Tachyonic neutrinos

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Abstract

In this paper, the possibility of the existence of tachyons is discussed.

If tachyons, particles moving faster than the speed of light, are to exist, there has to be a source able to create them. After the creation of a tachyon, a propagator is needed to describe the evolution of the particle through space-time. The reaction of the quantum field has to be great enough to allow for such particles to exist for long enough to be even named a particle.

The interactions of tachyons can be studied next, to see if these particles behave analogous to conventional particles. One of the processes best understood is the decay of a muon to an electron, a muon neutrino and an electron antineutrino. By substituting the two neutrinos by tachyonic neutrinos, it is possible to check wether this process can occur for tachyons or not.

Foreword

The choice for the subject of tachyons came from the results of the OPERA experiment, where tachyonic neutrinos were thought to have been observed. At the time of the conclusion of this report, these results were already discarded. Nevertheless the possibility of particles travelling faster than light speaks to the imagination and is still an interesting subject for further research.

This report is meant to be the conclusion of my bachelor education in physics. The complete research was done at the department of Theoretical High Energy Physics under the supervision of Prof. Dr. Ronald Kleiss at the Radboud University of Nijmegen.
Most of this report is based on and inspired by [1], for some of the basics in the first chapter, [2] is used. For the evaluation of the complex contour integrals in the same chapter [3] is used.

Some conventions that are used in this report are:

- The upper and lower indices at four-vectors will be dropped, meaning that a^μ will be written as a. Vectors will explicitly be denoted by a vector arrow, $\vec{a}$, and inner products between two four-vectors will be written as $a \cdot b$
- The speed of light, c, is taken to be 1 for simplicity

Contents

Chapter 1

Propagation of tachyons

1.1 Introduction

The very first question that comes to mind when a hypothetical particle is proposed, is whether or not it can propogate through space at all. When we consider elementary particles, a theory that includes relativity as well as quantum mechanics is required. Quantum field theory provides us with the tools necessary to investigate the creation, propagation and annihilation of such particles. A source is responsible the creation of particles, after that, the particle's behaviour is described by its propagator. When investigating the possibility for a particle to exist or not, it is vital to look at the reaction of the field to the creation and propagation of that particle.

In this chapter the most important properties of a tachyon are explained. After that, a source emitting such particles is introduced and its characteristics are described. The tachyon's propagator is given, which will give rise to the necessity to distinguish between two kinds of tachyons. For both cases the reaction of the quantum field will be calculated and the results will lead to conclusions that are discussed at the end of the chapter.

1.2 Tachyon properties

A tachyon is a hypothetical particle, which means that there is no experimental evidence of the existance of such particles. The study of tachyons is therefore purely theoretical.

The most important property of a tachyon is the fact that it's speed is faster than the speed of light. This immediately leads to some interesting consequences. The first thing to look at is the energy of a tachyon. Just like the energy of any other relativistic particle, it is given by

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1.1}$$

From Eq. (1.1) it can be seen that a tachyon, having a speed greater than the speed of light, must have an imaginary mass, because the energy must be real. Furthermore, one of the interesting facts is that the energy of a tachyon decreases as its speed increases. This leads to the fact that the speed of light is still a limit for tachyons, but where conventional particles can't go over the limit, tachyons are unable to go below it.

1.3 Sources, propagators and field reactions

The reaction of the quantum field to a tachyon is given by

$$\phi(x) = \frac{i}{\hbar} \int d^4y \; \Pi(x-y) J(y) \tag{1.2}$$

As can be seen from Eq. (1.2), a propagator describing the evolution of the tachyon through spacetime is needed, as well as a source, accounting for the creation of the particle. The propagator of a conventional particle in four-dimensional Minkowski space is given by

$$\Pi(x-y) = \frac{i\hbar}{(2\pi)^4} \int d^4k \; \frac{\exp(-ik \cdot (x-y))}{k \cdot k - m^2 + i\epsilon} \tag{1.3}$$

Using the knowledge that a tachyon has an imaginary mass, this propagator can be rewritten for tachyons by the substitution $m^2 \rightarrow -m^2$.

The source that is used to investigate the reaction of the field is one that is active for a period of $\frac{\sigma_0}{c}$ around a given time $t = 0$ and in a region of volume σ^3. So it shows a Gaussian behaviour in space and an exponentially decreasing behaviour in time.

$$J(x) \propto \exp\left(-\frac{|x^0|}{\sigma_0} - \frac{|\vec{x}|^2}{4\sigma^2} - \frac{i}{\hbar} \left(p^0 x^0 - \vec{x} \cdot \vec{p} \right) \right) \tag{1.4}$$

From the Fourier transform of this source,

$$J(k) \propto \left[\frac{1}{\sigma_0^2} + \left(k^0 - \frac{p^0}{\hbar}\right)^2\right]^{-1} \exp\left(-\sigma^2\left(\vec{k} - \frac{\vec{p}}{\hbar}\right)^2\right) \tag{1.5}$$

it can be seen that particles of all kinds of wave vectors are emitted by this source.

1.4 Two regimes

Looking back at the denominator of the propagator for a tachyon, we can define a quantity

$$\omega = \sqrt{|\vec{k}|^2 - m^2} \tag{1.6}$$

which immediately gives rise to two different regimes, namely $|\vec{k}|^2 > m^2$ where ω is real, and $|\vec{k}|^2 < m^2$ where ω is imaginary. Both of them have to be evaluated seperately.

1.4.1 The real regime

First, we look at the region where ω is real. The reaction of the field in this case is given by

$$\begin{aligned}\phi \propto \int d^4k \frac{\exp\left(-ik^0x^0 + i\vec{k}\cdot\vec{x}\right)}{(k^0)^2 - \omega^2 + i\epsilon} \times \\ \left[\frac{1}{\sigma_0^2} + \left(k^0 - \frac{p^0}{\hbar}\right)^2\right]^{-1} \exp\left(-\sigma^2\left(\vec{k} - \frac{\vec{p}}{\hbar}\right)^2\right)\end{aligned} \tag{1.7}$$

For positive times $x^0 > 0$, the contour has to be closed in the lower half complex-k^0 plane to make sure that the exponent vanishes. This integral displays two poles inside the contour, one at $k^0 = \omega - i\epsilon$ and one at $k^0 = \frac{p^0}{\hbar} - \frac{i}{\sigma_0}$. Doing the k^0 integral[1] leads to the following result

$$\begin{aligned}\phi(x) \propto \int d^3\vec{k}\ e^{i\vec{k}\cdot\vec{x} - \sigma^2\left(\vec{k} - \frac{\vec{p}}{\hbar}\right)^2} \times \\ \left[\frac{1}{2\omega}\frac{e^{-ix^0\omega}}{\left(\frac{p^0}{\hbar} - \omega\right)^2 + \frac{1}{\sigma_0^2}} + \frac{i\sigma_0}{2}\frac{e^{-\frac{ix^0p^0}{\hbar} - \frac{x^0}{\sigma_0}}}{\left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0}\right)^2 - \omega^2 + i\epsilon}\right]\end{aligned} \tag{1.8}$$

[1]The complete calculation can be found in appendix A

1.4.2 The imaginary regime

We now turn to the case $|\vec{k}|^2 < m^2$ where ω is imaginary. For aestatic reasons, we will use a notation where every quantity used is real. This means that in the following calculations ω will be replaced by $i\omega$.
Again, the contour has to be closed in lower half complex-k^0 plane, but one of the poles is different. The one located at $k^0 = \frac{p^0}{\hbar} - \frac{i}{\sigma_0}$ is still present, the other pole has moved to $k^0 = -(i\omega - i\epsilon)$. The integral[2] now leads to the result

$$\phi(x) \propto \int d^3\vec{k}\ e^{i\vec{k}\cdot\vec{x} - \sigma^2\left(\vec{k}-\frac{\vec{p}}{\hbar}\right)^2} \times \left[-\frac{1}{2i\omega} \frac{e^{-\omega x^0}}{\left(\frac{p^0}{\hbar} + i\omega\right)^2 + \frac{1}{\sigma_0^2}} + \frac{i\sigma_0}{2} \frac{e^{-\frac{ix^0p^0}{\hbar} - \frac{x^0}{\sigma_0}}}{\left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0}\right)^2 + \omega^2 + i\epsilon} \right] \tag{1.9}$$

1.5 Discussion

The reaction of the quantum field to the creation and propagation of tachyons depends on the characteristics of those particles. The absolute value squared of the wavevector of a tachyon can be either greater or smaller than its mass. We first consider the reaction of the field in the case where $|\vec{k}|^2 > m^2$. The expression is exactly the same as the reaction of the quantum field to a conventional particle, apart from the fact that the mass squared is negative in the case of tachyons.
The exponential function in the numerator of the second term in Eq. (1.8) consists of a real and an imaginary part. The real part can be seen to decay exponentially at a rate of $\frac{|x^0|}{\sigma_0}$, which is exactly the same as the rate at which the source decays. For the field to be free and unaffected by interactions, it is necessary to look at the situations where the source is no longer present. This means that the second term of Eq. (1.8) can be neglected in the study of the field.
For the field to be noteworthy, the denomenator of the first term has to be small, resulting in the fact that $\frac{p^0}{\hbar} \approx \omega$. From this it is possible, just like it is for conventional particles, to derive Newton's first law, leading to the

[2] The complete calculation can be found in appendix B

conclusion that a tachyon with $|\vec{k}|^2 > m^2$ might propagate through space.

We now turn to the second possibility for tachyons, namely $|\vec{k}|^2 > m^2$. Again, the second term decays exponentially with the source, so it can again be neglected. The denominator of the first term, however, cannot be small enough to allow signals to be transmitted over great distances. This is due to the fact that the factor ω in the denominator is now positive. Moreover, the exponential in the numerator is no longer imaginary, resulting in exponential decay of this term.

As concluded, a tachyon with $|\vec{k}|^2 > m^2$ might be created by a source and can be able to propagate through spacetime. These kind of tachyons would obey the same law[3] as conventional particles do and therefore it might nog be possible to, experimentally, distinguish between them.
A tachyon with $|\vec{k}|^2 < m^2$ leads to a reaction of the quantum field that goes to zero very fast, meaning that such a particle can not propagate very far. It might even be said that such a particle is not even able to exist at all because of this reason.
If tachyons would exist at all, there is at least one boundary for these particles, namely that $|\vec{k}|^2 > m^2$ should be obeyed.

[3]Newton's first law, that is

Chapter 2

Muon decay

2.1 Introduction

After a new particle has been proposed, and it's propagation through space and time has been established, the next thing to investigate is the interactions of these particles. One of the best known processes is the decay of a muon to an electron, an electron antineutrino and a muon neutrino. If tachyons are to exist, it might be possible for muons to decay to an electron and two tachyonic neutrinos.

In this chapter, the chance of tachyonic muon decay happening will be studied. To this end, the kinematics of the process are first studied. This will lead to an expression for the matrixelement of the process. Using this matrixelement, it is possible to obtain the decay width of the muon, which can than be compared to the decay width of a conventional muon.
At last, the decay of an electron to another electron is briefly studied. Conventionally, this process is impossible because of the conservation of energy and momentum, but it might be possible for tachyons to realise this decay.

2.2 Computing the matrixelement

The muon decay process can be described by a single Feynman diagram, containing one vertex where four fermions meet. The interaction will be studied from the frame in which the muon is at rest. The transition rate for this process is given by

$$\langle |\mathcal{M}|^2 \rangle = 64 G_F^2 \hbar^2 (q \cdot k_1)(p \cdot k_2) \tag{2.1}$$

Here, p, q, k_1 and k_2 are the four-vectors of the muon, the electron, the muon neutrino and the electron antineutrino respectively.

2.2.1 Conventional kinematics

To be able to compare the tachyonic case of muon decay to the conventional process, it is necessary first to determine the conventional matrix element. To compute this matrix element, the inner product of the four-vectors of the incoming particles has to be known, just like the inner product of the four-vectors of the outgoing particles.
The four-vectors of the muon and the electron are

$$p = \begin{pmatrix} E_\mu \\ \vec{p} \end{pmatrix}, \; q = \begin{pmatrix} E_e \\ \vec{q} \end{pmatrix} \tag{2.2}$$

Analogously, the four-vectors of the two neutrinos are

$$k_1 = \begin{pmatrix} E_1 \\ \vec{k_1} \end{pmatrix}, \; k_2 = \begin{pmatrix} E_2 \\ \vec{k_2} \end{pmatrix} \tag{2.3}$$

By using the condition that the particles have to lie on their mass shells and the fact that the muon has no impuls[1], the following four-vectors are obtained for the muon and the electron

$$p = \begin{pmatrix} m_\mu \\ \vec{0} \end{pmatrix}, \; q = \begin{pmatrix} \sqrt{m_e^2 + \vec{q}^2} \\ \vec{q} \end{pmatrix} \tag{2.4}$$

The four-vectors of the two neutrinos can be rewritten as

$$k_1 = \begin{pmatrix} E_1 \\ \sqrt{E_1^2 - m_1^2} \end{pmatrix}, \; k_2 = \begin{pmatrix} E_2 \\ \sqrt{E_2^2 - m_2^2} \end{pmatrix} \tag{2.5}$$

[1]We are considering the process in the rest system of the muon

The inner products can now be calculated

$$
\begin{aligned}
(p \cdot k_2) &= m_\mu E_2 && (2.6)\\
(q \cdot k_1) &= E_1\sqrt{m_e^2 + q^2} - \vec{q}\sqrt{E_1^2 - m_1^2} && (2.7)\\
&= \frac{1}{2}((q + k_1)^2 - (m_e^2 + m_1^2)) && (2.8)\\
&= \frac{1}{2}((p - k_2)^2 - (m_e^2 + m_1^2)) && (2.9)
\end{aligned}
$$

Where conservation of momentum is used at the last step.
By writing out the four-vectors, this expression can be rewritten as

$$
\begin{aligned}
(q \cdot k_1) &= m_\mu \left(\frac{(m_\mu^2 + m_2^2) - (m_e^2 + m_1^2)}{2m_\mu} - E_2 \right)\\
&\equiv m_\mu (K_c - E_2) \qquad (2.10)
\end{aligned}
$$

Plugging these in Eq. (2.1) results in the matrix element for the conventional case

2.2.2 Tachyonic kinematics

In the case where tachyonic neutrinos are used instead of conventional neutrinos, the four-vectors are changed to

$$
k_1 = \begin{pmatrix} E_1 \\ \sqrt{E_1^2 + m_1^2} \end{pmatrix}, \; k_2 = \begin{pmatrix} E_2 \\ \sqrt{E_2^2 + m_2^2} \end{pmatrix} \qquad (2.11)
$$

In this case the inner product $(q \cdot k_1)$ can be rewritten as[2]

$$
\begin{aligned}
(q \cdot k_1) &= m_\mu \left(\frac{(m_\mu^2 + m_1^2) - (m_e^2 + m_2^2)}{2m_\mu} - E_2 \right)\\
&\equiv m_\mu (K_t - E_2) \qquad (2.12)
\end{aligned}
$$

2.3 Phase space

The partial decay width of the muon is given by

$$
d\Gamma = \phi_\Gamma \langle |\mathcal{M}|^2 \rangle dV \qquad (2.13)
$$

[2] A full derivation can be found in appendix C

In this equation, ϕ_Γ is a collection of factors to account for the density of states of the incoming particle, etcetera. In this case $\phi_\Gamma = \frac{1}{2m_\mu}$. Having already calculated the matrix element, the only thing left to do is to determine the combined phase space integration element.
This integration element is defined by

$$dV(P;p_1,p_2,...,p_N) \equiv \left(\prod_{j=1}^{N} \frac{1}{(2\pi)^3} d^4p_j \delta(p_j^2 - m_j^2)\right) \times (2\pi)^4 \delta^4 \left(P - \sum_{j=1}^{N} p_j\right) \quad (2.14)$$

The m_j and the p_j are the masses and wavevectors respectively of the N particles in the final state and P is the wavevector of the incoming particle. The Dirac delta functions in the product makes sure the particles in the final state lie on their mass shell. The four-dimensional Dirac delta imposes the conservation energy and momentum.
In the case of tachyonic neutrinos, this phase space integration element is given by

$$dV(p:q,k_1,k_2) = \frac{1}{(2\pi)^5} d^4q\ d^4k_1\ d^4k_2\ \delta(k_1^2 + m_1^2)\ \delta(k_2^2 + m_2^2) \times \delta(q^2 - m_e^2)\ \delta^4(p - q - k_1 - k_2) \quad (2.15)$$

This can be rewritten as[3]

$$dV(p:q,k_1,k_2) = \frac{\pi^2}{(2\pi)^5} dE_1\ dE_2\ d\cos\theta_{1,2} \times \delta^4\left(\frac{-m_\mu^2 + k_1^2 + k_2^2 - m_e^2 + 2(m_\mu - k_1^0)(m_\mu - k_2^0)}{2|\vec{k_1}||\vec{k_2}|} - \cos\theta_{1,2}\right) \quad (2.16)$$

Using the fact that the cosine is bounded, the expression for the integration element that is left is

$$dV(p:q,k_1,k_2) = \frac{\pi^2}{(2\pi)^5} dE_1 dE_2 \quad (2.17)$$

[3]The complete computation can be found in appendix D

Where the following boundaries have to be imposed

$$-m_\mu^2 + k_1^2 + k_2^2 - m_e^2 + 2(m_\mu - k_1^0)(m_\mu - k_2^0) \leq 2|\vec{k_1}||\vec{k_2}| \quad (2.18)$$

$$-m_\mu^2 + k_1^2 + k_2^2 - m_e^2 + 2(m_\mu - k_1^0)(m_\mu - k_2^0) \geq -2|\vec{k_1}||\vec{k_2}| \quad (2.19)$$

Writing out all the wavevectors explicitly, these boundaries can be rewritten in terms of the energies of the two neutrinos and the masses of all the particles involved

$$\frac{m_\mu^2 - m_e^2 - m_1^2 - m_2^2 - 2m_\mu E_1 - 2m_\mu E_2 + 2E_1E_2}{2\sqrt{(E_1^2 + m_1^2)(E_2^2 + m_2^2)}} \leq 1 \quad (2.20)$$

$$\frac{m_\mu^2 - m_e^2 - m_1^2 - m_2^2 - 2m_\mu E_1 - 2m_\mu E_2 + 2E_1E_2}{2\sqrt{(E_1^2 + m_1^2)(E_2^2 + m_2^2)}} \geq -1 \quad (2.21)$$

2.4 Decay width

The decay width is now given by integration of the partial decay width over the energies of the two neutrinos

$$\Gamma = \int\int 32\ G_F^2\ \hbar^2\ m_\mu\ E_2(K_t - E_2)\frac{\pi^2}{(2\pi)^2} dE_1\ dE_2 \quad (2.22)$$

The boundaries for the integral over E_2 are given by fact that $\langle|\mathcal{M}|^2\rangle$ has to be positive. This implies that

$$(p \cdot k_2) = m_\mu E_2 \quad (2.23)$$

$$(q \cdot k_1) = m_\mu(K_t - E_2) \quad (2.24)$$

both have to be positive or negative.

If the first inner product is negative, then E_2 has to be negative. The fraction in the second inner product is always positive, because the masses of the two neutrinos are equal, and the mass of the muon is greater than that of the electron.

This means that, if the inner product is to be positive, E_2 has to be positive. This leaves only the possibility of $E_2 \geq 0$, the second inner product then forces $E_2 \leq K_t$.

So the integral over E_2 has to be taken from 0 to K_t

At last, the boundaries from Eq. (2.20) and Eq. (2.21) can be solved for E_1 by inserting values for the masses and using a computational program[4]. By taking the following masses

$$m_\mu = 100 \text{ MeV} \quad (2.25)$$

$$m_e = 0.5 \text{ MeV} \quad (2.26)$$

$$|m_1| = 1 \cdot 10^{-4} \text{ MeV} \quad (2.27)$$

$$|m_2| = 1 \cdot 10^{-4} \text{ MeV} \quad (2.28)$$

the boundaries can be rewritten to impose restrictions on E_1

$$E_1 \geq \frac{1}{-50 + E_2}\Bigg(5.0 \cdot 10^8 \Bigg(-5.0 \cdot 10^{10} + 1.5 \cdot 10^9 E_2 - 1.0 \cdot 10^7 E_2^2 + \sqrt{2.5 \cdot 10^9 - 10 \cdot 10^7 E_2 + 2.5 \cdot 10^{17} E_2^2 - 10 \cdot 10^{15} E_2^3 + 1.0 \cdot 10^{14} E_2^4}\Bigg)\Bigg) \equiv R_1 \quad (2.29)$$

$$E_1 \leq \frac{1}{-50 + E_2}\Bigg(5.0 \cdot 10^8 \Bigg(5.0 \cdot 10^{10} - 1.5 \cdot 10^9 E_2 + 1.0 \cdot 10^7 E_2^2 + \sqrt{2.5 \cdot 10^9 - 10 \cdot 10^7 E_2 + 2.5 \cdot 10^{17} E_2^2 - 10 \cdot 10^{15} E_2^3 + 1.0 \cdot 10^{14} E_2^4}\Bigg)\Bigg) \equiv R_2 \quad (2.30)$$

These boundaries fix the phase space, which is shown in Figure 2.1. The dark areas of the figure are the ones where both boundaries are satisfied.
First of all, for the square in the upper right corner to be allowed, the energy of the electron should be negative and therefore, this regime can be neglected.
The two regimes where one of the energies of the neutrinos is negative are only possible if $\langle|\mathcal{M}|^2\rangle$ is negative, which is impossible for the given matrix-element.

Now the decay width can be calculated by only integrating over the triangle in the positive quadrant

$$\Gamma = \int_0^{K_t} \int_{R_1}^{R_2} 32 G_F^2 \hbar^2 m_\mu E_2 (K_t - E_2) \frac{\pi^2}{(2\pi)^2} dE_1 dE_2 \quad (2.31)$$

[4]In this case, Maple was used

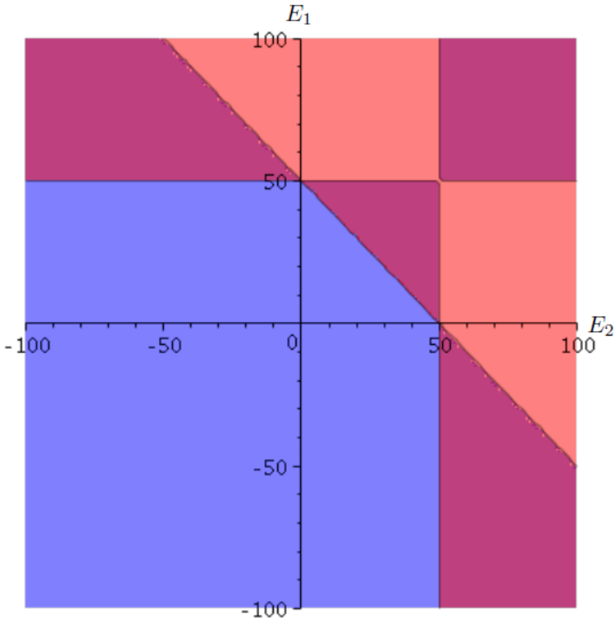

Figure 2.1: The phase space of the muon decay

Numerical calculation of these integrals[5] leads to the following result for the decay width

$$\Gamma = 1.68 \cdot 10^6 G_F^2 \hbar^2 \tag{2.32}$$

which is numerically equal to the decay width of the conventional muon decay

$$\Gamma = \frac{G_F^2 \hbar^2 m_\mu^5}{192\pi^3} \tag{2.33}$$

2.5 Electron decay

By changing the value of m_μ to that of the electron mass, this process can easily be changed to describe the possible decay of an electron. By doing this the phase space shown in Figure 2.2 is obtained and, by integrating over the phase space, while neglecting the cases where m_e or the matrix element is negative, the decay width is found to be equal to 0.

[5]Again, by the use of Maple

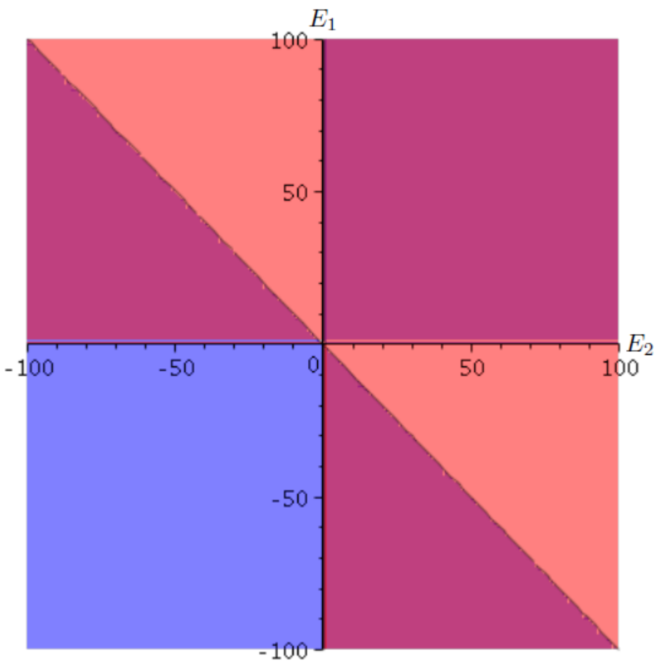

Figure 2.2: The phase space of the electron decay

2.6 Discussion

The matrix element for the process of a muon decaying to an electron and two tachyonic neutrinos is not known, so the best one can do is use the same matrix element that describes the process with conventional neutrinos.
The phase space can be divided into three parts. The part in the top right corner, which is kinematically forbidden because the electron energy in this regime would be negative, can be neglected.
The two parts in the quadrants where one of the energies of the neutrinos is negative can be seen to diverge to infinity. Finally, the triangular part in the positive quadrant is equal to the allowed phase space in the conventional case.

If the conventional matrix element is indeed the correct one to also describe the tachyonic process, the two diverging areas are forbidden because in these areas, the matrix element would be negative. Because this matrix element gives the chance for the muon decay process to happen, and a chance cannot

be negative, these areas are off limits.
If the conventional matrix element turns out not to be correct in the tachyonic case, and another matrix element that does describe the process can be found, these two diverging regimes cannot be neglected. But in this case the partial decay width has to be integrated over all phase space, which diverges, meaning that the total decay width will go to infinity in this case. The lifetime of a particle is proportional to $\frac{1}{\Gamma}$, so in the case of an infinite decay width, the lifetime of a particle is equal to 0, resulting in the fact that the particle is unable to exist at all.

When considering the possibility of an electron decaying to another electron and two tachyonic neutrinos, the phase space can be divided into two parts. The first part is the positive quadrant, where the electron energy has to be negative, so this part can again be neglected. The other part, just like in the case of muon decay, is impossible because the chance for this process would be negative in that regime. This leads to the conclusion that, just like in the conventional case, it is impossible for an electron to decay to another electron.

Appendix A

The field integral in the real regime

The integral in Eq. (1.7) will be worked out in some detail here. This reaction of the field can be rewritten as follows

$$\begin{aligned}\phi \propto \int d^4k \frac{\exp\left(-ik^0x^0 + i\vec{k}\cdot\vec{x}\right)}{(k^0)^2 - \omega^2 + i\epsilon} \times \\ \left[\frac{1}{\sigma_0^2} + \left(k^0 - \frac{p^0}{\hbar}\right)^2\right]^{-1} \exp\left(-\sigma^2\left(\vec{k} - \frac{\vec{p}}{\hbar}\right)^2\right) \end{aligned} \tag{A.1}$$

$$\begin{aligned}= \int d^3\vec{k}\ e^{i\vec{k}\cdot\vec{x}} e^{-\sigma^2\left(\vec{k}-\frac{\vec{p}}{\hbar}\right)^2} \times \\ \int \frac{e^{-ik^0x^0}\ dk^0}{\left(k^0 - (\omega - i\epsilon)\right)\left(k^0 + (\omega - i\epsilon)\right)\left(k^0 - \left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0}\right)\right)\left(k^0 - \left(\frac{p^0}{\hbar} + \frac{i}{\sigma_0}\right)\right)}\end{aligned} \tag{A.2}$$

To only integral that will be done here is the one over k^0. This integral can be closed in the lower half complex=k^0 plane.
Two poles are present in te contour

$$k^0 = \omega - i\epsilon \tag{A.3}$$

$$k^0 = \frac{p^0}{\hbar} - \frac{1}{\sigma_0} \tag{A.4}$$

To do this integral, the residues have to be calculated first.

The residue of the first pole, Eq. (A.3) can be calculated as follows

$$\lim_{k^0 \to (\omega - i\epsilon)} \frac{e^{-ik^0 x^0}}{(k^0 + (\omega - i\epsilon)) \left(k^0 - \left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0}\right)\right) \left(k^0 - \left(\frac{p^0}{\hbar} + \frac{i}{\sigma_0}\right)\right)} =$$

$$\frac{e^{-ix^0(\omega - i\epsilon)}}{((\omega - i\epsilon) + (\omega - i\epsilon)) \left((\omega - i\epsilon) - \left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0}\right)\right) \left((\omega - i\epsilon) - \left(\frac{p_0}{\hbar} + \frac{i}{\sigma_0}\right)\right)} =$$

$$\frac{e^{-ix^0(\omega - i\epsilon)}}{2(\omega - i\epsilon) \left((\omega - i\epsilon)^2 - (\omega - i\epsilon)\left(\frac{p^0}{\hbar} + \frac{i}{\sigma_0}\right) - (\omega - i\epsilon)\left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0}\right) + \left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0}\right)\left(\frac{p^0}{\hbar} + \frac{i}{\sigma_0}\right)\right)} =$$

$$\frac{e^{-ix^0(\omega - i\epsilon)}}{2(\omega - i\epsilon) \left(\omega^2 - 2i\epsilon\omega - \epsilon^2 - \frac{\omega p^0}{\hbar} - \frac{i\omega}{\sigma_0} + \frac{i\epsilon p^0}{\hbar} - \frac{\epsilon}{\sigma_0} - \frac{\omega p^0}{\hbar} + \frac{i\omega}{\sigma_0} + \frac{i\epsilon p^0}{\hbar} + \frac{\epsilon^2}{\sigma_0} + \frac{(p^0)^2}{\hbar} + \frac{1}{\sigma_0^2}\right)} =$$

$$\frac{e^{-ix^0(\omega - i\epsilon)}}{2(\omega - i\epsilon) \left(\omega^2 - \frac{2\omega p^0}{\hbar} + \frac{(p^0)^2}{\hbar^2} + \frac{1}{\sigma_0} - \left(2i\omega\epsilon + \epsilon^2 + \frac{2i\epsilon p^0}{\hbar}\right)\right)} =$$

$$\frac{e^{-ix^0\omega}}{2\omega \left(\omega - \frac{p^0}{\hbar} + \frac{i}{\sigma_0}\right)\left(\omega - \frac{p^0}{\hbar} - \frac{i}{\sigma_0}\right)} = \frac{e^{-ix^0\omega}}{2\omega \left(\frac{p^0}{\hbar} - \omega\right)^2 + \frac{1}{\sigma_0^2}} \tag{A.5}$$

The residue of the second pole, Eq. (A.4) is

$$\lim_{k^0 \to \left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0}\right)} \frac{e^{-ik^0 x^0}}{(k^0 - (\omega - i\epsilon))(k^0 + (\omega - i\epsilon)) \left(k^0 - \left(\frac{p^0}{\hbar} + \frac{i}{\sigma_0}\right)\right)} =$$

$$\frac{e^{-ix^0\left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0}\right)}}{\left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0} - (\omega - i\epsilon)\right)\left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0} + (\omega - i\epsilon)\right)\left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0} - \left(\frac{p^0}{\hbar} + \frac{1}{\sigma_0}\right)\right)} =$$

$$\frac{e^{-ix^0\frac{p^0}{\hbar} - \frac{x^0}{\sigma_0}}}{\left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0} - \omega + i\epsilon\right)\left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0} + \omega - i\epsilon\right)\left(-\frac{2i}{\sigma_0}\right)} =$$

$$\frac{i\sigma_0 e^{-i\frac{x^0 p^0}{\hbar} - \frac{x^0}{\sigma_0}}}{2\left(\frac{(p^0)^2}{\hbar^2} - 2\frac{ip^0}{\hbar\sigma_0} - \frac{1}{\sigma_0^2} - \omega^2 + 2i\epsilon\omega + \epsilon^2\right)} = \frac{i\sigma_0 e^{-i\frac{x^0 p^0}{\hbar} - \frac{x^0}{\sigma_0}}}{2\left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0}\right)^2 - \omega^2 + i\epsilon} \tag{A.6}$$

And the addition of these two residues results in Eq. (1.8)

Appendix B

The field integral in the imaginary regime

In this case, the poles are located at

$$k^0 = -(i\omega - i\epsilon) \tag{B.1}$$

$$k^0 = \frac{p^0}{\hbar} - \frac{1}{\sigma_0} \tag{B.2}$$

The residue of the first pole is now given by

$$\lim_{k^0 \to -(i\omega - i\epsilon)} \frac{e^{-ik^0 x^0}}{\left(k^0 - (i\omega - i\epsilon)\right)\left(k^0 - \left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0}\right)\right)\left(k^0 - \left(\frac{p^0}{\hbar} + \frac{i}{\sigma_0}\right)\right)} =$$

$$\frac{e^{i(i\omega - i\epsilon)x^0}}{-2\left(i\omega - i\epsilon\right)\left(-\left(i\omega - i\epsilon\right) - \left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0}\right)\right)\left(-\left(i\omega - i\epsilon\right) - \left(\frac{p^0}{\hbar} + \frac{i}{\sigma_0}\right)\right)} =$$

$$\frac{e^{-\omega x^0 + \epsilon x^0}}{-2\left(i\omega - i\epsilon\right)\left(\left(i\omega - i\epsilon\right)^2 + \left(i\omega - i\epsilon\right)\left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0}\right) + \left(i\omega - i\epsilon\right)\left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0}\right) + \left(\frac{p^0}{\hbar} - \frac{i}{\sigma_0}\right)\left(\frac{p^0}{\hbar} + \frac{i}{\sigma_0}\right)\right)} =$$

$$\frac{e^{-\omega x^0 + \epsilon x^0}}{-2\left(i\omega - i\epsilon\right)\left(-\omega^2 + 2\epsilon\omega - \epsilon^2 + \frac{2i\omega p^0}{\hbar} - \frac{2i\epsilon p^0}{\hbar} + \frac{(p^0)^2}{\hbar^2} + \frac{1}{\sigma_0^2}\right)} =$$

$$\frac{e^{-\omega x^0}}{-2i\omega\left(-\omega^2 + \frac{2i\omega p^0}{\hbar} + \frac{(p^0)^2}{\hbar^2} + \frac{1}{\sigma_0^2}\right)} = \frac{e^{-\omega x^0}}{-2i\omega\left(\left(\frac{p^0}{\hbar} + i\omega\right)^2 + \frac{1}{\sigma_0^2}\right)} \tag{B.3}$$

The residue of the second pole in this case is the same as the second pole in the real regime, except for the fact that ω^2 is now replaced by $-\omega^2$

Appendix C

Tachyonic kinematics

The inner product between the four-vectors

$$q = \begin{pmatrix} \sqrt{m_e^2 + \vec{q}^2} \\ \vec{q} \end{pmatrix}, \; k_1 = \begin{pmatrix} E_1 \\ \sqrt{E_1^2 + m_1^2} \end{pmatrix} \tag{C.1}$$

is to be calculated. The first thing to do is look at the following expression

$$\begin{aligned} (q + k_1)^2 &= \left(\sqrt{m_e^2 + \vec{q}^2} + E_1\right)^2 - \left(\vec{q} + \sqrt{E_1^2 + m_1^2}\right)^2 \\ &= m_e^2 - m_1^2 + 2\,(q \cdot k_1) \end{aligned} \tag{C.2}$$

This leads to

$$\begin{aligned} (q \cdot k_1) &= \frac{1}{2}\left((q + k_1)^2 - m_e^2 + m_1^2\right) \\ &= \frac{1}{2}\left((p - k_2)^2 - m_e^2 + m_1^2\right) \\ &= \frac{1}{2}\left((m_\mu - E_2)^2 - \left(-\sqrt{E_2^2 + m_2^2}\right)^2 - m_e^2 + m_1^2\right) \\ &= \frac{1}{2}\left(\left(m_\mu^2 + m_1^2\right) - \left(m_e^2 + m_2^2\right) - 2m_\mu E_2\right) \\ &= m_\mu \left(\frac{\left(m_\mu^2 + m_1^2\right) - \left(m_e^2 + m_2^2\right)}{2m_\mu} - E_2\right) \\ &\equiv m_\mu\,(K_t - E_2) \end{aligned} \tag{C.3}$$

Appendix D

Phase space

The phase space for the muon decay process reads

$$dV = \frac{1}{(2\pi)^5}\; d^4q\; d^4k_1\; d^4k_2\; \times$$
$$\delta(k_1^2 + m_1^2)\; \delta(k_2^2 + m_2^2)\; \delta(q^2 - m_e^2)\; \delta^4(p - q - k_1 - k_2) \tag{D.1}$$

First of all, let's examine the following expression

$$\begin{aligned}
d^4k\; \delta(k^2 + m^2) &= dk^0 d|\vec{k}|\; |\vec{k}|^2 d\Omega\; \delta\left(\left(k^0 - \sqrt{\vec{k}^2 - m^2}\right)\left(k^0 + \sqrt{\vec{k}^2 - m^2}\right)\right) \\
&= dk^0 d|\vec{k}|\; |\vec{k}|^2 d\Omega\; \frac{1}{k^0 + \sqrt{\vec{k}^2 - m^2}}\; \delta\left(k^0 - \sqrt{\vec{k}^2} - m^2\right) \\
&= d|\vec{k}|\; |\vec{k}|^2\; d\Omega\; \frac{1}{2\sqrt{\vec{k}^2 - m^2}} \\
&= d|\vec{k}|\; |\vec{k}|^2\; d\Omega\; \frac{1}{2k^0}
\end{aligned} \tag{D.2}$$

The phase space can now be written as:

$$dV = \frac{1}{(2\pi)^5} d^4q\; \delta(q^2 - m_e^2) \times$$
$$\frac{|\vec{k_1}|^2\; d|\vec{k_1}|\; d\Omega_1}{2k_1^0}\; \frac{|\vec{k_2}|^2\; d|\vec{k_2}|\; d\Omega_1}{2k_2^0}\; \delta^4(p - q - k_1 - k_2) \tag{D.3}$$

Evaluating the Dirac delta functions

$$\begin{aligned}
&d^4q\; \delta(q^2 - m_e^2)\; \delta^4(p - q - k_1 - k_2) = \delta^4((p - k_1 - k_2)^2 - m_e^2) \\
&= \delta^4((m_\mu - k_1^0 - k_2^0)^2 - (-\vec{k_1} - \vec{k_2})^2 - m_e^2) \\
&= \delta^4(m_\mu^2 - 2m_\mu k_1^0 - 2m_\mu k_2^0 + (k_1^0)^2 + (k_2^0)^2 + 2k_1^0 k_2^0 - m_e^2 \\
&\quad - (\vec{k_1})^2 - (\vec{k_2})^2 - 2|\vec{k_1}|\; |\vec{k_2}|\; \cos\theta_{1,2})
\end{aligned} \tag{D.4}$$

Furthermore:

$$d\Omega_1 = d\cos\theta_1 d\phi_1 \tag{D.5}$$

$$d\Omega_2 = d\cos\theta_{1,2} d\phi_{1,2} \tag{D.6}$$

The integral over $d\cos\theta_1 d\phi_1 d\phi_{1,2}$ leads to a factor $8\pi^2$, so we are left with:

$$dV = \frac{8\pi^2}{(2\pi)^5} d\cos\theta_{1,2} \frac{|\vec{k_1}|^2|\vec{k_2}|^2 d|\vec{k_1}|\ d|\vec{k_2}|}{4k_1^0 k_2^0} \times \\ \delta^4(m_\mu^2 - 2m_\mu k_1^0 - 2m_\mu k_2^0 + k_1^2 + k_2^2 - m_e^2 + 2k_1^0 k_2^0 - 2|\vec{k_1}||\vec{k_2}|\cos\theta_{1,2}) \tag{D.7}$$

Taking the derivative at both sides of

$$(p^0)^2 = m^2 + |\vec{p}|^2 \tag{D.8}$$

we find

$$2p^0 dp^0 = 2|\vec{p}|d|\vec{p}| \tag{D.9}$$

leading to the fact that

$$\frac{|\vec{p}|d|\vec{p}|}{p^0} = dp^0 \tag{D.10}$$

Using this, we obtain

$$\begin{aligned} dV = & \frac{\pi^2}{(2\pi)^5} dE_1 dE_2 d\cos\theta_{1,2} \times \\ & \delta^4\left(\frac{m_\mu^2 - 2m_\mu k_1^0 - 2m_\mu k_2^0 + k_1^2 + k_2^2 - m_e^2 + 2k_1^0 k_2^0}{2|\vec{k_1}||\vec{k_2}|} - \cos\theta_{1,2}\right) \\ = & \frac{\pi^2}{(2\pi)^5} dE_1 dE_2 d\cos\theta_{1,2} \times \\ & \delta^4\left(\frac{-m_\mu^2 + k_1^2 + k_2^2 - m_e^2 + 2(m_\mu - k_1^0)(m_\mu - k_2^0)}{2|\vec{k_1}||\vec{k_2}|} - \cos\theta_{1,2}\right) \\ = & \frac{\pi^2}{(2\pi)^5} dE_1 dE_2 \end{aligned} \tag{D.11}$$

the result of Eq. (2.17)

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