Particles and Sources

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Abstract

This paper deals with sources for creating particles in quantum field theory.

Particle propagation through spacetime is to be restricted since time is one directional. A simple source emits particles in both directions and one way propagators are needed. Analysis of two-point correlation functions provides information about correlations with future times only. This allows for constructing a source that emits particles one way through time, without restrictions on the propagator.

Particles with localized positions and momenta allow for a classical view. A Gaussian source however displays some difficulty with the calculation of the field. After creating an alternative source and using a different approach to the Gaussian, a classical interpretation of the emitted particles becomes possible. The kinematics of free particles emitted on their mass shell appear to be described by Newton's first law.

Foreword

As an undergraduate student I have done research in the field of quantum field theory. It is a type of preliminary work that serves as the conclusion of my bachelor education. The particular subject reflects the fact that I have an interest in theoretical physics and want to follow a master program in it.

The research was done under the tutorship of Prof. Dr. Ronald Kleiss at the department of Theoretical High Energy Physics, at the Radboud University of Nijmegen. One primary source is used for describing the setting and purpose of this research: sections 1.2, 2.2 and 2.5 are based on [1]. Furthermore, section 1.3 is based on [2]. The remaining sections contain material resulting from my work.

Throughout the paper several notations and conventions will be used:

- Fourvectors are simply denoted a and defined as $a^\mu = \left(a^0, \vec{a}\right)$, with $\vec{a} = \left(a^1, a^2, a^3\right)$. Inner products are evaluated as $a^\mu b_\mu = a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b}$; the square of a fourvector is given by $a^2 = \left(a^0\right)^2 - |\vec{a}|^2$.
- Planck's reduced constant $\hbar$ is written when emphasis is desired and omitted otherwise.
- Integrals run over all space, i.e. from -∞ to +∞.
- The Fourier transform $F(k)$ of a function $f(x)$ is defined as

  $$F(k) = \frac{1}{\sqrt{2\pi}} \int e^{ik \cdot x} f(x)\, dx$$

  where x and k are fourvectors. The inverse FT is then

  $$f(x) = \frac{1}{\sqrt{2\pi}} \int e^{-ik \cdot x} F(k)\, dk$$

  although this is not used anywhere.

Contents

Chapter 1

Particle creation and time flow

1.1 Introduction

In quantum field theory (QFT) physical activity is described using the notion of quantum fields. These fields assign values to all points in spacetime and can represent the existence and propagation of particles. The number of particles considered need not be held constant, QFT allows particles to be freely created or destroyed. Creation of particles is accounted for by the use of sources and will be the central topic of this chapter. After being created the evolution of a particle is described by an object called the propagator. The field representing the particle is dependent of both the source and the propagator which are used. However, the domain of the field, Minkowski spacetime, places no restriction on the direction in which particles evolve through time. The natural condition that life evolves forward through time, rather than backward, must therefore be enforced. For instance, this can be done using a retarded propagator. In this case only evolution in one time direction is considered[1].

The objective here is to define the direction of time flow using the source; an attempt is made to construct a source that creates only particles that evolve in the natural time direction, without restrictions on the propagator. First a simple source is considered to illustrate the issue. Then the problem is analyzed by considering correlations between fields and bases for creation and annihilation configurations. Next a possible solution is proposed. Finally a further discussion about the results and the direction of time flow follows.

[1] For more details, see the discussion at the end of the chapter.

1.2 Simple source

In this section the issue of time direction for the evolution of particles is illustrated. For this the propagator[2]

$$\Pi_\gamma(x) = \frac{i}{(2\pi)^4}\int d^4k \frac{e^{-ik\cdot x}}{k^2 - m^2 + i\gamma}$$

is used, with $\gamma > 0$. This parameter γ is related to the lifetime of the particles. Now consider a simple source that is active at one particular moment in time, say $t = 0$. With $x^0 = ct$ the source can then be written as

$$J(x) \propto \delta\left(x^0\right).$$

Calculating the response of the field yields

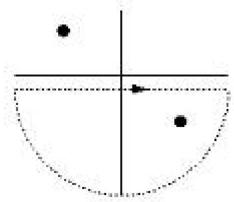
$$\begin{aligned}
\phi(x) &= i\int d^4y \Pi_\gamma(x-y) J(y) \\
&\propto \frac{-1}{(2\pi)^4}\int d^4y d^4k \frac{e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\gamma}\delta\left(y^0\right) \\
&= \frac{-1}{(2\pi)^4}\int d^3\vec{y} d^4k \frac{e^{-ik\cdot x - i\vec{k}\cdot\vec{y}}}{k^2 - m^2 + i\gamma} \\
&= \frac{-1}{(2\pi)^4}\int d^4k \frac{e^{-ik\cdot x}}{k^2 - m^2 + i\gamma}(2\pi)^3\delta^3\left(-\vec{k}\right) \\
&= \frac{-1}{2\pi}\int dk^0 \frac{e^{-ik^0x^0}}{\left(k^0\right)^2 - m^2 + i\gamma}.
\end{aligned}$$

This integral has two poles at

$$k^0 = \pm\sqrt{m^2 - i\gamma} \approx \pm\left(m - i\frac{\gamma}{2}\right)$$

where it is assumed that $\gamma \ll m^2$. For times later than $t = 0$ the contour can be closed along the lower half of the plane, as in the picture.

[2] It has been suggested to include an introductory section on the theory of Green's functions, i.e. propagators. This would certainly make the paper more standalone and accessible. Being aware it would be useful indeed, it is not my intention to explain this material in detail here. Instead I refer to standard literature on this matter.



Thus, for $x^0 > 0$ we find

$$\phi(x) \propto e^{-imx^0 - \frac{\gamma}{2m}x^0}.$$

In general the field is given by

$$\phi(x) \propto e^{-im\left|x^0\right| - \frac{\gamma}{2m}\left|x^0\right|}.$$

The probability of finding particles somewhere is related to $|\phi(x)|^2$. This is given by

$$|\phi(x)|^2 \propto e^{-\frac{\gamma}{m}\left|x^0\right|} = e^{-\frac{|t|}{\tau}}$$

where $\tau = \frac{m}{\gamma c}$ is the lifetime of the particles.

From this result it follows that the probability of finding particles anywhere decreases exponentially for particles of finite lifetime. This is precisely what is expected for unstable particles. However, the probability to find stable particles (infinite lifetime) anywhere does not decrease. Again as expected.

This result also demonstrates the anticipated problem. The probability of finding particles behaves just as desired for times in the future, relative to the moment of particle creation. But this probability is also nonzero for times in the past. Particles are found to propagate both forward and backward through time. In our normal view of the world this means that we see particles coming together and disappearing into one point in space, after which particles appear from it. The first part is unwanted and should be removed.

1.3 Correlations

In order to gain insight into the problem we now investigate the concept of correlations. Specifically we look at bases for creation and annihilation configurations and their correlations with each other and with quantum fields. It appears that these correlations are nonzero only from or until a specific moment in time. The goal is to shed light on the cause of these future and past correlations.

In this section the propagator will be taken as

$$\Pi(x) = \frac{i}{(2\pi)^4} \int d^4k \frac{e^{-ik\cdot x}}{k^2 - m^2 + i\epsilon}$$

with ϵ an infinitesimal number. This propagator works for stable particles, in contrast to the propagator with the γ from before. The two-point correlation function for a field is then

$$\langle \phi(y)\, \phi(x) \rangle = \Pi(y - x)\,.$$

At this point a basis for a creation configuration is introduced. It is written as[3]

$$a\left(\vec{k}, t\right)^* = -i \int d^3\vec{x} e^{-ik\cdot x} \overleftrightarrow{\partial_0} \phi(x)$$

where the operator $\overleftrightarrow{\partial_0}$ is defined by $A\overleftrightarrow{\partial_0}B = A\partial_0 B - (\partial_0 A)\, B$. The corresponding annihilation basis is given by

$$a\left(\vec{k}, t\right) = i \int d^3\vec{x} e^{ik\cdot x} \overleftrightarrow{\partial_0} \phi(x)\,.$$

Using the convention $k^0 = \omega\left(\vec{k}\right) = \sqrt{\left|\vec{k}\right|^2 + m^2}$ it can be shown that we have the right linear combination of $\phi(x)$ and $\partial_0 \phi(x)$ to ensure no correlations with earlier (or later) times occur. The correlation between the configuration bases is given by

$$\left\langle a\left(\vec{k}', t'\right) a\left(\vec{k}, t\right)^* \right\rangle = 2\omega\, (2\pi)^3\, \delta^3\left(\vec{k} - \vec{k}'\right) \theta\left(t' - t\right)$$

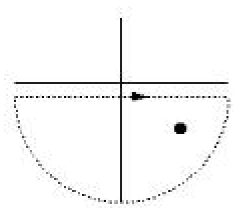
where we see a unit step function for the time: $\theta(t' - t)$. This result can be calculated

[3]It has been suggested to give more explanation of the material used here. On this I remark that the new language in this section is not used further in this paper. This section serves to learn how to make correlations with only earlier or later times.

in the same way as the correlation of a field with the creation basis, which is given by

$$\begin{aligned}\left\langle \phi(y)\, a\left(\vec{k},t\right)^{*}\right\rangle &= -i\int d^{3}\vec{x}\left\langle \phi(y)\, e^{-ik\cdot x}\overleftrightarrow{\partial}_{0}\phi(x)\right\rangle \\ &= -i\int d^{3}\vec{x}e^{-ik\cdot x}\left(\frac{\partial}{\partial x^{0}}+ik^{0}\right)\langle\phi(y)\,\phi(x)\rangle \\ &= \frac{1}{(2\pi)^{4}}\int d^{3}\vec{x}e^{-ik\cdot x}\int d^{4}l\left(\frac{\partial}{\partial x^{0}}+ik^{0}\right)\frac{e^{-il\cdot(y-x)}}{l^{2}-m^{2}+i\epsilon} \\ &= \frac{1}{(2\pi)^{4}}\int d^{3}\vec{x}d^{4}l\left(il^{0}+ik^{0}\right)\frac{e^{-i(k-l)\cdot x}e^{-il\cdot y}}{l^{2}-m^{2}+i\epsilon} \\ &= \frac{i}{2\pi}e^{-ik^{0}x^{0}+i\vec{k}\cdot\vec{y}}\int dl^{0}\frac{l^{0}+k^{0}}{\left(l^{0}\right)^{2}-\left(k^{0}\right)^{2}+i\epsilon}e^{-il^{0}\left(y^{0}-x^{0}\right)} \\ &= \frac{i}{2\pi}e^{-ik^{0}x^{0}+i\vec{k}\cdot\vec{y}}\int dl^{0}\frac{e^{-il^{0}\left(y^{0}-x^{0}\right)}}{l^{0}-k^{0}+i\epsilon}\end{aligned}$$

This integral has one pole at $l^{0}=k^{0}-i\epsilon$. An integration contour can be closed along the lower half of the complex plane.



The result of this integral is found by applying residue calculus for simple poles. After taking the limit $\epsilon\downarrow 0$ we find

$$\left\langle \phi(y)\, a\left(\vec{k},t\right)^{*}\right\rangle = e^{-ik\cdot y}\theta\left(y^{0}-x^{0}\right)$$

for the correlation.

This result contains a unit step function in time coordinates $\theta\left(y^{0}-x^{0}\right)$, resulting from the demand that the (infinitely large) curved part of the contour does not contribute to the integral. It means there is only a correlation when $y^{0}>x^{0}$. From the derivation one can see that this is the consequence of the cancellation of one pole before the integration over l^{0}. Since there are no poles in the upper half of the complex plane, closing the integration contour along the upper half gives a zero correlation for $y^{0}<x^{0}$. More precisely, the pole was removed by a specific linear combination of the field $\phi(x)$ and its derivative $\partial_{0}\phi(x)$. In the next section we will apply this idea to our source.

1.4 Special source

Armed with new insight into time correlations in this section we return to our original quest. The objective is to construct a source that creates particles, such that the fields now have correlations only with future times, relative to the moment of creation. To achieve this we use the idea of cancelling a pole for the last integration, which can be done by choosing a specific linear combination of the original source and its derivative.

As before the propagator will be

$$\Pi_\gamma(x) = \frac{i}{(2\pi)^4}\int d^4k \frac{e^{-ik\cdot x}}{k^2 - m^2 + i\gamma}$$

with $\gamma > 0$. In this case the source is constructed as a linear combination of the simple source and its derivative. With this combination hopefully the field will only show activity after the moment of particle creation. The source thus becomes

$$J(x) = \left(A\frac{d}{dx^0} + B\right)\delta\left(x^0\right)$$

with A and B coefficients. To perform the integration the delta is represented by the limit of a Gaussian function as

$$\delta(x) = \lim_{\epsilon\downarrow 0}\frac{1}{2\sqrt{\pi\epsilon}}e^{-x^2/4\epsilon}.$$

Calculating the field yields

$$\begin{aligned}
\phi(x) &= i\int d^4y\,\Pi_\gamma(x-y)\,J(y) \\
&= \frac{-1}{(2\pi)^4}\int d^4y\,d^4k\frac{e^{-ik\cdot(x-y)}}{k^2-m^2+i\gamma}\left(A\frac{d}{dy^0}+B\right)\delta\left(y^0\right) \\
&= \frac{-1}{2\pi}\int dy^0 dk^0\frac{e^{-ik^0\left(x^0-y^0\right)}}{\left(k^0\right)^2-m^2+i\gamma}\lim_{\epsilon\downarrow 0}\frac{1}{2\sqrt{\pi\epsilon}}\left(-\frac{A}{2\epsilon}y^0+B\right)e^{-\left(y^0\right)^2/4\epsilon} \\
&= \frac{-1}{2\pi}\int dk^0\frac{e^{-ik^0x^0}}{\left(k^0\right)^2-m^2+i\gamma}\lim_{\epsilon\downarrow 0}\frac{1}{2\sqrt{\pi\epsilon}}\int dy^0\left(-\frac{A}{2\epsilon}y^0+B\right)e^{-\left(y^0\right)^2/4\epsilon}e^{ik^0y^0}.
\end{aligned}$$

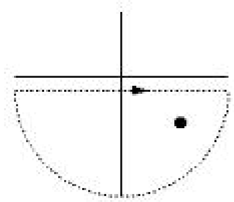
After integrating over y^0 and taking the limit $\epsilon\downarrow 0$ we have

$$\phi(x) = \frac{i}{2\pi}\int dk^0\frac{Ak^0+iB}{\left(k^0-\lambda\right)\left(k^0+\lambda\right)}e^{-ik^0x^0}$$

with $\lambda = \sqrt{m^2 - i\gamma}$. Choosing $A = 1$ and $B = -i\lambda$ gives a field of

$$\phi(x) = \frac{i}{2\pi} \int dk^0 \frac{e^{-ik^0 x^0}}{k^0 - \lambda}$$

This integral has one pole at $k^0 = \lambda$, which lies in the lower half of the complex plane. An integration contour can be closed around this half.



Finally the field is found to be

$$\phi(x) = e^{-i\lambda x^0} \theta\left(x^0\right).$$

The probability of finding particles anywhere is proportional to $|\phi(x)|^2$. This is given by

$$|\phi(x)|^2 \approx e^{-\frac{\gamma}{m} x^0} \theta\left(x^0\right) = e^{-\frac{t}{\tau}} \theta(t)$$

where $\tau = \frac{m}{\gamma c}$ is the lifetime.

As before the probability of finding particles anywhere decreases exponentially for unstable particles and remains constant for stable particles. However, the presence of the time step function cures the problem of the nonzero probability for past times. In fact, for $t < 0$ the field itself shows no activity and the probability of finding particles equals zero. Further discussion follows in the next section.

Note that the calculation of the field can be greatly simplified using the following theorem on the Fourier transformation, [3] eq. 3.55:

$$L_F\left[f'\left(x^0\right)\right] = -ik^0 L_F\left[f\left(x^0\right)\right] = -ik^0 F\left(k^0\right)$$

Here L_F denotes the Fourier transform operator. When Fourier transforming the derivative of a function this theorem may conveniently be applied. With this trick, representing the delta as the limit of a Gaussian function would have been unnecessary. The theorem will be used in the next chapter.

1.5 Discussion

In this chapter the issue of time correlations of quantum fields was investigated. A simple source that emits particles both into the future and the past was presented first. Through analysis of correlations between fields and creation and annihilation configuration bases the condition was found for future or past only activity. This was then used to adapt the simple source into a special source for the creation of only particles that evolve forward through time, without placing any restriction on the propagator.

The main point to be made is that the forward-only propagation is now determined completely by the source. The reason to let the source define the time direction of propagation is that the source may be seen as an actual physical object. Therefore it makes sense that this object defines all the properties of the particles it emits. This includes their mass and lifetime, but also their direction of evolution through time. With the result that the probability of finding (unstable) particles anywhere decreases as time goes by, we may now also say that the source *defines* the direction of time flow. It flows in the direction in which particles disappear rather than appear.

This can be set in contrast with the use of a one way propagator. For forward propagation it is also possible to use a retarded propagator

$$\Pi_{ret}(x) = \Pi(x)\,\theta\left(x^0\right).$$

This definition ensures that particles can only propagate for $t > 0$. Backward propagation can be enforced using an advanced propagator

$$\Pi_{adv}(x) = \Pi(x)\,\theta\left(-x^0\right)$$

which only allows propagation for $t < 0$. When such a one way propagator is used it restricts the direction in wich the particles can propagate. Effectively only one propagation direction is being considered.

Chapter 2

Kinematics and Newton's first law

2.1 Introduction

Quantum field theory (QFT) allows particles to be described by representing them as fields. In this view particles are subjected to all the conditions that any quantum theory implies, such as Heisenberg's uncertainty principle. In the classical limit these conditions are less restrictive. For instance, it is possible to measure times and energies, and positions and momenta, simultaneously with infinite precision. In order to consider particles as classical they need to be localized in both position and momentum space. Thus a restriction is placed on the source that emits the particles. The source must emit particles at positions, and with momenta, centered around certain values.

The goal in this chapter is to find a source that emits particles that are localized in position and momentum space, such that it allows a classical interpretation of these particles. The first attempt is a Gaussian source, but this gives difficulty with the calculation of the field integral. An alternative source is presented that does not suffer from this difficulty. Then the Gaussian case is studied in a different manner. After describing these sources a classical law of motion is derived for any physically acceptable source: Newton's first law. At last the classical validity of sources and the kinematic results are further discussed.

2.2 Gaussian source

As the first candidate source to allow a classical interpretation this section considers a Gaussian source. It is a reasonable choice, since the Gaussian function is a typical function with localization properties. The source is taken as

$$J(x) \propto e^{-\frac{\left(x^0\right)^2}{4\sigma_0^2}-\frac{|\vec{x}|^2}{4\sigma^2}-\frac{i}{\hbar}\left(p^0x^0-\vec{p}\cdot\vec{x}\right)}$$

which is active around $t = 0$ and $\vec{x} = 0$, with σ_0 and σ standard deviations. Its Fourier transform

$$J(k) \propto e^{-\sigma_0^2\left(k^0-\frac{p^0}{\hbar}\right)^2-\sigma^2\left(\vec{k}-\frac{\vec{p}}{\hbar}\right)^2}$$

shows that it emits particles with wavevectors $k^\mu = \left(k^0, \vec{k}\right)$ centered around $\frac{p^\mu}{\hbar}$ with momentum $p^\mu = \left(p^0, \vec{p}\right)$. Using the standard propagator

$$\Pi(x) = \frac{i}{(2\pi)^4}\int d^4k \frac{e^{-ik\cdot x}}{k^2 - m^2 + i\epsilon}$$

we obtain the field expression

$$\begin{aligned}
\phi(x) &= i\int d^4y \Pi(x-y) J(y) \\
&= \frac{-1}{(2\pi)^4}\int d^4k \frac{e^{-ik\cdot x}}{k^2 - m^2 + i\epsilon} J(k) \\
&\propto \frac{-1}{(2\pi)^4}\int d^4k \frac{e^{A(k)}}{(k^0)^2 - \omega^2 + i\epsilon}
\end{aligned}$$

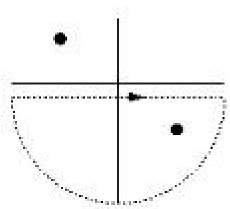
with

$$A(k) = -ik^0x^0 + i\vec{k}\cdot\vec{x} - \sigma_0^2\left(k^0 - \frac{p^0}{\hbar}\right)^2 - \sigma^2\left(\vec{k} - \frac{\vec{p}}{\hbar}\right)^2.$$

The integral over k^0 now has two poles at

$$k^0 = \pm\sqrt{\omega^2 - i\epsilon} = \pm\left(\omega - i\epsilon'\right)$$

where $k^0 \approx +\omega = \sqrt{\left|\vec{k}\right|^2 + m^2}$ is the relevant pole for $x^0 > 0$. At this point we would like to apply the method of contour integration using the contour as depicted.



We realise however that the integrand does not vanish everywhere along the curved part of the contour; when the (negative) imaginary part of k^0 becomes large the term $-\sigma_0^2\left(k^0-\frac{p^0}{\hbar}\right)^2$ in $A(k)$ becomes large and positive. This means it is not clear whether the contribution of the curved part to the integral vanishes. Closing the contour this way would therefore be unreliable; it is unclear what meaning to assign to the result.

The Gaussian source leads to a field expression for which the method of contour integration seems to fail. Without an explicit result of the integration over k^0 it is impossible to ensure that the source leads to an acceptable field expression. For this the integration over $\vec{k}$ should not diverge. In the next section an alternative source is presented, which does not suffer from this calculation difficulty.

2.3 Alternative source

We are looking for a possible source that allows for a classical interpretation of the emitted particles. For this it is required to be localized in both x-space and k-space. In addition we would like to ascertain that the field integral does not diverge.

Here we consider the source

$$J(x) \propto \frac{a}{a^2+(x^0)^2} e^{-\frac{|\vec{x}|^2}{4\sigma^2}-\frac{i}{\hbar}\left(p^0x^0-\vec{p}\cdot\vec{x}\right)}$$

which is centered around $t=0$ and $\vec{x}=0$. The positive parameter a describes the width of the time component of the distribution; the spatial part is left Gaussian and has σ as standard deviation. The Fourier transform is given by a combined exponential/Gaussian function

$$J(k) \propto e^{-a\left|k^0-\frac{p^0}{\hbar}\right|-\sigma^2\left(\vec{k}-\frac{\vec{p}}{\hbar}\right)^2}.$$

At this moment we take a break to apply our knowledge from chapter 1. Consider the expression

$$J(x) \propto \left(\frac{d}{dx^0}-i\omega\right)\left(\frac{a}{a^2+(x^0)^2} e^{-\frac{|\vec{x}|^2}{4\sigma^2}-\frac{i}{\hbar}\left(p^0x^0-\vec{p}\cdot\vec{x}\right)}\right)$$

with $\omega=\sqrt{\left|\vec{k}\right|^2+m^2}$. Here we invoke the theorem on the Fourier transformation

$$L_F\left[f'\left(x^0\right)\right] = -ik^0 L_F\left[f\left(x^0\right)\right] = -ik^0 F\left(k^0\right)$$

that was introduced at the end of section 1.4. Using this theorem the transform of the source becomes

$$J(k) \propto \left(k^0+\omega\right) e^{-a\left|k^0-\frac{p^0}{\hbar}\right|-\sigma^2\left(\vec{k}-\frac{\vec{p}}{\hbar}\right)^2}.$$

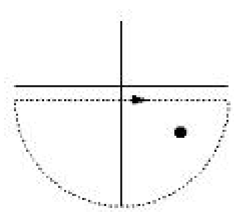
Although it is not essential for investigating the classical kinematics, we now have a source that defines the direction of time flow for the particles. Calculating the field gives

$$\begin{aligned}
\phi(x) &= i\int d^4y\,\Pi(x-y)\,J(y) \\
&= \frac{-1}{(2\pi)^4}\int d^4k\frac{e^{-ik\cdot x}}{k^2-m^2+i\epsilon}J(k) \\
&= \frac{-1}{(2\pi)^4}\int d^4k\frac{e^{-ik\cdot x}}{(k^0-\omega)(k^0+\omega)+i\epsilon}J(k) \\
&\propto \frac{-1}{(2\pi)^4}\int d^4k\frac{e^{B(k)}}{k^0-\omega+i\epsilon}
\end{aligned}$$

with

$$B(k) = -ik^0x^0+i\vec{k}\cdot\vec{x}-a\left|k^0-\frac{p^0}{\hbar}\right|-\sigma^2\left(\vec{k}-\frac{\vec{p}}{\hbar}\right)^2.$$

The integral over k^0 has one pole at $k^0=\omega-i\epsilon$ and an integration contour can be closed around the lower half of the complex plane.



Now the contour integration works and after taking the limit $\epsilon\downarrow 0$ we have the field

$$\phi(x) \propto \left\{\frac{i}{(2\pi)^3}\int d^3\vec{k}e^{C(\vec{k})}\right\}\theta\left(x^0\right)$$

with

$$C\left(\vec{k}\right) = -i\omega x^0+i\vec{k}\cdot\vec{x}-a\left|\omega-\frac{p^0}{\hbar}\right|-\sigma^2\left(\vec{k}-\frac{\vec{p}}{\hbar}\right)^2.$$

The integration over $\vec{k}$ is not straightforward, since ω also depends on $\vec{k}$. However, we can remark that the integration over $\vec{k}$ will converge. This conclusion can be drawn without calculating the actual result, because the modulus of the integrand exponentially (i.e. fast) goes to zero at infinity.

Essentially the phase of the integrand $e^{-i\omega x^0+i\vec{k}\cdot\vec{x}}$ is not relevant for this convergence. For the sake of argument it may well be taken to be stationary. Then it comes out of the integral and leads directly to a field expression for a free particle. This in fact leads to Newton's first law, as will be shown later.

Now it is clear that this source leads to a field that may be interpreted as a particle. Furthermore, due to the localization of the source, the field may be interpreted as a classical particle. Note that this argument holds for any value of the mass m.

2.4 Back to the Gaussian

In the case of the Gaussian source we found that our favoured method of contour integration fails. This was due to the fact that the contribution from the curved part of the contour does not seem to vanish. After finding a different source for which contour integration does work, we now analyze the Gaussian source again. The objective is to prove convergence of the field integral, which has not been achieved by contour integration. Here we will use a different approach, based on the complex error function.[1]

Here we will examine the field expression

$$\phi(x) \quad \propto \quad \frac{-1}{(2\pi)^4}\int d^4k \frac{e^{A(k)}}{k^0-\omega+i\epsilon}$$

with

$$A(k) = -ik^0x^0 + i\vec{k}\cdot\vec{x} - \sigma_0^2\left(k^0-\frac{p^0}{\hbar}\right)^2 - \sigma^2\left(\vec{k}-\frac{\vec{p}}{\hbar}\right)^2.$$

For simplicity only one pole is used, in the same way as demonstrated in the previous section. To simplify life further the phase of the integrand $e^{-ik^0x^0+i\vec{k}\cdot\vec{x}}$ will be taken stationary. The mass shell relation $k^0 = \sqrt{\left|\vec{k}\right|^2+m^2}$, however, follows from the pole of the integrand and will still be found. Now we require convergence of the remaining integral over k. The latter part is

$$\int dk^0 \frac{e^{-\sigma_0^2\left(k^0-\frac{p^0}{\hbar}\right)^2}}{k^0-\omega+i\epsilon}.$$

[1] This section was added after the previously described source had been constructed as an alternative to the Gaussian. With the aid of the complex error function and graphic demonstration by computer, the Gaussian case now seems to work after all, leaving the alternative source as a nice exhibition.

According to [4] (7.1.4) an integral of this type can be rewritten using the expression

$$w(z) \;=\; \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-t^2}dt}{z-t}$$

where $w(z)$ is the complex error function (Faddeeva function) and $\Im z > 0$. By substitution of variables $u = \sigma_0\left(k^0 - \frac{p^0}{\hbar}\right)$ the integral becomes

$$\int du \frac{e^{-u^2}}{u + \sigma_0\left[\frac{p^0}{\hbar} - \omega + i\epsilon\right]}$$

and by another careful substitution $t = -u$ we find

$$\int \frac{e^{-t^2}dt}{z-t}$$

with $z = \sigma_0\left(\frac{p^0}{\hbar} - \omega\right) + i\sigma_0\epsilon$. This expression meets the requirement $\Im z > 0$ and the integral over $\vec{k}$ becomes

$$\int d^3\vec{k}e^{-\sigma^2\left(\vec{k}-\frac{\vec{p}}{\hbar}\right)^2}\frac{\pi}{i}w(z).$$

The exponential term will definitely not cause divergence. For the function $w(z)$ we can turn to graphical illustration. In appendix A details are given and fortunately there appears to be no sign of diverging tendencies.

From the chain of arguments in this section we may now conclude that the Gaussian source is acceptable after all. We found that it gives a non-divergent field that can be interpreted as a classical particle.

2.5 Newton's first law

After considering possible sources for creating localized particles, now we investigate the classical kinematics that follow for the particles. For any acceptable, localized source $J(x)$ with Fourier transform $J(k)$ the field expression is

$$\begin{aligned}\phi(x) &= i\int d^4y\,\Pi(x-y)\,J(y)\\ &= \frac{-1}{(2\pi)^4}\int d^4k\frac{e^{-ik\cdot x}}{k^2-m^2+i\epsilon}J(k)\end{aligned}$$

where the source is active around $t = 0$ and $\vec{x} = 0$. In fact, we may as well use

$$\phi(x) = \frac{-1}{(2\pi)^4} \int d^4k \frac{e^{-ik\cdot x}}{k^0 - \omega + i\epsilon} J(k)$$

since we know how to remove the other pole. This integral has a pole near

$$k^0 = \omega\left(\vec{k}\right) = \sqrt{\left|\vec{k}\right|^2 + m^2}.$$

This means the integral gets its most important contributions from wavevectors around this value. However, if particles are emitted with wavevectors k centered around $\frac{p}{\hbar}$, the contribution is only appreciable when the momentum p behaves the same, so

$$\frac{p^0}{\hbar} = \omega\left(\frac{\vec{p}}{\hbar}\right).$$

Relating the zero component of the momentum to energy as

$$p^0 = \frac{E}{c}$$

we find the energy momentum relationship

$$E = \sqrt{|\vec{p}|^2 c^2 + M^2c^4}$$

with $m = \frac{Mc}{\hbar}$. This is the mass shell condition with M the mass of a particle moving freely through spacetime with energy E and momentum $\vec{p}$.

When a particle is emitted on its mass shell however, the field integral is not yet automatically large. The point is that the imaginary part of the factor $e^{-ik\cdot x}$ may lead to rapid oscillatory behaviour of the integrand and an essentially vanishing result, except when the phase of the integrand is stationary. Assuming that the term $J(k)$ does not add to the oscillatory behaviour, this happens if

$$\frac{\partial}{\partial\vec{k}}\left(-k^0x^0 + \vec{k}\cdot\vec{x}\right) = \frac{\partial}{\partial\vec{k}}\left(-\omega\left(\vec{k}\right)x^0 + \vec{k}\cdot\vec{x}\right) = -\frac{\vec{k}}{\omega\left(\vec{k}\right)}x^0 + \vec{x} = 0.$$

Realising that $\frac{\vec{k}}{k^0} = \frac{\vec{p}}{p^0}$ it follows that $\phi(x)$ is appreciable on a line in spacetime with

$$\vec{x} = \frac{\vec{p}}{p^0}ct.$$

Starting at $t = 0$ and from $\vec{x} = 0$, the (free) particle moves along a straight line with constant velocity $\frac{\vec{p}}{p^0}c$. This is Newton's first law!

2.6 Discussion

This chapter dealt with classical aspects of quantum field theory, especially finding a source for creating localized particles. A Gaussian source that shows some difficulty with field calculation was considered first. Then an alternative source was studied and found to allow a classical interpretation of the emitted particles. After some further consideration the same thing happened for the Gaussian case. Finally a classical law of motion, Newton's first law, was derived for particles emitted by any acceptable, localized source.

The central idea is that sources were found that allow for a classical interpretation of the particles. This is interesting because considering the classical limit is a good test for a quantum theory; being the more general theory it should reduce to something familiar from classical physics. In this case the kinematics of free particles turned out to be described by Newton's first law. Another advantage of considering this limit is that a classical view is easier to understand. It may help to see better what the theory actually describes.

Finally, a hypothetical word about integration. For the Gaussian source use was made of the complex error function to show convergence of the field integral. This was an alternative to the, in this case failing, method of contour integration. However, it seems very restrictive that a contour can be closed at infinity only when the integrand vanishes everywhere along the path. Perhaps circumstances may exist in which any contribution picked up along the curve is 'magically' cancelled by another contribution somewhere else. Then the residue theorem could be applied to such cases as well.

Appendix A

Complex error function

For Gaussian functions there does not exist a closed analytic form for the indefinite integral. Usually such an integral can be expressed using the error function. In the case of integration with a pole the function needs to be treated complex; the error function then becomes the so called complex error function. It is defined in [4] as

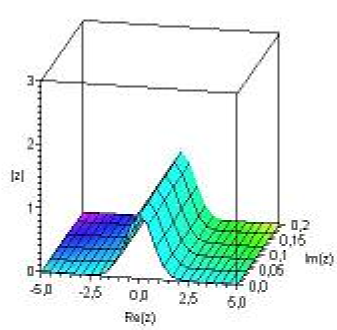
$$w(z) = e^{-z^2} erfc(-iz)$$

and is also known as the Faddeeva function. By the expression

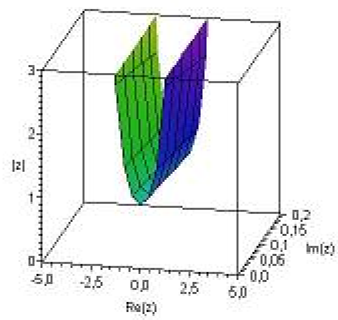
$$w(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{e^{-z^2} dt}{z - t}$$

with $\Im z > 0$, it is related to the Gaussian integral with a simple pole.

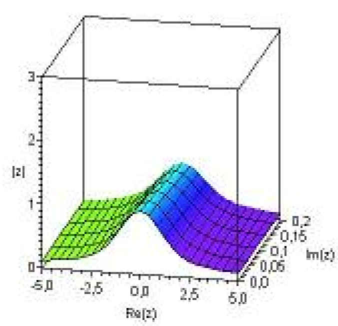
For purposes in this paper here some properties of the complex error function are demonstrated here, by graphical methods. From the definition we first look at the part e^{-z^2}. The modulus of this, for complex z looks like



where the imaginary part of z is kept small and positive. We here do not need to look at greater imaginary parts, although nothing very interesting happens anyway; the peak just shifts a bit. The modulus of the second factor in the definition, $erfc\left(-iz\right)$, looks like



for the same range of z. It is seen that the two factors making up the complex error function act against each other. The first part falls off to zero, the other goes to infinity, i.e. as a function of the real part of z. It is now interesting to see what happens to the function $w\left(z\right)$ in the same range of z. Its modulus looks like



and is seen to fall off to zero.

Practically, this result can be applied in the following sense. Imagine an integral of the type

$$\int dxw\left(z\right)$$

where $z = x + iy$ for some (small) constant y. This integral is now easily seen to converge, because of the properties of $w\left(z\right)$ displayed in the last picture above.

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