Radboud University

Bachelor project

The first law of Newton

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1 Introduction

”Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impresses thereon.”: Newton’s first law. In our everyday experience we all know that this is true. A car will not stop unless you use the break (ignoring the friction). But how does this work in quantum field theory? If we create a particle with a certain momentum, does its momentum stay the same if there is no force acting upon this particle?

1.1 The problem

The problem is the fact that in order to verify Newton’s law you have to know the position and momentum of the particle. You want to create a particle at a certain point with a certain momentum and want to see that after a while, the particle still has the same momentum. But this is in conflict with the Heisenberg uncertainty principle. So it is important that the particles that are created do not have a too precise position or momentum because then we can’t say anything about the relation between them.

1.2 Approach to tackle the problem

In this thesis I want to find out if the first law of Newton is still true in quantum field theory. First there will be some theoretical background. I investigate the properties of the Feynman propagator and which particle sources are appropriate. After this, I will consider each source separately. For each source the first step is to try to calculate the response of the quantum field and look for the possibilities of a non-negligible answer. The first source (chapter 3: Adiabatic source) is calculated mainly following chapter 3.3.3 in Pictures Paths Particles Processes [1]. For the second one (chapter 4: Gaussian source) another approach is needed.

2 Theoretical background

In this chapter concepts will be introduced and defined which are used in this thesis. Furthermore the properties of certain concepts will be investigated, so that later on results can be coupled to the properties of either the source or the propagator.

2.1 Feynman propagator

The Feynman propagator is the propagator used in free-particle theory. It is defined as the object that creates a mode at point y in space-time. This particle travels to point x in space-time and then the mode is annihilated again. The way the particle behaves in the intermediate time is described by function 1.

$$\Pi(x-y) = \frac{i\hbar}{(2\pi)^4}\int_{-\infty}^{\infty} d^4k \frac{\exp(-ik\cdot(x-y))}{k^2-m^2+i\epsilon} \tag{1}$$

Here k is the four-dimensional wave vector, m the rest mass of the created particle and ϵ is infinitesimal with $\epsilon > 0$. It is of course a four-dimensional integral since we are dealing with Minkowski space. The integral runs over all possible k-values for each dimension. The higher the value of k, the smaller is its contribution. This has to do with the fact that for this propagator the simplest probability density is used where there is no force acting upon the particle.

Now it is time to calculate the integral.
Use the Cauchy residue theorem on the time difference and subsequently change to polar coordinates:

$$\begin{aligned}
\Pi(X) &= \frac{i\hbar}{(2\pi)^4}\int_{-\infty}^{\infty} dk^0 d\vec{k}\ \frac{\exp(-ik^0\cdot X_0)\exp(i\vec{k}\cdot\vec{X})}{(k^0)^2-\vec{k}^2-m^2+i\epsilon} \\
&= \frac{-\hbar}{(2\pi)^3}\int_{-\infty}^{\infty} d\vec{k}\ \frac{\exp(-i\omega(k)\cdot X_0 + i\vec{k}\cdot\vec{X})}{2\omega(k)} \\
&= \frac{-i\hbar}{(2\pi)^2|\vec{X}|}\int_{-\infty}^{\infty} dk\ \frac{k\exp(-i\omega(k)\cdot X_0 + ik\cdot|\vec{X}|)}{2\omega(k)}
\end{aligned} \tag{2}$$

Here is $\omega(k) = \sqrt{k^2+m^2}$. The term $i\epsilon$ can now be neglected because we used the Cauchy residue theorem. We can now use the substitution $\eta = k + \omega(k)$, i.e.:

$$\begin{aligned}
k &= \frac{\eta^2-m^2}{2\eta} & \omega(k) &= \frac{\eta^2+m^2}{2\eta} \\
dk &= \frac{1}{2}(1+\frac{m^2}{\eta^2})d\eta & \int_{-\infty}^{\infty} &\rightarrow \int_{0}^{\infty}
\end{aligned} \tag{3}$$

We can insert this result in equation 2 and get:

$$\Pi(X) = \frac{-i\hbar}{(2\pi)^2|\vec{X}|}\int_0^\infty d\eta\ \frac{\eta^2-m^2}{4\eta^2}\ \exp\left(\frac{-i\eta}{2}(X_0-|\vec{X}|)-\frac{im^2}{2\eta}(X_0+|\vec{X}|)\right) \tag{4}$$

The next step is to put the exponential in the form of $\exp(-iA/2(t+1/t))$ and therefore we use the substitution:

$$\eta = \sqrt{\frac{X_0+|\vec{X}|}{X_0-|\vec{X}|}}\ m\cdot t \qquad\qquad \tilde{\eta} = \sqrt{\frac{X_0+|\vec{X}|}{|\vec{X}|-X_0}}\ m\cdot t' \tag{5}$$

Where η is used in the case that $X_0 > |\vec{X}|$ and $\tilde{\eta}$ when $|\vec{X}| > X_0$. This gives the following integrals for the propagator:

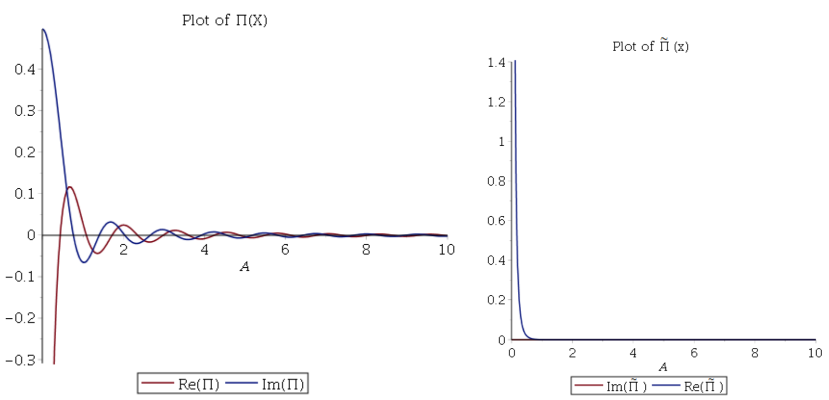
$$\Pi(X) = \frac{-i\hbar}{2(2\pi)^2}\int_0^\infty dt\ \frac{m}{\sqrt{X_0^2-\vec{X}^2}}\ \exp\left(\frac{-i}{2}\sqrt{(X_0^2-|\vec{X}|^2)}m(t+\frac{1}{t})\right) \tag{6}$$

$$\tilde{\Pi}(X) = \frac{-i\hbar}{2(2\pi)^2}\int_0^\infty dt'\ \frac{m}{\sqrt{\vec{X}^2-X_0^2}}\ \exp\left(\frac{i}{2}\sqrt{(\vec{X}^2-X_0^2)}m(t'-\frac{1}{t'})\right) \tag{7}$$

We plot this function for different values of A, by using that $A = \sqrt{|(X_0^2-\vec{X}^2)|}\ m$ and $\hbar = 1$. The results are given in figure 1. This plot looks like a Bessel function and with some trial and error we find that:

$$\Pi(X) = \frac{m^2}{8\pi A}\Big(i\text{Bessel}J_1(A)+\text{Bessel}Y_1(A)\Big), \qquad X^2 > 0 \tag{8}$$

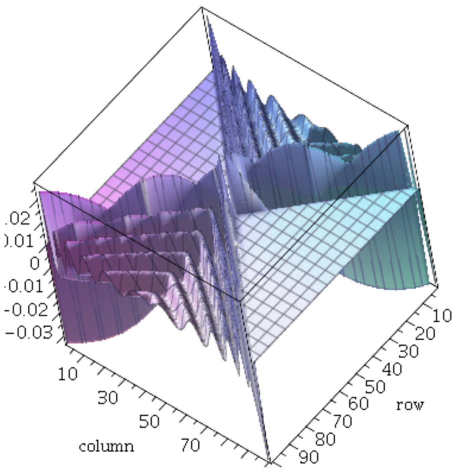
$$\Pi(X) = \frac{m^2}{4\pi^2 A}\text{Bessel}K_1(A), \qquad X^2 < 0 \tag{9}$$



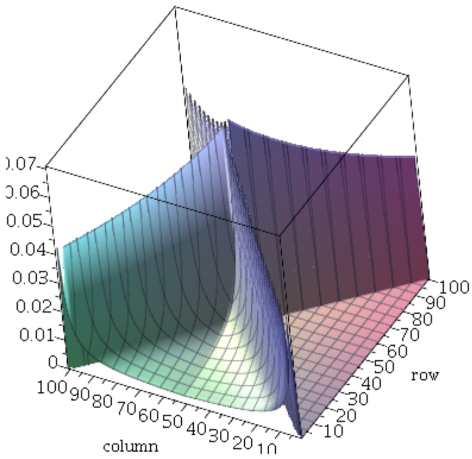
(a) (b)

Figure 1: Plots of the propagator functions. $A = \sqrt{|(\vec{X}^2 - X_0^2)|}\ m$

Now we have two functions that tell us how modes propagate trough space-time. One for modes that travel faster than light; the other for modes that are moving slower than light. It is nice to see that propagation faster than the speed of light is extremely suppressed, especially for large m which results in large A. There is a small probability to propagate faster than light but this is connected with the uncertainty principle between momentum and position. And we must not forget that we are using the propagator of particles with a mass. The limit of mass going to zero is more complicated than it looks. Moreover it is shown that particles that are on light-speed distance away from the origin correlate the strongest. The total propagator-field is depicted below:



(a) Real part of the propagator



(b) absolute value of the propagator

Figure 2: Plots of the propagator.

In figure 2 is used that $X^0 = (row - 50)/10$ and $\vec{X} = (column - 50)/10$. On the z-axis is shown the amount of correlation between the given point and the origin of the plot.

2.2 The source

The source creates a wave at a certain point in space-time. We will refer to this object as $J(x)$ where x is an point in space-time. This wave created on position x will propagate in space-time following the rules of the propagator. The form of $J(x)$ can be chosen as we like, there are no restrictions. The only thing we have to keep in mind is the uncertainty principle of Heisenberg. So if we know exactly at what place the source was active ($J(x) = \delta^3(\vec{x})$), we can determine the position of the particles very precisely. This means that we don't know anything about the momentum of the particle because each momentum is equally probable. In this case we can't find any law of motion. In the other case, when we know exactly at what time the source was active ($J(x) = \delta(x_0)$) we can determine the momentum of the particle very well but lose all knowledge of the position. In this case it is neither possible to find any law of motion. Thus it is important to determine a source which is appropriately spread out around the point x. In that case we don't know the exact position or momentum of the particle which means that we can determine an approximate law of motion because we can know them both more or less.

Good possible sources could be sources that are exponential or Gaussian oriented around a point in space-time. In this Thesis I first use a source which is Gaussian in position and exponential (i.e. sufficiently slow) in time. This source I will call the 'Adiabatic' source. Next I use a source which is Gaussian in both position and time. This source I will call the Gaussian source.

2.3 From source to field response

If we picture a one dimensional source in a two dimensional field and we let the field be discreet, we get something like figure 3. Every point of the source emits a wave in circles as pictured in 3. And if we are now interested in how the quantum field respond in position x, we have to take into account that all other places can influence point x. This contribution is mediated by the propagator that is used. For quantum field theory the limit to a continuous field has to be taken and hence we find Huygen's principle. Free quantum field theory satisfies this principle and therefore the response of a source can be found by equation 10:

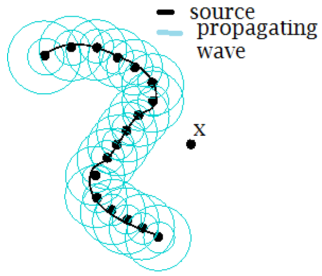


Figure 3: Huygen's priciple

$$\phi(x) = \frac{i}{\hbar} \int d^4y \Pi(x-y) J(y) \tag{10}$$

If we are interested in the time-evolution of our quantum field, we can use the Klein-

Gordon equation to find:

$$\begin{aligned}
& (\partial^\mu \partial_\mu + m^2)\phi(x) \\
= \quad & (\partial^\mu \partial_\mu + m^2)\left(\frac{-1}{(2\pi)^4}\int d^4y d^4k \frac{\exp(-ik\cdot(x-y))}{k^2-m^2+i\epsilon}J(y)\right) \\
= \quad & \frac{-1}{(2\pi)^4}\int d^4y d^4k \frac{\exp(-ik\cdot(x-y))}{k^2-m^2+i\epsilon}(m^2-k^2)J(y) \\
= \quad & \int d^4y \delta(x-y)J(y) = J(x)
\end{aligned} \tag{11}$$

This is exactly what one would expect because the only thing which gives time evolution is the source. So this gives a confirmation that the idea of $J(x)$ interpreted as a source is fine.

3 Adiabatic source

In the previous chapter we have seen how the quantum field reacts to certain sources and how the modes emitted by this source propagate through space-time. Now let us consider a special case where we choose the source to be a Gaussian in space and an exponential in time. i.e.:

$$J(y) \propto \exp\left(\frac{-|y^0|}{\sigma_0} - \frac{|\vec{y}|^2}{4\sigma^2} - \frac{i}{\hbar}(p^0 y^0 - \vec{y}\vec{p})\right) \tag{12}$$

In this case there is a certain spread in position and time such that we can say something about the relation between momentum and energy after the calculations. Also the momentum vector is not specified so there is no special relation between p^0 and $\vec{p}$ assumed. To solve this problem I will follow the steps of chapter 3 of [1].

3.1 Approach to the problem

In order to calculate the integral for $\phi(x)$ the first step is to do a Fourier transformation of the source to a function of k because the propagator is an integral over four-vector k. The time integral will be calculated first. This may be done with the Cauchy residue theorem because of the poles in the new function. After that there is only an integral over $\vec{k}$. This integral may be calculated but it is easier to find possible nonzero values and draw conclusions from that. I follow chapter 3 of [1] because my main research is about the calculations on the Gaussian source in chapter 4.

3.2 Four-Dimensional case

Starting from the source mentioned in equation 12 we can Fourier transform this in the case of four dimensions in terms of k. Were $k^2 = (k^0)^2 - \vec{k}^2$. The result is as follows:

$$\begin{aligned}
J(k) &\propto \int d^4y \exp\left(\frac{-|y^0|}{\sigma_0} - \frac{|\vec{y}|^2}{4\sigma^2} - \frac{i}{\hbar}(p^0 y^0 - \vec{y}\vec{p})\right)\exp(iky) \\
&= \int dy^0 \exp\left(\frac{-|y^0|}{\sigma_0} - iy^0\left(\frac{p^0}{\hbar} - k^0\right)\right)\int d\vec{y} \exp\left(\frac{-|\vec{y}|^2}{4\sigma^2} + i\vec{y}\left(\frac{\vec{p}}{\hbar} - \vec{k}\right)\right) \\
&\propto \left(\left(\frac{1}{\sigma_0} + i\left(\frac{p^0}{\hbar} - k^0\right)\right)^{-1} + \left(\frac{1}{\sigma_0} - i\left(\frac{p^0}{\hbar} - k^0\right)\right)^{-1}\right)\exp\left(-\sigma^2\left(\frac{\vec{p}}{h} - \vec{k}\right)^2\right) \\
&\propto \left(\frac{1}{\sigma_0^2} + \left(\frac{p^0}{\hbar} - k^0\right)^2\right)^{-1}\exp\left(-\sigma^2\left(\frac{\vec{p}}{h} - \vec{k}\right)^2\right)
\end{aligned}$$

The first integral is split into two parts, one from $-\infty$ to 0 and the other from 0 to ∞. For the second integral is used that $\int dx \exp\left(-ax^2 + ibx\right) \propto \exp\left(-b^2/4a\right)$.

We use this to obtain the integral for $\phi(x)$ over $k-$space instead of $y-$space. This leads to the integral:

$$\phi(x) \propto \int d^4k \frac{\exp(-ik^0x^0 + i\vec{k}\vec{x})}{(k^0)^2 - \vec{k}^2 - m^2 + i\epsilon} J(k)$$

This integral has four singular points: two due to the propagator; $k^0 = \pm(\omega(k) - i\epsilon)$ where $\omega(k) = \sqrt{\vec{k}^2 + m^2}$. The other two are due to the source; $k^0 = p^0/\hbar \pm i/\sigma_0$. To calculate this integral, we can use the Cauchy residue theorem. For positive time we have to close the integral on the lower half of the complex$-k^0$ plane because in that case the exponent will vanish at infinity. The contour is pictured in figure 4. The only two poles that are enclosed by this contour are $k^0 = \omega(k) - i\epsilon$ and $k^0 = p^0/\hbar - i/\sigma_0$. If we were interested in negative time, we would have to close the contour on the upper half and enclose the other singularities.

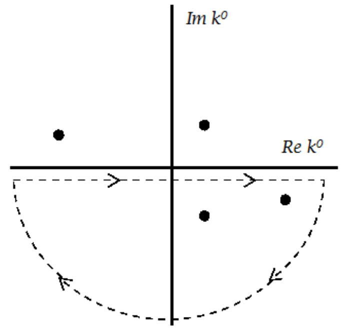


Figure 4: Contour that is taken for the integral over k^0 for $x^0 > 0$.

$$\phi(x) \propto \int d\vec{k} \exp\left(i\vec{k}\vec{x} - \sigma^2\left(\frac{\vec{p}}{\hbar} - \vec{k}\right)^2\right) \times \left[\frac{\exp(-i\omega(k)x^0)}{2\omega(k)\left(\frac{1}{\sigma_0^2} + \left(\frac{p^0}{\hbar} - \omega(k)\right)^2\right)} + \frac{i\sigma_0 \exp\left(-ix^0\left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0}\right)\right)}{2\left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0}\right)^2 - \omega(k)^2}\right] \quad (13)$$

The second term in the square brackets has an $\exp(-x^0/\sigma_0)$ which means that it vanishes concomitantly with the source. We are interested in times after the source is off because then the particle can move free. Therefore we can neglect the last term. Furthermore we see that $\vec{p}/\hbar \approx \vec{k}$ because otherwise the first factor will vanish and lead to negligible $\phi(x)$. And also $p^0/\hbar \approx \omega(k)$, because otherwise the first term in the square brackets will vanish. So:

$$\frac{E}{\hbar c} = \frac{p^0}{\hbar} \approx \omega\left(\frac{\vec{p}}{\hbar}\right) = \sqrt{\frac{\vec{p}^2}{\hbar^2} + m^2} \quad (14)$$

The only things that the source can send throughout space must be on the mass shell. It is easily seen that $m = Mc/\hbar$ where M is the mechanical mass of the particle. Therefore m can be interpreted as the inverse Compton wavelength. This requirement arises from the combination of the source and the field. The pole due to the propagator creates the first term in the square brackets of equation 13. But the source requires that $\vec{p}/\hbar$ and $p^0/\hbar$ are proportional to $\vec{k}$ and $\omega(k)$ respectively.
But there still is a complex phase in equation 13. Integration over $\vec{k}$ will lead to extremely rapid behavior if the particle is on the mass shell. Therefore the result will vanish. However, if the complex phase is stationary then the integration will not vanish. So for this reason the only non-vanishing results are such that:

$$\frac{\delta}{\delta\vec{k}}\left(\vec{x}\vec{k} - \omega(k)x^0\right) = \vec{x} - \frac{\vec{k}x^0}{\omega(k)} = 0 \quad (15)$$

so this means that the only particles that are possibly propagated are those that are on the mass shell and move through space-time described by:

$$\vec{x} = \frac{\vec{k}}{\omega(k)} x^0 = \frac{\vec{p}}{p^0} ct \tag{16}$$

This is the first law of Newton. The particles move in straight lines with constant speed.

3.3 Result

For an adiabatic source to have any lasting effect it is necessary to create particles that are on the mass shell and that these particles move through space-time following equation 16. The first law of Newton is true for this case. The fact that the particle has to be on the mass shell is due to the interaction of the Feynman propagator and the source. The trajectory of the particle is due to the requirement of stationary phase. The total field looks like figure 5. We see indeed that with these values $\vec{x} \approx \vec{p}/p^0 ct = \frac{2t}{3}$ when the particles are on the mass shell. The fact that this is not precisely the case is because for this plot $\vec{p}$ was relatively low so that the integral over $\vec{k}$ can be taken from -100 to 100. However, therefore the stationary condition becomes important for later time and spacial distances. Furthermore we see that particles cannot travel faster than light. This is one of the properties of the Feynman propagator.

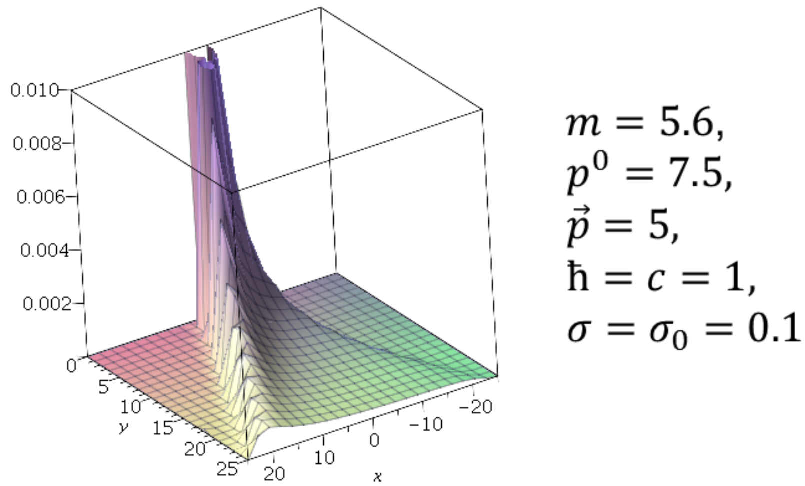


Figure 5: Adiabatic source acting on an operator field. $y = x^0$ and $x = \vec{x}$.

In figure 5 the absolute value of the response of the quantum field is depicted. The z-axis gives the probability to find the particle in that point of space-time. $\vec{p}$ is chosen to be in the same direction as $\vec{x}$. (This function is not normalised).

4 Gaussian source

The previous case of the Adiabatic source had a quite non-shocking result. But what happens if we take the time dependence of the source to be a non-adiabatic Gaussian too? In that case the source is:

$$J(y) \propto \exp\left(\frac{-|y^0|^2}{4\sigma_0^2} - \frac{|\vec{y}|^2}{4\sigma^2} - \frac{i}{\hbar}(p^0 y^0 - \vec{y}\vec{p})\right)$$

Or if we want it to depend on k we can Fourier transform it into:

$$J(k) \propto \exp\left(-\sigma_0^2\left(k^0 - \frac{p^0}{\hbar}\right)^2 - \sigma^2\left(\vec{k} - \frac{\vec{p}}{\hbar}\right)^2\right)$$

And therefore the integral for the field response on position x is given by:

$$\Phi(x) = \frac{i}{\hbar}\int_{-\infty}^{\infty} d^4k \frac{\exp\left(-\sigma_0^2\left(k^0 - \frac{p^0}{\hbar}\right)^2 - ik^0x^0 - \sigma^2\left(\vec{k} - \frac{\vec{p}}{\hbar}\right)^2 + i\vec{k}\vec{x}\right)}{(k^0)^2 - \omega(k)^2 + i\epsilon}$$

Because we now use a different source, it may not be necessary for this source to create particles that are on the mass shell. We have to re-analyse the field response for this source.

4.1 Approach to the problem

In the previous chapter we used the Cauchy residue theorem to calculate first the k^0-integral. Unfortunately in this case this is not possible. If we take a closer look at the exponential of the source we see that there is an $\exp(-\sigma_0^2 k^{0^2})$. This factor is problematic because, as shown in figure 6, if the integrand reaches large imaginary k^0 values the contribution of the integrand will become infinite. Therefore we cannot close the contour and another approach is needed. We consider the $(k^0)^2$ in the exponent and recognize the so-called W-function. The strategy is to rewrite the integral in a way that it is like the W-function (function 17) and then look what we can say about the possible values of z.

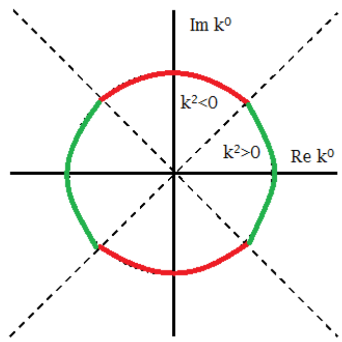


Figure 6: Complex k^0-plane where red areas have $k^{0^2} < 0$ and green $k^{0^2} > 0$.

$$W(z) = \frac{i}{\pi}\int_{-\infty}^{\infty} dt \frac{\exp(-t^2)}{z - t} = e^{-z^2}(1 - erf(-iz)), \qquad Im(z) > 0 \qquad (17)$$

4.2 Four-Dimensional case

The integral over k^0 can be written in the following way so that it resembles the form of the W-Function.

$$\Phi(x) = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} dk^0 \frac{\exp\left(-\sigma_0^2(k^0)^2 - \left(\frac{\sigma_0 p^0}{\hbar}\right)^2 + \frac{2\sigma_0^2 p^0 k^0}{\hbar} - ik^0x^0\right)}{(k^0)^2 - \omega(k)^2 + i\epsilon} F(\vec{k})$$

$$= \int_{-\infty}^{\infty} d^3k \exp\left(-\frac{(x^0)^2}{4\sigma_0^2} - \frac{ip^0x^0}{\hbar}\right) \int_{-\infty}^{\infty} dk^0 \frac{\exp\left(-\sigma_0^2\left(k^0 - \frac{p^0}{\hbar} + \frac{ix^0}{2\sigma_0^2}\right)^2\right)}{(k^0)^2 - \omega(k)^2 + i\epsilon} F(\vec{k})$$

$$F(\vec{k}) = exp\left(-\sigma^2\left(\vec{k} - \frac{\vec{p}}{\hbar}\right)^2 + i\vec{k}\vec{x}\right)$$

If we now want to substitute $u = k^0 - p^0/\hbar + ix^0/2\sigma_0^2$ we must take into account that we move the integral in the complex plane. In this case in the $-i$-direction for $x^0 > 0$ and in the $+i$-direction for $x^0 < 0$. This means that in the case of positive time the integral has crossed the pole $k^0 = \omega(k) - i\epsilon$ and for negative time the pole $k^0 = -\omega(k) + i\epsilon$. This is depicted in figure 7. We therefore get the following integral where $\Phi(x)$ is for positive times.

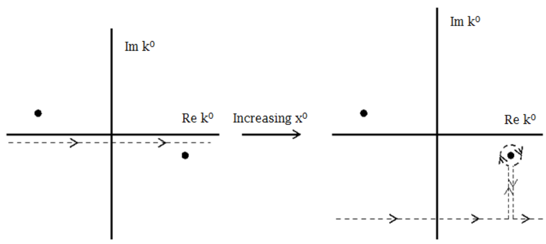


Figure 7: The change of the integral if x^0 is increasing and $x^0 > 0$.

$$\Phi(x) = \int_{-\infty}^{\infty} d^3k \exp\left(-\frac{(x^0)^2}{4\sigma_0^2} - \frac{ip^0x^0}{\hbar}\right) F(\vec{k}) \times \left(\oint_{-\epsilon} dk^0 \frac{\exp\left(-\sigma_0^2\left(k^0 - \frac{p^0}{\hbar} + \frac{ix^0}{2\sigma_0^2}\right)^2\right)}{(k^0 - \omega(k) + i\epsilon)(k^0 + \omega(k) - i\epsilon)} \right.$$

$$\left. - \int_{-\infty}^{\infty} du \frac{\exp\left(-\sigma_0^2 u^2\right)}{2\omega} \left(\frac{1}{u + \frac{p^0}{\hbar} - \frac{ix^0}{2\sigma_0^2} - \omega(k)} - \frac{1}{u + \frac{p^0}{\hbar} - \frac{ix^0}{2\sigma_0^2} + \omega(k)} \right) \right) \quad (18)$$

The circle integral is easily solved by the Cauchy residue theorem. And the integral over u will become more like the W-function. We substitute $t = \sigma_0 u$ to find for $x^0 > 0$:

$$\Phi(x) = \int_{-\infty}^{\infty} d^3k \exp\left(-\frac{(x^0)^2}{4\sigma_0^2} - \frac{ip^0x^0}{\hbar}\right) F(\vec{k}) \times \left[\frac{\exp\left(-\sigma_0^2\left(\omega(k) - \frac{p^0}{\hbar} + \frac{ix^0}{2\sigma_0^2}\right)^2\right)}{2\omega(k)} \right.$$

$$\left. - \int_{-\infty}^{\infty} dt \frac{\exp(-t^2)}{2\omega}\left(\frac{1}{t - \sigma_0\left(-\frac{p^0}{\hbar} + \frac{ix^0}{2\sigma_0^2} + \omega(k)\right)} - \frac{1}{t - \sigma_0\left(\frac{-p^0}{\hbar} + \frac{ix^0}{2\sigma_0^2} - \omega(k)\right)}\right)\right]$$

$$= \int_{-\infty}^{\infty} d^3k \frac{\exp\left(-\frac{(x^0)^2}{4\sigma_0^2} - \frac{ip^0x^0}{\hbar}\right)}{2\omega(k)} F(\vec{k}) \times \left[\exp\left(-\sigma_0^2\left(\omega(k) - \frac{p^0}{\hbar} + \frac{ix^0}{2\sigma_0^2}\right)^2\right) \right.$$

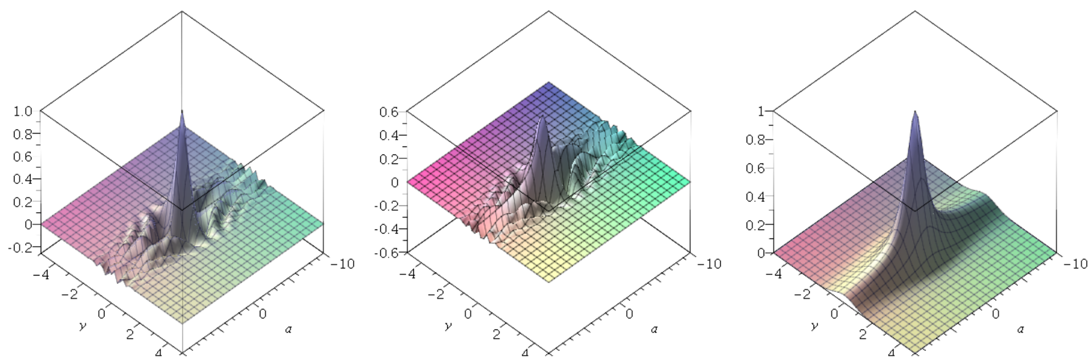
$$\left. - \frac{\pi}{i} W\left(-\sigma_0\left(\frac{p^0}{\hbar} + \omega(k)\right) + \frac{ix^0}{2\sigma_0}\right) + \frac{\pi}{i} W\left(-\sigma_0\left(\frac{p^0}{\hbar} - \omega(k)\right) + \frac{ix^0}{2\sigma_0}\right)\right] \tag{19}$$

And for $x^0 < 0$ we find:

$$\tilde{\Phi}(x) = \int_{-\infty}^{\infty} d^3k \frac{\exp\left(-\frac{(x^0)^2}{4\sigma_0^2} + \frac{ip^0|x^0|}{\hbar}\right)}{2\omega(k)} F(\vec{k}) \times \left[\exp\left(-\sigma_0^2\left(-\omega(k) - \frac{p^0}{\hbar} - \frac{i|x^0|}{2\sigma_0^2}\right)^2\right) \right.$$

$$\left. - \frac{\pi}{i} W\left(\sigma_0\left(\frac{p^0}{\hbar} - \omega(k)\right) + \frac{i|x^0|}{2\sigma_0}\right) + \frac{\pi}{i} W\left(\sigma_0\left(\frac{p^0}{\hbar} + \omega(k)\right) + \frac{i|x^0|}{2\sigma_0}\right)\right] \tag{20}$$

An interesting thing is that in both functions 19 and 20 the factor of $\exp(-(x^0/2\sigma_0)^2)$ extinguishes itself with the same speed as the source. The first therm in the square bracket counteracts this factor but what about the W-functions? We see that the part $\exp(-z^2)$ in equation 17 gives a factor of $\exp((x^0/2\sigma_0)^2)$in both cases. Therefore this part is not necessarily zero for large x^0. But the part of $1 - \text{erf}(-iz)$ in equation 17 can still become zero for large x^0. To illustrate this we can make a plot of the W-function multiplied with the exponential and get the following:



(a) Real part (b) Imaginary part (c) absolute value

Figure 8: plot of $W(-a + iy) * \exp(-(x^0/2\sigma_0)^2 - ip^0x^0/\hbar)$. $a = \sigma_0\frac{p^0}{\hbar} \pm \sigma_0\omega(k)$ and $y = (x^0/2\sigma_0)$.

These plots clearly show us that there is a spike around $a = 0$. This means that $p^0/\hbar \approx \pm\omega(k)$. But what also is clearly shown is that the response of the field goes to zero with time scale σ_0. This is before the source stops working. For this reason we can neglect the W-functions in equations 19 and 20. The surviving part of the equations are:

$$\Phi(x) = \int_{-\infty}^{\infty} d^3k \frac{\exp\left(-\sigma^2\left(\vec{k}-\frac{\vec{p}}{\hbar}\right)^2 - \sigma_0^2\left(\omega(k)-\frac{p^0}{\hbar}\right)^2 - i\Big(\omega(k)x^0 - \vec{k}\vec{x}\Big)\right)}{2\omega(k)} \tag{21}$$

$$\tilde{\Phi}(x) = \int_{-\infty}^{\infty} d^3k \frac{\exp\left(-\sigma^2\left(\vec{k}-\frac{\vec{p}}{\hbar}\right)^2 - \sigma_0^2\left(\omega(k)+\frac{p^0}{\hbar}\right)^2 - i\Big(\omega(k)|x^0| - \vec{k}\vec{x}\Big)\right)}{2\omega(k)} \tag{22}$$

The quantum field only gives a non-negligible response if $\vec{k} \approx \vec{p}/\hbar$ and $\omega(k) \approx \pm p^0/\hbar$. This will lead to the condition that the particles have to be on the mass shell. Furthermore the complex phase has to be stationary and therefore we find that particles move through space-time described by:

$$\vec{x} = \frac{\vec{k}}{\omega(k)}x^0 = \frac{\vec{p}}{p^0}ct \tag{23}$$

And hence we find the first law of Newton.

4.3 Result

For a Gaussian source to have any lasting effect it is necessary to create particles that are on the mass shell and that these particles move in space-time following equation 23. In figure 9 the absolute value of the response of the quantum field when a Gaussian source is acting upon it is depicted. This plot is not normalized and there is used that the particles has to be on the mass shell. There is shown that it is indeed most likely to travel in straight lines following more or less equation 23. The reason that it is not exactly this equation is because the chosen $\vec{p}$-value is relatively low so that the $\vec{k}$-integral can be chosen to run from -100 to 100. The stationary phase condition will then be less strong. Furthermore we see that particles cannot travel faster than the speed of light. This is one of the properties of the Feynman propagator.

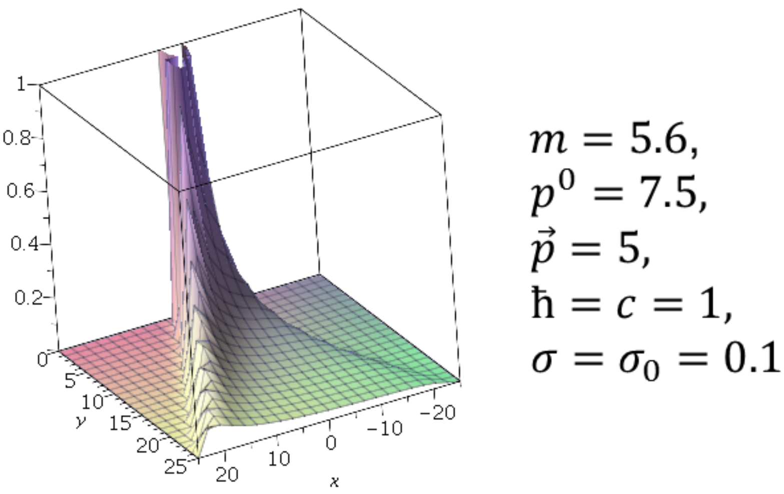


Figure 9: Gaussian source acting on an operator field. Here $\vec{p}$ is chosen to be in the same direction as $\vec{x}$, $y = x^0$ and $x = \vec{x}$.

5 Conclusion

In this chapter the question of whether Newton's first law is still true in quantum field theory will be answered. First the results of the two different sources will be compared and after that I will address the question.

5.1 Comparing the Results

If we compare the results of the Adiabatic and Gaussian source we find that for both sources only particles on the mass shell can propagate over macroscopic distances and survive longer than the duration of the source. The trajectories of these particles are straight lines. The fact that in both cases the particles are on the mass shell has to do with the interaction between the propagator and the source. The propagator links k^0 to $\omega(k)$ and both sources link it to the energy and momentum of the particle. The fact that the particles have to move in straight lines is completely due to the propagator and the fact that the particles are on the mass shell.

5.2 Does Newton's first law survive?

The first law of Newton does survive. I think it is reasonable to say that if a source requires the mass shell condition, its particles move following Newton's first law. By using the Feynman propagator and the flat minkowski space matric it is ensured that there is no force acting upon the particles and therefore the response of the quantum field for both sources are straight lines and that is exactly what the first law tells us.

5.3 Further possible investigations

- What if we use a field where there is a force acting. Can we find with this method other laws like $F = ma$?
- if we implement a gravitational field by using a different metric, can we find the gravitational force law?

References

[1] Prof.dr. R.H.P. Kleiss, Pictures Paths Particles Processes, Radboud University, Nijmegen, version of December 18, 2015.

[2] Prof. dr. W.J.P. Beenakker, Dictaat bij het college Kwantummechanica 3: de kwantummechanica van veeldeeltjessystemen, Radboud University, Nijemgen, Jaargang 2014-2015

[3] Mark Srednicki, Quantum Field Theory, University of California, Santa Barbara, 2006

[4] M. Abramowits and I. A. Stegun, Handbook of methematical functions,