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THEORETICAL HIGH ENERGY PHYSICS

Effectiveness of renormalisation in zero-dimensional $\varphi^3 + \varphi^4$ theory

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Abstract

In this research the asymptotic behaviour of the improvement factor of the numerical coefficients of the Connected Green's functions before and after renormalisation will be studied for φ^4 theory, φ^3 theory and a combination theory, $\varphi^3 + \varphi^4$ theory, that can function as the basis for a model for quantum chromodynamics. Using properties of factorially divergent series, which the Connected Green's functions are, a simple way of predicting the asymptotic of the improvement factor is found, $e^{\frac{-a}{c_f}}$. In order to find the two variables needed in the prediction, the series is cast into a new form using Stirling's approximation to find c_f and Feynman diagrams are used as an easy way to determine the first few Connected Green's functions used in the renormalisation process to eventually find a. For φ^4 theory this results in $e^{-\frac{15}{4}}$. For φ^3 this results in $e^{-\frac{10}{3}}$ without tadpole renormalisation and in $e^{-\frac{7}{3}}$ with tadpole renormalisation. For $\varphi^3 + \varphi^4$ theory, complications are discussed and a general form of c_f is found to be $c_f = \frac{(3-y)^3}{18(1-y)^2}$. The choice to renormalise λ_3 instead of λ_4 is made to work out a general form for a, resulting in $a = 5 + \frac{11}{6}z$ without tadpole renormalisation and $a = \frac{7}{2} + \frac{3}{2}z$ with tadpole renormalisation. Combining those, the improvement factors without and with tadpole renormalisation were found to be $e^{3\cdot\frac{-30+27y-19y^2}{(3-y)^3}}$ and $e^{9\cdot\frac{-7+5y-4y^2}{(3-y)^3}}$ respectively. An example is studied where y is chosen to be $\frac{1}{3}$, which results in an improvement factor of $e^{-\frac{117}{32}}$ without tadpole renormalisation and $e^{-\frac{351}{128}}$ with tadpole renormalisation.

Contents

1 Introduction

In this research a zero dimensional model for quantum field theory will be studied. In particular, the asymptotic behaviour of the improvement factor of the numerical coefficients of the Connected Green's functions before and after renormalisation will be predicted for some processes that have already been studied. This will also be studied for a process that might later function as the basis for quantum chromodynamics. This research was inspired by the research done by Michael Borinsky in [2], Dirk van Buul in [3] and Ilija Milutin in [4]. Where this was originally studied for multiple theories by Borinsky and later verified by explicit calculations for φ^4 theory by van Buul and for a model of quantum electrodynamics by Milutin. The next step was obviously to find a basis for studying a model for quantum chromodynamics, for which two theories need to be combined. I was interested in finding out how the underlying theories worked and what complications would arise in combining them. The approach in this research is mainly focused on the fact that the Connected Green's functions are factorially divergent series and using the properties of those series to study the asymptotic behaviour of the improvement factor.

2 Connected Green's functions and renormalisation

2.1 Quantum Field Theory

Quantum field theory (QFT) combines classical field theory with special relativity and quantum mechanics, and describes particles as an excited state of the underlying field. In our zero-dimensional case, fully explained in [1], the quantum field φ is a stochastic, it assigns a real number to a single point. Since this is a random variable we cannot say anything about it. However, something can be said about the probability density, $P(\varphi)$, and the collection of its moments, $G_n \equiv \langle \varphi^n \rangle$, called the Green's functions. The probability density can be written as

$$P(\varphi) \equiv N e^{-\frac{1}{\hbar} S(\varphi)} \tag{1}$$

where S is the action, specific to a quantum field theory, which we will come back to later. The normalization factor N can be written as follows

$$N^{-1} \equiv \int e^{-\frac{1}{\hbar} S(\varphi)} d\varphi \tag{2}$$

The Green's functions are then given by

$$G_n \equiv \langle \varphi^n \rangle \equiv N \int e^{-\frac{1}{\hbar} S(\varphi)} \varphi^n d\varphi \tag{3}$$

where the integral is taken from $-\infty$ to ∞[i] and G_0 must be 1. To make the function more manageable we define H_n as

$$H_n \equiv \int e^{-\frac{1}{\hbar} S(\varphi)} \varphi^n d\varphi, \tag{4}$$

so that of course $N = \frac{1}{H_0}$ and the Green's functions can be written as

$$G_n = \frac{H_n}{H_0}. \tag{5}$$

The same information contained in the Green's functions (moments) is also contained in the Connected Green's functions, C_n, which are the cumulants of $P(\varphi)$ and can be written as a function of powers of different Green's functions in the following way

$$C_n \equiv G_n - \sum_{m=1}^{n-1} \binom{n-1}{m-1} C_m G_{n-m} \, . \tag{6}$$

So, by looking at the Connected Green's functions something can be said about the quantum field theory under investigation. In order to do that we need to calculate H_n from which we can calculate G_n and eventually C_n. The easiest theory to take a look at is the free theory, where the action is defined as

$$S = \frac{\mu}{2} \varphi^2 \tag{7}$$

which is fairly easy to solve since we just get a Gaussian. It becomes a little more complicated when looking at higher order theories. For example, if we take a look at φ^4 theory we get an action of the form

$$S = \frac{\mu}{2} \varphi^2 + \frac{\lambda_4}{4!} \varphi^4 \, . \tag{8}$$

[i] This convention will be used throughout the thesis.

Since this is a lot more difficult to compute, we see this as a small deviation from the free theory. λ_4 is taken to be very small and perturbation theory is used to write

$$e^{-\frac{1}{\hbar}S(\varphi)} = e^{-\frac{\mu}{2\hbar}\varphi^2}\sum_{k\geq 0}\frac{1}{k!}\left(-\frac{\lambda_4}{24\hbar}\right)^k\varphi^{4k}\,. \tag{9}$$

Which can be inserted into the definition of H_n to yield

$$H_{2n} = \sum_{k\geq 0}\left(-\frac{\lambda_4}{24\hbar}\right)^k\left(\frac{\hbar}{\mu}\right)^{2k+n}\frac{(4k+2n)!}{(2k+n)!\,k!\,2^{2k+n}} \tag{10}$$

where we have used that

$$\int d\varphi\; e^{-\frac{\mu}{2\hbar}\varphi^2}\varphi^{2p} = \left(\frac{\hbar}{\mu}\right)^p\frac{(2p)!}{p!\,2^p}\,. \tag{11}$$

This only works with even powers of φ, hence the $2p$, in this case resulting in only even n and thus changing it to $2n$ in equation 10 for convenience. The same principle applies when looking at other theories. The power series expressions that are obtained for the Green's functions after this perturbation expansion are not convergent, but factorially divergent. This is an important property that we will get back to later.
In this thesis we will take a look at three different theories, φ^3, φ^4 and a combination theory. The actions are, respectively

$$S = \frac{\mu}{2}\varphi^2 + \frac{\lambda_3}{3!}\varphi^3 \tag{12}$$

$$S = \frac{\mu}{2}\varphi^2 + \frac{\lambda_4}{4!}\varphi^4 \tag{13}$$

and

$$S = \frac{\mu}{2}\varphi^2 + \frac{\lambda_3}{3!}\varphi^3 + \frac{\lambda_4}{4!}\varphi^4. \tag{14}$$

The H_n for φ^4 theory can be found in equation 10 and using the same principles as above we find H_n for φ^3 theory and the combination theory, respectively

$$H_n = \sum_{l\geq 0}\left(-\frac{\lambda_3}{6\hbar}\right)^l\left(\frac{\hbar}{\mu}\right)^{\frac{3l+n}{2}}\frac{(3l+n)!}{\left(\frac{3l+n}{2}\right)!\,l!\,2^{\frac{3l+n}{2}}}\theta(l+n\text{ even}) \tag{15}$$

and

$$H_n = \sum_{k,l\geq 0}\left(-\frac{\lambda_3}{6\hbar}\right)^l\left(-\frac{\lambda_4}{24\hbar}\right)^k\left(\frac{\hbar}{\mu}\right)^{\frac{3l+4k+n}{2}}\frac{(3l+4k+n)!}{\left(\frac{3l+4k+n}{2}\right)!\,l!\,k!\,2^{\frac{3l+4k+n}{2}}}\theta(l+n\text{ even}). \tag{16}$$

2.2 Renormalisation

Nothing is known about the parameters μ, λ_3 and λ_4 so the first few Connected Green's functions are used as measurement processes to determine these parameters, after which the rest of the Connected Green's functions can be predicted. In order to get the next higher order in perturbation theory for the prediction process, it needs to be done for the measuring process as well so a new fit of the parameters can be made and we get improved values for the prediction processes. This order by order improvement is called renormalisation. What we are then interested in is the improvement factor, the ratio between the coefficients of the Connected Green's functions before and after renormalisation. As can be seen in [2] and [3], in φ^4 theory, for high loop order the improvement factor goes asymptotically to a certain value for each of the Connected Green's functions. The same was shown for quantum electrodynamics in [2] and [4] as well. The rest of the thesis is focused on predicting these asymptotic values for three different theories.

3 Factorially divergent series

To understand the asymptotic behaviour of the improvement factor, it is fruitful to look at factorially divergent series because the Connected Green's functions are in this category, as can be seen in [1], [2] and references quoted therein. We shall take a look at this type of series and its properties in this section, based on their description in [1].

3.1 Comparisons

Consider $f(x)$ and $g(x)$, both asymptotic sums of the form

$$f(x)=\sum_{n\geq 0} x^n f_n\,, \tag{17}$$

where, for large n, we assume the coefficients f_n to be of the form

$$f_n \approx a_f c_f^n \Gamma(n+\beta_f)\,, \tag{18}$$

making $f(x)$ factorially divergent. We can then define some comparisons between these series $f(x)$ and $g(x)$,

$$f(x)\prec g(x) \qquad \text{if } \lim_{n\to\infty}\frac{f_n}{g_n}=0 \tag{19}$$

$$f(x)\succ g(x) \qquad \text{if } g(x)\prec f(x) \tag{20}$$

$$f(x)\sim g(x) \qquad \text{if } \lim_{n\to\infty}\frac{f_n}{g_n} \text{ is a finite nonzero constant} \tag{21}$$

$$f(x)\asymp g(x) \qquad \text{if } \lim_{n\to\infty}\frac{f_n}{g_n}=1\,. \tag{22}$$

$L(f,x)$ is defined to be the leading term of $f(x)$, it is the nonzero term in $f(x)$ with the lowest power of x. A series $f(x)$ is called proper factorially divergent (pfd) if $L(f,x)=1$.

3.2 Algebraics

Both $f(x)$ and $g(x)$ are factorially divergent, so

$$f(x)\succ g(x) \;\Rightarrow\; f(x)+g(x)\asymp f(x) \tag{23}$$

and

$$f(x)g(x)\asymp L(f,x)g(x)+L(g,x)f(x) \tag{24}$$

If $f(x)$ and $g(x)$ are pfd this changes into

$$f(x)g(x)\asymp f(x)+g(x)\,, \text{ leading to } \; f(x)^p\asymp p f(x) \tag{25}$$

If we then look at the case where both $f(x)$ and $g(x)$ are pfd and $g(x) \prec f(x)$ with $g_1 \neq 0$ we find

$$\begin{aligned} f\left(xg(x)\right) &= \sum_{n\geq 0} x^n f_n g(x)^n \\ &= \sum_{n\geq 0} x^n \left(f_n + f_{n-1}\left(g_1\binom{n}{1}\right) + f_{n-2}\left(g_1^2\binom{n}{2} + g_2\binom{n}{1}\right) + \dots\right) \\ &\asymp \sum_{n\geq 0} x^n \sum_{k=0}^{n} f_{n-k}\frac{n^k}{k!} g_1^k \\ &\asymp \sum_{n\geq 0} x^n f_n \sum_{k=0}^{n} \frac{1}{k!}\left(\frac{g_1}{c_f}\right)^k \\ f\left(xg(x)\right) &\asymp f(x)\; e^{\frac{g_1}{c_f}} \end{aligned} \tag{26}$$

where we used that for large n

$$f_{n-k}n^k \approx a_f c_f^{n-k}\Gamma(n-k+\beta_f)n^k \approx a_f c_f^{n-k}\Gamma(n+\beta_f)\,. \tag{27}$$

Another important thing to look at is inversion. If $f(x)$ is a pfd with $f_1 \neq 0$,

$$y = xf(x) \tag{28}$$

can be inverted to give

$$x = yg(y)\ , \tag{29}$$

where $g(x)$ is pfd with $g_1 = -f_1$ and

$$g(x) \asymp -f(x)\; e^{-\frac{f_1}{c_f}}\ . \tag{30}$$

3.3 Improvement factor

Every Connected Green's function can be written as a prefactor times a powerseries in u, where u is specific to each theory and dependent on $\hbar$ and the parameters μ, λ_3 and λ_4. This is denoted as

$$C_n = L(C_n,\hbar)\,\mathscr{C}_n(u) \tag{31}$$

where $\mathscr{C}_n(u)$ is pfd. In the renormalisation procedure u also gets expressed in terms of the renormalised parameters as $\hat{u}$[ii]. Using the lowest order Connected Green's functions[iii], $\hat{u}$ can be expressed as

$$\hat{u} = u\left(1+au+bu^2+\dots\right) \tag{32}$$

where the part between brackets is again pfd and we will denote it as $f(u)$, making this an equation of the form of equation 28. This means that we can invert this equation to yield

$$u = \hat{u}g(\hat{u}) \tag{33}$$

with $g_1 = -a$ and $g(u) \asymp -f(u)e^{-\frac{f_1}{c_f}}$. $\mathscr{C}_n(u(\hat{u}))$ is then of the form of equation 26, resulting in

$$\mathscr{C}_n(u(\hat{u})) \asymp \mathscr{C}_n(\hat{u})\; e^{\frac{-a}{c_f}}\ . \tag{34}$$

where the improvement factor is found to be $e^{\frac{-a}{c_f}}$.

[ii]The hat notation for renormalised parameters shall be used throughout the thesis.
[iii]This will be elaborated upon later.

4 Components of the improvement factor

In order to calculate the improvement factor we need to find a and c_f in equation 34.

4.1 Finding c_f

To find c_f as easily as possible some more properties of factorially divergent series are used. Just as with the Connected Green's functions, $H_n = L(H_n,\hbar)\mathscr{H}_n(u)$ and $G_n = L(G_n,\hbar)\mathscr{G}_n(u)$ with $\mathscr{H}_n(u)$ and $\mathscr{G}_n(u)$ pfd. Now we can use the comparisons in section 3.1 to find

$$\mathscr{H}_n+1 \succ \mathscr{H}_n \tag{35}$$

and

$$G_n = \frac{H_n}{H_0} = L(H_n,\hbar)\,\frac{\mathscr{H}_n(u)}{\mathscr{H}_0(u)} \asymp L(H_n,\hbar)\,\mathscr{H}_n(u) - \mathscr{H}_0(u) \asymp H_n \asymp H_0\,. \tag{36}$$

C_n is a combination of powers of G_m with $m = 2...n$, where the leading powers of $\hbar$ are the same in each term, meaning if we devide it out we get C_n in terms of $\mathscr{G}_m$. And since again

$$\mathscr{G}_n+1 \succ \mathscr{G}_n \tag{37}$$

and we get

$$C_n \asymp G_n \asymp H_n \asymp H_0\,. \tag{38}$$

This means that if we want to know the c_f for C_n, we can already find it by just looking at H_n or even just H_0, because all their c_f's are the same. Meaning H_n needs to be cast in the form of equation 17 in order to just read off the desired c_f.

4.2 Finding a

To find the a in equation 34, we take a look at the first few Connected Green's functions of our respective theory. Since a is the first non-trivial coefficient only the first loop order of the Connected Green's functions is needed, which is found easily using Feynman diagrams. We can express the Connected Green's funtions C_n in Feynman diagrams by writing out all connected diagrams with n external lines, where the Feynman rules shown in figure 1 are used.

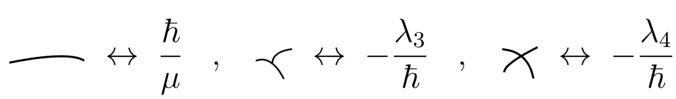

Figure 1: Feynman rules used in our model for φ^3, φ^4 and $\varphi^3+\varphi^4$ theory , taken from [1].

We use this to take a look at our first few Connected Green's functions for our different theories. Obviously for φ^3 theory we only have a three-point vertex and for φ^4 theory we only have a four-point vertex, whereas in the combination theory both are used.
The actual process of finding a is best shown using an example, here φ^4 theory is used. If we look at C_2, we get the series in figure 2 up to second loop order.

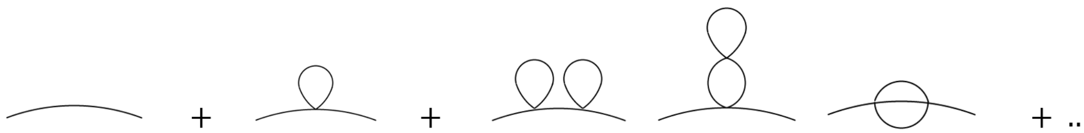

Figure 2: Diagrammatic representation of C_2 for φ^4 theory up to second loop order.

When taking symmetry factors and multiplicity into account this results in

$$C_2 = \frac{\hbar}{\mu} - \frac{\hbar^2\lambda_4}{2\mu^3} + \frac{\hbar^3\lambda_4^2}{\mu^5}\left(\frac{1}{4} + \frac{1}{4} + \frac{1}{6}\right) + \ldots = \frac{\hbar}{\mu}\left(1 - \frac{1}{2}u + \frac{2}{3}u^2 + \ldots\right) \tag{39}$$

with $u = \frac{\hbar\lambda_4}{\mu^2}$. The same can be done for C_4, for which the Feynman diagrams can be found in figure 3.

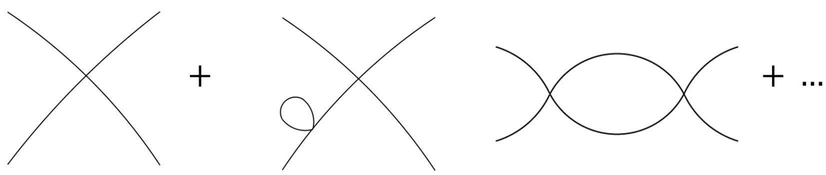

Figure 3: Diagrammatic representation of C_4 for φ^4 theory up to first loop order

The resulting series is then

$$C_4 = -\frac{\lambda_4\hbar^3}{\mu^4} + \frac{\lambda_4^2\hbar^4}{\mu^6}\left(\frac{4}{2} + \frac{3}{2}\right) + \ldots = -\frac{\hbar^3\lambda_4}{\mu^4}\left(1 - \frac{7}{2}u + \ldots\right). \tag{40}$$

In the renormalisation procedure these are set to be equal to their first term, but in renormalised parameters,

$$C_2 = \frac{\hbar}{\mu}\left(1 - \frac{1}{2}u + \frac{2}{3}u^2 + \ldots\right) = \frac{\hbar}{\hat{\mu}} \tag{41}$$

$$C_4 = -\frac{\hbar^3\lambda_4}{\mu^4}\left(1 - \frac{7}{2}u + \ldots\right) = -\frac{\hbar^3\hat{\lambda}_4}{\hat{\mu}^4}. \tag{42}$$

These can be used to determine $\hat{u}$,

$$\hat{u} = \frac{\hbar\hat{\lambda}_4}{\hat{\mu}^2} = -\frac{C_4}{C_2^2} = u\left(1 - \frac{5}{2}u + \ldots\right) \tag{43}$$

where we can just read off the constant a needed in equation 34 to be $-\frac{5}{2}$.

5 Improvement factors

In this section, we will work through finding the improvement factors for φ^4 theory, φ^3 theory and $\varphi^3+\varphi^4$ theory.

5.1 φ^4 theory

Since this theory was used as an example, a was already found to be $-\frac{5}{2}$. So now only c_f needs to be determined in order to calculate the improvement factor. As we have seen before,

$$H_{2n} = \sum_{k\geq 0}\left(-\frac{\lambda_4}{24\hbar}\right)^k \left(\frac{\hbar}{\mu}\right)^{2k+n} \frac{(4k+2n)!}{(2k+n)!\,k!\,2^{2k+n}}\,. \tag{44}$$

In order to find the c_f, H_{2n} needs to be of the form of equation 17 with coefficients of the form of equation 18. First, H_{2n} is rewritten as

$$H_{2n} = L(H_{2n},\hbar)\,\mathscr{H}_{2n}(u) = \left(\frac{\hbar}{\mu}\right)^n \frac{(2n)!}{n!\,2^n}\mathscr{H}_{2n}(u) \tag{45}$$

where $\mathscr{H}_{2n}$ is

$$\mathscr{H}_{2n} = \sum_{k\geq 0}\left(-\frac{u}{96}\right)^k \frac{(4k+2n)!}{(2k+n)!\,k!}\frac{n!}{(2n)!} \tag{46}$$

with of course $u=\frac{\hbar\lambda_4}{\mu^2}$. Then, in order to find the gamma function involved, we make an educated guess and use Stirling's approximation to see if it was correct and what factors are still missing. For the educated guess, one needs to look at the factorials where k becomes very large, so for a moment we ignore the ones with just n, in this case $\frac{(4k+2n)!}{(2k+n)!\,k!}$. For the guess we say $4k-2k-k=k$ and $2n-n=n$ so we guess $\Gamma(k+n)$. Turning our $\mathscr{H}_{2n}$ into

$$\mathscr{H}_{2n} = \sum_{k\geq 0}\left(-\frac{u}{96}\right)^k \Gamma(k+n)\frac{n!}{(2n)!}\frac{(4k+2n)!}{(2k+n)!\,k!\,\Gamma(k+n)} \tag{47}$$

$$\asymp \sum_{k\geq 0}\left(-\frac{u}{96}\right)^k \Gamma(k+n)\frac{n!}{(2n)!}\frac{64^k 8^n}{\sqrt{2}\pi} \tag{48}$$

$$= \sum_{k\geq 0}\left(-\frac{2u}{3}\right)^k \Gamma(k+n)\frac{n!}{(2n)!}\frac{8^n}{\sqrt{2}\pi} \tag{49}$$

$$\tag{50}$$

where Stirling's approximation was used. For a full derivation, see appendix A.1. We can now just read off $c_f=-\frac{2}{3}$. Consequently we can determine the improvement factor to be

$$e^{\frac{-a}{c_f}} = e^{\frac{5/2}{-2/3}} = e^{-\frac{15}{4}}\,. \tag{51}$$

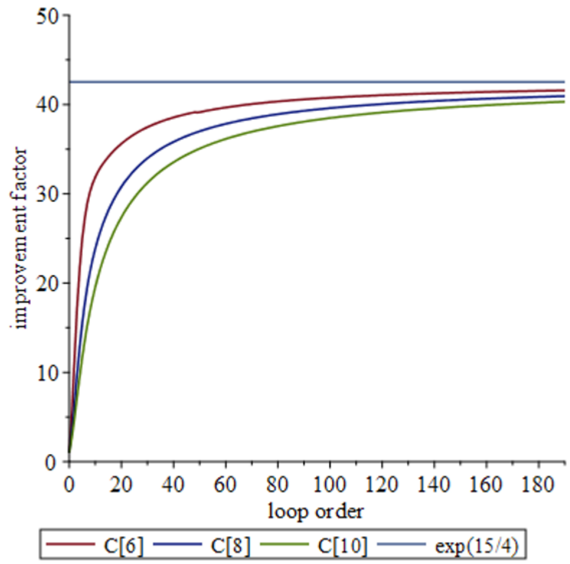

Figure 4: The improvement factors for φ^4 theory for increasing loop order for C_6, C_8 and C_{10} plotted with the predicted asymptotic value for the improvement factor.

In figure 4[iv] this prediction for the asymptotic value of the improvement factors is plotted together with the improvement factors for increasing loop order for C_6, C_8 and C_{10}. We see that the prediction is fairly accurate. It corresponds to what was found in [2] and [3]. If we take different Feynman rules, where the four-point vertex with λ_4 is positive instead of negative, the absolute values stay the same but c_f becomes positive and $-a$ becomes negative, resulting in the same improvement factor. We conclude from this that taking Feynman rules where the four-point vertex with λ_4 is positive has no effect on the resulting improvement factor and can therefore be done if desired.

[iv] In all the figures showing the asymptotic behaviour of the improvement factors included in this thesis, the improvement factor is taken as $\frac{\mathscr{C}(\hat{u})}{\mathscr{C}(u)}$, which results in taking away the minus sign in the exponent.

5.2 φ^3 theory

φ^3 theory varies somewhat from φ^4 theory in that it contains tadpoles. In our zero dimensional case that is not a problem, however when looking at higher dimensions sometimes these need to be divided out because they are unphysical. This process is called tadpole renormalisation and is done to make sure the expectation value of the vacuum field has the desired value. Since this is quite an important procedure that is often used, the expectation value shall be investigated both with and without tadpole renormalisation. However, the steps are very similar.

5.2.1 Without tadpole renormalisation

Again, we need to find a and c_f and enter them into our equation for the improvement factor $e^{\frac{-a}{c_f}}$. We shall start with finding a, for which we first take a look at C_2. In figure 5 a diagrammatic representation can be found.

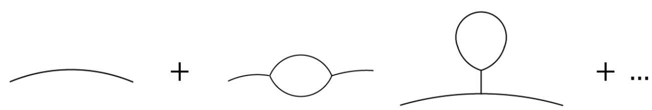

Figure 5: Diagrammatic representation of C_2 for φ^3 theory up to first loop order without tadpole renormalisation.

When taking symmetry factors and multiplicity into account this results in

$$C_2 = \frac{\hbar}{\mu} + \frac{\lambda_3^2 \hbar^2}{\mu^4}\frac{2}{2} + \ldots = \frac{\hbar}{\mu}\left(1 + u + \ldots\right) = \frac{\hbar}{\hat{\mu}} \tag{52}$$

with $u = \frac{\lambda_3^2 \hbar}{\mu^3}$. Now, the same is done for C_3, for which the diagrammatic representation can be found in figure 6.

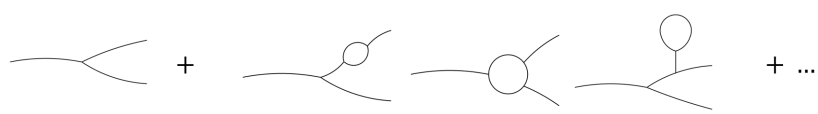

Figure 6: Diagrammatic representation of C_3 for φ^3 theory up to first loop order without tadpole renormalisation.

The resulting series is then

$$C_3 = -\frac{\lambda_3 \hbar^2}{\mu^3} - \frac{\lambda_3^3 \hbar^3}{\mu^6}\left(\frac{2 \cdot 3}{2} + 1\right) + \ldots = -\frac{\lambda_3 \hbar^2}{\mu^3}\left(1 + 4u + \ldots\right) = -\frac{\hat{\lambda}_3 \hbar^2}{\hat{\mu}^3}\,. \tag{53}$$

These are then used to determine $\hat{u}$,

$$\hat{u} = \frac{\hat{\lambda}_3^2 \hbar}{\hat{\mu}^3} = \frac{C_3^2}{C_2^3} = u\left(1 + 5u + \ldots\right) \tag{54}$$

so we end up with $a = 5$. Next, c_f needs to be determined, for which H_n needs to be of the form of equation 17, with coefficients of the form of equation 18. As seen before,

$$H_n = \sum_{l \geq 0} \left(-\frac{\lambda_3}{6\hbar}\right)^l \left(\frac{\hbar}{\mu}\right)^{\frac{3l+n}{2}} \frac{(3l+n)!}{\left(\frac{3l+n}{2}\right)!\, l!\, 2^{\frac{3l+n}{2}}} \theta(l + n \text{ even}) \tag{55}$$

and since $l+n$ needs to be even, we have two possibilities that we will look at separately, $n = 2p$ and $l = 2r$ or $n = 2p-1$ and $l = 2r+1$[v]. In the first case, H_n becomes

$$H_{2p} = \left(\frac{\hbar}{2\mu}\right)^p \sum_{r\geq 0} \left(\frac{\hbar\lambda_3^2}{288\mu^3}\right)^r \frac{(6r+2p)!}{(3r+p)!(2r)!} \tag{56}$$

$$= L\big(H_{2p},\hbar\big)\mathscr{H}_{2p}(u) = \left(\frac{\hbar}{2\mu}\right)^p \frac{(2p)!}{p!}\mathscr{H}_{2p}(u) \tag{57}$$

where $\mathscr{H}_{2p}(u)$ is

$$\mathscr{H}_{2p}(u) = \sum r \geq 0 \left(\frac{u}{288}\right)^r \frac{(6r+2p)!}{(3r+p)!(2r)!}\frac{p!}{(2p)!}\,. \tag{58}$$

For our Gamma function we guess $\Gamma(r+p)$, resulting in

$$\mathscr{H}_{2p}(u) = \sum r \geq 0 \left(\frac{u}{288}\right)^r \Gamma(r+p)\frac{p!}{(2p)!}\frac{(6r+2p)!}{(3r+p)!(2r)!\Gamma(r+p)} \tag{59}$$

$$\asymp \sum r \geq 0 \left(\frac{u}{288}\right)^r \Gamma(r+p)\frac{p!}{(2p)!}\frac{432^r 12^p}{2\pi} \tag{60}$$

$$= \sum r \geq 0 \left(\frac{3u}{2}\right)^r \Gamma(r+p)\frac{p!}{(2p)!}\frac{12^p}{2\pi} \tag{61}$$

where Stirling's approximation was used. For a full derivation, see appendix A.2. We can now just read off $c_f = \frac{3}{2}$. We shall now look at the case where $n = 2p-1$ and $l = 2r+1$ to see whether it results in a different c_f. In this case H_n becomes

$$H_{2p-1} = \left(-\frac{\lambda_3}{12\mu}\right)\left(\frac{\hbar}{2\mu}\right)^p \sum_{r\geq 0} \left(\frac{\hbar\lambda_3^2}{288\mu^3}\right)^r \frac{(6r+2p+2)!}{(3r+p+1)!(2r+1)!} \tag{62}$$

$$= L\big(H_{2p-1},\hbar\big)\mathscr{H}_{2p-1}(u) = \left(-\frac{\lambda_3}{12\mu}\right)\left(\frac{\hbar}{2\mu}\right)^p \frac{(2p+2)!}{(p+1)!}\mathscr{H}_{2p-1}(u) \tag{63}$$

where $\mathscr{H}_{2p-1}(u)$ is

$$\mathscr{H}_{2p-1}(u) = \sum r \geq 0 \left(\frac{u}{288}\right)^r \frac{(6r+2p+2)!}{(3r+p+1)!(2r+1)!}\frac{(p+1)!}{(2p+2)!}\,. \tag{64}$$

For our Gamma function we guess again $\Gamma(r+p)$, resulting in

$$\mathscr{H}_{2p-1}(u) = \sum r \geq 0 \left(\frac{u}{288}\right)^r \Gamma(r+p)\frac{(p+1)!}{(2p+2)!}\frac{(6r+2p+2)!}{(3r+p+1)!(2r+1)!\Gamma(r+p)} \tag{65}$$

$$\asymp \sum r \geq 0 \left(\frac{u}{288}\right)^r \Gamma(r+p)\frac{(p+1)!}{(2p+2)!}\frac{432^r 12^p 3}{\pi} \tag{66}$$

$$= \sum r \geq 0 \left(\frac{3u}{2}\right)^r \Gamma(r+p)\frac{(p+1)!}{(2p+2)!}\frac{12^p 3}{\pi} \tag{67}$$

where Stirling's approximation was used. For a full derivation, see appendix A.2. Again we find $c_f = \frac{3}{2}$, for our purpose the two approaches give the same result. Consequently we can determine the improvement factor to be

$$e^{\frac{-a}{c_f}} = e^{\frac{-5}{3/2}} = e^{-\frac{10}{3}}\,. \tag{68}$$

This corresponds to what has been found in [2].

[v] l cannot become smaller than 0.

5.2.2 With tadpole renormalisation

First we look for the different value for a. When looking at the diagrammatic representations, found in figure 7 and 8, we see a slight change due to the tadpole renormalisation.

Figure 7: Diagrammatic representation of C_2 for φ^3 theory up to first loop order with tadpole renormalisation.

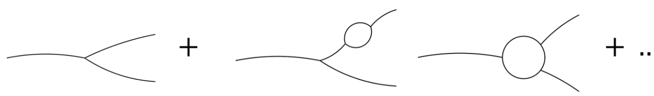

Figure 8: Diagrammatic representation of C_3 for φ^3 theory up to first loop order with tadpole renormalisation.

This also results in a slight change in the formulas

$$C_2 = \frac{\hbar}{\mu} + \frac{\lambda_3^2\hbar^2}{\mu^4}\frac{1}{2} + \ldots = \frac{\hbar}{\mu}(1 + +\ldots) = \frac{\hbar}{\hat{\mu}} \tag{69}$$

$$C_3 = -\frac{\lambda_3\hbar^2}{\mu^3} - \frac{\lambda_3^3\hbar^3}{\mu^6}\left(\frac{3}{2}+1\right) + \ldots = -\frac{\lambda_3\hbar^2}{\mu^3}\left(1 + \frac{5}{2}u + \ldots\right) = -\frac{\hat{\lambda}_3\hbar^2}{\hat{\mu}^3}\,. \tag{70}$$

$$\tag{71}$$

In the same way as before $\hat{u}$ is determined

$$\hat{u} = \frac{\hat{\lambda}_3^2\hbar}{\hat{\mu}^3} = \frac{C_3^2}{C_2^3} = u\left(1 + \frac{7}{2}u + \ldots\right)\,, \tag{72}$$

resulting in $a = \frac{7}{2}$. To determine c_f here, we include a tadpole counterterm in the action,

$$S(\varphi) = \frac{\mu}{2}\varphi^2 - \frac{\lambda_3}{6}\varphi^3 + T\varphi\,. \tag{73}$$

Where obviously we need to find out more about this extra term. T is tuned such that the tadpoles vanish so not only is $G_0 = 1$ as always, but also $G_1 = 0$. We can express all Green's functions in terms of T, for example $G_2 = \frac{2}{\lambda}T$. We already know that $L(G_2,\hbar) = \frac{\hbar}{\mu}$, from which we can determine $L(T,\hbar) = \frac{\hbar\lambda_3}{2\mu}$. Where we see that T starts with $\hbar$, meaning $T^2 \prec T$. In this way, using the other Green's functions, T can be determined up to increasing powers of $\hbar$. However, this is a lengthy task so we shall use a different method. We start again with a different field with a different action,

$$S(\psi) = \frac{m}{2}\psi^2 - \frac{\lambda_3}{6}\psi^3\,. \tag{74}$$

Where we look at the Schwinger-Dyson equation[vi] with source J,

$$m\psi(J) = J + \frac{\lambda_3}{2}\left(\psi(J)^2 + \hbar\psi'(J)\right)\,. \tag{75}$$

[vi] For an explanation of the origin of this equation, see [1].

This field ψ containes a tadpole, $t=\psi(0)$, so if we take $\phi(J)=\psi(J)-t$ it is free of tadpoles. When substituted into equation 75, we find

$$(m-\lambda_3 T)\phi(J)=J+\frac{\lambda_3}{2}\left(\phi(J)^2+\hbar\phi'(J)\right)-(mt-\frac{1}{2}\lambda_3 t^2)\ . \tag{76}$$

Where the tadpole-free action can be found if $\mu=m-\lambda_3 t$ and $T=mt-\frac{1}{2}\lambda_3 t^2$ are chosen. If $J=0$ is put into equation 75, a relation for t is found.

$$mt=\frac{\lambda_3}{2}\left(t^2+C_2\right)=\frac{\lambda_3}{2}\left(t^2+\frac{\hbar}{m}+\frac{\hbar\lambda_3}{m}\frac{\partial}{\partial m}t\right) \tag{77}$$

From its Feynman diagram, we know t must be of the form $t=\frac{\hbar\lambda_3}{2m^2}\tau(x)$ with $x=\frac{\hbar\lambda_3^2}{m^3}$. This can be put into the former equation, resulting in

$$\tau(x)=1+x\tau(x)+\frac{1}{4}x\tau(x)^2+\frac{3}{2}x^2\tau'(x)\ , \tag{78}$$

where it is easy to see that $\tau=1+\frac{5}{4}+...$, $c_\tau=\frac{3}{2}$ and $\beta_\tau=1$ and τ can be written as

$$\tau(x)\asymp\sum_n A\left(\frac{3}{2}\right)^n\Gamma(n+1)x^n\ . \tag{79}$$

Here, τ is still a function of x, which in turn is a function of m, but we want to know it as a function of μ, for which we use our regular $u=\frac{\hbar\lambda_3^2}{\mu^3}$. To make this switch, u is determined as a function of x as

$$u=x\left(1+\frac{x}{2}\tau(x)\right)^{-3}\asymp\sum_n A\left(\frac{3}{2}\right)^n\Gamma(n)x^n\ , \tag{80}$$

which is then inverted to yield $x=ug(u)$, where we have

$$g(u)\asymp-e^{-1}\sum_n A\left(\frac{3}{2}\right)^n\Gamma(n)u^n\prec\tau(u) \tag{81}$$

so $\tau(x)$ becomes

$$\tau(x)=\tau(ug(u))\asymp e^{-1}\tau(u) \tag{82}$$

meaning c_τ stays the same. Now with this $\tau(u)$ the original tadpole counterterm T can be determined as follows.

$$T=mt-\frac{\lambda_3}{2}t^2=\frac{\hbar\lambda_3}{2m}\tau(x)\left(1-\frac{x}{4}\tau(x)\right)\asymp e^{-1}\frac{\hbar\lambda_3}{2\mu}\tau(u) \tag{83}$$

which does not change the $c_T=\frac{3}{2}$. Now the H_n can be written down including this tadpole counterterm,

$$H_n=\sum_{l,k\geq 0}\left(\frac{\lambda_3}{6\hbar}\right)^l\left(-\frac{T}{\hbar}\right)^k\left(\frac{\hbar}{\mu}\right)^q\frac{1}{l!k!}\frac{(2q)!}{q!2^q}\theta(2q\text{ even}) \tag{84}$$

where $q=(3l+k+n)/2$. Since $c_T=\frac{3}{2}$ and $c_f=\frac{3}{2}$ without tadpole renormalisation, now $c_f=\frac{3}{2}$ as well. This is no surprise since full tadpole renormalisation and one-loop tadpole renormalisation result in the same asymptotic, as demonstrated in [1] and [4].

Since c_f is the same as without tadpole renormalisation, we get an improvement factor of

$$e^{\frac{-a}{c_f}}=e^{\frac{-7/2}{3/2}}=e^{-\frac{7}{3}}\ . \tag{85}$$

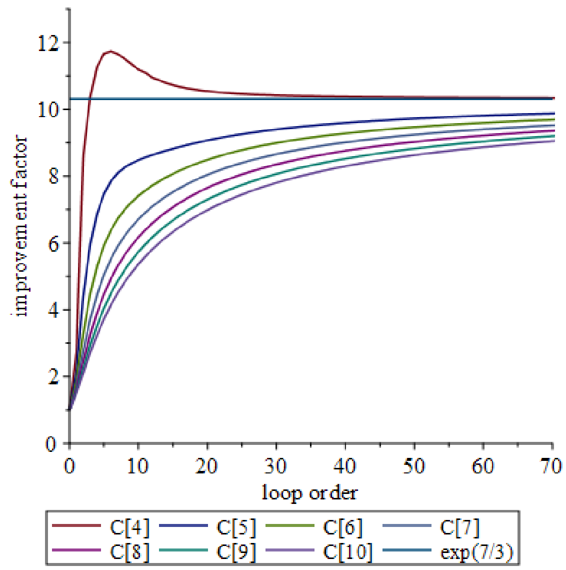

Figure 9: The improvement factors for φ^3 theory with tadpole renormalisation for increasing loop order for C_n, $n = 4, \ldots, 10$, plotted with the predicted asymptotic value for the improvement factor.

In figure 9 this prediction for the asymptotic value of the improvement factors is plotted together with the improvement factors for increasing loop order for C_n, $n = 4, \ldots, 10$. We see that the prediction is fairly accurate and it corresponds to what has been previously found in [2] and [4].

5.3 $\varphi^3+\varphi^4$ theory

In the combination theory we have both three-point vertices and four-point vertices, resulting in terms with λ_3 and λ_4, making this more complicated than the previous theories. These vertices might be connected through a ratio of their respective u's, or they might not be connected at all. Another issue is the renormalisation process, if the two are connected, should we renormalise λ_3 or λ_4, or if they are not connected, should we renormalise both? For this theory, we shall use Feynman rules where the four-point vertex is positive instead of negative, this will not affect the outcome, as said in section 5.1. To find a general c_f, we must take a look at H_n, which is in a slightly changed form from equation 16 due to the choice of positive four-point vertex,

$$H_n = \sum_{k,l\geq 0}\left(-\frac{\lambda_3}{6\hbar}\right)^l\left(\frac{\lambda_4}{24\hbar}\right)^k\left(\frac{\hbar}{\mu}\right)^{\frac{3l+4k+n}{2}}\frac{(3l+4k+n)!}{\left(\frac{3l+4k+n}{2}\right)!\,l!\,k!\,2^{\frac{3l+4k+n}{2}}}\theta(l+n\text{ even})\,. \tag{86}$$

From φ^3 theory we know it does not matter what we choose to make $l+n$ even so we choose $n=2p$ and $l=2r$, resulting in

$$H_{2p} = \left(\frac{\hbar}{2\mu}\right)^p\sum_{r,k\geq 0}\left(\frac{\lambda_3^2\hbar}{288\mu^3}\right)^r\left(\frac{\lambda_4\hbar}{96\mu^2}\right)^k\frac{(6r+2p+4k)!}{(3r+p+2k)!\,(2r)!\,k!}\frac{\Gamma(r+p+k)}{\Gamma(r+p+k)} \tag{87}$$

$$= L\left(H_{2p},\hbar\right)\mathscr{H}_{2p} = \left(\frac{\hbar}{2\mu}\right)^p\frac{(2p)!}{p!}\mathscr{H}_{2p} \tag{88}$$

$$\tag{89}$$

with

$$\mathscr{H}_{2p} = \sum_{r,k\geq 0}\left(\frac{u_3}{288}\right)^r\left(\frac{u_4}{96}\right)^k\Gamma(r+p+k)\frac{p!}{(2p)!}\frac{(6r+2p+4k)!}{(3r+p+2k)!\,(2r)!\,k!\,\Gamma(r+p+k)} \tag{90}$$

Since we do not know if either of the u will dominate for large r and k, we define $r=s-k$ and rewrite

$$\begin{aligned}\mathscr{H}_{2p} &= \sum_{s\geq 0}\sum_{k=0}^{s}\left(\frac{u_3}{288}\right)^{s-k}\left(\frac{u_4}{96}\right)^k\Gamma(s+p)\frac{p!}{(2p)!}\frac{(6s+2p-2k)!}{(3s+p-k)!\,(2s-2k)!\,k!\,\Gamma(s+p)}\\ &= \sum_{s\geq 0}\left(\frac{u_3}{288}\right)^s\Gamma(s+p)\sum_{k=0}^{s}\left(\frac{3u_4}{u_3}\right)^k\frac{p!}{(2p)!}\frac{(6s+2p-2k)!}{(3s+p-k)!\,(2s-2k)!\,k!\,\Gamma(s+p)}\end{aligned} \tag{91}$$

For simplicity, we shall say $\frac{3u_4}{u_3}=z$ and we will look at $\mathscr{H}_0$,

$$\mathscr{H}_0 = \sum_{s\geq 0}\left(\frac{u_3}{288}\right)^s\Gamma(s)\sum_{k=0}^{s}(z)^k\frac{(6s-2k)!}{(3s-k)!\,(2s-2k)!\,k!\,\Gamma(s)}\,. \tag{92}$$

We now define

$$C_{s,k} = \frac{(6s-2k)!\,z^k}{(3s-k)!\,(2s-2k)!\,k!\,\Gamma(s)\,288^s} \tag{93}$$

and

$$F_s = \sum_{k=0}^{s}C_{s,k}\,, \tag{94}$$

which contains everything needed to determine c_f. We want it to not have great influence if we have one more λ_4 relative to λ_3, so we want $C_{s,k+1}/C_{s,k} \asymp 1$,

$$\begin{aligned}\frac{C_{s,k+1}}{C_{s,k}} &= \frac{(6s-2k-2)!\,(3s-k)!\,(2s-2k)!\,k!\,z}{(6s-2k)!\,(3s-k-1)!\,(2s-2k-2)!\,(k+1)!} \\ &\asymp \frac{(3s-k)\,(2s-2k)^2\,z}{(6s-2k)^2\,k} \\ &= \frac{(s-k)^2 z}{k(3s-k)}\end{aligned} \tag{95}$$

which is 1 at $k = ys$, with y a constant. where we can see that

$$z = \frac{y(3-y)}{(1-y)^2} \tag{96}$$

$$y = \frac{3+2z-\sqrt{9+8z}}{2(1+z)}\,. \tag{97}$$

$$\tag{98}$$

Then we again use Stirling's approximation to evaluate $C_{s,k}$, for a full derivation see appendix A.3, resulting in

$$\begin{aligned}C_{s,k} &= (2\pi)^{-\frac{3}{2}}\,s^{-\frac{1}{2}}\,(3-y)^{3s}\,(1-y)^{-2s}\,(y(1-y))^{-\frac{1}{2}}\,18^{-s} \\ &= \left(\frac{(3-y)^3}{18(1-y)^2}\right)^s \frac{1}{\sqrt{(2\pi)^3\,s\,(y(1-y))}}\,.\end{aligned} \tag{99}$$

Which we can then use to determine the asymptotic behaviour of F_s,

$$F_s \asymp C_{s,ys}\sqrt{\frac{2\pi}{|(\ln(C_{s,ys}))''|}} \tag{100}$$

where

$$(\ln(C_{s,k}))'' = \ln\frac{C_{s,k+1}C_{s,k-1}}{C_{s,k}^2} \tag{101}$$

for which we first determine

$$\frac{C_{s,k+1}C_{s,k-1}}{C_{s,k}^2} = \frac{(6s-2k+1)(2s-2k)(2s-2k-1)k}{(6s-2k-1)(2s-2k+2)(2s-2k+1)(k+1)}\,. \tag{102}$$

Then, taking a series expansion of the logarithm of this we find

$$\ln\frac{C_{s,k+1}C_{s,k-1}}{C_{s,k}^2} = -\frac{y+3}{s(y-3)(y-1)y} + \ldots \tag{103}$$

$$\begin{aligned}F_s &\asymp \left(\frac{(3-y)^3}{18(1-y)^2}\right)^s \frac{1}{\sqrt{(2\pi)^3\,s\,(y(1-y))}}\sqrt{\frac{2\pi s(y-3)(y-1)y}{y+3}} \\ &= \left(\frac{(3-y)^3}{18(1-y)^2}\right)^s \frac{1}{2\pi}\left(\frac{3-y}{3+y}\right)^{\frac{1}{2}}\end{aligned} \tag{104}$$

where $c_f = \frac{(3-y)^3}{18(1-y)^2}$ is found as a function of $y = \frac{k}{s}$. If this ratio is known, it can be entered into the formula for c_f. To check that the correct asymptotic value for F_s was found, the value from the calculation of $F_s = \sum_{k=0}^{s} C_{s,k}$ was compared to the asymptotic value that was found.

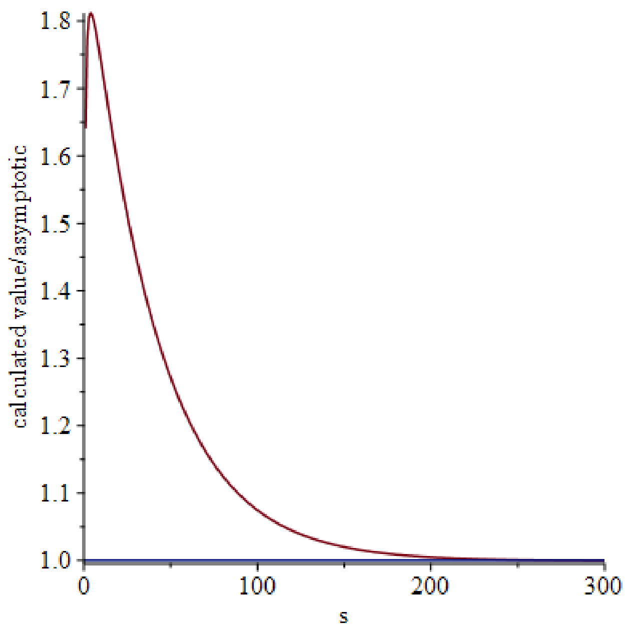

Figure 10: Comparison of the calculated value of F_s compared to the asymptotic.

A graph of this comparison can be found in figure 10. It can be seen that the calculated value matches the asymptotic very nicely, from which we conclude that our prediction is correct.
We can not determine a as straightforwardly as with the other theories since we need to choose what to renormalise on. If a choice is made, a can be determined as a function of y as well. We choose to renormalise λ_3 to show this derivation of a. The choice to renormalise λ_3 instead of λ_4 is based on a hunch that everything will work fine this way but the other way around may cause issues when looking at C_3. These issures might arise since for the renormalisation of λ_4 we need C_4 and renormalisation affects all the next Connected Green's functions but not the ones before it. However, it is important to keep in mind that this was not checked and the choice was made just to show the process.
In this theory, tadpoles play a role as well, so just as in φ^3 theory, we shall predict the improvement factor with and without tadpole renormalisation. As explained in section 5.2.2, this will not affect c_f, only a.

5.3.1 Without tadpole renormalisation

As always, first we consider the Feynman diagrams.

Figure 11: Diagrammatic representation of C_2 for $\varphi^3+\varphi^4$ theory up to first loop order.

In figure 11 a diagrammatic representation of C_2 can be found. When taking symmetry factors and multiplicity into account this results in

$$C_2 = \frac{\hbar}{\mu} + \frac{\lambda_4\hbar^2}{2\mu^3} + \frac{2}{2}\frac{\lambda_3^2\hbar^2}{\mu^4} + \ldots = \frac{\hbar}{\mu}\left(1+\frac{1}{2}u_4+u_3+\ldots\right) = \frac{\hbar}{\hat{\mu}} \tag{105}$$

where u_3 and u_4 are the same as before.

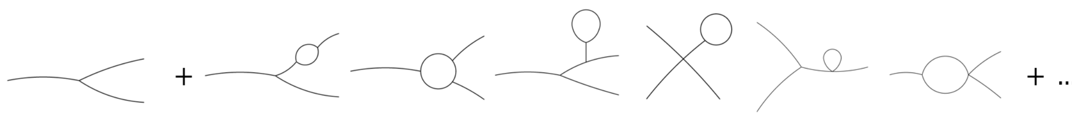

Figure 12: Diagrammatic representation of C_3 for $\varphi^3+\varphi^4$ theory up to first loop order.

In figure 12, the diagrammatic representation of C_3 can be found, for which we do the same as for C_2, resulting in

$$C_3 = -\frac{\lambda_3\hbar^2}{\mu^3} - \frac{\lambda_3^3\hbar^3}{\mu^6}\left(\frac{2\cdot 3}{2}+1\right) - \frac{\lambda_3\lambda_4\hbar^3}{\mu^5}\left(\frac{3}{2}+\frac{3}{2}+\frac{1}{2}\right) + \ldots = -\frac{\lambda_3\hbar^2}{\mu^3}\left(1+4u_3+\frac{7}{2}u_4+\ldots\right) = -\frac{\hat{\lambda}_3\hbar^2}{\hat{\mu}^3}\,. \tag{106}$$

Then, these are used to determine $\hat{u}_3$,

$$\hat{u}_3 = \frac{\hat{\lambda}_3^2\hbar}{\hat{\mu}^3} = \frac{C_3^2}{C_2^3} = u_3\left(1+5u_3+\frac{11}{2}u_4+\ldots\right) = u_3\left(1+\left(5+\frac{11}{6}z\right)u_3+\ldots\right) \tag{107}$$

where we used that $u_4 = \frac{z}{3}u_3$. Here, we find $a = 5+\frac{11}{6}z$ and using $z = \frac{y(3-y)}{(1-y)^2}$ we can determine

$$\frac{-a}{c_f} = \frac{-30+27y-19y^2}{6(1-y)^2}\cdot\frac{18(1-y)^2}{(3-y)^3} = 3\cdot\frac{-30+27y-19y^2}{(3-y)^3} \tag{108}$$

which leads to an improvement factor of $e^{3\cdot\frac{-30+27y-19y^2}{(3-y)^3}}$.
We will test our prediction by giving an example where we have set $z=2$, from which we can calculate $y=\frac{1}{3}$, resulting in an improvement factor of $e^{-\frac{117}{32}}$.

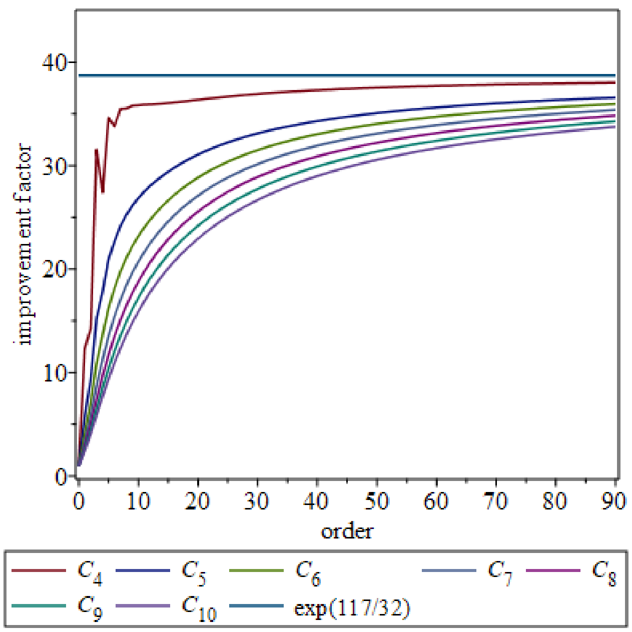

Figure 13: The improvement factors for $\varphi^3+\varphi^4$ theory without tadpole renormalisation for increasing loop order for C_n, $n = 4,...,10$, plotted with the predicted asymptotic value for the improvement factor.

In figure 13 the improvement factors for increasing loop order for C_n, $n = 4,...,10$, without tadpole renormalisation are plotted with the prediction for the asymptotic value of the improvement factor. Again, we see that the prediction is fairly accurate.

5.3.2 With tadpole renormalisation

Just as in φ^3 theory, tadpole renormalisation changes the first few Connected Green's functions in the following way.

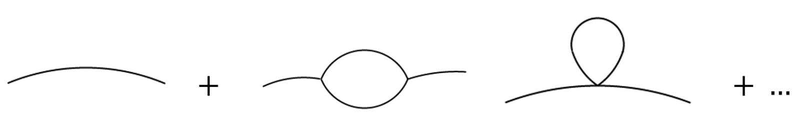

Figure 14: Diagrammatic representation of C_2 for $\varphi^3+\varphi^4$ theory up to first loop order when taking tadpole renormalisation into account.

To start, in figure 14 a diagrammatic representation of C_2 with tadpole renormalisation can be found. When taking symmetry factors and multiplicity into account this results in a slight change,

$$C_2 = \frac{\hbar}{\mu} + \frac{\lambda_4\hbar^2}{2\mu^3} + \frac{1}{2}\frac{\lambda_3^2\hbar^2}{\mu^4} + ... = \frac{\hbar}{\mu}\left(1 + \frac{1}{2}u_4 + \frac{1}{2}u_3 + ...\right) = \frac{\hbar}{\hat{\mu}} \tag{109}$$

where u_3 and u_4 are the same as before.

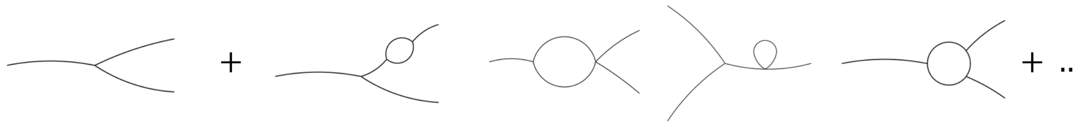

Figure 15: Diagrammatic representation of C_3 for $\varphi^3+\varphi^4$ theory up to first loop order when taking tadpole renormalisation into account.

In figure 15, the changed diagrammatic representation of C_3 when taking tadpole renormalisation is taken into account can be found, which leads to a slight change in the equation as well, resulting in

$$C_3 = -\frac{\lambda_3\hbar^2}{\mu^3} - \frac{\lambda_3^3\hbar^3}{\mu^6}\left(\frac{3}{2}+1\right) - \frac{\lambda_3\lambda_4\hbar^3}{\mu^5}\left(\frac{3}{2}+\frac{3}{2}\right) + \ldots = -\frac{\lambda_3\hbar^2}{\mu^3}\left(1+\frac{5}{2}u_3+3u_4+\ldots\right) = -\frac{\hat{\lambda}_3\hbar^2}{\hat{\mu}^3}\ . \tag{110}$$

These are once again used to determine $\hat{u}_3$,

$$\hat{u}_3 = \frac{\hat{\lambda}_3^2\hbar}{\hat{\mu}^3} = \frac{C_3^2}{C_2^3} = u_3\left(1+\frac{7}{2}u_3+\frac{9}{2}u_4+\ldots\right) = u_3\left(1+\left(\frac{7}{2}+\frac{3}{2}z\right)u_3+\ldots\right) \tag{111}$$

where we used $u_4 = \frac{z}{3}u_3$. Here, we find $a = \frac{7}{2}+\frac{3}{2}z$ and using $z = \frac{y(3-y)}{(1-y)^2}$ we can determine

$$\frac{-a}{c_f} = \frac{-7+5y-4y^2}{2(1-y)^2}\cdot\frac{18(1-y)^2}{(3-y)^3} = 9\cdot\frac{-7+5y-4y^2}{(3-y)^3} \tag{112}$$

which leads to an improvement factor of $e^{9\cdot\frac{-7+5y-4y^2}{(3-y)^3}}$.
The same example as before, with $z=2$ and $y=\frac{1}{3}$ is worked out, now taking tadpole renormalisation into account. The prediction for the improvement factor then becomes $e^{-\frac{351}{128}}$.

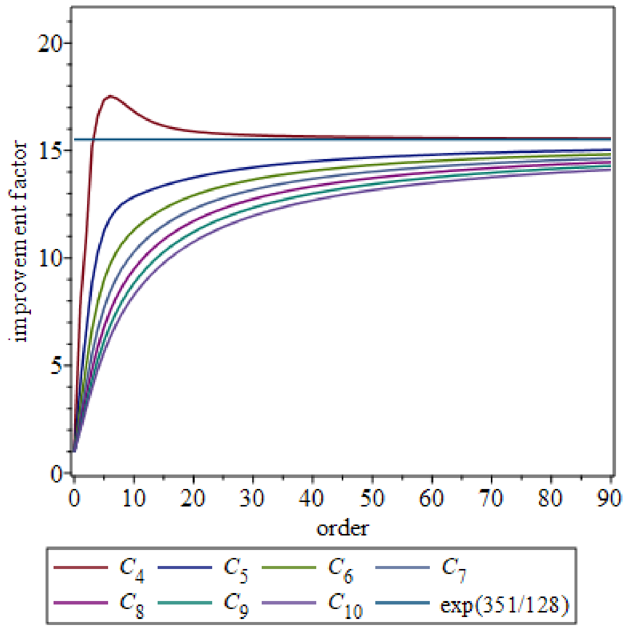

Figure 16: The improvement factors for $\varphi^3+\varphi^4$ theory with tadpole renormalisation for increasing loop order for C_n, $n = 4, \ldots, 10$, plotted with the predicted asymptotic value for the improvement factor.

In figure 16 the improvement factors for increasing loop order for C_n, $n = 4, \ldots, 10$, with tadpole renormalisation are plotted with the prediction for the asymptotic value of the improvement factor. As can be seen in this figure, our prediction for the improvement factor seems to be accurate.

6 Discussion

For $\varphi^3+\varphi^4$ theory it would be interesting to study the ratio between the couplings, y. As demonstrated, if a ratio is chosen the calculation of the improvement factor is fairly straightforward, however it might not make much sense if it is just chosen at random. Therefore it would be interesting to study what has the outcome closest to reality, perhaps in a quantum chromodynamics context.

For the same theory, we chose to renormalise λ_3 instead of λ_4, it would be interesting to see what impact this choice has and whether one is better than the other. A reason for the choice in this thesis was given, however that was not studied and it would be interesting to see if that reasoning is correct or not.

This combination theory, $\varphi^3+\varphi^4$ theory, is the basis for gluonic quantum chromodynamics (QCD), since it uses a three-point vertex and a four-point vertex. With the knowledge from our basic $\varphi^3+\varphi^4$ theory, a zero dimensional toy model for QCD can be studied, just as Michael Borinsky and Ilija Milutin did for quantum electrodynamics in [2] and [4] respectively. This might provide some insight on the best value of y to choose.

A Stirling's approximation

The two important things from Stirling's approximation we use are

$$\ln p! \asymp \left(p+\frac{1}{2}\right)\ln p - p + \ln\sqrt{2\pi} \tag{113}$$

$$\ln\Gamma(p) \asymp \left(p-\frac{1}{2}\right)\ln p - p + \ln\sqrt{2\pi} \tag{114}$$

for large p.

A.1 φ^4 theory

In equation 47 we take a look at the last part and therefore define

$$A \equiv \frac{(4k+2n)!}{(2k+n)!\,k!\,\Gamma(k+n)}\ , \tag{115}$$

since this is the part for which Stirling's approximation is used.

$$\begin{aligned}\ln A = &\left(4k+2n+\frac{1}{2}\right)\ln(4k+2n) - 4k - 2n + \ln\sqrt{2\pi}\\ &-\left(2k+n+\frac{1}{2}\right)\ln(2k+n) + 2k + n - \ln\sqrt{2\pi}\\ &-\left(k+\frac{1}{2}\right)\ln(k) + k - \ln\sqrt{2\pi}\\ &-\left(k+n-\frac{1}{2}\right)\ln(k+n) + k + n - \ln\sqrt{2\pi}\end{aligned} \tag{116}$$

$$\begin{aligned}\ln A = &\left(4k+2n+\frac{1}{2}\right)\left(\ln k + 2\ln 2 + \frac{n}{2k}\right) - 2n\\ &-\left(2k+n+\frac{1}{2}\right)\left(\ln k + \ln 2 + \frac{n}{2k}\right) + n\\ &-\left(k+\frac{1}{2}\right)\ln(k)\\ &-\left(k+n-\frac{1}{2}\right)\left(\ln k + \frac{n}{k}\right) + n\\ &-\ln 2\pi\end{aligned} \tag{117}$$

$$\begin{aligned}\ln A = &\ln k\left(4k+2n+\frac{1}{2}-2k-n-\frac{1}{2}-k-\frac{1}{2}-k-n+\frac{1}{2}\right)\\ &+2\ln 2\left(4k+2n+\frac{1}{2}\right)-\ln 2\left(2k+n+\frac{1}{2}\right)-\ln 2\pi\end{aligned} \tag{118}$$

$$\ln A = \ln 2\left(6k+3n+\frac{1}{2}\right)-\ln 2\pi \tag{119}$$

$$A \asymp 2^{6k}2^{3n}\sqrt{2}/2\pi = \frac{64^k 8^n}{\sqrt{2}\pi} \tag{120}$$

A.2 φ^3 theory

In equation 59 we take a look at the last part and just as in φ^4 theory, we define

$$A \equiv \frac{(6r+2p)!}{(3r+p)!\,(2r)!\,\Gamma(r+p)} \tag{121}$$

since this is the part for which Stirling's approximation is used.

$$\begin{aligned}\ln A = &\left(6r+2p+\frac{1}{2}\right)\ln(6r+2p)-6r-2p+\ln\sqrt{2\pi}\\ &-\left(3r+p+\frac{1}{2}\right)\ln(3r+p)+3r+p-\ln\sqrt{2\pi}\\ &-\left(2r+\frac{1}{2}\right)\ln(2r)+2r-\ln\sqrt{2\pi}\\ &-\left(r+p-\frac{1}{2}\right)\ln(r+p)+r+p-\ln\sqrt{2\pi}\end{aligned} \tag{122}$$

$$\begin{aligned}\ln A = &\left(6r+2p+\frac{1}{2}\right)\left(\ln r+\ln 6+\frac{p}{3r}\right)-2p\\ &-\left(3r+p+\frac{1}{2}\right)\left(\ln r+\ln 3+\frac{p}{3r}\right)+p\\ &-\left(2r+\frac{1}{2}\right)(\ln r+\ln 2)\\ &-\left(r+p-\frac{1}{2}\right)\left(\ln r+\frac{p}{r}\right)+p\\ &-\ln 2\pi\end{aligned} \tag{123}$$

$$\begin{aligned}\ln A = &\ln r\left(6r+2p+\frac{1}{2}-3r-p-\frac{1}{2}-2r-\frac{1}{2}-r-p+\frac{1}{2}\right)\\ &+\ln 3\left(6r+2p+\frac{1}{2}-3r-p-\frac{1}{2}\right)+\ln 2\left(6r+2p+\frac{1}{2}-2r-\frac{1}{2}\right)-\ln 2\pi\end{aligned} \tag{124}$$

$$\ln A = \ln 3\,(3r+p)+\ln 2\,(4r+2p)-\ln 2\pi \tag{125}$$

$$A \simeq 3^{3r}3^{p}2^{4r}2^{2p}/2\pi = 27^{r}16^{r}3^{p}4^{p}/2\pi = \frac{432^{r}12^{p}}{2\pi} \tag{126}$$

We want to do the same for H_{2p-1}, for which we take a look at the last part of equation 65 and again define

$$A \equiv \frac{(6r+2p+2)!}{(3r+p+1)!\,(2r+1)!\,\Gamma(r+p)} \tag{127}$$

since this is the part for which Stirling's approximation is used.

$$\begin{aligned}\ln A = &\left(6r+2p+\frac{5}{2}\right)\ln(6r+2p+2)-6r-2p-2+\ln\sqrt{2\pi}\\ &-\left(3r+p+\frac{3}{2}\right)\ln(3r+p+1)+3r+p+1-\ln\sqrt{2\pi}\\ &-\left(2r+\frac{3}{2}\right)\ln(2r+1)+2r+1-\ln\sqrt{2\pi}\\ &-\left(r+p-\frac{1}{2}\right)\ln(r+p)+r+p-\ln\sqrt{2\pi}\end{aligned} \tag{128}$$

$$\begin{aligned}
\ln A = & \left(6r + 2p + \frac{5}{2}\right)\left(\ln r + \ln 6 + \frac{p+1}{3r}\right) - 2p \\
& - \left(3r + p + \frac{3}{2}\right)\left(\ln r + \ln 3 + \frac{p+1}{3r}\right) + p \\
& - \left(2r + \frac{3}{2}\right)(\ln r + \ln 2) \\
& - \left(r + p - \frac{1}{2}\right)\left(\ln r + \frac{p}{r}\right) + p \\
& - \ln 2\pi
\end{aligned} \tag{129}$$

$$\begin{aligned}
\ln A = & \ln r \left(6r + 2p + \frac{5}{2} - 3r - p - \frac{3}{2} - 2r - \frac{3}{2} - r - p + \frac{1}{2}\right) \\
& + \ln 3 \left(6r + 2p + \frac{5}{2} - 3r - p - \frac{3}{2}\right) + \ln 2 \left(6r + 2p + \frac{5}{2} - 2r - \frac{3}{2}\right) - \ln 2\pi
\end{aligned} \tag{130}$$

$$\ln A = \ln 3 \left(3r + p + 1\right) + \ln 2 \left(4r + 2p + 1\right) - \ln 2\pi \tag{131}$$

$$A \asymp 3^{3r} 3^p 3 \cdot 2^{4r} 2^{2p} 2/2\pi = 27^r 16^r 3^p 4^p 6/2\pi = \frac{432^r 12^p 3}{\pi} \tag{132}$$

A.3 $\varphi^3+\varphi^4$ theory

The stirling approximation is used to evaluate $C_{s,k}$ in equation 93.

$$\begin{aligned}\ln C_{s,k} = k\ln z - \frac{3}{2}\ln 2\pi + \left(6s-2k+\frac{1}{2}\right)\left(\ln s + \ln 2 + \ln(3-y)\right) - 6s + 2k \\ - \left(3s-k+\frac{1}{2}\right)\left(\ln s + \ln(3-y)\right) + 3s - k \\ - \left(2s-2k+\frac{1}{2}\right)\left(\ln s + \ln 2 + \ln(1-y)\right) + 2s - 2k \\ - \left(k+\frac{1}{2}\right)\left(\ln s + \ln y\right) + k \\ - \left(s-\frac{1}{2}\right)\ln s + s - s\ln 288\end{aligned} \tag{133}$$

$$\begin{aligned}\ln C_{s,k} &= l\ln z - \frac{1}{2}\ln s - \frac{3}{2}\ln 2\pi \\ &+ \ln 2\left(6s-2k+\frac{1}{2}-2s+2k-\frac{1}{2}\right) \\ &+ \ln(3-y)\left(6s-2k+\frac{1}{2}-3s+k-\frac{1}{2}\right) \\ &- \ln(1-y)\left(2s-2k+\frac{1}{2}\right) - \ln y\left(k+\frac{1}{2}\right) - s\ln 288 \\ &= l\ln z - \frac{1}{2}\ln s - \frac{3}{2}\ln 2\pi \\ &+ 4s\ln 2 + \ln(3-y)\,(3s-k) - \left(2s-2k+\frac{1}{2}\right)\ln(1-y) - \left(k+\frac{1}{2}\right)\ln y - s\ln 2^4 - s\ln 18 \\ &= l\ln z - \frac{1}{2}\ln s - \frac{3}{2}\ln 2\pi \\ &+ s(3-y)\ln(3-y) - 2s(1-y)\ln(1-y) - \frac{1}{2}\ln(1-y) - \left(ys+\frac{1}{2}\right)\ln y - s\ln 18 \\ &= -\frac{1}{2}\ln s - \frac{3}{2}\ln 2\pi \\ &+ 3s\ln(3-y) - 2s\ln(1-y) - \frac{1}{2}\ln(y(1-y)) - s\ln 18 + ys\left(\ln z - \ln(3-y) + 2\ln(1-y) - \ln y\right) \\ &= -\frac{1}{2}\ln s - \frac{3}{2}\ln 2\pi + 3s\ln(3-y) - 2s\ln(1-y) - \frac{1}{2}\ln(y(1-y)) - s\ln 18\end{aligned} \tag{134}$$

$$C_{s,k} \asymp (2\pi)^{-\frac{3}{2}}\, s^{-\frac{1}{2}}\,(3-y)^{3s}\,(1-y)^{-2s}\,(y(1-y))^{-\frac{1}{2}}\,18^{-s} \tag{135}$$

References

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[3] Dirk van Buul, *"How effective is renormalization"*, bachelor thesis (2018).

[4] Ilija Milutin, *"Effectiveness of renormalization in zero-dimensional quantum electrodynamics"*, bachelor thesis (2019).