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Renormalization group flows of gravity and gravity-matter systems on foliated spacetimes

Master thesis

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Contents

This thesis will assume basic knowledge of both GR and QFT. Should the reader not be familiar with either of these topics please refer to [1] for an introduction to GR and [2] for an introduction to QFT.

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1 Introduction

The goal of a physical theory is to make predictions about processes in nature. Some of these prediction are intuitive for us like the prediction that an apple will fall towards the ground when dropped. Other prediction are more mysterious like the effect of gravitational lensing predicted in General Relativity (GR). Both predictions can be explained in the form of a physical theory written in mathematics. Though both are valid theories there is a large conceptual difference between them. The apple that falls from the tree was first observed by humans before it was explained in a physical theory by Newton. The order from observation to theory was the way that physics was done for centuries starting, as far as we know, with the ancient Greeks. However this changed during the 19^{th} century by the work of James Maxwell on the subject of electromagnetism. In his attempt to unify the fundamental equations of electromagnetism he decided to add a term to the already existing Ampère law on the basis of mathematics not observations. This step was crucial in the construction and consistency of the Maxwell equations and it was special because there was no observational reason to do so. Maxwell added the term purely on theoretical basis thus it was a prediction of what one should observe. This reversal of observation and prediction ties in to our second example which can be seen as a great success of this reversal. Einsteins theory of GR was mostly based on theoretical work as was its predecessor, Special Relativity (SR) Both contained predictions which at the time were considered very unlikely to be true, such as the prediction of the bending of light due to the presence of heavy objects, which we now know as gravitational lensing, was one of these predictions by GR. From a classical point of view this was impossible since gravity acts on the mass of a particle and light is massless. But due to the relation of curvature and gravity in GR, Einstein predicted that light would be influenced by gravity even though it is a massless particle. This prediction was found to be correct in 1919 by an observation preformed by Arthur Eddington. Thus once again a theoretical prediction was made before any observation of the effect, this sequence of events has become common in modern physics and has resulted in the mindset of building experiments to validate theories. The grandest of these is the Large Hadron Collider (LHC) at CERN in Geneva. For particle physics this methodology culminated, for now, in the standard model of particle physics (SM) which was completed with the discovery of the Higgs boson in 2012. However this discovery has lead to a rather peculiar situation in particle physics. There is no shortage of theoretical predictions that serve as an extension of the SM but there is a lack of experimental results to give a direction in which theoreticians should look for the next breakthrough. This breakthrough is needed since there are still a lot of question that are left unanswered by the SM such as the existence of dark matter, the existence of dark energy and the quantization of gravity. This thesis will focus on the last problem, the quantization of gravity.

In high energy physics we identify four different fundamental forces. Three of them are described in the SM: the strong interactions, the weak interactions and the electromagnetic force. The strong interactions describe the physics that is responsible for the formation of composite particles like the proton and neutron, the weak interactions gives among other things a description of β-decay and finally the electromagnetic force describing electromagnetism and thus also light. The fourth fundamental force which is not described by the SM is gravity. There

were many attempts to unify the SM and gravity but non have been successful for now. This leaves physicist in a difficult position. Do we throw away all that we learned in the construction of the SM or do we try to find an alternative way to unify the SM and gravity in the framework of QFT. The former choice is taken in theories like string theory. In this thesis we will use the theory known as Asymptotic Safety which stays much closer to the origin of QFT.
In section 2 we will introduce the idea for Asymptotic Safety and the techniques needed to do calculations in Asymptotic Safety. This is followed in section 3 about a more detailed description of the model we will be using and in section 4 the last step will be made in the calculations. In section 5 we will look at what our calculations have resulted in for both a pure gravity model and a simple matter model. Finally we will give a short summary and comment on our results in the conclusion.
It should be noted that the content of this thesis is based upon the work described in [3, 4].

To be as consistent as we can with existing literature we will use the standard convention for the fundamental constants of high energy physics

$$\hbar = 1$$
$$c = 1$$

We will also use the Einstein summation convention is for repeated indices.

$$T^\mu T_\mu = \Sigma^d_{\mu=0} T^\mu T_\mu$$

Here d is the number of spatial dimensions.
To distinguish between spatial dimensions and spactime dimensions we will use Latin letter for spatial indices, $i = (1, 2, 3)$, while we will use Greek letters for spacetime directions $\mu = (0, 1, 2, 3)$.

2 Asymptotic Safety

We stated before that there is a fundamental choice in the current search for a quantum gravity theory. This is due to the fact that gravity cannot be renormalized in the same way as all the other forces in the SM. The other forces are quantized by using perturbation theory which assumes a small coupling constant to have a convergent series of higher order corrections. When we try to use perturbation theory for gravity one finds that the corrections diverge. This is due to the negative mass dimension of the coupling constant of gravity, Newtons constant G_N. One can thus not use the same techniques that were used for the SM to try and quantize gravity.
There are other paths we can take though. One of these was put forward by Weinberg [5, 6] and relies on the existence of so called fixed points. To explain we have to realize that the coupling constants are not constant but functions of the renormalization scale k which can be identified as an energy scale. When we move along this scale we obtain a trajectory for the coupling constants as a function of k. By specifying an initial value for the coupling constant the entire trajectory becomes known. This can be done for multiple coupling constants by combining their trajectories. In these trajectories there might appear points which are known as fixed points. A fixed point in this sense is a specific point were the flow does not depend on k anymore, furthermore any trajectory near a fixed point is either attracted or repulsed for increasing k. Thus we can go to an arbitrary value for k while not having to worry about divergences when near such an attractive fixed point. A theory with an attractive fixed point is said to be Asymptotically safe. Thus the attempt to find such a fixed point for gravity to shield it from UV-divergences is called Asymptotic Safety.

2.1 Functional Renormalization Group Equation

In order to determine the fixed points in a theory we first have to calculate the trajectories that depend on k. To do this we use the Functional Renormalization Group Equation (FRGE) [7–10] which is also known as the Wetterich equation and encodes the entire energy-dependence of an action in a single equation.

$$k\partial_k \Gamma_k = \frac{1}{2}\,\mathrm{STr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} k\partial_k \mathcal{R}_k\right]. \tag{1}$$

Here Γ_k is the effective action and $\Gamma_k^{(2)}$ is the second variation of Γ_k also called the Hessian of Γ_k, k gives the renormalization group scale and $\mathcal{R}_k$ is the regulator. The FRGE implements an idea introduced by Wilson that the RG flow originates by integrating out the quantum fluctuations shell-by-shell in momentum space. To ensure that the FRGE is only determined by the fluctuations near a certain momentum k the regulator $\mathcal{R}_k$ is introduced. Since $\mathcal{R}_k$ appears both in the numerator and denominator it ensures that only momenta near k are considered if the regulator is chosen properly. For this thesis the Litim regulator will be used

$$\mathcal{R}_k \propto (k^2 - \Delta_i)\theta(k^2 - \Delta_i). \tag{2}$$

The FRGE is a functional equation that cannot be solved exactly. Thus we are forced to resort to the use of approximations and numerical methods to determine the flow of the coupling constant G_N and the cosmological constant Λ. The resulting equations that contain the information of the flow are called the β-functions. But first we will further explain the effective action since this concept is crucial in the understanding of the FRGE equation.

2.2 Effective action

The derivation that is shown here for the effective action is based upon [11]. The effective action is related to the classical action $S[\tilde{\varphi}]$ but differs in the fact that it also contains all quantum correction as we will now show. We start with the generating functional of the Green's functions $W[J]$.

$$e^{\frac{-W[J]}{\hbar}} = \int D\tilde{\varphi}\, e^{\frac{-1}{\hbar}\left(S[\tilde{\varphi}]+\int dx\, \tilde{\varphi}(x)J(x)\right)} \tag{3}$$

Here $\hbar$ is the Planck constant, which we will keep in this derivation despite stating that $\hbar = 1$ for purposes that will become clear later, and $J(x)$ is the source term. The classical action does not contain any quantum corrections in its definition, hence the name classical action. However these need to be accounted for in an action that describes a quantum theory. Therefore introduce the effective action

$$\Gamma[\phi] = W[J] - \int dx\, \phi(x)J(x) \tag{4}$$

Then (3) can be rewritten in terms of the effective action

$$e^{\frac{-1}{\hbar}\left(\Gamma[\phi]+\int dx\, \phi(x)J(x)\right)} = \int D\tilde{\varphi}\, e^{\frac{-1}{\hbar}\left(S[\tilde{\varphi}]+\int dx \tilde{\varphi}(x)J(x)\right)} \tag{5}$$

Here $\phi =< \tilde{\varphi} >$ is the mean field which is a constant background field and $J(x)$ acts as its source, thus it satisfies $J(x) = -\frac{\delta\Gamma[\phi]}{\delta\phi(x)}$. Next the field $\tilde{\varphi}$ can be split into the background field ϕ and the variation on this field φ as $\tilde{\varphi} = \phi + \varphi$. The background field is a constant as mentioned before but the variation is not a constant, therefore the integration variable $D\tilde{\varphi}$ transforms into $D\varphi$. It can also be seen that the extra term from the redefinition of $\tilde{\varphi}$ in the RHS drops out because it is not dependent on φ and thus can be placed outside of the integral. Furthermore the same term is on the LHS and can thus be cancelled.

$$e^{\frac{-1}{\hbar}\left(\Gamma[\phi]-\int dx\, \phi(x)\frac{\delta\Gamma[\phi]}{\delta\phi(x)}\right)} = \int D\varphi e^{\frac{-1}{\hbar}\left(S[\varphi+\phi]-\int dx(\varphi(x)+\phi(x))\frac{\delta\Gamma[\phi]}{\delta\phi(x)}\right)} \tag{6}$$

The functional $S[\phi+\varphi]$ can be expanded in a functional Taylor series such that S will only be a functional of the background field ϕ.

$$S[\phi+\varphi] = S[\phi] + \sum_{n=1}^{\infty}\frac{1}{n!}\int dx_1 \ldots dx_n S_n(x_1 \ldots x_n \mid \phi)\varphi(x_1)\ldots\varphi(x_n) \tag{7}$$

The classical vertex functions $S_n(x_1 \ldots x_n \mid \phi)$ are given by:

$$S_n(x_1 \ldots x_n \mid \phi) = \frac{\delta^n S[\phi]}{\delta\phi(x_1)\ldots\delta\phi(x_n)} \tag{8}$$

The expressions in (7) and (5) can be simplified by defining:

$$\begin{aligned} &\int dx_1 \ldots dx_n S_n(x_1 \ldots x_n \mid \phi)\varphi(x_1)\ldots\varphi(x_n) \equiv S^{(n)}[\phi]\varphi^n \\ &\int dx\, \varphi(x)\frac{\delta\Gamma[\phi]}{\delta\phi(x)} \equiv \varphi\Gamma^{(1)}[\phi] \end{aligned} \tag{9}$$

Using these relations in addition with the transformation $\varphi = \hbar^{-\frac{1}{2}}\varphi$, (5) can be written as:

$$e^{\frac{-1}{\hbar}\left(\Gamma[\phi]-S[\phi]\right)} = \int D\varphi\, e^{-\left(\frac{1}{2}S^{(2)}[\phi]\varphi^2+\sum_{n=3}^{\infty}\frac{\hbar^{\frac{n}{2}-1}}{n!}S^{(n)}[\phi]\varphi^n-\hbar^{-\frac{1}{2}}\varphi(\Gamma^{(1)}[\phi]-S^{(1)}[\phi])\right)} \tag{10}$$

The LHS and RHS still contain a combination of the effective action and classical action. Firstly we notice that $S^{(1)}$ vanishes since the fields ar onshell. Then by subtracting the classical action from the effective action we obtain $\bar{\Gamma}[\phi]$ which is defined as

$$e^{\frac{-1}{\hbar}\bar{\Gamma}[\phi]} = \int D\varphi\, e^{-\left(\frac{1}{2}S_2[\phi]\varphi^2+\sum_{n=3}^{\infty}\frac{\hbar^{\frac{n}{2}-1}}{n!}S_n[\phi]\varphi^n-\hbar^{-\frac{1}{2}}\varphi\bar{\Gamma}_1[\phi]\right)} \tag{11}$$

Now the exponent on the RHS only contains quantum corrections. The quantum corrections consist of a series of loop corrections and can be written as:

$$\bar{\Gamma}[\phi] = \sum_{n=1}^{n=\infty} \hbar^n \bar{\Gamma}^{(n)}[\phi] \tag{12}$$

The n denotes the order of the correction. A observation can be made about the presence of $\hbar$, from the equation it can be seen that each extra loop contributes a factor of $\hbar$. Thus the power of $\hbar$ before a term denotes with which number of loop it is associated. For example the $\hbar^2$ would only appear with terms that contain two loops. We see that the original statement about the effective action is true. It contains both the classical action and its quantum corrections by definition. This makes it suited to be used in our investigation of QG.

2.3 ADM-metric

The form of the effective action is dependent on the metric that is used in the calculation, which gives us a choice. Preferably we would like to work with a general Lorentzian metric, however this brings problems with it. Specifically that some mathematical tools we commonly use only work in Euclidean spacetimes, such as the use of heat kernels which will become important later in our calculation. In QFT one can use the Wick rotation to change the sign of the temporal part to rotate from a Lorentzian spacetime to Euclidean spacetime and vice versa. The ability to perform a Wick rotation depends on the metric and might therefore be problematic in a curved spacetime. This means using geometry to give causality is difficult to accomplish thus we look for an alternative.

Time is a rather peculiar problem in physics and invites the so called “problem of time” [12]. For us when we think about causality we come to the conclusion that it allows us to track time even without a coordinate system or in other

words it gives time a prefered direction. Thus to obtain causality we need a way to give a prefered direction even if we are in an Euclidean spacetime. One of the ways to achieve this is the use of the Arnowitt-Deser-Misner (ADM) decomposition [13,14]. This decomposition rewrites the metric in such a way that spacetime becomes foliated into spatial slices which are constant in time, thus moving through time is going from one slices to the other.
The general form of the metric in the ADM decomposition is

$$ds^2 = \gamma_{\alpha\beta}\, dx^\alpha dx^\beta = N^2 dt^2 + \sigma_{ij}\,(dx^i + N^i d\tau)(dx^j + N^j d\tau)\,. \qquad (13)$$

Here N is called the lapse function, N_i is the shift vector and σ_{ij} is the spatial metric, the geometry of the space is now completely determined by these three parameters. We can see in Fig.1 that there is indeed a distinct difference between the spatial and temporal part and thus we have given a prefered direction to the temporal part.

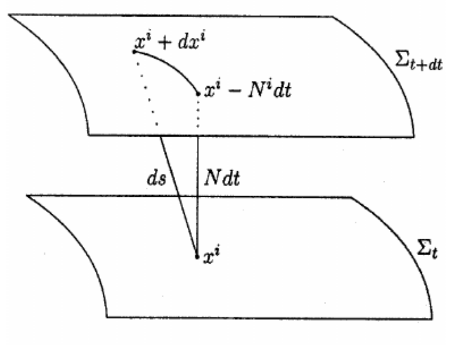

Figure 1: Graphical interpretation of the ADM decomposition where Σ_t is a spatial hypersurface.

The FRGE makes use of the background formalism thus we also want to use this in the ADM-formalism. This can simply be done by splitting N, N_i and σ_{ij} into the background and a fluctuations around this background. We choose a linear split following [14]

$$\begin{aligned} N &= \bar{N} + \hat{N}\,, \\ N_i &= \bar{N}_i + \hat{N}_i\,, \\ \sigma_{ij} &= \bar{\sigma}_{ij} + \hat{\sigma}_{ij}. \end{aligned} \qquad (14)$$

Every parameter with a bar is the background parameter and all the parameters with a hat are fluctuations. Since we want to describe gravity it is reasonable to use an action that describes GR, the Einstein-Hilbert action. It is important to note that the structure presented in this part is a causal structure but not a Lorentzian structure. It is causal since we have a distinct time direction, for a Lorentzian structure you need to have a very specific form of the metric which can be obtained by using Wick rotation. This has not been done here and therefore we cannot talk about a Lorentzian structure.
To complete our discussion of the ADM-formalism we will introduce the transformation properties of the components fields under a diffeomorphism transformation

$$\delta\gamma_{\alpha\beta} = \mathcal{L}_v \gamma_{\alpha\beta}. \qquad (15)$$

Were v can be decomposed into a temporal and spatial part

$$v^\alpha = \big(f(\tau, y), \zeta^i(\tau, y)\big)\,. \qquad (16)$$

Using (16) in (15) results in the transformation of the component fields

$$\begin{aligned}
\delta N &= \partial_\tau(fN) + \zeta^k\partial_k N - NN^i\partial_i f\,, \\
\delta N_i &= \partial_\tau(N_i f) + \zeta^k\partial_k N_i + N_k\partial_i\zeta^k + \sigma_{ki}\partial_\tau\zeta^k + N_k N^k\partial_i f + N^2\partial_i f\,, \\
\delta\sigma_{ij} &= f\partial_\tau\sigma_{ij} + \zeta^k\partial_k\sigma_{ij} + \sigma_{jk}\partial_i\zeta^k + \sigma_{ik}\partial_j\zeta^k + N_j\partial_i f + N_i\partial_j f\,,
\end{aligned} \tag{17}$$

and for completeness we also give the transformation of N^i

$$\delta N^i = \partial_\tau(N^i f) + \zeta^j\partial_j N^i - N^j\partial_j\zeta^i + \partial_\tau\zeta^i - N^iN^j\partial_j f + N^2\sigma^{ij}\partial_j f\,. \tag{18}$$

2.4 Einstein-Hilbert action

We can now introduce the approximation of the effective average action that we will be using, which will consist from several parts.Firstly there will of course be the gravitational part of the effective action given by the Einstein-Hilbert (EH) action. Secondly we will include non-interacting matter to study the influence of matter on the β-functions [15, 16].

The effective action for the matter sector is given by a combination of contributions from the possible forms of matter. For this thesis we will only consider non-interacting scalars, Dirac fermions and vector fields. This means that the matter action is

$$S^{\text{matter}} = S^{\text{scalar}} + S^{\text{vector}} + S^{\text{fermion}} \tag{19}$$

The separate parts of the matter actions are given by

$$\begin{aligned}
S^{\text{scalar}} =& \frac{1}{2}\sum_{i=1}^{N_S}\int d\tau d^dx N\sqrt{\sigma}\left[\,\phi^i\,\Delta_0\,\phi^i\right]\,, \\
S^{\text{vector}} =& \frac{1}{4}\sum_{i=1}^{N_V}\int d\tau d^dx N\sqrt{\sigma}\left[\,g^{\mu\nu}g^{\alpha\beta}F^i_{\mu\alpha}F^i_{\nu\beta}\right] \\
&+\frac{1}{2\xi}\sum_{i=1}^{N_V}\int d\tau d^dx \bar{N}\sqrt{\bar{\sigma}}\left[\,\bar{g}^{\mu\nu}\bar{D}_\mu A^i_\nu\right]^2 \\
&+\sum_{i=1}^{N_V}\int d\tau d^dx \bar{N}\sqrt{\bar{\sigma}}\left[\,\bar{C}^i\,\Delta_0\,C^i\,\right]\,, \\
S^{\text{fermion}} =& \, i\sum_{i=1}^{N_D}\int d\tau d^dx N\sqrt{\sigma}\left[\bar{\psi}^i\,\nabla\!\!\!/\,\psi^i\right]\,.
\end{aligned} \tag{20}$$

The index i functions as a summation index over the different matter types and we adopt $\xi = 1$ which is the standard Feynman gauge. Our approach for the scalar and vector sector is based on work in [15, 16] and [17–20] while we follow [17, 21] for our treatment of the Dirac fermions.

This is all we need to specify for the matter action. Thus we will now start to consider the gravitational action which we stated before is the EH action. In terms of the ADM-formalism it is given by

$$S^{EH} = \frac{1}{16\pi G}\int d\tau d^dy\, N\sqrt{\sigma}\left[K_{ij}\,\mathcal{G}^{ij,kl}\,K_{kl} - {}^{(d)}R + 2\Lambda\right]\,. \tag{21}$$

In this action K_{ij} is the extrinsic curvature while $K = K_{ij}\sigma^{ij}$, ${}^{(d)}R$ is the intrinsic curvature, σ the determinant of the spatial metric, Λ the cosmological constant and $\mathcal{G}^{ij,kl}$ is the Wheeler-de Witt metric given by

$$\mathcal{G}^{ij,kl} = \sigma^{ik}\sigma^{jl} - \lambda\sigma^{ij}\sigma^{kl}. \tag{22}$$

Here λ is a parameter that can be used to obtain different forms of the action, for EH $\lambda = 1$. The definition of K_{ij} is given by

$$K_{ij} = \frac{1}{2N}(\partial_\tau\sigma_{ij} - D_iN_j - D_jN_i) \tag{23}$$

We introduced the background formalism for the ADM-metric we can also use these definition to define $\bar{K}_{ij}$ and $\bar{K}$ as

$$\bar{K}_{ij} = \frac{1}{2\bar{N}}(\partial_\tau\bar{\sigma}_{ij} - \bar{D}_i\bar{N}_j - \bar{D}_j\bar{N}_i), \qquad \bar{K} = \bar{K}_{ij}\bar{\sigma}^{ij}. \tag{24}$$

We see that the entire EH action is now defined by the parameters from the ADM-formalism, Λ, G and ${}^{(d)}R$. This leaves the final ingredient before calculating the Hessian of the effective average action $\Gamma^{(2)}$, the background. The ADM-decomposition has a lot of similarities with cosmological perturbation theory, therefore we will also use a cosmological background. To simplify the calculations we would also like a background in which either $\bar{K}_{ij} = 0$ or ${}^{(d)}\bar{R} = 0$ which limits the number of choices we have some more. For this thesis the choice will be to use a Friedman-Robertson-Walker background (FRW). This metric is a exact solution of the field equations generated by GR and describes a homogeneous, isotropic expanding universe. It is often used in cosmological calculations because of its simplicity and relative accuracy for cosmological models. Using this as the background in the ADM decomposition gives the following values for the background.

$$\bar{N} = 1, \quad \bar{N}_i = 0, \quad \bar{\sigma}_{ij} = a(\tau)^2\delta_{ij}, \quad {}^{(d)}\bar{R} = 0. \tag{25}$$

3 Flow Projection

Having defined a background and the effective action we can begin to calculate the Hessian $\Gamma_k^{(2)}$ from (21). In order to simplify the calculation it is useful to split (21) in parts.

$$\begin{aligned}\Gamma_k^{grav} &= \int d\tau d^d y N\sqrt{\sigma}(K_{ij}K^{ij} - K^2 - {}^{(d)}R + 2\Lambda_k), \\ &= \quad I_1 - I_2 - I_3 + 2\Lambda_k I_4,\end{aligned} \tag{26}$$

where

$$\begin{aligned}I_1 &= \int d\tau d^d y N\sqrt{\sigma}K_{ij}K^{ij}, \qquad I_2 = \int d\tau d^d y N\sqrt{\sigma}K^2, \\ I_3 &= \int d\tau d^d y N\sqrt{\sigma}^{(d)}R, \qquad I_4 = \int d\tau d^d y N.\sqrt{\sigma}\end{aligned} \tag{27}$$

The Hessian is given by

$$\delta^2\Gamma_k = \delta^2 I_1 - \delta^2 I_2 - \delta^2 I_3 + 2\Lambda_k \delta^2 I_4. \tag{28}$$

Now we want to calculate all these components separately and then combine them again to obtain the full Hessian. Let us first define a few shortened notations

$$\int_x = \int d\tau d^d y\sqrt{\bar{\sigma}}, \quad \hat{\sigma} = \bar{\sigma}^{ij}\hat{\sigma}_{ij}. \tag{29}$$

These separate parts introduce the spatial Laplacian which is given by $\Delta = -\partial_i\partial^i$. The Hessians for the individual parts are

$$\begin{aligned}\delta^2 I_1 = \int_x \Big[& 2(\delta K_{ij})\bar{\sigma}^{ik}\bar{\sigma}^{jl}(\delta K_{kl}) + \tfrac{2}{d}\,\bar{K}\,\bar{\sigma}^{ij}\left(\delta^2 K_{ij} + (\delta K_{ij})(2\hat{N} + \hat{\sigma})\right) \\ & - \tfrac{8}{d}\,\bar{K}\,\hat{\sigma}^{ij}\,(\delta K_{ij}) + \tfrac{1}{d}\,\bar{K}^2\left(\tfrac{d-4}{d}\hat{N}\hat{\sigma} + \tfrac{d-8}{4d}\hat{\sigma}^2 - \tfrac{d-12}{2d}\hat{\sigma}_{ij}\hat{\sigma}^{ij}\right)\Big], \\ \delta^2 I_2 = \int_x \Big[& 2(\bar{\sigma}^{ij}\,\delta K_{ij})^2 + 2\bar{K}\,\bar{\sigma}^{ij}\left(\delta^2 K_{ij} + (\delta K_{ij})(2\hat{N} + \hat{\sigma})\right) \\ & - 4\bar{K}\hat{\sigma}^{ij}\,(\delta K_{ij}) + \bar{K}^2\left(\tfrac{d-4}{d}\hat{N}\hat{\sigma} + \tfrac{d^2-8d+8}{4d^2}\hat{\sigma}^2 - \tfrac{d-8}{2d}\hat{\sigma}_{ij}\hat{\sigma}^{ij}\right) \\ & - \tfrac{4}{d}\bar{K}\hat{\sigma}\bar{\sigma}^{ij}\delta K_{ij}\Big], \\ \delta^2 I_3 = \int_x \Big[& \left(2\hat{N} + \hat{\sigma}\right)\left(\partial_i\partial_j\hat{\sigma}^{ij} + \Delta\hat{\sigma}\right) - \tfrac{1}{2}\hat{\sigma}_{ij}\Delta\hat{\sigma}^{ij} - \tfrac{1}{2}\hat{\sigma}\Delta\hat{\sigma} \\ & + \left(\partial_i\hat{\sigma}^{ik}\right)\left(\partial_j\hat{\sigma}^j{}_k\right)\Big], \\ \delta^2 I_4 = \int_x \Big[& \hat{N}\hat{\sigma} + \tfrac{1}{4}\hat{\sigma}^2 - \tfrac{1}{2}\hat{\sigma}^{ij}\hat{\sigma}_{ij}\Big].\end{aligned} \tag{30}$$

We also note the following definitions in FRW

$$\begin{aligned}\delta K_{ij} &= -\hat{N}\bar{K}_{ij} + \frac{1}{2}(\partial_\tau\hat{\sigma}_{ij} - \partial_i\hat{N}_j - \partial_j\hat{N}_i) \\ \delta^2 K_{ij} &= \hat{N}^2 - \hat{N}(\partial_\tau\hat{\sigma}_{ij} - \partial_i\hat{N}_j - \partial_j\hat{N}_i + \hat{N}^k(\partial_i\hat{\sigma}_{jk} + \partial_j\hat{\sigma}_{ik} - \partial_k\hat{\sigma}_{ij})\end{aligned} \tag{31}$$

To simplify (30) it is useful to consider the contraction of the variations of the extrinsic curvatures and the background spatial metric.

$$\begin{aligned}\bar{\sigma}^{ij}(\delta K_{ij}) =& -\hat{N}\bar{K} + \frac{1}{2}\bar{\sigma}^{ij}(\partial_\tau \hat{\sigma}_{ij}) - \partial^i \hat{N}_i \\ \bar{\sigma}^{ij}(\delta^2 K_{ij}) =& \hat{N}^2\bar{K} - \hat{N}\bar{\sigma}^{ij}(\partial_\tau\hat{\sigma}_{ij}) + 2\hat{N}\partial^i\hat{N}_i + \hat{N}^k(2\partial^i\hat{\sigma}_{ik} - \partial_k\hat{\sigma})\end{aligned} \tag{32}$$

The expressions in (30) are still expressed in terms of δK_{ij} and $\delta^2 K_{ij}$ which is not useful for calculations therefore we want to rewrite these. Consider the following field decompositions of the fluctuations in $\hat{N}_i$and $\hat{\sigma}_{ij}$

$$\begin{aligned}\hat{N}_i =& u_i + \partial_i \frac{1}{\sqrt{\Delta}} B \\ ,\hat{\sigma}_{ij} =& h_{ij} - (\bar{\sigma}_{ij} + \partial_i\partial_j \frac{1}{\Delta})\psi + \partial_i\partial_j\frac{1}{\Delta}E + \partial_i \frac{1}{\sqrt{\Delta}} v_j + \partial_j\frac{1}{\sqrt{\Delta}} v_i,\end{aligned} \tag{33}$$

Here all vector fields are transverse and the tensor field is transverse traceless implying

$$\begin{aligned}\partial^i u_i =& 0, \qquad \partial^i v_i = 0 \\ \partial^i h_{ij} =& 0, \qquad \bar{\sigma}^{ij} h_{ij} = 0\end{aligned} \tag{34}$$

To test our setup in a simplified model we will first consider a flat background ($\bar{K} = 0$). Combining this background with (32) and expressing (30) in terms of (33) gives,

$$\begin{aligned}\delta^2 I_1 =& \int_x \Big[-\frac{1}{2}h^{ij}\partial_\tau^2 h_{ij} - \frac{d-1}{2}\psi\partial_\tau^2\psi - \frac{1}{2}E\partial_\tau^2 E - v^i\partial_\tau^2 v_i \\ & - 2B\sqrt{\Delta}\partial_\tau E - 2u^i\sqrt{\Delta}\partial_\tau v_i + 2B\Delta B + u^i\Delta u_i\Big], \\ \delta^2 I_2 =& \int_x \Big[-\frac{1}{2}((d-1)\psi + E)\partial_\tau^2((d-1)\psi + E) \\ & - 2B\sqrt{\Delta}\partial_\tau((d-1)\psi + E) + 2B\Delta B\Big], \\ \delta^2 I_3 =& \int_x \left[-\frac{1}{2}h^{ij}\Delta h^{ij} - 2(d-1)N\Delta\psi + \frac{(d-1)(d-2)}{2}\psi\Delta\psi\right], \\ \delta^2 I_4 =& \int_x \Big[-\frac{1}{2}h^{ij}h_{ij} - v^i v_i + \frac{(d-1)(d-3)}{4}\psi\psi - \frac{1}{4}EE \\ & + \frac{(d-1)}{2}\psi E - \hat{N}((d-1)\psi + E)\Big].\end{aligned} \tag{35}$$

Thus the Hessian in a flat background spacetime is,

$$\begin{aligned}16\pi G_k \delta^2\Gamma_k^{grav} =& \int_x \Big[\frac{1}{2}h^{ij}(-\partial_\tau^2 + \Delta - 2\Lambda_k)h_{ij} + v^i(-\partial_\tau^2 - 2\Lambda_k)v_i \\ & + E(-\Lambda_k)E - (d-1)\psi((d-2)(-\partial_\tau^2 + \Delta) - (d-3)\Lambda_k)\psi \\ & - \frac{(d-1)}{2}\psi(-\partial_\tau^2 - 2\Lambda_k)E + u^i(2\Delta)u_i - 2u^i\partial_\tau\sqrt{\Delta}v_i \\ & + 2(d-1)B\sqrt{\Delta}\partial_\tau\psi + 2(d-2)\hat{N}(\Delta - \Lambda_k)\psi \\ & + \hat{N}(-2\Lambda_k)E\Big]\end{aligned} \tag{36}$$

Let us introduce the wave operator $\Box = -\partial_\tau^2 + \Delta$. Any field with this term has a proper relativistic dispersion relation which is one of the demands for a proper theory of gravity. As can be seen in (36) most of the fields do not have the correct relativistic dispersion. This can be fixed via a gauge fixing.

3.1 Gauge fixing for the flat background

An important concept in modern high energy physics are symmetries. A symmetry is a change in a parameter which does not change the physical properties of the system. A simple example of this is a translation symmetry for an isolated system in an empty space. Since there is no external influence on the system it does not change any of its physics. Translation symmetry is a symmetry that one can easily understand. But there are symmetries which are less clear and can only be seen from the mathematics of the models, a prime example of this are gauge symmetries. Though gauge symmetries are often used in particle physics they can also be rather troublesome due to the fact that they can obscure the physics of the system. This can be illustrated by looking at the phase space of a theory. When determining the flow of the theory in phase space, one can come across lines where the parameters change but the theory does not, these lines are known as gauge orbits. One can then try to specify a gauge fixing to choose a specific point in these gauge orbits, this will result in a smaller phase space called the reduced phase space in which the gauge orbit has disappeared. Though this feels like the removal of information it can also be used to reveal more information. Considering (35) not all fields in this equation have the correct relativistic propagator but we can try to use a gauge fixing to find a form of the theory in reduced phase space where all fields do have the correct relation. We can do this because choosing a gauge fixing only shifts us on the gauge orbit where the physics remain the same as stated before. Thus we will try to find a gauge fixing which will give nice dispersion relations. Our gauge fixing will have the following form which is based on the work in [22]

$$\Gamma_k^{gf} = \frac{1}{32\pi G_k} \int_x \Big[F_i \bar{\sigma}^{ij} F_j + F^2 \Big]. \tag{37}$$

Here the functionals F_i and F are linear in the fluctuation fields and their most general form in a flat spacetime is

$$\begin{aligned} F =& c_1 \partial_\tau \hat{N} + c_2 \partial^i \hat{N}_i + c_3 \partial_\tau \hat{\sigma}, \\ F_i =& c_4 \partial_\tau \hat{N}_i + c_5 \partial_i \hat{N} + c_6 \partial_i \hat{\sigma} + c_7 \partial^j \hat{\sigma}_{ij}. \end{aligned} \tag{38}$$

The coefficients c_i can be used to obtain the missing parts of the wave operator and to reduce the number of cross terms between the fluctuation fields. Again for simplicity it is easier to calculate both terms separately and then combine them to obtain the complete gauge fixing

$$\delta^2 I_5 = \int_x F^2, \qquad \delta^2 I_6 = \int_x F_i \bar{\sigma}^{ij} F_j. \tag{39}$$

Here the δ^2 does not indicate that the variation needs to be taken but that the gauge fixing terms are quadratic in the fluctuation fields by construction. Thus we can write these two terms in terms of the fluctuation fields to obtain

$$\begin{aligned}\delta^2 I_5 = \int_x \Big[& c_2^2 B\Delta B - c_1^2 \hat{N}\Delta\hat{N} - c_3^2((d-1)\psi + E)\partial_\tau^2((d-1)\psi + E) \\ & - 2c_1c_2 B\sqrt{\Delta}\partial_\tau\hat{N} + 2c_1c_3\hat{N}\partial_\tau^2((d-1)\psi + E) \\ & + 2c_2c_3 B\sqrt{\Delta}\partial_\tau((d-1)\psi + E)\Big]\end{aligned} \tag{40}$$

and

$$\begin{aligned}\delta^2 I_6 = \int_x \Big[& c_5^2 \hat{N}\Delta\hat{N} - c_4^2 u^i\partial_\tau^2 u_i - c_4^2 B\partial_\tau^2 B + 2c_4c_5\hat{N}\partial_\tau\sqrt{\Delta}B \\ & - 2c_4c_6((d-1)\psi + E)\partial_\tau\sqrt{\Delta}B - 2c_4c_7 E\partial_\tau\sqrt{\Delta}B \\ & - 2c_4c_7 v^i\sqrt{\Delta}u_i - 2c_5c_6\hat{N}\Delta((d-1)\psi + E) - 2c_5c_7\hat{N}\Delta E \\ & + c_6^2((d-1)\psi + E)\Delta((d-1)\psi + E) + 2c_6c_7((d-1)\psi + E)\Delta E \\ & + c_7^2(E\Delta E + v^i\Delta v_i)\Big]\end{aligned} \tag{41}$$

We can combine the sum of the Hessian which was determined in (36) with (40) and (41).

$$\begin{aligned} & 32\pi G_k(\frac{1}{2}\delta^2\Gamma^{grav} + \Gamma_k^{gf}) = \\ & \int_x \Big[- \hat{N}(c_1^2\partial_\tau^2 - c_5^2\Delta)\hat{N} - B(c_4^2\partial_\tau^2 - c_2^2\Delta)B - 2B\sqrt{\Delta}(c_1c_2 + c_4c_5)\partial_\tau\hat{N} \\ & \quad + 2(d-1)\hat{N}(c_1c_3\partial_\tau^2 + (1 - c_5c_6)\Delta - \Lambda_k)\psi \\ & \quad + 2\hat{N}(c_1c_3\partial_\tau^2 - c_5(c_6 + c_7)\Delta - \Lambda_k)E \\ & \quad + 2B\sqrt{\Delta}(c_2c_3 + c_4(c_6 + c_7))\partial_\tau E \\ & \quad + 2(d-1)B\sqrt{\Delta}(1 + c_2c_3 + c_4c_6)\partial_\tau\psi \\ & \quad - (d-1)\psi((d-1)(c_3^2\partial_\tau^2 - c_6^2\Delta) + \frac{d-2}{2}(-\partial_\tau^2 + \Delta) - \frac{d-3}{2}\Lambda_k)\psi \\ & \quad + E((c_6 + c_7)^2\Delta - c_3\partial_\tau^2 - \frac{1}{2}\Lambda_k)E \\ & \quad + (d-1)\psi(2c_6(c_6 + c_7)\Delta(1 - 2c_3^2)\partial_\tau^2 + \Lambda_k)E - u^i(c_4^2\partial_\tau^2 - \Delta)u_i \\ & \quad + v^i(-\partial_\tau^2 + c_7^2\Delta - 2\Lambda_k)v_i - 2u^i\sqrt{\Delta}(1 - c_4c_7)\partial_\tau v_i \\ & \quad + \frac{1}{2}h^{ij}(-\partial_\tau^2 + \Delta - 2\Lambda_k)h_{ij}\Big]\end{aligned} \tag{42}$$

The coefficients c_i are then fixed as followed. Due to the normalization that we choose factors of $\sqrt{\Delta}$ appear in (42), these are not desirable since the square root of an operator is difficult to work with. Thus we will use the freedom provided by the gauge fixing to remove these terms. This fixes c_4,c_6,c_7 in terms of the remaining four coefficients.

$$\begin{aligned} c_4 &= -\frac{c_1c_2}{c_5}, & c_6 &= \frac{1 + c_2c_3}{c_1c_2}c_5, \\ c_7 &= -\frac{c_5}{c_1c_2}, & c_6 + c_7 &= \frac{c_2c_3}{c_1c_2}c_5. \end{aligned} \tag{43}$$

The $c_6 + c_7$ relation is given because it appears often in (42) and is useful to consider. Before we can complete the Hessian in a flat background spacetime we need to determine the ghost sector of the action. The ghost are the so-called Faddeev-Popov ghosts that are necessary to go from the path integral to a proper quantum field theory. The problem that is solved by the ghosts also deals with the unphysical states generated by the gauge symmetries in the action. By introducing the ghost fields these symmetries are broken and thus a consistent QFT is obtained. We do this by using the Faddeev-Popov method. We will consider two types of ghost firstly the scalar ghosts and secondly the vector ghosts. There are no tensor ghosts since the tensor fields are not altered by the gauge fixing as can be seen in (42). The scalar ghost sector is given by

$$\begin{aligned} S^{sgh} &= \int_x \bar{C} \frac{\delta F}{\delta \chi^i} \delta_{c,b_i}(\chi^i) \\ \chi^i &= \{\hat{N}, \hat{N}^i, \hat{\sigma}^{ij}\} \end{aligned} \tag{44}$$

Here δ_{c,b_i} are given by the (17) with the parameters f and ζ_i replaced by the scalar ghost field c and the vector ghost field b_i respectively. Doing the sum given in (44) gives the following result

$$S^{sgh} = \int_x \Big[\bar{C}(c_1 \partial_\tau^2 - c_2 \Delta) C + \bar{C}(c_2 + 2c_3) \partial_i \partial_\tau b^i \Big] \tag{45}$$

The last part of the ghost sector is now given by the vector part which is

$$\begin{aligned} S^{vgh} &= \int_x \bar{b}_i \frac{\delta F^i}{\delta \chi^j} \delta_{c,b_j}(\chi^j) \\ S^{vgh} &= \int_x \Big[\bar{b}_i (c_4 \partial_\tau^2 \delta^i_j - c_7 \Delta \delta^i_j + (2c_6 + c_7) \bar{D}^i \bar{D}_j) b_j \\ &\quad + \bar{b}_i (c_5 \bar{D}^i \partial_\tau + c_4 \partial_\tau \bar{D}^i) C \Big] \end{aligned} \tag{46}$$

Both (45) and (46) have cross terms which we do not want. The coefficients from the gauge fixing can be used to remove the cross terms, giving us two new constraints for the coefficients

$$\begin{aligned} c_2 &= -2c_3 \\ c_5 &= -c_4 \end{aligned} \tag{47}$$

Combing this with the constraints in (43) we obtain the new relation.

$$c_5 = -\frac{c_1 c_2}{c_4} \to c_5^2 = -2c_1 c_3 \tag{48}$$

At this stage we have two free coefficients left c_1,c_3. Before choosing the values of the coefficients let us take another look at (42). If we look at the $\hat{N}^2$ term in (42) we seen that this contain two coefficients c_1 and c_5. We want to obtain the wave operator for the $\hat{N}^2$ term and thus both coefficients need to be the same. Furtermore we see that all coefficients come in pairs, it becomes apparent that the coefficients can be split in two families on their dependence. Firstly c_1,c_2 and c_3 always come together and thus any minus sign between them is irrelevant and c_4,c_5,c_6 and c_7 also come in pairs thus the same property holds for them. We also notice that the two coefficients we saw in the $\hat{N}^2$ are each belong to a different family thus this combination determines all coefficients up to an arbitrary minus sign. For simplicity we choose $c_1 = \epsilon_1$ and $c_5 = -\epsilon_2$ where $\epsilon_1 = \pm 1$ and $\epsilon_2 = \pm 1$. This determines the coefficients to be

$$\begin{aligned} &c_1 = \epsilon_1, \quad c_2 = \epsilon_1, \quad c_3 = -\frac{1}{2}\epsilon_1 \\ &c_4 = \epsilon_2, \quad c_5 = -\epsilon_2, \quad c_6 = -\frac{1}{2}\epsilon_2, \quad c_7 = \epsilon_2 \end{aligned} \tag{49}$$

This gauge-choice is uniquely fixed up to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry which is irrelevant for the physics of the model. This unique choice is guaranteed by the conditions that we specified for the gauge-fixed action [3]. Then (42) becomes

$$\begin{aligned} &32\pi G_k(\frac{1}{2}\delta^2\Gamma^{grav} + \Gamma_k^{gf}) = \\ &\int_x \Big[\hat{N}\Box\hat{N} + B\Box B + (d-1)\hat{N}(\Box - 2\Lambda)\psi + \hat{N}(\Box - 2\Lambda_k)E \\ &\quad - \frac{(d-1)(d-3)}{4}\psi(\Box - 2\Lambda_k)\psi + \frac{1}{4}E(\Box - 2\Lambda_k)E \\ &\quad - \frac{1}{2}(d-1)\psi(\Box - 2\Lambda_k)E + u^i\Box u_i + v^i(\Box - 2\Lambda_k)v_i \\ &\quad + \frac{1}{2}h^{ij}(\Box - 2\Lambda_k)h_{ij}\Big] \end{aligned} \tag{50}$$

Furthermore the combined ghost sector becomes

$$S^{gh} = S^{sgh} + S^{vgh} = \int_x \Big[-\epsilon_1\bar{C}\Box C - \epsilon_2\bar{b}_i\Box b^i\Big] \tag{51}$$

We can see that there is a dependence on ϵ_1 and ϵ_2 in the ghost sector. This is no problem however since the ghost sector does not contain any relevant physics it is purely used to make sure that the complete effective action has the correct number of physical degrees of freedom. When we look at the complete gauge-fixed action in (50) we notice that all of the fluctuation fields now have a relativistic dispersion relation. This is a great feature since this means that we can calculate traces with the standard methods.
We can also notice that not all terms in (50) are diagonal, ψ,E and N still have some cross terms left. We would rather want all fields to be diagonal since this means that there are no interactions and thus the calculations that follow will be simplified. We can try to obtain the diagonal form by rotating to a new basis in field space which comes down to diagonalizing a 3-dimensional matrix $\mathbb{M}$. It turns out that it is impossible to obtain a diagonal basis for a non-zero cosmological constant, however for $\Lambda_k = 0$ it is possible to find a diagonal basis. The new fields in this "near-diagonal" basis become:

$$\begin{aligned} \chi_1 &= \hat{N} + \frac{1}{2}E\,, \\ \chi_2 &= -\tfrac{2}{C_d}\left(\hat{N} - \frac{1}{2}E - \tfrac{1}{4}(d-3+C_d)\psi\right)\,, \\ \chi_3 &= +\tfrac{2}{C_d}\left(\hat{N} - \frac{1}{2}E - \tfrac{1}{4}(d-3-C_d)\psi\right)\,. \end{aligned} \tag{52}$$

Where $C_d = \sqrt{d^2 - 6d + 25}$ which is positive definite for $d \geq 0$. Thus we can write the entire χ sector as

$$32\pi G_k \left(\frac{1}{2} \delta^2 \Gamma^{\text{grav}} + \Gamma_k^{\text{gf}} \right) \Big|_{\chi\chi} = \int_x \left\{ \chi_i \, \mathbb{M} \, \chi_j \right\}. \tag{53}$$

Where the $\mathbb{M}$ is given by

$$\begin{aligned}
\mathbb{M}_{11} =& \Delta - \frac{3}{2}\Lambda_k, \\
\mathbb{M}_{22} =& -\frac{d-1}{4} C_d \Delta + \frac{d^2 - 6d + 17 + (7d-5)C_d}{16} \Lambda_k, \\
\mathbb{M}_{33} =& \frac{d-1}{4} C_d \Delta + \frac{d^2 - 6d + 17 + (7d-5)C_d}{16} \Lambda_k, \\
\mathbb{M}_{12} =& \frac{d - 3 - C_d}{8} \Lambda_k, \\
\mathbb{M}_{13} =& \frac{d - 3 - C_d}{8} \Lambda_k, \\
\mathbb{M}_{23} =& -\frac{1}{2}\Lambda_k.
\end{aligned} \tag{54}$$

This completes the complete effective action for $\bar{K} = 0$ and no matters fields. The result is

$$\begin{aligned}
&32\pi G_k (\frac{1}{2}\delta^2 \Gamma^{grav} + \Gamma_k^{gf}) = \\
&\int_x \Big[B \Box B + u^i \Box u_i + v^i (\Box - 2\Lambda_k) v_i \\
&\quad + \frac{1}{2} h^{ij} (\Box - 2\Lambda_k) h_{ij} + \chi_i \mathbb{M}^{ij} \chi_j \Big].
\end{aligned} \tag{55}$$

3.2 Flow projection of EH on a curved background

We are not interested in the flat spacetime background since it is not appealing to investigate gravity on such a background. We would like to return to the scenario where $\bar{K} \neq 0$, this requires a redefinition of the Laplacian to correct for a curved spacetime since the Laplacian is defined as $-g^{\mu\nu} D_\mu D_\nu \psi = \Box \psi$. We have three types of fields in our calculations namely scalars, vectors and tensor fields. In a flat spacetime all of these fields have the same Laplacian however in a curved spacetime this is no longer the case due to the presence of Christoffel symbols in the covariant derivative, D_μ. We want to determine the following identities

$$-\bar{g}^{\mu\nu} \bar{D}_\mu \bar{D}_\nu \phi, \quad -\bar{g}^{\mu\nu} \bar{D}_\mu \bar{D}_\nu v_\lambda, \quad -\bar{g}^{\mu\nu} \bar{D}_\mu \bar{D}_\nu h_{\lambda\rho}. \tag{56}$$

We note that $\bar{D}_\mu$ is the covariant derivative with respect to the background. The effect of $\bar{D}_\mu$ on the scalar, vector and tensor fields is given by

$$\begin{aligned}
\bar{D}_\mu \phi =& \partial_\mu \phi \\
\bar{D}_\mu v_\nu =& \partial_\mu v_\nu - \Gamma^\lambda_{\mu\nu} v_\lambda \\
\bar{D}_\mu h_{\nu\lambda} =& \partial_\mu h_{\nu\lambda} - \Gamma^\rho_{\mu\nu} v_{\rho\lambda} - \Gamma^\rho_{\mu\lambda} v_{\nu\rho}
\end{aligned} \tag{57}$$

The Christoffel symbol is defined as

$$\Gamma^{\lambda}_{\mu\nu} \equiv \frac{1}{2} g^{\lambda\rho}(g_{\mu\rho,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}). \tag{58}$$

For our FRW background the components of the Christoffel symbol reduce to

$$\begin{aligned}\Gamma^{0}_{00} =& \Gamma^{0}_{0j} = \Gamma^{j}_{00} = \Gamma^{j}_{ik} = 0\\ \Gamma^{0}_{kj} =& -\frac{1}{d}\bar{K}\bar{\sigma}_{kj}\\ \Gamma^{j}_{0k} =& \frac{1}{d}\bar{K}\delta^{j}_{k}\end{aligned} \tag{59}$$

In order to calculate (56) we will first "lift" the tangent vectors from the spatial slices to the full D-dimensional spacetime. Therefore consider the projectors t^μ and e^μ_i which are defined as

$$t^{\mu} = \big(1\,,\,\vec{0}\big)\,, \qquad e_i{}^{\mu} = \big(\vec{0}\,,\,\delta^{j}_{i}\big)\,. \tag{60}$$

With this definition t^μ is always perpendicular to the spatial hypersurface. This projection allows us to "lift" the covariant derivative D_i which is initially well defined on the spatial hypersurface to the object D_μ which is the covariant derivative defined on the entirety of spacetime. Doing this gives a correction term in the Laplacian which indicates the causal setting of this model. We know of course that in the end we need to restrict the calculation to the spatial hypersurface. Doing this leads to the following calculation for the Laplacian that is acting on a scalar field.

$$\begin{aligned}\Delta_0\phi =& - g^{\mu\nu}\bar{D}_\mu\bar{D}_\nu\phi = -g^{\mu\nu}\bar{D}_\mu(\partial_\nu\phi)\\ =& - g^{\mu\nu}(\partial_\mu\partial_\nu\phi - \Gamma^{\lambda}_{\mu\nu}\partial_\lambda\phi)\\ =& \Box\phi + g^{\mu\nu}(\Gamma^{0}_{\mu\nu}\partial_\tau + \Gamma^{i}_{\mu\nu}\partial_i)\phi\\ =& \Box\phi + (\bar{\sigma}^{kj}\Gamma^{0}_{kj}\partial_\tau + g^{0j}\Gamma^{i}_{0j}\partial_i)\phi\\ =& \Box\phi + (-\bar{\sigma}^{kj}\bar{\sigma}_{kj}\frac{1}{d}\bar{K}\partial_\tau)\phi\\ =& \Big(\Box - \bar{K}\partial_\tau\Big)\phi\end{aligned} \tag{61}$$

The calculation for the Laplacian acting on a vector or tensor field is identical to the one for the scalar, but due to the appearance of objects with extra indices contains addiational terms in the Laplacian

$$\begin{aligned}\Delta_0\phi =& \Big(\Box - \bar{K}\partial_\tau\Big)\phi\\ \Delta_1 v_\mu|_{\mu=i} =& \Big(\Box - \frac{d-2}{d}\bar{K}\partial_\tau + \frac{1}{d}\dot{\bar{K}} + \frac{1}{d}\bar{K}^2\Big)v_\mu|_{\mu=i}\\ \Delta_2 h_{\mu\nu}|_{\mu=i,\nu=j} =& \Big(\Box - \frac{d-4}{d}\bar{K}\partial_\tau + \frac{2}{d}\dot{\bar{K}} + \frac{2(d-1)}{d^2}\bar{K}^2\Big)h_{\mu\nu}|_{\mu=i,\nu=j}.\end{aligned} \tag{62}$$

Now we repeat the same steps as for the flat spacetime but also try to obtain the appropriate Δ_i for each field. This will give extra terms in the Hessian of the form $\bar{K}\partial_\tau$,$\dot{\bar{K}}$ and $\bar{K}^2$. This is not a problem in principle since these do not impede the next step in the calculation they only make them slightly more

difficult.
We now return to the original form of the effective action given in (35). Again we will resort to some shorthand notations to make the resulting expressions more readable. Consider the kinetic part for I_1 and I_2 which we will call $\mathbb{K}_1$ and $\mathbb{K}_2$ respectively, these terms are given by

$$\mathbb{K}_1 \equiv 2 \int_x (\delta K_{ij})\, \bar{\sigma}^{ik} \bar{\sigma}^{jl} \,(\delta K_{kl})\,, \qquad \mathbb{K}_2 \equiv 2 \int_x (\bar{\sigma}^{ij}\, \delta K_{ij})^2\,. \tag{63}$$

These can be calculated by using the identities given in (32)

$$\begin{aligned}
\mathbb{K}_1 = \int_x \Big[&-\frac{1}{2}\, h^{ij} \left(\partial_\tau^2 + \tfrac{d-4}{d} \bar{K} \partial_\tau\right) h_{ij} - \tfrac{d-1}{2}\, \psi \left(\partial_\tau + \tfrac{d-2}{d} \bar{K}\right) \left(\partial_\tau + \tfrac{2}{d} \bar{K}\right) \psi \\
&- \frac{1}{2} E \left(\partial_\tau + \tfrac{d-2}{d} \bar{K}\right) \left(\partial_\tau + \tfrac{2}{d} \bar{K}\right) E + u_i \Delta u^i \\
&- 2\, B \sqrt{\Delta}\, \left(\partial_\tau + \tfrac{2}{d} \bar{K}\right) E - 2\, u^i \left(\partial_\tau + \tfrac{2}{d} \bar{K}\right) \sqrt{\Delta}\, v_i + 2 B \Delta B \\
&- \tfrac{4}{d} \bar{K} \hat{N} \sqrt{\Delta}\, B + \tfrac{2}{d} \bar{K} \hat{N}\, \left(\partial_\tau + \tfrac{2}{d} \bar{K}\right) \big((d-1)\psi + E\big) \\
&+ \tfrac{2}{d}\, \bar{K}^2\, \hat{N}^2 - v^i \left(\partial_\tau + \tfrac{d-3}{d} \bar{K}\right) \left(\partial_\tau + \tfrac{1}{d} \bar{K}\right) v_i \Big]\,, \\
\mathbb{K}_2 = \int_x \Big[&- \tfrac{1}{2} \big((d-1)\psi + E\big) \left(\partial_\tau + \tfrac{d-2}{d} \bar{K}\right) \left(\partial_\tau + \tfrac{2}{d} \bar{K}\right) \big((d-1)\psi + E\big) \\
&- 2 B \sqrt{\Delta} \left(\partial_\tau + \tfrac{2}{d} \bar{K}\right) \big((d-1)\psi + E\big) + 2 B \Delta B \\
&- 4 \bar{K} \hat{N} \sqrt{\Delta}\, B + 2 \bar{K} \hat{N}\, \left(\partial_\tau + \tfrac{2}{d} \bar{K}\right) \big((d-1)\psi + E\big) + 2 \bar{K}^2 \hat{N}^2 \Big]\,.
\end{aligned} \tag{64}$$

On this basis we find

$$\begin{aligned}
\delta^2 I_1 = \mathbb{K}_1 - \int_x \Big[&+ \tfrac{(d-4)(d-8)}{4d^2}\, \bar{K}^2 \hat{N}\, \big((d-1)\psi + E\big) \\
&+ \tfrac{4}{d}\, \bar{K}\, h^{ij} \big(\partial_\tau + \tfrac{d-12}{8d} \bar{K}\big) h_{ij} + \tfrac{4}{d}\, \bar{K}\, E \left(\partial_\tau + \tfrac{d+4}{8d}\, \bar{K}\right) E \\
&- \tfrac{4}{d}\, \bar{K}\, \left(u^k \sqrt{\Delta}\, v_k + B \sqrt{\Delta}\, E - v^i\, (\partial_\tau + \tfrac{d-8}{4d}\, \bar{K})\, v_i\right) \\
&- \tfrac{1}{d}\, \bar{K}\, \big((d-1)\psi + E\big) \big(\partial_\tau + \tfrac{1}{4} \bar{K}\big) \big((d-1)\psi + E\big) \\
&+ \tfrac{4(d-1)}{d}\, \bar{K}\, \psi \left(\partial_\tau + \tfrac{d+4}{8d}\, \bar{K}\right) \psi \Big]
\end{aligned} \tag{65}$$

and

$$\begin{aligned}
\delta^2 I_2 = \mathbb{K}_2 - \int_x \Big[&- \tfrac{d-4}{d}\, \bar{K}^2 \hat{N} \big((d-1)\psi + E\big) + 2\, \bar{K}\, h^{ij} \left(\partial_\tau + \tfrac{d-8}{4d}\, \bar{K}\right) h_{ij} \\
&- \bar{K}\, \big((d-1)\psi + E\big) \big(\tfrac{d-2}{d}\, \partial_\tau + \tfrac{d^2-8}{4d^2}\, \bar{K}\,\big) \big((d-1)\psi + E\big) \\
&+ 2\, \bar{K}\, E \left(\partial_\tau + \tfrac{1}{4}\, \bar{K}\right) E + 2(d-1) \bar{K}\, \psi \left(\partial_\tau + \tfrac{1}{4}\, \bar{K}\right) \psi \\
&+ 2 \bar{K} v^i \left(\partial_\tau + \tfrac{d-6}{2d}\, \bar{K}\right) v_i - \tfrac{4}{d} \bar{K}\, ((d-1)\psi + E)\, \sqrt{\Delta} B \Big]\,.
\end{aligned} \tag{66}$$

Finally, $\delta^2 I_3$ and $\delta^2 I_4$ written in terms of the component fields are

$$\begin{aligned}
\delta^2 I_3 &= \int_x \Big[\tfrac{(d-1)(d-2)}{2}\, \psi \Delta \psi - \frac{1}{2}\, h_{ij} \Delta h^{ij} - 2\, (d-1)\, \hat{N} \Delta \psi \Big], \\
\delta^2 I_4 &= \int_x \Big[\tfrac{(d-1)(d-3)}{4} \psi^2 + \tfrac{(d-1)}{2} \psi E - \tfrac{1}{4} E^2 - \hat{N} \big((d-1)\psi + E\big) \\
&\qquad - \frac{1}{2} h_{ij} h^{ij} - v_i v^i \Big].
\end{aligned} \tag{67}$$

The next thing we need is the gauge fixing, the original definitions of F and F_i missed the terms that contain $\bar{K}$ terms, these still need to be added. This gives the following definition of the gauge fixing

$$\begin{aligned} F =& c_1\partial_\tau \hat{N} + c_2\partial^i \hat{N}_i + c_3\partial_\tau \hat{\sigma} + c_8\bar{K}^{ij}\hat{N}_{ij} + c_9\bar{K}\hat{N} \\ F_i =& c_4\partial_\tau \hat{N}_i + c_5\partial_i \hat{N} + c_6\partial_i\hat{\sigma} + c_7\partial^j\hat{\sigma}_{ij} + dc_{10}\bar{K}_{ij}\hat{N}^j \end{aligned} \tag{68}$$

Here the factor d that appears before the new parameter c_{10} is inserted for later convenience. The new gauge fixing then becomes

$$\begin{aligned} \delta^2 I_5 = \int_x \Big[& c_2^2\, B\Delta B - \hat{N}\ \big(c_1\partial_\tau + (c_1 - c_9)\bar{K}\big)\big(c_1\partial_\tau + c_9\bar{K}\big)\,\hat{N} \\ & - \big((d-1)\psi + E\big)\big(c_3\partial_\tau + (c_3 - c_8)\bar{K}\big)\big(c_3\partial_\tau + c_8\bar{K}\big)\big((d-1)\psi + E\big) \\ & - 2\,c_2\,B\sqrt{\Delta}\ \big(c_1\,\partial_\tau + c_9\bar{K}\big)\ \hat{N} \\ & + 2\hat{N}\ \big(c_1\partial_\tau + (c_1 - c_9)\bar{K}\big)\big(c_3\partial_\tau + c_8\bar{K}\big)\big((d-1)\psi + E\big) \\ & + 2\,c_2\,B\sqrt{\Delta}\ \big(c_3\partial_\tau + c_8\bar{K}\big)\ \big((d-1)\psi + E\big)\Big] \end{aligned} \tag{69}$$

and

$$\begin{aligned} \delta^2 I_6 = \int_x \Big[& c_5^2\,\hat{N}\Delta\hat{N} - \ u^i\,\big(c_4\,\partial_\tau + (\tfrac{d-2}{d}c_4 - c_{10})\,\bar{K}\big)\big(c_4\partial_\tau + c_{10}\bar{K}\big)\,u_i \\ & - B\big(c_4\,\partial_\tau + (\tfrac{d-1}{d}c_4 - c_{10})\,\bar{K}\big)\big(c_4\partial_\tau + (\tfrac{1}{d}\,c_4 + c_{10})\,\bar{K}\big)\,B \\ & + 2\,c_5\,\hat{N}\big(c_4\partial_\tau + (\tfrac{2}{d}c_4 + c_{10})\,\bar{K}\big)\sqrt{\Delta}B \\ & - 2\,c_6\,\big((d-1)\psi + E\big)\big(c_4\partial_\tau + (\tfrac{2}{d}c_4 + c_{10})\bar{K}\big)\sqrt{\Delta}B \\ & - 2\,c_7\,E\big(c_4\partial_\tau + (\tfrac{2}{d}c_4 + c_{10})\bar{K}\big)\sqrt{\Delta}B \\ & - 2\,c_7\,v^i\sqrt{\Delta}\,(c_4\partial_\tau + c_{10}\bar{K})\,u_i - 2c_5c_6\,\hat{N}\Delta\,\big((d-1)\psi + E\big) \\ & - 2\,c_5\,c_7\,\hat{N}\Delta E + c_6^2\,\big((d-1)\psi + E\big)\Delta\big((d-1)\psi + E\big) \\ & + 2c_6c_7\,\big((d-1)\psi + E\big)\Delta E + c_7^2\,\big(E\Delta E + v^i\Delta v_i\big)\Big]\,. \end{aligned} \tag{70}$$

And just like we have done before we can now combine all I_i terms to obtain the complete gauge fixed Hessian

$$
\begin{aligned}
&32\pi G_k\Big(\frac{1}{2}\delta^2\Gamma^{\text{grav}}+\Gamma_k^{\text{gf}}\Big)=\\
&\int_x\Big\{-\hat N\,\Big[(c_1\partial_\tau+(c_1-c_9)\bar K)(c_1\partial_\tau+c_9\bar K)-c_5^2\,\Delta+\tfrac{2(d-1)}{d}\,\bar K^2\Big]\,\hat N\\
&\quad-B\,\big[(c_4\partial_\tau+(\tfrac{d-1}{d}c_4-c_{10})\bar K)(c_4\partial_\tau+(\tfrac{1}{d}c_4+c_{10})\bar K)-c_2^2\,\Delta\big]\,B\\
&\quad-2\,B\,\sqrt{\Delta}\,\Big[(c_1c_2+c_4c_5)\partial_\tau+(c_2c_9+c_4c_5\,\tfrac{d-2}{d}-c_5c_{10}-\tfrac{2(d-1)}{d})\bar K\Big]\,\hat N\\
&\quad+2\,\hat N\Big[(c_1\partial_\tau+(c_1-c_9)\bar K)(c_3\partial_\tau+c_8\bar K)-\tfrac{d-1}{d}\bar K\partial_\tau-c_5(c_6+c_7)\Delta\\
&\qquad-\tfrac{5d^2-12d+16}{8d^2}\bar K^2-\Lambda_k\Big]E\\
&\quad+2(d-1)\,\hat N\Big[(c_1\partial_\tau+(c_1-c_9)\bar K)(c_3\partial_\tau+c_8\bar K)-\tfrac{d-1}{d}\bar K\partial_\tau\\
&\qquad+(1-c_5c_6)\Delta-\tfrac{5d^2-12d+16}{8d^2}\bar K^2-\Lambda_k\Big]\psi\\
&\quad+2\,B\sqrt{\Delta}\,\big[\big(c_2c_3+c_4(c_6+c_7)\big)\partial_\tau+\big(c_2c_8+(c_6+c_7)(\tfrac{d-2}{d}c_4-c_{10})\big)\bar K\big]\,E\\
&\quad+2(d-1)B\sqrt{\Delta}\,\big[\big(1+c_2c_3+c_4c_6\big)\partial_\tau+\big(c_2c_8+\tfrac{d-2}{d}c_4c_6-c_6c_{10}\big)\bar K\big]\,\psi\\
&\quad-(d-1)\,\psi\,\Big[(d-1)\,\big((c_3\partial_\tau+(c_3-c_8)\bar K)(c_3\partial_\tau+c_8\bar K)-c_6^2\Delta\big)\\
&\qquad+\tfrac{d-2}{2}(-\partial_\tau^2+\Delta-\tfrac{2}{d}\dot{\bar K})+\tfrac{d^2-10d+14}{2d}\bar K\partial_\tau+\tfrac{d^2-8d+11}{4d}\,\bar K^2-\tfrac{d-3}{2}\Lambda_k\Big]\psi\\
&\quad+E\Big[(c_6+c_7)^2\Delta-(c_3\partial_\tau+(c_3-c_8)\bar K)(c_3\partial_\tau+c_8\bar K)\\
&\qquad-\frac{1}{2}\Lambda_k+\tfrac{d-1}{d}\bar K\partial_\tau+\tfrac{d-1}{4d}\,\bar K^2\Big]E\\
&\quad+(d-1)\,\psi\Big[2c_6(c_6+c_7)\Delta-2(c_3\partial_\tau+(c_3-c_8)\bar K)(c_3\partial_\tau+c_8\bar K)\\
&\qquad+\partial_\tau^2+\bar K\partial_\tau+\tfrac{d-1}{d}\dot{\bar K}+\tfrac{d-1}{2d}\bar K^2+\Lambda_k\Big]E\\
&\quad-u^i\,\big[\big(c_4\partial_\tau+(\tfrac{d-2}{d}c_4-c_{10})\bar K\big)\big(c_4\partial_\tau+c_{10}\bar K\big)-\Delta\big]\,u_i\\
&\quad+v^i\,\Big[-\partial_\tau^2+\tfrac{d-2}{d}\bar K\partial_\tau-\tfrac{1}{d}\dot{\bar K}+\tfrac{d^2-8d+11}{d^2}\bar K^2+c_7^2\,\Delta-2\Lambda_k\Big]\,v_i\\
&\quad-2\,u^i\,\big[\big(1-c_4c_7\big)\partial_\tau+c_7\big(c_{10}-\tfrac{d-2}{d}c_4\big)\bar K\big]\,\sqrt{\Delta}\,v_i\\
&\quad+\frac{1}{2}\,h^{ij}\,\Big[-\partial_\tau^2+\tfrac{3d-4}{d}\,\bar K\partial_\tau+\tfrac{d^2-9d+12}{d^2}\,\bar K^2+\Delta-2\Lambda_k\Big]\,h_{ij}\Big\}.
\end{aligned}
\tag{71}
$$

We again try to eliminate all terms with $\sqrt{\Delta}$ whose contributions are given by

$$
\begin{aligned}
&32\pi G_k\Big(\frac{1}{2}\delta^2\Gamma^{\text{grav}}+\Gamma_k^{\text{gf}}\Big)|_{\sqrt{\Delta}}=\\
&\int_x\Big\{-2\,B\,\sqrt{\Delta}\,\Big[(c_1c_2+c_4c_5)\partial_\tau+(c_2c_9+c_4c_5\,\tfrac{d-2}{d}-c_5c_{10}-\tfrac{2(d-1)}{d})\bar K\Big]\,\hat N\\
&\quad+2\,B\sqrt{\Delta}\,\big[\big(c_2c_3+c_4(c_6+c_7)\big)\partial_\tau+\big(c_2c_8+(c_6+c_7)(\tfrac{d-2}{d}c_4-c_{10})\big)\bar K\big]\,E\\
&\quad+2(d-1)B\sqrt{\Delta}\,\big[\big(1+c_2c_3+c_4c_6\big)\partial_\tau+\big(c_2c_8+\tfrac{d-2}{d}c_4c_6-c_6c_{10}\big)\bar K\big]\,\psi\\
&\quad-2\,u^i\,\big[\big(1-c_4c_7\big)\partial_\tau+c_7\big(c_{10}-\tfrac{d-2}{d}c_4\big)\bar K\big]\,\sqrt{\Delta}\,v_i\Big\}.
\end{aligned}
\tag{72}
$$

The next thing to do is to try and determine the values of the new parameters c_8, c_9 and c_{10}. We do not need to redetermine the old parameters since these ensure the presence of the wave operator in the equations. This is necessary since the wave operator still appears in the corrected Laplacians Δ_i. Thus we can eliminate these terms and remove the appropriate terms while investigating the remainder to constrict the new coefficients c_8 - c_{10}.

$$
\begin{aligned}
&32\pi G_k\Big(\frac{1}{2}\delta^2\Gamma^{\rm grav}+\Gamma_k^{\rm gf}\Big)|_{\sqrt{\Delta}} = \\
&\int_x \Big\{ -2\,B\sqrt{\Delta}\left[(\epsilon_1 c_9 - \tfrac{d-2}{d} + \epsilon_2 c_{10} - \tfrac{2(d-1)}{d})\bar{K}\right]\hat{N} \\
&\quad + 2\,B\sqrt{\Delta}\left[\big(\epsilon_1 c_8 + \frac{1}{2}\epsilon_2)(\tfrac{d-2}{d}\epsilon_2 - c_{10})\big)\bar{K}\right]E \\
&\quad + 2(d-1)B\sqrt{\Delta}\left[\big(\epsilon_1 c_8 - \frac{1}{2}\tfrac{d-2}{d} + \frac{\epsilon_2}{2}c_{10}\big)\bar{K}\right]\psi \\
&\quad - 2\,u^i\left[\epsilon_2\big(c_{10} - \tfrac{d-2}{d}\epsilon_2\big)\bar{K}\right]\sqrt{\Delta}\,v_i\Big\}
\end{aligned}
\tag{73}
$$

It can be seen that the following unique choice removes all terms in (73):

$$
c_8 = 0, \qquad c_9 = \tfrac{2\,(d-1)}{d}\,\epsilon_1, \qquad c_{10} = \tfrac{d-2}{d}\,\epsilon_2. \tag{74}
$$

Combining this with the coefficients determined in (49) gives the following complete set of coefficients

$$
\begin{array}{llll}
c_1 = \epsilon_1, & c_2 = \epsilon_1, & c_3 = -\frac{1}{2}\epsilon_1, & \\
c_4 = \epsilon_2, & c_5 = -\epsilon_2, & c_6 = -\frac{1}{2}\epsilon_2, & c_7 = \epsilon_2, \\
c_8 = 0, & c_9 = \frac{2\,(d-1)}{d}\epsilon_1, & c_{10} = \frac{d-2}{d}\epsilon_2. &
\end{array}
\tag{75}
$$

Substituting these values into (71) gives the curved spacetime equivalent to (50)

$$
\begin{aligned}
&32\pi G_k\Big(\frac{1}{2}\delta^2\Gamma^{\rm grav}+\Gamma_k^{\rm gf}\Big) = \\
&\int_x \Big\{\frac{1}{2}\,h^{ij}\left[\Delta_2 - 2\Lambda_k - \tfrac{2(d-1)}{d}\dot{\bar{K}} - \tfrac{d^2-d+2}{d^2}\bar{K}^2\right]h_{ij} \\
&\quad + u^i\left[\Delta_1 - \tfrac{d-1}{d}\dot{\bar{K}} - \tfrac{1}{d}\bar{K}^2\right]u_i + v^i\left[\Delta_1 - 2\Lambda_k - \dot{\bar{K}} - \tfrac{5d-7}{d^2}\bar{K}^2\right]v_i \\
&\quad + B\left[\Delta_0 - \tfrac{d-1}{d}\dot{\bar{K}} - \tfrac{d-1}{d^2}\bar{K}^2\right]B + \hat{N}\left[\Delta_0 - \tfrac{2(d-1)}{d}\dot{\bar{K}} - \tfrac{4(d-1)}{d^2}\bar{K}^2\right]\hat{N} \\
&\quad + \hat{N}\left[\Delta_0 - 2\Lambda_k - \tfrac{5d^2-12d+16}{4d^2}\bar{K}^2\right]\big((d-1)\psi + E\big) \\
&\quad - \tfrac{(d-1)(d-3)}{4}\,\psi\left[\Delta_0 - 2\Lambda_k - \tfrac{2(d-1)}{d}\dot{\bar{K}} - \tfrac{d-1}{d}\bar{K}^2\right]\psi \\
&\quad + \tfrac{1}{4}\,E\left[\Delta_0 - 2\Lambda_k - \tfrac{2(d-1)}{d}\dot{\bar{K}} - \tfrac{d-1}{d}\bar{K}^2\right]E \\
&\quad - \tfrac{1}{2}(d-1)\,\psi\left[\Delta_0 - 2\Lambda_k - \tfrac{2(d-1)}{d}\dot{\bar{K}} - \tfrac{d-1}{d}\bar{K}^2\right]E\Big\}
\end{aligned}
\tag{76}
$$

Now the final step is to try and use the "near-diagonal" basis in field space that we found in flat spacetime. As we have seen in all calculations in a curved spacetime the inclusion of curvature terms changes the form of the equations significantly. This also happens with the matrix in (54) which will now also contain $\bar{K}^2$, $\dot{\bar{K}}$ or $\bar{K}\partial_\tau$ terms. We will use partial integration to change the $\bar{K}\partial_\tau$ terms into $\bar{K}^2$ and $\dot{\bar{K}}$ terms because it is easier to use in the remaining calculation to obtain the β-functions. The general expression for (54) in curved spacetime then is

$$\begin{aligned}
\mathbb{M}_{11} =& \Delta_0 - \frac{3}{2}\Lambda_k - \frac{d-1}{d}\dot{\bar{K}} - \frac{3d^2-2d+8}{8d^2}\bar{K}^2 \\
\mathbb{M}_{22} =& -\frac{d-1}{4}C_d\Delta_0 + \frac{d^2-6d+17+(7d-5)C_d}{16}\Lambda_k \\
&+ \frac{d^4-12d^3+81d^2-274d+312+(15d^3-23d^2-14d+40)C_d}{64d^2}\bar{K}^2 \\
&+ \frac{(d-1)(d^2-2d-11+(3d-5)C_d}{8d}\dot{\bar{K}} \\
\mathbb{M}_{33} =& \frac{d-1}{4}C_d\Delta_0 + \frac{d^2-6d+17+(7d-5)C_d}{16}\Lambda_k \\
&+ \frac{d^4-12d^3+82d^2-274d+312-(15d^3-23d^2-14d+40)C_d}{64d^2}\bar{K}^2 \\
&+ \frac{(d-1)(d^2-2d-11-(3d-5)C_d}{8d}\dot{\bar{K}} \\
\mathbb{M}_{12} =& \frac{d-3-C_d}{8}\Lambda_k + \frac{(d-1)^2}{2d}\dot{\bar{K}} - \frac{(C_d-1)(d^2-5d+4)}{16d^2}\bar{K}^2 \\
\mathbb{M}_{13} =& \frac{d-3-C_d}{8}\Lambda_k + \frac{(d-1)^2}{2d}\dot{\bar{K}} + \frac{(C_d-1)(d^2-5d+4)}{16d^2}\bar{K}^2 \\
\mathbb{M}_{23} =& -\frac{1}{2}\Lambda_k + \frac{(d-1)(d^2-4d+7)}{4d}\dot{\bar{K}} \\
&- \frac{d^4-12d^3+63d^2-160d+144}{32d^2}\bar{K}^2
\end{aligned} \tag{77}$$

Implementing this new basis into (76) gives the following form of the Hessian

$$\begin{aligned}
&32\pi G_k\Big(\frac{1}{2}\delta^2\Gamma^{\text{grav}} + \Gamma_k^{\text{gf}}\Big) = \\
&\int_x \Big\{\frac{1}{2}\,h^{ij}\left[\Delta_2 - 2\Lambda_k - \tfrac{2(d-1)}{d}\dot{\bar{K}} - \tfrac{d^2-d+2}{d^2}\bar{K}^2\right]h_{ij} \\
&\quad + u^i\left[\Delta_1 - \tfrac{d-1}{d}\dot{\bar{K}} - \tfrac{1}{d}\bar{K}^2\right]u_i + v^i\left[\Delta_1 - 2\Lambda_k - \dot{\bar{K}} - \tfrac{5d-7}{d^2}\bar{K}^2\right]v_i \\
&\quad + B\left[\Delta_0 - \tfrac{d-1}{d}\dot{\bar{K}} - \tfrac{d-1}{d^2}\bar{K}^2\right]B \\
&\quad + \bar{b}^i\left[\Delta_1 + \frac{2}{d}\bar{K}\partial_\tau + \frac{1}{d}\dot{\bar{K}} + \frac{d-4}{d^2}\bar{K}^2\right]b_i + \bar{c}\left[\Delta_0 + \frac{2}{d}\bar{K}\partial_\tau + \dot{\bar{K}}\right]c \\
&\quad + \chi_i\mathbb{M}^{ij}\chi_j\Big\}.
\end{aligned} \tag{78}$$

Our original goal find an approximate solution to (1). After calculating the complete Hessian in (78) and defining the regulator in (2) we now have all the pieces to solve (1). The technique that we will be using is called the "universal RG machine" which was introduced in [23]. This paper stipulates a number of steps one can take to solve the FRGE and involves using the so called off-diagonal heat kernel expansion. Therefore we will now first introduce the concept of the heat kernel.

4 Heat kernel

The heat kernel can be defined by realizing the following equality.

$$\frac{1}{2}\operatorname{Tr}\log\mathcal{O} = \frac{1}{2}\sum_n \log\lambda_n = -\frac{1}{2}\sum_n \frac{d}{ds}\lambda_n^{-s}\Big|_{s=0} \tag{79}$$

Here λ_n are the eigenvalues of the operator, the sum can also be written as.

$$\sum_n \lambda_n^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\sum_n e^{-t\lambda_n} = \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\operatorname{Tr} e^{-t\mathcal{O}} \tag{80}$$

Combining (79) and (80) gives.

$$\frac{1}{2}\operatorname{Tr}\log\mathcal{O} = -\frac{d}{ds}\big(\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\operatorname{Tr} e^{-t\mathcal{O}}\big|\big) = -\frac{1}{2}\int_0^\infty \frac{dt}{t}\operatorname{Tr}\, e^{-t\mathcal{O}} \tag{81}$$

The final equation is thus.

$$\frac{1}{2}\operatorname{Tr}\log\mathcal{O} = -\frac{1}{2}\int_0^\infty \frac{dt}{t}\operatorname{Tr}\, e^{-t\mathcal{O}} \tag{82}$$

The only unknown is the equation above is the quantity $e^{-t\mathcal{O}} \equiv K_{\mathcal{O}}(t)$, furthermore it can be seen that the following equation also holds.

$$\partial_t K_{\mathcal{O}}(t) + \mathcal{O}K_{\mathcal{O}}(t) = 0 \tag{83}$$

This equation is known as the heat equation. With the boundary condition $K_{\mathcal{O}}(0) = 1$ it can be concluded that $K_{\mathcal{O}}(t)$ is the heat kernel for the operator $\mathcal{O}$. To link the quantum corrections with the heat kernel the corresponding operator must be identified. This operator depends on the theory that has to be renormalized. For EH gravity the operator is the Laplace operator. This can be seen from considering the action of a minimally coupled scalar field in a curved spacetime.

$$S = \frac{1}{2}\int dt d^d x N\sqrt{\sigma}\phi[\Box + m^2]\phi \tag{84}$$

When this action is derived twice with respect to ϕ it can be seen that the operator that is obtained corresponds to $\Box + m^2$. Then by using the Seeley-deWitt expansion the following expression can be found for $\operatorname{Tr} e^{-t\mathcal{O}}$.

$$\operatorname{Tr} e^{-t\Delta_i} = (4\pi)^{-\frac{D}{2}} t^{-\frac{D}{2}}\int d^D x\sqrt{g}\sum_{n\geq 0,i} t^n a_{2n,i}\mathcal{R}_{2n}^{(i)} \tag{85}$$

In this expression the $a_{2n,i}$ are numerical coefficients that are independent of D and the $\mathcal{R}_{2n}^{(i)}$ span a basis of curvature monomials built from $2n$ covariant derivatives. With this expression the heat kernel coefficients can be calculated.

4.1 Universal RG machine

With this understanding of the heat kernel we can now start using it. We will do this through the concept of the universal RG machine which was introduced in [23] and work done in [7, 24]. We briefly referenced the regulator in the beginning of this thesis but we will now start a more thorough discussion of the regulator. The function of the regulator is to constrain the fluctuations of the effective action close to a certain momentum k. This allows us to use the Wilsonian idea that the flow of the RG can be obtained by integrating out the quantum fluctuations shell-by-shell. A way to achieve the confinement of the fluctuations is the introduction of a mass term. Thus we are going to add a scale dependent mass term to the Laplacian which we call R_k

$$\Delta_s \to P_k \equiv \Delta_s + R_k \tag{86}$$

This procedure defines $\mathcal{R}_k$ uniquely. In general$\Gamma_k^{(2)}|_{\hat{\chi}_i\hat{\chi}_j}$ takes the form

$$\Gamma_k^{(2)}|_{\hat{\chi}_i\hat{\chi}_j} = (32\pi G_k)^{-\alpha_s} c(\Delta_s + w + v_1\bar{K}^2 + v_2\dot{\bar{K}} + v_3\bar{K}\partial_\tau). \tag{87}$$

Here $\alpha_s = 0, 1$ depending on whether the fields are ghosts or not, w indicates the possible presence of the cosmological constant in the matrix element and finally v_1,v_2 and v_3 are numerical coefficients whose values are dependent on the terms in (78). This then causes $\mathcal{R}_k$ to be of the form [24]

$$\mathcal{R}_k = (32\pi G_k)^{-\alpha_s} c R_k. \tag{88}$$

We can then split the sum of these two terms into the kinetic ($\mathcal{P}$) and interaction parts ($\mathcal{V}$).

$$(\Gamma_k^{(2)} + \mathcal{R}_k)|_{\chi\chi} = \mathcal{P} + \mathcal{V} \tag{89}$$

$\mathcal{P}$ contains all terms that have P_k and Λ_k in them while all terms with at least one power in $\bar{K}$ are slotted into $\mathcal{V}$. We are however looking for the inverse of (89) which can be written as a power series in $\mathcal{V}$

$$(\mathcal{P} + \mathcal{V})^{-1} = \mathcal{P}^{-1} - \mathcal{P}^{-1}\mathcal{V}\mathcal{P}^{-1} + \mathcal{P}^{-1}\mathcal{V}\mathcal{P}^{-1}\mathcal{V}\mathcal{P}^{-1} + \mathcal{O}(\bar{K}^3). \tag{90}$$

This forms a matrix that is usually block-diagonal in fields space. We assume that each block is determined by a single field, so h_{ij},B,v_i or u_i from the gravitational sector or one of the two ghost fields c or b_i. Then we can calculate the contribution of the fields separatly and then sum all their contributions into a final trace. Though it might be slightly confusing we will still use $\mathcal{P}$ and $\mathcal{V}$ to denoted the kinetic and potential part of each single block. Then we can determine from the Hessian that the $\mathcal{P}^{-1}$ is given by

$$\mathcal{P}^{-1} = (32\pi G_k)^{\alpha_s} c^{-1}(\Delta_s + R_k + w)^{-1}. \tag{91}$$

The constant c is determined by possible coefficients in the Hessian. The potential term can then be split into three separate pieces $\mathcal{V}_1$,$\mathcal{V}_2$ and $\mathcal{V}_3$ which are

$$\mathcal{V}_1 = (32\pi G_k)^{\alpha_s} c\bar{K}^2, \qquad \mathcal{V}_2 = (32\pi G_k)^{\alpha_s} c\dot{\bar{K}}, \qquad \mathcal{V}_3 = (32\pi G_k)^{\alpha_s} c\bar{K}\partial_\tau. \tag{92}$$

With all these identities we can write the right-hand-side of the flow equation in traces that are independent of the choice of cutoff function. It is now convenient

	S	V	T	TV	TTT
a_0	1	d	$\frac{1}{2}d(d+1)$	$d-1$	$\frac{1}{2}(d+1)(d-2)$
a_2	$\frac{d-1}{6d}$	$\frac{d-1}{6}$	$\frac{(d-1)(d+1)}{12}$	$\frac{d^3-2d^2+d+6}{6d^2}$	$\frac{d^4-2d^3-d^2+14d+36}{12d^2}$

Table 1: Heat-kernel coefficients for the component fields (33). Here S, V, T, TV, and TTT are scalars, vectors, symmetric two-tensors, transverse vectors, and transverse traceless symmetric matrices, respectively.

to define the profile function $R^{(0)}(\frac{\Delta_s}{k^2}) \equiv \frac{R_k}{k^2}$ in order to introduce the threshold functions.

$$\begin{aligned} \Phi^p_n(w) \equiv& \frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} \frac{R^{(0)}(z) - z R^{(0)\prime}(z)}{[z + R^{(0)}(z) + w]^p} \\ \tilde{\Phi}^p_n(w) \equiv& \frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} \frac{R^{(0)}(z)}{[z + R^{(0)}(z) + w]^p} \end{aligned} \tag{93}$$

As stated before we intent to use the Litim regulator given as $R_k = (k^2 - \Delta_s)(k^2 - \Delta_s)$. This allows us to evaluate the integrals in the threshold functions

$$\Phi^p_n(w) \equiv \frac{1}{\Gamma(n+1)} \frac{1}{(1+w)^p}, \qquad \tilde{\Phi}^p_n(w) \equiv \frac{1}{\Gamma(n+2)} \frac{1}{(1+w)^p}. \tag{94}$$

These threshold functions allow us to write the traces of the different part of the expansion in (90). Firstly the part that contains no $\mathcal{V}$

$$\begin{aligned} \mathrm{Tr}\left[\mathcal{P}^{-1} \partial_t \mathcal{R}_k\right] =& \frac{k^D}{(4\pi)^{D/2}} \int_x \Bigg[a_0 \left(2\Phi^1_{D/2}(\tilde{w}) - \eta \alpha_s \tilde{\Phi}^2_{D/2}(\tilde{w}) \right) \\ & + a_2 \left(2\Phi^1_{D/2}(\tilde{w}) - \eta \alpha_s \tilde{\Phi}^1_{D/2-1}(\tilde{w}) \right) \frac{\bar{K}^2}{k^2} \Bigg] \end{aligned} \tag{95}$$

Next we get the contributions from the terms with one $\mathcal{V}$

$$\begin{aligned} \mathrm{Tr}\left[\mathcal{V}_1 \mathcal{P}^{-2} \partial_t \mathcal{R}_k\right] =& \frac{k^D}{(4\pi)^{D/2}} \int_x a_0 \left(2\Phi^2_{D/2}(\tilde{w}) - \eta \alpha_s \tilde{\Phi}^2_{D/2}(\tilde{w}) \right) \frac{\bar{K}^2}{k^2}, \\ \mathrm{Tr}\left[\mathcal{V}_2 \mathcal{P}^{-2} \partial_t \mathcal{R}_k\right] =& - \frac{k^D}{(4\pi)^{D/2}} \int_x a_0 \left(2\Phi^2_{D/2}(\tilde{w}) - \eta \alpha_s \tilde{\Phi}^2_{D/2}(\tilde{w}) \right) \frac{\bar{K}^2}{k^2}, \\ \mathrm{Tr}\left[\mathcal{V}_3 \mathcal{P}^{-2} \partial_t \mathcal{R}_k\right] =& 0 \end{aligned} \tag{96}$$

The final contribution contains $\mathcal{V}^2$ and there is only a single term contributing to the flow. Applying the off-diagonal heat-kernel techniques gives

$$\mathrm{Tr}\left[(\mathcal{V})^2 \mathcal{P}^{-3} \partial_t \mathcal{R}_k\right] = \frac{-k^D}{2(4\pi)^{D/2}} \int_0 a_0 \left(2\Phi^2_{D/2+1}(\tilde{w}) - \eta \alpha_s \tilde{\Phi}^2_{D/2+1}(\tilde{w}) \right) \frac{\bar{K}^2}{k^2} \tag{97}$$

These traces contain two coefficients a_0 and a_2 which are spin-dependent heat-kernel coefficients, their values are given in Tab.1. Furthermore we use the dimensionless version of w which correspond to a possible cosmological contribution in the trace, it is defined as $\tilde{w} \equiv wk^{-2}$. We can use the traces defined in (95), (96) and (97) to determine the contribution from the diagonal fields in both the gravitational sector and the ghost fields. We now define

$$f_k \equiv (1 - 2\lambda_k)^{-1} \tag{98}$$

which is a contribution from the threshold functions if the diagonal matrix element has a Λ_k term. We also define the combination of the threshold functions as

$$q_n^p(w) \equiv 2\,\Phi_n^p(w) - \eta\,\widetilde{\Phi}_n^p(w)\,, \tag{99}$$

Another small detail is the fact that the trace itself can come with a factor due to the nature of the fields. In the case of a scalar (0), transverse vector (1T) or transverse, traceless tensor (2T) field the trace contributes an extra factor which is given by

$$\mathrm{tr}_0\,\mathbb{1} = 1\,, \qquad \mathrm{tr}_{1\mathrm{T}}\,\mathbb{1} = d-1\,, \qquad \mathrm{tr}_{2\mathrm{T}}\,\mathbb{1} = \tfrac{1}{2}(d+1)(d-2)\,. \tag{100}$$

4.2 Gravitational trace contributions

Using these definitions and the traces defined in (95), (96) and (97) gives the following traces for h_{ij}, u_i, v_i and B

$$\begin{aligned}
\mathrm{Tr}|_{hh} =& \tfrac{k^D}{2\,(4\pi)^{D/2}} \int_x \Big[\tfrac{(d+1)(d-2)}{2}\, q^1_{D/2}(-2\lambda) - \tfrac{(d-2)^2(d+1)^2}{2\,d^2}\, q^2_{D/2}(-2\lambda)\tfrac{\bar{K}^2}{k^2} \\
&+ \tfrac{d^4-2d^3-d^2+14d+36}{12d^2}\, q^1_{D/2-1}(-2\lambda)\,\tfrac{\bar{K}^2}{k^2} \Big]\,, \\
\mathrm{Tr}|_{vv} =& \tfrac{k^D}{2\,(4\pi)^{D/2}} \int_x \Big[(d-1)\, q^1_{D/2}(-2\lambda) + \tfrac{d^3-2d^2+d+6}{6d^2}\, q^1_{D/2-1}(-2\lambda)\,\tfrac{\bar{K}^2}{k^2} \\
&- \tfrac{(d-1)(d^2-5d+7)}{d^2}\, q^2_{D/2}(-2\lambda)\,\tfrac{\bar{K}^2}{k^2} \Big], \\
\mathrm{Tr}|_{uu} =& \tfrac{k^D}{2\,(4\pi)^{D/2}} \int_x \Big[(d-1)\, q^1_{D/2}(0) + \tfrac{d^3-2d^2+d+6}{6d^2}\, q^1_{D/2-1}(0)\,\tfrac{\bar{K}^2}{k^2} \\
&- \tfrac{(d-1)(d-2)}{d}\, q^2_{D/2}(0)\,\tfrac{\bar{K}^2}{k^2} \Big], \\
\mathrm{Tr}|_{BB} =& \tfrac{k^D}{2\,(4\pi)^{D/2}} \int_x \Big[q^1_{D/2}(0) + \tfrac{d-1}{6d}\, q^1_{D/2-1}(0)\,\tfrac{\bar{K}^2}{k^2} - \tfrac{(d-1)^2}{d^2}\, q^2_{D/2}(0)\,\tfrac{\bar{K}^2}{k^2} \Big].
\end{aligned} \tag{101}$$

Next is the ghost sector which can also be calculated by using (95), (96) and (97). For the scalar ghost field c the trace becomes

$$\mathrm{Tr}|_{\bar{c}c} = \tfrac{k^D}{(4\pi)^{D/2}} \int_x \Big\{ 2\,\Phi^1_{D/2} + \tfrac{\bar{K}^2}{k^2} \Big[\tfrac{d-1}{3d}\Phi^1_{D/2-1} + 2\Phi^1_{D/2} - \tfrac{4}{d^2}\,\Phi^1_{D/2+1} \Big] \Big\}\,, \tag{102}$$

For the vector ghosts b_i the trace reads

$$\mathrm{Tr}|_{\bar{b}b} = \tfrac{k^D}{(4\pi)^{D/2}} \int_x \Big\{ 2d\,\Phi^1_{D/2} + \tfrac{\bar{K}^2}{k^2} \Big[\tfrac{d-1}{3}\,\Phi^1_{D/2-1} + \tfrac{8}{d}\,\Phi^1_{D/2} - \tfrac{4}{d}\,\Phi^1_{D/2+1} \Big] \Big\}. \tag{103}$$

This leaves the χ sector which is not diagonal for a curved spacetime. This is where we need all the tools that the universal RG machine introduced. We already wrote down the matrix components of χ in (77). We now need to split this matrix in a $\mathcal{P}$ and $\mathcal{V}$ part. Firstly the kinetic part $\mathcal{P}$ is given by the matrix

$$\mathcal{P} =(32\pi G_k)^{-1} \begin{bmatrix} \Delta_0 & \frac{1}{2}(\Delta_0 - 2\Lambda) & \frac{d-1}{2}(\Delta_0 - 2\Lambda) \\ \frac{1}{2}(\Delta_0 - 2\Lambda) & \frac{1}{4}(\Delta_0 - 2\Lambda) & \frac{d-1}{4}(\Delta_0 - 2\Lambda) \\ \frac{d-1}{2}(\Delta_0 - 2\Lambda) & \frac{1-d}{4}(\Delta_0 - 2\Lambda) & \frac{(1-d)(d-3)}{4}(\Delta_0 - 2\Lambda) \end{bmatrix}, \tag{104}$$

while the matrix $\mathcal{V}$ is symmetric with entries

$$\begin{aligned} \mathcal{V}_{11} &= -\tfrac{2(d-1)}{d^2}\left(2\bar{K}^2 + d\dot{\bar{K}}\right), & \mathcal{V}_{12} &= -\tfrac{5d^2-12d+16}{8d^2}\bar{K}^2 \\ \mathcal{V}_{22} &= -\tfrac{d-1}{4d}\left(\bar{K}^2 + 2\dot{\bar{K}}\right), & \mathcal{V}_{13} &= -\tfrac{(d-1)(5d^2-12d+16)}{8d^2}\bar{K}^2 \\ \mathcal{V}_{33} &= \tfrac{(d-3)(d-1)^2}{4d}\left(\bar{K}^2 + 2\dot{\bar{K}}\right), & \mathcal{V}_{23} &= \tfrac{(d-1)^2}{4d}\left(\bar{K}^2 + 2\dot{\bar{K}}\right). \end{aligned} \tag{105}$$

The cutoff $\mathcal{R}_k$ in this sector is given by

$$\mathcal{R}_k = (32\pi G_k)^{-1} R_k \begin{bmatrix} 1 & \frac{1}{2} & \frac{d-1}{2} \\ \frac{1}{2} & \frac{1}{4} & -\frac{d-1}{4} \\ \frac{d-1}{2} & -\frac{d-1}{4} & -\frac{(d-1)(d-3)}{4} \end{bmatrix}. \tag{106}$$

The identities for the traces that we introduced in (95) and (96) still hold even when $\mathcal{P}$ and $\mathcal{V}$ are matrices. Using them we obtain the following contribution from the χ sector to the flow.

$$\begin{aligned} \mathrm{Tr}|_{\chi\chi} =& \tfrac{k^D}{2(4\pi)^{D/2}} \int_x \Big[2\, q^1_{D/2}(-2\lambda) + q^1_{D/2}\left(-\tfrac{d}{d-1}\lambda\right) \\ &+ \tfrac{d-1}{6d}\left(2\, q^1_{D/2-1}(-2\lambda) + q^1_{D/2-1}\left(-\tfrac{d}{d-1}\lambda\right)\right)\tfrac{\bar{K}^2}{k^2} \\ &- \left(\tfrac{2(d-1)}{d}\, q^2_{D/2}(-2\lambda) - \tfrac{3d^3+6d^2-16d+16}{4d^2(d-1)} q^2_{D/2}\left(-\tfrac{d}{d-1}\lambda\right)\right)\tfrac{\bar{K}^2}{k^2}\Big]. \end{aligned} \tag{107}$$

This completes the calculations of the contributions from the gravitational sector to the RG flow. To obtain the β-functions corresponding to the gravitational sector we need to use these expressions in the FRGE. This leaves us with the contribution from the matter sector which we will calculate now.

4.3 Matter trace contributions

The traces that contribute in the matter sector have all ready been calculated in previous works involving the metric formulation. We expand the matter fields around a vanishing background which will only result in variations of the matter fields and background Laplacians. To be as general as possible we introduce the parameter N_s which is the number of scalar fields, N_v the number of vector gauge fields and N_D the number of Dirac fields. This means that the contribution from the scalar fields ϕ becomes

$$\mathrm{Tr}|_{\phi\phi} = N_S\, \tfrac{k^D}{(4\pi)^{D/2}} \int_x \left\{\Phi^1_{D/2}(0) + \tfrac{d-1}{6d}\,\Phi^1_{D/2-1}(0)\,\tfrac{\bar{K}^2}{k^2}\right\} \tag{108}$$

For the gauge sector we have a contribution from A_μ and the ghost fields that are associated with A_μ which we call C and $\bar{C}$. Their contributions are given by

$$\begin{aligned} \mathrm{Tr}|_{AA} &= N_V \, \tfrac{k^D}{(4\pi)^{D/2}} \int_x \Big\{ (d+1)\, \Phi^1_{D/2}(0) + \tfrac{(d-1)(d^2+2d-11)}{6d\,(d+1)}\, \Phi^1_{D/2-1}(0)\, \tfrac{\bar{K}^2}{k^2} \Big\}, \\ -\mathrm{Tr}|_{\bar{C}C} &= N_V \, \tfrac{k^D}{(4\pi)^{D/2}} \int_x \Big\{ 2\, \Phi^1_{D/2}(0) + \tfrac{d-1}{3d}\, \Phi^1_{D/2-1}(0)\, \tfrac{\bar{K}^2}{k^2} \Big\}. \end{aligned} \tag{109}$$

Combining these two gives the total contribution from the gauge fields

$$\mathrm{Tr}|_{\mathrm{GF}} = N_V \, \tfrac{k^D}{(4\pi)^{D/2}} \int_x \Big\{ (d-1)\, \Phi^1_{D/2}(0) + \tfrac{(d-1)(d^2-13)}{6d\,(d+1)}\, \Phi^1_{D/2-1}(0)\, \tfrac{\bar{K}^2}{k^2} \Big\} \tag{110}$$

This leaves the Dirac fermions for whose contribution we follow the reasoning in [21] which results in

$$\begin{aligned} \mathrm{Tr}|_{\psi\psi} = -\, \tfrac{N_D\, 2^{(d+1)/2}\, k^D}{(4\pi)^{D/2}} \int_x \Big\{ & \\ \Phi^1_{D/2}(0) + \tfrac{d-1}{d} \Big[\big(\tfrac{1}{6} - \tfrac{r}{4}\big)\, \Phi^1_{D/2-1}(0) - \tfrac{1-r}{4}\, \Phi^2_{D/2}(0) \Big] \tfrac{\bar{K}^2}{k^2} \Big\} & \end{aligned} \tag{111}$$

The coefficient r depends on the precise implementation of the regulating function, for us we will choose $r = 0$ which corresponds with the Type I regulator that we are using.

By combining all the traces together we can obtain the right-hand-side of (1) which will enable us to determine the β-functions for both parameters that we are interested in Λ and G.

5 β-functions

We have now calculated all the traces needed to complete the right hand side of (1) this means we are able to determine the β-functions for the two relevant coupling constants in our theory Λ_k and G_k. However it is more useful to look at the dimensionless quantities that are associated with Λ_K and G_k which are given by

$$\lambda_k = \Lambda_k k^{-2}, \qquad g_k = G_k k^{d-1}. \tag{112}$$

There is another interesting parameter namely the anomalous dimension of G_k given by

$$\eta = (G_k)^{-1} k \partial_k G_k \tag{113}$$

In all these definitions k is the renormalization scale. The anomalous dimension η is already a derivative function and therefore can be seen as a β-function. This gives three function to look at β_λ, β_g and η. It is useful to define the following function before looking at the β-functions

$$B_{det}(\lambda) = (1 - 2\lambda)\,(d - 1 - d\lambda) \tag{114}$$

This function captures the λ-dependence occurring in the χ-sector. Using this function the β-functions become

$$\begin{aligned} \beta_g &= (d - 1 + \eta)\, g\,, \\ \beta_\lambda &= (\eta - 2)\lambda + \tfrac{2g}{(4\pi)^{(d-1)/2}}\, \tfrac{1}{\Gamma((d+3)/2)} \Big[N_S + (d-1)N_V - 2^{[(d+1)/2]}\, N_D \\ &\qquad - 2(d+1) + \big(d + \tfrac{d^2+d-4}{2(1-2\lambda)} + \tfrac{3d-3-(4d-2)\lambda}{B_{\rm det}(\lambda)}\big)\big(1 - \tfrac{\eta}{d+3}\big)\Big]. \end{aligned} \tag{115}$$

And the anomalous dimension of Newton's constant is

$$\eta = \frac{16\pi g\, B_1(\lambda)}{(4\pi)^{(d+1)/2} + 16\pi g\, B_2(\lambda)} \tag{116}$$

With

$$\begin{aligned} B_1(\lambda) \equiv &- \tfrac{d^5+17d^4+41d^3+85d^2+174d-78}{24\, d(d-1)\, \Gamma((d+5)/2)} + \tfrac{d^4-5d^2+16d+48}{12\, d(d-1)\, (1-2\lambda)\, \Gamma((d+1)/2)} \\ &- \tfrac{d^4-15d^2+28d-10}{2d(d-1)\, (1-2\lambda)^2\, \Gamma((d+3)/2)} + \tfrac{3d-3-(4d-2)\lambda}{6\, B_{\rm det}(\lambda)\, \Gamma((d+1)/2)} \\ &+ \tfrac{1}{6\, \Gamma((d+1)/2)} \Big[N_S + \tfrac{d^2-13}{d+1} N_V - \tfrac{1}{4}\, 2^{[(d+1)/2]} N_D \Big] \\ &+ \tfrac{c_{1,0}+c_{1,1}\lambda+c_{1,2}\lambda^2}{4\, d\, B_{\rm det}(\lambda)^2\, \Gamma((d+3)/2)} \end{aligned} \tag{117}$$

and

$$\begin{aligned} B_2(\lambda) = &\tfrac{d^4-10d^3+21d^2+6d+6}{24\, d(d-1)\, \Gamma((d+5)/2)} + \tfrac{d^4-5d^2+16d+48}{24\, d(d-1)\, (1-2\lambda)\, \Gamma((d+3)/2)} \\ &+ \tfrac{3d-3-(4d-2)\lambda}{12\, B_{\rm det}(\lambda)\, \Gamma((d+3)/2)} + \tfrac{c_{2,0}+c_{2,1}\lambda+c_{2,2}\lambda^2}{8\, d\, B_{\rm det}(\lambda)^2\, \Gamma((d+5)/2)} \\ &- \tfrac{d^4-15d^2+28d-10}{4\, d(d-1)\, (1-2\lambda)^2\, \Gamma((d+5)/2)} \end{aligned} \tag{118}$$

Where the coefficients $c_{i,j}$ are polynomials depending on d which are given by

$$\begin{aligned} c_{1,0} &= -5d^3 + 22d^2 - 24d + 16\,, & c_{1,1} &= 4\left(d^3 - 10d^2 + 16d - 16\right)\,, \\ c_{1,2} &= 4\left(d^3 + 6d^2 - 16d + 16\right)\,, & c_{2,0} &= -5d^3 + 22d^2 - 24d + 16\,, \\ c_{2,1} &= c_{1,1}\,, & c_{2,2} &= c_{1,2}\,. \end{aligned} \tag{119}$$

The equations (115) and (116) give the complete RG flow of the system when projected on Einstein-Hilbert action. They do still depend on the number of matter fields and the number of spacetime dimensions $D = d + 1$. We can look at the flow of the dimensionless coupling constants with the RG scale k. This defines the so-called RG flow of the theory, the properties of the RG flow will be discussed in the next section.

5.1 Pure gravitational flow

We will now consider the RG flow of pure gravity, $N_S = N_V = N_D = 0$ in the physical case of $D = 4$. The β-functions we obtained are not analytically solvable, which means we have to resort to numerical methods. This includes the concept of stability analysis which allows the study of the long-term behaviour of the system. In this analysis there may appear some points in which the β-functions are zero. These points (g_*,λ_*) are called fixed points (FP) in the RG flow. In terms of the β-functions this means

$$\beta_g(g_*, \lambda_*) = 0\,, \qquad \beta_\lambda(g_*, \lambda_*) = 0 \tag{120}$$

The fact that the β-functions are zero means that any RG flow that goes into this point will never get out of it again even for $k \to \infty$ or $k \to 0$. It will thus not develop any divergences and can be said to be complete. By linearizing (115) around the FP we can obtain the k-dependence of the coupling constants near the FPs.

This allows us to determine the critical exponents of the FP. These are the eigenvalues of the Jacobi matrix which is also known as the “stability matrix” of the linearized system. The critical exponents determine the relevant directions of the FP solutions and tell us whether the flow is attracted or repelled near the FP. So our next goal is to determine the FP and its critical exponents.

It is a well known feature of the EH renormalization that it possesses a Gaussian fixed point (GFP) at the origin. This FP repels the flows approaching the origin and causes gravity to be non-renormalizable in perturbative renormalization. This GFP is also found in our model and is also located at the origin. Another feature in Asymptotic safety is a non-Gaussian fixed point (NGFP) which determines the UV limit. [25–56]. Both features are also found in our model and the location of the NGFP is given by:

$$\text{NGFP:} \qquad g_* = 0.785\,, \qquad \lambda_* = 0.315\,, \qquad g_*\lambda_* = 0.248 \tag{121}$$

And the critical exponents by

$$\theta_{1,2} = 0.503 \pm 5.377i \tag{122}$$

Any critical exponent which is a couple of complex conjugate numbers gives rise to a spiral behaviour of the flow around the fixed point. Again this has been

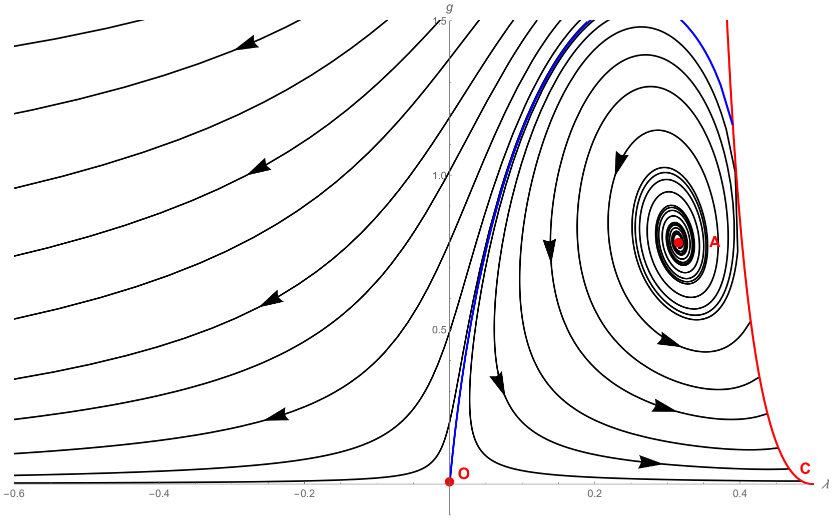

Figure 2: RG flow in $D = 4$, the arrows indicate the flow direction of the coupling constants for lower k. Only the flow for a positive g is shown since this is the physically relevant part of the theory space. The RG flow is determined by the GFP O and the NGFP A. The blue line is a "Type IIa" trajectory, and it is a separatrix between the fixed points at the locations A and O. These trajectories end their flow in a singular curve (red line), where the β-functions are divergent.

seen in previous studies of Einstein-Hilbert actions in Asymptotic Safety. The complete flow of the dimensionless Newtons constants and the dimensionless cosmological constant can be seen in Fig.2

In Fig.2 we notice the existence of a singular line which is caused by a divergence in the β-functions. Due to the appearance of both g and λ appearing in the denominators of both β_λ and η. The divergence of η is given by a d-dependent function which for the specific cases of $d = 2$ and $d = 3$ become

$$\begin{aligned} d = 2: \qquad & \eta^{\text{sing}}: \qquad g = -\frac{45\pi(1-2\lambda)^2}{2\,(76\lambda^2-296\lambda+147)}, \\ d = 3: \qquad & \eta^{\text{sing}}: \qquad g = -\frac{144\pi(6\lambda^2-7\lambda+2)^2}{144\lambda^4-1884\lambda^3+3122\lambda^2-1688\lambda+279}. \end{aligned} \tag{123}$$

The divergences for β_λ are completely determined by the λ and are given by

$$\lambda_1^{\text{sing}} = \frac{1}{2}, \qquad \lambda_2^{\text{sing}} = \frac{d-1}{d} \tag{124}$$

The positions of these divergences for both the $d = 2$ and $d = 3$ cases are shown in Fig.3

If we look at the flow of λ_k in the region $g \geq 0$ we notice that a difference can be observed between the cases shown in Fig.3. In $d = 2$ we encounter the fixed singularity of β_λ first and thus η remain finite throughout the flow. For $d = 3$ it is exactly the opposite and we encounter the singularity in η while the singularity in β_λ is screened. We notice that the positions of these singularities are completely independent of N_S,N_V and N_D. Finally we note that the point

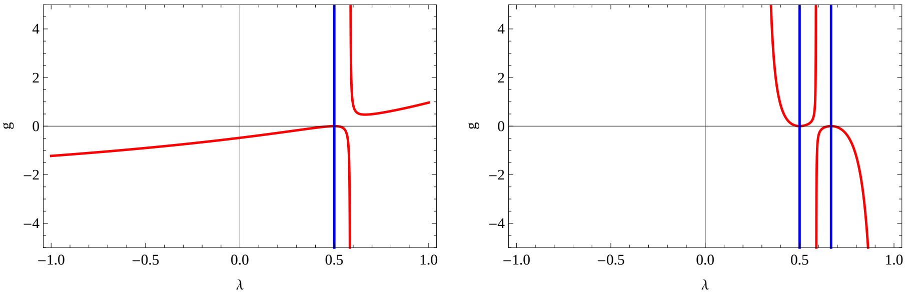

Figure 3: Singularities of the beta functions (115) in the λ-g–plane in $d = 2$ (left diagram) and $d = 3$ (right diagram). The blue lines are fixed singularities of β_λ, while the red lines are the curves where η develops a singularity.

$(\lambda, g) = (1/2, 0)$ is a special point where the β-functions are of the form $0/0$. Whether this point is finite or not depends on the direction from which the point is approached. Therefore even though this point can satisfy $\beta_g = \beta_\lambda = 0$ for certain cases we do not call it a fixed point, instead we will call it a "quasi-fixed point" and it will be denoted as the point $C = (1/2, 0)$.
In order to be consistent with existing literature [26] we also adopt the following conventions satisfying the solutions displayed in Fig.2 by their end-points

$$\begin{aligned} &\text{Type Ia:} && \lim_{k\to 0}(\lambda_k, g_k) = (-\infty, 0)\,, && \Lambda_0 < 0\,, \\ &\text{Type IIa:} && \lim_{k\to 0}(\lambda_k, g_k) = (0, 0)\,, && \Lambda_0 = 0\,, \\ &\text{Type IIIa:} && \text{terminate at } \eta^{\text{sing}} && \Lambda_{k_{\text{term}}} > 0 \end{aligned} \tag{125}$$

The Type Ia trajectories are those that flow towards the left in Fig.2, Type IIIa trajectories flow towards the right and Type IIa are those on the separatrix between the GFP and the NGFP. We know that in the classical regime both G and Λ are essentially k-independent thus we might look for an area in the flow where the same characteristics hold, it turns out that this happens near the GFP "O". When this happens depends on which type of trajectory we consider. For Type Ia this regime extends to $k = 0$ while the Type IIIa trajectories are terminated in the singularity η^{sing} at a finite k_{term}. The high-energy and low-energy regimes are connected by a crossover of the flow. Some of these crossovers go through the red line though which marks the divergence in η this is due to the critical exponent values $\theta_{1,2}$ being extremely small causing the spiraling to be less compact in comparison to similar calculation done earlier, this feature does not show in comparative work that has been done [3, 13, 14, 26] We therefore conclude that this is probably an artifact in our computation rather than a physical feature.

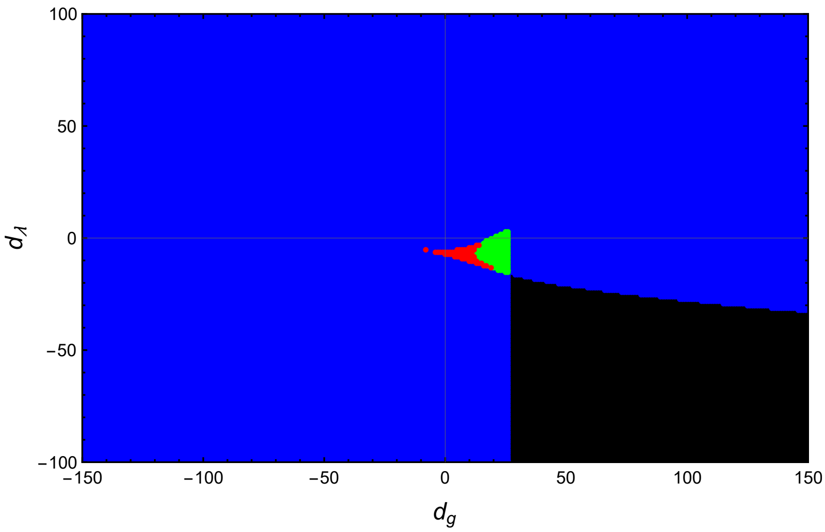

Figure 4: The number of NGFPs occurring in the beta functions (115) as a function of the parameters d_g and d_λ. The colors black, blue, green, and red represent zero, one, two, and three NGFPs for the physical relevant region of $g_* > 0$, $\lambda_* < 1/2$, respectively.

5.2 Gravity-matter flow

Up until now we have considered pure gravity while we initially also introduced matter in the β-functions given in (115). For simplicity we introduce the following notations for number of scalar fields N_s, vector fields N_V and Dirac spinors N_D

$$\begin{aligned} d_g &\equiv N_S + \frac{d^2-13}{d+1} N_V - \tfrac{1}{4}\, 2^{(d+1)/2} N_D \,, \\ d_\lambda &\equiv N_S + (d-1) N_V - 2^{(d+1)/2} N_D \end{aligned} \tag{126}$$

These two combination capture the contribution of the matter content in both β_λ and η. We can simplify these two definitions by using $d = 3$ which is the physically more interesting case.

$$d_g = N_S - N_V - N_D \,, \quad d_\lambda = N_S + 2N_V - 4N_D \tag{127}$$

It should be noted that N_S and N_V are both positive integers including zero while N_D are half-integers in order to include chiral fermions. This results in d_g and d_λ to be half-integer and integer respectively, which we covers the entire $d_g - d_\lambda$-plane. With these definitions we can construct a coordinate system for different matter sectors such as the SM for which the number of fields is $N_S = 4$, $N_V = 12$ and $N_D = 45/2$, a similar discussion was done in [17]. The coordinates for the SM in Fig.4 are $(d_g, d_\lambda) = (-61/2, -62)$ and the point has also been marked in Fig.5.

We see in Fig.4 that the β-functions can support a diverse set of NGFPs with a small dependence on the matter sector. We can however wonder whether

class	NGFPs	NGFP_1	NGFP_2	NGFP_3	color code
Class 0	0	–	–	–	black region
Class Ia	1	UV, spiral	–	–	blue region
Class Ib	1	UV, real	–	–	green region
Class Ic	1	saddle	–	–	magenta region
Class Id	1	IR, spiral	–	–	red region
Class Ie	1	IR, real	–	–	orange region
Class IIa	2	UV, real	IR, real	–	open circle
Class IIb	2	UV, real	IR, spiral	–	open square
Class IIc	2	UV, spiral	IR, spiral	–	open triangle
Class IId	2	UV, spiral	UV, real	–	open diamond
Class IIIa	3	UV, real	saddle	IR, real	filled circle
Class IIIb	3	UV, real	saddle	IR, spiral	filled square
Class IIIc	3	UV, spiral	saddle	IR, spiral	filled triangle

Table 2: Color-code for the fixed point classification provided in Fig.5. The column NGFPs gives the number of NGFPs while the subsequent columns give their behaviour in terms of 2 UV-attractive directions (UV), one UV-attractive and one IR-attractive direction (saddle) and 2 IR-attractive directions (IR) with real (real) and complex (spiral) critical exponents.

there is a maximum number of NGFPs that we can have in a theory. Let us therefore analyze the structure of the β-functions again. We know that a NGFP entails $\beta_g|_{g=g^*} = 0$ which in turn means that we get a restriction on the anomalous dimension $\eta_* = -2$. This restriction can be solved analytically to determine the coordinates of the fixed points as coordinates of λ and d_g, $g_*(\lambda_*, d_g)$. This solves the first equation. To solve the complete equation we also need $\beta_\lambda|_{g=g_*} = 0$ which leads to a fifth order polynomial in $\lambda(d_g, d_\lambda)$. Every root of this polynomial is a candidate NGFP which means that a β-function can at most have five NGFPs independent on the matter content in the theory.

We have specified in Fig.4 the number of NGFPs that appear as a function of the particle content of the theory. It would be useful though to also investigate what kind of NGFPs appear in the β-functions. Therefore let us split the regions we found in Fig.4 into smaller regions classified by their stability coefficients i.e. UV,IR or saddle point-NGFP (SP-NGFP). This classification can be seen in Tab.2, and a visual representation is given in Fig.5

We can see that Asymptotic Safety allows for many different sets of NGFPs depending on the matter content. Especially the occurrence of zero or one NGFP are rather common for both complex and real critical exponents. The appearance of two or three NGFPs are slightly more rare and is mainly restricted in an area around the origin. Furthermore this area is logically more diverse in

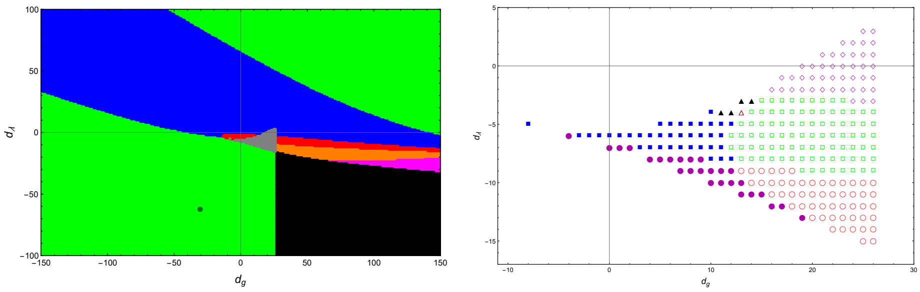

Figure 5: Classification of the NGFPs arising from the beta functions (115) in the d_g-d_λ–plane, using the color-code given in Tab.2. On the left the stability behaviour of the one-fixed point sector. Note that, the black region does not support any NGFP while the regions giving rise to a single, UV-attractive NGFP with complex and real critical exponents are marked in blue and green, respectively. The matter content of the standard model is placed in the lower-left part of the figure at $(d_g, d_\lambda) = (-61/2, -62)$ and it is marked with a bold green dot. The gray area, supporting multiple NGFPs is magnified in the right diagram with empty and filled symbols indicating the existence of two and three NGFPs, respectively. The precise nature of these NGFP can be found in Tab.2

model	N_S	N_D	N_V	d_g	d_λ
pure gravity	0	0	0	0	0
Standard Model (SM)	4	$\frac{45}{2}$	12	$-\frac{61}{2}$	-62
SM, dark matter (dm)	5	$\frac{45}{2}$	12	$-\frac{59}{2}$	-61
SM, $3\,\nu$	4	24	12	-32	-68
SM, $3\,\nu$, dm, axion	6	24	12	-30	-66
MSSM	49	$\frac{61}{2}$	12	$+\frac{13}{2}$	-49
SU(5) GUT	124	24	24	$+76$	$+76$
SO(10) GUT	97	24	45	$+28$	$+91$

Table 3: Particle content of the SM and some of its extensions, here d_g and d_λ give the position of the model in Fig.5. Most of the theories are located in the lower-left corner of Fig.5 with the exception of the MSSM and GUT theories which are located in the lower-right corner and the top-right corner respectively.

the combinations of the types of NGFP one finds. Due to the interplay of the additional NGFPs it would be interesting to study theories in this small area around the origin.

A couple of interesting features can be observed from Tab.3 and Tab.4, firstly all of the models considered have a single UV-NGFP with real critical exponents. The values of these critical exponents do differ drastically between two sets of models. Firstly we can look at the SM and the other models that reside in the

model	g_*	λ_*	θ_1	θ_2
pure gravity	0.78	$+0.32$	$0.50 \pm 5.38\,i$	
Standard Model (SM)	0.75	-0.93	3.871	2.057
SM, dark matter (dm)	0.76	-0.94	3.869	2.058
SM, $3\,\nu$	0.72	-0.99	3.884	2.057
SM, $3\,\nu$, dm, axion	0.75	-1.00	3.882	2.059
MSSM	2.26	-2.30	3.911	2.154
SU(5) GUT	0.17	$+0.41$	25.26	6.008
SO(10) GUT	0.15	$+0.40$	19.20	6.010

Table 4: The values for g_* and λ_* for the models given in Tab.3 together with the corresponding critical exponents. All models with matter contain a single UV-NGFP with real critical exponents. The variation in the critical exponents is rather small when compared with the SM for most models. Only the GUT models have vastly different critical exponents.

lower-left corner together with the MSSM. All these models have very similar critical exponents $\theta_1 \simeq 3.8$, $\theta_2 \simeq 2.0$ and they also all have a characteristic product $g_*\lambda_* < 0$. This indicates that these models are rather robust under a change of the number of particles. This is different in the GUT models whch, beside their different position in the (d_g, d_λ)-plane also exhibit a very different value of its critical exponents $\theta_1 > 19$ and their characteristic product is also positive $g_*\lambda_* > 0$. These values are much larger than those seen in the SM and its extensions, furthermore the critical exponents are also more dependent on the particle content. This can be seen in the fact that the critical exponents between SU(5) and SU(10) are also rather different. This might be a indication that the SM-type theories have more predictive powers than GUT theories when we look at a lower number of relevant coupling constants in the gravitational sector.
Let's now focus on the RG flow of the SM which means the following values for the particle number parameters $(d_g, d_\lambda) = (-61/2, -62)$. We can then insert this in the β-functions given in (115) to obtain the RG flow shown in Fig.6. The RG flow in Fig.6 is qualitatively similar to the pure gravity flow in Fig.2 as there is an interplay between the NGFP which is now situated at $(g_*, \lambda_*) = (0.75, -0.93)$ and the GFP which is still located in the origin. The NGFP still controls the high-energy behaviour while the GFP determines the low-energy behaviour. The classification of the flow that was given in (125) also still holds as the singular line in the anomalous dimension is still present. These were all the similarities now let us look at the differences. The first difference is the fact that the NGFP now has real critical exponents, thus there is no spiraling behaviour which was observed for the pure gravity case. Also it is important to notice that λ_* has shifted to be negative for the NGFP. As we stated before most extensions of the SM gave very similar critical exponents as in the SM, thus we can now state that the flow in Fig.6 serves as a showcase for

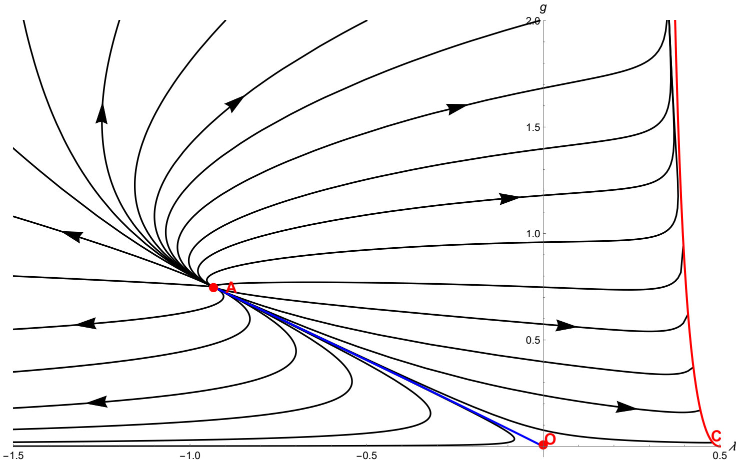

Figure 6: RG flow of gravity coupled to the minimally interacting matter content of the SM in $D = 3+1$. Again the phase diagram is dominated by the interplay of the NGFP (point “A”) controlling the flow for ultra-high energies and the GFP (point “O”) determining the low-energy behaviour. Again the singular line in the anomalous dimension is given by (123) and is depicted by the red line. Finally just as the flow in Fig.2 arrows point towards lower values of k.

all of these flows.

Finally we will investigate a scenario where $N_V = N_D = 0$ since these models are interesting for cosmological model building. In order to not neglected the models in the lower-left corner of Fig.5 we will also allow for $N_S < 0$. There will always be an UV-NGFP (two UV attractive eigendirections) and thus we want to study the behaviour of this NGFP under a change of N_S. In Fig.7 the dependence of the location and critical exponents on N_S is given. There are a couple of interesting features in Fig.7, firstly we notice a sudden change in the position of the NGFP around $N_S \simeq -6$. On the left of this point we see that $\lambda_* < 0$ this is the opposite from $N_S > -5$ where $\lambda_* > 0$. Secondly for $N_S \to \infty$ the fixed point approaches $C \equiv (1/2, 0)$ which for large N_S is really a fixed point and not a QFP. If we look at the bottom two pictures in Fig.7 we can also see that there are two interesting values $N_S = -6$ and $N_S = 46$. Before $N_S = -6$ we have real critical exponents which changes to be a complex pair at $N_S = -6$ and finally at $N_S = 46$ they become real again. We can then again consider Tab.3 and notice that even though all theories have real critical exponents they do fall in the different regions. This means that the Asymptotic Safety mechanism appearing in that generate these models are different in nature.

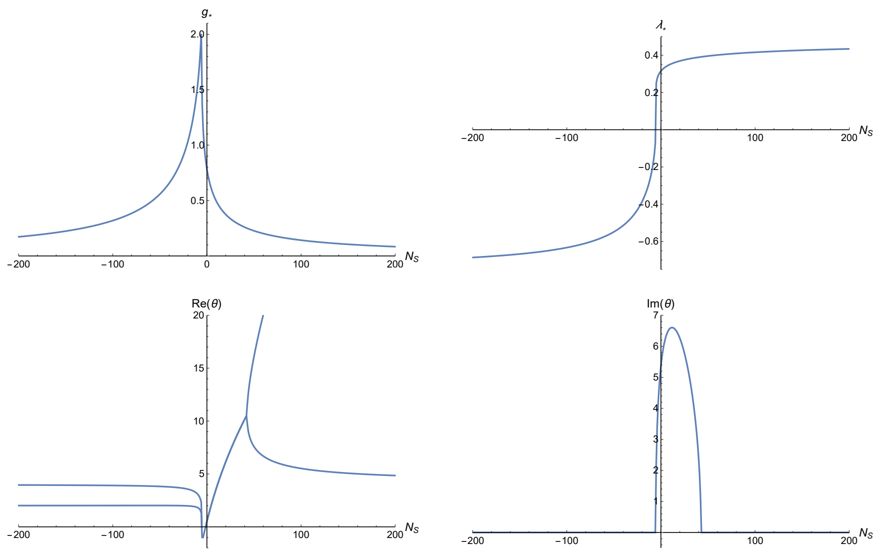

Figure 7: Location (top) and critical exponents (bottom) of the UV NGFPs appearing in gravity-scalar systems as a function of N_S. The fixed point structure undergoes a qualitative change at $N_S \approx -6$ and $N_S \approx 46$ where the critical exponents change from real to complex values.

6 Conclusion

In this thesis we used the functional renormalization group equation (FRGE) for the effective Einstein-Hilbert action in combination with the Arnowitt-Deser-Misner (ADM) decomposition for the metric to investigate the renormalization group flow of the coupling constants of gravity, Newtons constant G and the cosmological constant Λ, in a general gravity-matter model. In our investigation we made a conceptual advancement in the construction of a natural foliation of spacetime which results in a distinguished "time"-direction and thus it may be used for the implementation of a Wick rotation to transform Euclidean spacetime into Minkowski spacetime.
By using the ADM decomposition we encoded the metric degrees of freedom in terms of three fields, the Lapse function N, the shift vector N_i and the spatial metric σ_{ij}. This decomposition introduces a complication in the calculation of the off-shell flow equation due to the fact that N and N_i appear as Lagrange multipliers. Therefore the use of the "standard" gauge choice, the temporal gague [57], which sets both of these parameters to zero leads to non-canonical propagators for the remaining fluctuation fields. In order to circumvent this problem we introduced a new Feynman-type gauge scheme for our ADM-fields. This new gauge scheme ensured that all fields obtain a proper relativistic dispersion relation. Furthermore they also all travel with the same speed of light when the dispersion relations are evaluated on a Minkowski background. In the new gauge scheme the demand of a proper dispersion relation fixes all free parameters up to a irrelevant $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. The entire gauge scheme exhibits a structure that reminds us of the Feynman gauge in quantum electrodynamics

which is also used to obtain a suitable kinetic term.
A important difference with the existing literature is that we do not need a compact time-direction for our computations. Which is one of the main limitation is calculations done in the Matsubara formalism [13, 14].
The central result of our investigation is the phase diagram which is shown in Fig.2 and the matter NGFP landscape shown in Fig.4. The phase diagram for pure gravity qualitatively matches the on found in the metric formulation [26, 27, 29, 30] and from the evaluation of Lorentzian RG flows based on the Matsubara-formalism [13, 14]. The key element of Asymptotic Safety, a UV-attractive non-Gaussian fixed point with complex critical exponents, is present in all these calculations. The landscape we found for the gravity-matter system shows that the matter contribution to the gravitational β-functions can be given in terms of two parameters that deform the β-functions of pure gravity. The occurrence of NGFPs seems rather generic in the landscape of Fig.5. Notably the occurrence of a NGFP with real critical exponent is present for the matter content of the SM and some of its common extensions as shown in Tab.3 and Tab.4. This is an extension of earlier studies that were based on the metric formulation by showing that the NGFPs responsible for Asymptotic Safety in the Standard Model and Grand Unified Theories are qualitatively different in nature.
The results of this thesis are twofold, firstly it acts as a stepping stone in the future development of Asymptotic Safety in a gravitational setting by introducing the ability to remove one of the main obstructions for computing transition amplitudes between spatial geometries for different time instances. On the phenomenological side it provides a setup which is very close to the setup used in cosmic perturbation theory. We showed that the addition of matter fields does not break the Asymptotic Safety mechanism. This makes our setup predestined for the study of the scale-dependence of cosmic perturbations within the Asymptotic Safety program. For future research directions the use of asymptotically safe single-field inflationary models constitute a natural starting point.

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