde{f}_{-4}$ from equation (4.48). This makes it easier to find the minima of the potential described by (4.46), since it is now given by

$$24\frac{f_2}{f_0} + 2\tilde{f}_{-4}\frac{f_2}{f_0} = (\chi_0')^2\left(24 - \frac{63}{5}\tilde{f}_{-4} + \frac{31}{30}\tilde{f}_{-4}^2\right) \quad , \tag{4.49}$$

which yields

$$(\chi_0')^2 = \frac{2f_2}{f_0}\frac{12 + \tilde{f}_{-4}}{24 - \frac{63}{5}\tilde{f}_{-4} + \frac{31}{30}\tilde{f}_{-4}^2} \quad . \tag{4.50}$$

From this it is clear that $(\chi_0')^2 \propto \frac{f_2}{f_0}$, as in Chapter 1, example 1.2. The natural interpretation is that $\varkappa_{-4} = \mathcal{O}(\varkappa_0^3)$, so that $\tilde{f}_{-4}$ is now just a number.
From this we can derive a concrete expression for $f_{-2}$ in terms of the other constants:

$$f_{-2} = -\frac{\tilde{f}_{-4} f_0^2}{600 f_2} \frac{(7\tilde{f}_{-4} - 186)(31\tilde{f}_{-4}^2 - 378\tilde{f}_{-4} + 720)}{(\tilde{f}_{-4} + 12)^2} \quad . \tag{4.51}$$

We now re-examine the potential from Proposition 3.1 to demonstrate the effect of the tadpole consistency condition. This potential is obtained by changing to a constant field $\Phi$, where we have furthermore substituted $f_{-2}$ from equation (4.51) and $f_{-4}$ using (4.48). For the dependence on the minimum $\chi_0'$ of the potential in equations (4.48) and (4.51) have used equation (4.50). This potential for a constant field $\Phi$ is then given by

$$\begin{aligned} V(\Phi, f_2, f_0, \tilde{f}_{-4}) &= -4 f_2 \Phi^2 + 2 f_0 \Phi^4 \\ &+ \frac{\tilde{f}_{-4} f_0^2}{f_2} \frac{(7\tilde{f}_{-4} - 186)(31\tilde{f}_{-4}^2 - 378\tilde{f}_{-4} + 720)}{900(\tilde{f}_{-4} + 12)^2} \Phi^6 \\ &+ \frac{\tilde{f}_{-4} f_0^3}{f_2^2} \frac{(31\tilde{f}_{-4} - 378\tilde{f}_{-4} + 720)^2}{21600(\tilde{f}_{-4} + 12)^2} \Phi^8 \quad , \end{aligned} \tag{4.52}$$

where we have removed a factor $\Lambda^4/(4\pi)^2$ and the constant $f_4$-term is left out. And as expected, the solution (4.50) yields the minimum of the potential

$$0 = \frac{\mathrm{d}V}{\mathrm{d}\Phi}[\chi_0'] \quad .$$

By rescaling $\Phi \to \Phi\sqrt{\frac{f_2}{f_0}}$ we see that the shape of this potential only depends on $\tilde{f}_{-4}$. An idea of the shape of this potential can be found in Figures 4.2-4.5. For negative values of $\tilde{f}_{-4}$ the potential is unbounded from below, although there remains a local minimum. Furthermore, there are two branches of deeper minima which start around $\tilde{f}_{-4} = 1, 10$.
The final task is to see whether we can retrieve the $\beta$-functions from Chapter 1, equations (1.39)-(1.42). Rewriting the $\beta$-functions from equations (4.38)-(4.43) using equation (4.48) and the tadpole consistency condition yields at lowest order

$$\begin{aligned} \beta_{\varkappa_0} &= -\frac{26784 - \tilde{f}_{-4}(43416 + 7\tilde{f}_{-4}(349\tilde{f}_{-4} - 9270))}{4(-6 + \tilde{f}_{-4})^2(-6 + 7\tilde{f}_{-4}) f_0^2} \quad ; \\ \beta_{\varkappa_{-2}} &= -\frac{135(\chi_0')^2}{f_0^2 \tilde{f}_{-4}} \frac{45(12 + \tilde{f}_{-4})^2(12 + \tilde{f}_{-4}(220 + 7\tilde{f}_{-4}))}{(186 - 7\tilde{f}_{-4})^2(\tilde{f}_{-4} - 6)^2(7\tilde{f}_{-4} - 6)} \quad ; \\ \beta_{\varkappa_{-4}} &= -\frac{135(\chi_0')^4}{2 f_0^2 \tilde{f}_{-4}} \frac{(12 + \tilde{f}_{-4})(6 + 5\tilde{f}_{-4})}{(\tilde{f}_{-4} - 6)^2(7\tilde{f}_{-4} - 6)} \quad ; \\ \beta_{f_2} &= \frac{135 f_2}{2 f_0} \frac{(12 + \tilde{f}_{-4})(6 + 5\tilde{f}_{-4})}{(-6 + \tilde{f}_{-4})^2(-6 + 7\tilde{f}_{-4})} \\ &\quad + \frac{(\chi_0')^2}{2} \frac{(13392 + \tilde{f}_{-4}(19332 + \tilde{f}_{-4}(94 - 149\tilde{f}_{-4})))}{(\tilde{f}_{-4} - 6)^2(12 + \tilde{f}_{-4})} \quad ; \\ \beta_{\chi_0'} &= \frac{135\chi_0'}{4 f_0} \frac{\tilde{f}_{-4} + 12}{(\tilde{f}_{-4} - 6)^2} \quad ; \\ \gamma_\varphi &= \frac{135}{4 f_0} \frac{(12 + \tilde{f}_{-4})(6 + 5\tilde{f}_{-4})}{(\tilde{f}_{-4} - 6)^2(7\tilde{f}_{-4} - 6)} \quad . \end{aligned}$$

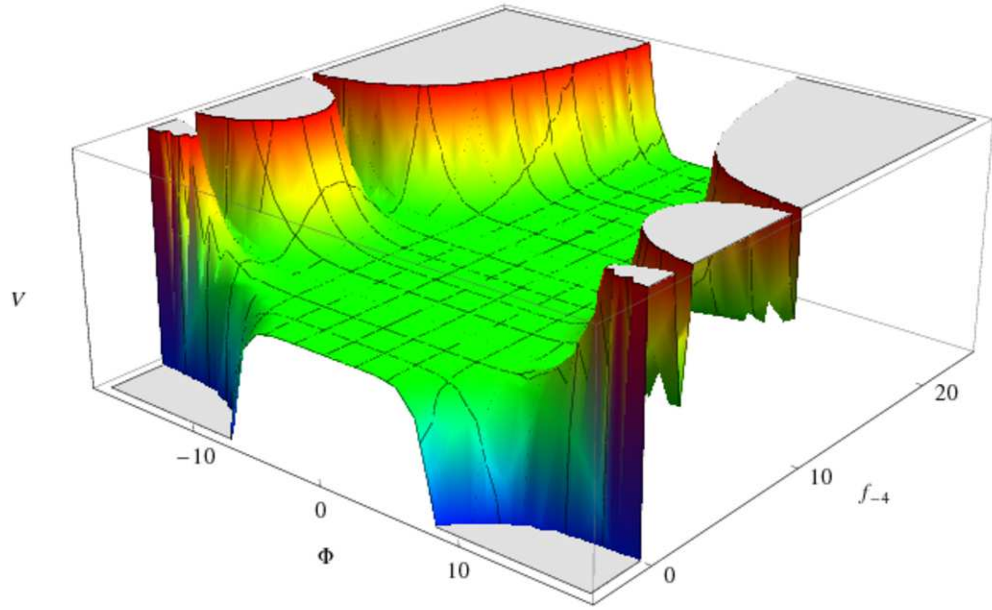


Figure 4.2: The potential of the renormalizable ersHL. On one axis the constant field $\Phi$, on the other $\tilde{f}_{-4}$.

From these it can be seen relatively easily that there is no choice of $\tilde{f}_{-4}$ for which these simplify to the $\beta$-functions of Chapter 1, equations (1.39)-(1.42). This means that the ersHL behaves differently under renormalization from the rsHL and cannot simply be seen as an extension of it.
However, these interpretations of the coupling constants together with the $\beta$-functions show that $\beta_{\varkappa_{-2}}$ and $\beta_{\varkappa_{-4}}$ are of higher order. This we have determined via the propagator, but it is a statement on the various higher-point interactions as well. These higher dimensional field operators are not expected to renormalize, which we have thus confirmed to a certain extent.

## Concluding remarks

Instead of deriving the tadpole consistency condition we could try to renormalize the theory by adding the running of the mass scale $\beta_\Lambda$. However, as in remark 1.20, it turns out that this is not independent of the other parameters and therefore forms no alternative.
The problem that the one-point function does not immediately vanish under the renormalization group equation could be fixed by arranging somehow that $N_{2,3} = -28$, so that $\beta_{\chi_0'} = 0$. This remarkable observation may be interpreted as a sign that the subtraction operator used alters the kinetic structure of the theory in such a way that renormalizability is not preserved, contrary to the rsHL-case. Since we have no evidence for this statement and no alternative techniques at our disposal, we cannot go any deeper into this.

We may wonder whether the obtained $\beta$-functions of Proposition 4.2 are the same as

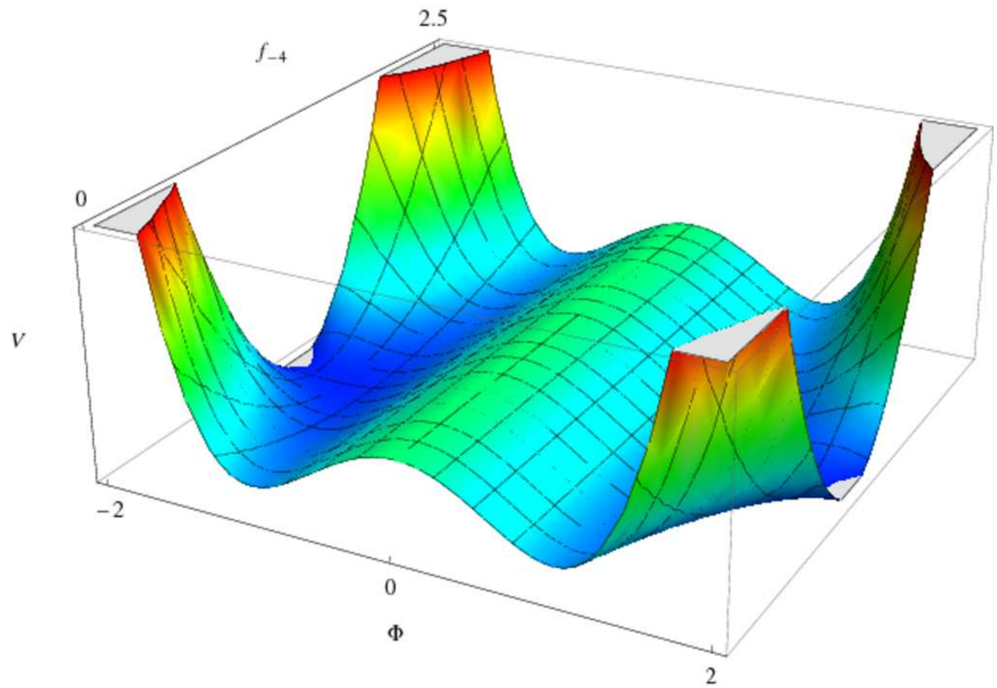


Figure 4.3: A detail of Figure 4.2. The potential of the renormalizable ersHL. On one axis the constant field $\Phi$, on the other $\tilde{f}_{-4}$.

those of the $\overline{\mathfrak{T}}_1$-renormalized theory. The answer is no. First it should be noticed that the equations obtained in paragraph 4.4 for the lowest order $\beta$-functions using $\overline{\mathfrak{T}}_2$-subtraction would be obtained as well for $\overline{\mathfrak{T}}_1$. In addition to those, the $\overline{\mathfrak{T}}_1$-renormalized theory has a renormalization group equation for the $(6,2)$-vertex as well, see Proposition 3.2. There are no divergent diagrams for this vertex in Figure 3.2. This means that the renormalization group equation on this vertex gives $\beta_{\varkappa_{-4}} = -6\varkappa_{-4}\gamma_\varphi$, which combined with equation (4.30) yields that both $\gamma_\varphi$ and $\beta_{\varkappa_{-4}}$ are equal to zero. This cannot be made compatible with the rsHL.
Considering the spectral action experimentally our approach is pointless, since an amplitude's momentum dependence on the external lines is then determined by the expansion length chosen for the spectral action. The expansion length is given by the upper bound of the sum in action (2.3).
Although it is possible to obtain $\beta$-functions at lowest order using hybrid renormalization, they are not simply the same as in the $\overline{\text{eMS}}$-scheme. In paragraph 1.4.4 we explained that when two ways of renormalizing do not produce the same outcome, then at least one of the two is not applicable. From physical arguments it is clear that $\overline{\mathfrak{T}}_2$ is not relevant. However, both are well-defined subtraction operators and should give sensible renormalization theories. The fact that the tadpole consistency condition is needed to complete the renormalization is more a feature of the spectral action than of the renormalization techniques used here.

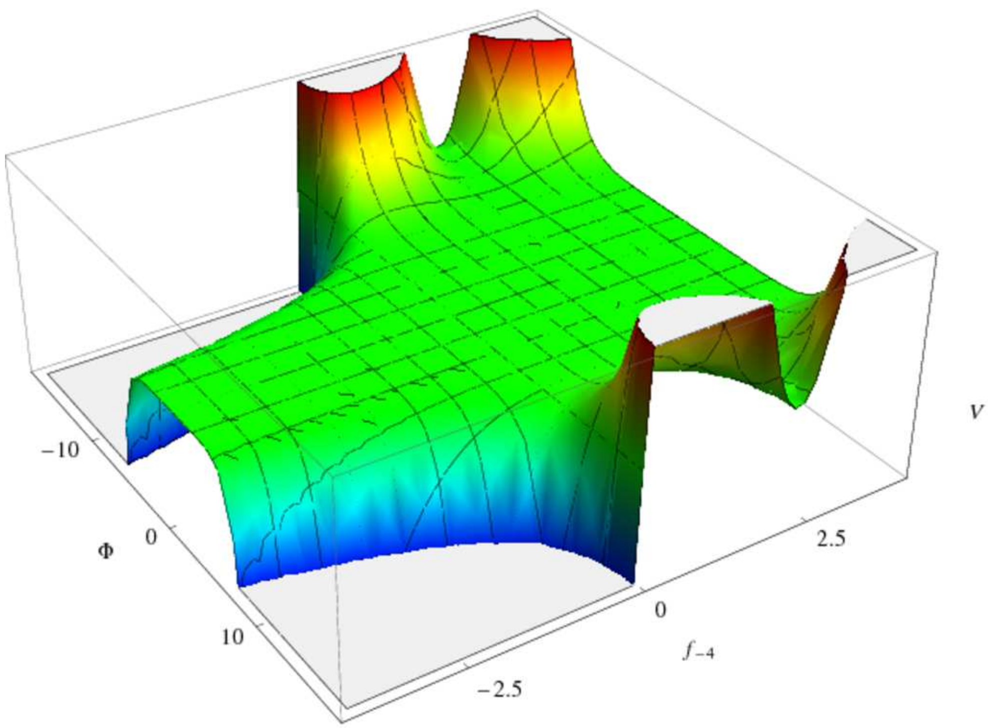


Figure 4.4: A detail of Figure 4.2. The potential of the renormalizable ersHL. On one axis the constant field Φ, on the other $\tilde{f}_{-4}$.

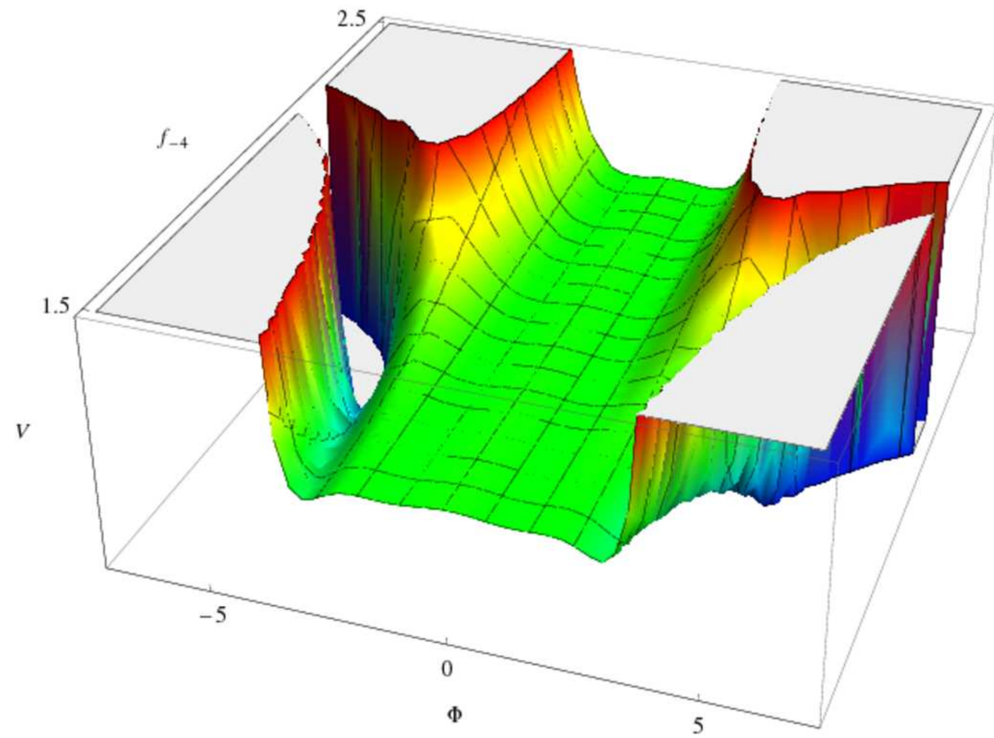


Figure 4.5: A detail of Figure 4.2. The potential of the renormalizable ersHL. On one axis the constant field $\Phi$, on the other $\tilde{f}_{-4}$.

# 5 Conclusions

In this work we have studied the renormalization properties of the scalar spectral action on a compact manifold with vanishing Christoffel symbols and without a gauge connection. This action comes naturally with a mass parameter $\Lambda$ that acts as an eigenvalue cutoff. By a heat kernel expansion the scalar spectral action asymptotically yields Higgs-like Lagrangians. For two of these Lagrangians we have determined the $\beta$-functions, which describe the UV-behaviour of the quantum field theory.
The lowest order real scalar Higgs Lagrangian (rsHL) is a renormalizable theory that consists of relevant and marginal terms only. This means that all Lagrangian terms are of importance for the renormalization of this theory. The renormalization is done by a BPHZ-subtraction scheme.
Regularizing with the substitution operator, the mass parameter $\Lambda$ is removed and substituted for a dimensional constant $\Lambda_0$. This scheme reduces in practice to $\overline{\mathrm{MS}}$ with dimensional regularization. Alternatively, the subtraction operator yields a hybrid regularization scheme that substitutes the mass parameter in the parts relevant for the renormalization group equations, but subtracts the irrelevant contributions. The obtained $\beta$-functions for the rsHL are the same at lowest order for both regularization schemes, although the amplitudes are not.
Although definitely not all physical properties can be determined using this hybrid renormalization, obtaining the $\beta$-functions does not appear to be impossible. Furthermore, since it is well-defined, its renormalization should make sence, which makes it suitable to study the renormalization properties of the spectral action.
These substitution or subtraction schemes allow the mass-parameter to be taken to infinity as required by the asymptotic expansion. Since a finite shift of this natural regularization scheme is allowed to have physical implications, there is no need to circumvent this limit to preserve a mass term.
This subtraction operator simplifies calculations that much that in principle any loop-calculation can be done. This makes it possible to analyze a higher order Higgs Lagrangian (ersHL). The higher order theory studied in this work is an extension of the rsHL with irrelevant interactions, which are expected to have a trivial UV-behaviour. These interactions make the ersHL superficially a superrenormalizable higher derivative field theory. From this we see that higher order spectral action terms can have a naturally regularizing quality. However, these higher derivative terms alter the propagator such that calculations become more difficult in general. Due to this change in the propagator it is not possible to renormalize this theory multiplicatively. Therefore, only by using the simplifying subtraction operator we have been able to determine loop amplitudes.
The correspondence of the $\beta$-functions at lower order in the heat kernel expansion gives us an argument to seek for the $\beta$-functions at higher order. However, it appears that the ersHL is not straightforwardly renormalizable. It becomes so under the tadpole consistency condition that relates the extension parameters to each other order by order. Unlike usual higher derivative field theory, higher order field theories from the spectral action appear to be renormalizable. Furthermore, there is no hybridly

renormalized spectral extension of the RSHL such that the $\beta$-functions for both theories coincide.
Even though these outcomes are not directly valid for other renormalization prescriptions, it makes it clear that in the renormalization of extended spectral Lagrangians surprises may be expected.

# Bibliography


[1] P.I. Pronin and K. Stepanyantz. One loop counterterms for higher derivative regularized lagrangians. *Phys. Lett. B*, 414:117–122, 1997.

[2] W.D. van Suijlekom. Renormalization of the asymptotically expanded yang-mills spectral action. *Commun. Math. Phys.*, 312:883–912, 2012.

[3] C.P. Martin and F. Ruiz Ruiz. Higher covariant derivative regulators and non-multiplicative renormalization. *Phys. Lett. B.*, 343:218–224, 1995.

[4] M. Srednicki. *Quantum Field Theory.* Cambridge University Press, 2007.

[5] J. Zinn-Justin. *Quantum Field Theory and Critical Phenomena.* Clarendon Press, 2002.

[6] R.P. Feynman. Space-time approach to non-relativistic quantum mechanics. *Rev. Mod. Phys.*, 20:367–387, Apr 1948.

[7] D. Fujiwara. Remarks on convergence of the feynman path integrals. *Duke Math. J.*, 47(3):559–600, 1980.

[8] M.E. Peskin and D.V. Schroeder. *An Introduction To Quantum Field Theory.*

[9] B. Delamotte. A hint of renormalization. *Am. J. .Phys.*, 72(2):170–184, 2004.

[10] G. Baxter. An analytic problem whose solution follows from a simple algebraic identity. *Pacific J. Math.*, 10:731–742, 1960.

[11] G.C. Rota. Baxter algebras and combinatorial identities i,ii. *Bull. Amer. Math. Soc.*, 75:325–329, 1969.

[12] K. Ebrahimi-Fard, L. Guo, and D. Kreimer. Integrable renormalization i: the ladder case. *J. Math. Phys.*, 45(10):3758–3769, 2004.

[13] A. Das. *(Lectures on) Quantum Field Theory.*

[14] J.C. Collins. *Renormalization: An Introduction to Renormalization, the Renormalization Group and the Operator-Product Expansion.*

[15] A.H. Chamseddine and A. Connes. The spectral action principle. *Commun. Math. Phys.*, 186:731–750, 1997.

[16] J.B. Conway. *A course in functional analysis.*

[17] M. Reed and B. Simon. *I: Functional Analysis.*

[18] G.K. Pedersen. *Analysis Now.*

[19] N. Berline, E. Getzler, and M. Vergne. *Heat Kernels and Dirac Operators.*

[20] A. Connes. *Noncommutative geometry.*

[21] A. Connes and M. Marcolli. *Noncommutative geometry, quantum fields and motives.*

[22] K. van den Dungen and W.D. van Suijlekom. Particle physics from almost commutative spacetimes. *Rev.Math.Phys.*, 24:1230004, 2012.

[23] J.M. Gracia-Bondia, J.C. Varilly, and H. Figueroa. *Elements of noncommutative geometry.*

[24] G. Landi. *An introduction to noncommutative spaces and their geometries.*

[25] P.B. Gilkey. *Invariance Theory: The Heat Equation and the Atiyah-Singer Index Theorem.* Taylor and Francis, 1994.

[26] W.D. Van Suijlekom. Perturbations and operator trace functions. *J. Funct. Anal.*, 260(8):2483 – 2496, 2011.

[27] Getzler E. and Szenes A. On the chern character of a theta-summable fredholm module. *J. Funct. Anal.*, 84(2):343 – 357, 1989.

[28] I.G. Avramidi. A covariant technique for the calculation of the one-loop effective action. *Nuclear Phys. B*, 355:712–754, 1991.

[29] S.F.J. Wilk, Y. Fujiwara, and T.A. Osborn. N-body green's functions and their semiclassical expansion. *Phys.Rev.A*, 24:2187–2202, 1981.