Functional Renormalization Group for scalar field theories

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Abstract

Renormalization is the method used to treat infinities and end up with finite physical results in Quantum Field Theories. In this thesis we will focus on the nonperturbative renormalization of the ϕ^4 theory by means of the Renormalization Group (RG). We will start off with some general classical and quantum field theory, and apply these principles to the ϕ^4 theory. The perturbative renormalization procedure will be briefly summarized, after which we will continue to the nonperturbative analysis. In this analysis an equation of the RG flow will be derived and used to study the behaviour of the ϕ^4 theory under RG flow in various dimensions of spacetime.

Contents

1 Introduction

In non-relativistic quantum mechanics, the time evolution of a state is given by the Schrödinger equation. An equivalent[1] formulation is developed by Dirac and Feynman, in which the probability amplitude $U(x_i, t_i, x_f, t_f) = \langle \psi(x_i, t_i) | \psi(x_f, t_f) \rangle$ dictates how the wavefunction ψ changes from (x_i, t_i) to (x_f, t_f). The probability amplitude is dependent on all possible paths starting at (x_i, t_i) and ending at (x_f, t_f), by means of the following path integral

$$U(x_f, t_f, x_i, t_i) = \int Ds(t)\ e^{\frac{i}{\hbar} S(x)}. \tag{1}$$

Here the integral over $Ds(t)$ stands for integration over all possible trajectories $s(t)$ from (x_i, t_i) to (x_f, t_f), and S is the classical action of this path.
We can perform a Wick rotation, which simply speaking makes the substitution $t \to -it$. The exact nature of this is more subtle: according to the Osterwalder-Schrader Theorem[1] one can go from an Euclidean quantum field theory to a Minkowskian (with Wightman axioms) quantum field theory by Wick rotation, given certain axioms are satisfied in the Euclidean case. We will not go into further details on this and simply assume analytic continuation is possible. This Wick rotation results in the Euclidian path integral

$$U_{\text{Eucl.}}(x_f, t_f, x_i, t_i) = \int Ds(t) e^{-\frac{1}{\hbar} S(x)}. \tag{2}$$

In the classical limit ($\hbar \to 0$) the integral is determined by the minimum of S according to the saddle point method[2]. The minimum of S coincides with the path for which $\delta S = 0$, which is Hamilton's principle, yielding classical physics.

In classical field theory, we no longer work with particles. The fields in our case will be scalar fields, which are continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$, which vanish at infinity. We will denote the set of all fields by

$$\mathcal{F} = \{\phi : \mathbb{R}^d \to \mathbb{R}\ \ |\phi \text{ is continuous } \wedge \lim_{||x|| \to \infty} \phi(x) = 0\}. \tag{3}$$

The equation of motion is obtained by minimising the action

$$S[\phi] = \int d^d x L[\phi, \partial_\mu \phi].$$

With L the Lagrangian (density), for suitable boundary conditions of ϕ this yields the Euler-Lagrange equation

$$\partial_\mu \frac{\partial L[\phi, \partial_\mu \phi]}{\partial(\partial_\mu \phi)} - \frac{\partial L[\phi, \partial_\mu \phi]}{\partial \phi} = 0. \tag{4}$$

Quantum field theory (QFT) arises from the combination of classical (relativistic) field

[1]For a proof of Feynman's path integral to the Schrödinger equation, see [2], for a proof of the Schrödinger equation to Feynman's path integral, see [3].

[2]Also called method of steepest descent or stationary phase method, see [4] for details.

theory with quantum mechanics. The dynamical variables are quantized, and time evolution is based on the path integral formulation with the action. In the following we will use natural units, setting $\hbar = c = 1$.

In this thesis, we will start off with some classical field theory, followed by quantum field theory for scalar fields. After some general definitions, this will be analysed for both the free theory and a ϕ^4 interacting theory. This will include some perturbative calculations and a brief overview of how perturbative renormalization works. After which we will go into the main focus of the thesis, which is nonperturbative renormalization for the ϕ^4, in different spacetime dimensions.

2 Classical Field Theory for scalar fields

In this chapter we introduce the notion of functional derivatives, and use these to derive equations of motion for the free and interacting ϕ^4 theories. The solution of the equation of motion for the interacting theory is analysed perturbatively by using Green's functions.

2.1 Functional Calculus

The objects in quantum field theory are functions from $\mathbb{R}^d$ to $\mathbb{R}$, and the actions in turn are functions on the fields, called functionals. Given a functional $F[\phi]$, the derivative of F with respect of v (or in the direction of v), where v is also a scalar field, is given by:

$$\delta_v F[\phi] = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(F[\phi + \epsilon v] - F[\phi]), \tag{5}$$

which has the same form as a derivative of a usual function. Disregarding questions of convergence, we can thus conclude ordinary calculus rules like product and chain rule hold for this derivative as well. In its local form this is defined as:

$$\frac{\delta}{\delta\phi(y)} F[\phi] = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(F[\phi + \epsilon\delta(\cdot - y)] - F[\phi]) \tag{6}$$

The action S is an example of such a functional. S can take on different forms depending on the interaction between fields.

2.2 Free scalar field

We start off with a non-interacting (free) scalar field, and derive the equation of motion. For a non-interacting field, the action is given by a quadratic functional of ϕ. We consider the special case of

$$S^{\text{free}}[\phi] = \frac{1}{2} \int d^d x [-\phi(x)\partial_x^2 \phi(x) + m^2 \phi(x)^2] \tag{7}$$

Here we define $\partial_x^2 = g_{\mu\nu}\partial_x^\mu \partial_x^\nu$, where $g_{\mu\nu}$ is the metric, generalising the equation to both flat Euclidean and Minkowskian[3] spacetime. In both cases the volume element is

[3] We use the convention $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.

$\int d^dx\sqrt{|g|} = \int d^dx$.
For the equation of motion, we need that the derivative $\delta S = 0$. This can be found by calculating the directional derivative as in (5) and demanding it to be 0 for all v. Let us begin by calculating $S[\phi + \epsilon v]$ first.

$$\begin{aligned} S^{\text{free}}[\phi + \epsilon v] &= \frac{1}{2}\int d^dx[-(\phi+\epsilon v)(x)\partial_x^2(\phi+\epsilon v)(x) + m^2(\phi+\epsilon v)(x)^2] \\ &= \frac{1}{2}\int d^dx[-\phi(x)\partial_x^2\phi(x) - \epsilon\phi(x)\partial_x^2 v(x) - \epsilon v(x)\partial_x^2\phi(x) - \epsilon^2 v(x)\partial_x^2 v(x) \\ &\qquad + m^2\phi(x)^2 + 2m^2\epsilon v(x)\phi(x) + m^2\epsilon^2 v(x)^2] \end{aligned}$$

We recognize $S[\phi]$ in this and order terms by power of ϵ.

$$\begin{aligned} S^{\text{free}}[\phi + \epsilon v] = S^{\text{free}}[\phi] &+ \epsilon\frac{1}{2}\int d^dx[-\phi(x)\partial_x^2 v(x) - v(x)\partial_x^2\phi(x) + 2m^2 v(x)\phi(x)] \\ &+ \epsilon^2\int d^dx[-v(x)\partial_x^2 v(x) + m^2 v(x)^2)] \end{aligned}$$

All terms with power of ϵ greater than 1 vanish when taking the limit $\epsilon \to 0$ in the functional derivative. Therefore the derivative of S^{free} is

$$\delta_v S^{\text{free}}[\phi] = \int d^dx[-\frac{1}{2}\phi(x)\partial_x^2 v(x) - \frac{1}{2}v(x)\partial_x^2\phi(x) + m^2 v(x)\phi(x)]$$

We can use partial integration two times on the first term. There are no boundary terms because by definition, the fields go to 0 at infinity. So that

$$\delta_v S^{\text{free}}[\phi] = \int d^dx[-v(x)\partial_x^2\phi(x) + m^2 v(x)\phi(x)] \tag{8}$$

If we demand that $\delta_v S^{\text{free}}[\phi] = 0$ for all v we obtain the Klein-Gordon equation

$$[-\partial_x^2 + m^2]\phi = 0 \tag{9}$$

Which admits plane wave solutions $\phi(x) = e^{ip_\mu x^\mu}$, with $p^2 + m^2 = 0$. In Minkowskian spacetime we recover the famous formula $E^2 = \vec{p}^2 + m^2$ of special relativity.

2.3 Non-free scalar fields

The free theory might not contain the complete picture. It describes non-interacting fields and gives us a setting in which exact calculations are possible. Most cases are interacting however, being described by what is called a non-free theory. As an example, we add a ϕ^4 interaction term to the Lagrangian. The expression for the action then becomes

$$S[\phi] = S^{\text{free}}[\phi] + S^{\text{int}}[\phi], \tag{10}$$

with the additional interaction

$$S^{\text{int}}[\phi] = \int d^dx\frac{\lambda}{4!}\phi^4(x). \tag{11}$$

Next, we compute the derivative of $S[\phi]$. Since δ_v is linear, we start by calculating the derivative of the ϕ^4 term, and add it to the derivative we previously found in eq. (8)

$$\delta_v S^{\text{int}}[\phi] = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int d^d x[\frac{\lambda}{4!}((\phi+\epsilon v)(x))^4 - \frac{\lambda}{4!}\phi(x)^4]$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int d^d x\frac{\lambda}{4!}[\phi(x)^4 + 4\epsilon v(x)\phi(x)^3 + 6\epsilon^2 v(x)^2\phi(x)^2 +$$
$$4\epsilon^3 v(x)^3\phi(x) + \epsilon^4 v(x)^4 - \phi(x)^4]$$

Rearranging terms to order of ϵ and taking the limit

$$\delta_v S^{\text{int}}[\phi] = \lim_{\epsilon\to 0}\int d^d x\frac{\lambda}{4!}[4v(x)\phi(x)^3 + 6\epsilon v(x)^2\phi(x)^2 + 4\epsilon^2 v(x)^3\phi(x) + \epsilon^3 v(x)^4]$$
$$= \int d^d x\frac{\lambda}{3!}v(x)\phi(x)^3.$$

Combining this with (8) we obtain the derivative of S:

$$\delta_v S[\phi] = \int d^d x[-v(x)\partial_x^2\phi(x) + m^2 v(x)\phi(x) + \frac{\lambda}{3!}v(x)\phi(x)^3]. \tag{12}$$

Finding the equation of motion for this is similar to before, the result being:

$$(-\partial_x^2 + m^2)\phi = -\frac{\lambda}{6}\phi^3. \tag{13}$$

This equation is non-linear unlike the Klein Gordon equation, but it still has exact solutions involving Jacobi elliptic functions[5]. However we will approximate the solution by expanding in powers of λ, considering the interaction term to be a perturbation. Solutions of (13) can be written as

$$\phi(x) = \sum_{n=0}^{\infty}\lambda^n\phi_n(x). \tag{14}$$

This changes the differential equation into:

$$(-\partial_x^2 + m^2)\sum_{n=0}^{\infty}\lambda^n\phi_n = -\frac{1}{6}\lambda\sum_{r,s,t=0}^{\infty}\lambda^{r+s+t}\phi_r\phi_s\phi_t \tag{15}$$

Comparing powers of λ for the cases $n = 0, 1, 2$ we get

$$[-\partial_x^2 + m^2]\phi_0 = 0$$
$$[-\partial_x^2 + m^2]\phi_1 = -\frac{1}{6}\phi_0^3$$
$$[-\partial_x^2 + m^2]\phi_2 = -\frac{1}{2}\phi_1\phi_0^2$$

Green's Functions
In principle, one can directly solve these (now linear) differential equations step by step. We will follow a different approach and proceed to use a Green's function[6], which is a function[4] G that solves

$$[-\partial_x^2 + m^2]G(x-y) = \delta(x-y) \tag{17}$$

Solving this can be done with a Fourier transform, so that

$$\int d^dq \frac{1}{(2\pi)^{d/2}}[-\partial_x^2 + m^2]\tilde{G}(q)e^{iqx} = \int d^dq \frac{1}{(2\pi)^d}\, e^{iqx}$$
$$\int d^dq \frac{1}{(2\pi)^{d/2}}[q^2 + m^2]\tilde{G}(q)e^{iqx} = \int d^dq \frac{1}{(2\pi)^d}\, e^{iqx}$$

So the Fourier transform of G is

$$\tilde{G}(q) = \frac{1}{(2\pi)^{d/2}(q^2+m^2)}. \tag{18}$$

And so $G(x)$ is found by inverse Fourier transform

$$G(x) = \int d^dq \frac{e^{iqx}}{(2\pi)^d(q^2+m^2)}. \tag{19}$$

With the Green's function, we can construct a solution to any differential equation of the form

$$[-\partial_x^2 + m^2]\phi(x) = u(x).$$

With $u(x)$ a (nice enough) function. The solution will be given by convolution with G:

$$\phi(x) = \int d^dy\; u(y)G(x-y) \tag{20}$$

Indeed, substituting (20) in the LHS of the field equation yields:

$$\begin{aligned}
[-\partial_x^2 + m^2]\phi(x) &= [-\partial_x^2 + m^2]\int d^dy\; u(x)G(x-y) \\
&= \int d^dy\; u(y)[-\partial_x^2 + m^2]G(x-y) \\
&= \int d^dy\; u(y)\delta(x-y) = u(x)
\end{aligned}$$

In this sense, the Green's function can be considered as a right-inverse of the differential operator $-\partial_x^2 + m^2$. Applying this to the second differential equation (the first is just the Klein Gordon equation), the solution becomes

$$\phi_1(x) = \int d^dy\; d^dq\; \frac{e^{iq(x-y)}}{(2\pi)^d(q^2+m^2)}\frac{-1}{6}\phi_0(y)^3 \tag{21}$$

Where ϕ_0 is the most general solution to the Klein Gordon equation, i.e. a superposition of plane waves with the energy relation constraint,

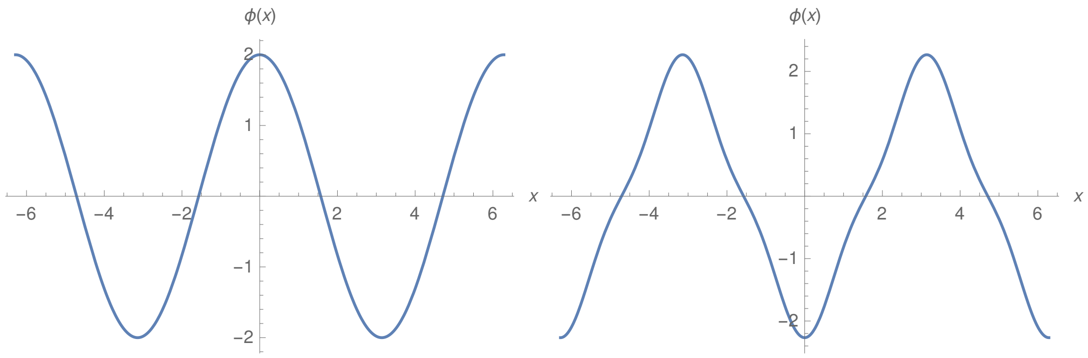
$$\phi_0(x) = \int d^dp\; \widetilde{\phi}(p)e^{ipx}\,\delta(p^2+m^2).$$

[4]To be precise, G is a distribution.

For instance, we can choose initial conditions such that $\phi_0(x) = 2\cos(\vec{p} \cdot \vec{x})$[5]. The solution up to first order of λ is then given by

$$\phi(x) = \phi_0(x) + \lambda\phi_1(x) = 2\cos(\vec{p} \cdot \vec{x}) - \lambda\frac{\cos(\vec{p} \cdot \vec{x})}{\vec{p}^2 + m^2} - \frac{1}{3}\lambda\frac{\cos(3\vec{p} \cdot \vec{x})}{9\vec{p}^2 + m^2}$$

We see that for $\lambda \geq 2(\vec{p}^2 + m^2)$, the second term dominates, suggesting that it is no longer a small perturbation and more terms of the expansion are necessary. This is illustrated in the following graphs, where the 1-dimensional case is plotted for $m = p = 1$, with $\lambda = 1$ and $\lambda = 8$.



$\lambda = 1$ $\lambda = 8$

Figure 1: First order approximation of the solution of the EOM for ϕ^4 theory with $m = p = 1$ in $d = 1$. The left graph has $\lambda = 1$ while the right one has $\lambda = 8$. The relatively strong interaction ($\lambda = 8$) clearly leads to a strong deviation from the Klein Gordon solution, with a change of sign. This means that the perturbative solution no longer holds.

[5]Note that they do not satisfy the boundary conditions, though.

3 Quantum Field Theory for scalar fields

We approach Quantum Field Theory from the path integral formulation, in which transition amplitudes are calculated by evaluating a functional integral (still called a path integral) over all possible field configurations, similarly as we calculate an integral over all paths in quantum mechanics. The position $\vec{x}$ is no longer a dynamical operator, but a label, since it indicates where the field $\phi(x)$ (which is a dynamical variable) is evaluated.

The path integral is formally written as

$$Z[J] = C \int d\phi e^{-S[\phi]} e^{J[\phi]}. \tag{22}$$

Still one has to make sense of $\int d\phi\, e^{-S[\phi]}$, either by discretization or more formally as in [7]. This is the vacuum to vacuum probability amplitude given an external source field J. Here $C = 1/\int d\phi e^{-S[\phi]}$ so that $Z[0] = 1$. J is the source term, which is the dual vector of ϕ and thus a linear functional

$$J[\phi] = \int d^d x J(x)\phi(x). \tag{23}$$

We distinguish the free and non-free cases with

$$Z^{\text{int}}[J] = C \int d\phi e^{-\int d^d x [\frac{1}{2}(-\phi(x)\partial_x^2\phi(x) + m^2\phi(x)^2) + \frac{\lambda}{4!}\phi(x)^4]} e^{J[\phi]} \tag{24}$$

and

$$Z^{\text{free}}[J] = C \int d\phi e^{-\int d^d x \frac{1}{2}(-\phi(x)\partial_x^2\phi(x) + m^2\phi(x)^2)} e^{J[\phi]} \tag{25}$$

So that Z^{free} is the measure based on the free action and Z^{int} is based on the action with the ϕ^4 interaction term.

Free scalar field

In order to construct correlation functions, which tell us how the theory "behaves", we need to find its derivatives, starting with the free theory

$$\begin{aligned}
\delta_K Z^{\text{free}}[J] &= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int d\phi e^{-S[\phi]} (e^{(J+\epsilon K)[\phi]} - e^{J[\phi]}) \\
&= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int d\phi e^{-S[\phi]} e^{J[\phi]} (-1 + e^{\epsilon K[\phi]}) \\
&= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int d\phi e^{-S[\phi]} e^{J[\phi]} (\epsilon K[\phi] + \mathcal{O}(\epsilon^2)) \\
&= \int d\phi e^{-S[\phi]} e^{J[\phi]} K[\phi]
\end{aligned}$$

Applying this to the case $K(x) = \delta(x - y)$ this is by definition equal to:

$$\frac{\delta}{\delta J(y)} Z^{\text{free}}[J] = \int d\phi e^{-S[\phi]} e^{J[\phi]} \phi(y) = \langle \phi(y) \rangle_{S,J} \tag{26}$$

If J is not written in the subscript of the expectation value, it is set to 0. Iterations of this result give in general

$$\langle \phi(x_1)\phi(x_2)...\phi(x_n)\rangle_{S,J} = \frac{\delta^n}{\delta J(x_1)\delta J(x_2)...\delta J(x_n)} Z^{\text{free}}[J]. \tag{27}$$

In order to calculate $Z^{\text{free}}[J]$ for the free theory as in equation (7):

$$Z^{\text{free}}[J] = C\int d\phi e^{-S[\phi]}e^{J[\phi]} = C\int d\phi e^{-\frac{1}{2}\int d^dx[-\phi(x)\partial_x^2\phi(x)+m^2\phi(x)^2]}e^{\int d^dx J(x)\phi(x)}$$

we define the self-adjoint operator

$$A = -\partial_x^2 + m^2 \tag{28}$$

and substitute this in the expression above, which then has the form of a gaussian integral with linear term:

$$Z^{\text{free}}[J] = C\int d\phi e^{-\frac{1}{2}\phi A\phi}e^{J[\phi]},$$

with A as in (28) and J as in (23). The generating functional $Z_{\text{free}}[J]$ then assumes the general form

$$\int d\phi e^{-\frac{1}{2}\phi A\phi}e^{J[\phi]} = \sqrt{\frac{1}{\det(A)}}e^{\frac{1}{2}JA^{-1}J}. \tag{29}$$

The factor with the determinant cancels because of the normalization constant C, so that

$$Z^{\text{free}} = e^{\frac{1}{2}JA^{-1}J}. \tag{30}$$

Interacting scalar field

In the ϕ^4 case, the term with ϕ^4 can be formally expanded with Taylor's formula

$$e^{\int d^dx\frac{\lambda}{4!}\phi^4} = 1 + \int d^dx\frac{\lambda}{4!}\phi(x)^4 + \frac{1}{2}\left(\int d^dx\frac{\lambda}{4!}\phi(x)^4\right)^2 + ...$$

Using (27) this can be expressed in terms of derivatives of $Z[J]$:

$$\begin{aligned} Z[J] &= Z^{\text{free}}[J] + \int d^dy\frac{\lambda}{4!}\langle\phi^4(y)\rangle_{S,J} + \frac{1}{2}\int d^dy\frac{\lambda}{4!}\langle\phi(y)^4\rangle_{S,J}\int d^dz\frac{\lambda}{4!}\langle\phi(z)^4\rangle_{S,J} + ... \\ &= Z^{\text{free}}[J] + \left(\frac{\lambda}{4!}\right)\int d^dy\frac{\delta^4}{\delta J(y)^4}Z^{\text{free}}[J] \\ &\qquad + \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2\int d^dy\frac{\delta^4}{\delta J(y)}\int d^dz\frac{\delta^4}{\delta J(z)^4}Z^{\text{free}}[J] + ... \end{aligned}$$

Assuming λ to be sufficiently small, one usually terminates the series after a few terms of the expansion.

Schwinger functional & Effective action
We define the Schwinger functional

$$W[J] = \ln(Z[J]) \tag{31}$$

and the effective action by the Legendre-Fenchel transformation of W,

$$\Gamma[\chi] = \sup_{J\in\mathcal{F}}[J[\chi] - W[J]]. \tag{32}$$

Hereby χ is the connected expectation value of ϕ defined as

$$\chi = \langle\phi\rangle^c_{S,J} \equiv \frac{\delta}{\delta J}W[J] \tag{33}$$

And in general for arbitrary functional derivatives we obtain

$$\langle\phi^n\rangle^c_{S,J} \equiv (\frac{\delta}{\delta J})^n W[J] \tag{34}$$

For the first two n these relate to the ordinary expectation value as

$$\langle\phi\rangle^c = \frac{\delta}{\delta J}\ln Z[J] = \frac{1}{Z[J]}\frac{\delta}{\delta J}Z[J] \tag{35}$$

$$\begin{aligned}\langle\phi^2\rangle^c_{S,J} &= \frac{\delta^2}{\delta J\delta J}\ln Z[J] = \frac{\delta}{\delta J}(\frac{1}{Z[J]}\frac{\delta}{\delta J}Z[J]) \\ &= -\Big(\frac{1}{Z[J]}\frac{\delta}{\delta J}Z[J]\Big)^2 + \frac{1}{Z[J]}\frac{\delta^2}{\delta J\delta J}Z[J] \end{aligned}\tag{36}$$

We first compute the effective action for the free theory,

$$W^{\text{free}}[J] = \ln(Z^{\text{free}}[J]) = \frac{1}{2}JA^{-1}J. \tag{37}$$

The supremum can be found by setting the derivative of $J[\chi] - W[J]$ to 0.

$$\begin{aligned}\partial_k[J[\chi] - W[J]] &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}((J+\epsilon K)[\chi] - W^{\text{free}}[J+\epsilon K] - J[\chi] + W^{\text{free}}[J]) \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}(\epsilon K[\chi] - W^{\text{free}}[J+\epsilon K] + W^{\text{free}}[J])\end{aligned}$$

We examine the $W^{\text{free}}[J+\epsilon K]$ term

$$W^{\text{free}}[J+\epsilon K] = \frac{1}{2}(J+\epsilon K)A^{-1}(J+\epsilon K) = \frac{1}{2}(JA^{-1}J + \epsilon(JA^{-1}K + KA^{-1}J) + \epsilon^2 KAK)$$

Because A is self-adjoint, we have $KA^{-1}J = JA^{-1}K$, giving us

$$W^{\text{free}}[J+\epsilon K] = \frac{1}{2}(J+\epsilon K)A^{-1}(J+\epsilon K) = \frac{1}{2}(JA^{-1}J + 2\epsilon JA^{-1}K + \epsilon^2 K^2 A)$$

The first term will cancel with the $W^{\text{free}}[J]$ term, and the last term will be 0 in the limit $\epsilon \to 0$. Setting the derivative to 0 then gives

$$\partial_k[J[\chi] - W[J]] = K[\chi] - JA^{-1}K = 0$$

For solutions of this equation the following then holds

$$K[\chi] - J_{\max} A^{-1} K = 0 \qquad \forall K$$

From this it follows that

$$J_{\max} = A[\chi] \tag{38}$$

So the effective action is given by

$$\Gamma^{\text{free}}[\chi] = J_{\max}[\chi] - W^{\text{free}}[J_{\max}] = \frac{1}{2}\chi A \chi = S^{\text{free}}[\chi] \tag{39}$$

Which coincides with the bare free action. This is special for the free theory though.

3.1 Perturbation Theory

For this section we reintroduce factors of $\hbar$ in order to later expand in powers of $\hbar$. Equation (22) then becomes

$$Z[J] = C \int d\phi e^{-S[\phi]/\hbar} e^{J[\phi]/\hbar} = \exp(W[J]/\hbar) \tag{40}$$

And the effective action

$$\Gamma[\chi] = J_{\max}[\chi] - W[J_{\max}] \tag{41}$$

with

$$\chi = \langle \phi \rangle_J^c = \frac{\delta}{\delta J} W[J] \Big|_{J=J_{\max}}$$

Taking the derivative of the effective action with respect to χ gives

$$\begin{aligned} \frac{\delta}{\delta \chi}\Gamma[\chi] &= \chi \frac{\delta}{\delta \chi} J_{\max} + J_{\max} \frac{\delta}{\delta \chi}\chi - \frac{\delta}{\delta \chi} W[J_{\max}] \\ &= J_{\max} + \chi \frac{\delta}{\delta \chi} J_{\max} - \frac{\delta}{\delta J_{\max}} W[J_{\max}] \frac{\delta}{\delta \chi} J_{\max} = J_{\max}. \end{aligned}$$

Inserting this in Z results in

$$\begin{aligned} Z[J_{\max}] &= C \int d\phi e^{-S[\phi]/\hbar + \int d^d x \delta_\chi \Gamma[\chi] \phi/\hbar} \\ &= \exp\Big[W[J_{\max}]/\hbar\Big] = \exp\Big[-\Gamma[\chi]/\hbar + J_{\max}[\chi]/\hbar\Big] \\ &= \exp\left[-\Gamma[\chi]/\hbar + \int d^d x \delta_\chi \Gamma[\chi] \chi/\hbar\right] \end{aligned}$$

Solving this expression for $\Gamma[\chi]$ we obtain the following integro-differential equation:

$$\exp[-\Gamma[\chi]/\hbar] = C \int d\phi e^{-S[\phi]/\hbar + \int d^d x (\phi - \chi) \delta_\chi \Gamma[\chi]/\hbar}. \tag{42}$$

We now formally expand the effective action in powers of $\hbar$

$$\Gamma[\chi] = S[\chi] + \sum_{L=1}^{\infty} \hbar^L \Gamma^{[L]}[\chi]$$

We know that the 0th order is the classical action, because setting $\hbar = 0$ ought to result in classical physics. Using (42) and expanding S up to second order of $\hbar$ gives

$$\begin{aligned}\Gamma[\chi] &= -\hbar \ln\left(C \int d\phi e^{-S[\phi]/\hbar + \int d^d x \delta_\chi \Gamma[\chi](\phi-\chi)/\hbar}\right) \\ &= -\hbar \ln\left(C \int d\phi e^{-S[\chi]/\hbar} e^{-\frac{1}{2}\delta^2 S[\chi](\phi-\chi)^2/\hbar + \int d^d x \delta_\chi \Gamma[\chi](\phi-\chi)/\hbar} + \mathcal{O}(\hbar^{3/2})\right)\end{aligned}$$

We do the substitution $\psi = \frac{\phi-\chi}{\sqrt{\hbar}}$ and integrate over ψ. We also neglect any terms with order of $\hbar$ higher than 1, since the calculation is up to first order.

$$\begin{aligned}\Gamma[\chi] &= -\hbar \ln\left(e^{-S[\chi]/\hbar} C \int d\psi \sqrt{\hbar}\, e^{-\frac{1}{2}\delta^2 S[\chi]\psi^2 + \int d^d x \delta_\chi \Gamma]\chi]\psi/\sqrt{\hbar}}\right) \\ &= S[\chi] - \hbar \ln\left(\sqrt{\frac{1}{\det \delta^2 S[\chi]}} e^{\frac{1}{2}\frac{\delta_\chi \Gamma[\chi]}{\sqrt{\hbar}}(\delta^2 S)^{-1}\frac{\delta_\chi \Gamma[\chi]}{\sqrt{\hbar}}}\right) + \hbar \ln(C') \\ &= S[\chi] + \frac{\hbar}{2}\mathrm{Tr}\ln(\delta^2 S[\chi]) + \hbar \ln(C') + \mathcal{O}(\hbar^2)\end{aligned}$$

Where the integral is solved with (29) and the identity $\ln\det(A) = \mathrm{Tr}\ln(A)$ is used. C' is constant with respect to the fields and thus unimportant to physics.

This can then be expanded in λ. These expansions in $\hbar$ and λ restrict us to low loop order and weakly interacting situations. The process of renormalization then goes as follows[8]: start computations with a regulator (upper bound to integrals) for an expression containing the bare mass m, bare coupling constant λ and the upper bound Λ. Compute the physical mass and coupling constant m_p and λ_p, which are based on experiment. All these expressions are then combined and m and λ are eliminated in favour of m_p and λ_p. The final expression should then be independent of the cutoff Λ, which can then be taken to be infinite (UV complete theory). A theory is called (perturbatively) renormalizable, if only a finite number of bare couplings are needed to absorb occuring infinites. With this method it can be found that the ϕ^4 is renormalizable in spacetime dimensions $d \leq 4$.

4 Nonperturbative Renormalization

The method for renormalization is based on the behaviour of coupling constants (like mass or λ in ϕ^4 scalar theory) with changing energy scale. These coupling constants define the theory, and so a tuple of them define a point in theory space. In this chapter we use a nonperturbative approach to investigate renormalization.
The following procedure's goal is to find the Renormalization Group (RG) flow with respect to the energy scale of this theory space, and determine renormalizability according to this flow. Fixed points (a scale invariant theory) of the RG flow are of great importance: convergence to a fixed point ensures that a theory's coupling constants do not go to infinity for increasing energy scale, but its UV behaviour is determined by the scale invariant theory of the fixed point. These theories are called UV complete. usually, there is a trivial fixed point, namely the free theory. A fixed point is called Gaussian

if it corresponds to a free theory, and non-Gaussian if it corresponds to an interacting theory. A theory is called asymptotically free if it converges to a Gaussian fixed point for increasing energy scale[6], and asymptotically safe if it converges to a non-Gaussian fixed point in the same limit.

The set of all points that converge to a fixed point for increasing energy scale is called the UV critical hypersurface. All theories within this hypersurface are (non)perturbatively renormalizable and therefore candidates for a Quantum Field Theory. The dimension of this critical surface is equal to the amount of relevant/UV attractive couplings. Hence a lower hypersurface dimension means that less experiments are necessary to determine the right theory and so its predictivity increases. This also means that an infinite dimensional critical hypersurface cannot lead to a physically predictive fundamental theory.

4.1 Functional Renormalisation Group Equation

Up until now, the generating functional $Z[J]$ has integrated over all modes of the field ϕ. However experiments are on some energy scale. We describe the theory at that scale k by integrating over all modes of the field with momentum $p^2 > k^2$. This is done by introducing a suppression factor as a smooth cutoff. The new functional is then

$$Z_k[J] = C\int d\phi e^{-S[\phi]}e^{-\Delta S_k[\phi]}e^{J[\phi]} \tag{43}$$

where

$$\Delta S_k[\phi] = \frac{1}{2}\int d^dx\phi(x)R_k(-\partial_x^2)\phi(-x) \tag{44}$$

Here $R_k(q^2)$ is arbitrary aside from meeting the following demands: $R_k(q^2)$ is a monotonically decreasing function of q^2 and a monotonically increasing function of k, $R_k(q^2) \to 0$ for $q \to \infty$ faster than any polynomial, and $R_k(q^2) \to k^2$ for $q \to 0$. The Schwinger functional and Legendre transform are then applied to give

$$\hat{\Gamma}_k[\chi] = \sup_J[J[\chi] - W_k[J]], \tag{45}$$

where similarly to before

$$\langle\phi\rangle^c_{S,J,\Delta S_k} = \frac{\delta}{\delta J}W_k[J]\Big|_{J=J_{\max}} \equiv \chi. \tag{46}$$

This transform is done however with the cutoff included, we subtract this term to define the effective average action

$$\Gamma_k[\chi] = \hat{\Gamma}_k[\chi] - \Delta S_k[\chi] \tag{47}$$

Derivation of the Functional Renormalization Group Equation
The goal is to investigate the effect of the energy scale k on this object, so we compute

[6] A famous example of this is quantum chromodynamics (QCD).

the derivative of the average effective action with respect to k. The first relevant term in this calculation is W_k.

$$\begin{aligned}
\partial_k W_k[J] &= \partial_k \ln(Z_k[J]) = \frac{1}{Z_k[J]}\partial_k Z_k[J] \\
&= -\frac{1}{Z_k[J]}\int d\phi e^{-S[\phi]-\Delta S_k[\phi]+J[\phi]}\partial_k \Delta S_k[\phi] \\
&= -\frac{1}{Z_k[J]}\int d\phi e^{-S[\phi]-\Delta S_k[\phi]+J[\phi]}\frac{1}{2}\int d^d x \phi(x)\partial_k R_k(-\partial_x^2)\phi(-x) \\
&= -\frac{1}{Z_k[J]}\frac{1}{2}\mathrm{Tr}[\langle\phi\phi\rangle_{S,J,\Delta S_k}\,\partial_k R_k(-\partial_x^2)]
\end{aligned}$$

where the trace replaces coordinate integration. The trace of an operator $\widehat{\mathrm{O}}$ (in the trace class) is defined as

$$\mathrm{Tr}[\widehat{\mathrm{O}}] = \sum_i \langle \psi_i, \widehat{\mathrm{O}}\psi_i\rangle\,. \tag{48}$$

Where $(\psi_i)_i$ is an orthonormal basis set. Using (35) and (36), this becomes

$$\begin{aligned}
&-\frac{1}{2}\mathrm{Tr}\big[\frac{\delta^2}{\delta J\delta J}W_k[J]\partial_k R_k + \frac{1}{Z_k[J]^2}\langle\phi\rangle_{S,J,\Delta S_k}\,\partial_k R_k\,\langle\phi\rangle_{S,J,\Delta S_k}\big] \\
&= -\frac{1}{2}\mathrm{Tr}\big[\frac{\delta^2}{\delta J\delta J}W_k[J]\partial_k R_k + \langle\phi\rangle^c_{S,J,\Delta S_k}\,\partial_k R_k\,\langle\phi\rangle^c_{S,J,\Delta S_k}\big] \\
&= -\frac{1}{2}\mathrm{Tr}\big[\frac{\delta^2}{\delta J\delta J}W_k[J]\partial_k R_k\big] + \partial_k S_k[\chi]
\end{aligned}$$

The second term then conveniently cancels because of the definition of the effective average action

$$\begin{aligned}
\partial_k \Gamma_k[\chi] &= -\partial_k W_k[J] - \partial_k \Delta S_k[\chi] \\
&= \frac{1}{2}\mathrm{Tr}\big[\frac{\delta^2}{\delta J\delta J}W_k[J]\partial_k R_k\big]
\end{aligned}$$

This can be computed using

$$\frac{\delta}{\delta\chi}\widehat{\Gamma}_k[\chi] = J.$$

Hence, using expression (46) we obtain:

$$\frac{\delta^2}{\delta\chi\delta\chi}\widehat{\Gamma}_k = \frac{\delta}{\delta\chi}J = \big(\frac{\delta}{\delta J}\chi\big)^{-1} = \big(\frac{\delta^2}{\delta J\delta J}W_k\big)^{-1}.$$

With this we finally arrive at the Functional Renormalization Group Equation[9]

$$k\partial_k\Gamma_k[\chi] = \frac{1}{2}\mathrm{Tr}\left[\left(\frac{\delta^2}{\delta\chi\delta\chi}\Gamma_k + R_k\right)^{-1} k\partial_k R_k\right]. \tag{49}$$

Where $\frac{\delta^2}{\delta\chi\delta\chi}\Gamma_k$ is the operator corresponding to that functional. This is implicitly done when we replace integration with the trace. An example with some extra details is given in the appendix.

4.2 Deriving the beta-functions for the ϕ^4 theory

In the ϕ^4 theory, we focus on an effective average action of the form

$$\Gamma_k[\phi] = \int d^dx[-\frac{1}{2}z_k\phi(x)\partial_x^2\phi(x) + \frac{1}{2}m_k^2\phi(x)^2 + \frac{\lambda_k}{4!}\phi(x)^4] \tag{50}$$

Where coupling constants z_k, m_k and λ_k are now dependent on the energy scale.

Left-hand-side of the FRGE
To find the beta functions of these couplings, we take the derivative with respect to k of (50). This gives

$$k\partial_k\Gamma_k = k\int d^dx[-\frac{1}{2}\partial_k z_k\phi(x)\partial_x^2\phi(x) + \frac{1}{2}\partial_k m_k^2\phi(x)^2 + \frac{1}{4!}\partial_k\lambda_k\phi(x)^4]. \tag{51}$$

We can then extract their corresponding beta functions using projecting operations of the following form

$$\Pi_{i,j}F[\phi] = \frac{1}{i!}\frac{1}{j!}\partial_{\phi_c}^i\partial_q^j\left(F[\phi]\big|_{\phi=\phi_c e^{iqx}}\right)\big|_{\phi_c=0,q=0} \tag{52}$$

To extract z_k, we need $(i,j) = (2,2)$, m_k and λ_k have operations with $(i,j) = (2,0)$ and $(i,j) = (4,0)$ respectively. Applying these leads to the following expressions,

$$\Pi_{2,2}k\partial_k\Gamma_k[\phi] = \int d^dx\ [\frac{1}{2}k\partial_k z_k - x^2\partial_k m_k^2] \tag{53a}$$

$$\Pi_{2,0}k\partial_k\Gamma_k[\phi] = \int d^dx\ k\frac{1}{2}\partial_k m_k^2 \tag{53b}$$

$$\Pi_{4,0}k\partial_k\Gamma_k = \int d^dx\ \frac{1}{4!}k\partial_k\lambda_k \tag{53c}$$

With the FRGE (49), we can find different expressions for these objects and compare.

Right-hand-side of the FRGE
We now compute the FRGE for (50), given the truncation where we neglect all powers of ϕ higher than 4, and all other derivatives. We can refer to (74) to find the expression for the Hessian of $\delta_\phi^2\Gamma_k$, with some adjustment to factors. We find as a result

$$k\partial_k\Gamma_k = \frac{1}{2}\mathrm{Tr}\left[\frac{1}{-z_k\partial_x^2 + m_k^2 + \frac{\lambda_k}{2}\phi(x)^2 + R_k(-\partial_x^2)}k\partial_k R_k(-\partial_x^2)\right] \tag{54}$$

We formally can expand the operator on the right in a Neumann series, which is of the form

$$(\mathbb{1} - T)^{-1} = \sum_{k=0}^{\infty} T^k. \tag{55}$$

For T an operator for which this series converges in norm. To rewrite our previous result in this form, we solve the general case. Here, A, B, C are invertible operators.

$$\begin{aligned}(A-B)^{-1} &= C(\mathbb{1}-T)^{-1} \qquad\qquad C = D^{-1}\\ &= D^{-1}(\mathbb{1}-T)^{-1} = ((\mathbb{1}-T)D)^{-1} = (D-TD)^{-1}\end{aligned}$$

So that we find $D = A$, so $C = A^{-1}$ and $T = BA^{-1}$.

Using this, with $A = -z_k\partial_x^2 + m_k^2 + R_k(-\partial_x^2)$ and $B = -\frac{\lambda_k}{2}\phi(x)^2$, we can expand in powers of ϕ, up to ϕ^4. For convenience, we define $P(-\partial_x^2) = -z_k\partial_x^2 + m_k^2 + R_k(-\partial_x^2)$.

$$\begin{aligned}k\partial_k\Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\Big[\frac{1}{P(-\partial_x^2)}\Big\{&\mathbb{1} - \frac{\lambda_k}{2}\phi(x)^2(P(-\partial_x^2))^{-1}\\ &+ \frac{\lambda_k^2}{4}\phi(x)^2(P(-\partial_x^2))^{-1}\phi(x)^2(P(-\partial_x^2))^{-1}\Big\}k\partial_k R_k(-\partial_x^2)\Big]\end{aligned}$$

We use the cyclic property of the trace, i.e. $\mathrm{Tr}[AB] = \mathrm{Tr}[BA]$, to move the operator $P(-\partial_x^2)^{-1}$ to the right

$$\begin{aligned}k\partial_k\Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\Big[\Big(&\mathbb{1} - \frac{\lambda_k}{2}\phi(x)^2(P(-\partial_x^2))^{-1} +\\ &\frac{\lambda_k^2}{4}\phi(x)^2(P(-\partial_x^2))^{-1}\phi(x)^2(P(-\partial_x^2))^{-1}\Big)k\partial_k R_k(-\partial_x^2)\frac{1}{P(-\partial_x^2)}\Big]. \qquad (56)\end{aligned}$$

The trace can now be evaluated by introducing a complete orthonormal basis. We choose the Fourier basis $\psi_p = (\frac{1}{2\pi})^{d/2}e^{ipx}$, for which the expression becomes

$$\begin{aligned}k\partial_k\Gamma_k[\phi] = \frac{1}{2}\frac{1}{(2\pi)^d}\int d^dxd^dp\, e^{-ipx}\Big\{&\mathbb{1} - \frac{\lambda_k}{2}\phi(x)^2(P(-\partial_x^2))^{-1}\\ &+ \frac{\lambda_k^2}{4}\phi(x)^2(P(-\partial_x^2))^{-1}\phi(x)^2(P(-\partial_x^2))^{-1}\Big\}\\ &k\partial_k R_k(-\partial_x^2)\frac{1}{P(-\partial_x^2)}e^{ipx}\end{aligned}$$

Next we set $\phi = \phi_c e^{iqx}$, as is the first step in projections (52), which results in

$$\begin{aligned}k\partial_k\Gamma_k[\phi_c e^{iqx}] = \frac{1}{2}\frac{1}{(2\pi)^d}\int d^dxd^dp\, e^{-ipx}\Big\{&\mathbb{1} - \frac{\lambda_k}{2}\phi_c^2 e^{2iqx}(P(-\partial_x^2)\Big\}^{-1}\\ &+ \frac{\lambda_k^2}{4}\phi_c^2 e^{2iqx}(P(-\partial_x^2))^{-1}\phi_c^2 e^{2iqx}(P(-\partial_x^2))^{-1}\Big\}\\ &k\partial_k R_k(-\partial_x^2)\frac{1}{P(-\partial_x^2)}e^{ipx}.\end{aligned}$$

This makes it so that by commuting the final e^{ipx} every term with ∂_x^2 instead gives a term with the combined factor in the exponential to its right-hand-side, squared, so that we find

$$\begin{aligned}k\partial_k\Gamma_k[\phi_c e^{iqx}] = \frac{1}{2}\frac{1}{(2\pi)^d}\int d^dxd^dp\,\Big\{&P(p^2)^{-1} - \frac{\lambda_k}{2}\phi_c^2 e^{2iqx}(P(p^2))^{-2}\\ &+ \frac{\lambda_k^2}{4}\phi_c^4 e^{4iqx}(P(p^2))^{-2}(P((p+2q)^2))^{-1}k\partial_k R_k(p^2)\Big\}\end{aligned}$$

In the final step we apply the projecting operations (52) to this equation, yielding:

$$\Pi_{2,0} k\partial_k \Gamma_k = -\frac{1}{(2\pi)^d} \int d^d x d^d p \; \frac{\lambda_k}{4} \left(z_k p^2 + m_k^2 + R_k(p^2) \right)^{-2} k\partial_k R_k(p^2) \tag{57a}$$

$$\Pi_{4,0} k\partial_k \Gamma_k = \frac{1}{(2\pi)^d} \int d^d x d^d p \; \frac{\lambda_k^2}{8} \left(z_k p^2 + m_k^2 + R_k(p^2) \right)^{-3} k\partial_k R_k(p^2) \tag{57b}$$

$$\Pi_{2,2} k\partial_k \Gamma_k = \frac{1}{(2\pi)^d} \int d^d x d^d p \; \frac{\lambda_k}{2} x^2 \left(z_k p^2 + m_k^2 + R_k(p^2) \right)^{-2} k\partial_k R_k(p^2) \tag{57c}$$

Comparing coefficients of these with equations (53a,b,c), and solving for all x rather than integrating over x, we find.

$$\frac{1}{2} k\partial_k z_k - x^2 k\partial_k m_k^2 = \frac{1}{(2\pi)^d} \int d^d p \; \frac{\lambda_k}{2} x^2 (z_k p^2 + m_k^2 + R_k(p^2))^{-2} k\partial_k R_k(p^2) \tag{58a}$$

$$\frac{1}{2} k\partial_k m_k^2 = -\frac{1}{(2\pi)^d} \int d^d p \; \frac{\lambda_k}{4} (z_k p^2 + m_k^2 + R_k(p^2))^{-2} k\partial_k R_k(p^2) \tag{58b}$$

$$\frac{1}{4!} k\partial_k \lambda_k = \frac{1}{(2\pi)^d} \int d^d p \; \frac{\lambda_k^2}{8} (z_k p^2 + m_k^2 + R_k(p^2))^{-3} k\partial_k R_k(p^2) \tag{58c}$$

By Combining the first 2 equations, we find that the beta function of the wavefunction renormalization factor vanishes:

$$k\partial_k z_k = 0. \tag{59}$$

To explicitly find the other 2, we need to choose a cutoff function R_k. The choice we make is the so-called optimized cutoff[11]

$$R_k(s) = z_k k^2 \left(1 - \frac{s}{k^2} \right) \Theta(1 - \tfrac{s}{k^2}) \tag{60}$$

Where Θ is the Heaviside distribution, with the convention that $\Theta(0) = 0$. Its derivative is a delta function of the same argument. By use of the chain and product rule we find

$$\begin{aligned} \int d^d p f(p) k\partial_k R_k(p^2) &= \int d^d p f(p) (z_k (2k^2 \Theta(1 - \frac{p^2}{k^2}) + 2p^2 (1 - \frac{p^2}{k^2}) \delta(1 - \frac{p^2}{k^2})) \\ &\quad + k(k^2 - p^2) \partial_k z_k \Theta(1 - \frac{p^2}{k^2})) \\ &= \int_{p^2 < k^2} f(p)(2k^2 z_k + k(k^2 - p^2)\partial_k z_k) = \int_{p^2 < k^2} f(p) 2k^2 z_k \end{aligned}$$

for any function $f(p)$, where the last equality holds due to (59). This result implies we only need to evaluate the integral for the region where $p^2 < k^2$. For this the cutoff becomes

$$R_k(p^2)|_{p^2 < k^2} = z_k (k^2 - p^2). \tag{61}$$

And so the operator P simplifies to

$$P(p^2)|_{p^2 < k^2} = z_k p^2 + m_k^2 + z_k (k^2 - p^2) = z_k k^2 + m_k^2 \tag{62}$$

Substituting this back into equation (58b,c) we find.

$$\frac{1}{2}k\partial_k m_k^2 = -\frac{1}{(2\pi)^d}\int_{p^2<k^2} d^dp\, \frac{\lambda_k}{4}(z_k k^2 + m_k^2)^{-2}2k^2 z_k \tag{63}$$

$$\frac{1}{4!}k\partial_k \lambda_k = \frac{1}{(2\pi)^d}\int_{p^2<k^2} d^dp\, \frac{\lambda_k^2}{8}(z_k k^2 + m_k^2)^{-3}2k^2 z_k \tag{64}$$

Both integrands have become p independent, and the integration domain is the d dimensional ball with radius k, of which the volume is given by

$$V_d(k) = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}k^d. \tag{65}$$

Where $\Gamma(t)$, not to be confused with the effective action, is the Gamma function, which is an extension of the factorial function. It is defined by

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx. \tag{66}$$

This finally leads to the beta functions for the dimensionful couplings in our ansatz

$$k\partial_x z_k = \beta_{z_k} = 0 \tag{67a}$$

$$k\partial_k m_k^2 = \beta_{m_k^2} = \frac{-\lambda_k k^{d+2} z_k}{2^d\pi^{d/2}\Gamma(\frac{d}{2}+1)(z_k k^2 + m_k^2)^2} \tag{67b}$$

$$k\partial_k \lambda_k = \beta_{\lambda_k} = \frac{6\lambda_k^2 k^{d+2} z_k}{2^d\pi^{d/2}\Gamma(\frac{d}{2}+1)(z_k k^2 + m_k^2)^3} \tag{67c}$$

These equations are the basis for the following discussion. Note that these equations hold for general spacetime dimension d.

4.3 Dimensionless coupling constants

So far the coupling constants m_k and λ_k were dimensionful. In natural units, every dimension reduces to a mass dimension. The effective action is dimensionless in natural units, which we can see as it is defined by the sum of W (which is a logarithm, so dimensionless) and $J[\chi]$ (which is seen in an exponent in Z, and thus must be dimensionless). Furthermore, z_k is dimensionless, as the dimension of that term is already given by ∂_x^2. In short, counting mass units we have $[x] = -1$, $[\partial_x] = 1$, $[\Gamma_k] = 0$, and $[z_k] = 0$. Integration over spacetime thus gives $-d$ mass units: $[d^dx] = -d$
From the kinetic term we can conclude that $[\phi] = \frac{d-2}{2}$, which then implies that $[\lambda_k] = 4-d$ and, unsurprisingly, $[m_k^2] = 2$. These are the so called canonical dimensions of the couplings.

Furthermore, k denotes an energy scale, and so $[k] = 1$. We can use this to make our coupling constants dimensionless, by multiplying with a suitable power of k.

$$\widetilde{m_k^2} = k^{-2}m_k^2 \tag{68a}$$

$$\widetilde{\lambda_k} = k^{d-4}\lambda_k \tag{68b}$$

Next, let us substitute the dimensionless couplings into the beta functions to find the dimensionless beta functions

$$k\partial_k \widetilde{m_k^2} = \beta_{\widetilde{m_k^2}} = \frac{-\widetilde{\lambda}_k z_k}{2^d \pi^{d/2} \Gamma(\frac{d}{2}+1)(z_k + \widetilde{m_k^2})^2} - 2\widetilde{m_k^2} \tag{69a}$$

$$k\partial_k \widetilde{\lambda}_k = \beta_{\widetilde{\lambda}_k} = \frac{6\widetilde{\lambda}_k^{\,2} z_k}{2^d \pi^{d/2} \Gamma(\frac{d}{2}+1)(z_k + \widetilde{m_k^2})^3} - (4-d)\widetilde{\lambda}_k \tag{69b}$$

$$k\partial_k z_k = \beta_{z_k} = 0 \tag{69c}$$

Which no longer have an explicit k dependence, making this an autonomous system of coupled ODEs.

4.4 RG Analysis

Equations (69a,b,c) give us the information we need to to do our nonperturbative analysis. The differential equations are solved numerically rather than analytically. Furthermore, fixed points are found and their convergence is analysed

4.4.1 General attributes

Because the coupling z_k can be fully absorbed in a redefinition of the field and the other couplings, this coupling becomes redundant[12], in the sense that we can redefine the field as $\phi \to \sqrt{z_k}\phi$ and the other couplings as $\frac{\widetilde{m_k}^2}{z_k}$ and $\frac{\widetilde{\lambda}_k}{z_k^2}$ to remove this coupling. We effectively do so by choosing $z_k = 1$. It is important that z_k is k independent if we want to do this, otherwise the field will become k dependent. Indeed this is the case as we have found in (69c). This means we only have to look at the 2-dimensional theory space of the couplings $\widetilde{m_k^2}$ and $\widetilde{\lambda}_k$.

The RG flow has a singularity for $\widetilde{m_k}^2 = -z_k = -1$. However, as we shall see, the fixed points are all to the right of the singularity, so every theory in the space left of the singularity is not UV complete. Therefore we can restrict ourselves to only look at the space to the right of the singularity.

Furthermore, there is a dividing line $\widetilde{\lambda}_k = 0$, which makes it impossible for a theory with positive $\widetilde{\lambda}_k$ for some k, to change to negative $\widetilde{\lambda}_k$ for some other k, and vice versa. Meaning that a (renormalizable) theory with attractive potential (negative λ) cannot become repulsive at short distances (high energy scale). In addition, $\widetilde{\lambda}_k = 0$ for some $k = k_0$ implies that $\lambda_k = 0$ for all k.

4.4.2 Fixed points

To find fixed points, we set both beta functions to 0. One fixed point, where both couplings are 0, is immediately obvious. This is called the Gaussian fixed point, as it corresponds to a (massless) free theory[7]. Another one, called the Wilson-Fisher fixed

[7] From here on, the mass term will be considered part of the potential. Note that however the mass term need not be 0 for the theory to have no interaction.

point (in $d = 3$), is given by the following set of fixed point values

$$z_* = 1 \tag{70a}$$

$$\widetilde{m_*}^2 = \frac{d-4}{16-d} \tag{70b}$$

$$\widetilde{\lambda}_* = \frac{9 \cdot 2^{d+5} \pi^{d/2} \Gamma(\frac{d}{2}+1)(4-d)}{(16-d)^3} \tag{70c}$$

Note that when setting $d = 4$, this non-trivial fixed point will coincide with the Gaussian fixed point. For d between 4 and 16, $\widetilde{\lambda}_*$ is negative, which is problematic for reasons discussed in the following section.

4.4.3 RG flow for $d = 1, 2, 3$

The RG flow around a fixed point is linearized by the matrix $(B_{\alpha\gamma}) = (\partial_\gamma \beta_\alpha(u_*))$, where u is a point in theory space and u_* in particular is the fixed point. The linearized flow is given by

$$k\partial_k u_\alpha = \sum_\gamma B_{\alpha\gamma}(u_\gamma(k) - u_*), \tag{71}$$

with general solution

$$u_\alpha(k) = u_{*\alpha} + \sum_i C_i V_\alpha^i \left(\frac{k_0}{k}\right)^{\theta_i}. \tag{72}$$

Where V^i's are eigenvectors with eigenvalues $-\theta_i$, k_0 is a reference scale, C_i's are integration constants and $-\theta_i$ are called the critical exponents. From this equation we can see that UV attractive directions correspond to $\theta_i > 0$ and UV repulsive (IR attractive) correspond to $\theta_i < 0$.
Using this method, we find that the Gaussian fixed point only has UV attractive directions. The flow is then numerically solved starting from the eigenvector near the Wilson-Fisher fixed point which is UV repulsive.

d	1	2	3
$\widetilde{m_*}^2$	$-1/5$	$-1/7$	$-1/13$
$\widetilde{\lambda}_*$	$\frac{32\pi}{125} \approx 0.804$	$\frac{288\pi}{325} \approx 2.784$	$\frac{1728\pi^2}{2197} \approx 7.763$
$\theta_{1,\text{WF}}$	-3.91548	-2.51915	-1.1759
$\theta_{2\text{WF}}$	1.91548	1.85248	1.84256
$\theta_{1,\text{Gauss}}$	2	2	2
$\theta_{2,\text{Gauss}}$	3	2	1

Table 1: Table containing Wilson-Fisher fixed point values and critical exponents for both Gaussian and Wilson-Fisher fixed points, for spacetime dimensions $d = 1, 2$ and 3.

We see that the Wilson Fisher fixed point has positive λ and negative m^2 for dimensions 1, 2, 3. In general we can characterize the 4 quadrants using the equation of motion (13). When looking for constant solutions to this equation, we find

$$\phi(x) = 0 \quad \wedge \quad \phi(x) = \pm\phi_c = \pm\sqrt{\frac{-6m^2}{\lambda}}.$$

Because ϕ is a real scalar field, m^2 and λ must be of opposite sign for the second solution to be a valid one. To check whether these solutions are maxima or minima, we take the second derivative of the potential with respect to ϕ

$$m^2 + \frac{\lambda}{2}\phi^2.$$

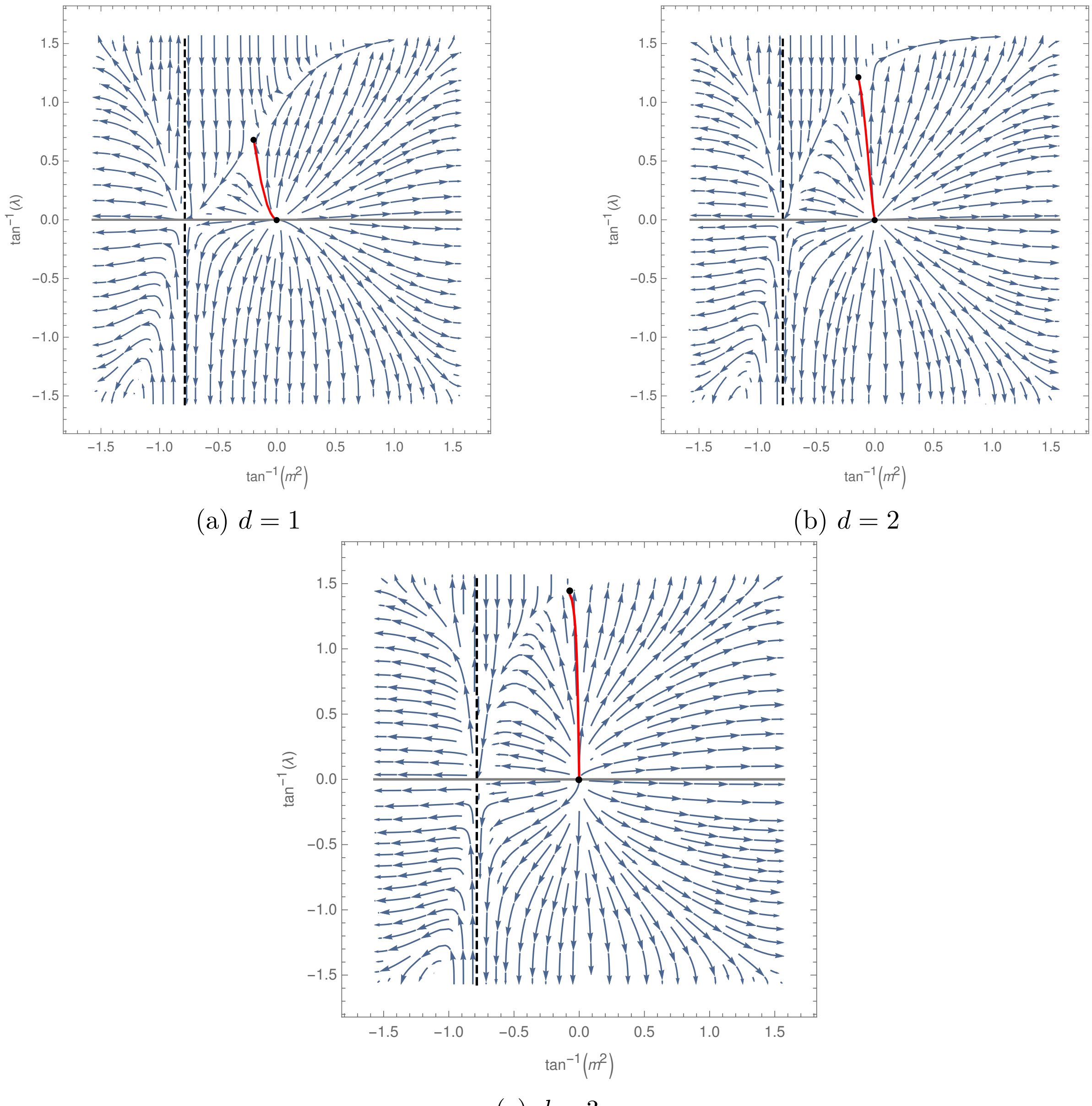
Positivity of this implies a minimum, and negativity a maximum. The following table (2) summarizes the results for different quadrants.

	$\lambda < 0$	$\lambda > 0$
$m^2 > 0$	Loc. minimum at 0, maxima at $\pm\phi_c$	Minimum at $\phi = 0$
$m^2 < 0$	Maximum at 0	Loc. maximum at 0, minima at $\pm\phi_c$

Table 2: Overview of minima and maxima for different possibilities of m^2 and λ, sorted by quadrant.

Whenever $\lambda < 0$, there is no global minimum of the potential. So the potential (and therefore the Hamiltonian) is not bounded from below and thus there is no ground state, leading to an unphysical theory. The fixed points found have $m^2 < 0$ and $\lambda > 0$, so that the potential has two global minima. This leads to spontaneous breaking of the symmetry $\phi \to -\phi$, which is the basic mathematical phenomena responsible for the Higgs mechanism.

The following plots (Fig. 2) show the flow of the couplings $(\widetilde{\lambda_k}, \widetilde{m_k^2})$ for different dimensions, where a parametrization with the inverse tangent is made to make the domain finite. The dashed line represents the singularity $\widetilde{m_k}^2 = -z_k = -1$, the grey line is the dividing line $\widetilde{\lambda_k} = 0$, and the red curve indicates the trajectory of a theory which goes from the Wilson-Fisher fixed point to the Gaussian fixed point for increasing k. Furthermore, the arrows point towards the IR direction (decreasing k).



(a) $d = 1$

(b) $d = 2$

(c) $d = 3$

Figure 2: RG flow in cases $d = 1, 2, 3$. The arrows point in the IR direction, the dividing line $\widetilde{\lambda}_k = 0$ is grey, and the singularity line $\widetilde{m_k}^2 = -1$ is dashed. The red line indicates the trajectory of the theory with both UV and IR limits.

4.4.4 RG flow for $d = 4$

In $d = 4$ spacetime dimensions, the former Wilson-Fisher and the Gaussian fixed point coincide. Hence, a ϕ^4 scalar theory which has a fixed point in the UV and is controlled by a fixed point in the IR does not exist in 4 dimensional spacetime. The beta function of $\widetilde{\lambda_k}$ is always positive (right of the singularity line), so theories with positive $\widetilde{\lambda}$ do not have a well defined UV limit, while negative $\widetilde{\lambda}$ theories are physically not allowed. Any UV complete theory thus necessarily has $\widetilde{\lambda_k} = 0$. Since $\widetilde{\lambda_{k_0}} = 0$ for some k_0 implies $\widetilde{\lambda_k} = 0\ \forall k$, there is no interaction generated by the RG flow. This is known as quantum triviality. This is problematic because the Higgs mechanism in the Standard Model of particle physics is described by a ϕ^4 scalar field theory. It remains an open problem [13][14], though the coupling to gravity shows promise in solving this (see for instance [16]). Furthermore, all RG trajectories with nonzero $\widetilde{\lambda_k}$ exhibit a Landau Pole. In standard classification [15], there are three cases: if $\beta_{\widetilde{\lambda_k}}$ has a zero at $\widetilde{\lambda_{k_0}}$ then $\widetilde{\lambda_k} \to \widetilde{\lambda_{k_0}}$ for $k \to \infty$; if $\beta_{\widetilde{\lambda_k}} \sim \widetilde{\lambda_k}^{\alpha}$ with $\alpha < 1$ for large $\widetilde{\lambda_k}$ and $\beta_{\widetilde{\lambda_k}}$ is not alternating, then $\widetilde{\lambda_k}$ goes to infinity as k goes to infinity; if $\beta_{\widetilde{\lambda_k}} \sim \widetilde{\lambda_k}^{\alpha}$ with $\alpha > 1$ for large $\widetilde{\lambda_k}$, then $\widetilde{\lambda_k}$ will diverge for some finite k_0. Since $\beta_{\widetilde{\lambda_k}} \sim \widetilde{\lambda_k}^{2}$ for large $\widetilde{\lambda_k}$, we are in the case where $\widetilde{\lambda_k}$ diverges for some finite k_0.

Once again we have a plot (Fig 3) of the RG flow in the IR direction, with dividing and singularity lines. There is no trajectory between fixed points.

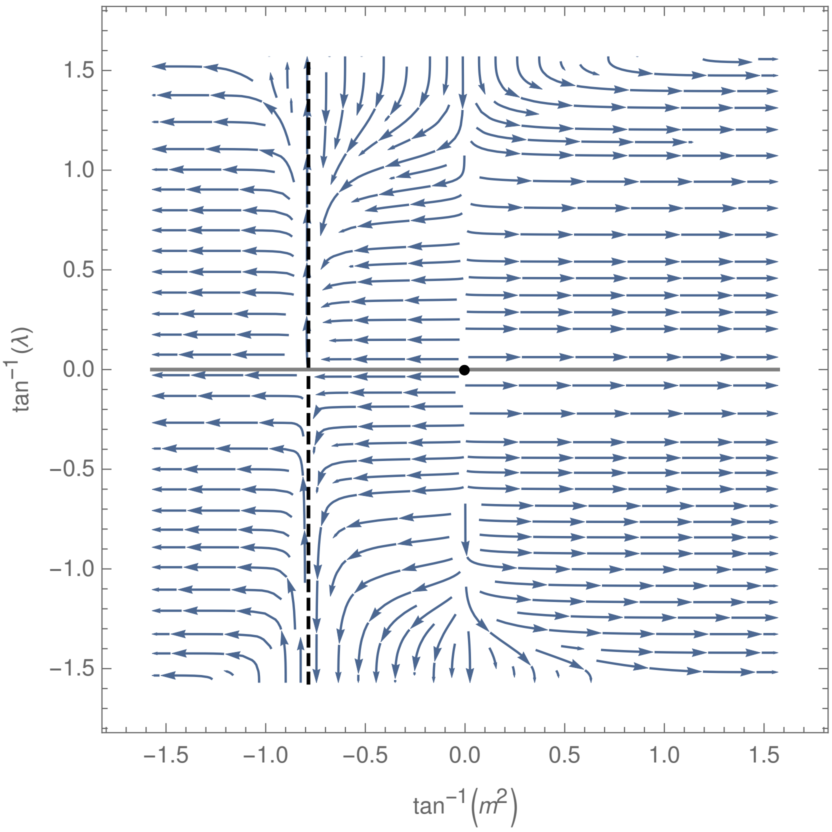


Figure 3: RG flow in $d = 4$ case. The arrows point in the IR direction, the dividing line $\lambda_k = 0$ is grey, and the singularity line $m_k^2 = -1$ is dashed.

5 Conclusion

We first started with some general classical and quantum field theory for a ϕ^4 interacting theory. After giving a brief sketch of the perturbative renormalization procedure, we proceeded to the nonperturbative analysis. Here (a version of) the FRGE was derived and used to study the RG flow for spacetime dimensions 1 through 4. We found that while for dimensions $1, 2, 3$ non-Gaussian fixed points and thus a renormalizable UV complete ϕ^4 theory exist, in 4 dimensional spacetime there exists only a Gaussian fixed point, indicating that quantum triviality remains even on the nonperturbative level. This indicates that a classical interacting (ϕ^4 in this case) theory becomes non-interacting due to quantum phenomena, since in 4 dimensions the coupling λ is necessarily 0 for all energy scales for UV complete theories.

Possible further research could involve taking higher order terms into account, coupling the system to different matter and gravitational sectors, and doing a careful study of the cutoff function dependence of the results. Furthermore this effect can be studied in the case of fermionic fields, instead of the scalar fields we used.

A The Hessian Operator of the ϕ^4 theory

Calculating the second derivative of the ϕ^4 action

$$\delta_v S[\phi+\epsilon w] = \int d^d x[-\frac{1}{2}(\phi+\epsilon w)(x)\partial_x^2 v(x) - \frac{1}{2}v(x)\partial_x^2(\phi+\epsilon w)(x) + m^2 v(x)(\phi+\epsilon w)(x) + \frac{\lambda}{3!}v(x)(\phi+\epsilon w)(x)^3]$$

Expanding the brackets gives

$$= \int d^d x[-\frac{1}{2}(\phi+\epsilon w)(x)\partial_x^2 v(x) - \frac{1}{2}v(x)\partial_x^2(\phi+\epsilon w)(x) + m^2 v(x)(\phi+\epsilon w)(x) + \frac{\lambda}{3!}v(x)(\phi(x)^3 + 3\epsilon\phi(x)^2 w(x) + 3\epsilon^2\phi(x)w(x)^2 + \epsilon^3 w(x)^3)]$$

For the second derivative $\delta_w\delta_v S[\phi] = \lim_{\epsilon\to 0}\frac{1}{\epsilon}(\delta_v S[\phi+\epsilon w] - \delta_v S[\phi])$, all terms with ϵ^0 cancel between the two terms, while all terms with power of ϵ greater than 1 vanish in the limit. Only terms scaling linearly with ϵ are left. This leaves us with the equation.

$$\delta_w\delta_v S[\phi] = \int d^d x[-\frac{1}{2}w(x)\partial_x^2 v(x) - \frac{1}{2}v(x)\partial_x^2 w(x) + m^2 v(x)w(x) + \frac{\lambda}{2}\phi(x)^2 v(x)w(x)] \tag{73}$$

Using partial integration twice on the second term gives, assuming w and v go to zero on the boundary, the following result

$$\delta_w\delta_v S[\phi] = \int d^d x[-v(x)\partial_x^2 w(x) + m^2 v(x)w(x) + \frac{\lambda}{2}\phi(x)^2 v(x)w(x)] \tag{74}$$

We can rewrite this as

$$\delta_w\delta_v S[\phi] = \int d^d x\ v(x)\widehat{\text{Hess}}\ w(x)$$

With the Hessian given by

$$\widehat{\text{Hess}} = -\partial_x^2 + m^2 + \frac{\lambda}{2}\phi(x)^2. \tag{75}$$

The Riesz Representation Theorem states that every linear functional can be written uniquely as an inner product[8] with a specific vector[17]. Writing the second derivative of S, which is linear in both v and w, in this way shows that the Hessian is the unique operator $\widehat{\text{Hess}}$ so that

$$\delta_w\delta_v S[\phi] = \langle v, \widehat{\text{Hess}}\ w\rangle\,. \tag{76}$$

The operator $\widehat{\text{Hess}}$ is unique because, for every w, the theorem states that the vector $\widehat{\text{Hess}}\ w$ is unique.

[8] In our case, the fields inheret the L^2 inner product.

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