Radboud University Nijmegen

Faculty of Science

Gravitational waves emitted in an EMRI

Thesis BSc Physics and Astrophysics

Supervisor:
Béatrice Bonga

Author:
Jimmy Raijmann

Second reader:
Timothy Budd

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Abstract

A recent paper that considers an elemental integral series representation for Heun functions has shed new light on gravitational wave research. It offers a new promising approach to gravitational wave calculus and offers us a representation of the solution to the Teukolsky equation. In this thesis, we look at an Extreme Mass Ratio Inspiral, with one very light black hole spiralling on a Kerr spacetime. We will also derive the Teukolsky equation and use the new Heun approach to offer a representation of this equation. We demonstrate the essence of the Heun technique by giving an example and using the code written for this example to analyze the Teukolsky equation. We provide plots and data of this representation.

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1 Introduction

1.1 Concerning Gravity

For a long time it was thought that sir Isaac Newton created a theory of gravity that describes gravitational motion very accurately. This theory describes gravity as a force of attraction between two masses. Here on earth, this theory seems to hold for most physical cases. However, in more extreme environments the theory of Newtonian gravity shows some flaws. We already see this happening in our own solar system, which is a relatively non-extreme environment. For example, one of the tests of general relativity for light bending around our very own sun. This is why Einstein developed a new theory in 1915: The theory of general relativity. In modern physics this is still the best theory we have concerning gravity. The theory of general relativity states that massive objects curve space and time. They change the geometry of 'spacetime' this results in a 'gravitational attraction' between two massive objects.

In this thesis we will use some very important properties that directly follow from Einstein's theory of general relativity. A curved spacetime also takes the definition of a straight line into consideration (or the fastest route between two points in spacetime). This kind of motion can now be described by geodesic motion, which describes the trajectory of a free particle in a certain spacetime. Now that we have a tool to analyze motion in curved spacetimes, we still need to know how the curvature behaves and how we can calculate distances. Here we make use of the metric tensor (or line-element) of a spacetime. Combining these two properties, we can perform calculations on motion of free particles in certain spacetimes as we will also see later in this thesis.

1.2 Extreme environments

General relativity also predicts the existence of the most extreme environments in the universe: black holes. These are extremely compact objects, where the gravitational field is very strong. Another prediction of general relativity is the existence of gravitational waves, these are small ripples in the structure of spacetime. They can be formed by massive accelerating objects rotating around each other.

For a very long time, black holes were only a theoretical prediction. However, through the years more evidence on the existence of black holes has been collected. Our telescopes have collected a lot of data throughout the years that supports the existence of black holes. For example:

1. When we look at the center of our universe and observe the orbits of stars, we see orbits that imply the existence of a massive central object (i.e. black hole).
2. Looking at X-ray binaries, we can observe a black hole eating a nearby star emitting X-rays.
3. We have observed radiation coming from an accretion disc around a black hole.

This is of course very promising and is another hint towards the existence of black holes.

What really makes us believe that general relativity is still the best gravitational theory are the two major discoveries of this past decade. First in 2015 LIGO and

Virgo detected gravitational waves, which confirms one of the two major predicitons of general relativity. Therefore this was a huge step towards proving general relativity as a whole. A few years later in 2019 another huge breakthrough was confirmed by the Event Horizon Telescope team, they succeeded in making a picture of a black hole using some out of the box thinking.[1] This picture confirms the existence of black holes and shows us that general relativity's predictions still hold.

1.3 Extreme mass ratio inspirals

The detection of gravitational waves motivates our research. In this thesis we look at EMRIs (Extreme mass ratio inspirals). An EMRI is the situation where two black holes rotate around each other, but one of the black holes has a very large mass compared to the other black hole. While performing this inspiral, the two black holes create gravitational waves. These are not the same as the gravitational waves that LIGO and Virgo detected in 2015, the gravitational waves resulting from an EMRI have a way smaller frequency then the gravitational waves detected at LIGO and Virgo (30-1000 Hz). This is why in 2034 a new gravitational wave detector is set to be launched into space. LISA has way longer 'arms' than detectors on earth and is therefore able to detect different gravitational wavelengths. It would be a huge breakthrough if LISA would be able to detect gravitational waves coming from EMRIs.

1.4 Different spacetimes

Before all of the observable discoveries were made in the past century, a lot of theoretical work had already been done. This work, based on the work of Einstein gives us different solutions for different sorts of black holes. The first solution discovered in 1915 was the Schwarzschild solution, this solution discovered by Johannes Droste and Karl Schwarzschild is the most simple one and describes a static black hole. A few years later in 1918 the Reissner-Nordström solution was developed by Hans Reissner and Gunnar Nordström. This solution describes the spacetime of a static black hole, but this time with an electric field.

It is only until 1963 that Roy Kerr found a solution for rotating black holes. Many scientists believe this theoretical solution to be the one that describes reality the closest. Finally, just like with the Schwarzschild solution, there is also a solution for rotating black holes with charge. This solution is called the Kerr-Newman solution. Another reason why the Kerr solution seems to be the most likely one, is the fact that it is very difficult for extremely large objects to preserve a charge. Therefore it is thought that black holes described by the Reissner-Nordström and Kerr-Newman solutions do not even exist in our universe. In this thesis we will only use the Kerr spacetime and the properties following from the theory of general relativity.

1.5 Mathematical solution to a physical problem

The purpose of this research was to write a code with which we could analyze gravitational waves and make a working, faster program for future gravitational wave detection. Our research was boosted by a paper on Heun functions this past fall. This paper showed us a mathematical way to analyze gravitational waves using elementary functions and helped us interpret their properties. We used the techniques shown in this paper to write a code based on the mathematical foundation of differential equations and used our code to see if we could accurately analyze gravitational waves and compare them to their numerical solution.

2 The Kerr spacetime and its properties

The Kerr spacetime describes the fabric of space and time of a rotating, uncharged axially symmetric black hole. This makes it different and more difficult to interpret than the Schwarzschild spacetime, which concerns a non-rotating, uncharged spherically symmetric black hole. These properties, are of course approximations of a real physical situation. In a situation where we would have an actual black hole, it would not have constant values and have certain fluctuations that would alter the axial symmetry and rotation. The difficulty of this solution becomes even more clear when we look at the years in which both spacetimes where solved for the first time, for Schwarzschild the solution was discovered in 1915. However for the Kerr spacetime it was only discovered in 1963. In Boyer-Lindquist coordinates (t, r, θ, ϕ), the line-element of this spacetime takes the following form

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \left(\frac{4Mar\sin^2\theta}{\Sigma}\right)dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \\ &+ \left((r^2 + ct^2)^2 - a^2\sin^2\theta\Delta\right)\frac{\sin^2\theta}{\Sigma}d\phi^2, \end{aligned} \tag{1}$$

where

$$\begin{aligned} \Sigma &:= r^2 + a^2\cos^2\theta \\ \Delta &:= r^2 - 2Mr + a^2 \end{aligned} \tag{2}$$

This solution has some very astonishing properties. For example, the Kerr spacetime does not have one, but two horizons, where one is actually inside of the black hole. This property can eventually make us exit the black hole into another universe (although very interesting, not the main subject of this thesis). In this chapter we look at some of the properties of the Kerr spacetime relevant to this thesis. [2]

2.1 Geodesic motion

2.1.1 The Hamilton-Jacobi equation

To help us gain a better understanding, we start by looking at some of the most interesting and most useful properties of these black holes: the equations of motion for freely falling particles (Also known as the geodesic equation). The reason that the geodesic equation is of great importance for this thesis has to do with the physical situation we investigate. We study a situation where a small black hole rotates around a large black hole, with a very big mass ratio. Which means that we approximate the small black hole as if it is a test particle in the spacetime (It feels the curvature of the large black hole, but does not alter it). Making this approximation, the trajectory of the black hole actually becomes a geodesic in the spacetime of a massive black hole. The assumptions made in the definition of the black hole (axial symmetry, uncharged and rotating) leaves us with some much simpler differentials in the end and is therefore very important in actually obtaining a solution.

There are two ways in which we can determine the equations of motion for freely falling particles. The first one relies on using Lagrangian methods. The first technique yields second order differential equations. The second technique which we used in this thesis, deploys Hamiltonian techniques. This technique yields a set of first-order differential equations. As we will see in the derivation of the equations of motion for freely falling particles, we have enough known variables to solve these equations. To begin with the derivation, we first need the Hamilton-Jacobi equation:

$$H(x^\mu, \frac{\partial S}{\partial x^\mu}) + \frac{\partial S}{\partial \lambda} = 0, \tag{3}$$

where, x^μ are the coordinates of the system, $\dot{x}^\mu$ is the time evolution of the coordinates of the system and S is Hamilton's principal function.

Before we can actually use this equation, we need to define some basic properties. We start of by defining the Lagrangian for a free particle moving in a curved spacetime with metric $g_{\mu\nu}$,

$$\mathcal{L}(x^\mu, \dot{x}^\mu) = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu. \tag{4}$$

This is of course a generalization of a free particle from classical mechanics. Now using the definition of the Lagrangian, we are able to define the conjugated momentum as

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = g_{\mu\nu} \dot{x}^\nu. \tag{5}$$

By inverting this equation, we obtain an expression for $\dot{x}^\mu$ as a function of the momentum. Now we can write the Hamiltonian as a function of the coordinates $x^\mu(\lambda)$ and their conjugated momentum $p_\nu(\lambda)$, where λ is some affine parameter along the geodesic.

$$H(x^\mu, p_\nu) = p_\mu \dot{x}^\mu(p_\nu) - \mathcal{L}(x^\mu, \dot{x}^\mu(p_\nu)) \tag{6}$$

This equation can, using the definitions provided above, be written as

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \tag{7}$$

Finally, using the Hamilton-Jacobi equations, we obtain the time-evolution for coordinates along the trajectory as well as the time-evolution for their moments.

$$\dot{x}^\mu = \frac{\partial H}{\partial p_\mu} \tag{8}$$

$$\dot{p}_\mu = -\frac{\partial H}{\partial x^\mu} \tag{9}$$

We do not have the solution to the equations above yet, because we need one more variable to be able to obtain a full solution. This is why we return to the Hamilton-Jacobi (Eq. (1)) equation to derive one more constant. Now let's solve this equation using the fact that: $S = S(x^\mu, \lambda)$ is a know solution to the Hamilton-Jacobi equation, which we can therefore use to solve this equation and obtain some important properties of the spacetime.

Let us continue with defining some of the consequences that arise from the core definition of the Kerr spacetime and take these assumptions into account to greatly simplify the problem. There are three things we already know that can help us. First, we can express the Hamiltonian in terms of κ, which is a constant that can take the values $(-1, 0, 1)$ depending on the type of geodesic (timelike, null-like, spacelike) at the core of the problem.:

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu = \frac{1}{2}\kappa. \tag{10}$$

Second, the approximation that the Kerr spacetime is stationary, which means that we can always pick one (or more) time-coordinates for which the spacetime looks the same. In short this states that for every time-slice the spacetime looks the same. In order to obtain our second constant, we need a little help using a killing vector K^μ. If K^μ is applied to a certain metric and the metric does not change, we call K^μ a killing vector[3]. Now applying both, we obtain:

$$p_t = p_\mu K^\mu = -E, \tag{11}$$

where, E is the total energy of the freely falling particle.

For this thesis, the constant E can be interpreted as the total energy of the small black hole orbiting the large black hole.

Finally, the next property that also gives us a constant is the axial symmetry of the Kerr spacetime, which simply means that the curvature of the spacetime is the same for every co-latitude.

$$p_\phi = L, \tag{12}$$

where, he constant L is the total angular momentum of the freely falling particle.

For this thesis, the constant L can be interpreted as the total angular momentum of the small black hole orbiting the large black hole.

Using the conditions given above, we can now compose an equation for our trial solution S:

$$S = -\frac{1}{2}\kappa\lambda - Et + L\phi + S^{(r\theta)}(r,\theta) \tag{13}$$

In this equation, the last term is still unknown, however we will look for a seperable solution to this equation so we will use separation of variables to determine S as a function of r and θ. We make the following ansatz for S:

$$S^{(r\theta)} = S^{(r)}(r) + S^{(\theta)}(\theta) \tag{14}$$

Substituting this into equation 13, we obtain:

$$S = -\frac{1}{2}\kappa\lambda - Et + L\phi + S^{(r)}(r) + S^{(\theta)}(\theta) \tag{15}$$

Now that we have constructed this equation for S, we can use the Hamilton-Jacobi equation together with this S to try and obtain a solution to Eq. (3). We have:

$$\begin{aligned} -\kappa + \frac{\Delta}{\Sigma}\left(\frac{dS^{(r)}}{dr}\right)^2 + \frac{1}{\Sigma}\left(\frac{dS^{(\theta)}}{d\theta}\right)^2 - \frac{1}{\Delta}\left[r^2 + a^2 + \frac{2Ma^2r}{\Sigma}\sin^2\theta\right]E^2 \\ + \frac{4Mra}{\Sigma\Delta}EL + \frac{\Delta - a^2\sin^2\theta}{\Sigma\Delta\sin^2\theta}L^2 = 0. \end{aligned} \tag{16}$$

To simplify this complicated differential equation, we can use the convenient relation

$$(r^2 + a^2) + \frac{2Mra^2}{\Sigma}\sin^2\theta = \frac{1}{\Sigma}\left[(r^2+a^2)^2 - a^2\sin^2\theta\Delta\right], \tag{17}$$

if we use this relation and we take the product of the Hamilton-Jacobi equation with Σ , we then obtain:

$$\begin{aligned} &-\kappa(r^2 + a^2\cos^2\theta) + \Delta\left(\frac{dS^{(r)}}{dr}\right)^2 + \left(\frac{dS^{(\theta)}}{d\theta}\right)^2 \\ &-\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]E^2 + \frac{4Mra}{\Delta}EL + \left(\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right)L^2 = 0. \end{aligned} \tag{18}$$

Now this already looks a lot more promising, because we can rewrite this equation so that the left side of the equation only depends on r and the right side of the equation only depends on θ.

$$\Delta\left(\frac{dS^{(r)}}{dr}\right)^2 - \kappa r^2 - \frac{(r^2+a^2)^2}{\Delta}E^2 + \frac{4Mra}{\Delta}EL - \frac{a^2}{\Delta}L^2 + a^2E^2 + L^2 = \\ -\left(\frac{dS^{(\theta)}}{d\theta}\right)^2 + ka^2\cos^2\theta + a^2\cos^2\theta E^2 - \frac{\cos^2\theta}{\sin^2\theta}L^2 \tag{19}$$

Both sides of the equation depend on a different variable, but must be equal to one another. This of course implies that both sides must be equal to a certain constant. This constant is called the Carter constant, because the Carter constant makes the geodesic motion integrable, which allows us to solve the differential equations.

2.1.2 The Carter constant and geodesic motion

Looking at Eq. (19), we define the Carter constant C as

$$\left(\frac{dS^{(\theta)}}{d\theta}\right)^2 - \cos^2\theta\left[(k+E^2)a^2 - \frac{1}{\sin^2\theta}L^2\right] = C, \tag{20}$$

Consequently the radial part is

$$\Delta\left(\frac{dS^{(r)}}{dr}\right)^2 - \kappa r^2 + (L-aE)^2 - \frac{1}{\Delta}\left[E(r^2+a^2) - La\right]^2 = -C. \tag{21}$$

The two equations above and the Carter constant now give us a very nice way of defining two functions that are only dependent on one variable, which becomes for r and θ respectively:

$$R(r) := \Delta\left[-C + kr^2 - (L-aE)^2)\right] + \left[E(r^2+a^2) - La\right]^2 \tag{22}$$

$$\Theta(\theta) := C + \cos^2\theta\left[(k+E^2)a^2 - \frac{1}{\sin^2\theta}L^2\right] \tag{23}$$

This finally gives us the option to fill in the last blanks that were missing in the solution of the Hamilton-Jacobi equation, because from the definition of the equations above it follows that

$$\left(\frac{dS^{(r)}}{dr}\right)^2 = \frac{R}{\Delta^2} \tag{24}$$

$$\left(\frac{dS^{(\theta)}}{d\theta}\right)^2 = \Theta \tag{25}$$

Now finally filling this into our trial solution for the Hamilton-Jacobi equation, we obtain our final solution for the equation:

$$S = -\frac{1}{2}\kappa\lambda - Et + L\Phi + \int\frac{\sqrt{R}}{\Delta}\,dr + \int\sqrt{\Theta}\,d\theta \tag{26}$$

Now that we finally have the solution to the Hamilton-Jacobi equation in terms of the four constants κ, E, L, C, we can analyze and solve the geodesic motion in the Kerr spacetime. We do this by using the property of our Hamiltonian defined earlier in this derivation. We use that by definition

$$\frac{\partial S}{\partial x^\mu} = p_\mu \tag{27}$$

From this property and using our solution to the Hamilton-Jacobi equation, we can finally determine the conjugated momentum in this spacetime

$$p_r^2 = \left(\frac{\Sigma}{\Delta}\dot{r}\right) = \frac{R(r)}{\Delta^2} \qquad (28)$$

$$p_\theta^2 = \left(\Sigma\dot{\theta}\right) = \Theta(\theta) \qquad (29)$$

These equations finally gives us the geodesic equations

$$\dot{\theta} = \pm\frac{1}{\Sigma}\sqrt{\Theta} \qquad (30)$$

$$\dot{r} = \pm\frac{1}{\Sigma}\sqrt{R} \qquad (31)$$

This is the result we wanted to achieve, because with these two equations we can analyze the geodesic motion in the Kerr spacetime. We can even go one step further and parametrize the geodesic using the 'Mino time' so we can then define bound orbits in the spacetime and analyze those for this special case.[4] Using these two coupled equations, we can visualize the geodesics around a Kerr black hole using the BHP-Toolkit in Mathematica[5], this is shown in the figure below.

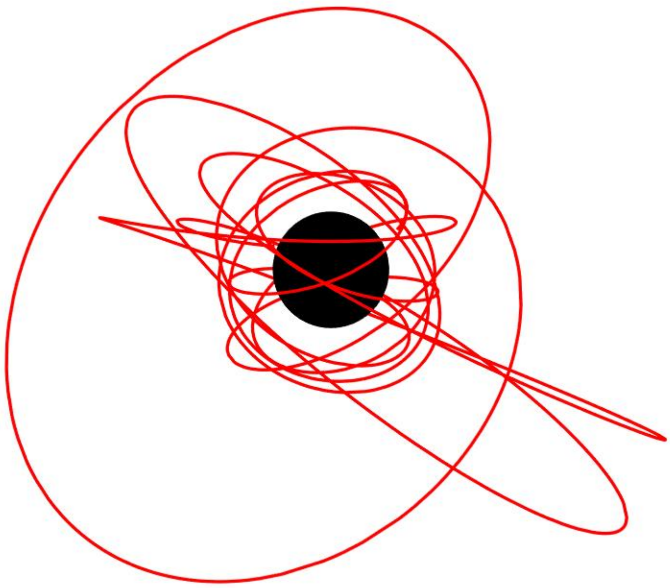

Figure 1: This figure shows geodesic motion around a Kerr black hole which is located in the center of the figure, the red lines are the geodesic.

2.2 Spin Coefficients and the Weyl tensor

Another important property of a spacetime are the so called spin coefficients, they are of extreme importance in the field of black hole perturbation theory and the observation and calculation of gravitational waves. There are several ways to write down the Einstein equations, one of them is using the spin-coefficients. This technique is especially useful in calculating small perturbations in rotating black hole spacetimes, because it leads to a seperable wave equation, whereas in terms of metric coefficients the equations are not seperable. That is why we will use this technique later on in this thesis and now we will derive the spin-coefficients.

2.2.1 The Newman-Penrose formalism

The first step in determining the spin coefficients of the Kerr spacetime, is to construct a null basis. The Newman-Penrose formalism helps us in making the right choice for such a null basis. The underlying concept of this formalism is the choice of a null basis, this must consist of 2 real vectors (l, n) and 2 complex conjugates $(m, \bar{m})$. These vectors must obey the following orthogonality conditions:

$$l \cdot m = l \cdot \bar{m} = n \cdot m = n \cdot \bar{m} = 0. \tag{32}$$

Additionally arising from the fact that the vectors have to be null

$$l \cdot l = n \cdot n = m \cdot m = \bar{m} \cdot \bar{m} = 0. \tag{33}$$

We add two more normalization conditions to the null vectors, because we would also like the vectors to be normalized

$$l \cdot n = -1 \tag{34}$$

$$m \cdot \bar{m} = 1 \tag{35}$$

Using all of these restrictions on the basis vectors, we construct the Kinnersly tetrad for the Kerr spacetime in BL-Coordinates [6, 7]

$$l^\mu = \left[\frac{r^2 + a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right] \tag{36}$$

$$n^\mu = \left[\frac{r^2 + a^2}{2\Sigma}, -\frac{\Delta}{2\Sigma}, 0, \frac{a}{2\Sigma}\right] \tag{37}$$

$$m^\mu = \left[\frac{ia\sin\theta}{\sqrt{2}(r + ia\cos\theta)}, 0, \frac{1}{\sqrt{2}(r + ia\cos\theta)}, \frac{i}{\sqrt{2}(r + ia\cos\theta)\sin\theta}\right] \tag{38}$$

$$\bar{m}^\mu = \left[\frac{-ia\sin\theta}{\sqrt{2}(r - ia\cos\theta)}, 0, \frac{1}{\sqrt{2}(r - ia\cos\theta)}, -\frac{i}{\sqrt{2}(r - ia\cos\theta)\sin\theta}\right] \tag{39}$$

2.2.2 The spin coefficients

With the Kinnersly tetrad, it is now relatively easy to compute the spin coefficients of this spacetime. The spin coefficients are related through the product of the tetrad vectors and a contraction using the covariant derivative. For example, one of the twelve spin coefficients is κ and the relation that gives it is

$$\kappa = \gamma_{131} = -m^{\alpha} l^{\beta} \nabla_{\beta} l_{\alpha} \tag{40}$$

In this paper we used Mathematica to calculate the actual spin coefficients using exactly the contraction as described above. Finally the coefficients have also been checked using the results in the literature.[8]

2.2.3 A representation of the Weyl tensor

Another application of the Kinnersly tetrad is the representation of the Weyl tensor by five scalars[3]. The Weyl tensor is a part of the Riemann tensor, with all of its contractions removed. Which means that the Weyl tensor contains pure geometry, while the Ricci tensor, for example, only contains terms that have information about matter in it. This property of the Weyl tensor makes it very useful for research on gravitation waves. We computed the Weyl tensor using the GeneralRelativityTensors Package from the BHP-Toolkit in Mathematica. Next we computed the five scalars which represent the ten different components of the Weyl tensor using the null-basis we constructed in the previous section. The scalars are defined by [7]

$$\Psi_0 = C_{\alpha\beta\gamma\delta} l^{\alpha} m^{\beta} l^{\gamma} m^{\delta} \tag{41a}$$

$$\Psi_1 = C_{\alpha\beta\gamma\delta} l^{\alpha} n^{\beta} l^{\gamma} m^{\delta} \tag{41b}$$

$$\Psi_2 = C_{\alpha\beta\gamma\delta} l^{\alpha} m^{\beta} \bar{m}^{\gamma} n^{\delta} \tag{41c}$$

$$\Psi_3 = C_{\alpha\beta\gamma\delta} l^{\alpha} n^{\beta} \bar{m}^{\gamma} n^{\delta} \tag{41d}$$

$$\Psi_4 = C_{\alpha\beta\gamma\delta} n^{\alpha} \bar{m}^{\beta} n^{\gamma} \bar{m}^{\delta} \tag{41e}$$

Further on in section 3 these scalars will be used in the calculation of the Teukolsky equation.

3 Gravitational Waves

3.1 The Teukolsky master equation

Having set the stage, we finally arrive at the main purpose of this thesis. The beauty of gravitational waves is that they can be described by one single master equation, the Teukolsky master equation. This equation is a separable wave equation and is constructed using the spin-coefficients and Weyl scalars form the previous chapter.

First, we need to filter out the Weyl scalars that are of importance for the calculation of gravitational waves. The ones that need to be considered are Eq. (41a) and Eq. (41e). In the rest of this derivation we will try to use Ψ_0 to eventually obtain an equation for Ψ_4, which will tell us something about emitted gravitational waves.

From Pirani (1964) we take the three non-vacuum equations in the Newman-Penrose formalism. These equations are very general and hold for all space.

$$\begin{aligned}(\bar{\delta} - 4\alpha + \pi)\Psi_0 - (D - 4\rho - 2\epsilon)\Psi_1 - 3\Psi_2 = (\delta + \bar{\pi} - 2\bar{\alpha} - 2\beta)\Phi_{00} \\ - (D - 2\epsilon - 2\bar{\rho})\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02},\end{aligned} \tag{42a}$$

$$\begin{aligned}(\Delta - 4\gamma + \mu)\Psi_0 - (\delta - 4\tau - 2\beta)\Psi_1 - 3\sigma\Psi_2 = (\delta + 2\bar{\pi} - 2\beta)\Phi_{01} \\ - (D - 2\epsilon + 2\bar{\epsilon} - 2\bar{\rho})\Phi_{02} - \bar{\lambda} + 2\sigma\Phi_{00} + 2\sigma\Phi_{11} - 2\kappa\Phi_{12},\end{aligned} \tag{42b}$$

$$(D - \rho - \rho - \bar{3}\epsilon + \bar{\epsilon})\sigma - (\delta - \tau + \bar{\pi} - \bar{\alpha} - 3\beta)\kappa - \Psi_0 = 0. \tag{42c}$$

In these equations the right hand side of these equations contains some Ricci scalars, these scalars are all given in terms of the stress-energy tensor by the Einstein field equations

$$\Phi_{00} = -\frac{1}{2}R_{\mu\nu}l^{\mu}l^{\nu} = 4\pi T_{\mu\nu}l^{\mu}l^{\nu} = 4\pi T_{ll} \tag{43}$$

Where $R_{\mu\nu}$ is the Ricci tensor and $T_{\mu\nu}$ is the stress-energy tensor.

3.1.1 Perturbation theory

After defining the three equations in the preliminaries of this chapter, we start by looking at two important properties of the Weyl scalars and some spin-coefficients

$$\Psi_0^{(0)} = \Psi_1^{(0)} = \Psi_2^{(0)} = \Psi_3^{(0)} = \Psi_4^{(0)} = 0,$$

$$\kappa^{(0)} = \sigma^{(0)} = \nu^{(0)} = \lambda^{(0)} = 0.$$

where the 'zero' between the brackets means that this is the unperturbed case. In the remainder of this chapter we will continue indicating the degree of perturbation with brackets.

Adding a small perturbation into the Pirani equations will start our derivation of the Teukolsky equation

$$\begin{aligned}l &= l^{(0)} + l^{(1)} \\ n &= n^{(0)} + n^{(1)}.\end{aligned} \tag{44}$$

The same goes for m and $\bar{m}$ and we add the same perturbation to the Weyl scalars

$$\Psi_1 = \Psi_1^{(0)} + \Psi_1^{(1)} \tag{45}$$

Similarly for the other four Weyl scalars. If we simply fill in the perturbations and we rewrite the Ricci scalars into stress energy tensor terms in Eq. (42a) (Also note that we drop all quadratic terms because we only look at the linear part of perturbation theory), we will obtain the following

$$\begin{aligned}
&((\bar{\delta} - 4\alpha + \pi)^{(0)} + (\bar{\delta} - 4\alpha + \pi)^{(1)})(\Psi_0^{(0)} + \Psi_0^{(1)}) - ((D - 4\rho - 2\epsilon)^{(0)} \\
&+ (D - 4\rho - 2\epsilon)^{(1)})(\Psi_1^{(0)} + \Psi_1^{(1)}) - 3(\kappa^{(0)} + \kappa^{(1)})(\Psi_2^{(0)} + \Psi_2^{(1)}) = \\
&4\pi((\delta + \bar{\pi} - 2\bar{\alpha} - 2\beta)^{(0)} + (\delta + \bar{\pi} - 2\bar{\alpha} - 2\beta)^{(1)})(T_{ll}^{(0)} + T_{ll}^{(1)}) \\
&- 4\pi((D - 2\epsilon - 2\bar{\rho})^{(0)} + (D - 2\epsilon - 2\bar{\rho})^{(1)})(T_{ln}^{(0)} + T_{ln}^{(1)}) \\
&+ 8\pi(\sigma^{(0)} + \sigma^{(1)})(T_{nl}^{(0)} + T_{nl}^{(1)}) - 8\pi(\kappa^{(0)} + \kappa^{(1)})(T_{nn}^{(0)} + T_{nn}^{(1)}) \\
&- 4\pi(\bar{\kappa}^{(0)} + \bar{\kappa}^{(1)})(T_{lm}^{(0)} + T_{lm}^{(1)}).
\end{aligned} \tag{46}$$

At first glance, this looks like a very complicated equation, but some of the spin-coefficients that are zero and the unperturbed Weyl scalars ($\Psi_n^{(0)}$) that also give zero, we can simplify this equation significantly. The equation reduces to

$$\begin{aligned}
&(\delta - 4\bar{\alpha} + \pi)^{(0)}\Psi_0^{(1)} - (D - 4\rho - 2\epsilon)^{(0)}\Psi_1^{(1)} - 3\kappa^{(1)}\Psi_2^{(0)} = \\
&4\pi(\delta + \bar{\pi} - 2\bar{\alpha} - 2\beta)^{(0)}T_{ll}^{(1)} - 4\pi(D - 2\epsilon - 2\bar{\rho})^{(0)}T_{ln}^{(1)}.
\end{aligned} \tag{47a}$$

For the second (42b) and third equation (42c) respectively we can apply exactly the same techniques as demonstrated above for the first equation and we obtain the following two equations

$$\begin{aligned}
&(\Delta - 4\gamma + \mu)^{(0)}\Psi_0^{(1)} - (\delta - 4\tau - 2\beta)^{(0)}\Psi_1^{(1)} - 3\sigma^{(1)}\Psi_2^{(1)} = \\
&4\pi(\delta + 2\bar{\pi} - 2\beta)^{(0)}T_{ln}^{(1)} - 4\pi(D - 2\epsilon + 2\bar{\epsilon} - \bar{\rho})^{(0)}T_{lm}^{(1)},
\end{aligned} \tag{47b}$$

$$(D - \rho - \bar{\rho} - 3\epsilon + \bar{\epsilon})^{(0)}\sigma^{(1)} - (\delta - \tau + \bar{\pi} - \bar{\alpha} - 3\beta)^{(0)}\kappa^{(1)} - \Psi_0^{(1)} = 0. \tag{47c}$$

Now that we have these three equations, the goal from now on is to eliminate Ψ_1 and Ψ_2 from the equations. We want to do this because this way we will obtain an equation only containing terms with Ψ_0 in the end. Once we have reached the equation described above, we can use the mapping of the spin-coefficents to rewrite this equation to an equation only containing Ψ_4, which was our goal from the beginning.

The first step in eliminating Ψ_2 from our equations is to introduce some properties of the spin coefficients in combination with the partial derivative operators.

$$D^{(0)}\Psi_2^{(0)} = 3\rho^{(0)}\Psi_2^{(0)}, \tag{48a}$$

$$\delta^{(0)}\Psi_2^{(0)} = 3\tau^{(0)}\Psi_2^{(0)}. \tag{48b}$$

After defining these two relations we can rewrite Eq. (47c)

$$\begin{aligned}
&(D - 4\rho - \bar{\rho} - 3\epsilon + \bar{\epsilon})^{(0)}\Psi_2^{(0)}\sigma^{(1)} \\
&- (\delta - 4\tau + \bar{\pi} - \bar{\alpha} - 3\beta)^{(0)}\Psi_2^{(0)}\kappa^{(1)} - \Psi_0^{(1)}\Psi_2^{(0)} = 0.
\end{aligned} \tag{49}$$

At first it might seem strange to add more terms to this equation that we do not wish to use in the end, but in fact the relation described above will come in handy later in the derivation of the Teukolsky equation.

The next step is to eliminate Ψ_1 from equations (47a) and (47b), for this step we will use a known commutator relation, which will just like the relation derived above, come in handy later on in the derivation to greatly simplify the equations. The following commutator relation will be used

$$\begin{aligned}&[D-(p+1)\epsilon+\bar{\epsilon}+q\rho-\bar{\rho}](\delta-p\beta+q\tau)\\&-[\delta-(p+1)\beta-\bar{\alpha}+\bar{\pi}+q\tau](D-p\epsilon+q\rho)=0.\end{aligned} \tag{50}$$

Where p and q are integers.

After deriving the first part and defining these relations, we are already close to the solution, the only thing that remains is some algebra, therefore we will start by operating with $(D-3\epsilon+\bar{\epsilon}-4\rho-\bar{\rho})$ on Eq. (47b).

$$\begin{aligned}&(\Delta-4\gamma+\mu)^{(0)}\Psi_0^{(1)}(D-3\epsilon+\bar{\epsilon}-4\rho-\bar{\rho})^{(0)}\\&-(\delta-4\tau-2\beta)^{(0)}\Psi_1^{(1)}(D-3\epsilon+\bar{\epsilon}-4\rho-\bar{\rho})^{(0)}\\&-3\sigma^{(1)}\Psi_2^{(0)}(D-3\epsilon+\bar{\epsilon}-3\rho-\bar{\rho})^{(0)}=\\&4\pi[(\delta+2\bar{\pi}-2\beta)^{(0)}T_{ln}^{(1)}(D-3\epsilon+\bar{\epsilon}-4\rho-\bar{\rho})^{(0)}\\&-(D-2\epsilon+2\bar{\epsilon}-\bar{\rho})^{(0)}T_{lm}^{(1)}(D-3\epsilon+\bar{\epsilon}-4\rho-\bar{\rho})^{(0)}].\end{aligned} \tag{51}$$

Next we will apply the same technique by operating with $(\delta-\bar{\pi}-\bar{\alpha}-3\beta-4\tau)$ on Eq. (47a)

$$\begin{aligned}&(\bar{\delta}-4\alpha+\pi)^{(0)}\Psi_0^{(1)}(\delta-\bar{\pi}-\bar{\alpha}-3\beta-4\tau)^{(0)}\\&-(D-4\rho-2\epsilon)^{(0)}\Psi_1^{(1)}(\delta-\bar{\pi}-\bar{\alpha}-3\beta-4\tau)^{(0)}\\&-3\kappa^{(1)}\Psi_2^{(0)}(\delta-\bar{\pi}-\bar{\alpha}-3\beta-4\tau)^{(1)}=\\&4\pi[(\delta-\bar{\pi}-\bar{\alpha}-3\beta-4\tau)T_{ll}^{(1)}(\delta-\bar{\pi}-\bar{\alpha}-3\beta-4\tau)^{(0)}\\&-(D-2\epsilon-2\bar{\rho})^{(0)}T_{ln}^{(1)}(\delta-\bar{\pi}-\bar{\alpha}-3\beta-4\tau)^{(0)}].\end{aligned} \tag{52}$$

To continue the derivation we again use some algebra to substract Eq.51 and Eq 52 from one another.

$$\begin{aligned}&((D-3\epsilon+\bar{\epsilon}-4\rho-\bar{\rho})^{(0)}(\Delta-4\gamma+\mu)^{(0)}\\&-(\delta-\bar{\pi}-\bar{\alpha}-3\beta-4\tau)^{(0)}(\bar{\delta}-4\alpha+\pi)^{(0)})\Psi_0^{(1)}\\&+(-(D-3\epsilon+\bar{\epsilon}-4\rho-\bar{\rho})^{(0)}(\delta-4\tau-2\beta)^{(0)}\\&-(\delta-\bar{\pi}-\bar{\alpha}-3\beta-4\tau)^{(0)}(D-4\rho-2\epsilon)^{(0)})\Psi_1^{(1)}\\&-3((D-3\epsilon+\bar{\epsilon}-4\rho-\bar{\rho})^{(0)}\sigma^{(1)}+(\delta-\bar{\pi}-\bar{\alpha}-3\beta-4\tau)^{(0)}\kappa^{(1)})\Psi_2^{(0)}=\\&4\pi[(\delta-\bar{\pi}-\bar{\alpha}-3\beta-4\tau)^{(0)}((D-2\epsilon-2\bar{\rho})^{(0)}T_{ln}^{(1)}-(\delta+\bar{\pi}-2\bar{\alpha}-2\beta)^{(0)}T_{ll}^{(0)})\\&+(D-3\epsilon+\bar{\epsilon}-4\rho-\bar{\rho})^{(0)}((\delta+2\bar{\pi}-2\beta)^{(0)}T_{ln}^{(1)}-(D-2\epsilon+2\bar{\epsilon}-\bar{\rho})^{(0)}T_{lm}^{(1)})].\end{aligned} \tag{53}$$

From now on it will save us a lot of writing to call the right hand side of the equation $4\pi T_0$, because on this side of the equation, nothing will change anymore, so it does not matter if we simply give it a name and write it out in short.

As was mentioned earlier in this paragraph we derived to equations that would simplify equation 53 greatly. We will now apply the relations (49) and (50). We start off by using the commutator relation (Eq.(50)) to eliminate Ψ_1 from our equation, which will take us one step closer to obtaining our final equation. With some puzzling the integers in the commutator relation in this case are $p = 2$ and $q = -4$. Now simply collecting terms from Eq. (53) and rewriting them as a commutator, we can see that some parts of the equation will become zero, which leaves us with

$$\begin{aligned} &((D - 3\epsilon + \bar{\epsilon} - 4\rho - \bar{\rho})^{(0)}(\Delta - 4\gamma + \mu)^{(0)} \\ &- (\delta - \bar{\pi} - \bar{\alpha} - 3\beta - 4\tau)^{(0)}(\bar{\delta} - 4\alpha + \pi)^{(0)})\Psi_0^{(1)} \\ &- 3(D - 3\epsilon + \bar{\epsilon} - 4\rho - \bar{\rho})^{(0)}\sigma^{(1)} + (\delta - \bar{\pi} - \bar{\alpha} - 3\beta - 4\tau)^{(0)}\kappa^{(1)} = 4\pi T_0. \end{aligned} \tag{54}$$

Notice that the terms $\kappa^{(1)}$ and $\sigma^{(1)}$ can be eliminated using our other derived relation: Eq. (49). We obtain our final equation for Ψ_0:

$$\begin{aligned} &((D - 3\epsilon + \bar{\epsilon} - 4\rho - \bar{\rho})^{(0)}(\Delta - 4\gamma + \mu)^{(0)} \\ &- (\delta - \bar{\pi} - \bar{\alpha} - 3\beta - 4\tau)^{(0)}(\bar{\delta} - 4\alpha + \pi)^{(0)}\Psi_0^{(1)} \\ &- 3\Psi_2^{(0)})\Psi_0^{(1)} = 4\pi T_0. \end{aligned} \tag{55}$$

Now that we have this equation, we can now use the symmetries of the Newman-Penrose formalism and the relations between the spin-coefficients to rewrite this equation and obtain an equation for Ψ_4, which is also know as the Teukolsky equation:

$$\begin{aligned} &((\Delta + 3\gamma - \bar{\gamma} + 4\mu + m\bar{u}u)^{(0)}(D + 4\epsilon - \rho)^{(0)} \\ &- (\bar{\delta} - \bar{\tau} + \bar{\beta} + 3\alpha + 4\pi)^{(0)}(\delta - \tau + 4\beta)^{(0)} - 3\Psi_2^{(0)})\Psi_4^{(1)} = 4\pi T_4. \end{aligned} \tag{56}$$

Now we are finally ready to derive the Teukolsky equation in Boyer-Lindquist coordinates. We do this by using the spin-coefficients, the equation for Ψ_4 and the relations D, Δ and δ. By simply filling them out and doing a lot of algebra, in the end we obtain the following equation[9]:

$$\begin{aligned} &\left[\frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2\theta\right]\frac{\partial^2(\rho^{-4}\Psi_4)}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2(\rho^{-4}\Psi_4)}{\partial t \partial\phi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2(\rho^{-4}\Psi_4)}{\partial\phi^2} \\ &- \Delta^2\frac{\partial}{\partial r}\left(\Delta^{-1}\frac{\partial(\rho^{-4}\Psi_4)}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial(\rho^{-4}\Psi_4)}{\partial\theta}\right) \\ &+ 4\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial(\rho^{-4}\Psi_4)}{\partial\phi} + 4\left[\frac{M(r^2 - a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial(\rho^{-4}\Psi_4)}{\partial t} \\ &+ (4\cot^2\theta + 2)\rho^{-4}\Psi_4 = 8\pi\rho^{-4}T_4. \end{aligned} \tag{57}$$

3.2 Separation of Variables

The Teukolsky equation has the very useful property that it is a seperable wave equation. Therefore, the technique "separation of variables" can be used to find two coupled equations. These two equations respectively can tell us a lot about gravitational waves and their behaviour.

3.2.1 The homogeneous solution

For convenience we start by deriving the homogeneous solution ($T = 0$). This solution then gives us some extra tools to solve the non-homogeneous Teukolsky equation, which we are interested in. We start by making the following ansatz

$$\Psi = \sum_{l,m} e^{i\omega t} e^{im\phi} S_{ln}(\theta) R_{ln}(r), \tag{58}$$

here ω are the frequency modes of vibration and m are the angular modes.

Substituting our ansatz into Eq. (57) yields the following two solutions to the homogeneous case

$$\Delta^2 \frac{d}{dr}\left(\Delta^{-1}\frac{dR}{dr}\right) + \left(\frac{K^2 + 4i(r-M)K}{\Delta} - 8i\omega r - \lambda\right) R = 0, \tag{59a}$$

$$\begin{aligned} &\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) \\ &+ \left(a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta} + 4a\omega\cos\theta + \frac{4m\cos\theta}{\sin^2\theta} - 4\cot^2\theta - 2 + A\right) S = 0, \end{aligned} \tag{59b}$$

here $K = (r^2 + a^2)\omega - am$, $\lambda = A + a^2\omega^2 - 2am\omega$ and A is the separation constant.

3.2.2 The non-homogeneous solution

After finding the radial and angular equations for the homogenous case, we can now determine the radial and angular equation for a non zero stress-energy tensor ($T \neq 0$). This case corresponds to the case we study in this thesis, the stress-energy tensor corresponds to a point particle like source (the small black hole rotating the massive black hole). In this procedure we use the eigenfunctions of Eq. (59b) to expand Ψ and $4\pi T_4$

$$4\pi\Sigma T = \int d\omega \sum_{l,m} G(r)\, {}_{-2}S_l^m(\theta) e^{im\phi} e^{-i\omega t}, \tag{60a}$$

$$\Psi = \int d\omega \sum_{l,m} R(r)\, {}_{-2}S_l^m(\theta) e^{im\phi} e^{-i\omega t}, \tag{60b}$$

where $G(r)$ is the source term, ${}_{-2}S_l^m$ are the eigenfunctions of Eq. (59b) and if we have a special case where $a\omega = 0$ we can write that ${}_{-2}Y_l^m = {}_{-2}S_l^m e^{im\phi}$ are the spin-weighted spheroidal harmonics.

Now that we have the solutions to the non-homogeneous equation, we would like to analyze some of the boundary conditions concerning this case. These boundary conditions will come in handy when we would like to constrain our solutions for gravitational wave emission to certain properties. When analyzing the radial equation boundary conditions, it is useful to make the following transformations

$$Y = \Delta^{-1}(r^2 + a^2)^{\frac{1}{2}} R, \tag{61a}$$

$$\frac{dr^*}{dr} = \frac{(r^2 + a^2)}{\Delta}. \tag{61b}$$

Now using the expansions and transformations as defined above, we obtain a new equation for the non homogeneous radial part

$$Y_{,r^*r^*} + \left[\frac{K^2 + 4i(r - M)K - \Delta(8ir\omega + \lambda)}{(r^2 + a^2)^2} - G^2 - G_{,r^*}\right] Y = 4\pi T_4, \tag{62}$$

where $, r^*$ means a partial derivative with respect to r^* and $G = -2\frac{(r-M)}{(r^2+a^2)} + \frac{r\Delta}{(r^2+a^2)^2}$.

Taking the limit $r \to \infty$ and consequently also $r^* \to \infty$, Eq. (62) reduces to

$$Y_{,r^*r^*} + (\omega^2 - \frac{i\omega}{r})Y \approx 4\pi T_4. \tag{63}$$

So for our solution at $r \to \infty$ we obtain the asymptotic solutions $Y \sim r^{\pm -2} e^{\mp i\omega r^*}$, which yields two asymptotic solutions for R

$$R \sim \frac{e^{-i\omega r^*}}{r}, \tag{64a}$$

$$R \sim \frac{e^{i\omega r^*}}{r^{-3}}. \tag{64b}$$

We can also analyze our boundary conditions at the outer horizon of the black hole for $r \to r_+$ and consequently $r* \to -\infty$. If we take this limit we obtain the following equation from Eq. (62)

$$Y_{,r^*r^*} + \left(\frac{K^2 + 4i(r_+ - M)k}{2Mr_+} - 4\frac{(r_+ - M^2)^2}{(2Mr_+)^2}\right) Y \approx 4\pi T_4, \tag{65}$$

where $k = \omega - m\omega_+$ and $\omega_+ = \frac{a}{2Mr_+}$. So for $r \to r_+$ we obtain the asymptotic solutions $Y \sim \Delta^{\pm -1} e^{ikr^*}$, which yields

$$R \sim e^{ikr^*}, \tag{66a}$$

$$R \sim \Delta^2 e^{-ikr^*} \tag{66b}$$

4 The Confluent Heun equation

An important property of the Teukolsky radial equation is that by using transformations, we can transform the radial equation into Confluent Heun form. A general representation of elementary integrands for Heun functions (second order Fuchsian equations with four regular singularities) has long been an unsolved problem in modern physics, until recently when an explicit integral representation in terms of exponentials and polynomials was found [10]. In particular, the expression for the confluent Heun equation gives us new insight in finding a representation of the solution to the Teukolsky equation. The newly discovered expression for the Confluent Heun can then be used to analyze the Teukolsy equation in Heun form to obtain a solution in terms of elementary functions.

4.1 A simple example

Before we can use the Teukolsky equation in Heun form, we first write code for a simple example. We can then use this code to analyze our problem.

The Confluent Heun equation is given by

$$\frac{d^2H(z)}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + 4\rho\right)\frac{dH(z)}{dz} + \frac{4\alpha\rho z - \sigma}{z(z-1)}H(z) = 0, \tag{67}$$

and we define the boundary conditions as $H(z_0) = H_0$ and $H'(z_0) = H'_0$. The equation also has three singularities, two regular singularities (at $z = 0$ and $z = 1$) and one irregular singularity (at $z = \infty$), we therefore also require that $z_0 \neq 0$ and $z_0 \neq 1$. The domain for which the function applies, is defined by the z_0 we choose. If we choose $z_0 < 0$ the domain in which the function is valid will be $I \in]-\infty, 0[$. For $0 < z_0 < 1$, $I \in]0, 1[$ and for $z_0 > 1$, $I \in]1, \infty[$. All the remaining terms are parameters (we will get to those later), except for the function $H(z)$, which is given by

$$H(z) = H_0 + H_0 \int_{z_0}^{z} G_1(\zeta, z_0)\, d\zeta + (H'_0 - H_0)\left(e^{z-z_0} - 1 + \int_{z_0}^{z} (e^{z-\zeta} - 1) G_2(\zeta, z_0)\, d\zeta\right), \tag{68}$$

where $G_i = \sum_{n=1}^{\infty} K_i^{*n}$ is the resolvent of the function K_i, with $i = 1, 2$ and K_i^{*n} means that K_i is integrated n times.

The two functions for K_1 and K_2 are given by

$$K_1(z,z_0) = 1 + e^{-z}\int_{z_0}^{z}\left(\frac{e^{\zeta\epsilon}\zeta^{\gamma}(\zeta-1)^{\delta}}{e^{z\epsilon}z^{\gamma}(z-1)^{\delta}}e^{\zeta}\left(\frac{q-\alpha\zeta}{(\zeta-1)\zeta}-\frac{\gamma}{\zeta}-\frac{\delta}{\zeta-1}-\epsilon-1\right)\right)d\zeta, \quad (69a)$$

$$K_2(z,z_0) = \left(\frac{q-\alpha z}{(z-1)z}-\frac{\gamma}{z}-\frac{\delta}{z-1}-\epsilon-1\right)e^{z-z_0}-\frac{q-\alpha z}{(z-1)z}. \quad (69b)$$

Now having defined these functions, the next step is to determine a representation around $z_0 = -4$. In order to obtain this representation, we use several mathematical techniques. The algorithm below can be used to reconstruct our code for $z > z_0$.

Step 1:

Discretize the functions K_1 and K_2 with step size S and $range = 3.5$ and convert them into a matrix. The matrices will be triangular due to the fact that we only look at $z > z_0$ (So every component $z_0 < z$ will give 0). The triangular matrices make our calculations faster.

Step 2:

Convert the two functions now given by the triangular matrices into the functions G_1 and G_2. Do this by using the resolvent technique. The resolvent of the functions K_i is given by

$$(I_n - K_i * S)^{-1}, \quad (70)$$

where I_n is the identity matrix which takes on the length $\frac{range}{S} + 1$.

Step 3:

In the two integrals in the Heun function, there are more terms that have to be integrated (1 and $e^{z-\zeta} - 1$). Use the same technique as discussed in step 1 two create two more triangular matrices containing these terms. Finally use the dot product to integrate the terms under the two integrals.

Step 4:

Add every term together and obtain the matrix for $H(z)$. Next obtain the right row (for $z_0 = -4$) out of the matrix, which in our case is the first row.

For $z < z_0$ the same procedure can be applied, but the range has to be changed.

Applying this algorithm gives the following figure, which indeed shows that for this simple example the code definitely works. The graph gives us a perfect fit onto the numerical calculation around $z_0 = -4$ with boundary conditions $H_0 = 0$ and $H_0' = 1$. Parameters are:

$\alpha = 5$
$q = 1$
$\gamma = 3$
$\delta = 2/3$
$\epsilon = 4$

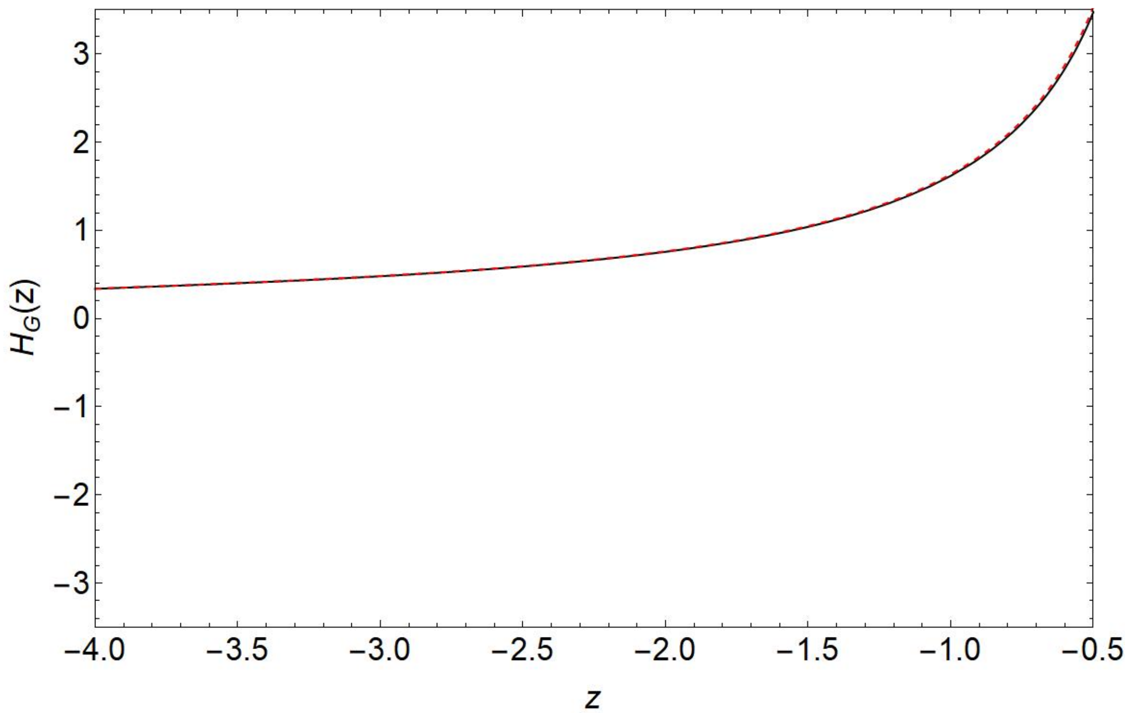

Figure 2: This figure shows the comparison between the solution of our algorithm and the numerical solution to the Heun equation around $z_0 = -4$. Here the red dotted line consists of the results coming from our algorithm. The black line gives the numerical solution to the Heun equation. All results from this figure are only analyzed for $z_0 < z$. The computation takes $33.30sec$ on this laptop to obtain this figure.

4.2 From radial to Heun

In order to build on our example in the previous paragraph we first need to transform the Teukolsky radial equation into Heun form.

The Teukolsky radial equation has three singularities, one irregular singularity at $r = -\infty$ and two regular singularities at the roots of $\Delta = 0$, which gives us

$$r_\pm = M \pm \sqrt{M^2 - a^2}, \tag{71}$$

where $r_\pm$ are the event and the Cauchy horizon respectively.

We start the transformation by rewriting the radial equation to the following form

$$R(r) = (r - r_+)^\xi (r - r_-)^\eta e^{\zeta r} H(r), \tag{72}$$

with parameters $\zeta = \pm i\omega$, $\xi = -1 \pm \frac{(-2+2i\sigma_+)}{2}$, $\eta = 1 \pm \frac{(-2-2i\sigma_-)}{2}$ and $\sigma_\pm = \frac{-2Mr_\pm - ma}{r_+ - r_-}$.

We will now make another transformation using the dimensionless variables

$$\bar{r} \equiv \frac{r}{M}, \tag{73a}$$

$$\bar{a} \equiv \frac{a}{M}, \tag{73b}$$

$$\bar{\omega} \equiv M\omega, \tag{73c}$$

$$\bar{\zeta} \equiv M\zeta, \tag{73d}$$

therefore we can also transform the radial coordinate into the dimensionless variable z:

$$z = \frac{r - r_-}{r_+ - r_-} = \frac{\bar{r} - \bar{r}_-}{\bar{r}_+ - \bar{r}_-}. \tag{74}$$

Finally, by using all of our transformations and definitions above and doing a lot of algebra, we obtain a confluent Heun equation with auxiliary function $H(z)$. This equation takes the same form as Eq. (67) with parameters

$$\rho = (\bar{r}_+ - \bar{r}_-)\frac{\bar{\zeta}}{2} \tag{75a}$$

$$\gamma = 1 + \eta \tag{75b}$$

$$\delta = 2\xi - 1 \tag{75c}$$

$$\alpha = \xi + \eta - 2\bar{\zeta} - 2\frac{i\bar{\omega}}{\bar{\zeta}} - 1 \tag{75d}$$

$$\sigma = {}_sA_{lm}(\bar{a}\bar{\omega}) + \bar{a}^2\bar{\omega}^2 - 8\bar{\omega}^2 + \rho(2\alpha + \gamma - \delta) + (-1 - \frac{\gamma + \delta}{2})(-2\frac{\gamma + \delta}{2}) \tag{75e}$$

We can now rewrite our algorithm to implement the parameters as shown above. Now there is only one thing left to make the calculations for our program work in a real physical situation. Applying certain boundary conditions will make this possible. We choose to use "ingoing" boundary conditions[11], which is illustrated in the Kruskal diagram below. We calculated these boundary conditions on $R(r)$ using the BHP-Toolkit in Mathematica [5]. To apply these conditions to our Heun equation we simply transformed the boundary conditions the same way we transformed all of the variables, using Eq. (72).

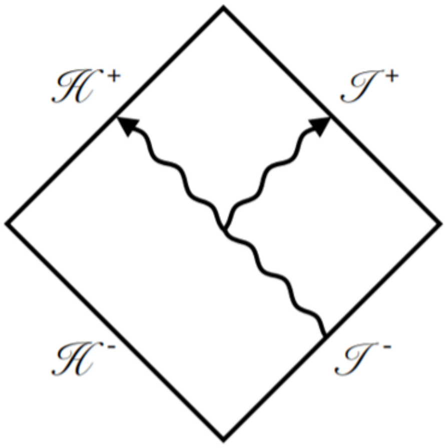

Figure 3: This figure shows the "ingoing" boundary conditions we used for our calculations. As you can seein this Kruskal diagram, these boundary conditions are defined by a wave coming in from past infinity. It then scatters at the horizon, one part travels towards the future horizon and one scatters out towards future infity.This picture was taken from [11]

4.3 The results

For our final result we use all of the work we have done before. We start by defining a point in our spacetime we want to analyze, in our research we picked $r = 3$. The next step was to use the previous paragraph to transform this point into Heun coordinates. The same is done for the ingoing boundary conditions around this point. All the used parameters to obtain the final result are defined below:

$s = -2$
$l = 2$
$m = 2$
$a1 = 0.3$
$\Omega = 0.1$

Now we have obtained all our information to start running the algorithm. We run it the same way we have done previously with our simple case and give our new data as input. It is important to realize that in our simple case one of the integrals in the Heun function always returned zero. This made the calculation of the simple case a lot faster. Another important property that arises not only from our boundary conditions, but also from the foundation of gravitational waves is the fact that we do not only get a real solution to our problem, but also an imaginary solution. Therefore the output of the algorithm gives us two graphs which both describe the gravitational waves. The total calculation time of these two graphs was 1 hour and 20 minutes on this laptop.

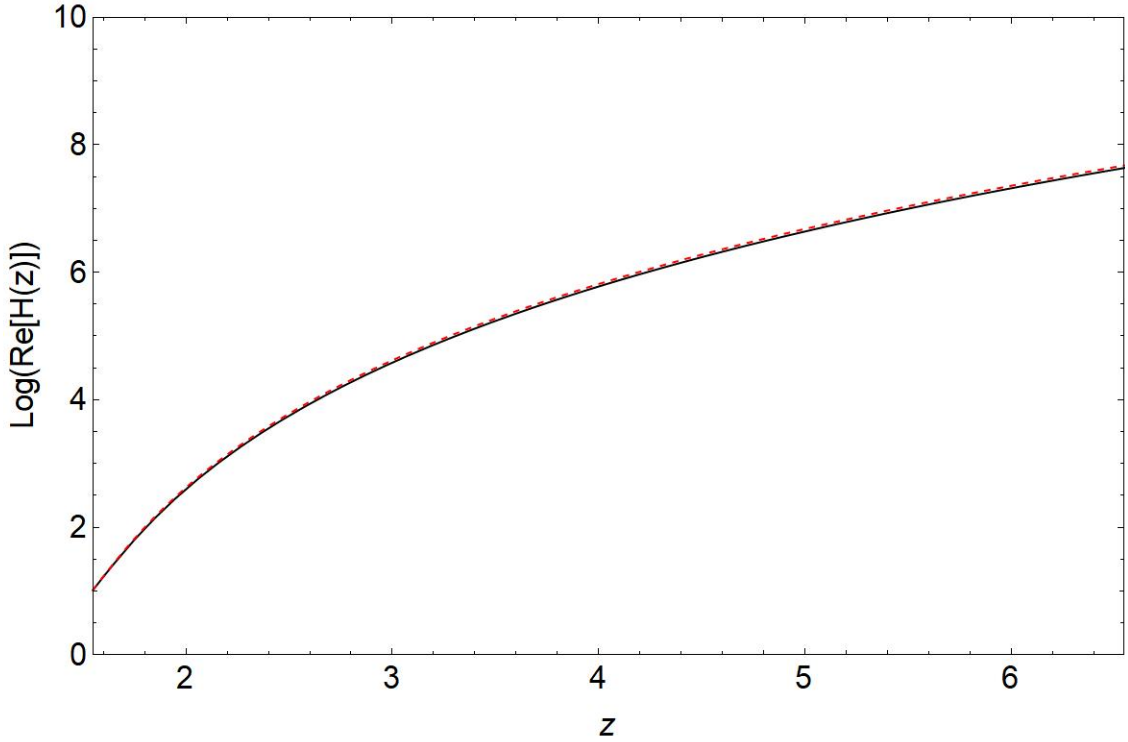

Figure 4: The solution for the real part of the radial Teukolsky equation around $r = 3$ in Heun form. Here the red dotted line consists of the real part of the results coming from our algorithm. The black line gives the numerical solution to the Heun equation around $r = 3$. This means that we look at $z_0 = 1.54828$.

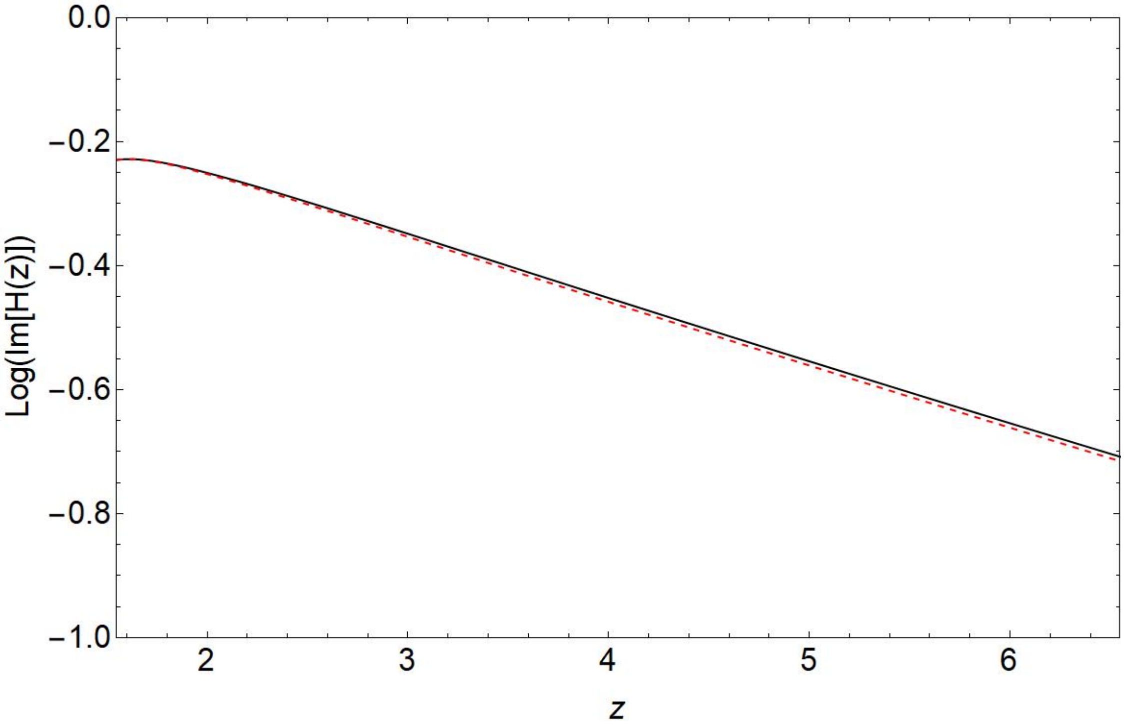

Figure 5: The solution for the imaginary part of the radial Teukolsky equation around $r = 3$ in Heun form. Here the red dotted line consists of imaginary part of the results coming from our algorithm. The black line gives the numerical solution to the Heun equation around $r = 3$. This means that we look at $z_0 = 1.54828$.

Once again these graphs seem to be an almost perfect fit which means that not only in the simple case, but also in our physical case around $r = 3$, this algorithm still works.

5 Conclusion and discussion

The work in this thesis provides a working algorithm to analyze the real and complex parts of gravitational waves emitted in an EMRI. In the beginning of this thesis we looked at different spacetimes and in particular the Kerr spacetime. The next step was to analyze and understand this spacetime so that we could develop a set of equations with which we could determine the geodesic motion. We finally obtained two coupled equations using Hamilton-Jacobi techniques and illustrated the behaviour of geodesics around a Kerr black hole.

The next step was to define a convenient basis to work on, this basis is called the Newman-Penrose formalism. From this formalism, we defined the spin-coefficients, which are of great importance in calculations concerning gravitational waves. What we could also determine from the Newman-Penrose formalism, is a representation of the Weyl tensor, which gives us a lot of information on the geometry of a certain spacetime and thus also about gravitational waves.

We then moved on from the basis of the spacetime, to the derivation of the Teukolsky master equation, which provides us with an equation containing all the information about gravitational waves. We started this derivation by defining some non-vacuum Newman-Penrose equations that hold for all space our basis defines. Next we added a small perturbation to the spin coefficients and Weyl scalars. After that we used a lot of algebra and the symmetry of the Newman-Penrose formalism and the relations between the spin-coefficients to obtain the Teukolsky equation.

Before being able to analyze this equation, we needed to separate it into an angular and radial part. We did this using the simple technique separation of variables and showed both the homogeneous and non-homogeneous solution.

In chapter four we then finally arrived at the core of our research: Constructing an algorithm that can be used to analyze gravitational waves. We started by trying to construct an algorithm that could analyze a simple case of the Heun equation. In this example one integral would always be zero, so the calculations were a lot easier for us and for the computer. Then we moved on to converting our separated Teukolsky radial equation into Heun form including the boundary conditions and transforming them into Heun form. Finally everything was ready for an input. We choose $r = 3$, converted everything into the right form and ran the algorithm. In the end we can conclude that the code works quite well, it is fast and gives a good approximation of the numerical solution concerning an EMRI.

In this thesis we wanted to construct a working piece of code to help gravitational wave detection in the future. It is important to realize that gravitational wave detection is not as simple as detecting some other basic physical quantities. In order to detect a gravitational wave, the measured data has to be submitted to a certain model which calculates the properties of this specific gravitational wave. This is important, because otherwise we would not be able to tell if the detected wave was just noise or an actual gravitational wave. The purpose of our research was to create such a model for gravitational waves emitted in an EMRI. In this thesis we succeeded in creating such model using new mathematical techniques. Although our code still lacks some physical relevance. One of the things that still needs to be improved in order for our code to work on real gravitational waves is the addition of a source term. We only wrote our code for the homogeneous solution to the Teukolsky radial equation, in a real EMRI however, these equations would be non-homogenous. The small black hole does also have a source term which is non-zero.

Another important thing that needs to be pointed out is that in the end we had to calculate one of the integrals in the Heun function numerically. This cost a lot of time in our program, which made it a lot slower. If this problem could somehow be fixed in the future, the algorithm would be much more efficient.

Nevertheless, this algorithm is a great basis to further build on, and is a great step forward in creating a fast and accurate way to analytically model the Teukolsky radial equation.

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