Radboud universiteit Nijmegen
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Bachelor thesis

Finding a minimum in a not so flat universe

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In this thesis I attempt to give a description on how to treat a specific quantum field theory in a non flat background. We will discover different effects due to dynamical backgrounds, including black hole radiation, and the non uniqueness of the vacuum. At the end drawing an utter important conclusion and shortly reviewing physical consequences of it.

Contents

1 Introduction

One of today's major problems in theoretical physics is the description of a theory that describes both quantum mechanics (QM) and general relativity (GR). Many attempts have been made towards constructing such a theory. Main candidates at this moment are: loop quantum gravity (LQG) and string theory.
In this thesis we would like to take you through an attempt of trying to do quantum field theory in a non-trivial background. We will start off by finding a proper description for a quantum field that obeys the relativistic energy-momentum relation and finding the solutions of the equation of motion. After that we try to make a jump into a world that is not Minkowskian anymore by introducing a general metric tensor. Subsequently we can work out all equations of motion and can do interesting physics.
In the world of curved spacetimes we will very often encounter the term Bogolyubov transformation. This will be a transformation applied to the so called mode functions that describe the wave character of the fields. This transformation contains information about the particle interpretation of the theory. We can for example consider particles in one frame of reference, but as it turns out this does not have to be a particle in another frame. So we can pick our own preferred particle interpretation. This will play an important role in finding the vacuum state, which brings me to the main subject of this thesis. We are mostly interested in not only setting up the quantum field theoretical 'rules' but rather finding a good description of the vacuum state. Do we actually have a vacuum? And if we do, what does it look like? Do the energy expectation values of the flat and curved spacetime vacua agree with each other? All these kind of questions will be addressed and hopefully be answered during this thesis.

2 The real Klein-Gordon field

2.1 Derivation

We will start off by introducing the Klein-Gordon equation (KGE). Since at first we are looking for a relativistic[1] quantum mechanical description of free particles we have to require that at all equal time slices the relativistic energy-momentum relation is fulfilled.

$$E^2 = \vec{p}^2 c^2 + m^2 c^4 \tag{1}$$

If we follow the same procedure for quantisation as one does with the Schrödinger equation, i.e. using the operator substitutions $E \to i\hbar\frac{\partial}{\partial t}$ and $\vec{p} \to -i\hbar\vec{\nabla}$ and applying these to a wavefunction $\phi(x) \equiv \phi(t, \vec{r})$, we find:

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2 c^2}{\hbar^2})\phi(x) = 0 \tag{2}$$

Using a Minkowski metric signature of (+,-,-,-) we note that the combination of the temporal and spatial derivatives combine into the four dimensional scalar derivative $\partial_\mu \partial^\mu$, which is most often written as ∂^2. The equation then becomes:

$$(\partial^2 + m^2)\phi(x) = 0 \tag{3}$$

where we have used natural units $c = \hbar = 1$.

2.2 Solutions

Now we have found the KGE, we can try to construct the solutions. Throughout this thesis we will actually focus on the real solutions for the Klein-Gordon field. We started at the relativistic energy-momentum relation, but we still need to assure that the principle of relativity is built in. Namely, that this equation is the same for all observers. We therefore perform a Lorentz transformation on the equation. We observe that ∂^2 is scalar and therefore Lorentz invariant. The m^2 term in the equation is also scalar and Lorentz invariant. We conclude that $\phi(x)$ is scalar and thus corresponds to a zero spin eigenvalue.

Finding real scalar solutions to the differential equation is easy. We immediately recognise a wave equation, so plane wave solutions are a good ansatz.

$$\phi(x) = e^{-i(Et - \vec{p}\cdot\vec{r})} = e^{-ip\cdot x} \tag{4}$$

where we have used the inner product $p \cdot x = p^\mu x_\mu = Et - \vec{p} \cdot \vec{r}$ with $p^\mu = (E, \vec{p})$, $E = \sqrt{\vec{p}^2 + m^2}$ and $x^\mu = (t, \vec{r})$. By plugging this in we find that:

$$\partial^2 \phi(x) = (\vec{p}^2 - E^2)\phi(x) \tag{5}$$

[1]Read 'Lorentz invariant'

which of course is exactly $-m^2\phi(x)$.

This is just one type of solution, namely a *positive energy solution*. Since the energy-momentum relation is quadratic we can also have *negative energy solutions*. We get the same solution if we replace E by $-E$ and at the same time $\vec{p}$ with -$\vec{p}$. These two different solutions will be given the interpretation of belonging to particles and anti-particles. By summing over all Fourier modes we can now write down a full solution, namely

$$\phi(x) = \sum_{\vec{p}} A_p e^{-ip\cdot x} + B_{-p} e^{ip\cdot x} \tag{6}$$

It is however more convenient to work with wavevectors instead of momenta in QFT and since we are working in a continuous spacetime and have no discrete momenta:

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4}(A(k)e^{-ik\cdot x} + B(-k)e^{ik\cdot x}) \tag{7}$$

where $A(k)$ and $B(k)$ are Fourier coefficients to be determined. What we can however say, is that for a real Klein-Gordon field $\phi(x) \in \mathbb{R}$ we have that $B(-k) = A(k)^*$. Plugging this solution back into the KGE yields:

$$\int \frac{d^4k}{(2\pi)^4}(m^2 - k^2)(A(k)e^{-ik\cdot x} + A(k)^* e^{ik\cdot x}) = 0 \tag{8}$$

which in general can only be true if $k^2 = m^2$. Hence, we can rewrite the coefficients in a δ function form as: $A(k) = 2\pi\delta(k^2 - m^2)\Theta(k^0)\phi(k)$, such that we can write:

$$\int \frac{d^4k}{(2\pi)^3}\delta(k^2 - m^2)\Theta(k^0)(\phi(k)e^{-ik\cdot x} + \phi(k)^* e^{ik\cdot x}) = 0 \tag{9}$$

From the definition of the δ function we know that still only values of $k^2 = m^2$ will contribute. We can now calculate the temporal part of the integral, for which we find the following expression:

$$\delta(k^2 - m^2) = \delta(k^{02} - (\vec{k}^2 + m^2)) = \frac{1}{2\omega_{\vec{k}}}(\delta(k^0 + \omega_{\vec{k}}) + \delta(k^0 - \omega_{\vec{k}})) \tag{10}$$

where we have defined $\omega_{\vec{k}} = (\vec{k}^2 + m^2)^{1/2}$. If we plug this result back into our solution for the wavefunction and normalise to $2\omega_{\vec{k}}$, i.e. $\phi(k) = \sqrt{2\omega_{\vec{k}}}a_{\vec{k}}$. We finally find:

$$\phi(x) = \int \frac{d\vec{k}}{(2\pi)^3\sqrt{2\omega_{\vec{k}}}}(a_{\vec{k}}e^{-ik\cdot x} + a^*_{\vec{k}}e^{ik\cdot x}) \tag{11}$$

with k^0 fixed at the on-shell value $\omega_{\vec{k}}$.

In the quantum mechanical description of the Klein- Gordon field the Fourier coefficients $a^*_{\vec{k}}$ and $a_{\vec{k}}$ are interpreted as the usual creation and annihilation operators $\hat{a}^\dagger_{\vec{k}}$ and $\hat{a}_{\vec{k}}$ from the harmonic oscillator. Now that $\hat{\phi}(x)$ is an operator field that creates or annihilates particles with four-momenta k with a plane wave character. These operators satisfy the commutation relations:

$$[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}] = 0$$

$$[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] = (2\pi)^3\delta(\vec{k} - \vec{k}')\hat{I} \tag{12}$$

with $\hat{I}$ the unit operator. In the next section we will show how to interpret this result more physically and what can be done with it.

2.3 Lagrangian and Hamiltonian density

Having this solution is great, but if we want to calculate observables of the system we will need some more information. The first thing we want to know is the Hamiltonian describing this system, in terms of a general $\phi(x)$ and in terms of our solution. Recall the KGE $(\partial^2 + m^2)\phi(x) = 0$. In the Lagrangian formalism this should coincide with the Euler-Lagrange equation (ELE):

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{13}$$

where $\mathcal{L}$ denotes the Lagrangian density.[2] With this being known, we can reconstruct the Lagrangian density from the KGE. First of all we need both the derivative and the m^2 term acting on a term ϕ. Therefore, in $\mathcal{L}$ they have to come up as a quadratic term in ϕ. We can also say that $\mathcal{L}$ should contain a kinetic- and potential term, the free potential term being the rest-mass selfcoupling term proportional to $m^2\phi^2$, whereas the kinetic term should be something with derivatives of ϕ. As we can see in the KGE there is a second derivative, so if we use first derivatives of ϕ in $\mathcal{L}$ this will be fulfilled. The predicted lagrangian density becomes:

$$\mathcal{L} \sim \partial_\mu\phi\partial^\mu\phi - m^2\phi^2 \tag{14}$$

As we can see this gives extra factors of 2 when taking derivatives. Fixing this yields the correct Lagrangian density for the free real scalar field:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{15}$$

[2] In QFT one rather works with Lagrangian densities than strict Lagrangians or even Hamiltonians. This is because a Lagrangian density treats space and time on equal footing, and is therefore a Lorentz scalar.

From here it is easy to find the Hamiltonian density, using the Legendre transform:

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} \tag{16}$$

where π is called the conjugate momentum corresponding to the field ϕ and defined by the following expression

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} \tag{17}$$

In terms of the Fourier decomposition, $\pi(x)$ reads:

$$\pi(x) = \dot{\phi}(x) = \int \frac{d\vec{k}}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\vec{k}}}{2}}(a_{\vec{k}}e^{-ik\cdot x} - a^*_{\vec{k}}e^{ik\cdot x}) \tag{18}$$

If we calculate the Hamiltonian densities by rewriting the Lagrangian density in terms of temporal and spatial derivatives we find easily:

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \tag{19}$$

Since this is the Hamiltonian density, we just integrate over all of space to get the real Hamiltonian H:

$$H = \int d\vec{x}(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2) \tag{20}$$

We may now use this result to calculate[3] the Hamiltonian operator $\hat{H}$ for our quantized solution of the KGE.

2.4 Constructing the Hamiltonian for the free real scalar field

Absorbing the time dependence in our Fourier coefficients, i.e. $\hat{a}_{\vec{k}}(t) = \hat{a}_{\vec{k}}e^{-i\omega_{\vec{k}}t}$ we can write:

$$\hat{\phi}(x) = \int \frac{d\vec{k}e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^3\sqrt{2\omega_{\vec{k}}}}(\hat{a}_{\vec{k}}(t) + \hat{a}^\dagger_{-\vec{k}}(t)) \tag{21}$$

$$\hat{\pi}(x) = \int \frac{d\vec{k}(-i\omega_{\vec{k}})e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^3\sqrt{2\omega_{\vec{k}}}}(\hat{a}_{\vec{k}}(t) - \hat{a}^\dagger_{-\vec{k}}(t)) \tag{22}$$

$$\nabla\hat{\phi}(x) = \int \frac{d\vec{k}(i\vec{k})e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^3\sqrt{2\omega_{\vec{k}}}}(\hat{a}_{\vec{k}}(t) + \hat{a}^\dagger_{-\vec{k}}(t)) \tag{23}$$

For computational convenience we have taken $\vec{k} \to -\vec{k}$ in the negative energy parts of the expressions. Writing out all parts and taking equal times we find the Hamiltonian to be:

$$\hat{H} = \int d\vec{x} \int \frac{d\vec{k}d\vec{k}'e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}}{(2\pi)^6 4\sqrt{\omega_{\vec{k}}\omega_{\vec{k}'}}}\big[- \omega_{\vec{k}}\omega_{\vec{k}'}(\hat{a}_{\vec{k}}(t) - \hat{a}^\dagger_{-\vec{k}}(t))(\hat{a}_{\vec{k}'}(t) - \hat{a}^\dagger_{-\vec{k}'}(t))$$

[3]By replacing all classical quantities by their respective operator, i.e. the second quantization process

$$+(m^2 - \vec{k}\cdot\vec{k'})(\hat{a}_{\vec{k}}(t) + \hat{a}^\dagger_{-\vec{k}}(t))(\hat{a}_{\vec{k'}}(t) + \hat{a}^\dagger_{-\vec{k'}}(t))] \tag{24}$$

Performing the $\vec{x}$ integral gives $(2\pi)^3\delta(\vec{k}+\vec{k'})$, resulting in a sum of oscillators

$$\hat{H} = \int \frac{d\vec{k}}{(2\pi)^3}\frac{1}{2}\omega_{\vec{k}}(\hat{a}_{\vec{k}}\hat{a}^\dagger_{\vec{k}} + \hat{a}^\dagger_{-\vec{k}}\hat{a}_{-\vec{k}}) \tag{25}$$

which is time-independent because of the combinations of $\hat{a}_{\vec{k}}(t)$ and $\hat{a}^\dagger_{\vec{k}}(t)$. Just like expected from the time-translation invariance of the Klein-Gordon theory. Using the bosonic commutator $[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k'}}] = (2\pi)^3\delta(\vec{k}-\vec{k'})\hat{I}$ we find that $\hat{H}$ nicely decouples into two terms, one so called counting part and a ground state term that is independent of time:

$$\hat{H} = \int \frac{d\vec{k}}{(2\pi)^3}\omega_{\vec{k}}\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}} + \int d\vec{k}\frac{1}{2}\omega_{\vec{k}}\delta(\vec{0})\hat{I} \tag{26}$$

We arive at the final answer for the Hamiltonian, defining the counting operator $\hat{N}_{\vec{k}} = \hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}}$ we have easily:

$$\hat{H} = \int \frac{d\vec{k}}{(2\pi)^3}\omega_{\vec{k}}(\hat{N}_{\vec{k}} + \frac{1}{2}(2\pi)^3\delta(\vec{0})\hat{I}) \tag{27}$$

Please note the $\delta(\vec{0})$ in the second term that represents the ground state energy. This means that the groundstate energy is infinite. It is due to the fact that in all of spacetime $\hat{H}$ receives a zero-point contribution from all oscillator modes, which all have a finite energy but collectively result in an infinite ground state energy.[4] From this we can easily read off the vacuum expectation value:

$$\langle 0|\hat{H}|0\rangle = \int d\vec{k}\frac{1}{2}\omega_{\vec{k}}\delta(\vec{0}) \tag{28}$$

with $|0\rangle$ being the state of zero particles.[5] So no particles have been created by the field yet. This is thus the energy of the vacuum, in the absence of any oscillators. The rest of this thesis will focus on a way to find this expectation value in a non-Minkowskian metric space.

[4]This kind of infinity is called an infra-red divergence corresponding to large distances or low energies. Also note that the integral over $\omega_{\vec{k}}$ will diverge, this is called an ultra-violet divergence corresponding to short distances or high energies. The ground state energy density is UV-divergent only, as a result of dividing out the infinite spatial volume from the total energy.

[5]Such that for *all* $\hat{a}_{\vec{k}}$, $\hat{a}_{\vec{k}}|0\rangle = 0$

3 Approaching non flat metric spaces using simple substitutions

3.1 substitution approach

To set up the same Klein-Gordon theory in a non Minkowskian metric space we will start with the same set up. But this time using all ingredients that are used to describe curved spacetimes as in GR. Three examples of those substitutions are:

1. Replacing all Minkowski metrics with general metrics: $\eta_{\mu\nu} \to g_{\mu\nu}$
2. Using covariant derivatives instead of the usual partial derivative as these follow the general manifolds via the connection[6]. So: $\partial_\mu \to \nabla_\mu$
3. Using a covariant volume element, which is thus observer or system independent, rather than just a Lorentz-boost invariant volume element. In other 'words': $d^4x \to d^4x\sqrt{-g}$, where g is the determinant of the metric tensor.

These ingredients will assure the manifold we are working on to be locally Minkowskian. Let's start by writing down a Lagrangian density that incorporates these. Our old Lagrangian density had a kinetic term of the form $\partial^\mu\phi\partial_\mu\phi$. We can directly use point 2 to transform this into: $\nabla^\mu\phi\nabla_\mu\phi$. Since the Lagrangian actually originates from varying the action functional $S = \int \mathcal{L}d^4x$ we can see that also the covariant volume element will find its way into the Lagrangian density. Also writing out the summation convention in a general metric, we find again for a free real scalar field:

$$\mathcal{L} = \sqrt{-g}(\frac{1}{2}g_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi - \frac{1}{2}m^2\phi^2) \tag{29}$$

Using the ELE we derive the modified KGE very naturally as:

$$(\nabla_\mu\nabla^\mu + m^2)\phi(x) = (g_{\mu\nu}\nabla^\mu\nabla^\nu + m^2)\phi(x) = 0 \tag{30}$$

This equation might be written in an even more particular form by noting that the covariant derivative only acts on vector or higher order fields. But our theory describes a real scalar field, therefore the first derivative will simply be a partial derivative. The second one however will need a connection since the derivative or 'gradient' will differ for different geometries. So in this theory $\nabla_\mu\nabla^\mu\phi(x) = \nabla_\mu\partial^\mu\phi(x)$. To not write out the complicated Christoffel symbols contained in the covariant derivatives we will use the identity:

$$\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g} = \Gamma^\lambda_{\lambda\mu} \tag{31}$$

[6]The covariant derivative is given by: $\nabla_\mu v^\rho = \partial_\mu v^\rho + \Gamma^\rho_{\mu\sigma}v^\sigma$ for a contravariant vector field v^σ, where $\Gamma^\rho_{\mu\sigma}$ denotes the Christoffel symbols. Higher order terms may be needed for other derivatives (on tensors etc.)

Using this we can write down an equation that is completely determined by the choice of metric:

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) + m^2\phi = 0 \tag{32}$$

Equations (30) and (32) are completely general and equal according to (31).

3.2 Coupling approach

In the previous subsection we have shown the modified KGE in the case of a general metric. We now have to consider if this is all that is needed for having a single particle in an empty space. We could leave this for later discussion but it makes more sense to discuss it right now. The curvature can make an appearance in two possible ways here:
1) through the metric.
2) through a mass-like selfcoupling term.

The metric appearance is built in. This however only takes care of the background metric dependence of the theory. There are no 'interactions' built in. The particle can propagate through spacetime and keep interacting with itself by coupling to the scalar curvature of space. We can therefore add another potential term to the Lagrangian density that is proportional to $\mathcal{R}\phi^2$. This leaves us with the Lagrangian density:

$$\mathcal{L} = \sqrt{-g}(\frac{1}{2}g_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{2}\alpha\mathcal{R}\phi^2) \tag{33}$$

with $\alpha \geq 0$ being a real coupling parameter[7], which is zero for minimal coupling. Working out the ELE gives us the modified KGE to work with:

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) + (m^2 + \alpha\mathcal{R})\phi = 0 \tag{34}$$

4 The Friedman-Lemaitre-Robertson-Walker metric

Now that we have found our equation of motion for the particle moving in the curved background, we can make our choice of metric. We could for example consider a particle near a black hole using a Schwartzschild metric. This however turned out to be extremely complicated and non trivial using this approach. Using this approach would get us nowhere since a Schwartzschild metric is *defined* to be Ricci flat. The metric that can however be solved analytically turns out to be the conformally flat Friedman-Lemaitre-Robertson-Walker, or FLRW metric. The FLRW metric basically describes a spatially scaled Minkowski metric. This makes it conformally flat and therefore suitable for a realistic approach, since the universe is measured to be fairly flat at large scales. At the quantum scale nobody really knows what spacetime does, but we will stick to this reasoning. The FLRW metric takes the form:

$$ds^2 = dt^2 - a(t)^2 d\vec{x}^2 \tag{35}$$

[7]It has no mass dimension because the Ricci scalar has mass dimension 2 from all metric derivatives.

This equation describes an *expanding* universe. Therefore $a(t)$ is called the *scaling factor*. Since the universe is expanding, it is important to handle matters like distance a bit different. The (real) distance between objects might change due to the expansion. To cancel this effect we can define a 'comoving time' η, usually called the *conformal time*. The conformal time is defined as $\eta = \int_0^t \frac{dt'}{a(t')}$. Then we can write:

$$ds^2 = a(\eta)^2(d\eta^2 - d\vec{x}^2) \tag{36}$$

One can see that this implies that using the conformal time we have that $g_{\mu\nu}(\eta, \vec{x}) = a^2(\eta)\eta_{\mu\nu}$ with $\eta_{\mu\nu}$ the usual Minkowski metric. So at fixed η the FLRW universe is fully flat. In the next part we will continue using $a(t) = a$ instead because it is easier while dealing with the metric. The spatial element $d\vec{x}$ is to be taken in spherical coordinates with a reduced radius. This means $d\vec{x}^2$ is given as:

$$d\vec{x}^2 = \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \tag{37}$$

where $d\Omega^2 = d\theta^2 + sin^2(\theta)d\phi^2$ and k being a parameter that indicates the curvature of the background with $k \in \{-1, 0, 1\}$ describing an open, flat and closed universe respectively. This causes $g_{\mu\nu}$ has to be diagonal. Its elements can be read off by using $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Taking $x^\mu = (t, r, \theta, \phi)$ we find:

$$g_{tt} = 1$$

$$g_{rr} = -a^2/(1 - kr^2)$$

$$g_{\theta\theta} = -a^2 r^2$$

$$g_{\varphi\varphi} = -a^2 r^2 sin^2(\theta) \tag{38}$$

The inverse metric $g^{\mu\nu}$ has its components inverted. Using these we can find all Christoffel symbols for the FLRW metric and the Ricci tensor $R_{\mu\nu}$. The Ricci scalar $\mathcal{R}$ is then found by contracting the indices as $\mathcal{R} = R^\mu_\mu = g^{\mu\nu}R_{\mu\nu}$.The Ricci scalar for the FLRW metric is given by $\mathcal{R} = \frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + k)$, with $\dot{a} = \frac{da}{dt}$ and $\ddot{a} = \frac{d^2a}{dt^2}$. Calculating the square root of the determinant gives:

$$\sqrt{-g} = \left[-1 \cdot (\frac{-a^2}{f(r)^2}) \cdot (-a^2r^2) \cdot (-a^2r^2sin^2(\theta))\right]^{1/2} = \frac{a^3r^2sin(\theta)}{f(r)} \tag{39}$$

where $f(r) = \sqrt{1 - kr^2}$.

4.1 Solving the KGE in the FLRW metric with spherical coordinates in the flat static case

We will now solve the KGE in FLRW space. The first step is to simply substitute our metric into our modified KGE, resulting in:

$$\frac{3\dot{a}}{a}\frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial t^2} - \frac{1}{a^2}\nabla^2_{FLRW}\phi + (m^2 + \alpha\mathcal{R})\phi = 0 \tag{40}$$

with the Laplacian given by:

$$\nabla^2_{FLRW} \equiv \frac{f(r)}{r^2}\frac{\partial}{\partial r}\big(f(r)r^2\frac{\partial}{\partial r}\big) + \frac{1}{r^2 sin(\theta)}\frac{\partial}{\partial \theta}\big(sin(\theta)\frac{\partial}{\partial \theta}\big) + \frac{1}{r^2 sin^2(\theta)}\big(\frac{\partial^2}{\partial \varphi^2}\big) \tag{41}$$

In the static case that we want to study in this section, for which $\dot{a} = 0$, the first term of (40) drops out. Note that time and space nicely decouple here so that for a solution we may use seperation of variables:

$$\Phi(t, r, \theta, \varphi) = R(r)A(\theta, \varphi)e^{-i\omega t} \tag{42}$$

Observe that the angular part of the FLRW Laplacian is equal to the one in spherical coordinates for flat spacetime. It is therefore reasonable to make the assumption that the angular solutions are spherical harmonics $Y_l^m(\theta, \varphi)$. The radial part is different and depends on our choice of 'universe', i.e. the value of k. Let us first solve it in a completely flat (zero curvature) universe, i.e. $k = 0$. We expect to find relativistic particles that will travel in straight lines rather than a trajectory that is spacetime dependent. We first define a new variable that is analogous to r but will become a more general 'angle' of the system, namely

$$d\chi = \frac{dr}{\sqrt{1 - kr^2}} = \frac{dr}{f(r)} \tag{43}$$

This has different solutions for different values of k. For $k = 0$ we have that $f(r) = 1$ and thus $r = \chi$.

Writing out the radial part of the modified KGE and inserting $\frac{\nabla^2_{FLRW}A(\theta,\phi)}{A(\theta,\phi)} \equiv \frac{-l(l+1)}{r^2}$ we find:

$$\big[\frac{f(r)}{r^2}\frac{\partial}{\partial r}\big(f(r)r^2\frac{\partial}{\partial r}\big) - \frac{l(l+1)}{r^2} + a^2(\vec{k}^2 - \alpha\mathcal{R})\big]R(r) = 0 \tag{44}$$

where $\vec{k}^2 = \omega^2 - m^2$. The ω comes from the time derivative. Note that we have kept the $\mathcal{R}$ parameter here, which actually vanishes since we consider $k = 0$ and $\dot{a} = 0$. We can rewrite this expression and plug in all we know, like $f(r) = 1$, and substitute $\beta^2 = a^2(\vec{k}^2 - \alpha\mathcal{R})$. We then find:

$$\big[\chi^2\frac{\partial^2}{\partial \chi^2} + 2\chi\frac{\partial}{\partial \chi} + \chi^2\beta^2 - l(l+1)\big]R(\chi) = 0 \tag{45}$$

We recognize this equation as a *Bessel equation*. This has fairly simple solutions:

$$R(\chi) = c_1 J_l(\beta\chi) + c_2 Y_l(\beta\chi) \tag{46}$$

where $J_l(\beta\chi)$ and $Y_l(\beta\chi)$ are Bessel functions of the first and second kind respectively. The second kind is not normalisable and therefore not physical. We therefore throw away the second kind solutions and write down the full solution to the scalar field in the FLRW metric as:

$$\Phi(x) = Constant \cdot J_l(a|\vec{k}|r)Y_l^m(\theta, \varphi)e^{-i\omega t} \tag{47}$$

We can nicely see how inside the radial solution the FRLW metric comes up as being a scaling of space.

4.2 Solving the KGE in the FLRW metric with spherical coordinates in the flat dynamical case

As we have seen in the previous section, for the static case where $\dot{a} = 0$, we found a nice solution to the modified KGE. Let us now consider the case that $\dot{a} \neq 0$. In this case we get from equation (40) the extra term $\frac{3\dot{a}}{a}\frac{\partial\phi}{\partial t}$. Inspired by the current epoch of our universe, also called the dark-energy dominated era, we define the ratio $\frac{\dot{a}}{a} \equiv H$ as the *Hubble parameter*. This means that in the dark-energy dominated era $a(t) \sim e^{Ht}$. Our new modified KGE then reads:

$$3H\frac{\partial\phi}{\partial t} + \frac{\partial^2\phi}{\partial t^2} - \frac{1}{a^2}\nabla^2_{FLRW}\phi + (m^2 + \alpha\mathcal{R})\phi = 0 \tag{48}$$

Let us continue by using separation of variables: $\phi(x) = \psi(t)\xi(\vec{r})$. If we leave the term $(m^2 + \alpha\mathcal{R})\phi$:

$$\frac{3H}{\psi(t)}\dot{\psi}(t) + \frac{\ddot{\psi}(t)}{\psi(t)} + m^2 + \alpha\mathcal{R} = -\frac{\lambda^2}{a(t)^2} \tag{49}$$

with $-\lambda^2 \equiv \dfrac{\nabla^2_{FLRW}\xi(\vec{r})}{\xi(\vec{r})}$.[8] Multiplying both sides by $\psi(t)$ and taking λ^2 to the other side we find:

$$\ddot{\psi}(t) + 3H\dot{\psi}(t) + (m^2 + \alpha\mathcal{R} + \frac{\lambda^2}{a^2(t)})\psi(t) = 0 \tag{50}$$

Next we plug in $a(t) = e^{Ht}$ and for the sake of simplicity we calculate the Ricci scalar $\mathcal{R} = \frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + k) = \frac{6}{a^2}(a^2H^2 + a^2H^2 + 0) = 12H^2$:

$$\ddot{\psi}(t) + 3H\dot{\psi}(t) + (m^2 + 12\alpha H^2 + \lambda^2 e^{-2Ht})\psi(t) = 0 \tag{51}$$

We see that this temporal part has a time dependent frequency

$$\omega_\lambda^2(t) = m^2 + \alpha\mathcal{R} + a^{-2}\lambda^2 \tag{52}$$

Note that λ takes the role of our usual wave number $|\vec{k}|$. To find solutions we will use the Runge-Kutta method used by Wolframalpha[5]. Since everything is constant except for the e^{-2Ht}, we can write the equation in the following form:

$$\ddot{\psi}(\tau) + 3\dot{\psi}(\tau) + (\frac{m^2}{H^2} + 12\alpha + \frac{\lambda^2}{H^2}e^{-2\tau})\psi(\tau) = 0 \tag{53}$$

[8] $-\lambda^2$ is chosen because it makes the equation recognizable.

with $\tau = Ht$. For the special case $\frac{m^2}{H^2} + 12\alpha = \frac{\lambda^2}{H^2} = 1$, this equation has the following numerical solution:

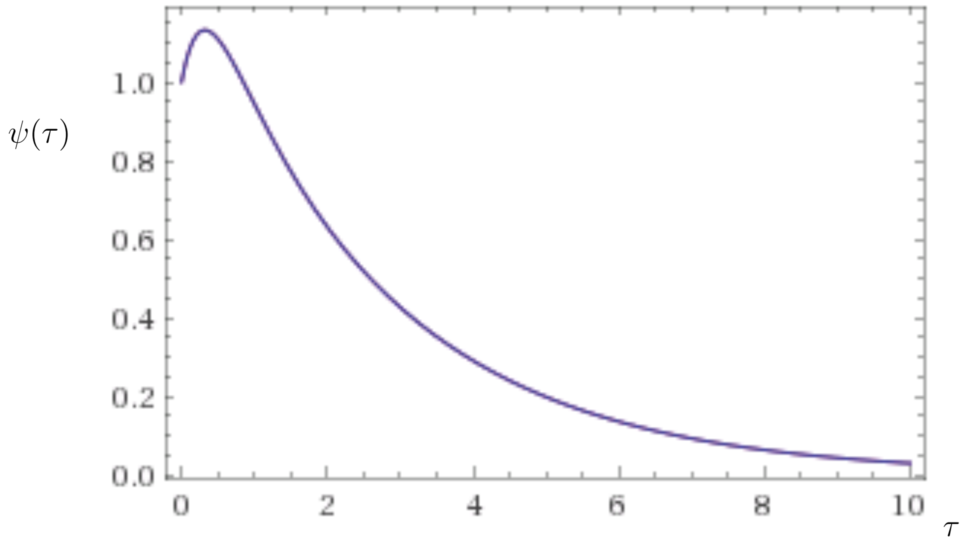

Figure 1: This graph shows the solution $\psi(\tau)$. We have taken $\psi(\tau = 0) = 1$ as an inital condition

Observe that this graph represents the behaviour of a damped harmonic oscillator with a damping factor of $3H$ and a time dependent frequency ω_λ. We conclude that due to the expansion of the universe and constant flat curvature, the field value $|\psi(t)|$ is not constant in time anymore. We will adress this effect later on again.
Finally, solving the spatial function $\xi(\vec{r})$ will not differ from section 4.1. In the next section we will see what happens if we take a curvature into account, but don't expand the universe.

4.3 Solving the KGE in the FLRW metric with spherical coordinates in the closed static case

Let us now consider the exact same equation (40) but this time using $k = 1$. This results in the fact that $d\chi = \frac{dr}{\sqrt{1-r^2}}$, with solution: $r = sin(\chi)$. Plugging this in yields the new radial equation:

$$\Big[\frac{1}{sin^2(\chi)}\frac{\partial}{\partial\chi}\big(sin^2(\chi)\frac{\partial}{\partial\chi}\big) - \frac{l(l+1)}{sin^2(\chi)} + \beta^2\Big]R(\chi) = 0 \tag{54}$$

with β defined as earlier. Solving this equation is rather non-trivial and has been done in [1]. Throughout the derivation in [1] a few definitions were made that facilitated the quantization of this equation. First of all, in the derivation we have the requirement that $\beta^2 = n_B^2 - 1$, where $n_B = 1, 2, 3\ldots$ is the principal quantum number for the bosonic case. This can be linked to a quantum number called the *radial quantum number*

$n_r = 0, 1, 2, 3\ldots$ through $n_B = l + 1 + n_r$. We can directly derive the energy dispersion from this:

$$\omega = \sqrt{\frac{n_B^2 - 1}{a^2} + \alpha\mathcal{R} + m^2} \tag{55}$$

For the closed static case with $\dot{a} = 0$ and $k = 1$ we have that $\mathcal{R} = \frac{6}{a^2}$. Hence,

$$\omega = \sqrt{\frac{(n_B^2 - 1) + 6\alpha}{a^2} + m^2} \tag{56}$$

For the groundstate with the lowest ω we take $n_B = 1$ and we find the energy to be:

$$\omega = \sqrt{\frac{6\alpha}{a^2} + m^2} \tag{57}$$

So we observe that our energy spectrum gets shifted by the scaling factor a. This means that our vacuum will have more energy. This energy is due to the curved background, and will come back later if we try to interpret this vacuum energy as a quasi particle density.
Returning to equation (54), the solutions are given by:

$$R(\chi) = Constant \cdot sin^l(\chi) C_{n_r}^{l+1}(cos(\chi)) \tag{58}$$

where the $C_{n_r}^{l+1}(cos(\chi))$ are Gegenbauer polynomials of degree n_r. These polynomials represent generalised Legendre polynomials and are normalisable. Since the spherical harmonics are normalised it is again enough to normalise the radial solution. From [1] we have the normalisation constant:

$$Constant = \pi^{l+\frac{1}{2}}\Gamma(l+1)\sqrt{\frac{n_B(n_B - l - 1)!}{2\omega\Gamma(n_B + l + 1)}} \tag{59}$$

In terms of the Gamma-function Γ. The full solution to the KGE equation in FLRW space becomes:

$$\Phi(x) = \pi^{l+\frac{1}{2}}\Gamma(l+1)\sqrt{\frac{n_B(n_B - l - 1)!}{2\omega\Gamma(n_B + l + 1)}} sin^l(\chi) C_{n_r}^{l+1}(cos(\chi)) Y_l^m(\theta, \varphi) e^{-i\omega t} \tag{60}$$

We could now create a full solution (superposition) and do the same trick as before trying to find the Hamiltonian of this KGE. We will have a new definition of our creation and annihilation operators suitable for this solution. This however is one big mess and there is a much nicer way to find this. Due to the curvature our interpretation of *particles* will be a bit more complicated, or change. The flat vacuum was defined by zero particles, but since we added a curved background this is no longer the case. It is now possible to create 'particles' purely because the background is different. In the next section I will discuss a different approach to the same problem, without analytical solutions, but with important insights into the quantisation of curved background fields.

5 A quasi-particle interpretation

Let us now consider the particle quantisation aspects of a free real scalar field on the FLRW metric. At first let's consider the gravitationally coupled free real scalar field defined on Minkowski space. We were able to write down the solutions as a sum of two independent subsets of solutions. For some solution $f_{\vec{k}}(x)$ to the flat KGE with wave vector $\vec{k}$ we could write down that the full KG field operator looks like:

$$\hat{\phi}(x) = \sum_{\vec{k}} (\hat{a}_{\vec{k}} f_{\vec{k}}(x) + \hat{a}^\dagger_{\vec{k}} f^*_{\vec{k}}(x)) \tag{61}$$

These functions $f_{\vec{k}}(x)$ and $f^*_{\vec{k}}(x)$ were picked as positive and negative energy solutions to form a complete basis, and $\hat{a}_{\vec{k}}$ and $\hat{a}^\dagger_{\vec{k}}$ are annihilation and creation operators of a mode with wave vector $\vec{k}$.

5.1 Fock space dependence

These solutions are picked quite naturally because we used them as an ansatz to the KGE. This was possible because of the translational invariance of space, being isotropic, that the Hamiltonian commutes with the momentum operator. Therefore we had momentum eigenfunctions as energy eigenfunctions. Minkowski space exhibits Poincaré symmetry, generated by the Poincaré group. Meaning:

1. Translation invariance: because Minkowksi space is a flat spacetime all physics described within is invariant under the transformation $x_\mu \to x_\mu + a_\mu$ for some constant four-vector a_μ

2. Rotation invariance: similarly, spatial rotations of spacetime should not change the physical outcome. All equations have to be *form invariant* under rotations.

3. Boost invariance: these are generalized hyperbolic rotations specific for special relativity.

These three invariances form the heart of the principle of relativity, which states that all of physics should be the same for all intertial observers. If we now continue on a curved spacetime geometry it is easy to conclude that Poincaré symmetry is broken. We could for example consider a free particle moving through spacetime. The path this particle will follow is described by the *geodesic equation*:

$$\frac{d^2\gamma^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{d\gamma^\mu}{d\tau}\frac{d\gamma^\nu}{d\tau} = 0 \tag{62}$$

with τ a parameter along the geodesic curve γ. Since curvature may vary along the geodesic it is possible that a momentum eigenfunction will no longer be an eigenfunction of the Hamiltonian. So in this case $[H, \hat{\vec{p}}] \neq 0$, because translational invariance is broken. This sums up to the fact that solutions of the flat KGE don't have to be solutions to the curved KGE. Even worse, it is now not possible to pick a fixed basis of functions since these might vary with time. If we can't pick a desired basis we need to pick a general one. A general form of a basis is generated by performing a Bogolyubov transformation in terms of the old basis. If we can however not define a basis, this implies that we are unable to define a state of 'no particles'. Our interpretation has to change such that we *are* able to define a so called 'quasi vacuum' state which is then characterized by the absence of so called *quasi particles*[9]. A direct consequence of this is that the vacuum is dependent on the choice of basis. In one basis there might be particles that are not measurable in the other basis, all due to a different background. We will now show what this means quantitatively.

5.2 Particle creation in curved spacetimes.

Let us now take a look at the same modified KGE as earlier, but this time not to solve it explicitly but rather discuss the implications that this system describes. We first realize that the scaling factor was actually time dependent. This means that $a(t)$ can have different values for states at different times.

A field $\hat{\phi}(x)$ can always be written in terms of a set of basis functions and corresponding creation/annihilation operators as done in (61). It is easy to show that if we impose the following relations,

$$\hat{a}_{\vec{k}} \rightarrow u^*_{\vec{k}}\hat{b}_{\vec{k}} - v_{\vec{k}}\hat{b}^\dagger_{-\vec{k}}$$

$$\hat{a}^\dagger_{\vec{k}} \rightarrow u_{\vec{k}}\hat{b}^\dagger_{\vec{k}} - v^*_{\vec{k}}\hat{b}_{-\vec{k}} \tag{63}$$

and use that every mode function $f_{\vec{k}}$ can be written as:

$$f_{\vec{k}} \rightarrow u_{\vec{k}}\psi_{\vec{k}} + v^*_{\vec{k}}\psi^*_{-\vec{k}} \tag{64}$$

the field $\hat{\phi}(x)$ is invariant if we require that $|u_{\vec{k}}|^2 - |v_{\vec{k}}|^2 = 1$ and $u_{-\vec{k}} = u_{\vec{k}}$, $v_{-\vec{k}} = v_{\vec{k}}$[10] This type of transformation is called a *Bogolyubov* transformation. Being able to write the old operators in terms of two new ones, also means that we change our 'particle interpretation'. We are now not creating or annihilating the old particles, but rather quasi particles that will match the theory[11] We have written our field $\hat{\phi}(x)$ in terms of new creation and annihilation operators $\hat{b}$ and $\hat{b}^\dagger$ and a new basis $\psi_{\vec{k}}$. The requirement $|u_{\vec{k}}|^2 - |v_{\vec{k}}|^2 = 1$ and $u_{-\vec{k}} = u_{\vec{k}}$, $v_{-\vec{k}} = v_{\vec{k}}$ is analogous with saying that $\hat{b}$ and $\hat{b}^\dagger$ also

[9] A quasi particle is interpreted as a particle interpretation that is observer, or basis, dependent. In one basis there might be particles to be observed, while in the other there are not. Those particles are then called quasi particles.

[10] Which are requirements for a valid Bogolyubov transformation and the last two also make sense for a isotropic universe.

[11] In fermionic theory this could for example be a 'hole', which is a quasi particle related to an unoccupied fermion state.

satisfy the bosonic commutation relations like $\hat{a}$ and $\hat{a}^\dagger$. The field is then expressed as:

$$\hat{\phi}(x) = \sum_{\vec{k}} (\hat{b}_{\vec{k}} \psi_{\vec{k}}(x) + \hat{b}^\dagger_{-\vec{k}} \psi^*_{-\vec{k}}(x)) \tag{65}$$

So let us consider a universe where a long time ago (asymptotic limit backwards in time) the field was described by $\hat{a}$ and $\hat{a}^\dagger$ and the scaling factor was given by a_{in}. In the far future the field will be described by $\hat{b}$ and $\hat{b}^\dagger$ and the scaling factor a_{out}. In the present we have our well known FLRW universe. As we are not considering any timescale at the moment we can distinguish two different scenarios of switching between a_{in} and a_{out}. If we transform our two vacua into each other instantaneously, there is no 'evolving time' for the old vacuum to slowly adjust. Later we will discuss a more 'adiabatic' kind of transformation. We can then write down two steady vacuum states $|0_a\rangle$ and $|0_b\rangle$.

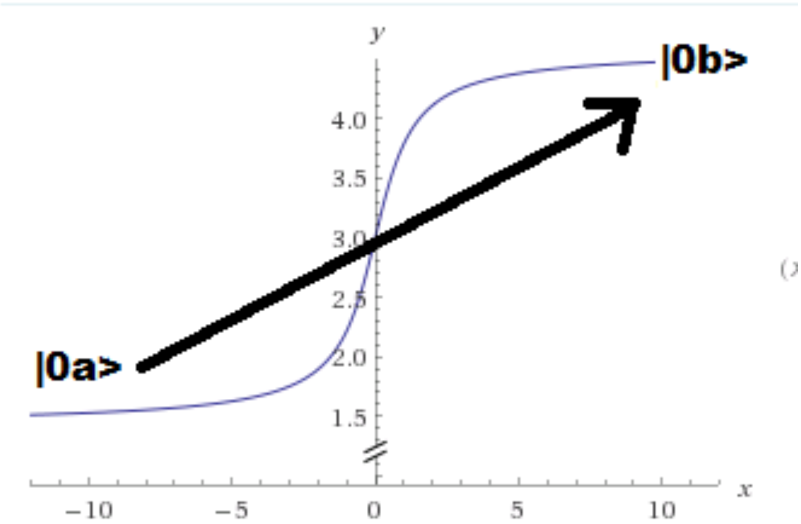

Figure 2: This figure representents a schematic view on how we can transform between two different interpretations. On the horizontal axis we have the time and on the vertical axis we have the evolution of the scaling factor with corresponding vacuum. In the past we have a vacuum $|0_a\rangle$ and in the future we have a vacuum state $|0_b\rangle$

In Fig. 2 we have a schematic representation of this. Each of the vacuum states will be annihilated by the corresponding annihilation operator. Either way, to go from the old vacuum to the new vacuum something has to happen in the present. Suppose that our current state is $|0_a\rangle$; now if we would like to count the amount of particles in this state we would of course find 0. In the future, however, the new quasi particles are counted by $\hat{N}^{(b)}_{\vec{k}} = \hat{b}^\dagger_{\vec{k}} \hat{b}_{\vec{k}}$. This means that from the perspective of a future observer with vacuum $|0_b\rangle$ the old vacuum $|0_a\rangle$ contains $\langle 0_a|\hat{b}^\dagger_{\vec{k}} \hat{b}_{\vec{k}}|0_a\rangle$ quasi particles in the $\vec{k}$th mode. In general this does not have to be zero. Consider a function $\hat{\mathcal{G}}$ that smoothly transforms between the two vacua:

$$|0_b\rangle = \hat{\mathcal{G}}|0_a\rangle \tag{66}$$

Since we want to find a function that creates a new vacuum, it would be useless if it depends on the annihilation operator $\hat{a}$. Focussing on a single $\vec{k}$ and $-\vec{k}$ combination we only want to create particles with momenta $\vec{k}$ and $-\vec{k}$ i.e. $\hat{\mathcal{G}} = \hat{\mathcal{G}}(\hat{a}^\dagger_{\vec{k}}, \hat{a}^\dagger_{-\vec{k}})$. So if we apply $\hat{b}_{\pm\vec{k}}$ on both sides of (66) we get an equation with 0 on the left-hand side. If we plug in the Bogolyubov transformation for the $\hat{b}_{\pm k}$ we find:

$$(u_{\vec{k}}\hat{a}_{\pm\vec{k}} + v_{\vec{k}}\hat{a}^\dagger_{\mp\vec{k}})\hat{\mathcal{G}}(\hat{a}^\dagger_{\vec{k}}, \hat{a}^\dagger_{-\vec{k}})|0_a\rangle = 0 \tag{67}$$

Since the first factor on the left-hand side depends on creation and annihilation operators just like $\hat{\mathcal{G}}$, we have to take care of the commutators. Using the identity: $\hat{a}_{\pm\vec{k}}\hat{\mathcal{G}}|0_a\rangle = [\hat{a}_{\pm\vec{k}}, \hat{\mathcal{G}}]|0_a\rangle = \frac{\partial\hat{\mathcal{G}}}{\partial\hat{a}^\dagger_{\pm\vec{k}}}|0_a\rangle$ and noting that $|0_a\rangle \neq 0$, we find the sufficient operator identity[12]:

$$u_{\vec{k}}\frac{\partial\hat{\mathcal{G}}}{\partial\hat{a}^\dagger_{\mp\vec{k}}} + v_{\vec{k}}\hat{a}^\dagger_{\pm\vec{k}}\hat{\mathcal{G}} = 0 \tag{68}$$

with simple solution:

$$\hat{\mathcal{G}} = Ae^{-\frac{v_{\vec{k}}}{u_{\vec{k}}}\hat{a}^\dagger_{\vec{k}}\hat{a}^\dagger_{-\vec{k}}} \tag{69}$$

We can write this exponential in its series expansion to find the normalisation constant to be:

$$A = \frac{1}{u_{\vec{k}}} \tag{70}$$

There is one thing left to do. In the derivation we counted all states twice, namely all $\vec{k}, -\vec{k}$ as well as $-\vec{k}, -(-\vec{k})$. We therefore have to divide by 2 in the exponential. Another thing to mention is that we found the answer for just one single $\vec{k}$ value. To get the full solution we need to take the product state[13]:

$$|0_b\rangle = \prod_{\vec{k}} \frac{1}{u_{\vec{k}}} e^{-\frac{v_{\vec{k}}}{2u_{\vec{k}}}\hat{a}^\dagger_{\vec{k}}\hat{a}^\dagger_{-\vec{k}}}|0_a\rangle \tag{71}$$

involving all $\vec{k}$-modes. Using the Bogolyubov transformation, a simple calculation now leads to the amount of b- particles created in the a-particle vacuum state.

$$\langle 0_a|\hat{b}^\dagger_{\vec{k}}\hat{b}_{\vec{k}}|0_a\rangle = \langle 0_a|(u^*_{\vec{k}}\hat{a}^\dagger_{\vec{k}} + v^*_{\vec{k}}\hat{a}_{-\vec{k}})(u_{\vec{k}}\hat{a}_{\vec{k}} + v_{\vec{k}}\hat{a}^\dagger_{-\vec{k}})|0_a\rangle \tag{72}$$

Writing out yields two terms that have $\hat{a}_{\vec{k}}$ on the right-hand side and one term with $\hat{a}^\dagger_{\vec{k}}$ on the left-hand side. These terms annihilate the vacuum and we are left with:

$$\langle 0_a|\hat{b}^\dagger_{\vec{k}}\hat{b}_{\vec{k}}|0_a\rangle = |v_{\vec{k}}|^2\langle 0_a|\hat{a}_{-\vec{k}}\hat{a}^\dagger_{-\vec{k}}|0_a\rangle \tag{73}$$

[12] When $\hat{a}_{\pm\vec{k}}$ is commuted to the right it annihilates the vacuum.

[13] In this result it is obvious why we divided by the factor 2, since $\vec{k}$ and $-\vec{k}$ give the same state.

Making use of the commutator implies that the operators will change position and again annihilate the vacuum, so we are left with a unit operator. Indicating the expectation value for the amount of b-particles with momentum $\vec{k}$ in the state $|0_a\rangle$ by $n_{b\vec{k}}$, we can write down:

$$n_{b\vec{k}} = |v_{\vec{k}}|^2 \tag{74}$$

This is usually a non-zero number, which means that we create particles due to the choice of basis. So what we did was reinterprete the old a-particle vacuum in terms of the new b-type particles. In normal flat QFT we have that the vacuum state is an eigenstate of the Hamiltonian, which depends on the system we are considering. However, since the Hamiltonian is time-independent in that case one should be able to define a vacuum that minimizes the energy at every time. In the FLRW case we can't do this, as we are switching between different particle interpretations[14]. In the next sections we will quantitatively describe this.

5.3 Solving the massless KGE in the FLRW metric with spherical coordinates for the conformally coupled closed dynamical case

Let us go back to equation (40) and rethink the situation for $k = 1$ and $\dot{a} \neq 0$. We are working in an expanding universe so we could also think of the field as 'expanding' in terms of a. Therefore we introduce a new field $\chi(x)$ as a scaled field $\phi(x)$ according to $\chi(x) = a(t)\phi(x)$. We write out all terms:

$$\frac{\partial \phi}{\partial t} = \frac{1}{a(t)}\frac{\partial \chi}{\partial t} + \chi\frac{\partial}{\partial t}\frac{1}{a(t)} = \frac{1}{a(t)}\frac{\partial \chi}{\partial t} - \frac{\chi}{a(t)}H$$

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{1}{a(t)}\frac{\partial^2 \chi}{\partial t^2} - \frac{2H}{a(t)}\frac{\partial \chi}{\partial t} + \frac{H^2}{a}\chi \tag{75}$$

where again $\frac{\dot{a}}{a} = H = constant$. The whole equation becomes:

$$a^2\ddot{\chi} + a^2 H\dot{\chi} - 2a^2H^2\chi - \nabla^2_{FLRW}\chi + a^2(m^2 + \alpha\mathcal{R})\chi = 0 \tag{76}$$

Again we impose separation of variables $\chi(x) = \xi(\vec{r})T(t)$. Taking the separation constant to be $-\sigma^2 \equiv \frac{\nabla^2_{FLRW}\xi(\vec{r})}{\xi(\vec{r}}$, we find:

$$\ddot{T}(t) + H\dot{T}(t) + (m^2 + \alpha\mathcal{R} - 2H^2 + \frac{\sigma^2}{a^2(t)})T(t) = 0 \tag{77}$$

We can rewrite this using the expression for $\mathcal{R}$, using $k = 1$:

$$\alpha\mathcal{R} - 2H^2 = 12\alpha H^2 + 6\alpha a^{-2} - 2H^2 = 12H^2(\alpha - \frac{1}{6}) + \frac{6\alpha}{a^2} \tag{78}$$

[14] In that case the Bogolyubov transformation will prove handy.

This then gives the equation:

$$\ddot{T}(t) + H\dot{T}(t) + \big(m^2 + 12H^2[\alpha - \frac{1}{6}] + (\sigma^2 + 6\alpha)e^{-2Ht}\big)T(t) = 0 \tag{79}$$

Again we see the time dependent frequency. Writing $\tilde{\sigma}^2 = \sigma^2 + 6\alpha$ it is given by:

$$\omega_{\tilde{\sigma}}^2 = m^2 + 12H^2[\alpha - \frac{1}{6}] + \frac{\tilde{\sigma}^2}{a^2} \tag{80}$$

We now can see the term proportional to H^2 cancelling if we take $\alpha = \frac{1}{6}$. The cancellation is possible because of our choice of χ. We call this choice for α 'conformal coupling'[15]. Considering the massless and conformally coupled case, i.e. $m = 0$ and $\alpha = \frac{1}{6}$, the equation reduces to:

$$\ddot{T} + H\dot{T} + \tilde{\sigma}^2 e^{-2Ht} T = 0 \tag{81}$$

This equation has exact real solutions of the form

$$T_{\tilde{\sigma}}(t) = A\big[e^{(\frac{i\tilde{\sigma}e^{-Ht}}{H})} + e^{(\frac{-i\tilde{\sigma}e^{-Ht}}{H})}\big] \tag{82}$$

with A a real constant. Now remember our definition of χ, we then have for the temporal part of $\phi(x)$:

$$\phi(x) = a(t)^{-1}\chi(x) = a(t)^{-1}T(t)\xi(\vec{r}) \equiv \psi(t)\xi(\vec{r}) \tag{83}$$

The functions

$$\psi_{\tilde{\sigma}}(t) = A\Big[e^{\frac{i\tilde{\sigma}e^{-Ht}}{H} - Ht} + e^{(\frac{-i\tilde{\sigma}e^{-Ht}}{H}) - Ht}\Big] \tag{84}$$

which are called the mode functions of the field. How do we interpret this temporal solution?

[15]Because the action for this field is then invariant

The temporal function $\psi_{\tilde{\sigma}}(t)$ looks like:[6]

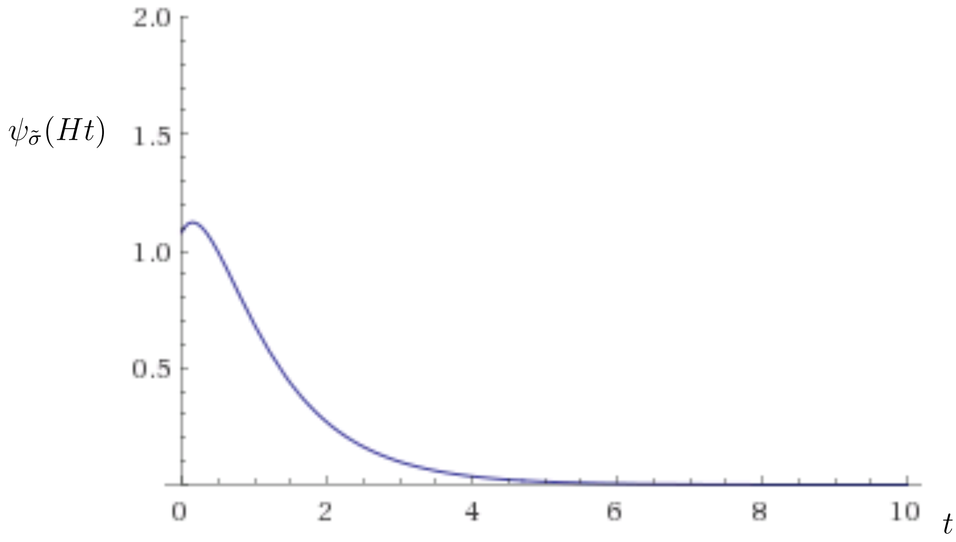

Figure 3: The function $\psi_{\tilde{\sigma}}$ in arbitrary units for $\tilde{\sigma} = H$

As we see for large times the temporal part goes to zero. This means that the field amplitude also goes to zero effectively. A common scenario where this happens is in particle decays. So this might hint at decaying particles due to the curvature. This is also theorized in [2] for particles like pions and protons. So even if the lifetime of a proton is very large, it could decay in accelerated frames.
However, we have left out an important part in plotting this function. Is it really the field amplitude that is going to zero? No. In fact, it has to decrease in order to keep our normalization fixed. If we consider this ($m = 0$, $\alpha = \frac{1}{6}$) case and we normalize like usual we have a normalization constant that looks like: $\frac{1}{\sqrt{\omega_{\vec{k}}}}$. If we write out we find:

$$\frac{1}{\sqrt{\omega_{\vec{k}}}} = \frac{1}{[m^2 + 12H^2(\alpha - \frac{1}{6}) + \frac{\tilde{\sigma}^2}{a(t)^2}]^{1/4}} = \sqrt{a(t)/\tilde{\sigma}} = \frac{e^{\frac{Ht}{2}}}{\sqrt{\tilde{\sigma}}} \tag{85}$$

Observe the time dependent normalization 'constant' which keeps growing in time. We therefore need a decreasing field amplitude in order to prevent the field from blowing up. If we are normalizing the field we do this at one point in time. The universe however keeps expanding, so we are considering an expanding spatial slice on which the field lives at this moment. So normalization takes place at different 'levels' as we have to take the expansion, which gives a $1/a$ term for every dimension, in to account and we have the value from above. All together we end up with having a term a^{-1} left. This is exactly found in the definition of $\xi(x)$. As a result of such the field amplitudes are accordingly decreasing.

Returning to the solutions of the differential equation, we still have to solve for the spatial part $\xi(\vec{r})$ which satisfies the equation:

$$\nabla^2_{FLRW}\xi(\vec{r}) = -\sigma^2\xi(\vec{r}) \tag{86}$$

Luckily this was already done in section 4.3. The spatial part is effectively the same, save for the different separation constant σ^2 which is quantized according to:

$$\sigma^2 = N^2 - 1 \tag{87}$$

with $N = 1, 2, 3\cdots$ We have now solved most of the cases analytically, let us now go back to quantization using the most general case.

5.4 Quantizing using the scaled field in curved spacetime with scalar coupling

In this section we will try to quantize the field equations in mode functions just like the Klein-Gordon case. Starting with the Lagrangian density:

$$\mathcal{L} = \sqrt{-g}\Big[\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}(m^2 + \alpha\mathcal{R})\phi^2\Big] \tag{88}$$

we again consider FLRW space, such that $g_{\mu\nu}(\eta) = a^2(\eta)\eta_{\mu\nu}$. This time we will replace $a(\eta) = a$ for simplicity. Let us work out the equation of motion like we did earlier. The resulting equation of motion was:

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) + (m^2 + \alpha\mathcal{R})\phi = 0 \tag{89}$$

Because of our choice of $g_{\mu\nu}$ we easily find: $g^{00} = a^{-2}$ and $g^{jj} = -a^{-2}$ as well as $\sqrt{-g} = a^4$. Plugging this in we find a new equation of motion in the conformal FLRW metric:

$$\phi'' + \frac{2a'}{a}\phi' - \nabla^2\phi + a^2(m^2 + \alpha\mathcal{R})\phi = 0 \tag{90}$$

Here a prime stands for differentiation with respect to the conformal time. Let's now use the scaled field from the previous section such that $\phi = \frac{\chi}{a}$. Plugging this in gives rise to a simple equation to work with:

$$\chi'' - \nabla^2\chi + \chi(a^2m^2 + \alpha a^2\mathcal{R} - \frac{a''}{a}) = 0 \tag{91}$$

We can define $m^2_{eff} = a^2m^2 + \alpha a^2\mathcal{R} - \frac{a''}{a}$, in order to write this equation in a compact way:

$$(\partial^2 + m^2_{eff})\chi = 0 \tag{92}$$

where $\partial^2 = \frac{\partial^2}{\partial\eta^2} - \nabla^2$ is the Klein-Gordon operator in terms of the conformal time. This is just a KGE! We know how to solve these, but we have to take care of the *time dependence* of the mass. This will have consequences as we will see later. The spatial part however can again be solved easily. As always a good ansatz is decomposition in Fourier modes:

$$\chi(\vec{x},\eta) = \int \frac{d\vec{k}}{(2\pi)^3}\chi_{\vec{k}}(\eta)e^{i\vec{k}\cdot\vec{x}} \tag{93}$$

Plugging this solution back into the KGE we find the so called Mukhanov-Sasaki equation for the mode functions:[16]:

$$\chi''_{\vec{k}}(\eta) + (\vec{k}^2 + m^2a^2(\eta) + \alpha a^2(\eta)\mathcal{R} - \frac{a''(\eta)}{a(\eta)})\chi_{\vec{k}}(\eta) = \chi''_{\vec{k}}(\eta) + \omega^2_{\vec{k}}(\eta)\chi_{\vec{k}}(\eta) = 0 \tag{94}$$

where we again find a time dependent frequency:

$$\omega^2_{\vec{k}}(\eta) = \vec{k}^2 + m^2_{eff}(\eta) \tag{95}$$

Because of the time dependence, we have to treat this equation a bit differently to find a general solution for $\chi_{\vec{k}}(\eta)$. We start by finding $a(\eta)$. Using the definition of the conformal time η and the expression for $a(t)$ yields:[17]

$$\eta = \int_{\infty}^{t} \frac{dt'}{e^{Ht'}} = \frac{e^{-Ht}}{-H} = \frac{-1}{aH} \longrightarrow a(\eta) = \frac{-1}{H\eta} \equiv a \tag{96}$$

So we can write m^2_{eff} in the following form:

$$m^2_{eff} = (\frac{m^2}{H^2\eta^2} + \frac{\alpha\mathcal{R}}{H^2\eta^2} - \frac{2}{\eta^2}) = (\frac{m^2}{H^2} + \frac{\alpha\mathcal{R}}{H^2} - 2)\frac{1}{\eta^2} \tag{97}$$

For massless particles we have that $\frac{m^2}{H^2} = 0$ whereas for massive particles $\frac{m^2}{H^2} \gg 1$. Let us again consider the former case and afterwards comment on the latter. Putting $m = 0$ we find:

$$m^2_{eff} = (\frac{\alpha\mathcal{R}}{H^2} - 2)\frac{1}{\eta^2} \tag{98}$$

Our frequency then reads:

$$\omega^2_{\vec{k}}(\eta) = \vec{k}^2 + (\frac{\alpha\mathcal{R}}{H^2} - 2)\frac{1}{\eta^2} \tag{99}$$

Just to illustrate we now consider a minimally coupled field[18], so we set $\alpha = 0$. Using the relation between $\vec{k}$ and the particle's wavelength $\lambda = \frac{2\pi}{|\vec{k}|}$ we can write down the following relation:

$$\omega^2_{\vec{k}}(\eta) = \vec{k}^2 - \frac{2}{\eta^2} = \vec{k}^2(1 - \frac{\lambda^2}{2\pi^2\eta^2}) \tag{100}$$

[16] $\chi_{\vec{k}}(\eta)$ are called the mode functions of the system. If we are able to solve for these, we can write the full field solution in terms of these.

[17] We set $t = +\infty$ at the far future and use this as a reference, since for $t = -\infty$ the expression for η blows up. We then pick up an extra minus sign

[18] Which is analagous to saying that $\alpha\mathcal{R} \ll H^2$, just like for extremely small masses.

For the mode functions $\chi_{\vec{k}}(\eta) \equiv \chi_{\vec{k}}$ we found the harmonic equation:

$$\chi''_{\vec{k}} + \omega^2_{\vec{k}}(\eta)\chi_{\vec{k}} = 0 \tag{101}$$

Now we can split the solutions in two parts corresponding to the two extrema for short and long wavelenghts, i.e. large and small values for $|\vec{k}|$.

1. <u>Short wavelengths</u>, i.e. $\lambda \ll |\eta|$. In units of λ we are dealing with large negative conformal times and therefore $\omega^2_{\vec{k}}(\eta) \sim \vec{k}^2$. It follows that the mode functions are $\chi_{\vec{k}} \sim e^{-i|\vec{k}|\eta}$ and $\chi^*_{\vec{k}}$. These are just oscillating phases and *unaffected* by the dynamics of spacetime. This scenario describes an adiabatic change in mode functions, as they are unaffected. This means that the field ϕ still corresponds to $\phi = \frac{\chi}{a}$.

2. <u>Long wavelengths</u>, i.e. $\lambda \gg |\eta|$. In units of λ we are dealing with small conformal times. Now the other term in $\omega^2_{\vec{k}}$ is dominant and we find $\omega^2_{\vec{k}}(\eta) \sim \frac{-2}{\eta^2}$. Therefore the kind of mode functions satisfying this are growing or decreasing functions of the form $\chi_{\vec{k}} \sim \frac{1}{\eta}$. Note that we have thrown away the $\chi_{\vec{k}} \sim \eta^2$ solution, because for small η this is strongly surpressed. These long wavelength solutions represent real time dependence and are *strongly deformed* by the dynamics of spacetime. This is a diabatic scenario, as the mode functions can not follow the deformation of spacetime. Observe that $\phi = \frac{\chi}{a} \sim \frac{\frac{1}{\eta}}{a} \sim \eta^0 =$ *constant*. So we find a solution that is constant in time and only shows spatial dependence. This means that the mode functions are not waves, but static solutions. This means we can not expand in wave modes like we always did and we therefore lose the idea of 'particle -wave duality'.

This sounds logical, since short wavelenghts can kind of 'feel' the dynamical background and can adiabatically adjust to changes in the geometry. If however the wavelength is very large it is possible that spacetime has huge jumps over the conformal time interval needed to traverse a single wavelength and you get some kind of deformation effect on these quantum modes. One can think of this as a sort of uncertainty in wavelengths of the modes when measuring the time.

It is possible that we can define a so called 'horizon' at which there is a transition between the above two extrema. Observe that the long wavelength limit has an ambiguity. Take a look at the dispersion relation (100). For later times, so smaller η, we see that the absolute value of the frequency will diverge. Actually, before this happens it is possible to have $\omega^2_{\vec{k}} < 0$, and therefore producing imaginary freqencies. So we can conclude that this dispersion relation can not hold for long wavelenghts, since parts of the 'wave' will not be in causal contact anymore, i.e. the wave can not be traversed anymore in a finite amount of time. Let us now take a quick look at the massive case. We return to the effective dispersion relation and now only take $\alpha = 0$. We have:

$$\omega^2_{\vec{k}}(\eta) = \vec{k}^2 + (\frac{m^2}{H^2} - 2)\frac{1}{\eta^2} \tag{102}$$

The case with the most prominent difference that will differ from before will be the long wavelength case. The short wavelength case will not differ too much since now $\vec{k}$ is battling m^2a^2. This however will turn out positive for the $\vec{k}$ term. There will be only a small contribution which will not make the small wavelength case have real solutions like the long wavelength case from earlier. They will still be oscillating phases. Returning to the Mukhanov-Sasaki equation:

$$\chi''_{\vec{k}} + ((\frac{m^2}{H^2} - 2)\frac{1}{\eta^2})\chi_{\vec{k}} = 0 \tag{103}$$

We solve this equation with the ansatz: $\chi_{\vec{k}} = \eta^A$. If we plug this in we get:

$$A(A-1) + \frac{m^2}{H^2} - 2 = 0 \tag{104}$$

Under the assumption that $\frac{m^2}{H^2} \gg 1$ we find A to be:

$$A = \frac{1}{2} \pm i\frac{m}{H} \tag{105}$$

such that the mode functions are:

$$\chi_{\vec{k}} = \eta^{\frac{1}{2} \pm i\frac{m}{H}} \tag{106}$$

We see that $\phi = \frac{\chi}{a} = \frac{\eta^{\frac{1}{2}\pm i\frac{m}{H}}}{\frac{-1}{H\eta}} = -H\eta^{\frac{3}{2}\pm i\frac{m}{H}}$. Observe that we regain a complex function in contrast to the massless case where we had a constant field. Now we have $\phi \sim a^{-3/2}$ and a complex part that is a phase. So this results in the fact that for a massive particle the long wavelength era past the horizon does give nice solutions.
We will now continue working with the minimally coupled massless case.

We are able to write down an ansatz for the mode functions. Each mode function is composed of two terms like earlier for the usual Klein-Gordon field:

$$\chi_{\vec{k}} = \frac{1}{\sqrt{2}}(a_{\vec{k}}v_{\vec{k}}(\eta) + a^*_{-\vec{k}}v^*_{-\vec{k}}(\eta)) \tag{107}$$

Then the real solution $\chi(x)$ reads:

$$\chi(x) = \int \frac{d\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{2}}(a_{\vec{k}}v_{\vec{k}}(\eta)e^{i\vec{k}\cdot\vec{x}} + a^*_{\vec{k}}v^*_{\vec{k}}(\eta)e^{-i\vec{k}\cdot\vec{x}}) \tag{108}$$

We have found an expansion of mode functions for a field in curved spacetime just like the Klein-Gordon field. This however is still a classical field, we need to quantize the solution by promoting all relevant quantities to operators: $a_{\vec{k}}$ and $a^*_{\vec{k}} \longrightarrow \hat{a}$ and $\hat{a}^\dagger$ as well as the field $\chi(x)$ itself. Using second quantization, which I assume is still valid in this case, we are able to derive the commutators for equal time fields $\hat{\chi}(t, \vec{x})$ and

$\hat{\chi}(t,\vec{y})$. This follows from the causality postulate of relativity. Defining the conjugate momentum $\hat{\Pi}(x) \equiv \hat{\chi}(x)' = \frac{\partial \hat{\chi}(x)}{\partial \eta}$

$$[\hat{\chi}(t,\vec{x}), \hat{\Pi}(t,\vec{y})] = i\delta(\vec{x}-\vec{y})\hat{I}$$

and

$$[\hat{\chi}(t,\vec{x}), \hat{\chi}(t,\vec{y})] = [\hat{\Pi}(t,\vec{x}), \hat{\Pi}(t,\vec{y})] = 0 \tag{109}$$

This[19] is equivalent to imposing the following commutation relations for the coefficients $\hat{a}$ and $\hat{a}^\dagger$:

$$[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] = (2\pi)^3\delta(\vec{k}-\vec{k}')\hat{I}$$

$$[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}] = [\hat{a}^\dagger_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] = 0 \tag{110}$$

Observe that these commutators are just the usual raising and lowering operators from the harmonic oscillator. After second quantization we find:

$$\hat{\chi}(x) = \int \frac{d\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{2}}(\hat{a}_{\vec{k}}v_{\vec{k}}(\eta)e^{i\vec{k}\cdot\vec{x}} + \hat{a}^\dagger_{\vec{k}}v^*_{\vec{k}}(\eta)e^{-i\vec{k}\cdot\vec{x}}) \tag{111}$$

The field $\hat{\chi}(x)$ now is an operator field that describes a scalar particle in a curved or dynamical background and creates or annihilates particles with a wave character of the form $v_{\vec{k}}(\eta)e^{i\vec{k}\cdot\vec{x}}$. Also note that we consider a hermitian scalar field operator. This means that our theory does not have antiparticles and therefore only electrically neutral particles. We also find the same kind of Hamiltonian for this field. We find:

$$\hat{H} = \int \frac{1}{2}(\hat{\Pi}^2 + (\nabla\hat{\chi})^2 + m^2_{eff}\hat{\chi}^2)d\vec{x} \tag{112}$$

This time, however, we see that $\hat{H}$ is *not* time independent due to the absence of time translation invariance. Therefore we will have a hard time finding a suitable description of a vacuum state. If we perform a suitable Bogolyubov transformation to the mode functions and the creation and annihilation operators, we can write $\hat{\chi}(x)$ in the exact same way but described by different creation and annihilation operators, resulting in the non-uniqueness of the vacuum. Let us work out the Hamiltonian just like we did for the Klein-Gordon field. We need the following ingredients:

$$\hat{\Pi}(x) = \frac{1}{\sqrt{2}}\int \frac{d\vec{k}}{(2\pi)^3}(\hat{a}_{\vec{k}}v'_{\vec{k}}e^{i\vec{k}\cdot\vec{x}} + \hat{a}^\dagger_{\vec{k}}v'^*_{\vec{k}}e^{-i\vec{k}\cdot\vec{x}})$$

$$\nabla\hat{\chi}(x) = \frac{1}{\sqrt{2}}\int \frac{d\vec{k}(i\vec{k})}{(2\pi)^3}(\hat{a}_{\vec{k}}v_{\vec{k}}e^{i\vec{k}\cdot\vec{x}} - \hat{a}^\dagger_{\vec{k}}v^*_{\vec{k}}e^{-i\vec{k}\cdot\vec{x}})$$

[19]This is very much like the fundamental commutators between position and momentum operators.

$$(113)$$

By working out the squares we find the following Hamiltonian:

$$\hat{H} = \frac{1}{4}\int d\vec{x}\int\int \frac{d\vec{k}d\vec{k'}}{(2\pi)^6}e^{-i(\vec{k}-\vec{k'})\cdot\vec{x}}\big((\hat{a}_{-\vec{k}}v'_{-\vec{k}} + \hat{a}^{\dagger}_{\vec{k}}v'^{*}_{\vec{k}})(\hat{a}_{\vec{k'}}v'_{\vec{k'}} + \hat{a}^{\dagger}_{-\vec{k'}}v'^{*}_{-\vec{k'}})$$

$$+\vec{k}\cdot\vec{k'}(\hat{a}_{-\vec{k}}v_{-\vec{k}} + \hat{a}^{\dagger}_{\vec{k}}v^{*}_{\vec{k}})(\hat{a}_{\vec{k'}}v_{\vec{k'}} + \hat{a}^{\dagger}_{-\vec{k'}}v^{*}_{-\vec{k'}})$$

$$+m^2_{eff}(\hat{a}_{-\vec{k}}v_{-\vec{k}} + \hat{a}^{\dagger}_{\vec{k}}v^{*}_{\vec{k}})(\hat{a}_{\vec{k'}}v_{\vec{k'}} + \hat{a}^{\dagger}_{-\vec{k'}}v^{*}_{-\vec{k'}})\big) \qquad (114)$$

By performing the $\vec{x}$-integral we again get a delta function of $\vec{k}-\vec{k'}$. If we replace $\vec{k}$ by $-\vec{k}$, use that $\omega^2_{\vec{k}} = \vec{k}^2 + m^2_{eff}$ and assume that $v_{\vec{k}} = v_{-\vec{k}}$ we find:

$$\hat{H} = \frac{1}{4}\int \frac{d\vec{k}}{(2\pi)^3}\big(\hat{a}_{\vec{k}}\hat{a}_{-\vec{k}}(v'^2_{\vec{k}}+\omega^2_{\vec{k}}v^2_{\vec{k}})+\hat{a}^{\dagger}_{\vec{k}}\hat{a}^{\dagger}_{-\vec{k}}(v'^{*2}_{\vec{k}}+\omega^2_{\vec{k}}v^{*2}_{\vec{k}})+(2\hat{a}^{\dagger}_{\vec{k}}\hat{a}_{\vec{k}}+(2\pi)^3\delta(\vec{0}))(|v'_{\vec{k}}|^2+\omega^2_{\vec{k}}|v_{\vec{k}}|^2)\big) \qquad (115)$$

For notational convenience let us very suggestively use the following shorthand notations:

$$E_{\vec{k}} = (|v'_{\vec{k}}|^2 + \omega^2_{\vec{k}}|v_{\vec{k}}|^2)$$

$$F_{\vec{k}} = (v'^2_{\vec{k}} + \omega^2_{\vec{k}}v^2_{\vec{k}}) \qquad (116)$$

so that $\hat{H}$ simply becomes:

$$\hat{H} = \frac{1}{4}\int \frac{d\vec{k}}{(2\pi)^3}\big(\hat{a}_{\vec{k}}\hat{a}_{-\vec{k}}F_{\vec{k}} + \hat{a}^{\dagger}_{\vec{k}}\hat{a}^{\dagger}_{-\vec{k}}F^{*}_{\vec{k}} + (2\hat{a}^{\dagger}_{\vec{k}}\hat{a}_{\vec{k}} + (2\pi)^3\delta(\vec{0}))E_{\vec{k}}\big) \qquad (117)$$

Observe that both $E_{\vec{k}}$ and $F_{\vec{k}}$ are still time dependent. Let's define an *instantaneous* vacuum at one time η_0 as $\hat{a}_{\vec{k}}|0_{\eta_0}\rangle = 0$. We then see that for every mode $\vec{k}$ the following expectation value holds:

$$\langle 0_{\eta_0}|\hat{H}|0_{\eta_0}\rangle = \delta(\vec{0})\int \frac{1}{4}E_{\vec{k}}(\eta)d\vec{k} \qquad (118)$$

To find the vacuum from here we have to investigate $E_{\vec{k}}$ a little bit more. Since we consider the vacuum, it would be good to have $E_{\vec{k}}$ minimized for every mode $\vec{k}$ at some time η_0. What we did not mention yet is the normalization of the mode functions $v_{\vec{k}}$. To do so one uses the *Wronskian* of the two linearly independent solutions $v_{\vec{k}}$ and $v^{*}_{\vec{k}}$:

$$W[f,g] \equiv f^{'}g - fg^{'} \qquad (119)$$

One finds that:

$$W[v_{\vec{k}}, v^*_{\vec{k}}] = v'_{\vec{k}} v^*_{\vec{k}} - v_{\vec{k}} v^{*'}_{\vec{k}} \tag{120}$$

If we rewrite the mode functions according to $v_{\vec{k}} = x + iy$, we find:

$$v'_{\vec{k}} v^*_{\vec{k}} - v_{\vec{k}} v'^*_{\vec{k}} = 2i(xy' - x'y) = -2iW[x, y] \tag{121}$$

Therefore we normalize the mode functions with the constant $2i$, i.e. $W[x, y] = -1$. We can use this as a boundary condition to minimize E_k in the following way. Suppose we have the following initial condition at conformal time η_0:

$$v_{\vec{k}}(\eta_0) = a$$

$$v'_{\vec{k}}(\eta_0) = b \tag{122}$$

with $a, b \in \mathbb{C}$. Because we need to minimize $E_{\vec{k}}$, which is only dependent on the magnitude of the mode functions, we have phase freedom as is used in [9]. We will use this to make a real. Then rewriting b as $b = b_1 + ib_2$ we find:

$$a = \frac{2i}{b - b^*} = \frac{1}{b_2} \tag{123}$$

Rewriting $E_{\vec{k}}$ yields:

$$E_{\vec{k}} = b_1^2 + b_2^2 + \frac{\omega^2_{\vec{k}}(\eta_0)}{b_2^2} \tag{124}$$

Now we can minimize with respect to the coefficients $b_{1,2}$. In the following calculation we assume $\omega_{\vec{k}} > 0$. This is only true for modes that have $|\vec{k}|\eta > \sqrt{2}$. We are unable to do this for modes with imaginary frequencies as these do not define quantized particles.

$$\frac{\partial E_{\vec{k}}}{\partial b_1} = 0 \rightarrow b_1 = 0$$

$$\frac{\partial E_{\vec{k}}}{\partial b_2} = 0 \rightarrow b_2 = \pm\sqrt{\omega_{\vec{k}}(\eta_0)} \tag{125}$$

Assuming $b_2 > 0$ we find for the instantaneous mode functions:

$$v_{\vec{k}}(\eta_0) = \frac{1}{b_2} = \frac{1}{\sqrt{\omega_{\vec{k}}(\eta_0)}}$$

$$v'_{\vec{k}}(\eta_0) = ib_2 = i\sqrt{\omega_{\vec{k}}(\eta_0)} = i\omega_{\vec{k}}(\eta_0) v_{\vec{k}}(\eta_0) \tag{126}$$

and:

$$E_{\vec{k}} = 2\omega_{\vec{k}}(\eta_0)$$

$$F_{\vec{k}} = 0 \tag{127}$$

We see for this minimized $E_{\vec{k}}$ that the Hamiltonian is diagonal at some time η_0, namely:

$$\hat{H}(\eta_0) = \int \frac{d\vec{k}}{(2\pi)^3}(a^\dagger_{\vec{k}} a_{\vec{k}} + \frac{1}{2}(2\pi)^3\delta(\vec{0}))\omega_{\vec{k}}(\eta_0) \tag{128}$$

So we find completely analogous to the usual KG field that,

$$\langle 0_{\eta_0}|\hat{H}|0_{\eta_0}\rangle = \delta(\vec{0})\int d\vec{k}\frac{1}{2}\omega_{\vec{k}}(\eta_0) \tag{129}$$

This means we can indeed find a unique state that at one given time diagonalizes the Hamiltonian, it is therefore often called: *the vacuum state of instant diagonalization*. We can however *not* guarantee that this state will be a vacuum at some later time because $\omega_{\vec{k}}$ generally depends on η, making it hard to find eigenstates valid for every time. As we have seen for our instant vacuum, we found that $F_{\vec{k}} = 0$. What if we require this to hold at all times? In that case we can write down the following differential equation for the mode functions:

$$v'^2_{\vec{k}} + \omega^2_{\vec{k}} v^2_{\vec{k}} = 0 \tag{130}$$

with solutions of the form:

$$v_{\vec{k}}(\eta) = Ae^{\pm i\int \omega_{\vec{k}}(\eta)d\eta} \tag{131}$$

This however does not agree with the Mukhanov-Sasaki equation (101) for the mode functions $v_{\vec{k}}$ if $\omega_{\vec{k}}$ is not constant over time. We can therefore not define one unique set of mode functions that would define a vacuum eigenstate at all times! A nice try to go around this problem could be to rethink our way of 'defining' a vacuum. Untill now we have always tried to minimize the energy of the system in order to find the vacuum. What if one tries to minimize the actual field amplitude fluctuations? Let's have a look at the following expectation value of the squared fluctuation of a vacuum state at some time ν:

$$\langle 0_\nu|\int \hat{\chi}(x)^2 d\vec{x}|0_\nu\rangle \tag{132}$$

Working out yields:[20]

$$\int \hat{\chi}(x)^2 d\vec{x} = \frac{1}{2}\int \frac{d\vec{k}}{(2\pi)^3}(\hat{a}_{-\vec{k}}\hat{a}_{\vec{k}} v^2_{\vec{k}} + \hat{a}_{\vec{k}}\hat{a}^\dagger_{\vec{k}}|v_{\vec{k}}|^2 + \hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}}|v_{\vec{k}}|^2 + \hat{a}^\dagger_{-\vec{k}}\hat{a}^\dagger_{\vec{k}} v^{*2}_{\vec{k}}) \tag{133}$$

After using the annihilation of the a-vacuum we find:

$$\langle 0_\nu|\int \hat{\chi}(x)^2 d\vec{x}|0_\nu\rangle = \frac{1}{2}\delta(\vec{0})\int d\vec{k}|v_{\vec{k}}|^2 \tag{134}$$

[20] Again, assuming that $v_{-\vec{k}} = v_{\vec{k}}$. This is justified since it assumes isotropy in the mode functions, which is true because our FLRW universe is isotropic by definition.

This gives a physical meaning to the mode functions, namely the vacuum fluctuation amplitude. If this is the starting point of defining a vacuum, we now have to minimize $|v_{\vec{k}}|^2$. This however is pointless since we are free to choose our initial condition on the mode functions at some conformal time η_0. We are thus able to pick $v_{\vec{k}}(\eta_0)$ as small as we want as long as the Heisenberg uncertainty principle is *not* violated between $\hat{\chi}(x)$ and $\hat{\Pi}(x)$. We are only able to take the limit to zero of the mode functions, since absolute zero would make the conjugate momenta blow up. It is thus safe to say that it is *impossible* to pick a set of modes with the 'smallest' fluctuation amplitude! We have seen that the only mode function that minimizes the energy is of the form:

$$v_{\vec{k}}(\eta_0) = \frac{1}{\sqrt{\omega_{\vec{k}}(\eta_0)}} e^{i\omega_{\vec{k}}(\eta_0)\eta_0}$$

and

$$v'_{\vec{k}}(\eta_0) = i\omega_{\vec{k}}(\eta_0) v_{\vec{k}}(\eta_0) \tag{135}$$

However, these mode functions minimize the vacuum expectation value only at that one time. It is not given that at a later time η_1 these mode functions still define a vacuum as we have seen above. This is because energy is not conserved in the system. It goes into the interaction field, namely the gravitational field. We therefore observe a decaying field amplitude. We can however explore one interesting part of this, namely the solutions that have short wavelenghts. Well below the so called 'horizon' in spacetime these solutions were not affected by the dynamical background and therefore should describe a perfect vacuum. We have seen that $\omega_{\vec{k}}^2 \sim \vec{k}^2$ in this case. The Mukhanov-Sasaki equation becomes simply

$$v''_{\vec{k}} + \vec{k}^2 v_{\vec{k}} = 0 \tag{136}$$

yielding the positive frequency solution:

$$v_{\vec{k}}(\eta) = \frac{1}{\sqrt{|\vec{k}|}} e^{-i|\vec{k}|\eta} \tag{137}$$

which minimizes $E_{\vec{k}}$ for large enough $|\eta|$. After we impose the boundary condition at η_0 the following limit should hold for modes with $|\vec{k}|\eta_0 \gg 1$.

$$\lim_{\eta \to -\infty} v_{\vec{k}} = \frac{1}{\sqrt{\omega_{\vec{k}}(\eta_0)}} e^{i|\vec{k}|\eta_0} \tag{138}$$

Now we can return to the other extremum where we have large wavelengths in our massless field. In that asymptotic regime the Mukhanov- Sasaki equation reads:

$$v''_{\vec{k}} - \frac{2}{\eta^2} v_{\vec{k}} = 0 \tag{139}$$

As we are now considering *late* times, we take the dominant term of the solutions to be:

$$v_{\vec{k}} \sim A\frac{1}{\eta} \tag{140}$$

these are solutions highly deformed by the background. These are the prefered modes, but again, they are not defining a unique vacuum. We have found both extrema. A solution that satisfies both limits is called a *Bunch-Davies vacuum*. In cosmology this state describes a vacuum, least-energy or 'zero'- particle state, that is observed by someone moving on a geodesic in an expanding universe.

We can conclude that it is not possible to define a vacuum based on the Hamilton operator in a curved background. However we were able to find prefered modes for certain eras on the universe's time scale. These mode functions will vary with time and it is only possible to define an instantaneous vacuum for the time dependent Hamiltonian. Also note that for late-time-modes, the super horizon solutions can have imaginary frequencies, if $\vec{k}^2 < \frac{2}{\eta^2}$, and therefore have a tachyon-like particle interpretation. These modes however do *not* have a wave like character as their wavelengths are so large that parts of the 'wave' are out of causal contact with each other because of the rate of expansion of the universe. The wave can not be traversed anymore in a finite amount of time. Therefore the super horizon modes can not be expanded in waves. The sub horizon modes however are real and have 'well defined' solutions. We will stop here and discuss another way to 'tackle' the same problem of defining a vacuum and finding its expectation values and also real consequences.

5.4.1 Massless minimally coupled zero mode

There is one ambiguity left. So far we have seen field(mode) expansions in operators that create/annhihilate particles with a wave factor of the type $v_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}$. But if we take a close look at the equations of motion from the previous sections, we can construct functions that purely depend on time and are solutions to the field equations. Considering the massless minimally coupled mode ($m = \alpha = 0$), we have the equation:

$$\ddot{\phi} - a^{-2}\nabla^2_{FLRW}\phi = 0 \tag{141}$$

If we take a purely time dependent field $\phi(x) = \phi(\vec{x}, t) = \phi(t)$ of the form: $\phi(t) = a + bt$, the equation is satisfied. This solution is *not* contained in our solution space, but must also be quantized. However, this kind of field is not normalizable since it behaves like a free particle, and not like a harmonic oscillator. Moreover it just can not be quantized in the usual way in view of the absence of spatial dependence. We can thus not quantize or expand these solutions in creation and annihilation operators and therefore they are not correct Fock space solutions.[21] The quantization on a four-sphere has been done in [4].

[21] At least in this context of (wave) mode expansion QFT.

6 The Stress energy tensor in semiclassical gravity

There is one completely different way to tackle the background dependence in a QFT. This is to work the other way around. Instead of trying to build curvature into a quantum theory we could as well try to build a quantum way of thinking into a gravitational theory. Einstein's field equations that describe the curvature of spacetime due to the presence of energy in empty space, including a cosmological constant Λ, read:

$$R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{142}$$

Here $R_{\mu\nu}$ is the Ricci curvature tensor and $T_{\mu\nu}$ the so called 'stress energy tensor'. In GR these are 16 coupled differential equations due to the 3+1 dimensions of spacetime. In our spin zero theory however we need scalars that couple to gravity. So a first ansatz could be to look at the stress energy tensor, which in a dynamical metric is defined as:

$$T^{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi - m^2\phi^2) \tag{143}$$

Using the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu}$ and $\frac{8\pi G}{c^4} = \kappa$ we can write:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{144}$$

This is still a valid equation from a geometric point of view, but quantum mechanically we can not use it since it doesn't mean anything for a quantum state. If we look at the components of the stress energy tensor, however, we could calculate the expectation value of each of the components for some state Ψ.

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa\langle\Psi|T_{\mu\nu}|\Psi\rangle \tag{145}$$

From this equation we are able to calculate components for the curvature tensor and make a geometric picture for some state, for example the vacuum state. One would then also expect geometric fluctuations to occur, just like the usual vacuum quantum fluctuations. What indeed has been found is that the above expectation value *locally* is proportional to the metric. This can then be assigned to the term with cosmological constant Λ and therefore hint at vacuum energy, or the expansion of the universe being a quantummechanical effect.

7 Other effects

Of course, all this theory has its consequences in the real world. There are some effects that might occur, other than 'derived' by me. The first one is the so called 'Unruh effect'.

7.1 Unruh effect

In previous sections we have considered an expanding universe. Let us now consider that not the universe is expanding but we have a stable and time independent universe with a uniformly accelerating observer, called Bob, looking at the vacuum or some excited state of it. I will give a simplified 'derivation' for the effect. Let us again set $c = 1$. Then an acceleration $\dot{v}$ in the not moving frame is related to Bobs acceleration a by:

$$\dot{v} = a(1 - v^2)^{3/2} \tag{146}$$

therefore we find the velocity to be: $v(t) = \frac{at}{\sqrt{1+a^2t^2}}$ if $v(0) = 0$. Now we use the proper-time relation $dt = \frac{d\tau}{\sqrt{1-v^2}}$, to find the velocity of Bob in the non-moving frame:

$$v(\tau) = tanh(a\tau) \tag{147}$$

We can now combine this result with the expressions earlier to find $t(\tau)$:

$$t(\tau) = \frac{1}{a} sinh(a\tau) \tag{148}$$

To find Bob's position we calculate the differential $\frac{dx}{dt}$ and find:

$$x(\tau) = \frac{1}{a} cosh(a\tau) \tag{149}$$

Where we have chosen x to be the direction in which Bob moves.[22] If some source radiates at a frequency $\omega_{\vec{k}}$, then Bob will measure a Doppler shifted frequency given by:

$$\omega'_{\vec{k}} = \frac{\omega_{\vec{k}}[1 - tanh(a\tau)]}{\sqrt{1 - tanh^2(a\tau)}} = \omega_{\vec{k}} e^{-a\tau} \tag{150}$$

with $|\vec{k}| = \omega_{\vec{k}}$. Since the frequency measured by Bob is time dependent we get an overal phase factor $\phi(\tau)$ to the frequency spectrum that he can measure. The distribution for all frequencies Ω is then proportional to the following expression by a Fourier transform:

$$|\int_{-\infty}^{\infty} d\tau e^{i\Omega\tau} e^{i\phi(\tau)}|^2 \tag{151}$$

with $\phi(\tau) = \int_{-\infty}^{\infty} \omega'_{\vec{k}}(\tau')d\tau'$. Performing this integral yields the following expression according to [3]:

$$Spectrum \sim \frac{2\pi}{\Omega a} \frac{1}{e^{2\pi\Omega/a} - 1} \tag{152}$$

We get a nice Bose-Einstein distribution for the frequency spectrum as observed by Bob. Also taking $\hbar = k_B = 1$ we find that this is actually a spectrum that radiates at a temperature $T = \frac{a}{2\pi}$. Restoring all constants we find:

$$T = \frac{\hbar a}{2\pi k_B c} \tag{153}$$

[22] These are called Rindler coordinates

so that even a vacuum that is being observed by an accelerated observer will radiate at some finite temperature T, leaving the vacuum actually not vacuum anymore since there must be (classically) particles that create this temperature. We can calculate the acceleration that Bob has to undergo in order to observe a $1K$ spectrum. Plugging in all numbers we find that $a = 2.466 \cdot 10^{20}$ m/s^2 which is pretty huge and not obtainable with today's standards.
However, our derivation is not quite correct as we were only considering one specific frequency mode of the 'field /source'. A more sophisticated way would be to expand the field again in Fourier modes running over all possible frequencies and then calculate the vacuum expectation value. This computation however gives the exact same result as can be seen in [3]. We can also look at the typical acceleration scale of our expanding universe. If we define the acceleration scale as:

$$a_0 = Hc \tag{154}$$

the corresponding temperature T_0 is then given by:

$$T_0 = \frac{\hbar a_0}{2\pi k_B c} = \frac{\hbar H}{2\pi k_B} \tag{155}$$

which for current measurements of H is a very small temperature: $3 \cdot 10^{-30} K$. This is about zero, and would represent the temperature of an expanding universe neglecting cosmic background radiation.
If we would be dealing with spin 1/2 particles like electrons, the distribution found in (152) would become a Fermi-Dirac distribution rather than a Bose-Einstein distribution as we have to ensure that the particle's spin is conserved along the geodesic.

7.2 Black hole radiation

A realistic example where the Unruh effect happens is on the event horizon of a black hole. Einstein has told us in GR that a gravitational effect from curvature is not distiguishable from an acceleration and we can see this happening if we calculate the temperature of a black hole at its surface. In natural units and $G = 1$, the radius of a blackhole is equal to $2M$. From Newtons law of gravitation we find that the surface acceleration then is:

$$a \equiv g_{BH} = \frac{1}{4M} \tag{156}$$

From the Unruh temperature we then simply have

$$T_{BH} = \frac{1}{8\pi M} \tag{157}$$

which indeed corresponds to the Hawking temperature for black holes. Black holes therefore create a 'nice' background for quantum fields [23]. Black hole radiation is a pure

[23] From this temperature one can also derive the Power-law for black hole radiation and it turns out that a solar mass black hole radiates at around $\sim 10^{-29}$ Watts. So it is justified to call that a black hole indeed.

quantum mechanical effect, namely the creation and annihilation of particles due to the (extreme) curvature of space near a black hole. Also observe that T_{BH} is extremely small, but finite, as it is inversely proportional to the black hole's mass.

8 Conclusion

We have come to an end of finding a description of the behaviour of a scalar field in a curved background. More specifically a scalar field in an expanding universe, the Friedmann -Lemaitre -Robertson -Walker universe, which is much like the well known 'de Sitter' background. We have solved Klein-Gordon equations for different backgrounds in order to show different effects and also the analytic solutions. We have seen the flat and closed cases in the stable and expanding universes. In most cases we observed a decaying character of the quantum field amplitude which was due to a fixed normalization constant in a curved background.

Then we jumped to the second quantization approach to quantize the fields in terms of harmonic oscillators. At first we saw the Bogolyubov transformation, which describes the relation between two different interpretations of one system. This gave rise to the non uniqueness of the vacuum state. Then we went on quantizing the scalar field in a Friedmann-Lemaitre-Robertson-Walker metric. When quantizing we have found that there is a time dependent term that couples mass-like to the field, we interpreted this as an 'effective mass' which depends on time in the Hamiltonian/Lagrangian formalism. We split up the solutions in effectively early- and late time solutions by comparing the conformal time to the wavelength of a particular mode. We discovered that solutions with $|\eta| \gg \lambda$ have a mode expansion and can adiabatically adjust to the dynamical background. Contrary to the solutions with $|\eta| \ll \lambda$ which are deformed by the background and do not yield nice solutions that can be expanded in wave modes. These solutions could not be used as parts of the 'wave' were out of causal contact with each other. From here we have used the Hamiltonian formalism to calculate the vacuum expectation value and found a spacetime dependent expression. For the massless minimally coupled case we recover the original expression for the Klein-Gordon vacuum but with a time dependent frequency. We conclude that the vacuum energy is spacetime dependent. This has had its implications in for example black hole theory.

9 Outlook

As the major result of this research might be the vacuum energy expectation value we discuss a few aspects, implications to other fields of research and why it is of great interest to do more research on this.

For the vacuum energy expectation value in a dynamical background we have found that it is spacetime dependent, this means that if you are at another point in space you will measure a different energy. We also have a strict time dependence which means that if we let our system evolve in time we will also measure different energies, which are lower and therefore we do not have a unique lowest energy state. We can see this from the vacuum energy expectation value for the massless minimally coupled case. The expectation value becomes larger if $\omega_{\vec{k}}$ increases. We have that $\eta = \frac{1}{H}e^{-Ht}$, so η decreases for later times, this means the frequency goes down because we have an extra minus sign, and therefore the energy as well. This means that we can not uniquely define a global vacuum, only locally, which is unbounded from below. As these two differ and there are also differences between the local vacua, we can for example describe a system with multiple subsystems with different vacua. We are unable to define one fixed vacuum for the system. For example, consider a galaxy with a massive black hole in the middle. Around this black hole one might locally define a vacuum as we have seen given by the local curvature. We can then go the boundary of the galaxy and define a different vacuum. Since these two possess different lowest energies we can calculate a gradient, which then gives rise to a force. We could try to interpret this physically and it could hint at things like dark matter or dark energy. Since these effects already occur in accelerating frames it could drive the expansion of the universe or result in the measured radial velocity of galaxies being not the ones predicted by theory. We have also seen that for the two branches of solutions, short- and long wavelengths, we lose our possibility of expanding in wave modes for the long wavelength case. This means that we lose our general description of the 'wave-particle duality'. We thus need more general objects than plane waves to expand in.

At the end of this thesis we presented an outlook to tackle the problem, namely using Einsteins equations of general relativity. These make use of geodesics which are already more general than plane waves and are geometry dependent. This eventually implies that the Hamiltonian and Lagrangian formalisms are not a suitable way of tackling the problem of quantizing a quantum field in a dynamical background. We therefore need a more general object, we have seen that the stress energy tensor might be the way to go. We thus need to do more research using these Einstein field equations to quantize in a more adequate way.

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