Investigating $f(R)$-gravity on hyperbolic backgrounds

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1 Introduction

1.1 A theory for quantum gravity

For over a hundred years, we have a theory of gravity called general relativity \([1]\). Predictions made by the theory in all but extreme situations are highly accurate. General relativity, however, is incompatible with quantum mechanics. This poses problems when gravity is examined at very small (sub-Planckian) length scales. Examples of situations where this becomes apparent are the inflation period of the very early universe (until \(10^{-34}\) seconds after the Big bang \([2]\)) and inside black holes. One of the greatest challenges of modern day physics is to unite general relativity and quantum mechanics and find a consistent theory of quantum gravity. This search implies extending the length scales (or, equivalently via \(E = h f = \frac{hc}{\lambda}\), the energy scales) where the theory is well-defined. We say we want to 'complete' the theory of gravity, meaning, extend it to all domains.

Let us first look at a description of general relativity and then review a situation where general relativity is insufficient to give us a hint at where we might start finding a quantum theory of gravity.

For any conservative theory, the dynamics of a system can be obtained from the action via the principle of least action. This principle states that the equations of motion of a system are determined by the solution to the variation of the action with respect to the position and velocity,

\[ \delta S = 0, \]  

where the action is defined as the time integral of the Lagrangian

\[ S = \int_{t_1}^{t_2} dt L \]  

and the Lagrangian is \( L = T - V \), the kinetic energy \( T \) minus the potential energy \( V \).

At a classical level, knowing the action and the solution to the principle of least action is thus equivalent to a description of the dynamics of a system. An action, therefore, defines a theory. For general relativity with a cosmological constant, we have the Einstein-Hilbert action

\[ S = \frac{c^4}{16\pi G} \int d^4x \sqrt{-g}(R - 2\Lambda) \]  

where \( G \) is Newton’s constant, \( g = det(g_{\mu\nu}) \) is the determinant of the metric tensor, that is used to define the causal structure of spacetime, \( \Lambda \) is the cosmological constant, and \( R \) is the scalar curvature. The integration is carried out over the whole spacetime. Here the variation is to be taken with respect to the inverse metric.

From this action and the principle of least action, the Einstein field equations

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \]  

1
can be derived \[^{3}\]. Here \(R_{\mu\nu}\) is the Ricci curvature tensor and \(T_{\mu\nu}\) the stress-energy tensor. These equations describe the gravitational interaction as a curvature of spacetime.

We will use the action to investigate the theory of quantum gravity introduced in the next subsection.

### 1.2 \(f(R)\)-gravity

Now, we review a situation where general relativity by itself is insufficient and needs an additional ingredient: the inflation period. To explain the extreme expansion of the universe very shortly after the Big Bang an inflaton field has to be introduced. Figure 1 shows the experimentally permitted potential of the inflaton field, found by the Planck 2015 collaboration \[^{4}\] and values predicted by different inflation models. The orange dots at the center of the ‘blobs’ represent a different model, proposed by Starobinsky \[^{5}\]. In this model, no inflaton field is introduced, but instead, general relativity is modified to include quantum corrections that become important on these scales. The modification, whose structure can be derived using quantum field theory, is implemented by adding a curvature-squared term in the Lagrangian

\[
S = \frac{c^4}{16\pi G} \int d^4x \sqrt{-g} \left( -2\Lambda + R + \frac{R^2}{6M^2} \right),
\]

where \(M\) has the dimension of mass. The Starobinsky model thus takes quantum corrections to the Einstein-Hilbert action, implemented as a squared scalar curvature dependence, as the source of the inflaton field. Figure 1 shows that the Starobinsky model works incredibly well.

**Figure 1:** The various colored ‘blobs’ indicate the experimentally permitted potential of the inflaton field. Most connected dots represent different models for this inflaton field; the orange dots represent the Starobinsky model.

In our search for a quantum theory of gravity, we generalize the inclusion of quantum corrections by adding higher order scalar curvature terms to the Lagrangian. This quantum gravity model is called \(f(R)\)-gravity: the Lagrangian is an undetermined general function of the scalar curvature
In this class of models $f(R) \propto -2\Lambda + R$ corresponds to classical general relativity and $f(R) \propto -2\Lambda + R + R^2/6M^2$ to the Starobinsky model. For $f(R)$-gravity we have the action

$$S = \frac{c^4}{16\pi G} \int d^4 x \sqrt{-g} f(R).$$

(6)

We will investigate this model for an arbitrary and to be determined function $f(R)$ using methods introduced in the next subsections.

### 1.3 Asymptotic Safety

A standard method to investigate a theory that needs a small adjustment is perturbation theory. Unfortunately, we cannot consider quantum effects as perturbations of a free field theory of gravity. The reasons are of quantum field theoretical origin and beyond the scope of this paper. The essential problems are infinite expressions in loop diagrams that make general relativity perturbatively nonrenormalizable [6].

An alternative approach to investigate the theory is Asymptotic Safety: a non-perturbative scenario proposed by Weinberg [7]. The key idea of this scenario is to obtain a theory where all quantities are finite for all energy scales.

Because due to the averaging out of microscopic details a system may change when zooming out. Therefore the physics (e.g., the action) can possess a scale dependency. This dependency is captured in the renormalization group flow equation, or flow equation, that will be more deeply discussed in a later subsection. The evolution with respect to the scale gives a vector field on theory space. Every point in this space corresponds to a possible action and can be characterized by coefficients $g_i$ that are the coupling constants of the theory. The dynamics of the system are described by the beta-functions

$$\beta_{g_i}(\{g_n\}) = \partial_t g_i = \frac{\partial g_i}{\partial \ln(k)}, \quad i = 0, 1, ...$$

(7)

where $k$ is the energy scale, $t = \ln(k)$ is the `renormalization group time' and $\{g_n\}$ denotes the set of coefficients.

As we want to complete the theory of gravity, we want the high energy limit $k \to \infty$ to exist. The central feature of Asymptotic Safety is, therefore, a non-Gaussian fixed point $\{g^*_n\}$ of the flow equation. At this fixed point solution all beta-functions vanish

$$\beta_{g_i}(\{g^*_n\}) = \partial_t g^*_i = 0 \quad \text{for all } i.$$  

(8)

This ensures that the theory does not unphysically diverge for the high energy scales where quantum effects come into play. This solution is a function $f(R)$ that we still have to determine.
1.4 Stability of the fixed point solutions

After finding a solution to (8), we investigate its behaviour in the theory space. More specifically, we want to find its attractive surface in this space. Energy-scale trajectories on this surface are attracted to the fixed point and therefore well-defined as the energy scale is increased, giving raise to a complete theory. This attractive surface is characterized by a set of stable eigendirections in the theory space. To find these stable eigendirections, we investigate the effect of a small perturbation $\xi_i = g_i - g_i^*$ from one coefficient of the fixed point. If the perturbation grows with time the eigendirection corresponding to the coefficient of the fixed point is unstable. If the perturbation diminishes with time, points on the corresponding eigendirection in the vicinity of the fixed point are attracted to this fixed point and the eigendirection is stable. We thus investigate the dynamics of $g_i = g_i^* + \xi_i$:

$$\partial_t g_i = \beta_{g_i}(\{g_n\})$$

$$= \beta_{g_i}(\{g_n^*\}) + \sum_j \frac{\partial \beta(g_i)}{\partial g_j} \bigg|_{\{g_n^*\}} \xi_j + O(\xi^2_j) \tag{9}$$

where we can ignore the $\xi^2_j$ and higher-order terms in the last step because the perturbation is small. The dynamics of the perturbation is therefore governed by

$$\partial_t \xi_i = \sum_j B_{ij} \xi_j \tag{10}$$

where, as a 'linearization-coefficient', we have

$$B_{ij} = \bigg|_{\{g_n^*\}} \frac{\partial \beta(g_i)}{\partial g_j} \tag{11}$$

If then $\lambda$ is an eigenvalue of $B_{ij}$ -thus satisfying $B \mathbf{v} = \lambda \mathbf{v}$ for an eigenvector $\mathbf{v}$- a solution to equation (10) is

$$\xi = C \mathbf{v} e^{\lambda t}. \tag{12}$$

Or, as we can have complex eigenvalues $\lambda = a \pm ib$ and corresponding eigenvectors $\mathbf{v} = u \pm i w$ we get a pair of solutions

$$\xi_+ = e^{at} (u \sin(bt) + w \cos(bt)), \tag{13}$$

$$\xi_- = e^{at} (u \cos(bt) - w \sin(bt)). \tag{14}$$

In fact, any solution to equation (10) is a linear combination of these solutions. Thus, the long term behavior of perturbations from the fixed point is governed by the real part of the eigenvalues $a$. The imaginary part only determines oscillatory behavior and therefore not the long term stability. The stability coefficients $\theta$ are defined as minus the eigenvalues of $B_{ij}$. A negative real part of the
stability coefficient thus means the perturbation will continue to grow with time and the eigen-
direction is unstable. A positive real part of the stability coefficient, on the other hand, means the
perturbation will diminish.

We are interested in fixed points where the linearized equation \((9)\) has a finite number of stable
directions (and thus a finite number of positive real parts of \(\theta's\)). We say the fixed point then has
a finite-dimensional attractive surface or UV critical surface. See a schematic overview of a two-
dimensional UV critical surface in Figure 2. All flows not on this surface can be disregarded as their
high energy limit diverges to infinity. The dimensionality of the UV critical surface is the number
of measurements needed to determine which trajectory is realized in nature.

Figure 2: The attractive surface in theory space. The arrows point from high to low energy scales.
All but the green colored trajectories are attracted to the fixed point and therefore stable, the green
trajectory diverges. Obtained from [8].

1.5 The flow equation

We now examine the flow equation. The effect of a changing energy scale is captured in the flow
equation:

\[
\partial_k \Gamma_k = \frac{1}{2} \text{STr} \left[ (\Gamma_k^{(2)} + R_k)^{-1} \partial_k R_k \right]
\]  

where \(\Gamma_k\) is the effective average action, a field theory analog of the classical action and \(\partial_k\) denotes
a derivative with respect to the scale \(k\). Furthermore, \(R_k\) is a regulator responsible for an infrared
cutoff, providing a minimal value for the energy and \(\Gamma_k^{(2)}\) is the second variation of \(\Gamma_k\). A subscript
of \(k\) means the function depends on the scale \(k\) and STr is the supertrace, an operation related to
the trace. The effective average action \(\Gamma_k\) is a functional of all fluctuating fields and a background
field. This background metric is taken to be maximally symmetric, such that we can have spheri-
cally symmetric \((R > 0)\) and hyperbolically symmetric \((R < 0)\) background metric. In search of a
global solution, it is interesting to investigate solutions on the hyperbolically symmetric background in addition to the investigations that have been carried out on spherically symmetric background solutions.

Equation (15) equates the change in effective physics to the contribution of quantum fluctuations (that is zero in the classical case). Thereby it determines the energy dependent flow of the effective action from $\Gamma_{k \to 0} \to \Gamma_{k \to \infty}$ in theory space. Evaluating this flow equation is beyond the scope of this project. We will work with already established differential equations for different backgrounds.

The work [9] constructed the partial differential equations that give the flow equation on three-dimensional spherical as well as hyperbolic backgrounds. In both equations the dimensionless quantities $r \equiv R/k^2$ and $\varphi_k(r) \equiv f_k(R)/k^d = f_k(k^2r)/k^3$ are used.

For spherical backgrounds the flow equation is

$$\dot{\varphi}_k + 3\varphi_k - 2r\varphi'_k = \frac{r^{3/2}}{24\sqrt{6}\pi^2} \sum_{n \geq 1} n^2 \theta(1 - \zeta) \frac{c_1\varphi'_k + c_2\varphi''_k + c_3\varphi'_k + c_4(\varphi''_k - 2r\varphi''_k)}{3\varphi_k + 4(1 - r)\varphi'_k + 4(2 - r)^2\varphi''_k}$$

(16)

with coefficients

$$c_1 = 4(3 - \zeta), \quad c_2 = 16(3 + \zeta^2) - 8r(3 + \zeta), \quad c_3 = 4(1 - \zeta), \quad c_4 = 16(1 - r + \zeta)(1 - \zeta),$$

(17)

and

$$\zeta = \frac{1}{6}(n^2 - 1)r.$$  

(18)

Here $n$ is the number of times a particle circles the sphere before returning to its starting point. The dot ($\dot{\varphi}_k$) and apostrophe ($\varphi'_k$) denote the derivatives with respect to renormalization group time $t = \ln(k)$ and dimensionless curvature $r$, respectively. The latter is due to the curvature-dependence of the Lagrangian ($\varphi_k(r)$) and the former is the energy-scale-dependence that we will investigate.

Note that, compared to [9], we take $e$ to be zero, such that "the r.h.s. is non-trivial on the entire interval $r \in [0, \infty]$ while, at the same time, the constant mode is not integrated out as the flow reaches $r \to \infty$" [9].

For hyperbolic backgrounds the flow equation looks similar:

$$\dot{\varphi}_k + 3\varphi_k - 2r\varphi'_k = \frac{1}{4\pi^2} \left(1 + \frac{r}{6}\right)^{3/2} \frac{\tilde{c}_1\varphi'_k + \tilde{c}_2\varphi''_k + \tilde{c}_3\varphi'_k + \tilde{c}_4(\varphi''_k - 2r\varphi''_k)}{3\varphi_k + 4(1 - r)\varphi'_k + 4(2 - r)^2\varphi''_k}$$

(19)

with coefficients

$$\tilde{c}_1 = \frac{4}{45}(36 + r), \quad \tilde{c}_2 = \frac{8}{189}(432 - 234r + 5r^2), \quad \tilde{c}_3 = \frac{4}{45}(6 + r), \quad \tilde{c}_4 = \frac{64}{945}(180 - 108r - 23r^2).$$

(20)

We have again taken $e = 0$ such that $r \in [-6, 0]$ and "the flow equation integrates out the whole spectrum of fluctuations on $H^3$. This means it "provides a good description of the RG flow in the realm of negative scalar curvature" [9].
1.6 This project

Fixed point solutions for $f(R)$-gravity have been found on spherically symmetric backgrounds [10–12]. However, while Asymptotic Safety has been studied extensively, there has not yet been many studies on this scenario for hyperbolic backgrounds. In particular, stable solutions for hyperbolic backgrounds have not yet been found. In this project, we search for these solutions. As it is not possible to just continue the flow equations to spacetimes of negative curvature, it is essential to study them separately. It is expected that this will also give information about the "background independence at the level of topology" [13].

The goal of this project is to find stable and converging solutions to the fixed point equation

$$\dot{\varphi}_k(r) = 0$$

by approximating $\varphi_k(r)$ using a power series and Chebyshev series expansion of different orders, which we will discuss in the next section. This means we require the number of positive $\theta$’s to remain the same when the order of our approximation is changed. An approximation where the behavior depends on the order is unreliable. We would also like to observe a convergence in the solution as well as the stability coefficients as the order of the approximation is increased.

In [13] it was found that for the flow equation on a four-dimensional hyperbolic background there is a poor convergence of the stability coefficients when using the Taylor series expansion of the solution around $R = r = 0$. We will investigate the flow equation on a three-dimensional hyperbolic background [19] and demonstrate that there exists a converging solution for this case.

The plan for the rest of this thesis is as follows: In the next section, we introduce the idea and framework of solving differential equations using spectral methods. In section 3 we extend this framework to the flow equation for a four-dimensional spherically symmetric background of which the solutions are known [14]. In section 4 we use this algorithm to find the desired solutions of the equation on the three-dimensional hyperbolic background. The discussion, conclusion, and outlook can be found in section 5. In appendix A the Chebyshev polynomials and their properties are defined.
2 Spectral methods: a hands-on example

2.1 Spectral methods

The tools we use for investigating the differential equation (19) are spectral methods. The foundation of these methods is that almost any function can be represented as an infinite sum of basis functions $\psi_n(x)$ [15]:

$$f(x) = \sum_{n=0}^{\infty} a_n \psi_n(x). \quad (21)$$

Unfortunately, we cannot work with infinite sums, so our spectral methods rely on the assumption that any function can be approximated by $N + 1$ of these basis functions [16]:

$$f(x) \approx \sum_{n=0}^{N} a_n \psi_n(x). \quad (22)$$

We thus assume the contributions from $a_{N+1}$ and higher order coefficients are negligible.

In this thesis, we use different sets of basis functions which give two kinds of approximations: the power series approximation using $\psi_n(x) = x^n$ and the Chebyshev series approximation using $\psi_n(x) = T_n(x)$. In the latter, $T_n(x)$ is the Chebyshev polynomial of degree $n$, defined in Appendix A. Both choices of basis functions have their respective (dis)advantages. An advantage of a power series expansion around $R = 0$ is the direct relation between its coefficients and the coupling constants of the theory: from equation (6) and equation (21) for $\psi_n(r) = r^n$ it can be seen that for the dimensionless Newton’s constant $g = G \cdot k^2$ and dimensionless cosmological constant $\lambda = \Lambda/k^2$ we have the relations:

$$g = -\frac{1}{16\pi a_1}, \quad \lambda = -\frac{a_0}{2a_1}. \quad (23)$$

Note that we went from Lorentzian to a Euclidean signature here. A disadvantage of a power series expansion is the dependence of the radius of convergence on singularities that we will discuss later. Chebyshev polynomials, on the other hand, converge on the whole domain. Their disadvantage is the operator mixing due to the defining recurrence relation (64c); adding a higher order to the approximation affects the lower order coefficients, as we will see later. This is also the reason there is no direct relation between the coefficients of the Chebyshev expansion and the coupling constants.

We will now use both basis sets to solve a differential equation.

2.2 The example equation and its analytic solution

To investigate the application of power series and Chebyshev series approximations for solving differential equations we designed an ordinary first order linear differential equation, the example equation:

$$\left(x^2 + \frac{1}{4}\right)^{-1} y(x) + y'(x) = 0. \quad (24)$$
Here \( y'(x) \) denotes the derivative of \( y(x) \) with respect to \( x \). The equation is designed to have a first order complex pole at \( x = \pm i/2 \). This gives us the opportunity to investigate the influence of a singularity on the radius of convergence of a power series expansion.

We can solve this differential equation analytically. Rearranging gives:

\[
\frac{y'(x)}{y(x)} = -\frac{1}{x^2 + \frac{1}{4}}. \tag{25}
\]

Integrating both sides gives

\[
\ln(y(x)) = -2\arctan(2x) + C \tag{26}
\]

which gives the solution

\[
y(x) = \hat{C} e^{-2\arctan(2x)}. \tag{27}
\]

For practical reasons we will choose \( \hat{C} \) to be equal to one. This means \( y(0) = 1 \). The solution is well-defined on the entire real axis as \( y(\pm\infty) = e^{\mp\pi} \) and is shown as the blue curve in Figure 4.

### 2.3 Power series approximation

We will now write \( y(x) \) as a power series:

\[
y(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{28}
\]

The derivative can be taken termwise and is

\[
y'(x) = \sum_{n=0}^{\infty} a_n n x^{n-1} \tag{29}
\]

where the sum can start at \( n = 0 \) because this term does not contribute. Filling in the power series representations in our example equation (24) we obtain

\[
\sum_{n=0}^{\infty} a_n x^n + \sum_{n=2}^{\infty} a_{n-1}(n-1)x^n + \frac{1}{4} \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n = 0. \tag{30}
\]

We can now change the dummy-indices to write all terms as sums of \( x^n \)

\[
0 = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=2}^{\infty} a_{n-1}(n-1)x^n + \frac{1}{4} \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n
\]

\[
= (a_0 + \frac{1}{4}a_1) + (a_1 + \frac{1}{2}a_2)x + \sum_{n=2}^{\infty} \left( a_n + a_{n-1}(n-1) + \frac{1}{4}a_{n+1}(n+1) \right) x^n. \tag{31}
\]

Every coefficient of \( x^n \) has to equal zero, which gives rise to relations for the coefficients

\[
n = 0 : \quad a_0 + \frac{1}{4}a_1 = 0, \tag{32a}
\]

\[
n = 1 : \quad a_1 + \frac{1}{2}a_2 = 0, \tag{32b}
\]

\[
n > 1 : \quad a_n + a_{n-1}(n-1) + \frac{1}{4}a_{n+1}(n+1) = 0. \tag{32c}
\]
Together with the normalization condition $a_0 = 1$ (as we chose $y(0) = 1$) these relations determine the coefficients.

For a series approximation truncated after the $N^{th}$ term, the normalization condition together with $N$ relations is sufficient to determine the approximation.

A power series with five basis points ($N = 4$) gives the following approximate solution for the example equation (24):

$$y(x) \simeq 1 - 4x + 8x^2 - 5.3333x^3 - 10.6667x^4 + O(x^5).$$

(33)

This result is shown as the yellow curve in Figure 4. Note that the coefficients agree with the ones obtained from a Taylor series expansion of the exact solution at $x = 0$.

A power series will converge for some values of $x$ and may diverge for others. The Cauchy-Hadamard theorem [17] states that the radius of convergence of a power series is given by

$$\frac{1}{r_{\text{con}}} = \limsup_{n \to \infty} \left( |a_n|^{1/n} \right)$$

(34)

where $r$ is the radius of convergence and lim sup denotes the upper limit. This radius then, can be computed by

$$r_{\text{con}} = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|.$$  

(35)

We have plotted $|a_{n-1}/a_n|$ of the power series solution around $x = 0$ for $n = 1$ to $n = 19$ in Figure 3.

The plot shows a convergence to the theoretically expected radius of convergence of 0.5, derived from Darboux’s principle for ordinary linear differential equations. This principle states that a regular spectral series converges on the largest domain that is free of complex singularities [16]. For a power series this domain is a disk, thus the radius of convergence of a power series is equal to the distance from the center to the nearest singular point. The complex singular point of the example equation is located at $x = \pm i/2$, where the denominator goes to zero.

We can also calculate the radius of convergence from (35) and the recursive relations for the coefficients (32c):

$$r_{\text{con}} = \lim_{n \to \infty} \sqrt[n]{\left| \frac{a_n}{a_{n+2}} \right|} = \lim_{n \to \infty} \sqrt[n]{\frac{1}{n} \left( -a_{n+1} - \frac{1}{2} a_{n+2} (n + 2) \right)} = \sqrt[n]{\frac{-a_{n+2}}{a_{n+2}}} = \frac{1}{2}.$$  

(37)

We thus see that the power series approximate solution only converges from $-0.5 < x < 0.5$, where the analytic solution is well-defined on the entire real axis. This is a clear disadvantage of this approximation method.
Figure 3: The ratio of $a_n$ over $a_{n+1}$ for $n = 1$ to $n = 19$ for the power series approximation coefficients.

We can also perform a power series expansion around the center $x = 1$. The power series then becomes
\[ y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n. \]  
(38)

We can use the same strategy as above, utilizing that $\left( \frac{1}{4} + x^2 \right) = (x-1)^2 + 2(x-1) + \frac{5}{4}$ to obtain a similar equation:
\[
\left( a_0 + \frac{5}{4}a_1 \right) + \left( 3a_1 + \frac{5}{2}a_2 \right)(x-1) + \sum_{n=2}^{\infty} \left( (n-1)a_{n-1} + (2n+1)a_n + \frac{5}{4}(n+1)a_{n+1} \right)(x-1)^n = 0.
\]
(39)

Again, equating every coefficient to zero gives our recursive relations
\[
\begin{align*}
\text{n = 0:} & \quad a_0 + \frac{5}{4}a_1 = 0, \\
\text{n = 1:} & \quad 3a_1 + \frac{5}{2}a_2 = 0, \\
\text{n > 1:} & \quad (n-1)a_{n-1} + (2n+1)a_n + \frac{5}{4}(n+1)a_{n+1} = 0.
\end{align*}
\]
(40a, 40b, 40c)

As for the series expansion, we use the initial condition in the expansion point: $y(1) = a_0 = e^{-2\arctan(2)} \approx 0.1092$. This gives the following approximate solution for the truncation at $N = 4$:
\[
y(x) \approx 0.1092 - 0.0874(x-1) + 0.1049(x-1)^2 - 0.1165(x-1)^3 + 0.1212(x-1)^4 + O(x^5)
\]
(41)
\[= 0.5392 - 1.1313x + 1.1814x^2 - 0.6012x^3 + 0.1212x^4 + O(x^5).
\]
(42)

This result is plotted as the green curve in Figure 4.
2.4 Chebyshev series approximation

We can also write \( y(x) \) as a series of Chebyshev polynomials. The defining relations for these polynomials can be found in Appendix A. We write

\[
y(x) = \sum_{n=0}^{\infty} b_n T_n(x).
\]

Using relations (70) and (71) we write our example equation (24) in terms of Chebyshev polynomials of the second kind:

\[
\sum_{n=0}^{\infty} \frac{b_n}{2} (U_n(x) - U_{n-2}(x)) + \sum_{n=0}^{\infty} \frac{n}{4} b_n (U_{n+1}(x) + 3U_{n-1}(x) + U_{n-3}(x)) = 0
\]

where we used the recurrence relation (67c) twice to write

\[
\left( \frac{1}{4} + x^2 \right) \sum_{n=0}^{\infty} b_n n U_{n-1}(x) = \sum_{n=0}^{\infty} b_n \frac{n}{4} ((U_{n+1}(x) + 3U_{n-1}(x) + U_{n-3}(x)).
\]

As for the power series this allows us to equate to zero the coefficients for each \( n \). It is important to take relation (48) into account. We obtain the recurrence relations

\[
U_0 : \quad b_0 + \frac{b_1}{2} - \frac{b_2}{2} + \frac{3b_3}{4} = 0
\]

\[
U_n : \quad \frac{n-4}{4} b_{n-4} + \frac{b_{n-3}}{2} + \frac{3(n-2)}{4} b_{n-2} - \frac{b_{n-1}}{2} + \frac{n}{4} b_n = 0
\]

Together with the normalization condition \( y(0) = \sum_{n=0}^{N} b_n T_n(0) = 1 \), \( N \) relations are sufficient to determine a \( N \)th order approximation.

A Chebyshev series with five basis points gives the following approximate solution:

\[
y(x) \simeq 2.8987 T_0(x) - 4.2532 T_1(x) + 1.6203 T_2(x) + 0.0506 T_3(x) - 0.2785 T_4(x) + O(x^5)
\]

\[
= 1 - 4.4051 x + 5.4684 x^2 + 0.2025 x^3 - 2.2279 x^4 + O(x^5)
\]

This result is shown as the red line in Figure 1. The relative errors of the different approximations are plotted on a logarithmic scale in Figure 2. These plots embody the main result from this section.

For Chebyshev polynomials the aforementioned convergence domain is an ellipse with foci at \( x = \pm 1 \). A Chebyshev series thus has (infinite order) convergence if all its derivatives are bounded on the interval \([-1, 1]\). We can rescale every finite interval to this domain. From Figure 1 and 2 it can be seen that the Chebyshev series converges on the whole domain, which is the reason we want to use this approximation in our project.
Figure 4: From the top down at $x = -1$: the analytic solution of the example equation in blue, the Chebyshev series approximation for $N = 4$ in red, the power series approximation around $x = 0$ for $N = 4$ in yellow and the power series approximation around $x = 1$ for $N = 4$ in green.

Figure 5: Logarithmic plot of the relative errors. From the top down at $x = -1$: the difference between the exact solution and the power series approximation around $x = 1$ for $N = 4$ in yellow, between the exact solution and the power series approximation around $x = 0$ for $N = 4$ in blue and between the exact solution and the Chebyshev series approximation for $N = 4$ in green.
3 The test: on a four-sphere background

We will now go on to test this method for solving differential equations with spectral methods on a flow equation for which the power series solution has already been constructed [14].

3.1 Finding fixed point solutions

We extend the framework developed above to cope with the flow equation on a four-dimensional spherically symmetric background. As the power series solution to this equation is known, we can test and evaluate our algorithm.

The fixed point equation under investigation, which was derived in [14], is

\[ 768\pi^2 (2\varphi(r) - r\varphi'(r)) = \frac{\dot{c}_1 \varphi'(r) + \dot{c}_2 \varphi''(r) + \dot{c}_3 (\varphi'(r) - r\varphi''(r))}{\varphi(r) + (1 - \frac{2}{3})\varphi'(r)} + 10r^2 + \frac{607r^2}{60} - 6r - 36 + \frac{511r^2}{90} - 4r - 12 \tag{50} \]

with coefficients

\[
\begin{align*}
\dot{c}_1 &= \left(\frac{7249r^3}{2268} - \frac{136r^2}{9} - 30r + 80\right), & \dot{c}_2 &= -2r \left(\frac{7249r^3}{4536} - \frac{271r^2}{36} - 5r + 10\right), \\
\dot{c}_3 &= 5r^2 \left(\frac{2r}{3} + 3\right), & \dot{c}_4 &= -\left(\frac{37r^3}{756} + \frac{29r^2}{15} + 18r + 48\right), \\
\dot{c}_5 &= -\left(\frac{37r^4}{756} - \frac{29r^3}{10} - \frac{121r^2}{5} - 12r + 216\right), & \dot{c}_6 &= 2r \left(-\frac{181r^4}{3360} - \frac{29r^3}{30} - \frac{91r^2}{20} + 27\right), \\
\end{align*}
\]

which is a third order nonlinear differential equation. It is, on the other hand, ordinary since it is the fixed point equation: \( \dot{\varphi}(r) \) and its derivatives are put to zero. The domain is \( 0 < r < 6 \). Note that compared to [14] we took all Heaviside stepfunctions \( \theta(x) \) to be equal to one, which is valid if all arguments are bigger than zero and therefore we restrict the domain to \( 0 < r < 3 \).

We extend the framework for solving differential equations from the previous section. We use a power series approximation, as this was also done in the paper [14], allowing us to compare the results. The most natural point to expand about is \( R = r = 0 \), the small curvature expansion.

Solving (50) for \( \varphi''(r) \) gives a denominator \( 2r \left(\frac{-181r^4}{3360} - \frac{29r^3}{30} - \frac{91r^2}{20} + 27\right) \) giving rise to singularities at \( r = 0, r = -4.97634 \pm 0.468514i, r = 9.99855 \) and \( r = 2.00648 \). The last singularity is closest to \( r = 0 \) and therefore sets the radius of convergence \( r_{con} = 2.00648 \).

As before, we want to obtain equations for every power of \( r \). However, due to the complexity and nonlinearity of this differential equation we cannot find a general recursion relation for the coefficients. Therefore, we use a power series approximation for \( \varphi(r) \) truncated at \( N \):

\[
\varphi(r) = \sum_{n=0}^{N} a_n r^n. \tag{52}
\]
Since we are dealing with a third-order differential equation, we will start with a series expansion
with three basis points: \( N = 2 \). We obtain our equations for different powers of \( r \) by substituting
this approximation into the fixed point equation \((50)\), differentiating zero, one and two times and
then putting \( r \) to zero. These equations can then be solved for the coefficient \( a_0 \), \( a_1 \) and \( a_2 \).

We obtain seven nonzero real solutions, of which one, \( a_0 = 0.003328 \), \( a_1 = -0.01251 \), \( a_2 = 0.001490 \),
corresponds to the one found in the paper, see Table 1. The coefficients of the solutions have to be
real because they correspond to physical coupling constants (e.g., via the relations in equation \((23)\)).

<table>
<thead>
<tr>
<th>( N )</th>
<th>( g^*_0 )</th>
<th>( g^*_1 )</th>
<th>( g^*_2 )</th>
<th>( g^*_3 )</th>
<th>( g^*_4 )</th>
<th>( g^*_5 )</th>
<th>( g^*_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00523</td>
<td>-0.0202</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.00333</td>
<td>-0.0125</td>
<td>0.00149</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.00518</td>
<td>-0.0196</td>
<td>0.00070</td>
<td>-0.0104</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.00505</td>
<td>-0.0206</td>
<td>0.00026</td>
<td>-0.0120</td>
<td>-0.0101</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.00506</td>
<td>-0.0206</td>
<td>0.00023</td>
<td>-0.0105</td>
<td>-0.0096</td>
<td>-0.00455</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.00504</td>
<td>-0.0208</td>
<td>0.00012</td>
<td>-0.0110</td>
<td>-0.0109</td>
<td>-0.00473</td>
<td>0.00238</td>
</tr>
</tbody>
</table>

Table 1: Power series expansion coefficients of the fixed point solution \((50)\) for different values of \( N \),
from \([14]\). We write \( g^*_i \) for the coefficients from the literature to tell them apart from the coefficients
\( a_i \) found in this research.

Now we want to know how good our approximation \( a_0 + a_1 r + a_2 r^2 \) works. To examine this, we
increase the order of the approximation and investigate how the coefficients change in the next
subsections.

### 3.2 First method: fixed found coefficients

There are two ways to obtain these higher order approximate solutions. One method is to keep the
first three coefficients as we found them and see how big the contribution from the higher order
coefficients is. We find these higher order coefficients by filling in the higher order power series
approximation, differentiating \( N \) times and filling in the already found coefficients \( a_0 \), \( a_1 \), \ldots, \( a_{N-1} \).

We find that this gives a very good convergence, see Table 2. The solutions found in the paper,
on the other hand, do not show this convergence, see the plot in Figure 6 and Table 1. Where the
solutions for \( N = 2 \) are equal, the higher order approximations differ a lot in value and convergence.

This motivates us to use another method for obtaining the higher order approximations in the next
subsection.

### 3.3 Second method: optimizing all coefficients

In this method the already obtained coefficients can be adjusted in the higher order approximation.
The search for a higher order solution starts at the values found earlier and at zero for the new
Table 2: Power series expansion coefficients for different values of $N$ of the fixed point solutions of equation (50) obtained with our first method. The solution seems to converge very well, but also note the extraordinary high value of $a_5$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$a_0$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.00333</td>
<td>-0.01251</td>
<td>0.00149</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.00333</td>
<td>-0.01251</td>
<td>0.00149</td>
<td>-0.00005</td>
<td>0.00027</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.00333</td>
<td>-0.01251</td>
<td>0.00149</td>
<td>-0.00005</td>
<td>0.00027</td>
<td>0.00073</td>
</tr>
<tr>
<td>5</td>
<td>0.00333</td>
<td>-0.01251</td>
<td>0.00149</td>
<td>-0.00005</td>
<td>0.00027</td>
<td>0.00073</td>
</tr>
<tr>
<td>6</td>
<td>0.00333</td>
<td>-0.01251</td>
<td>0.00149</td>
<td>-0.00005</td>
<td>0.00027</td>
<td>0.00073</td>
</tr>
</tbody>
</table>

Figure 6: Comparing the solutions for different values of $N$ from [14] and our first method. The thick line are the indistinguishable solutions from our method for $N = 2, 3, 4, 5$, and the solution from the paper for $N = 2$. Below are the higher order solutions from [14]. Plotted for $0 < r < r_{con}$.

coefficient, but the algorithm keeps all $a_i$’s as free parameters that are determined from the polynomial equations. We do have to mention that we have to give the algorithm a hint to start looking for $a_3$ at -0.001, in stead of the standard value of zero, where it does give a very different solution. This gives us valuable information. There are multiple higher order solutions, of which one is in accordance to the one found in [14], as can be seen in Figure 7 and Table 3. However, the selection algorithm does not automatically selects this solution. Without knowledge of the correct solution from the literature the algorithm would have given a different solution. We have to keep this in mind in our further investigations.

This is the central result from this section. We will now go on to investigate the stability of the found solution.

3.4 Stability and convergence

We analyze the stability of our solution by calculating the stability coefficients $\theta$. This is done by substituting the power series

$$\dot{\phi}(r) = \sum_{n=0}^{N} \beta_n(a_n)r^n$$

(53)
Figure 7: Comparing the solutions for different values of $N$ from [14] and our second method. We just observe five lines for four different values of $N$. The solutions from the paper and the second method are indistinguishable.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$a_0$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_5$</th>
<th>$a_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.003328</td>
<td>-0.012513</td>
<td>0.001490</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.005185</td>
<td>-0.020644</td>
<td>0.000226</td>
<td>-0.010460</td>
<td>-0.010109</td>
<td>-0.004565</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.005048</td>
<td>-0.02065</td>
<td>0.000256</td>
<td>-0.010528</td>
<td>-0.009598</td>
<td>-0.004556</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.005057</td>
<td>-0.020823</td>
<td>0.000125</td>
<td>-0.011015</td>
<td>-0.010985</td>
<td>-0.004729</td>
<td>0.002383</td>
</tr>
</tbody>
</table>

Table 3: Power series expansion coefficients of the approximate solutions of the fixed point equation (50) for different values of $N$. Obtained using our second method. The results agree with Table 1.

Comparing equal powers of $r$ gives us equations that we solve for the beta-functions. Using equation (11) we construct the matrix $B$. Substituting in our fixed point solution and generating the eigenvalues of the matrix then gives the stability coefficients, found in Table 4.

\[
384\pi^2 (\dot{\varphi}(r) + 4\varphi(r) - 2r\varphi'(r)) = \frac{d_1\varphi''(r) + d_2\varphi'(r) + d_3\varphi'(r) + \frac{607r^2}{60} - 6r - 36}{(1 - \frac{r}{3}) \varphi'(r) + \varphi(r)} + \frac{511r^2}{12} - 4r - 12 + \frac{4536}{1 - \frac{r}{3}} \frac{d_4\varphi'''(r) + d_5\varphi''(r) + d_6\varphi'(r) + d_7\varphi''(r) + d_8\varphi'(r)}{9 (1 - \frac{r}{3})^2 \varphi''(r) + 3 (1 - \frac{2r}{3}) \varphi'(r) + 2\varphi(r)}
\]

with coefficients

\[
\begin{align*}
    d_1 &= -\frac{14809r^4}{2268} - \frac{539r^3}{18} + 10r^2 - 20r, & d_2 &= \frac{14809r^3}{2268} + \frac{134r^2}{9} - 30r + 80, \\
    d_3 &= \frac{14809r^4}{4356} + \frac{539r^2}{36} - 5r + 10, & d_4 &= \frac{181r^5}{1680} + \frac{29r^4}{15} + \frac{91r^3}{10} - 54r, \\
    d_5 &= -\frac{37r^4}{756} - \frac{29r^3}{10} - \frac{121r^2}{5} - 12r + 216, & d_6 &= \frac{37r^3}{756} + \frac{29r^2}{30} + 18r + \frac{1469}{30}, \\
    d_7 &= -\frac{181r^4}{3360} - \frac{29r^3}{30} + \frac{91r^2}{20} + 27, & d_8 &= \frac{37r^3}{1512} + \frac{29r^2}{60} + 3r + 6,
\end{align*}
\]

that is obtained from [14]. Note that we have again put all Heaviside stepfunctions to one, again restricting the domain to $0 < r < 3$.

Comparing equal powers of $r$ gives us equations that we solve for the beta-functions. Using equation (11) we construct the matrix $B$. Substituting in our fixed point solution and generating the eigenvalues of the matrix then gives the stability coefficients, found in Table 4.
Table 4: The stability coefficients for $N = 2$ to $N = 6$. The first two coefficients are a complex pair $\theta = \theta' \pm i\theta''$. The stability coefficients from the literature can be found in Table 5.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\theta'$</th>
<th>$\theta''$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>$\theta_4$</th>
<th>$\theta_5$</th>
<th>$\theta_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.163</td>
<td>-2.053</td>
<td>29.131</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>2.851</td>
<td>-2.213</td>
<td>2.71</td>
<td>-4.286</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2.964</td>
<td>-2.323</td>
<td>2.181</td>
<td>-3.497+1.644 i</td>
<td>-3.497-1.644 i</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2.056</td>
<td>-2.304</td>
<td>3.688</td>
<td>-3.707</td>
<td>1.487 +5.508 i</td>
<td>1.487 -5.508 i</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>2.438</td>
<td>-2.148</td>
<td>3.355</td>
<td>-6.277</td>
<td>-0.149+7.681 i</td>
<td>-0.149-7.681 i</td>
<td>-4.067</td>
</tr>
</tbody>
</table>

Table 5: The stability coefficients for $N = 2$ to $N = 6$ from [14]. The first two coefficients are a complex pair $\theta = \theta' \pm i\theta''$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\theta'$</th>
<th>$\theta''$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>$\theta_4$</th>
<th>$\theta_5$</th>
<th>$\theta_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.26</td>
<td>-2.44</td>
<td>27.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2.67</td>
<td>-2.26</td>
<td>2.07</td>
<td>-4.42</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2.83</td>
<td>-2.42</td>
<td>1.54</td>
<td>-4.28</td>
<td>-5.09</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2.57</td>
<td>-2.67</td>
<td>1.73</td>
<td>-4.40 -3.97 + 4.57 i</td>
<td>-3.97 - 4.57 i</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>2.39</td>
<td>-2.38</td>
<td>1.51</td>
<td>-4.16 -4.67 + 6.08i</td>
<td>-4.67 - 6.08i</td>
<td>-8.67</td>
<td></td>
</tr>
</tbody>
</table>

When comparing the stability coefficients from our computations in Table 4 and the literature in Table 5 we notice that they roughly agree. The clearest differences are the complex pair we find for $N = 4$ that does not appear in the literature and the sign of the real part of the complex pair for $N = 5$. Apart from this latest pair, the number of positive (real parts of the) found stability coefficients is constant and equal to that of the literature.

### 3.5 Using Chebyshev series expansions

We will now go on to analyze the same flow equation using a Chebyshev series expansion. The main reason is the singularity at $r = 2.00648$ limiting the convergence domain. To be able to use Chebyshev polynomials we shift the domain from $r \in [0,3]$ to $x \in [-1,1]$ thus introducing a new variable $x$ by $x = 2r/3 - 1$.

After substituting the approximation $\varphi(x) = \sum_{n=0}^{N} a_n T_n(x)$ in the fixed point flow equation we want to obtain equations for every order of the Chebyshev polynomials. To retrieve these, we make use of the orthogonality equation [65]: we multiply the flow equation with the Chebyshev series approximation with Chebyshev polynomials of different orders and integrate from -1 to 1. As before, we start with a third order approximation ($N = 2$) and then multiply with $T_0(x)$, $T_1(x)$ and $T_2(x)$ to obtain three equations. Solving these equations for the coefficients gives us fifteen nonzero real solutions.

Because of the operator mixing in Chebyshev polynomials using our second method to obtain higher order approximations is more challenging. We can, however, make series expansions in $x$ of the lower order approximations and all found higher order approximations and compare these. Then
the optimal higher order approximation can be selected. For some solutions the same higher order approximation is selected, giving nine unique solutions for $N = 6$. Plotting the sixteen solutions on the domain where we know the power series solution from the literature in Figure 8 and comparing these to the known solution makes us select one solution to investigate further: the third solution. In general of course, we do not have a solution to compare with and need different methods to evaluate the solutions. In this situation, however, the objective is to compare the solutions we found using the Chebyshev expansions to the one from the literature that uses power series expansions.

Figure 8: The sixteen solutions found using the Chebyshev expansions for $N = 2$ to $N = 6$. Plotted on the same domain and range as used in Figure 7.

A closer look at the third solution and the power series from the literature can be seen in Figure 9.

Figure 9: Comparison of the third solution found using Chebyshev expansions and the power series solutions from [14], for $N = 2$ to $N = 6$.

We observe some strong similarities, mainly from $N = 4$ on and up to around $r = 1.5$. A clear difference is that the Chebyshev solutions 'climb' for higher $N$, while the power series solutions from the literature only show this behaviour from $N = 5$ onward. Motivated by this observation, we also calculate and plot the power series solutions for $N = 7, 8$ -as the literature solutions are unknown here- and the Chebyshev series solution for $N = 7, 8$ in Figure 10. Here we evaluate the solutions on the whole domain $r \in [0, 3]$. 
We can clearly see the influence of the radius of convergence at \( r = 2.00648 \) on the power series solutions. The Chebyshev solutions do not have this dependence, which is the reason we are using them. Furthermore, we observe that both the Chebyshev solutions and the power series solutions from \( N = 5 \) onward 'climb' for increasing \( N \). The Chebyshev solutions have a less steep decline, but this is probably because they do not diverge towards the radius of convergence like the power series solutions. Because of this difference between the different methods it is also more difficult to compare the Chebyshev solutions to the power series solutions from the literature and to rate them by doing this. We conclude that the solutions are similar on the interval \( r \in [0,1] \) and that the global character of the Chebyshev solutions gives us enough reason to also apply them in the next section.

We can determine the stability coefficients for this third solution, see Table 6. Unlike for the stability coefficients of the power series expansions there is no converging complex pair or constant number of positive theta’s. No clear convergence is observed at all.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \theta_1 )</th>
<th>( \theta_2 )</th>
<th>( \theta_3 )</th>
<th>( \theta_4 )</th>
<th>( \theta_5 )</th>
<th>( \theta_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>30.425</td>
<td>5.844</td>
<td>-0.226</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4.469 +0.658 i</td>
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<td>-4.952+3.007 i</td>
<td>-4.952-3.007 i</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>17.617</td>
<td>4.022</td>
<td>2.105</td>
<td>-8.206+4.072 i</td>
<td>-8.206-4.072 i</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>4.005 +2.009 i</td>
<td>4.005 -2.009 i</td>
<td>-3.338+10.512 i</td>
<td>-3.338-10.512 i</td>
<td>-6.118</td>
<td>-21.34</td>
</tr>
</tbody>
</table>

Table 6: The first six stability coefficients for \( N = 2 \) to \( N = 6 \) from our third Chebyshev expansion solution.

Computing the higher order stability coefficients takes a lot of computing power and time and does not add valuable information as we can already see the stability coefficients are not converging.
From this testing section we conclude that we are able to find solutions that agree with those found in the literature. Unfortunately, we have no infallible selection procedure to select these correct (higher order) solutions. We will now go on to apply our method to a flow equation where no literature solutions exist, bearing in mind the findings from this section.
4  Fixed points on three-dimensional hyperbolic spaces

4.1  Finding fixed point solutions

We now investigate the flow equation on hyperbolic spaces in conformally reduced gravity in three Euclidean dimensions \([10]\). In previous studies of conformally reduced flow equations \([50, 51, 52, 53, 54]\) it was found that they "lead to results quite similar to the full case, where the contributions of all metric fluctuations are included" \([18]\).

The fixed point equation is obtained by putting \(\dot{\phi}(r)\) and its derivatives to zero:

\[
(3\phi(r) - 2\phi'(r)) \left(4(1-r)\phi'(r) + 4(2-r)^2\phi''(r) + 3\phi(r)\right) = \frac{c_1^* \phi'(r) + c_2^* \phi''(r) + c_3^* \phi'''(r)}{4\pi^2} \tag{56}
\]

with

\[
c_1^* = \frac{8}{15} \left(\frac{r}{6} + 6\right),
\]

\[
c_2^* = \left(\frac{r}{6} + 1\right)^{3/2} \left(-\frac{16}{63} + 12 \left(\frac{r^2}{36} - \frac{r}{6} - 6\right) + 49r\right),
\]

\[
c_3^* = \frac{128}{315} \left(\frac{23r}{2} - 15\right) \left(\frac{r}{6} + 1\right)^{5/2} r
\]

on the domain \(-6 < r < 0\).

Thus, we have again a third order nonlinear ordinary differential equation to solve. Due to its success in the previous section, we start with a power series expansion around \(r = 0\) and apply the second method. Solving equation \([56]\) for \(\phi'''(r)\) gives a denominator \(32 \left(\frac{r}{6} + 1\right)^{5/2} r \left(\frac{23r}{2} - 15\right)\). We therefore have fixed singularities situated at \(r = 0\), \(r = -6\) and \(r = 1.30435\). The last determines the radius of convergence.

As it is again a third order differential equation, we again start using three basis points. This results in three nonzero real solutions. In the previous section, where there were seven solutions, we chose the one corresponding to the solution from the literature. Now we evaluate all three solutions, using the second method to raise the order of the approximation up to \(N = 11\). The three solutions for all orders of the approximation are shown in Figure 11. All solutions are plotted on the domain \(-1.3 < r < 0\) because of the singularity at \(r = 1.30435\).

4.2  Analyzing the solutions

Now, we analyze the solutions. We will start with the second and the third solution and subsequently turn to the first solution. The second and third solution have similar features. Both solutions show rapid convergence (from \(N = 3\) and \(N = 4\) onward, respectively) and both are continuously increasing.

We will now look at the physical interpretation of these features. As can be seen from equation \([23]\), both the second and the third solution account for a negative Newtons constant and therefore a repulsive gravity. We have no observations of gravity in this trans-Planckian regime, but the sign
Figure 11: The fixed point solutions of equation (56) using the power series approximation for $N = 2$ to $N = 11$. For the second and third solution, the approximations for $N = 4$ to $N = 11$ lie on top of each other on the middle and lower line, respectively. The first solution ‘climbs’ with every increase of the order.

of Newton’s constant is unchanged along the flow in a polynomial $f(R)$ treatment. Therefore, this implies we have the wrong sign for Newton’s constant in Einsteins equations, too. We do not expect this and discard these solutions on physical grounds.

Now we examine the first solution, whose coefficients can be found in Table 7. This solution shows a particular behavior when changing the order of the approximation: where $a_0$ seems to stay the same, and $a_3$ and higher order coefficients do not seem to contribute much, $a_1$ runs from positive to negative values and $a_2$ from negative to positive. The latter however, becomes smaller for higher order where the behaviour is mainly determined by $a_0$ and $a_1$. When increasing the order above $N = 8$, we seem to observe a convergence, see Figure 12. The diminishing of the highest coefficients means that the new higher order contributions are negligible and our truncation is justified.

Here the solution for $N = 13$ is selected manually. The algorithm picks out a solution that looks very different from the lower approximations as well as the higher order that is derived from it. This is likely because the algorithm selects a solution where the coefficients $a_2, a_3, \ldots, a_{12}$ are in great accordance to the ones for $N = 12$, but where $a_0$ and $a_1$ are not. As these two mainly determine the shape of the solution we manually selected another solution. The approximation for $N = 13$ still seems a bit off, but the convergence is very apparent.

This does not disqualify the solution or make it less accurate. In the previous section we had already encountered that the automatic selection is not infallible and a manual selection may be
Figure 12: Fixed point solution 1 for $N = 8$ to $N = 14$, where the solution for $N = 13$ was selected manually. The solution seems to converge.

needed to find the best solution. We are investigating if there exists a solution that converges when the order of the approximation is increased, not if the algorithm selects this solution automatically. We thus have a solution with a clear convergence and -from $N = 6$ onward- a positive Newton’s constant. We will analyze the stability and convergence of this solution in the next subsection.

<table>
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<tr>
<th>$N$</th>
<th>$a_0$</th>
<th>$a_1$</th>
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<th>$a_4$</th>
<th>$a_5$</th>
<th>$a_6$</th>
<th>$a_7$</th>
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<td></td>
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</tr>
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<td>0.00008</td>
<td>0.00002</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 7: The first eight coefficients of fixed point solution one.

4.3 Stability and convergence

We again analyze the stability of our solution by calculating the stability coefficients $\theta$. We now substitute the power series

$$ \dot{\varphi}(r) = \sum_{n=0}^{N} \beta_n(a_n)r^n $$

(58)
into the flow equation (19) and calculate the stability coefficients as before. For our first fixed point solution these are found in Table 8.

\[
\begin{array}{cccccccc}
N & \theta_1 & \theta_2 & \theta_3 & \theta_4 & \theta_5 & \theta_6 & \theta_7 \\
2 & 2.906 & 0.827 & -1.532 & & & & \\
3 & 2.841 & 0.975 & -1.425 & -4.543 & & & \\
4 & 2.741 & 1.094 & -1.685 & -5.164 & -10.152 & & \\
7 & 1.91+0.386i & 1.91-0.386i & -3.374 & -7.458+5.256i & -7.458-5.256i & -15.457+4.792i & \\
8 & 1.906+0.683i & 1.906-0.683i & -4.063 & -6.669+5.615i & -6.669-5.615i & -14.14+7.74i & \\
9 & 1.904+0.894i & 1.904-0.894i & -4.378 & -6.258+5.691i & -6.258-5.691i & -12.816+9.029i & \\
12 & 2.474+1.924i & 2.474-1.924i & 0.023 & -4.569+5.638i & -4.569-5.638i & -10.181+5.512i & \\
\end{array}
\]

Table 8: The first six stability coefficients for the first fixed point solution using the second method, for \(N = 2\) to \(N = 12\). The higher stability coefficients have been omitted for clarity reasons. They do not constitute essential information at this point.

For all orders but \(N = 12\), there are two positive real parts of the theta’s, giving a two-dimensional attracting surface in theory space (like in Figure 2). Up to \(N = 6\) the first stability coefficient decreases and the second increases. Then there is a jump. From \(N = 7\) onward, the first two coefficients are a complex pair, where up to \(N = 11\) the real part resides between 1.904 and 1.955.

The stability coefficients of the second and third fixed point solutions, found in Table 9 and Table 10 respectively, show a much clearer convergence. Both have only real-valued stability coefficients, and both have two positive real parts of the \(\theta’s\) for all orders.

\[
\begin{array}{cccccccc}
N & \theta_1 & \theta_2 & \theta_3 & \theta_4 & \theta_5 & \theta_6 & \theta_7 & \theta_8 \\
2 & 6.706 & 1.686 & -5.251 & & & & & \\
3 & 6.914 & 1.532 & -0.815 & -106.846 & & & & \\
4 & 6.844 & 1.576 & -0.662 & -5.246 & -203.079 & & & \\
\end{array}
\]

Table 9: The first nine stability coefficients for the second fixed point solution using the second method, for \(N = 2\) to \(N = 12\).

The coefficients of the second solution converge to values just off 7, 1.5, -1, -4, -7, -10, -13, a difference of three most of the time. The coefficients of the third solution converge to values just off 4, 2, -2, -5, -8, -11, -14, again a difference of three almost every time. The real parts of the coefficients of the first solution may converge to values just off 2, 2, -2, -6, -6, -10, -10, the pairs due to the
complex eigenvalues. From the dimensional analysis carried out below we would expect a different pattern for a physical solution.

The classic action is dimensionless: 
\[
\mathcal{S} = \int d^3x \sqrt{g(\tilde{a}_0 + \tilde{a}_1 R + \tilde{a}_2 R^2 + ...)} = 0,
\]
where we used the dimensionful coefficients \(\tilde{a}_i\). We know the mass-dimensions \([d^3x] = -3\), \([g] = 0\) and \([R] = 2\).

This means we also know the mass-dimensions of the dimensionful coefficients: \([\tilde{a}_j] = 3 - 2j\). The relation of these coefficients to the dimensionless coefficients \(a_i\) in Table 2 and Table 7 is thus

\[
a_i = \frac{\tilde{a}_i}{k^{3-2i}}. \tag{59}
\]

This means
\[
\partial_t a_0 = \frac{\partial}{\partial \ln(k)} \frac{\tilde{a}_0}{k^3} = k \left( \frac{\partial}{\partial k} \frac{\tilde{a}_0}{k^3} \right) = k \left( -3\frac{\tilde{a}_0}{k^4} \right) = -3a_0. \tag{60}
\]

which is an eigenvalue equation for \(a_0\) with eigenvalue \(\lambda = -3\). When compared to the linearized flow equation (9), we recognize this implies that \(\theta_1 = 3\). Extending this calculation gives that the classical value of the stability coefficients is \(\theta_i = 5 - 2i\). Because of quantum effects, it is expected that stability coefficients for the fixed point solutions receive corrections to these values.

We seem to observe these values for the first solution up to \(N = 6\). On the other hand, up to \(N = 6\) this solution gives us a negative Newton’s constant. For the higher orders it is not clear what values the stability coefficients are converging to.

This concludes our investigation of the solutions found using the power series expansion. We go on to apply the Chebyshev series expansion in the next subsection.

4.4 Using Chebyshev series expansions

Finally, we use Chebyshev series approximations for the fixed point equation (56). As in the previous section, the main reason is the singularity of this equation at \(r = 1.30435\) and the subsequent radius of convergence. This means all solutions of the power series expansion around \(r = 0\) are only converging on approximately \(\frac{1}{4\pi}\) of the domain, where solutions of the Chebyshev expansion will

<table>
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<tr>
<th>(N)</th>
<th>(\theta_1)</th>
<th>(\theta_2)</th>
<th>(\theta_3)</th>
<th>(\theta_4)</th>
<th>(\theta_5)</th>
<th>(\theta_6)</th>
<th>(\theta_7)</th>
<th>(\theta_8)</th>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 10: The first nine stability coefficients for the third fixed point solution using the second method, for \(N = 2\) to \(N = 12\).
converge on the whole domain. To be able to use Chebyshev polynomials we again shift the domain, now from \( r \in [-6, 0] \) to \( x \in [-1, 1] \), introducing the new variable \( x \) by \( r = 3(x - 1) \).

We then proceed with the same procedure as before to give us four nonzero real solutions for \( N = 2 \). We also use the same method as in the previous section to obtain the higher order solutions for \( N = 3 \) and \( N = 4 \) two of the solutions become the trivial solution with all coefficients equal to zero. We therefore discard these solutions. The two solutions that are nonzero for all \( N \) are are shown up to \( N = 10 \) on the domain \( r \in [-6, 0] \) in Figure 13.

![Graph](image1.png)

(a) The first fixed point solution.

![Graph](image2.png)

(a) The second fixed point solution.

Figure 13: The fixed point solutions of equation (56) using the Chebyshev series approximation.

The solutions show a convergence -on the whole domain-, but both have the problem of a negative Newton’s constant. Contrary to the solutions found using power series expansions no solution shows a climbing behaviour, converging to a solution with a positive Newton’s constant.

The stability coefficients for the two solutions can be found in Table 11 and 12 respectively.
Table 11: The first six stability coefficients first Chebyshev solution

<table>
<thead>
<tr>
<th>N</th>
<th>( \theta_1 )</th>
<th>( \theta_2 )</th>
<th>( \theta_3 )</th>
<th>( \theta_4 )</th>
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Table 12: The first six stability coefficients second Chebyshev solution

<table>
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</table>

We have noted in the previous section that, even when we do find the solutions corresponding to the literature, it is not straightforward that these solutions are selected. This also means that we do not know for sure if the solutions we obtain for the flow equation in this section are selected correctly. It is expected that the power series expansion and Chebyshev series expansion give -on the radius of convergence for the power series expansion- the same solutions, something we have not obtained. Motivated by this, we will have another look at our selection mechanism in the next subsection.

4.5 A different selection method

We want to investigate if there is a physical solution to the flow equation that justifies our truncation (i.e. that is converging), not if this solution is automatically selected by the algorithm. Here we will search for this solution. In Figure [14] we plotted all possible solutions for the Chebyshev expansion for \( N = 2 \) to \( N = 10 \). It seems that, apart from the converging solution with a negative Newton’s constant selected above, there is are possible solutions with a positive Newton’s constant present as well. To investigate this, we look at all possible solutions for each \( N \) of the Chebyshev expansions that satisfy \( a_0 > 0 \) and \( a_1 < 0 \), as this means they have a positive Newton’s constant and Cosmological constant \((23)\). For \( N = 2 \), \( N = 3 \) and \( N = 4 \) no such solutions exist and we will come to these orders later. For the other orders these solutions do exist and furthermore, there seems to be a converging solution. In fact, there are two possible solutions, differing for \( N = 8, 9, 10 \). We have plotted all physical solutions with the manually selected solutions highlighted in Figure [15]. The solutions are selected on their convergence behaviour and, when multiple options were available, on their stability coefficients that we will discuss later.
Figure 14: All possible solutions for the Chebyshev expansion for $N = 2$ to $N = 10$.

Figure 15: All possible solutions with $a_0 > 0$ and $a_1 < 0$ for $N = 5$ to $N = 10$ are dashed and the manually selected solutions are thick-lined. For $N = 8, 9, 10$ there are two possible solutions: the first in yellow and the second in blue.
For $N = 2, 3, 4$ there are no solutions possible with $a_0 > 0$ and $a_1 < 0$, so we select the solutions from the unrestricted set that are closest to the ones of Figure 15 and have to closest stability coefficients, see Figure 16.

![Figure 16](image)

Figure 16: All possible solutions and the selected solutions for $N = 2, 3, 4$.

As mentioned above, we have two possibilities for a converging solution. The first solution shows no divergence at all on the domain, but has a 'jump' at $r = 0$ from $N = 7$ to $N = 8$. The second solution does not have this jump and looks very good on the domain and range from the first power series solution in Figure 11 but shows some divergence towards $r = 6$ for the highest order solutions. See Figure 17. The second solution is also similar to the first power series solution for $N = 8, 9, 10$, see Figure 18.

![Figure 17](image)

Figure 17: The two manually selected solutions, for $N = 2$ to $N = 10$. Plotted on both the power series domain as the full domain.

The stability coefficients can give us more insight in the quality of this manually selected set of solutions. They can be found in Table 13. The coefficients obey the dimensional analysis very well. It looks like we have now found the
Figure 18: The second manually selected Chebyshev series expanded solution and the first power series expanded solution for $N = 7$ to $N = 10$ on the power series domain.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>$\theta_4$</th>
<th>$\theta_5$</th>
<th>$\theta_6$</th>
<th>$\theta_7$</th>
</tr>
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<tr>
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<td>-1.326</td>
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<tr>
<td>3</td>
<td>3.557</td>
<td>0.562</td>
<td>-1.198+0.878 i</td>
<td>-1.198-0.878 i</td>
<td></td>
<td></td>
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<tr>
<td>4</td>
<td>2.989</td>
<td>1.003</td>
<td>-0.99</td>
<td>-2.998</td>
<td>-4.977</td>
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<td>5</td>
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<td>1.004</td>
<td>-1.117</td>
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<td>-8.776</td>
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<td>1.037</td>
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<td>-9.377+2.87 i</td>
<td>-9.377-2.87 i</td>
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<tr>
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<td>-1.415</td>
<td>-1.632</td>
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</tr>
<tr>
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<td>1.002</td>
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<td>-10.902+5.019 i</td>
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<td>-3.137-2.38 i</td>
<td>-5.374</td>
<td>-8.332+6.807 i</td>
</tr>
</tbody>
</table>

Table 13: The first seven stability coefficients for the manually selected solution. For $N = 8$ the first row comes from the first solutions and the second row from the second.

Unfortunately, we are not able to calculate the stability coefficients for $N > 8$ that could help us to select one of the two solutions because this takes too much computation power and time.

For the power series solution we are not able to find a manual solution with stability coefficients obeying the dimensional analysis as good as those above. We will therefore keep the solution we had already obtained, encouraged by the fact that this solution is similar to our second solution on its convergence domain for high $N$. 

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5 Discussion and outlook

We have tried two approximation methods to find fixed point solutions of the three-dimensional flow equation on hyperbolic spaces. The solutions obtained from the power series approximations seemed to have the problem of a negative Newton’s constant, making them unphysical. One solution, however, admitted a Newton’s constant that ‘walked’ from negative to positive values and converged to a positive value. This walking is probably the reason a converging solution on hyperbolic backgrounds was not found before. This solution - dubbed ‘fixed point solution one’- looks promising, although the stability coefficients do not show as strong a convergence as we would like and do not obey the dimensional analysis as well. The two-dimensional critical surface on the other hand, is again promising.

Using Chebyshev series approximations initially only rendered solutions with a negative Newton’s constant too. We were however able to manually select a converging solution with a positive Newton’s constant. A solution with two possibilities for higher orders. This solution is similar to the power series solution on its convergence domain, but itself converges on the whole domain. It also obeys the dimensional analysis very well.

We can compare these fixed point solutions to solutions of the fixed point equation for the three-dimensional spherical background, from [9]:

\[
3\varphi(r) - 2r\varphi'(r) = \frac{\hat{c}_1\varphi'(r) + \hat{c}_2\varphi''(r) + \hat{c}_3\varphi'''(r)}{1260\pi^2 (4((r - 2)^2\varphi''(r) - (r - 1)\varphi'(r)) + 3\varphi(r))}
\]  

(61)

with

\[
\hat{c}_1 = 7 (r^2 + 15r + 144), \\
\hat{c}_2 = 20 (r^3 - 14r^2 - 126r + 288), \\
\hat{c}_3 = -4r (-17r^3 - 35r^2 - 308r + 480).
\]

(62)

We use the same algorithm as used to obtain the fixed point solution of the power series on the hyperbolic background to acquire four real non-zero solutions. The second one has a Newton’s constant ‘walking’ from negative to positive values, converging to a positive value. Due to it requiring immense computing power and time we are not able to obtain the fixed point solutions on the spherical background using Chebyshev expansions. We therefore compare this power series solution to our power series and Chebyshev series solutions in Figure[19]

The continuation for the power series looks very smooth. The Chebyshev solutions are less smooth, but in this case also a different method is used. It is important to note that we are comparing different topologies with very different features here. A hyperbolic background gives very different physics than a spherical background.
We conclude that we have found very promising fixed point solutions for the flow equation on hyperbolic backgrounds in three dimensions. Our approximation method does not give error bars, but for approximations of the ninth order of the power series solution and higher, we observe a clear convergence. This means the new higher order coefficients do not contribute much and our assumption (22) is valid. Using our manual selection method we also find a converging fixed point solution for the Chebyshev expansion. Both solutions have the desired finite-dimensional UV critical surface, and for the Chebyshev solution the stability coefficients follow the classical scaling relations.

The application of the algorithm to flow equations on three- as well as four-dimensional spherical backgrounds in this project has also shown that it is possible to extend the algorithm. For the flow equation on the four-sphere our power series reproduced solutions from the literature and our Chebyshev solution could reproduce these to some extend, converging on the whole domain. On the three-sphere both methods also gave, after a climbing behaviour and using a manual selection respectively, similar converging solutions. The Chebyshev solutions again converging on the whole domain and obeying the dimensional analysis. Hence it is interesting to apply both methods to other flow equations as well. Herein the Chebyshev series expansion might be a valuable tool in finding global fixed point solutions.

The methods used in this project can for example be helpful in the investigation of flow equations for gravity including matter\footnote{An investigation in flow equations including matter fields was successfully carried out during the completion of this project.}. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure19.png}
\caption{A fixed point solution of the flow equations on hyperbolically \cite{19} and spherically \cite{16} curved backgrounds. For $r < 0$ the dashed graphs are the second choice of Chebyshev series solutions: $N = 7$ in purple, $N = 8$ in brown, $N = 9$ in blue and $N = 10$ in yellow. The solid lines are the power series solutions for both backgrounds: $N = 7$ in blue, $N = 8$ in yellow, $N = 9$ in green and $N = 10$ in red.}
\end{figure}
Acknowledgements

My gratitude towards Dr. Saueressig diverges to $\infty$. This project truly was an incredible journey through (theory) space and time, with Dr. Saueressig in a Gandalf-like role of adviser-companion. Constantly, it felt like an epic adventure, with his guiding and teaching where needed. Because of his myriad of -also off-topic- advice and all the interesting discussions we had, my gratefulness is definitely asymptotically unsafe.
A  Chebyshev polynomials

All relations and definitions below can be found in standard mathematical textbooks, like [19].

The Chebyshev polynomials, introduced by Pafnuty Chebyshev [20] are the unique polynomials satisfying

\[ T_n(\cos(\theta)) = \cos(n\theta) \]  

for \( n = 0, 1, 2, 3, \ldots \) on the domain \( |x| \leq 1 \). They are defined by the recurrence relation

\[ T_0(x) = 1 \]  
\[ T_1(x) = x \]  
\[ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \]  

The Chebyshev polynomials are a set of orthogonal polynomials, satisfying

\[ \int_{-1}^{1} T_n(x)T_m(x) \frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 & n \neq m \\ \pi & n = m = 0 \\ \frac{\pi}{2} & n = m \neq 0 \end{cases} \]  

We also use Chebyshev polynomials of the second kind, that satisfy

\[ U_{n-1}(\cos(\theta)) \cdot \sin(\theta) = \sin(n\theta) \]  

for \( n = 0, 1, 2, 3, \ldots \) also on the domain \( |x| \leq 1 \). They are defined by a similar recurrence relation as the polynomials of the first kind

\[ U_0(x) = 1 \]  
\[ U_1(x) = 2x \]  
\[ U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x). \]  

Note that for \( n \leq 0 \) this means that

\[ U_{-1} = 0, \]
\[ U_{-2}(x) = -U_0(x), \]
\[ U_{-3}(x) = -U_1(x), \]
\[ U_{-4}(x) = -U_2(x). \]  

The Chebyshev polynomials are also orthonormal with respect to a weight

\[ \int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2} \frac{dx}{x} = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m. \end{cases} \]  

The Chebyshev polynomials of the first and second kind are related via

\[ T_n(x) = \frac{1}{2}(U_n - U_{n-2}) \]
and the derivative of a Chebyshev polynomial of the first kind is
\[
\frac{dT_n(x)}{dx} = nU_{n-1}.
\] (71)

Note that a series of Chebyshev polynomials
\[
y(x) = \sum_{n=0}^{\infty} b_n T_n(x)
\]
(72)
is equal to a Fourier series with \(x = \cos(\theta)\).
References


