A Study of Dark Energy Driven MOND

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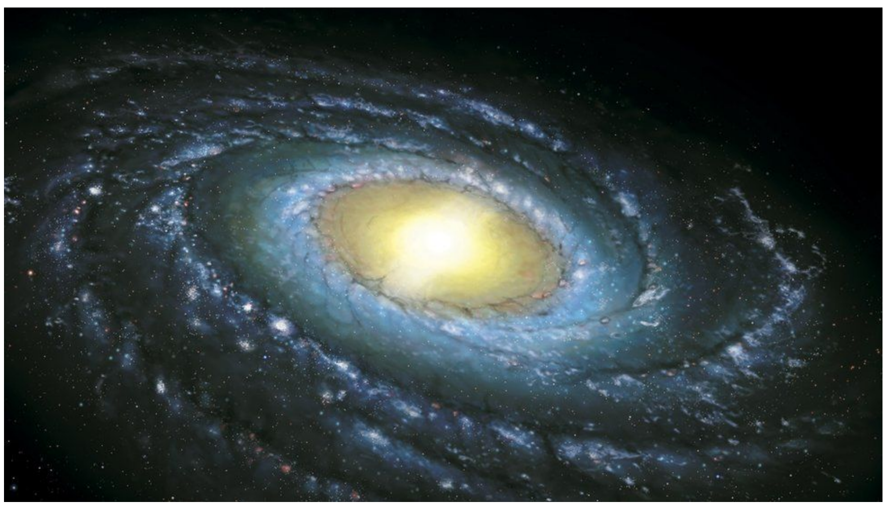

Figure 1: An artist's rendition of the Milky Way galaxy [1].

Abstract

In this bachelor internship and resulting thesis, research has been done on whether dark energy could function as a causal mechanism behind the discrepancy between theoretical predictions and empirical findings for galactic rotation curves. For this purpose, a model has been built that modifies classical N-body (and specifically two-body) orbital mechanics, such that a prediction can be given for the relation between the orbital speed of objects within a galaxy and the distance from the galactic center. The formula $v(r) = \sqrt{\frac{GM}{r} + \frac{a}{r^b}}$ is then fitted for parameters 'a' and 'b' to the rotation curve of the Milky way galaxy, resulting in $a = (1.4 \pm 0.7) \cdot 10^6 \frac{km^{2.1}}{s^2}$ and $b = 0.10 \pm 0.02$, which is generally in line with the behaviour one would expect of galactic rotation curves. Finally, some empirical conditions are derived that have to be satisfied in order for the model to also make correct predictions on the scale of the solar system. We conclude that given the assumptions and approximations made, the first results corroborate the theory for now. However, further research is needed for a stronger epistemological judgment of the model.

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1 Introduction

Contemporary physics is still plagued by many problems. Two of the most prevalent of those are the problem of galactic rotation curves and the problem of the accelerated expansion of the universe. The former lays out the mismatch between theoretical predictions and observational data related to the speed and position at which objects revolve in a galaxy. Meanwhile, the latter concerns the question as to why we observe an accelerated expansion of the universe. As physicists do not (yet) have much of a clue about the cause of this process, it is commonly dubbed 'dark energy'.

While several hypotheses exist attempting to explain the problem of galactic rotation curves, they usually treat the aforementioned problems as distinct in nature. However, this bachelor thesis attempts to begin to investigate the possibility of a Modified Newtonian Dynamics (MOND) approach in which dark energy is not just the cause underlying the accelerated expansion of the universe, but, in addition, fulfills this role for the observed rotation curves of galaxies. That is to say that, while we will refrain from an in-depth investigation as to how this could mechanistically be explained at the level of fundamental physics, the main focus will be on whether such an approach could be in the realm of possibility when considering the resulting mathematical models of such a premise compared to empirical reality. If true, this could open a new direction in which physicists could look for solutions to some of the most important problems in contemporary physics. For this purpose a number of approximations will be made. This is followed by a study of whether adding potential or force terms resulting from the aforementioned causal mechanism to a two-body problem could produce galactic rotation curves that are mathematically consistent with what is in fact observed.

To this end, the thesis will start by explaining more thoroughly the problem of galactic rotation curves and of the accelerated expansion of the universe. Next, we will lay out the approach to solving the problem of galactic rotation curves pursued in this thesis. Continuing, the physical and mathematical theory of N-body problems shall be briefly laid out, after which the new concept will be introduced to this theory. The next step consists in going over how exactly galactic rotation curves are set up. We will then go on to do precisely that for our model. Lastly, it will be investigated whether this model corresponds to empirical reality. A further note is that within this thesis, some tools and methods will be presented that can be used for more extensive research on the topic.

2 Research Question

Given the introduction above, the main research question will be formulated as follows:
'Can a coherent MOND-model in which dark energy serves as the causal mechanism behind observed data of galactic rotation curves quickly be shown not to explain the existing discrepancy between theory and experiment?'

One important comment is that a negative answer to this question does not in any way prove that such a causal mechanism is indeed behind the given discrepancy. It merely corroborates the possibility of such an approach and might serve to legitimise research into it in the future. Conversely, an affirmative answer would show that such an approach is extremely likely not to be fruitful and must be rejected in favour of other research programs. This would narrow the search allowing for more resources to be allocated to actually promising theories.
The word 'quickly' is added here, since a bachelor student thesis may not find an affirmative answer to this question even though that might very well be the true answer to it, and this would be shown by deeper and more extensive research and analysis.
The word 'coherent' refers to the fact that we should desire of such a model that it also produces accurate predictions for systems other than galaxies, such as the solar system.

3 The general problems and common approaches towards solving them

In this section, we will more thoroughly go through the problems of galactic rotation curves and the accelerated expansion of our universe. It will be described why these are problems, what kind of theories are invoked by physicists in trying to provide a solution to them and the gaps that still remain in such theories. First, the problem of galactic rotation curves will be covered, after which the attention will be turned towards the problem of the accelerated expansion of the universe.

3.1 The problem of galactic rotation curves

Observations of the rotational speed[1] of bodies (such as stars and their planets or gas clouds) in a galaxy show significant discrepancies with theoretical predictions for both classical and relativistic physics alike. These discrepancies can be found when looking at so-called 'rotation curves'. A rotation curve is a plot where the orbital speed of a body rotating within a galaxy is expressed against the distance from the center of that galaxy. As we move away from the galactic center further out into a galaxy, we classically expect the orbital speed to fall off. This can be shown qualitatively by considering a body in circular orbit around a much heavier one, where the inward gravitational force on the body balances the resulting centripetal force: $|\vec{F}_g| = F_{cf}$. From this it follows that $\frac{GMm}{r^2} = \frac{mv^2}{r}$, and therefore $v \propto \frac{1}{\sqrt{r}}$, where 'v' is the orbital speed of the body and 'r' its distance from the heavier body. As can be seen, one would expect the orbital speed to drop by the inverse square root of the body's distance. However, the following figure shows observations of galaxies, which seem to imply quite a different relationship:

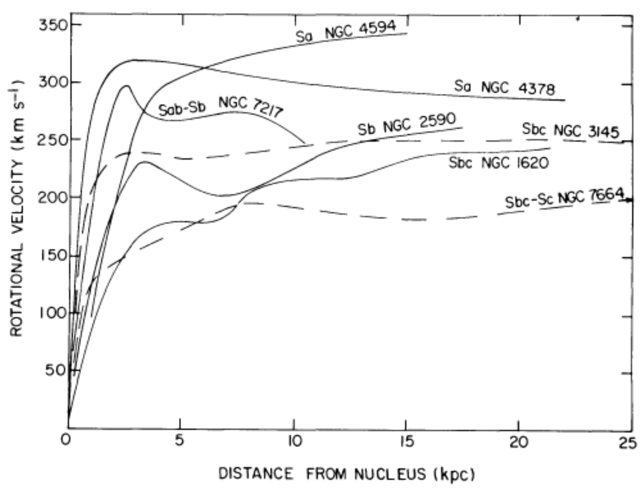


Figure 2: The rotation curves for a number of galaxies for which they were measured [2]. The differences between galaxies can be attributed to factors like their mass, structure and age. In general however, the pattern of an initial increase and roughly constant behaviour after that can be seen for all these galaxies, just as it can be seen that most rotation curves end up somewhere between 150 and 350 km/s.

[1] Speed refers to the magnitude of the velocity, and what we are interested in right now is specifically the magnitude of the angular component of the velocity of bodies in orbit. Therefore we will often refer to the orbital 'speed', unless a more directional analysis is required, as will be the case later.

We see an initial increase followed by the observation that for all galaxies in this figure, their orbital speed stays roughly constant after this phase. There are some differences between the galaxies due to factors like mass, structure and age[2], but generally they all show the roughly constant behaviour that is not in agreement with classical predictions. In addition, after the initial increase they all seem to display a constant orbital speed in the same range of somewhere between 150 and 350 km/s.

A reasonable first question would concern the basis of our expectation. Indeed, the aforementioned relation $v \propto \frac{1}{\sqrt{r}}$ presupposes a number of things that are not entirely correct. These are:

- There being two bodies, rather than the hundred million stars found in an average galaxy[3], all ordered in a specific way [3].
- Circular orbits.
- The adequacy of classical mechanics.

While these approximations do matter, they do not do so enough or in the exact way as to explain away the deviating expectation. Since this has the added benefit of explaining a number of important features of the rotation curves, we will comment on them in the same order:

- While the average amount of stars in a galaxy is closer to a hundred million than to just two bodies, their distribution also matters when considering the gravitational force on any given body. Galaxies are not ideal spheres of uniform density for which Gauss's law can be applied, but there are still structures that somewhat legitimise this approach. A galaxy can often be modelled as consisting of a relatively small central bulge and a large disk [4]. The fact that figure 2 shows an initial increase in orbital speed can be understood by considering the approximate high-density bulge. In this part of the graph, one can think of an object radially moving from the center of the bulge to its edge. Due to the approximate spherical symmetry of the bulge, one can apply Gauss' law for (Newtonian) gravity and Newton's shell theorem to make sense of the initial increase of the orbital speed as the distance from a galactic center increases. After this phase, while there will definitely remain significant deviations from this clean cut picture due to the only approximate spherical symmetry of the bulge and due to the mass in the disk also exerting gravitational force, we can classically still expect the orbital speed to decrease as we move away enough from the bulge mass and towards the outer areas of a galaxy. This is predicted to happen approximately with the inverse of the square root of the distance to the galactic center, a prediction also made by numerical computation based on observed galactic mass distributions [4]. Apparently, the visible mass distributions in galaxies are insufficient for explaining the behaviour of galactic rotation curves.
- While it is indeed true that orbits are rarely circular and tend to be more or less elliptical, most orbits are only of very small eccentricity. We will return to this fact to discuss its implications more thoroughly later in this thesis. In any case, this means that given the rough approximation we are dealing with, we should still expect a decreasing relationship as with circular orbits.
- General relativity is a more accurate theory than classical mechanics. Yet, relativistic effects will tend to show up in one of two cases, the first of which is nearing the speed of light. This is, as can be seen in figure 2, not at all the case. The second one is in the case of strong gravitational fields. However, on the length scales we are considering, the gravitational fields are generally not so strong as to necessitate a GR-treatment, especially in the outer galactic regions we are more concerned with [5]. The relativistic effect of frame dragging also turns out to be unable to explain galactic rotation curves [6].

[2]Or rather, the consequences thereof, such as having less elliptical orbits as we will explore later.
[3]Let alone sources of mass such as interstellar dust, black holes and planets.

In conclusion, since the expected pattern is theoretically legitimate, we are left with a discrepancy that cannot be explained by the standard model of physics. The orbital speed, which is expected to decrease like $v \propto \frac{1}{\sqrt{r}}$, instead seems to remain constant as distance increases. This calls for a different solution. We will study some of the major proposed solutions to this problem in the following subsections.

3.2 The dark matter approach

The problem of galactic rotation curves introduced in the previous subsection can be explained as a problem of 'missing (visible) mass'. The orbital speed of a body in a galaxy is not determined merely by its distance from the galactic center (or, for the sake of completeness, all other bodies in the system), but the other variable at play is the mass (and its distribution) pulling on the body. In fact, one can successfully construct rotation curves that resemble measured ones by adding a particular 'extra' mass distribution on top of the mass already observed to be present in the galaxy in question [4].

Thus, perhaps unsurprisingly, the most common solution to the problem of galactic rotation curves is that of so-called 'dark matter'. In very general terms, dark matter consists of mass that does not interact with light for us to detect, or at least it barely does so. Dark matter is usually subdivided into two categories: MACHOs (Massive Astrophysical Compact Halo Objects) and WIMPs (Weakly Interacting Massive Particles) [7]. Examples of MACHOs may be black holes, rogue planets or brown dwarfs. While all of these objects certainly exist and are known to emit little to no light, observations seem to indicate that these objects are not present to the degree that would enable them to explain the amount of 'missing mass' required for the discrepancy between the theoretical predictions and empirical reality of rotation curves to vanish [7]. That brings us to the WIMPs. Most research into dark matter, and indeed into the entire problem of galactic rotation curves, is focused on them [8]. As the name suggests, WIMPs are hypothetical massive particles that interact gravitationally and possibly with the weak nuclear force, but not (or extremely weakly) with the electromagnetic force and the strong nuclear force. These conditions make sense in light of what we are looking for. We need gravitational interactions in order to explain the central problem of galactic rotation curves. Weak interactions are acceptable, since they are too feeble to make dark matter visible [9]. We cannot, however, have electromagnetic interactions, or at least not to any noticeable degree. If this were the case, we could simply observe the light emitted by those particles, contradicting the meaning of the word 'dark' used to denote the concept of dark matter in the first place. Lastly we cannot have strong nuclear interactions since this would enable dark matter to be bound to a nucleus, which is contrary to our knowledge of matter. If this were to be the case, we would, for example, at least once in a while measure a hydrogen atom much heavier than a 'regular' one.

These hypothetical particles would, if found, be an addition to the standard model. WIMPs may be a new type of particle we have no idea about yet, but there is also research being done on right-handed neutrino's (referring to the particle's helicity). These neutrino's (also called 'sterile' neutrino's) would be much heavier than regular neutrino's, with energies in the order of keVs and higher [10]. Nevertheless, researchers are not yet fully sold on any one WIMP dark matter candidate.

Dark matter (most notably through WIMPs) has had some success in explaining all kinds of phenomena in the universe. This success is not limited to rotation curves alone. Let us list four examples in this context that provide evidence for the existence of dark matter. Firstly, the degree to which light is gravitationally bent when passing a galaxy implies the presence of much more mass than one would classically predict [11]. Secondly, this same gravitational lensing effect is observed for entire clusters of galaxies as well, showing that the effect can also be observed at greater length scales [12]. Thirdly, the motion of galaxies in clusters of galaxies exceeds the expected velocities based on the visible mass in the cluster. Like the rotation curve on the galactic scale, the analogous curves for intergalactic motion are also off with respect to classical predictions. Dark matter, however, could explain this [13].

Finally, the presence of more (and specifically configured) mass than is ordinarily assumed allows one to explain certain footprints left in the structure of the cosmic microwave background [14]. There are more examples we could list, but their existence understandably strengthens researchers' commitment to theories on dark matter, specifically WIMPs.
At the same time, dark matter is not (yet) the definitive solution to the problem of galactic rotation curves. There are at least three arguments for wanting to look into other hypotheses as well:

1. There are a number of issues where dark matter predictions do not correspond with observations or where other theories seem to do better. An example can be found in observations of dwarf galaxies formed by the ejection of baryonic matter from the tidal interaction or merger of two bigger galaxies. Under dark matter hypotheses you would expect a rotation curve more in line with those we would classically predict. However, those galaxies still seem to deviate from that, in the same way we have seen before [15]. The approach considered in the next subsection happens to be better suited to deal with this particular example.

2. After decades of intensive experimental research to directly or indirectly detect WIMPs, the search has so far been unsuccessful [8]. While this is not an argument against the existence of WIMPs per se, the absence of strong evidence for them after such a time frame at the very least should serve to legitimise the search for other possible explanations for the observed rotation curves. We are still outside of the domain of any notion of scientific certainty here.

3. While perhaps more of a philosophical argument, scientists *should* be engaging in different research programs. The existence of several parallel research programs could serve to stimulate the existence of a competitive, dynamic and critical scientific culture, which may in turn be more productive for reaching our goals. Moreover, the world's true physical constitution has continued to surprise us at every turn. It can hardly be the best course of action to put all of our eggs in one basket.

Dark matter is without a doubt a strong contender in the race to solve the problem of galactic rotation curves, and rightfully so. Yet, for the reasons given above, we will take a look at probably the second most popular approach in the next subsection.

3.3 The Modified Newtonian Dynamics approach

Another approach to try and solve the problem of galactic rotation curves is that of modified Newtonian dynamics (MOND). As the name implies, the strategy here is not to try and build upon our current understanding of physics by adding something new as is the case with dark matter, but rather modify our existing, supposedly well-established, understanding of physics.

MOND does this by modifying Newton's second law [16][17] such that:

$$F = ma \cdot \mu\left(\frac{a}{a_0}\right)$$

Here 'F', 'm' and 'a' are as usual, but 'μ' is a function of the real acceleration 'a' and a fitting parameter 'a_0'. Note that the Newtonian acceleration is still $\frac{F}{m}$, and consequently we can already expect that $\mu \to 1$ for everyday accelerations. However, when $\mu \neq 1$, it is not the quantity $\frac{F}{m}$ that yields the real acceleration, but the equation must first be solved for 'a' to find it[4]. That is, the quantity $\frac{F}{m}$ is still relevant for determining the acceleration but is not its sole determining factor.
There are a number of options for μ, but as an illustration of the method behind MOND we will continue by

[4] From here on out, I shall leave out the quotation marks. Mathematical and physical quantities will always be displayed in Italic.

using the so-called 'simple interpolating function' [17], where $\mu\left(\frac{a}{a_0}\right) = \frac{1}{1+\frac{a_0}{a}}$. With this it is possible to rewrite $F = ma \cdot \mu\left(\frac{a}{a_0}\right)$ as a quadratic equation and solve for a. This yields:

$$a = \frac{\left(\frac{F}{m} + \sqrt{\left(\frac{F}{m}\right)^2 + 4a_0\frac{F}{m}}\right)}{2}$$

Here the usual $\pm$ in front of the square root is taken positive as to agree with the boundary condition that we should find the regular $a = \frac{F}{m}$ at everyday accelerations. Note that a is the value the acceleration will take in reality, while the quantity $\frac{F}{m}$ is the Newtonian acceleration we had expected to be the correct one but, according to MOND, is in fact not the complete picture.

At this point it is important to mention that most MOND theories hypothesise that we start seeing the physics change at very low accelerations, which in practice means the fitting parameter a_0 will be around the $10^{-10}\frac{m}{s^2}$ order of magnitude. According to MOND's founder, Prof. Mordehai Milgrom (1946), classical mechanics has only been tested for much higher accelerations than this order of magnitude [16]. Such small accelerations are those that we tend to see further out in a galaxy. After all, approximating the orbits as circles (with constant orbital speed), we know the centripetal acceleration is proportional to the inverse of the distance to the galactic center. We also see this behavior in the expression of the acceleration for the simple interpolating function. Given the order of magnitude of a_0, and considering accelerations of order of magnitude $10^0\frac{m}{s^2}$ that we tend to see on a daily basis, we find that $a \approx \frac{F}{m}$, as we should. However, when the classical acceleration $\frac{F}{m}$ is very small compared to a_0 (such that $\frac{F}{m} \ll a_0$), the terms $\frac{F}{m}$ and $\left(\frac{F}{m}\right)^2$ can be neglected with respect to $\sqrt{4a_0\frac{F}{m}}$ and $4a_0\frac{F}{m}$ respectively. We then find that $a \approx \sqrt{a_0\frac{F}{m}}$.

Now, as we did before with the expected rotation curve, let us consider the situation of balance between the resulting centripetal and (Newtonian) gravitational force. Rewriting the approximate acceleration value for low accelerations given above in terms of the force, we find a resultant force of $F = m\frac{a^2}{a_0}$. In addition, we know that for circular orbits, the magnitude of the acceleration is equal to that of its centripetal component, and we can write $a = \frac{v^2}{r}$. Rearranging the terms in the equality $|\vec{F}_g| = F_{cf}$ then yields:

$$\frac{GMm}{r^2} = \frac{m}{a_0}\left(\frac{v^2}{r}\right)^2$$

which after rearranging produces:

$$v^4 = GMa_0$$

So, rather than $v = \sqrt{\frac{GM}{r}}$, we find an orbital speed *independent of the body's distance to the other body*. It can easily be seen how this, at least qualitatively, may result in a picture consistent with the observed rotation curves. In the inner parts of galaxies, where the acceleration is relatively large compared to a_0, we do not find this effect. However, as we move further away from the galactic center, where the acceleration becomes relatively small compared to a_0, the orbital speed becomes constant. This matches with empirical data, which suggests that the orbital speed of bodies in a galaxy remains roughly constant once we go further out.

While MOND's approach seems to make sense on a qualitative level and has seen some success under scrutiny[5],

[5]Sometimes even in areas where dark matter has a harder time, such as point 1 in the previous subsection [15].

it certainly has its own collection of problems. Many attempts to test MOND in practice do not support its predictions [18], it is unable to explain an array of other observed dynamical features [19][20] and it does not necessarily fit nicely when considering how it would carry over to general relativity [21]. In addition, dark matter has the advantage over MOND of being able to explain many other phenomena aside from just that of galactic rotation curves, such as gravitational lensing or certain footprints left in the cosmic microwave background. And even when MOND is applied to galaxies, there still seems to be 'missing mass' left. Given the many problems mentioned here, it is perhaps unsurprising that most physicists would bet their money on dark matter, even though it still has its own, albeit fewer, issues.

One may notice that above we looked at a very general MOND overview where the physics is to change at small accelerations. However, is it possible to construct a coherent MOND theory that changes the physics at *high* accelerations? In the next section, we will come back to this question.

3.4 The problem of the accelerated expansion of the universe

The previous sections concern the problem of galactic rotation curves and some common attempts to solve it. However, in popular media, this problem is often named in one breath with another (in)famous problem plaguing contemporary astrophysics: that of the accelerated expansion of the universe.

Classically, one would expect that gravity would eventually decrease the rate of expansion of the universe. Another theory would be that the expansion could be 'frozen', so-called 'contra-gravity'. Yet modern astrophysical observations indicate that the rate of the expansion of the present-day universe is in fact increasing, after having previously experienced a period of slowing down. The most important evidence for this idea is found by observing the light emitted by faraway galaxies. In general, when the source of a wave is moving away from or towards a certain point, this will, respectively, decrease or increase the frequency of the wave observed at the point. This is of course the famous Doppler effect, and astronomers use it all the time to infer all kinds of data from light. When a galaxy is moving away from us, we will see its spectrum as being shifted towards a lower frequency range, hence this is called 'redshifted'[6]. Similarly, the spectrum observed from a galaxy moving towards us is 'blueshifted'. Using these principles, astronomer Edwin Hubble showed in 1927 that galaxies are moving away from us [22].

However, more information is needed to conclude that this expansion is in fact accelerating. Since light has a finite speed and it therefore takes time for it to reach us, we know that the further away a galaxy is, the older the light we observe from it is. Yet, this does not tell us how to determine the distance of a galaxy, since we measure the light's luminosity. That means that we cannot infer just from the light itself whether a galaxy is dim but close by, or bright but farther away. Luckily, there exists an event that can effectively serve as an objective signpost, because its real brightness is more or less the same in all its occurrences. This event is a type Ia supernova. Because of this special property, these star-exploding events allow astronomers to figure out how far away galaxies in which they occur are from the apparent brightness of the light they emit [23]. We know, after all, how brightness dims over distance. Combining the Doppler shift and the stated proportionality between the distance of a galaxy and the time in the past at which the light we now see was emitted, it allows one to figure out an exact relationship between the speed at which a galaxy is moving away from us at different times in the history of the universe. In conclusion, it turns out that the older starlight is that reaches us, the more redshifted it is. In fact, we can infer that the corresponding rate of expansion of the universe is observed to be decreasing until roughly 5 billion years ago, when expansion rates seemed to have started accelerating again. In other words, contrary to expectation, galaxies are moving away from each other faster and faster as time goes by. Therefore, we infer that the universe is expanding and that it is doing so in an accelerated fashion at late times [24].

[6]As red is on the low frequency side of the visible part of the electromagnetic spectrum.

Now one might object that the universe 'expanding' does not explain how the distance between galaxies 'within' the universe gets larger. However, this criticism would be grounded upon a false interpretation of what it means to say that the 'universe is expanding'. Albeit understandable given that 'expansion' often refers to such a process in everyday life, it is not meant in the sense that the edges of the universe are expanding into something[7], but rather that the metric of space (or actually, spacetime) itself is changing. That is, space itself is expanding at each and every point in the universe. Without any of the fundamental forces holding matter together, even our bodies themselves would drift apart as the space between every organ, every cell, every molecule, grows larger. The rate of this expansion is completely negligible on the scale of human life. However, the cumulative effect of expanding space over the astronomical distances between galaxies is very much observable indeed.

Thus, as explained in this subsection, we once again find ourselves in a position of radical divergence between theoretical prediction and empirical reality. The question as to why the universe is expanding in an accelerated fashion is therefore similarly open to a lot of theorising. The mechanism behind the accelerated expansion is usually denoted as 'dark energy', and in the next subsection we will discuss it and its effects on the smallest of scales rather than the largest.

3.5 Dark energy and the cosmological constant

The nature of dark energy is still very much up for debate, more so than its dark matter counterpart. What is known is that dark energy is a form of energy that permeates the entire universe and, according to the Λ-CDM model, makes up about 68% of the total energy content of the universe[8] [22]. This form of energy is hypothesised to be responsible for the accelerated expansion of the universe through its property of exerting a 'negative pressure', pushing objects away rather than toward each other (such as gravity) [24].

Dark energy is most often (but not exclusively) described in terms of a 'cosmological constant'. For the introduction of the cosmological constant to make sense it is helpful to have a brief word about its origin in general relativity by Albert Einstein. Einstein's modified field equations can be written as:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

When considering the tensors, $R_{\mu\nu}$ expresses the Ricci curvature tensor, $g_{\mu\nu}$ expresses the metric tensor and $T_{\mu\nu}$ the stress-energy tensor. The Ricci curvature tensor says something about the degree some space differs from Euclidean space. The metric tensor contains all information about the way in which spacetime is curved. Lastly, the stress-energy tensor tells you all about the matter and energy present within a considered spatio-temporal domain. Aside from the tensors, two scalar quantities of interest in the equation are the scalar curvature R, which expresses the amount of curvature with respect to flat spacetime, and the cosmological constant 'Λ'. Einstein's field equations relate the matter and energy in a system to the curvature of spacetime. While the matter and energy make spacetime curve, this simultaneously changes their trajectories of motion as these quantities exist in spacetime.
When Einstein derived the field equations, they seemed to imply something he did not expect. Namely, that the universe is not static, unchanging in size. Since Einstein, at the time, was a proponent of the belief that the universe is static, he added a term to his field equations in order to arrive at this result. This is how the concept of a cosmological constant was born. The cosmological constant functioned as a measure of energy intrinsic to space itself. In other words: vacuum energy. After Hubble discovered that the universe is in fact expanding, Einstein abandoned the idea of the cosmological constant. However, in contemporary physics, the cosmological constant has returned.

[7] Which would supposedly not even make sense, since there is nothing for the universe to expand into.
[8] Compared to 27% dark matter and just 5% visible matter described by the standard model of particle physics.

This time it does not function as a means to make the universe static, but rather as a term representing dark energy's effect of accelerating the expansion of the universe [25].

The concept of the cosmological constant thus implies that all space has an intrinsic constant energy density. This means that as the universe expands, the total amount of vacuum energy increases. At the same time, the total amount of matter and non-vacuum energy remains the same. When we consider the density of matter in the universe as compared to the vacuum energy per unit volume, we find an enlightening tug of war between the two. When the universe was small, all matter was relatively close together. This meant that the density of matter in the universe was relatively large. However, as the universe expanded, this density dropped. Here, we are considering mostly the universe at large, rather than specific local pockets. When 'zooming-out', the universe can be modelled as a fluid that is isotropic and homogeneous, meaning that it has the same properties in all directions. This fact is called the 'cosmological principle'. In cosmology, the scale factor[9] S is important as it is a measure of how much the universe expands (or contracts). If we consider a simple measuring rod, a scale factor of $S = 2$ would double the rod's size. Since the expansion rate of the universe has changed over time, S can be expressed as a function of time: $S = S(t)$. Without expanding too much on the physics determining the scale factor here[10], we know that (cold) matter dilutes proportional to S^{-3} as space expands while radiation and relativistic matter do so as S^{-4}. This makes sense since there are three spatial dimensions. For radiation it even dilutes to the power of four since there is also the redshift component that has to be taken into account. So, with an ever-growing scale factor, the matter density of the universe continues to dilute, while the vacuum energy density is proportional to S^0. Being an intrinsic property of space, it stays constant. This all means that while the universe was expanding, at some point the energy density of matter dropped below that of the vacuum. This was an inflection point for the function $S(t)$, as rather than slowing down the initial expansion rate $\frac{dS}{dt}$ of the universe, it now began to increase. Because of the cosmological constant, it now seems that rather than slowing down to eventually go on to contract, the universe will continue to accelerate its expansion forever [26].

There is one more problem that arises for the cosmological constant, and it is known simply as the cosmological constant problem. It arises when one attempts to calculate the zero-point energy of the vacuum on the basis of quantum field theory. According to quantum field theory, vacuum energy can be modelled as an infinite collection of harmonic oscillators in the ground state, where the energy of the ground state is called the zero-point energy. When this is done for flat spacetime it is possible to do a calculation of how much vacuum energy there is, given that you introduce some cut-off that removes any quantum fields below a certain wavelength from the integration. In essence, you are differentiating between different domains of physics, in the sense that we know quantum field theory dominates the small world of particles and general relativity dominates the large world of astronomical objects. Note how both working with flat spacetime and cut-offs are approximations that neglect important features of the physical system. As long as the stress-energy tensor is not zero, which it is not since we are considering a system with energy present, there will be some degree of curvature of spacetime. At the same time, cut-offs are not well-established techniques in physics but rather an expression of our ignorance as to how to connect the quantum world with that of general relativity. The result of the calculation of the vacuum energy with these assumptions is a staggering disagreement between the theoretical prediction and what is actually observed. Concretely, the theoretical prediction is off by 120 orders of magnitude. By using renormalisation[11] one can somewhat ameliorate the situation, but a large discrepancy nevertheless remains. Given this result, the cosmological constant problem can be said to be tied to the problem of a lack of an adequate theory of quantum gravity. If there would be such a theory, the two aforementioned assumptions implicit in the calculation of the vacuum energy would presumably

[9] Usually the scale factor is denoted as a, but since we are already using this letter for the acceleration we will call it S here.

[10] For which we would need to invoke Friedmann's equation.

[11] Renormalisation is a technique that is used in quantum field theory to deal with infinities arising in calculations of quantum corrections by expressing observables in terms of other observables, essentially switching to a better eigenvector basis for performing perturbation theory.

no longer be necessary, and the discrepancy at the heart of the cosmological constant problem would vanish [27].

In this section, we have described the problem of galactic rotation curves, that of the accelerated expansion of the universe, the most commonly proposed solutions to these problems and the difficulties still remaining. In the next section we will use this information to propose a new approach with which to look at these problems.

4 A new approach: dark energy as an explanation for galactic rotation curves

Matter and energy curve spacetime. Yet when calculating the vacuum energy, we usually assume spacetime is always flat. We use perturbation theory under the assumption that flat spacetime is the unperturbed state. Therefore, it is not surprising that such an approach fails to deliver. This is the case because Λ is based on flat spacetime, but if you have Λ there cannot be flat spacetime [27]. It results into a kind of 'chicken or the egg' scenario. Instead, we should consider flat spacetime as having zero energy and then relate vacuum energy to the scalar curvature of spacetime such that $\Lambda = \Lambda(R)$, where Λ is again the cosmological constant and R the scalar curvature. For our intents and purposes all we need to know is that there is a positive relationship between Λ and R: increasing R means increasing Λ. That is, curved spacetime will contain more of this intrinsic vacuum energy we have discussed. This statement is the most important assumption in this thesis and behind the model that will be developed from this later. It follows that in places with a high matter and energy content, we find more dark energy effects, because due to spacetime being curved more in such regions the Λ-value will be higher. That is, the expansion rate of space is relatively high in such regions. In contrast, for flat space, $\Lambda = 0$.

We can relate the inclusion of the curvature effect described above to the problem of galactic rotation curves through ascribing both a primary and a secondary effect to it. This section will describe these effects on a very qualitative level, such that the central idea is explained. After this, quantitative effects of these ideas will be studied. The idea of the primary effect is as follows. Within the central bulge of a galaxy, where the density of matter and energy is high, there will be more curvature and thus more vacuum energy (since we established a positive relationship between Λ and R). This means the expansion rate of space is locally higher in such a place. Suppose an object has a roughly circular orbit within the central bulge. Now due to the increasingly high density the further you move to the center of the bulge, space is expanding relatively rapidly there. As more space is rapidly created[12], the body is slightly drifted (or 'pushed' if you will) away from the center. Since the bulge can be modelled as a spherically symmetric mass distribution, Newton's shell theorem tells us only that the mass inside this sphere exerts a force on the body in orbit. Thus, modelling this big inner mass as a point mass M and the orbiting mass as a much smaller mass m, we can intuitively see that the body will now need less orbital speed to stay in a stable orbit than you would expect on the basis of classical models without this dark energy effect. It is, after all, 'pushed outward' a little bit. Consequently, it does not fall towards the bigger mass if its speed is lowered somewhat compared to the situation without this 'outward push' effect, due to space being more rapidly added inside the body's orbit than outside of it. This effect of bodies in high matter density areas orbiting with slower orbital speeds is the primary effect of our established relationship between Λ and R.

However, this primary effect also causes a secondary effect. On the basis of the formula for the orbital speed in classical mechanics $v = \sqrt{\frac{GM}{r}}$ we can see that for a given distance between two bodies, a lower orbital speed implies a smaller value for the mass that is being orbited. Thus, the idea is that if we do not incorporate this dark energy effect into our equations, but rather just use Newtonian mechanics, we will underestimate the amount of mass contained in the galaxy. We determine the mass by measuring the orbital speed, and if this is lower than the mass in fact warrants (since we do not take the dark energy effect into account), our classical equations will mistakenly lead us to conclude that the amount of mass is smaller than it in fact is. Thus, as bodies move further away from the galactic center and out of the central bulge, the effect becomes clearer as the high density sources of mass causing space to expand more rapidly locally are more and more united within the body's orbit. Therefore, more outer-positioned orbits move in accordance to the true amount of mass hidden within a galaxy. The idea is that this amount of mass is much higher than we think and that it can explain the constant behaviour of rotation curves, just like dark matter provides additional mass that can potentially solve the problem. Yet, the crux of the model is that

[12] Where it is important to remember that what is meant by that is that the metric of spacetime is changing, and 'rapidly' should very much be interpreted as rapidly *relative to* lower density areas, and not rapid in some imagined absolute sense.

there is not actually an invisible form of matter that we have not yet discovered, but rather that the 'missing mass' has been there all along, we have just measured observable properties in a way which does not include that mass. Another argument that can be made on qualitative grounds is that this effect could potentially explain other observed phenomena such as the degree of gravitational lensing of light passing around a galaxy, normally attributed to dark matter. This idea of there being more mass than we think, and consequently the net extra attraction this causes on bodies in the galaxy, is the secondary effect of the relationship between Λ and R.

In this sense, the model can be viewed as a variant of MOND reproducing a conclusion one also reaches when working with dark matter. Namely, the idea that we have missed a substantial portion of mass in our deliberations on these problems, and that this can explain many observations we think to be problematic.
At the same time, the model clearly remains within the domain of modified Newtonian dynamics as it is effectively adding a new force to the classically established theory of gravitational dynamics. Yet it differentiates itself from most theories of MOND, as the (primary) effect manifests itself in the domain of high accelerations within the dense area of the galactic center, rather than for the low accelerations typically found very far out in a galaxy. If you want to express this in the mathematical terms of MOND as we have covered in section 3.3: $\mu(\frac{a}{a_0}) \to 1$, if $a \ll a_0$. Therefore, as compared to the large scale at which most MOND-theories apply, this version of it encompasses a more local MOND version, meaning one where we are looking at the galactic center where high accelerations are to be found. In addition, dark energy is the reason as to why this is the case.

The question then becomes whether a model containing the premises we have discussed in this subsection can come up with a self-consistent quantitative, empirically promising and on all distance domains viable description of galactic rotation curves. It might make some qualitative sense, but that leaves open the question as to whether it is actually possible for it to stand on its feet when placed under scrutiny. For this purpose, we will have to add a term to the Hamiltonian or Langrangian of a representative system in orbital mechanics. In the next section, we will repeat some of the most important and relevant parts of the theory of orbital mechanics, which then enables us to attempt and set up such a dark energy model in the subsequent section.

5 N-body problems in orbital mechanics

In this section, we will review the most important results of classical orbital mechanics. We will start with a general layout on N-body problems, as galaxies are enormously complex instances of such problems. After this, we will move on to describe the $N = 2$ case in more detail, as it remains a relevant case and approximation scheme while having the benefit of still having analytical solutions. Lastly, we will see how orbital mechanics can get us to expressions for the orbital speed as a function of distance between bodies, which is useful for rotation curves.

5.1 N-body problem

Suppose we have a system of N-bodies, all with kinetic energy and gravitational potential energy. The Lagrangian for one point-like body of mass m in such a system will have the form:

$$\mathscr{L} = \frac{1}{2} m \vec{v}^2 - U(\vec{r})$$

Here, $\vec{v}^2$ is the inner product of the body's velocity with itself and $\vec{r}$ is its position vector with respect to the origin. In cylindrical coordinates[13] the inner product of the body's velocity with itself can be written as $\vec{v}^2 = \dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2$. The dot denotes the time derivative of that quantity, two dots denote the second time derivative, and so on. As for the gravitational potential energy, we need to sum the potential for all other bodies in the system, such that

$$U(\vec{r}) = -Gm \sum_{i=1}^{N} \frac{m_i}{|\vec{r} - \vec{r}_i|}$$

where the m_i are the masses of the other bodies in the system and the $\vec{r}_i$ are the positions of the other bodies with respect to the origin.
Since we are expressing our terms in cylindrical coordinates, we can also do so for the vectors in the denominator. Writing it out in some steps, we eventually arrive at the following expression:

$$|\vec{r} - \vec{r}'| = \sqrt{\vec{r}^2 + (\vec{r}')^2 - 2\vec{r} \cdot \vec{r}'} = \sqrt{r^2 + z^2 + (r')^2 + (z')^2 - 2(xx' + yy' + zz')} =$$

$$\sqrt{r^2 + z^2 + (r')^2 + (z')^2 - 2(rr' \cos(\phi)\cos(\phi') + rr' \sin(\phi)\sin(\phi') + zz')} = \sqrt{r^2 + z^2 + (r')^2 + (z')^2 - 2(rr' \cos(\phi - \phi') + zz')}$$

Using the trigonometric identity $\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) = \cos(\alpha - \beta)$.
This means that so far, the Lagrangian for the considered body, as expressed in cylindrical coordinates, is:

$$\mathscr{L} = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) + Gm \sum_{i=1}^{N} \frac{m_i}{\sqrt{r^2 + z^2 + (r_i)^2 + (z_i)^2 - 2(rr_i \cos(\phi - \phi_i) + zz_i)}}$$

Next, we will put the problem in its galactic context. First, it is important to note that a galaxy like the Milky Way contains hundreds of billions of stars. That is a lot of bodies to sum over, and it does not even account for all the planets, interstellar gas or black holes in our galaxy. Luckily, it is also the case that many galaxies, to some extent, have symmetries in their mass distribution that can be taken advantage of. Finally, while within galaxies there is certainly a lot of empty space and bodies seem to be divided in a discrete manner, when zooming out and looking at the enormous structures called galaxies in their entirety, it can make sense to model them as continuous mass distributions. For these reasons, this is often done when people do calculations on galactic properties. One example of this is Camiel Pieterse, who in his bachelor thesis 'Galaxy Rotation Curves of a Galactic Mass distribution'

[13] It will soon become clear why this is a useful coordinate system to choose.

modelled our galaxy as a continuous mass distribution with a central bulge and two disks, both with cylindrical symmetry (and hence the cylindrical coordinate system). This mass density of the galaxy was taken as follows [4]:

$$\rho(r,z) = \frac{\rho_{b,0}}{\left(1+\frac{\sqrt{r^2+\left(\frac{z}{q}\right)^2}}{r_0}\right)^{\alpha}} e^{-\frac{r^2+\left(\frac{z}{q}\right)^2}{r_{cut}^2}} + \frac{\sigma_{d,0,1}}{2Z_{d,1}} e^{-\left(\frac{|z|}{Z_{d,1}}+\frac{r}{R_{d,1}}\right)} + \frac{\sigma_{d,0,2}}{2Z_{d,2}} e^{-\left(\frac{|z|}{Z_{d,2}}+\frac{r}{R_{d,2}}\right)}$$

where the first term constitutes the density of the central bulge and the other two that of the disks. As can be seen, the density model contains several (known) fitting parameters, and the free cylindrical coordinates r and z. The fitting parameters will not be given here as they will not be used in this thesis, but they are included for the sake of completeness.

It should however be noted that while this mass distribution is far more accurate than modelling the galaxy as a two-body problem, it still has its own fair share of problems. In reality, galaxies are not made up of perfectly symmetric shapes (bulge and disks), but there exist many asymmetries. One example would be the spiral arms of the Milky Way galaxy, but there are many other non-axisymmetric structures. The study of these asymmetries, that is, the morphological qualities of a galaxy, can further improve the quality of data. It is therefore an important object of astronomical research. Similarly, the galaxy is not actually a continuous mass distribution, however accurate such a model might be.

Given the continuous mass distribution, we can change the gravitational potential energy sum in the Lagrangian to an integral, by taking the limit $N \to \infty$ and $\Delta m_i \to 0$. We can then substitute $dm = \rho(r',z')d^3\vec{r}\,' = \rho(r',z')r'dr'dz'd\phi'$, given that we now denote the position of a mass element in the distribution as $\vec{r}\,'$. This leaves us with the following Lagrangian:

$$\mathscr{L}(r,\phi,z,\dot{r},\dot{\phi},\dot{z}) = \frac{1}{2}m(\dot{r}^2+r^2\dot{\phi}^2+\dot{z}^2)+Gm\int_{-\infty}^{\infty}dz'\int_0^{2\pi}d\phi'\int_0^{\infty}\frac{r'\rho(r',z')}{\sqrt{r^2+z^2+(r')^2+(z')^2-2(rr'\cos(\phi-\phi')+zz')}}dr'$$

where we integrate over all of space[14].

Due to the enormous complexity of this Lagrangian, it is not possible to find an analytical solution for the time evolution $\vec{r}(t)$ of the position of the body at position $\vec{r}$ when inserting it in the Lagrange equation[15]. Continuing from this point would demand numerical analysis, which, while very much helpful, will not be done here. Rather, we will now turn our attention to the much simpler two-body problem, which will provide us with an analytical solution.

5.2 The N = 2 case

The Lagrangian of a two-body system under the influence of a gravitational field can be written as follows:

$$\mathscr{L} = \frac{1}{2}m_1(\dot{\vec{r}}_1)^2 + \frac{1}{2}m_2(\dot{\vec{r}}_2)^2 + \frac{Gm_1m_2}{|\vec{r}_1-\vec{r}_2|}$$

[14]As it should be, we know that in the limit that r and z approach infinity, the density approaches zero. In practice, given that the many parameters in the density function are fitted to observational data, the density will quickly approach zero when taking $\vec{r}\,'$ outside the dimensions of the galaxy. So while mathematically correct, in reality the actual contributions to the integral are not coming from all of space.

[15]One would reach the same conclusion with the Hamiltonian formalism.

The problem, however, becomes much easier when introducing [28] relative coordinates $\vec{r} = \vec{r}_1 - \vec{r}_2$ and $\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}$, where $M := m_1 + m_2$. We also define the 'reduced mass'[16] $\mu := \frac{m_1 m_2}{M}$. Physically, $\vec{R}$ can be interpreted as the position of the center of mass of the two bodies, while $\vec{r}$ can be interpreted as the relative coordinate of which the absolute value is the distance between the two bodies. The Lagrangian can now be rewritten in terms of these coordinates such that:

$$\mathscr{L} = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 + \frac{GM\mu}{|\vec{r}|}$$

Since $\mathscr{L}$ does not depend on $\vec{R}$, we immediately notice that $\vec{R}$ is an ignorable coordinate. Therefore, $\dot{\vec{R}}$ is constant, meaning that the center of mass of the two bodies moves through space at a constant speed that becomes zero if you choose a coordinate system in which the center of mass is at rest. The interesting part of the motion therefore lies in the relative coordinate $\vec{r}$ between the two bodies. This motion lies in a plane ($z = 0$), and since we are doing *orbital* mechanics we can express $\vec{r}$ in terms of polar coordinates. The Lagrangian now becomes:

$$\mathscr{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{GM\mu}{r}$$

Since the Lagrangian is independent of ϕ, this coordinate is also ignorable. Because the physical consequence of that is a little more subtle than was the case with $\vec{R}$, it will be applied directly to the Lagrange equation. As a brief reminder: the Lagrange equation is a differential equation whose solution allows one to solve the system for its coordinates, and ideally express them as a function of time so as to know the position of the body in question at all times[17]. The Lagrange equation takes the form:

$$\frac{\partial\mathscr{L}}{\partial q} = \frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\dot{q}}\right)$$

where q denotes a generalised coordinate.
We can solve the Lagrangian for the ϕ coordinate. This yields:

$$\frac{d}{dt}(\mu r^2\dot{\phi}) = 0 \implies \mu r^2\dot{\phi} = l$$

The first equation states that the quantity $\mu r^2\dot{\phi}$ is constant, but this quantity can also be identified as the angular momentum of the system, thus it follows that the quantity is equal to l. Therefore, it simply states that angular momentum is conserved. We then know that $\dot{\phi} = \dfrac{l}{\mu r^2}$, which can be substituted into the Lagrangian. We now turn our attention to the more difficult radial equation, which results from applying the Lagrange equation to the r-component of the Lagrangian. This yields:

$$\mu\ddot{r} = \frac{l^2}{\mu r^3} - \frac{GM\mu}{r^2} = -\frac{d}{dr}\left(\frac{l^2}{2\mu r^2} - \frac{GM\mu}{r}\right)$$

which is effectively Newton's second law, both written out in terms of forces and in terms of its effective potential, showing that the net force is conservative. Unfortunately, however, this is a non-linear differential equation, without an analytical solution. Luckily, it can be solved if one aims to express r as a function of ϕ, which is, as will be seen, still a useful endeavour [28]. To solve the differential equation in this manner, we change the time derivative

[16] If $m_1 << m_2, \mu \approx m_1$ and $M \approx m_2$. Given that our model encompasses one body within a galaxy, these approximations effectively apply. Yet while it is good to realize this, the gained generality of using (μ, M) over (m_1, m_2) does not come with any real disadvantages, so we will continue to use it.

[17] Given that we know two initial conditions.

operator using the chain rule such that $\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \frac{l}{\mu r^2}\frac{d}{d\phi}$, where we utilise that $\dot{\phi} = \frac{l}{\mu r^2}$. In addition, we substitute $u := \frac{1}{r}$. Combining this substitution, the differential operator and the radial (force) equation, this yields:

$$\frac{d^2 u}{d\phi^2} = -u + \frac{GM\mu^2}{l^2}$$

which is the familiar differential equation of a harmonic oscillator with an extra constant term. Its solution is no less familiar:

$$u(\phi) = A\cos(\phi - \delta) + \frac{GM\mu^2}{l^2}$$

Since δ is merely a phase shift, the coordinate system can be directed in a way such that it is zero. The symbol A is a remaining integration constant. When r is substituted back in, this equation can be written as:

$$r(\phi) = \frac{c}{1 + \epsilon\cos(\phi)}$$

where $c := \frac{l^2}{GM\mu^2}$ and $\epsilon = Ac$.
This familiar result is the equation of an ellipse[18] in polar coordinates, and all standard geometrical knowledge of ellipses applies, such as the fact that the perihelion[19] can be written as $\frac{c}{1+\epsilon}$ and that the semi-major axis can be written as $\frac{c}{1-\epsilon^2}$. Notably, the geometrical interpretation of ϵ is the eccentricity of the elliptical orbit. The eccentricity contains information about the shape of the ellipse, specifically its elongation. As can be seen from the equation for $r(\phi)$, $\epsilon = 0$ corresponds to a circle of radius c, since it results in a constant separation between the bodies. When ϵ lies between 0 and 1 (such that $0 < \epsilon < 1$), the orbit is elliptical, becoming more elongated as ϵ approaches 1. The case of $\epsilon = 1$ results in a parabola and $\epsilon > 1$ in a hyperbola. These orbits are no longer bounded[20]. This can be seen from the fact that if $\epsilon \geq 1$ then $0 \in 1 + \epsilon\cos(\phi)$. In other words, as the denominator approaches zero, the distance between the bodies approaches infinity.

The expression for $r(\phi)$ consists of several different terms. In order to get as clear an understanding of the kind of orbits we are talking about and avoid any confusion, we will briefly go over them. When writing all terms out, these are: r, ϕ, G, M, μ, l and A. Here r and ϕ are our coordinates of interest, G is the universal gravitational constant and M and μ are constants, essentially representing the masses of the bodies. This leaves the angular momentum l and the constant of integration A. To determine this constant of integration, some initial condition is needed for the system. The angular momentum can similarly not be calculated on an a-priori basis. One more piece of information that will come in handy later can be derived. This can be done from the Hamilton-Jacobi equation, but it is also possible to do so from the fact that at the perihelion, the energy of the system is equal to its potential energy. After all, at this point the bodies are closest together, so they will reverse their relative motion, and the kinetic energy will then be zero at that point. Thus, using the expression for the total potential energy present in the radial equation, $E = \frac{l^2}{2\mu r_{min}^2} - \frac{GM\mu}{r_{min}}$, and knowing that $r_{min} = \frac{c}{1+\epsilon}$, we can, after writing out c, derive an expression relating the energy of the system to its angular momentum and eccentricity:

$$E = \frac{G^2 M^2 \mu^3}{2l^2}(\epsilon^2 - 1)$$

[18] As will be shown in a minute, this need not lay out an ellipse but will do so in the case that $0 < \epsilon < 1$. Yet since this is the orbital shape of most interest to us, we will mostly refer to it as such.

[19] The point within an elliptical orbit where the two bodies are closest together

[20] While bounded orbits correspond to negative energies, unbounded orbits correspond to positive energies. This makes sense, as in the former case the gravitational potential energy is greater than the kinetic energy, while this is the other way around for the latter case.

This again tells us something interesting. In the case of a circle ($\epsilon = 0$ and $\dot{\phi}$ constant) with a given fixed radius r_i, the energy and angular momentum of the system are determined, since $l = \mu r_i^2 \dot{\phi}$ shows that the angular momentum is fixed by the orbital radius and the equation above relates l to E. In the case of a parabola ($\epsilon = 1$), the energy is zero, which is also to be expected as this marks the point where the kinetic energy is equal to the negative of the potential energy. However, in the elliptical case, there seem to be, in some sense, two degrees of freedom to an orbit. While the energy expression above allows one to express the eccentricity as $\epsilon = \epsilon(E, l)$, knowing the eccentricity will, unlike in the circular and parabolic case, not determine the energy and/or angular momentum of the system. This makes sense, as elliptical orbits do not just have one fixed radius r_i, and many quite different elliptical orbits can take on the same inter-body distance r_i somewhere during their orbit.

After establishing the solutions to the equations of motion for the classical two-body problem, we now move on to a quantity of particular interest for our purposes: the orbital speed.

5.3 Rotation curves

In the end, we are interested in an expression for the orbital speed of a body as a function of the distance between two bodies. Concretely, such a function $v(r)$ will provide an expression for the rotation curves of galaxies, here modelled as two body problem of a star in the disk orbiting a bulge.
We can express the orbital speed in terms of the polar coordinates we worked in before as:

$$v = \sqrt{\dot{r}^2 + r^2\dot{\phi}^2}$$

where $\dot{r}$ is the speed in the radial direction and $r\dot{\phi}$ is the speed in the angular direction.
When also considering the direction of the motion, which follows from the orbital velocity vector, we can conclude that the magnitude of its angular component will clearly be much larger than the radial component. Yet, we are dealing with elliptical orbits, so for the sake of completeness we will look at the total orbital speed, including both the motion in the angular as well as the radial direction. We can now work out both of these terms before substituting them back in the above expression of the orbital speed.

To find $\dot{r}^2$, it stands to reason that we first take the time derivative of the expression $r(\phi)$ found in the last subsection. This yields:

$$\dot{r} = \frac{d}{dt}\left(\frac{c}{1+\epsilon\cos(\phi)}\right) = \frac{d\phi}{dt}\cdot\frac{d}{d\phi}\left(\frac{c}{1+\epsilon\cos(\phi)}\right) = \frac{l}{\mu r^2}\cdot\frac{-c}{(1+\epsilon\cos(\phi))^2}\cdot -\epsilon\sin(\phi) = \frac{lc\epsilon\sin(\phi)}{\mu r^2}\cdot\frac{r^2}{c^2} = \frac{l\epsilon\sin(\phi)}{c\mu}$$

where we use the chain rule, that $r(\phi) = \frac{c}{1+\epsilon\cos(\phi)}$ and what we learned from the ϕ-equation in the previous subsection, namely that $\mu r^2\dot{\phi} = l$.
Our next step will then be to square this result, which brings us to the following expression:

$$\dot{r}^2 = \left(\frac{l\epsilon\sin(\phi)}{c\mu}\right)^2 = \left(\frac{l\epsilon}{c\mu}\right)^2(1-\cos^2(\phi)) = \frac{l^2\epsilon^2}{c^2\mu^2}\left(1-\frac{1}{\epsilon^2}\left(\frac{c}{r}-1\right)^2\right) = \frac{l^2}{\mu^2c^2}(\epsilon^2-1) + \frac{2l^2}{\mu^2 c}\cdot\frac{1}{r} - \frac{l^2}{\mu^2}\cdot\frac{1}{r^2}$$

where we used the well-known identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and the function $r(\phi)$ rewritten in terms of the cosine such that $\cos(\phi) = \frac{1}{\epsilon}\left(\frac{c}{r}-1\right)$.

Having found our expression for $\dot{r}^2$, the term $r^2\dot{\phi}^2$ will be a breeze. We can express it as follows:

$$r^2\dot{\phi}^2 = r^2\left(\frac{l}{\mu r^2}\right)^2 = \frac{l^2}{\mu^2}\cdot\frac{1}{r^2}$$

where we once again used that $\mu r^2\dot{\phi} = l$.

Combining these two expressions to find an expression for the orbital speed as a function of the distance between the bodies, we get:

$$v(r) = \sqrt{\frac{l^2}{\mu^2 c^2}(\epsilon^2 - 1) + \frac{2l^2}{\mu^2 c}\cdot\frac{1}{r} - \frac{l^2}{\mu^2}\cdot\frac{1}{r^2} + \frac{l^2}{\mu^2}\cdot\frac{1}{r^2}} = \sqrt{\frac{l^2}{\mu^2 c^2}(\epsilon^2 - 1) + \frac{2l^2}{\mu^2 c}\cdot\frac{1}{r}}$$

To get to our final, more elegant and familiar expression, some substitutions will be helpful.
First of all, one may remember that the term $(\epsilon^2 - 1)$ is also found in the energy of the orbit $E = \frac{G^2M^2\mu^3}{2l^2}(\epsilon^2 - 1)$. This gets rid of the eccentricity in the expression. Secondly, we defined $c = \frac{l^2}{GM\mu^2}$. These substitutions yield:

$$v(r,E) = \sqrt{\frac{l^2}{\mu^2}\cdot\frac{G^2M^2\mu^4}{l^4}\cdot\frac{2l^2E}{G^2M^2\mu^3} + \frac{2l^2}{\mu^2}\cdot\frac{GM\mu^2}{l^2}\cdot\frac{1}{r}} = \sqrt{\frac{2E}{\mu} + \frac{2GM}{r}}$$

With this, we have found the orbital speed of elliptical orbits of the classical two-body problem. It does not just depend on r but also on the energy. This makes sense, as unlike with the circular case, even when we know a particular r_i-value is crossed during the ellipse's orbit, it would not be possible to infer the speed from just that. For example, it does not say anything about the eccentricity (related to the energy) of an orbit. Orbits of different eccentricity and speed might both find the inter-body distance to be a specific value r_i a few times during their orbit. Imagine, for instance, a large circular orbit. Inside this circle we place an ellipse whose two values r_{max} both touch the circle. Kepler's second law would dictate that the object following the elliptical orbit would move rather slow at this r-value, while this is not the case for the circular orbit. The motion of a given object following an elliptical orbit is almost never fully in the angular direction as would be with a circular orbit, but rather, the orbital velocity has an angular and radial component, the ratio of their magnitudes changing during the elliptical orbit.
Therefore, both the energy and distance between bodies are needed to express the orbital speed of a body in a two-body problem.

Lastly, we can do checks to see if this result is correct. The first of these would be that we expect the limiting condition to hold that $v \to 0$ as $r \to \infty$. That is, when the bodies become infinitely separated, their orbital speed approaches zero. While at first glance the energy term within the square root of the $v(r)$ expression seems to prevent this, one should remember that $E = \frac{G^2M^2\mu^3}{2l^2}(\epsilon^2 - 1)$ and $l = \mu r^2\dot{\phi}$. Consequently, $E = \frac{G^2M^2\mu}{2\dot{\phi}^2 r^2}(\epsilon^2 - 1) \propto \dot{\phi}^{-2} r^{-4}$. Here, the angular speed[21] has been left in the denominator for a reason. As the separation between any two considered bodies approaches infinity, the angular speed will simultaneously approach zero. Thus, to show that $v \to 0$ as $r \to \infty$ we must first come to the conclusion that the denominator $\dot{\phi}^2 r^4 \to \infty$ as we take the simultaneous limits $\dot{\phi} \to 0$ and $r \to \infty$. This is true due to the powers of the two variables at play. Since r^4 will grow more quickly than $\dot{\phi}^2$ shrinks, the limits will produce a product that approaches infinity rather than zero[22]. In conclusion, the orbital speed indeed approaches zero as the distance between two bodies approaches infinity.
Our second test is that the orbital speed should reduce to the familiar $v(r) = \sqrt{\frac{GM}{r}}$ if we consider, specifically, the circular case. The circular case, of course, corresponds to zero eccentricity, which means the energy term becomes $E = -\frac{G^2M^2\mu^3}{2l^2}$. At the same time, for a circle, the distance r between bodies is constant by definition. Remembering the expression for the r-equation in the last subsection, we saw that $\mu\ddot{r} = \frac{l^2}{\mu r^3} - \frac{GM\mu}{r^2}$. Since r is constant, $\ddot{r} = 0$, and

[21]It is specifically the magnitude of the angular velocity that is of importance here, thus it is referred to as 'angular speed' in this case.
[22]While it should be stated that this can be proven rigorously through the use of mathematical analysis, this will for brevity reasons not be done here.

we can rewrite this expression in terms of the angular momentum such that $l = \mu\sqrt{GMr}$. While seemingly at odds with the expectation that $l \propto r$ for circular orbits, this paradox can be resolved. This follows from the definition of angular momentum $\vec{l} = \vec{r} \times \vec{p}$ which, in the circular case, reduces to $\mu v r$ because of the right angle between the two vectors[23]. However, when combining this expression with the well-known relation $v = \sqrt{\frac{GM}{r}}$ that we gain from balancing the gravitational force with the resulting centripetal force in such a situation, we find that by substituting for v we get $l = \mu\sqrt{GMr}$, and we now understand where the counter-intuitive $l \propto \sqrt{r}$ in this situation comes from. Substituting this expression of the angular momentum into the one for the energy of a circular orbit, we find that $E_{circle} = -\frac{G^2M^2\mu^3}{2(\mu\sqrt{GMr})^2} = -\frac{GM\mu}{2r}$. When we substitute this back into the orbital speed $v(r, E)$ we find:

$$v_{circle}(r) = \sqrt{\frac{2}{\mu} \cdot \frac{-GM\mu}{2r} + \frac{2GM}{r}} = \sqrt{\frac{GM}{r}}$$

which is exactly what we know to be the case for a circle.

The above gives us all information on orbital mechanics that we will need in order to apply it to our dark energy model. In the next section, we will do this by means of introducing a modified Lagrangian in the context of orbital mechanics, accounting for the dark energy contribution.

[23]While technically the regular expression m should be used for the mass of the orbiting body here, this is more or less equivalent to μ in the particular domains we are looking at. After all, one mass is many order of magnitudes larger than the other, so we can for all intents and purposes just use the reduced mass here to avoid confusion in switching between m and μ.

6 The orbital mechanics of a dark energy Lagrangian

In this section we will attempt to build a model within the framework of orbital mechanics that includes an effective dark energy force. First, we will extensively focus on the two-body problem, after which we briefly turn our attention to its galactic N-body generalisation.

6.1 The contribution to the potential from the dark energy effect

In section 4, we mentioned how the higher expansion rate of space inside and around the central bulge could yield a new term in the Lagrangian. This term is not aimed to describe the aforementioned primary effect dark energy has on the galactic system, but rather the secondary effect whereby outer orbits are *effectively* subjected to a greater inward force than would be the case without this effect[24]. We now introduce our Ansatz for the additional potential in the Lagrangian as:

$$U_\Lambda(r) = -\beta\left(\frac{r}{r_0}\right)^{-\delta}$$

Here Λ is simply communicating the nature of the potential we are considering and β is a constant determining the strength of the effect. For Newtonian gravity, this would be $\frac{Gm_1m_2}{r_0}$. Since it is likely that the magnitude of the dark energy effect also depends on, among other factors, the masses of the bodies involved, β can be interpreted as a composite expression. We will return to this idea later.

The parameter δ is the power with which the potential related to the force falls off with distance. Since $U_\Lambda(r) \propto r^{-\delta}$, we know that $F_\Lambda(r) \propto r^{-(\delta+1)}$, and we can immediately state that the condition $0 < \delta + 1 < 2$ should hold for this parameter. If $\delta + 1 \leq 0$, this would defy the necessary condition that $F_\Lambda \to 0$ as $r \to \infty$, since the force between the bodies would grow infinitely large as the distance approaches infinity. On the other hand, if $\delta + 1 \geq 2$, the effect would fall off more rapidly than gravity itself and could not possibly dominate the dynamics of bodies in the outer parts of galaxies, as it should if it is to explain the structure of rotation curves.

Lastly, r_0 can be interpreted as a characteristic length, just as regular MOND has a fitting parameter (acceleration) a_0 in the function $\mu\left(\frac{a}{a_0}\right)$. However, it can be absorbed into the other constants. That is, in essence, $U_\Lambda(r) = ar^{-b}$, and only two initial conditions are needed to determine the potential. We will stick to r_0 out of convention.

6.2 The dark energy contribution in terms of force

In order to extend our toolbox, it may also be useful to know how this starting point for the potential can be expressed as a force. This can be done in various ways, some less obvious than others, however not less useful. First of all, one could just take the derivative of the potential given above. This yields:

$$F_\Lambda(r) = -\frac{d}{dr}U_\Lambda(r) = -\frac{\beta\delta}{r_0}\left(\frac{r}{r_0}\right)^{-(1+\delta)}$$

However, another possibility is to come to a force that corresponds to a logarithmic potential. A logarithmic potential yields a force inversely proportional to the distance, which is easy to work with. Yet, it would rather restrictive to introduce our Ansatz that way, as we do not know whether $F \propto r^{-1}$. We do not know how this effective dark energy force falls off with distance, so we may as well leave many options open. There is, however, a way in which we can still profit from the easiness of working with a logarithmic potential, while still being physically more realistic than

[24] Note that if we were to describe the primary effect here, whereby bodies in inner orbits are somewhat 'pushed outward' due to the locally more rapid expansion of space within their orbit, the sign of such a modelled force would have to be opposite to that of gravity. This is not the case for the secondary effect.

just taking the force to be proportional to the inverse of the distance between bodies. For this, we assume that δ is small, being very close to zero. We now first expand the potential as below:

$$U_\Lambda(r) = -\beta\left(\frac{r}{r_0}\right)^{-\delta} = -\beta e^{-\delta\ln(\frac{r}{r_0})} \approx -\beta\left(1-\delta\ln(\frac{r}{r_0})+\frac{1}{2}\delta^2\ln^2(\frac{r}{r_0})-\frac{1}{6}\delta^3\ln^3(\frac{r}{r_0})+\mathcal{O}(\delta^4)\right)$$

Now let us reflect on δ once more. If $\delta \leq 0$, that would correspond to a divergent potential as $r \to \infty$, which, as will be shown later, corresponds to an infinite orbital speed for bodies as $r \to \infty$. This would be physically absurd[25]. At the same time, if the model is to be successful, we do not expect δ to be all that close to 1. In that case, it would fall off almost as quickly as Newtonian gravity, even though we should expect this process to be significantly slower so as for the orbital speed to appear constant on a galactic scale. Therefore, it is much more likely that δ will turn out to be a positive but small number when applied to galactic rotation curves, which provides a strategical justification for our expansion. It is this characteristic that makes a logarithmic potential feasible. This also means higher orders of δ may be negligible. We now take the derivative of the potential to arrive at the force:

$$F_\Lambda(r) = -\frac{d}{dr}U_\Lambda(r) \approx -\frac{\beta\delta}{r}\left(1-\delta\ln\left(\frac{r}{r_0}\right)+\mathcal{O}(\delta^2)\right)$$

The reason we come to this expression of the force is that we can study the $\delta = 0$ case, which is effectively what we are looking at when taking the first term of the approximation and neglect all higher order terms of δ. Yet for the sake of a general starting point it makes more physical sense to arrive at this conclusion from our Ansatz, rather than to just take $F \propto r^{-1}$ as a bold starting point. Due to the small δ we could also study a logarithmic potential.

6.3 The two-body dark energy Lagrangian

Given the potential, we can now formulate a new Lagrangian:

$$\mathcal{L} = \mathcal{L}_{classical} + \mathcal{L}_\Lambda = \left(\frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{GM\mu}{r}\right) + \left(\beta\left(\frac{r}{r_0}\right)^{-\delta}\right)$$

We now rewrite the radial equation as given in the previous section, incorporating the dark energy term:

$$\mu\ddot{r} = \frac{-GM\mu}{r^2} + \frac{l^2}{\mu r^3} - \frac{\beta\delta}{r_0}\left(\frac{r}{r_0}\right)^{-(1+\delta)} = -\frac{d}{dr}\left(\frac{-GM\mu}{r} + \frac{l^2}{2\mu r^2} - \beta\left(\frac{r}{r_0}\right)^{-\delta}\right)$$

When it comes to solving this differential equation, we at least know that a solution is possible for the values $\delta = 1$ and $\delta = 2$. After all, the radial equation has a solution without the dark energy term ($\beta = 0$), and the cases $\delta = 1$ and $\delta = 2$ simply correspond with the powers of the other two terms in the equation. While we have established before that one can claim that δ must be smaller than these values, we will very briefly look at them as these will at least yield some analytical solution allowing us to see how the dark energy term influences the system in these cases. In particular, it is only for these values of δ that the method of u-substitution, used to solve the regular radial equation, yields the possibility for analytical solutions. To see this, we need only to remind ourselves that for $\delta = 1$ and $\delta = 2$ we can uniquely express the above differential equation as a harmonic oscillator differential equation.
Choosing $\delta = 1$ effectively serves to increase the strength of gravity, as the coefficient $-GM\mu$ now becomes $-GM\mu - \beta r_0$. This binds bodies more strongly. The effect is visible in the solution to the differential equation, which then becomes $r(\phi) = \frac{c_\beta}{1+\epsilon_\beta\cos(\phi)}$, with $c_\beta = \frac{l^2}{\mu(GM\mu+\beta r_0)}$ and $\epsilon_\beta = Ac_\beta$, where A again denotes the integration constant.

[25]For $\delta = 0$, the orbital speed will not diverge but will approach a constant as $r \to \infty$, which also does not make physical sense, as the orbital speed of two bodies should approach zero as their distance approaches infinity. There can be no interaction between them causing some orbital speed in this limit.

Here we see two effects. Firstly, relating to the eccentricity, we see that the denominator becomes larger, such that the eccentricity will be smaller. That is, the stronger the effect, the more circular an orbit becomes. Secondly, since we have seen before that $E \propto (\epsilon^2 - 1)$, this means that the energy of the system will decrease and thus the bodies will be more strongly bound to each other. This all is to be expected for a force aligned to gravity in direction and here chosen to be similar in its falling-off behavior (in this case being an inverse square law).
On the other hand, choosing $\delta = 2$ effectively serves to weaken the centrifugal force $\frac{l^2}{\mu r^3}$. The addition of the dark energy term leads to two different solutions. To see this, notice that the differential equation after u-substitution for $\delta = 2$ will take the following form:

$$\frac{d^2u}{d\phi^2} = -\left(1 - \frac{2\mu\beta r_0^2}{l^2}\right)u + \frac{GM\mu^2}{l^2}$$

Thus we can expect two solutions[26], one in the case $\frac{2\mu\beta r_0^2}{l^2} < 1$ and the other in the case $\frac{2\mu\beta r_0^2}{l^2} > 1$.
Solving the differential equation for the case $\frac{2\mu\beta r_0^2}{l^2} < 1$ then yields:

$$r(\phi) = \frac{c}{1 + \epsilon\cos\left(\phi\sqrt{1 - \frac{2\mu\beta r_0^2}{l^2}}\right)}$$

This would appear to 'mess up' the elliptical orbit. For example, the maxima formerly seen at $\cos(\phi) = 0$, i.e. $\phi = \pi(n + \frac{1}{2}), n \in \mathbb{Z}$, are now seen at $\phi = \frac{\pi(n+\frac{1}{2})}{\sqrt{1-\frac{2\mu\beta r_0^2}{l^2}}}$, which means that any orbiting body will not carve out the same elliptical line in space each period 2π. Rather, we will see maxima at different angles for consecutive periods of 2π.
The case $\frac{2\mu\beta r_0^2}{l^2} > 1$ will result in exponentials:

$$r(\phi) = \left(c_1 e^{\phi\sqrt{\frac{2\mu\beta r_0^2}{l^2} - 1}} + c_2 e^{-\phi\sqrt{\frac{2\mu\beta r_0^2}{l^2} - 1}} - \frac{GM\mu^2}{2\mu\beta r_0^2 - l^2}\right)^{-1}$$

where c_1, c_2 are constants of integration.
This would be a rather silly solution yielding unstable and unbounded orbits. We will not discuss it further as this solution by itself is not relevant for our current purposes, which now brings us to smaller values of δ.

6.4 On the possibility of analytical solutions by means of the integral method

As stated in section 6.2, the δ-values of physical interest lie in the range[27] $0 < \delta < 1$. These appear not to have easy solutions. To show this, let us try to cast the differential equation with the dark energy term in its u-form. Again substituting $u := \frac{1}{r}$ and using the fact that $\frac{d}{dt} = \frac{l}{\mu r^2}\frac{d}{d\phi}$ yields:

$$\frac{-l^2u^2}{\mu}\frac{d^2u}{d\phi^2} = \frac{l^2}{\mu}u^3 - GM\mu u^2 - \beta\delta r_0^\delta u^{1+\delta}$$

[26]Technically there are three solutions, as $\frac{2\mu\beta r_0^2}{l^2} = 1$ is not included here. This choice was made because this would be a trivially unrealistic scenario. It would cause uniformly accelerated motion for u as we increase ϕ leading to an exponential spiraling away movement between the bodies.
[27]While it was also argued that δ values closer to zero are likely to be more probable, we will start out more generally by accounting for all values at least physically conceivable for our model.

Rearranging:

$$\frac{d^2u}{d\phi^2} = -u + \frac{GM\mu^2}{l^2} + \frac{\beta\delta r_0^\delta \mu}{l^2} u^{\delta-1}$$

Crucially, we can see that the harmonic oscillator differential equation is lost when choosing our parameter such that $0 < \delta < 1$. Moreover, this range will necessitate that the dark energy term will have a negative power of u, since $(\delta - 1) \in (-1, 0)$. In conclusion, we are left with a second-order non-linear ordinary differential equation. Such equations tend not to have an analytical solution. Even for the most 'elegant' case of $\delta = \frac{1}{2}$, *Wolfram Alpha* does not produce a solution. The same holds true when we use the Taylor expansion of the force given in section 6.2 and neglect all δ terms, leaving only a r^{-1} term. This 'effective $\delta = 0$ term' also provides no analytical solution.

Another method that can be tried is to cast this differential equation in integral form. In order to do this, we first substitute $v := \frac{du}{d\phi}$, and then use the chain rule such that $\frac{d^2u}{d\phi^2} = \frac{dv}{d\phi} = \frac{dv}{du}\frac{du}{d\phi} = \frac{dv}{du}v$. We now substitute this expression for $\frac{d^2u}{d\phi^2}$ in our differential equation, which yields:

$$v\frac{dv}{du} = -u + \frac{GM\mu^2}{l^2} + \frac{\beta\delta r_0^\delta \mu}{l^2} u^{\delta-1}$$

This is a separable differential equation, so it can be integrated on both sides to yield[28]:

$$\frac{1}{2}v^2 + c_1 = -\frac{1}{2}u^2 + \frac{GM\mu^2}{l^2}u + \frac{\beta\mu r_0^\delta}{l^2}u^\delta$$

where c_1 is a constant of integration. Now we might think of a boundary condition to determine c_1, since its value is relevant for eventual solutions to the equation. One candidate is the condition that $v \to 0$ as $u \to 0$. This is true since $v = \frac{du}{d\phi} = \frac{d}{d\phi}\frac{1}{r} = -\frac{1}{r^2}\frac{dr}{d\phi} = -u^2\frac{dr}{d\phi}$. Since the term $\frac{dr}{d\phi}$ represents the change in the distance between the bodies as we vary ϕ by an infinitesimal amount, this must be finite for any continuous smooth elliptical curve, which we know must be the case since it is what we observe in reality. However, there is another layer to the argument. Taking the limit of $u \to 0$ amounts to taking the limit of $r \to \infty$, due to them being each other's inverse. Thus, physically, the limit $u \to 0$ is a standard boundary condition where we study the state of the system at infinite separation. As the distance between the bodies approaches infinity, the quantity $-\frac{1}{r^2}\frac{dr}{d\phi}$ as a whole will approach zero as well, regardless of the specific behaviour of $\frac{dr}{d\phi}$ in this limit[29]. This is the case due to a similar argument as seen in section 5.3. In the limit of $r \to \infty$, the quantity r^{-2} will due to its power shrink more rapidly than the quantity $\frac{dr}{d\phi}$ could hypothetically grow. Thus, if we look at the condition that $v \to 0$ as $u \to 0$ through a more physical lense by substituting the expressions for u and v in there, we come to the conclusion that on the basis of another infinite separation boundary condition argument we find that $-\frac{1}{r^2}\frac{dr}{d\phi} \to 0$ as $r \to \infty$, which we can now put to use to determine c_1. Looking back at the solution to our separable differential equation, we know that due to the condition that $0 < \delta < 1$, u^δ has a positive power and therefore will approach zero in the limit we are taking. Since this obviously also holds for the other u and v terms in the limit, this boundary condition leads us to conclude definitively that $c_1 = 0$.

The next step is to substitute u back for v. Since $v^2 = \left(\frac{du}{d\phi}\right)^2$, we can rearrange the equation such that:

$$\left(\frac{du}{d\phi}\right)^2 = -u^2 + \frac{2GM\mu^2}{l^2}u + \frac{2\beta\mu r_0^\delta}{l^2}u^\delta$$

[28]Note how the effective $\delta = 0$ term discussed above, resulting from neglecting all δ terms in the expansion of the effective dark energy force, would yield a logarithmic term as we integrate over u. This has the additional effect of not leading to $c_1 = 0$.

[29]While $\frac{dr}{d\phi}$ by itself could conceptually be argued to approach zero as we approach infinite separation, the point is that the argument holds even without this assumption.

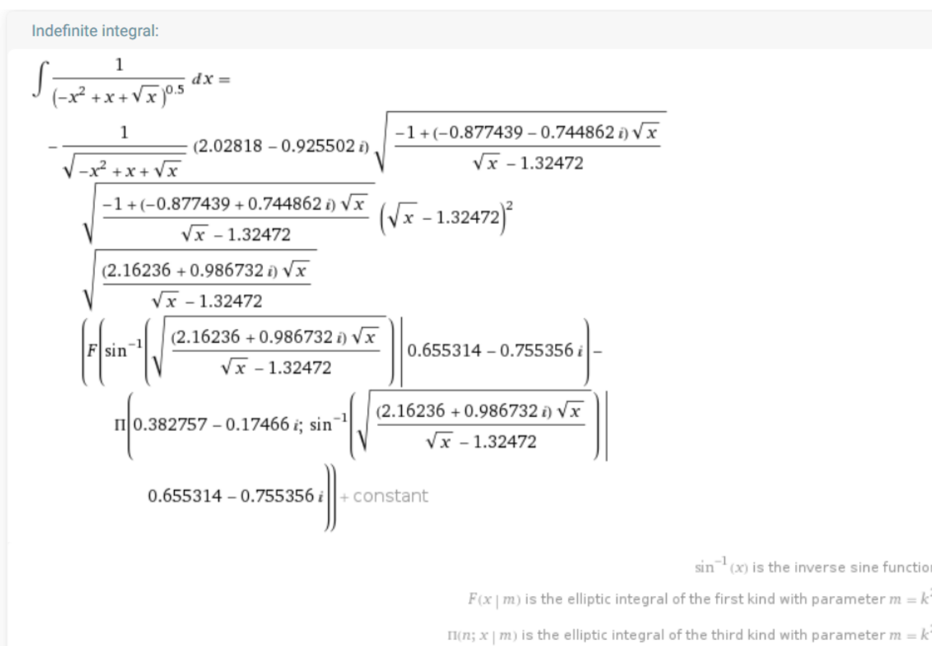
After taking the square root of both sides we again find a separable differential equation that can be written as:

$$\phi + \phi_0 = \int \left(-u^2 + \frac{2GM\mu^2}{l^2} u + \frac{2\beta\mu r_0^\delta}{l^2} u^\delta \right)^{-\frac{1}{2}} du$$

Here ϕ_0 is a constant of integration, that can be set to zero, since it is a phase shift that can be removed by orienting the coordinate system in a specific way.
Alternatively, it is possible to use the aforementioned equation $\frac{du}{d\phi} = -u^2 \frac{dr}{d\phi}$ and the definition $u := \frac{1}{r}$, to then go on and apply the same trick as for the previous integral in order to express it in terms of r:

$$\phi = \int \left(-r^2 + \frac{2GM\mu^2}{l^2} r^3 + \frac{2\beta\mu r_0^\delta}{l^2} r^{4-\delta} \right)^{-\frac{1}{2}} dr$$

In theory, the two above integrals give us a way to express ϕ as a function of r or u, after which these could ideally be rewritten as $r = r(\phi)$.
This has been tested by evaluating the integrals for the values $\delta = \frac{1}{10}, \frac{1}{5}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, since these are the most reasonable values within the domain of δ for which we can expect a solution.
After entering the integrals in *Wolfram Alpha*, one finds that solutions do exist in the case $\delta = \frac{1}{2}$, as mathematical intuition might already have judged to be the most probable value. However, the resulting functions are rather messy, as can be seen in the figures below which portray both the u-integral and r-integral for $\delta = \frac{1}{2}$.


Indefinite integral:

$$\int \frac{1}{(-x^2+x+\sqrt{x})^{0.5}}\, dx = -\frac{1}{\sqrt{-x^2+x+\sqrt{x}}} (2.02818 - 0.925502\, i) \sqrt{\frac{-1+(-0.877439-0.744862\, i)\sqrt{x}}{\sqrt{x}-1.32472}} \sqrt{\frac{-1+(-0.877439+0.744862\, i)\sqrt{x}}{\sqrt{x}-1.32472}} \left(\sqrt{x}-1.32472\right)^2 \sqrt{\frac{(2.16236+0.986732\, i)\sqrt{x}}{\sqrt{x}-1.32472}} \left(F\left(\sin^{-1}\left(\sqrt{\frac{(2.16236+0.986732\, i)\sqrt{x}}{\sqrt{x}-1.32472}}\right) \middle|\, 0.655314-0.755356\, i\right) - \Pi\left(0.382757-0.17466\, i;\, \sin^{-1}\left(\sqrt{\frac{(2.16236+0.986732\, i)\sqrt{x}}{\sqrt{x}-1.32472}}\right) \middle|\, 0.655314-0.755356\, i\right)\right) + \text{constant}$$

$\sin^{-1}(x)$ is the inverse sine function

$F(x \mid m)$ is the elliptic integral of the first kind with parameter $m = k^2$

$\Pi(n; x \mid m)$ is the elliptic integral of the third kind with parameter $m = k^2$

Figure 3: The u-integral for $\delta = \frac{1}{2}$. The integral was performed by Wolfram Alpha. *The resulting expression is rather impractical and cannot be rewritten as $r = r(\phi)$ (or even $u = u(\phi)$).*

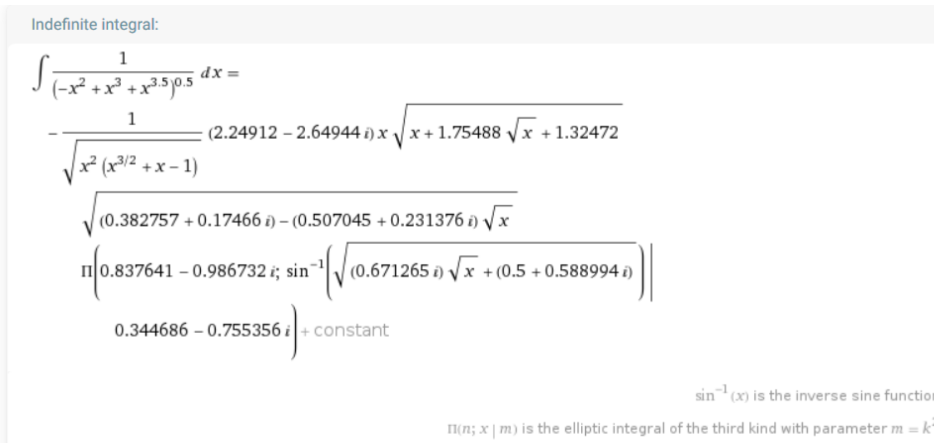


Figure 4: The r-integral for $\delta = \frac{1}{2}$. The integral was performed by Wolfram Alpha*. The resulting expression is rather impractical and cannot be rewritten as $r = r(\phi)$.*

Apart from being messy, these results are also complex-numbered[30], contain a Π-function (which is an elliptic integral of the third kind [29]) and due to their size are impractical to work with. Lastly and perhaps most importantly, they cannot be cast in the form $r = r(\phi)$. Alternatively, one could Taylor expand the function in the integral. This attempt has been made, but since a lot of terms are needed for an accurate approximation of the function, the same criticisms of impracticality and inability to express the result in terms of $r = r(\phi)$ remain, in addition to the fact that expanding this function to the large degree necessary could be done much faster and more accurately by a computer, at which point one might as well use numerical analysis directly on the integral anyway. In conclusion, while the above integrals can be correctly used, this should be done numerically. While this leaves us without a clean analytical solution to our problem, it does provide an opportunity for further numerical analysis of our model[31].

6.5 The circular approximation

As found in the last section, there is no analytical solution to the two-body problem when including the Ansatz for the effective dark energy force in the Lagrangian. This brings us to another, surprisingly legitimate, approximation, namely the circular approximation. As the name implies, we model galactic orbits as circles rather than ellipses. The reason for the legitimacy of this approximation can be found in the following plot:

[30] It is however possible to find a real number from the integral, the root just needs to be positive such that (considering the u-integral) $\frac{2GM\mu^2}{l^2}u + \frac{2\mu\beta\sqrt{r_0}}{l^2}\sqrt{u} > u^2$. Since the coefficients in front of the positive terms are large, this condition will hold well into a range where u is so large that the integrand's contribution becomes negligible, and the integral from 0 to the u-value that is the solution of the equation $\frac{2GM\mu^2}{l^2}u + \frac{2\mu\beta\sqrt{r_0}}{l^2}\sqrt{u} = u^2$ can be approximated as the integral from 0 to ∞. However, we are in need of the primitive of the function, not a numerical value.

[31] It should be noted that the same integral method has also been applied to the Taylor expanded dark energy force introduced in section 6.2. However, this leaves one with a logarithmic term in the u- and r-integral, which also does not produce an analytical solution. In addition, the aforementioned limiting condition that $v \to 0$ as $u \to 0$ will cause the logarithmic term to blow up such that the integration constant c_1 remains a factor in the subsequent u- and r-integral.

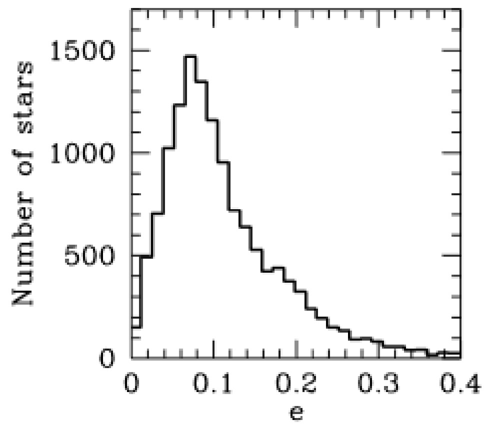


Figure 5: Distribution of eccentricity values of orbits in a data set of the entire Milky Way galaxy [30].

In this plot of the distribution of eccentricity values of orbits in the Milky Way galaxy, it can clearly be seen that lower eccentricities dominate. The most probable eccentricity as well as the average eccentricity can be seen to be $\epsilon \approx 0.1$. Moreover, most higher-eccentricity orbits can be found in old stars in the central bulge [31], which are generally not the orbits we are most interested in when considering galactic rotation curves anyway.
This finding makes sense, as very non-circular orbits would over time lead to collisions and dissipation of energy, such that the disks settle increasingly close to circular orbits.
It is also important to note that the approximation is based on more than just stating the eccentricity to be close to zero. We should remember that the ratio of the semi-minor axis b to the semi-major axis a of an ellipse can be expressed in terms of the eccentricity as follows:

$$\frac{b}{a} = \sqrt{1-\epsilon^2}$$

Thus, this ratio, which is of course 1 for circles, will be much closer to 1 than just $(1-\epsilon)$. Namely, $\frac{b}{a} = \sqrt{1-0.1^2} \approx 0.995$. This is extremely close to the circular ratio of 1, and therefore the given distribution can legitimately be said to warrant a circular approximation.

From section 6.3, we know that we can express the radial equation as:

$$\mu\ddot{r} = \frac{-GM\mu}{r^2} + \frac{l^2}{\mu r^3} - \frac{\beta\delta}{r_0}\left(\frac{r}{r_0}\right)^{-(1+\delta)} = -\frac{d}{dr}\left(\frac{-GM\mu}{r} + \frac{l^2}{2\mu r^2} - \beta\left(\frac{r}{r_0}\right)^{-\delta}\right)$$

However, since we are now working with circular orbits, by definition $\ddot{r} = 0$. That means, just considering the force part of the differential equation and rearranging slightly:

$$-GM\mu^2 r + l^2 - \beta\mu\delta r_0^{\delta} r^{2-\delta} = 0$$

While this equation still does not seem to grant an expression for the distance between bodies[32] we must take a step back and remember that we ultimately want to know the effect of the dark energy term on the predicted rotation

[32] It does so for $\delta = 1$, however as established before $\delta \in (0,1)$. That is, 0 and 1 themselves are excluded. One might be temped to approximate $\delta \approx 0$, but since δ is part of the constant in front of the $r^{2-\delta}$ term, this would just make the dark energy effect disappear from the equation and we would get the radius of a classical circular orbit, which is not what we are looking for.

curves. In other words, we are interested in an expression for $v(r)$, where v refers to the orbital speed of a body relative to an origin (in this thesis often the center of a galaxy) while the scalar r refers to the distance between two bodies (see section 5). The expression given above does not yield a direct expression for r but as we will see later it will come in handy when we try to determine an expression for the orbital speed of a body in the galaxy.
However, before we continue this discussion, we lastly turn our attention to a more complete presentation of our model, without the rough two-body approximation.

6.6 The N-body generalisation of the dark energy Lagrangian

In section 5.1, we derived the Lagrangian of one body within a galaxy. For the sake of completeness, we will now proceed to incorporate the term resulting from the secondary dark energy effect into this study.
We know that $U_\Lambda(r) = -\beta\left(\frac{r}{r_0}\right)^{-\delta}$ describes the potential energy between two bodies due to the force resulting from the secondary effect of dark energy. However, galaxies are large N-body problems, not two-body problems. This leads to the following sum:

$$U_\Lambda(r) = -r_0^\delta \sum_{i=1}^{N} \frac{\beta_i}{(r^2+z^2+(r_i)^2+(z_i)^2-2(rr_i\cos(\phi-\phi_i)+zz_i))^{\frac{1}{2}\delta}}$$

where the denominator represents the quantity $|\vec{r}-\vec{r}_i|^\delta$ in polar coordinates, as established in section 5.1.

When we introduced β we noted how it is composed of a number of different quantities. At this point, it is important to distinguish the relative or system-dependent quantities making up β from the universal or system-independent quantities. With universal quantities I mean to refer to the quantities that are the same for any two bodies, such as the fundamental strength constant of the force. Relative quantities, on the other hand, refer to factors such as the mass of the source body we are considering the effect of. 'Source body' here refers to one of the bodies we are summing over. This very much differs per situation. As an example, the gravitational force strength is determined by Gm_1m_2, where G is a universal quantity while m_1 and m_2 are relative quantities.
Knowing this, we can infer that the mass of the source body is an important relative quantity since the mass of a body is instrumental in the determination of curvature, and thereby the effect we consider. There may well be other relevant relative quantities one may think of, such as the mass and exposed surface area of the body whose motion we are considering, but also the volume of a given source body. However, given that we are summing the potentials the considered body is subject to as caused by many different source bodies, these latter quantities will be common to all terms, since the considered body is always a part of it. Moreover, we will be modelling the galaxy as a continuous mass distribution, turning the sum over all kinds of differently shaped discrete objects into an integral over a 'fluid' with varying mass-density. This, in approximation, justifies just 'extracting' the mass of the source body as a term from β_i and leaving all else in a β factor that is common to each term in the sum. A more in-depth discussion on the identity of β can be found in section 9.
In conclusion, this means that we will now substitute $\beta_i \to \beta m_i$, where m_i represents the mass of a randomly considered source body in a galaxy[33].

We now model the galaxy as a continuous mass distribution and we can replace the sum by an integral with cylindrical infinitesimals $d^3\vec{r'} = r'dr'd\phi'dz'$ and integrate over all of space. The total Lagrangian now becomes:

$$\mathscr{L}(r,\phi,z,\dot{r},\dot{\phi},\dot{z}) = \frac{1}{2}m(\dot{r}^2+r^2\dot{\phi}^2+\dot{z}^2)+Gm\int_{-\infty}^{\infty}dz'\int_0^{2\pi}d\phi'\int_0^{\infty}\frac{r'\rho(r',z')}{\sqrt{r^2+z^2+(r')^2+(z')^2-2(rr'\cos(\phi-\phi')+zz')}}dr'$$

[33] In reality this substitution presupposes a direct proportionality $\beta \propto m_i$ between β and m_i. However, even if this is not the case, the same method can be used to get to a result. Different relationships between β and m_i will be covered in section 9

$$+\beta r_0^\delta \int_{-\infty}^{\infty} dz' \int_0^{2\pi} d\phi' \int_0^{\infty} \frac{r'\rho(r',z')}{(r^2+z^2+(r')^2+(z')^2-2(rr'\cos(\phi-\phi')+zz'))^{\frac{1}{2}\delta}} dr'$$

The above Lagrangian then provides a way to do more accurate research on the effect of the addition of the secondary dark energy force through numerical analysis. While this method will not be pursued in this thesis, it is included here to show, for possible further research, a more accurate and less approximation-heavy model than the one we go on to work with.

7 How rotation curves come to be

This section will focus on two things. First, we will derive an expression for the orbital speed as a function of the distance between bodies on the basis of the two-body problem with the effective dark energy force. Secondly, we will take a brief step back to look at how the rotation curve of our galaxy is found in practice. With these two pieces of knowledge, we will then be able to apply our model to the real data for the rotation curve of the Milky Way galaxy.

7.1 Deriving the orbital speed for the dark energy model

To find a relation between the orbital speed of a body in the galaxy and the distance between the body and the bulge center, we can use tricks similar to those used in section 5.3.
First of all, we note that for the circular orbits we work with, the orbital speed only has an angular component ($\dot{r} = 0$ after all). Thus, $v = \sqrt{\dot{r}^2 + r^2\dot{\phi}^2} = r\dot{\phi}$. Invoking once more that $l = \mu r^2\dot{\phi}$, we find that:

$$v = \frac{l}{\mu r}$$

However, this is not where we want to end up, as the angular momentum can also be identified. As before, we can find it by considering the differential equation, now extended with the dark energy term. For this, we should pick up on the cliffhanger from section 6.5, where we stated that for circular orbits:

$$-GM\mu^2 r + l^2 - \beta\mu\delta r_0^\delta r^{2-\delta} = 0$$

Rewriting this in terms of the angular momentum we find:

$$l = \sqrt{GM\mu^2 r + \beta\mu\delta r_0^\delta r^{2-\delta}}$$

Finally, substituting the angular momentum into the expression for the orbital speed, we find that:

$$v(r) = \sqrt{\frac{GM}{r} + \frac{\beta\delta r_0^\delta}{\mu} \cdot \frac{1}{r^\delta}}$$

As can be seen, the dark energy effect essentially appears as an addition to the classical term $\frac{GM}{r}$ in the square root. Before analysing this result, one should note that the same result could be derived in perhaps a more direct way, by simply writing out Newton's second law such that $\frac{GM\mu}{r^2} + \frac{\beta\delta r_0^\delta}{r^{1+\delta}} = \frac{\mu v^2}{r}$ and rearranging in terms of v.

The result, at least on a qualitative level, seems to be in line with what is expected. It has been stated before that δ is likely to be a small number on the lower end of the range $0 < \delta < 1$. Consequently, the second term will be one to fall off with distance in a much slower fashion than the classical term does. The coefficient of the classical term is likely to be bigger, but when divided by an r typical for the length scales found in the galaxy this term will become very small. The curve will, when focusing on the galactic domain, appear to be constant due to the small power of the second term. Finally, we note that we should eventually find that v becomes roughly constant at an orbital speed of approximately a few hundred kilometers per second. This fact is common to all observed galaxies [2].

While this is a good sign at a qualitative level, it remains to be seen whether the predictions of such a formula agree with empirical data to an acceptable degree. We will explore this question in the next section, but first we shall briefly dive into 'the making of' rotation curves.

7.2 How rotation curves are set up in reality

The process of setting up rotation curves might quickly raise two important questions that we will address in this subsection:

1. How does one measure the distance from the center and the orbital speed of a body at a certain position in a galaxy?

2. Orbits of bodies in a galaxy are never *perfect* circles. As stated before, most have an eccentric orbit with $\epsilon \approx 0.1$. Moreover, as can be seen in figure 5, there are just as well bodies with higher eccentricities. Rotation curves on the other hand seem at first to imply that for some definite r-value, we can assign a definite v-value, as if every body is a perfect circle with constant r and v. Yet, if some r-value is an element of the orbits of many objects with different eccentricities, what orbital speed do we assign to it? Or, alternatively posed, if I am studying a particular orbit, which r and v value do I give it in a rotation curve, if neither of these values are not constant throughout the body's orbit? The question thus becomes how we construct accurate rotation curves given the elliptical nature of the orbits.

7.2.1 How astronomers measure distances and orbital speeds in galaxies

Let us start with the first question. In this subsection, we will start by explaining how we can, firstly, determine the orbital speed of bodies in other galaxies and then, secondly, how we can determine their position relative to the center of their galaxy. After this, we will look specifically at how to do this in our Milky Way galaxy, where being a part of the same galaxy makes some aspects of this process a little more difficult.

We have already discussed in section 3.4 how one can measure the distance from us to other galaxies using standard candles such as type Ia supernovas. How the orbital speed of a body within another galaxy is determined however, is a different question. This is usually done using spectroscopical techniques. More specifically, one studies the spectral emission lines of neutral hydrogen. A typical considered wavelength is the HI spectral line (also called the 21 cm line, corresponding to its associated wavelength), created in a common and specific hydrogen energy state transition. One big advantage here is the abundance of neutral hydrogen within galaxies, enabling researchers to detect such emissions throughout them. Since we know the emission spectrum of hydrogen, we can compare that to the spectrum we measure being emitted by an object in a galaxy. Here, the Doppler effect[34] is relevant to us again. When the object is moving away from us, the frequency at which we observe the electromagnetic radiation will be slightly lower, just as with the pitch of an ambulance whose siren has just passed you. The converse happens when a body is moving towards us. It is therefore possible to measure quantitatively how much observed spectral lines are shifted compared to those of stationary hydrogen emissions. Once you know this, it is possible to infer the speed of bodies in another galaxy, using the well-known equations describing the relativistic Doppler effect [32][33].

Two nuances must be added to this explanation. First of all, in naming only the spectroscopical technique we are implicitly assuming that we are looking at another galaxy with a convenient side-view, from which stars orbiting in that galaxy are always moving away or towards us. In other words, the plane of the galaxy is parallel to our line of sight. However, galaxies can be tilted in all kinds of ways, and in the most extreme case the galaxy might be tilted fully perpendicular with respect to us. Since the motion of the bodies in there then exists in a plane perpendicular to us, we cannot use the aforementioned technique as the bodies are not moving towards or away from us. Given that there are about *two trillion* galaxies[34] in the observable universe, however, the easiest way to determine orbital speeds within them is to pick mainly those with a relatively clear side-view.

[34] In this case, we should actually refer to it as the relativistic Doppler effect, since as opposed to sound waves, light travels at the speed of light. This makes it so that relativistic effects come into play when considering motion relative to this.

The second nuance when considering other galaxies is that not only the stars within a galaxy may, for example, be moving away relative to us, but as we have discussed in section 3.4, the entire galaxy likely is. We can however compensate for this by measuring the orbital speed of a star moving towards us on the other side of that galaxy. To make this concrete through an example, let us consider a galaxy of which, for the sake of clarity, we have a side-view. If we measure a star on one side of the galaxy moving towards us to have a speed (parallel to our line of sight) of v_-, and another star on exactly the other side moving away from us to have a speed v_+, we know that these both must contain a term V representing the speed at which the entire galaxy is moving away from us. However, let us call the orbital speed with that these two stars move with respect to the center of mass of their own galaxy v_T. This means that $v_- = V - v_T$ and $v_+ = V + v_T$, from where it follows that $V = \frac{v_+ + v_-}{2}$ and $v_T = \frac{v_+ - v_-}{2}$. Knowing this, it is possible to subtract V from measurements and finally infer the orbital speed of bodies in different galaxies. A final nuance within this nuance is that this example is non-relativistic. It will work for galaxies comparatively close by, such as the Andromeda galaxy. However, galaxies much farther away will move away from us with large velocities increasingly close to the speed of light. At that point, it is no longer possible to just subtract velocities like in the non-relativistic example above, and the equations from special relativity just be used.

That may explain how we arrive at orbital speeds, but it is also necessary to know how far a particular body in a galaxy is removed from its center. This is, at least conceptually, a simpler process. The diameter of that galaxy can be determined using simple geometry by measuring the distances from the edges of the galaxy to us (for example by using standard candles) and then using the angle between these two observed points. One can then choose at what distance from that galaxy's center you want to study a particular body and measure its Doppler shift. This yields both the values of r and v needed to construct a rotation curve.

For the Milky Way galaxy, this is a bit more difficult as we are part of this galaxy ourselves. Let us first consider how to determine the distance of objects from us. For this purpose, parallax techniques are the most direct. We then observe how much a star changes apparent position as the Earth orbits the sun, and from that small angle (in the order of arcseconds) use basic trigonometric arguments to determine the distance of a star in the Milky Way to us. However, parallax measurements only work when studying distances up to about 400 lightyears from Earth, as the shifts become too small to measure [35]. At that point, astronomers often measure the emitted electromagnetic spectrum of a star, as this correlates with its real brightness. As we can measure the star's apparent brightness, we can work out the distance from this as we know how brightness dilutes over distance [37]. Lastly, astronomers can also use cepheid variables. These are stars that pulsate with a measurable period that is related to the real brightness of these stars. From this, astronomers can then work out their distance. They are a type of standard candle just as the type Ia supernovas discussed before, although these have a higher absolute brightness than the more frequently observed cepheids such that they remain useful further out in the universe [36]. Note that these techniques give us the distance from a star in the Milky Way galaxy to Earth. We must first take into account the position of Earth within our galaxy to derive the position of other stars relative to the galactic center from our measurements.

The question however remains how we measure the orbital speeds of objects in the Milky Way. Here we must keep in mind that a body in orbit has a velocity with two components: a radial and angular one. The former refers to the component of the motion of the star towards or away from the center of our galaxy. The latter refers to the component of the motion that does not change the distance between the galactic center and the object (the circular component of the motion). Both components are measured in different ways, and for both it is important to keep in mind that our measurements are taken from Earth, which is itself in motion relative to the galactic center.
While the radial component can just as well be measured by the spectroscopical tools laid out above, this is not possible for the angular one as the Doppler shift is derived from motion towards or away from us as observers. Before getting to how we measure the angular component however, we should note that we can thus only use Doppler shift to measure the radial component of star motion relative to us as observers, not relative to the Milky Way galaxy

itself. We can of course derive motion relative to the galaxy itself after the fact, but the distinction is important to keep in mind.
To find the angular component of the motion of stars relative to us, the 'proper motion' of an object is measured. The proper motion is defined as the angular change of an object in the sky over time. Often, the proper motion is measured by looking at the angular change of an object in the sky over many years, as the effect is cumulative and this makes the measurement easier and more accurate. One more thing to note is the fact that a star closer to us traverses a smaller distance than a star far away from us even if they have the same proper motion. Therefore, we need to take the distance of these stars to us into account to truly work out the angular velocity, which can be done by multiplying the proper motion with the distance a star is from us [38].
Knowing both the value of the angular and radial components of motion of a star relative to us yields the velocity of that star relative to us. After compensating for the motion of Earth within the galaxy, the actual orbital velocity of stars within the Milky Way galaxy can be derived. Thus, the toolkit to measure rotation curves is now complete even for our own galaxy.

7.2.2 How astronomers construct rotation curves given elliptical orbits

The second question above concerns the elliptical nature of orbits. Orbits are usually elliptical, even more so, and inevitably to some extent, in galaxies with morphologies that cannot with great accuracy be modelled as an axisymmetric structure with central bulge and a disk (or disks). However, as has been noted before, in practice most orbits only have a very small eccentricity, especially in the disk. If this was not the case, objects would frequently crash into each other. This does not necessarily apply to stars, because they are very small compared to the large distances in a galaxy. We should rather think of, for example, the friction between the many interstellar gas particles that can be found throughout galaxies. These will tend to cancel relative velocities (radial or differences in angular velocities), bringing the system ever closer to a circular configuration over large cosmic timespans [39].
Thus, when studying an ensemble of objects orbiting at a certain approximate distance from the center, one can find an average rotational velocity whose magnitude (speed) also happens to be almost completely in the angular direction. Therefore, we can assign an average speed to ensembles at a given distance, since the speeds do not change all that much at cosmic scales given the low eccentricity. If we would find many different velocities at a certain distance and angle, that would imply highly elliptical orbits, as the velocities then differ very much throughout an orbit.
Nevertheless, differences do exist. In the central bulge, with more elliptical orbits, there will be higher differences between the rotational velocities measured in an ensemble in some relatively small space in the bulge [40]. To take this into account, the concept of *velocity dispersion* is introduced. The velocity dispersion is a measure of the spread in velocities of an ensemble of bodies in orbit at an approximate radius and angle [41]. Higher velocity dispersion can therefore mean more elliptic orbits (for example deep within the bulge), but also just less accurate measurements (for example at the outer edges of our galaxy). Think of it as the standard deviation of the velocity as known in statistics, telling us something about the accuracy of the v-value in the rotation curve at a given r-value.
Even taking this into account, orbits are circular enough and have almost exclusively angular components to their orbital velocity, all closely together in magnitude, to such a degree that it justifies the construction of rotation curves where for each specific value of r, a specific value of v is assigned.

The above justifies why rotation curves can be made with sufficient accuracy and how this is done. We will now move on to real data of the Milky Way galaxy, where we also take the velocity dispersion into account.

8 Setting up a rotation curve with the dark energy Lagrangian

In this section, we will set up a rotation curve on the basis of the model laid out in the previous sections. In order to do this, we use data from the Milky Way galaxy, as it has been measured quite well these days, is accessible and it would be a good start to see how our model holds up in our own neighbourhood. The section will be structured in a couple of subsections describing the phases of this endeavour.

8.1 Goal and Expectations

The goal is to see if our formula $v(r)$ can produce an adequate fit of the data points (v, r) of the Milky Way galaxy. In section 7.1 we covered how we in principle would expect this.

On the other hand, the approximations made to get to our result must also not be forgotten. The most important one is the two-body approximation, which has an impact through both neglecting entirely the spatial configurations of bodies in a galaxy relative to each other (that is, the shape, or morphology, of the galaxy) as well as simply the amount of bodies we consider.

Starting with the first of these two points, the Milky Way galaxy does not perfectly portray the morphology we have previously referred to, namely a spherical bulge and a (couple of overlapping) disk(s). Rather, the Milky Way has spiral arms, with which it loses some of the axisymmetrical properties presupposed by the bulge-disk(s) model. Nevertheless, as has been seen before when covering the N-body problem, our galaxy can be modelled as such to a sufficient degree.

This brings us to the second important point to keep in mind, which is the fact that we only consider two bodies. As mentioned before, the disks cannot just be reasoned away by arguments from symmetry, and they are also far more massive than the bulge. One may argue that the bulge is much smaller and therefore has such a large density that this increases the approximation's merit, but this argument falls flat when actually plugging in the numbers. More specifically, if we calculate the ratio of the volumes [42][43] of the disk (modelled as a cylinder) and the bulge (modelled as a sphere), we find that $\frac{V_{disk}}{V_{bulge}} \approx \frac{\pi h_{disk} r_{disk}^2}{\frac{4}{3}\pi r_{bulge}^3} \approx 25$, where h_{disk} is the thickness of the disk and r_{disk} and r_{bulge} are the radii of the disk and the bulge respectively. Meanwhile, the ratio of their masses [44][45] is such that $\frac{M_{disk}}{M_{bulge}} \approx 75$. Under the assumption of a uniform mass distribution, the ratio of their densities is then $\frac{\rho_{disk}}{\rho_{bulge}} = \frac{M_{disk}}{M_{bulge}} \cdot \frac{V_{bulge}}{V_{disk}} = \frac{75}{25} = 3$. In conclusion, the bulge may have a high density, but the disk is just so much more massive that it *still* has a higher *averaged* density than the bulge, adding to the point that it is an important player in intragalactic dynamics.
Still, at any point in the galaxy the gravitational pull of a specific 'mass element' in the disk is at least to some degree mitigated by a mass element on the other side pulling in the opposite direction. Due to there being at least some of this 'force cancellation' for a body in the disk in addition to the fact that we will introduce some additional tricks in the next subsection to minimise its distorting impact on the approximation, the approximation is justified for the purpose at hand. After all, we want to have a rather general look at the dark energy effect and what it does, rather than strive for quantitative perfection.
The important take-away message here is that the eventual fit values we find will not be perfect and are certainly subject to further research. Nevertheless, it gives us a good first idea of what results our model produces. While we will not expect numerical perfection, we will expect and indeed strive to produce an adequate fit that displays the general trends needed to be compatible with the behaviour of actual measured rotation curves.

8.2 Method

A brief word on our methodology is in place. We will structure this as a chronological list of considerations:

1. We use data from an extensive research paper from the University of Tokyo [46]. The data gives us a large set of points (r, v, σ), where σ denotes the velocity dispersion and r, v the usual. The units used are kpc for the distances and km/s for the orbital speeds. The data is, as stated before, from measurements of the Milky Way galaxy and will show a constant orbital speed at a couple of hundred kilometers per second, as can be seen for many rotation curves.

2. One important note is that we cut off a part of the data for the fit, namely the data points before the 6.2kpc mark. This is because these points correspond, roughly, to points inside the bulge. At that point the orbital speed is still increasing as you move out of the bulge. But our model presupposes two bodies, where a body in the disk orbits the bulge. Thus, we are interested in whether we can somewhat capture the behaviour of bodies outside the bulge, where we would expect a similar inverse square root falling-off behaviour. Taking points inside of the bulge into account would then mess with the data and not correspond to the falling off behaviour our function $v(r)$ predicts for the two-body problem.
   It should be noted that the radius of the real bulge is only about half of the value given here [42]. Thus, we are working with a larger 'effective bulge', which works better for the two-body assumption we are making, since there is still somewhat of an increase in the orbital speed data points until that point. This makes 'our bulge' a little more massive than the real bulge and not a perfect spherically symmetric bulge, but it somewhat limits the distorting influence of the disk.

3. To find for what parameters our function $v(r)$ fits the data best, we need fitting software. For this purpose, *SciDAVis* was used. This software can fit data in many ways. Specifically, it allows us to take into account the velocity dispersion in our fitting procedure. It stands to reason that if one point has a far larger velocity dispersion, it should not count as strongly in the fitting procedure, as this data is of less quality and might send the curve off in the wrong way. Thus, error bars representing the velocity dispersion are added to the data points and these are weighed in the fitting procedure.

4. Lastly, we need to explicitly give a formula to fit. In our case, this will be $v(r) = \sqrt{\frac{GM_{bulge}}{r} + \frac{a}{r^b}}$. It can easily be computed that $GM_{bulge} = 2.7 \cdot 10^{21} \frac{km^3}{s^2}$ from known data.[35] The parameters a and b will be our fitting parameters. We can infer their identities from what we know $v(r)$ to be, namely $a = \frac{\beta\delta r_0^\delta}{\mu}$ and $b = \delta$. So, while the former tells us something about the strength of the additional dark energy term, the latter tells us about how it will fall off with distance.

With this, our fitting toolkit is complete.

8.3 Results

For the fitting parameters we find the following values:

$$a = \frac{\beta\delta r_0^\delta}{\mu} = (1.4 \pm 0.7) \cdot 10^6 \frac{km^{2.1}}{s^2}$$

$$b = \delta = 0.10 \pm 0.02$$

[35] Since the data points have been expressed in kilometers, so will GM. Other units could be chosen later, but kilometers are the most elegant way to express at least the orbital speed.

Below, the rotation curve can be found as given by *SciDAVis*.

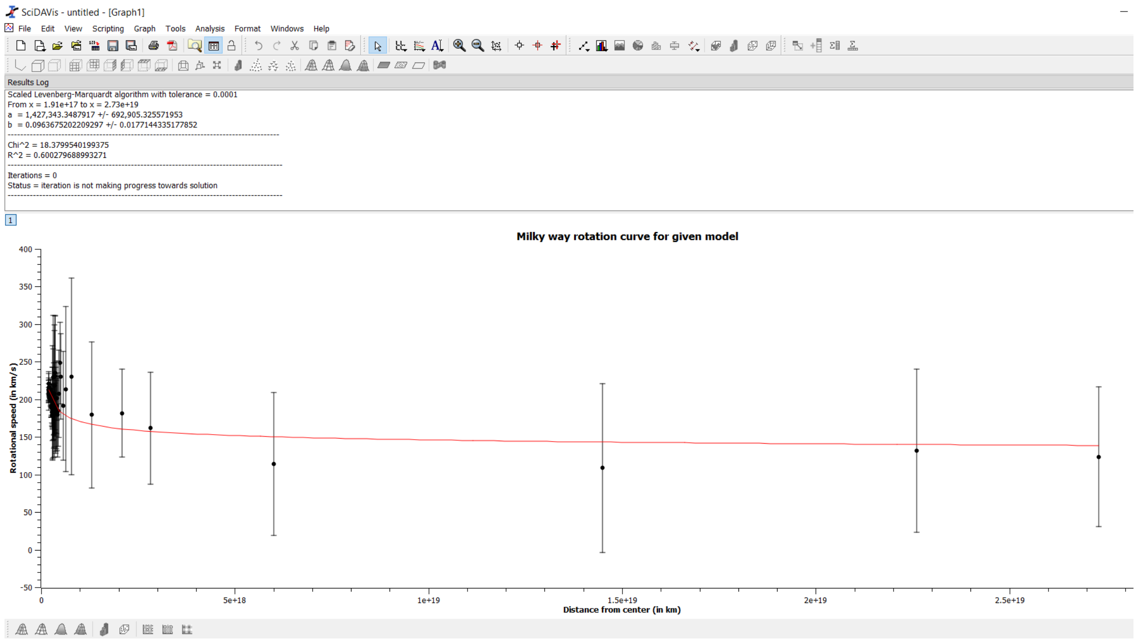


Figure 6: Fit of the rotation curve function $v(r) = \frac{GM_{bulge}}{r} + \frac{a}{r^b}$ *to data of the Milky Way galaxy outside of the (effective) bulge. Errors are taken into account in the fitting procedure, which minimises the difference between data points and the curve value. The fitting parameters a and b yield values* $a \approx 1.4 \cdot 10^6 \frac{km^{2.1}}{s^2}$ *and* $b \approx 0.1$.

Due to many data points being concentrated at the lower end of the range[36], a second figure is included that more clearly shows the higher number of data points in the $r \sim 10^{17}$km range. This also allows us to see very clearly how the data points on the lower end have a relatively small velocity dispersion, while the further out in the galaxy we get, the greater the dispersion becomes.

[36] To see the falling off behaviour and compare it to traditional rotation curves, I chose to similarly to regular rotation curves like those in figure 2 use linear scaling on both axes, regardless of this causing data points on the lower end of the range to be visibly cluttered.

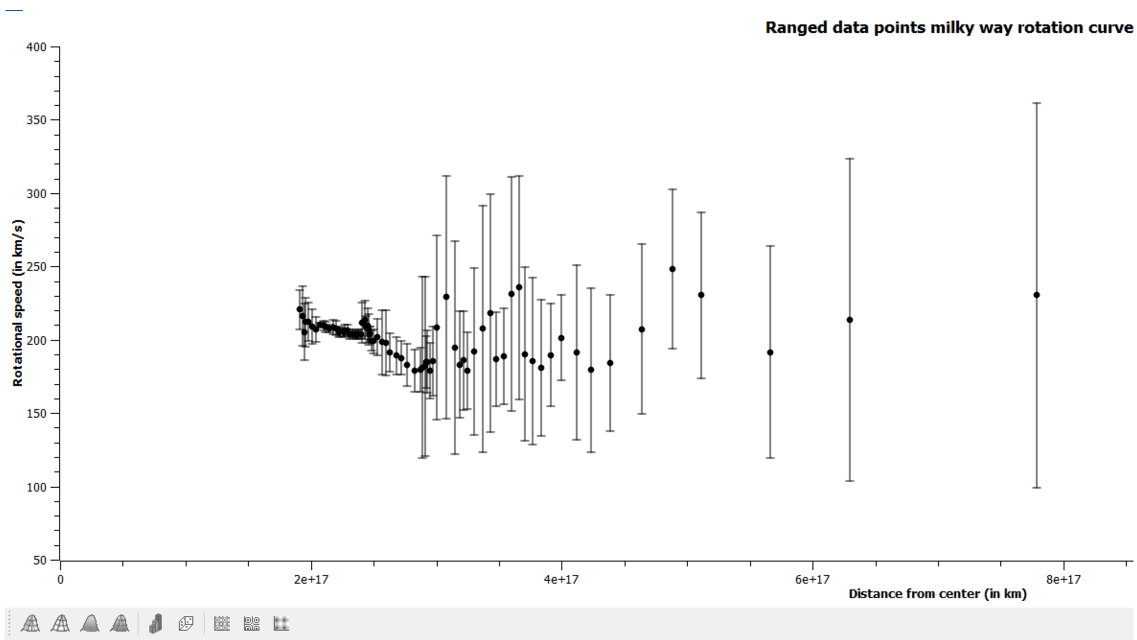


Figure 7: Zoomed in version of the fit. It better shows the high concentration of data points in the inner parts of the disk, just outside the bulge. The higher accuracy of these points can also be inferred from their smaller velocity dispersion, with some of the data points further out being included also to display the increasing velocity dispersion at greater distances.

Entering the numbers[37] into the expression for $v(r)$ yields:

$$v(r) = \sqrt{\frac{2.7 \cdot 10^{21}}{(r/km)} + \frac{1.4 \cdot 10^{6}}{(r/km)^{0.1}}}\,\frac{km}{s}$$

This is the resulting formula for our rotation curve.

8.4 Discussion of the results

When looking at the rotation curve given in the previous subsection, two things are immediately noticeable.

- The high density collection of data points just outside the effective bulge has a significantly lower velocity dispersion than points further out in the galaxy. This could partially be due to a higher degree of circularity in orbits, but it can most likely be attributed to the closeness of these points to Earth, which makes for easier measurements, both generally due to close distance and due to the fact that techniques such as parallax measurements can be used in that range. Quantitatively, the range from $2 \cdot 10^{17}$km to $3 \cdot 10^{17}$km seen in the graph corresponds to the range of 6.5 kpc to 9.7 kpc. The Earth can be found at a distance of about 7.9 kpc from the bulge's center [47]. Therefore it is situated, as may be expected, approximately in the middle of these high-precision measurements.
  As a consequence of their accuracy, or in other words, their low velocity dispersion, these points are weighed somewhat more strongly when determining the fit parameters. This is not unreasonable, as the point density

[37] Here we are neglecting the given standard deviations due to the primary importance of the order of magnitude of the values.

decreases due to measurement difficulty as we move further out to the edges of the galaxy (and even out of the galaxy). For this purpose, it is good to remember that the diameter of the galaxy is roughly 10^{18}km [43]. Points beyond half of that are therefore objects observed outside of the galaxy, which are quite useful until you get too much interference from Andromeda located at about $2.5 \cdot 10^{19}$km [48]. Aside from interference due to neighbouring galaxies' gravitational pull, these data points external to the Milky Way are by themselves rather useful, as they enable one to check whether the curve really stays constant even as we move out of the galaxy. This possibility thus counts as another argument to use the Milky Way galaxy as our fitting source, since such points external to the galaxy can be found more easily (relatively) close by. Unfortunately, the high velocity dispersion is a practical problem for these specific data points. Since our fitting formula assumes the Milky Way to be a two-body system, it does not take other galaxies like Andromeda into account, which is an additional reason as to why objects external to the Milky Way galaxy should not be counted as strongly.

- The curve seems approximately constant on a galactic scale, while, as can be seen from the formula, still falling off in the long run due to the positive value $\delta = 0.1$. Consequently, the boundary condition that $v \to 0$ as $r \to \infty$ is kept in place.

We then get to the big question as to whether the fit parameters make sense. The expression for $v(r)$ is shown to have two terms, namely two quotients with a constant numerator, diluted over distance by the denominator. Since we found that $b = \delta \approx 0.1$, this happens much slower for the term associated with the dark energy effect. This is in line with our expectation. We expected that $0 < \delta < 1$, likely on the lower end, and that it thereby falls off much more slowly than the classical $v(r) \propto \frac{1}{\sqrt{r}}$ term, at a pace such that the curve appears as good as constant on the galactic scale.
Let us also look at whether a makes sense. Suppose we cross the (hard to define) 'edge' of the galaxy such that, roughly, $r \sim 10^{18}$km. We then find that $v(r = 10^{18}km) = \sqrt{2.7 \cdot 10^3 + 2.2 \cdot 10^4} = \sqrt{2.5 \cdot 10^4} \approx 160 \frac{km}{s}$. Firstly, this value is what we would expect for the orbital speed. This is unsurprising, since we fitted our curve to real data. Secondly, and more interestingly, we see that at this r-value, the contribution of the classical term becomes relatively small, being one order of magnitude smaller than the dark energy term. Therefore, it contributes less and less compared to the very slowly decreasing dark energy term as r increases. This dark energy term keeps around the same orbital speed value on the scale of the galaxy. This is in line with expectations. To see the effect at play somewhat better, let us calculate the orbital speed for a number of relevant r-values and summarise them in a small table:

Distance order of magnitude (in km)	sum in square root (in km/s)	orbital speed (in km/s)
10^0	$\sqrt{2.7 \cdot 10^{21} + 1.4 \cdot 10^6}$	$5.2 \cdot 10^{10}$
10^{15}	$\sqrt{2.7 \cdot 10^6 + 4.4 \cdot 10^4}$	1656
10^{16}	$\sqrt{2.7 \cdot 10^5 + 3.5 \cdot 10^4}$	552
10^{17}	$\sqrt{2.7 \cdot 10^4 + 2.8 \cdot 10^4}$	235
10^{18}	$\sqrt{2.7 \cdot 10^3 + 2.2 \cdot 10^4}$	157
10^{19}	$\sqrt{2.7 \cdot 10^2 + 1.8 \cdot 10^4}$	135
10^{20}	$\sqrt{2.7 \cdot 10^1 + 1.4 \cdot 10^4}$	118
10^{30}	$\sqrt{2.7 \cdot 10^{-9} + 1.4 \cdot 10^4}$	37.4

The first column shows the order of magnitude of the distance we are looking at. The second is useful to include since it gives us a feeling of how the ratio of the classical term and the effective dark energy term in the square root changes over distance. Finally, the third column shows the orbital speed.

The first and final row can be interpreted as the cases $r \ll r_{MW}$ and $r \gg r_{MW}$, respectively. Here, r_{MW} denotes the radius of the Milky Way. In the domain much smaller than the galactic scale (exemplified as $r = 1$km), the classical term $\frac{GM_{bulge}}{r}$ dominates, and we get large orbital speeds. One could be inclined to attribute this to the high orbital speeds of stars around the central black hole of the galaxy[38], but in fact, the orbital speed increases when moving out of the bulge, not closer towards its center. This is merely a consequence of our inaccurate portrayal of the galaxy as a two-body model, which leads us to mathematically find a limit whereby $v \to \infty$ as $r \to 0$. Thus, this does not correspond to the reality, but it is not the domain of r we are interested in anyway. In the second case we are in the domain much larger than the galactic one (exemplified by $r = 10^{30}$km). We find a much smaller orbital speed than in the galaxy, namely $37.4\,\frac{km}{s}$. Yet, considering we went up ten orders of magnitude from the last value, decreasing only one order of magnitude in terms of orbital speed constitutes quite a small effect. This, however, is to be expected given the dominance of the slowly decreasing $\frac{1}{r^{0.1}}$ term in the square root in this domain. Slow as it is, the term will likewise approach 0 as the distance approaches infinity. It is also good to note that at the length scale of 10^{30}km, we are looking at a distance far greater than the diameter of the observable universe ($\sim 10^{24}$km), so it is a physically irrelevant value to the extent that we cannot measure it. Even if it were not, at that scale the Milky Way galaxy would be completely irrelevant in determining the dynamics of a body at that distance, compared to other structures in the universe.

Now let us look at the scale of interest for the Milky Way galaxy and its neighbourhood, very broadly defined as the range $r \in [10^{15}km, 10^{20}km]$. We see that while the orbital speed is decreasing, it does so rather slowly. In our plot, we can see data points for roughly 10^{17}km to 10^{19}km. In the first of these three orders of magnitude, we still see a somewhat higher orbital speed. This corresponds to the two-body formula we are using, since 10^{17}km is about 3 kpc, which is not far out of the bulge. In reality, the value for the orbital speed at 3 kpc is likely somewhat smaller than the value we find, due to the real rotation curve increasing until roughly the start of this post-bulge domain, while our model is decreasing from the get-go due to the two-body formula setup. In the bulge domain, our model therefore overestimates the orbital speed.
It then falls off a little, and as we look at bodies further out we barely see any change as we transition from 10^{18}km to 10^{19}km, especially taking into account that the velocity dispersion at these scales tends to be around $\sigma \sim 100\,\frac{km}{s}$, therefore being about five times higher than the decrease between those two length scales predicted by the formula. The same can be said for the transition towards 10^{20}km, at which point we are already at the order of magnitude of the Andromeda galaxy anyway, such that other galaxies can no longer be neglected, and we are left with a complicated web of forces on each body.

Therefore, given that values smaller than $r = 10^{17}$km do not correspond greatly to reality due to the two-body approximation scheme we use, and given that values of $r = 10^{19}$km or greater also do not greatly correspond to reality as other galaxies become relevant here, our strongest judgments should be directed towards the domain $r \in [10^{17}, 10^{19}]$, where the measurements do roughly correspond to the predictions of our model as noted in our comments above, regardless of inaccuracies due to our assumptions.

In conclusion, this means that despite being far from quantitative perfection, the predicted curve does have the right shape to be able to be fitted to rotation curve data at these scales. Moreover, the values for the fitting parameters are also compatible with the data.

Another way to test our model, however, is to see if it also produces acceptable results when applied to other domains. The question as to whether the model can stand its ground in such a situation is the topic of the next section.

[38]Neglecting for a moment that at $r = 1$ km, you would already be deep inside the central black hole Sagittarius A*.

9 On the viability of the model on different length scales

A common ideal in physics is that physical laws should have explanatory and predictive power on all length scales. Regardless of its later discovered flaws, Newton's law of universal gravitation did not just explain the dropping of apples on Earth, but also the orbit of Earth around the sun. The fact that quantum field theory, governing the small, and general relativity, governing the big, cannot be mixed into a grand theory making accurate predictions on all domains is considered perhaps the biggest problem in physics. Therefore, we should desire that the application of our model to the scale of the solar system[39] will reproduce classical equations known to work well there, just as the equations in special relativity will reproduce those of classical mechanics in the limit $v \ll c$. So far we have only studied our model on the scale of a galaxy, but the question now is whether it can still produce accurate results, in line with classical mechanics, when applied to other astronomical domains. If so, the model remains viable for now, in the sense that it cannot be refuted by displaying a lack of correspondence between such domains. If not, the model likely leads to a dead end. For the sake of completeness, general relativity may also be said to be 'wrong' as it does not work on the smallest scales, but at the very least it works perfectly on large scales and it has bulks of evidence in its favour, which is why we do not simply reject it. Clearly, it does many things right. Our model as discussed here however is still child's play. There is not a wide array of evidence for its accuracy, nor is it without, at least right now, stronger competitors on that front. This makes that non-correspondence on the solar system scale can both in practice and on probabilistic theoretical grounds be said to be a fatal blow to our model. Thus, the remainder of this section will investigate the preconditions for success of our model on a solar system scale.

9.1 The problem of β

One may suggest to test the function $v(r)$ with fitted parameters for the solar system by just changing the mass of the orbited body (that is, the sun rather than the bulge) and entering the relevant r-values. Even more optimistically: the solar system can to a far greater (and generally very satisfying) degree of accuracy be modelled as a two-body problem, vastly improving the correspondence of our model with reality. While the latter is certainly true, it is unfortunately not the case that we only need to substitute $M_{bulge} \rightarrow M_{\odot}$, where $M_{\odot} = 2 \cdot 10^{30} kg$, denoting the mass of the sun. To see this, let us once more consider the identity of the fitting parameters in the dark energy term $\frac{a}{r^b}$. First of all, $b = \delta$, and we found that $\delta = 0.1$. This is a reasonable result and we expect, perhaps demand, no differences there for the solar system. However, the other parameter was stated to be $a = \frac{\beta \delta r_0^{\delta}}{\mu}$ and consists of multiple terms. The Milky Way fit gave us $a = 1.4 \cdot 10^6 \frac{km^{2.1}}{s^2}$, but on closer inspection we find that this cannot be the case for the solar system.

Let us therefore take a deeper look at the parameter a. First of all, we can enter $\delta = 0.1$, as we know this to be the case. The characteristic length r_0 has always been somewhat of an 'extra', as already said during its introduction. It is conventional and it can be useful to define such a characteristic length, but at a more fundamental level there are only two constants in the Ansatz for the potential of the dark energy force, and the characteristic length can just be absorbed into them. Right now, it is actually less elegant to explicitly introduce it, and so we will absorb $r_0^{0.1}$ into β for now.

Then there is μ, the reduced mass of the system. In the systems we are considering (where one mass is much bigger than the other) this can be interpreted as the smaller mass of the two bodies. While this could be the mass of a star in the case of the Milky Way galaxy as a whole, this could, for example, be the mass of a planet when considering the scale of the solar system.

Last, but certainly not least, there is β. When β was introduced, it was already stated that it is a composite expression regulating the strength of the secondary dark energy effect, like Gm_1m_2 for Newtonian gravity. We then also

[39] One could also look at, for example, the scale of galaxy clusters, but the solar system will make for an easier yet just as important case study.

argued that the physics behind the effect shows that β must hold some relation to the mass of the bigger object, as it determines the extent to which space is curved, and thus at what rate it will locally expand. On the other hand, β might also depend on some yet unknown fundamental strength constant related to this effect, just as Newtonian gravity depends on the universal gravitational constant G.
Knowing the above, we will now introduce the following expression for β:

$$\beta = \mu \cdot \gamma(M, x_1, ..., x_n, y_1, ..., y_m)$$

where μ is the reduced mass and γ an expression depending on the variables in the brackets.
The reason μ is in there is due to physical necessity. It ensures that μ is cancelled from the dark energy term in the $v(r)$ function, as can be seen from the definition of the fit parameter a. Suppose μ is not in our expression for β. Then the orbital speed depends on the mass of the smaller body. This leads to absurd results, where the orbital speed around the sun (or any body) would approach infinity as the mass of the orbiting object approaches zero (due to it being in the denominator of the dark energy term in the orbital speed expression). If this were the case, we could observe it, but we have long known that the acceleration of objects in a gravitational field is independent of that object's mass. The same holds here. Thus, μ is in there because had it not been, it would make predictions strongly contradicting empirical reality.
Then there is the expression $\gamma(M, x_1, ..., x_n, y_1, ..., y_m)$. This expresses the real strength constant for the effective dark energy force. It can be interpreted as β's true identity. Let us take a brief look at what the terms within its brackets mean:

- M: this is just the mass of the heavier object in the two-body problem. When we are to look at the solar system, this will be the sun. In the context of our model, it obviously must be a part of γ in one way or another, for physical reasons already explained above.
  The other variables making up γ discussed next refer back to the discussion on universal or system-independent quantities versus relative or system-dependent quantities.

- $x_1, ..., x_n$: these stand for n relative or system-dependent quantities that γ depends on. As a matter of fact, M itself belongs to this group, but the reason for separating it will become clear later. System-dependent quantities will be different when looking at the solar system setup than when looking at the galactic setup. Examples could be the volume of the source body (the sun instead of the bulge), the rotational kinetic energy of the source body (while the sun is not so much, there are plenty of rapidly spinning stars) and the surface area of the subjugated body (a planet instead of a star in the galaxy). Referring to the example of Newtonian gravity, where the force constant is Gm_1m_2, again, we see that m_1 and m_2 are the system-dependent quantities. While we could for physical reasons argue that some system-dependent quantities are likely to be a part of γ, a stronger mathematical and fundamental physics theory behind our model could predict what variables are elements of γ and which ones are not. This has the advantage of possibly enhancing the opportunity of empirical research on the effect, as we can look at systems where these variables have different values. One example could be to look at a large amount of galaxies selected such that they differ almost only in mass, and then to see if the prediction of the effect of mass on the orbital speed is consistent with observations.

- $y_1, ..., y_m$: these stand for m universal or system-independent quantities within γ. These could be a composition of familiar fundamental constants such as G or the speed of light c. There may or may not also be a new fundamental constant regulating the strength of this dark energy effect in there. Perhaps these could be derived from a complete physical foundation of the theory behind our model. In Newtonian gravity, G would play the role of the system-independent quantity.

Substituting the above expression for β in $v(r)$, we find that:

$$v(r) = \sqrt{\frac{GM}{r} + \frac{0.1\gamma(M, x_1, ..., x_n, y_1, ..., y_m)}{r^{0.1}}}$$

With the above distinction and notation in mind, we can study different cases for what γ looks like. Such a case will then reveal a condition that must be satisfied in order for the model to be consistent at the level of the solar system.

9.2 Case I: the simple M case

This case is probably the simplest one and a good one to start with.
For this case, we make three assumptions about $\gamma(M, x_1, ..., x_n, y_1, ..., y_m)$.

1. There are no system-dependent variables other than M.
   $\Rightarrow \gamma(M, x_1, ..., x_n, y_1, ..., y_m) = \gamma(M, y_1, ..., y_m)$
2. γ can be factorised in a M-dependent and a y_i-dependent part.
   $\Rightarrow \gamma(M, y_1, ..., y_m) = A(M) \cdot B(y_1, ..., y_m)$
3. The function A(M) is just a power of M.
   $\Rightarrow A(M) = M^k, k \in \mathbb{R}$

Here, $B(y_1, ..., y_m)$ is a constant, as it is by definition just some composition of more elementary constants.
For $A(M)$ we could in principle have chosen any function of M that we wanted. However, in physics we often find functions with strength constants that involve just powers of quantities, so at least on probabilistic grounds one could say that this is more likely than, for instance, $A(M) = c_1 \sin(c_2 \ln(c_3 \sqrt{c_4 M + c_5 e^{c_6 M}}))$. At the same time, powers of quantities are easy to work with and the method used here can just as well be applied to some other function of M. Lastly, we chose $k \in \mathbb{R}$ since we have no a-priori reason to restrict k-values further.
More system-dependent variables could be added later, but again, this serves as a good starting point. The assumption of factorisability of γ is, like the fact that $A(M)$ is taken to be a power of M, common as well as something that expands the possibilities of what we can do here. Thus, we now find:

$$\gamma(M, y_1, ..., y_m) = M^k B(y_1, ..., y_m)$$

and therefore,

$$v(r) = \sqrt{\frac{GM}{r} + \frac{0.1 M^k B(y_1, ..., y_m)}{r^{0.1}}}$$

Since $B(y_1, ..., y_m)$ is by definition made up of system-independent parameters, and since we know the system-dependent parameters (in this case just M), we can use our fit of the Milky Way galaxy to figure out the value of $B(y_1, ..., y_m)$. Denoting the value of the fit parameter a we got for the Milky Way as a_{MW}, and remembering that we absorbed r_0 into β, we know that $a_{MW} = \frac{\beta\delta}{\mu} = 0.1 M_b^k B(y_1, ..., y_m) \Rightarrow B(y_1, ..., y_m) = \frac{10 a_{MW}}{M_b^k}$, where M_b denotes the mass of the bulge. If we substitute this into the expression for the orbital speed specific for the solar system, we find:

$$v(r) = \sqrt{\frac{GM_\odot}{r} + \left(\frac{M_\odot}{M_b}\right)^k \frac{a_{MW}}{r^{0.1}}}$$

Thus, we find the expression for the orbital speed of, for example, a planet in the solar system, for which the only unknown is the free-to-chose variable r, just as we wanted.

We can now analyse this result. One should note that we expect to find that $v(r) = \sqrt{\frac{GM_\odot}{r} + \left(\frac{M_\odot}{M_b}\right)^k \frac{a_{MW}}{r^{0.1}}} \approx \sqrt{\frac{GM_\odot}{r}}$. In other words, the classical term should be much larger than the dark energy term. After all, classical mechanics adequately describes the dynamics of the solar system[40]. It is only on scales beyond the solar system that $v(r) = \sqrt{\frac{GM_\odot}{r}}$ no longer suffices, since other star systems then provide relevant forces too.
We now plug in the numbers, so we can study the quantitative effect in more detail:

$$v(r) = \sqrt{\frac{1.3 \cdot 10^{11}}{(r/km)} + (5 \cdot 10^{-11})^k \cdot \frac{1.4 \cdot 10^6}{(r/km)^{0.1}}} \frac{km}{s}$$

where the orbital speed is expressed in km/s.
We can see that for small r, the classical term dominates (as it should) and the dark energy term is negligible. The question is: is this still the case for the largest r-value that is still a part of the solar system? That is, does the formula still adequately describe the very outer part of the solar system? If at any point it does not we expect it to produce a higher orbital speed than measured, since the dark energy part falls off much slower than the classical part does as r increases.
A good candidate for a body to look at is a dwarf-planet that lies beyond Pluto, called Makemake. We know that classical mechanics describes its orbit, it is one of the outermost major bodies in our solar system and it has a relatively circular orbit[41] compared to other trans-Plutonian dwarf-planets[42], with $\epsilon \approx 0.16$. The semi-major axis of Makemake is about $6.8 \cdot 10^9$km, which, using our circular approximation again, we will now take to be its distance from the sun during its entire orbit [49].
Thus, knowing that the orbit of Makemake can be accurately described by classical mechanics, we should demand of our model that even for this outer orbit we *still* find that the classical term in $v(r)$ is much larger than the dark energy associated term. That is, we expect that $v(r = 6.8 \cdot 10^9 km) \approx \sqrt{\frac{GM_\odot}{r}} = 4.4 \frac{km}{s}$, which is the number in agreement with real observational data of Makemake's orbital speed [49].
Entering this r-value into the formula for the orbital speed and purely looking at the two terms within the square root, we find the condition:

$$19.1 \gg 1.5 \cdot 10^5 \cdot (5 \cdot 10^{-11})^k$$

Namely, the classical term at this r-value must be much larger than the dark energy term in order to find that the actual orbital speed is approximately equal to the orbital speed predicted by our model. How much larger it should be depends on the accuracy with which our measurement devices can determine the orbital speed of Makemake. It could, for example, be the case that the dark energy term needs to be at least $\frac{1}{1000}$ of the classical term in order to notice a deviation from classical predictions that cannot be ascribed to the imperfection of our measurement devices. Moreover, the denominator of a fraction like this would not be set in stone, but would increase over time, as due to technological progress, our measurements become more precise. Therefore, we will be able to gain a more specific condition on k if we define a 'precision parameter' p in such a way that measurement accuracy is accounted for[43]. The precision parameter is thus the fraction of the classical term the dark energy term must at least be to be able to notice a deviation from classical predictions. It follows that $0 < p < 1$. Combining this method with the inequality given above leads to the following condition on k:

$$1.5 \cdot 10^5 \cdot (5 \cdot 10^{-11})^k = 19.1p$$

[40] Of course general relativity is even more accurate, as it also accounts for features like the slight deviation in Mercury's orbit from classical predictions. But due to this being a small effect, classical mechanics is adequate in its description for our purposes.
[41] Since $v(r)$ assumes circular orbits, we would do well not to take a *too* eccentric orbit as an example.
[42] For example, Eris, which is even further out, has a very elliptical orbit [50] of $\epsilon = 0.44$.
[43] We will not introduce a specific time dependence $p = p(t)$, as increasing accuracy of our measurement devices is likely, but not a natural law or inevitability.

This, when solving for k, yields:

$$k = \log_{(5\cdot 10^{-11})}(1.27\cdot 10^{-4} p)$$

This is a rather messy logarithm with base $5\cdot 10^{-11}$ and a precision parameter that can be freely chosen by a hypothetical astronomer to represent the accuracy of their measurement devices.

We must take one brief step back, however. What we want to know, is from what k-value onward the dark energy term would produce a deviation that can be accurately measured to give the wrong prediction of the orbital speed of Makemake. The value of k in the equality above is such that if it becomes any smaller, our model would produce inaccurate predictions. So, the condition on k that our model has to abide by in order not to contradict empirical reality is:

$$k \geq \log_{(5\cdot 10^{-11})}(1.27\cdot 10^{-4} p)$$

Having finally arrived at our condition, let us make a brief remark and then give one example. We can remark that we stated that $k \in \mathbb{R}$, but one can see from the original inequality $19.1 \gg 1.5\cdot 10^5 \cdot (5\cdot 10^{-11})^k$ that if it were to be that $k \in \mathbb{N}$, then this condition would probably[44] already be satisfied for $k > 0$. In other words, if the power is a natural number[45], it must be a non-zero positive integer. Specifically, we can see that a direct proportionality between γ and M, i.e. $k = 1$, also makes the cut, which is what we assumed when looking at the N-body problem in section 6.6. This, thereby, adds legitimacy to the N-body model constructed there.

For the sake of intuition and a bit of a cleaner result than the logarithmic one in the condition for k arrived at above, we shall now take an example with a precision parameter value of $p = \frac{1}{1000}$. This yields:

$$k \geq \log_{(5\cdot 10^{-11})}\left(1.27\cdot 10^{-4}\cdot \frac{1}{1000}\right) = 0.67 \approx \frac{2}{3}$$

Hence, assuming this specific p-value, and keeping in mind that we made many assumptions such as Makemake having a circular orbit and the three assumptions for the simple M case, we find that our model corresponds to the solar system domain if $k \geq 0.67$. That is, if you derive a formula for the rotation curve from the theoretical foundations underlying the model, and you find that $k \ngeq 0.67$, the theory must be wrong[46]. The condition on k therefore serves as an empirical test that must be passed by any theory of the effect we study here. If it does not, the theory does not correspond to reality. It is good to have such checks in place. This ensures that our model is testable and falsifiable. With the empirical demand on k we can either ascertain that the theory is incorrect or slightly increase the probability of it being correct. Nevertheless, this would in no way prove that it is.

9.3 Case II: the many system-dependent quantities case

As mentioned before, one could argue that γ likely depends on more system-dependent variables than just the mass of the bigger object. For example, the volume of the bigger object might be relevant for the effect, as the matter density is diluted over more space. The surface area of the lighter body may or may not also be argued to play a role, as when this is larger, this leaves more area to be slightly 'pushed away' by the extra space being created between the bodies. The possible rotation of the bigger object might also add to the effect, as this increases its (rotational kinetic) energy and therefore curves spacetime more, increasing the effect. While it is hard to predict exactly what variables are most relevant without a solid study of the fundamental theory behind the assumed effect in our model, it seems at least likely that more system-dependent variables are relevant for γ other than just the larger body's mass. Thus, we will drop the first assumption of the previous section and very briefly and generally look at how this affects the work done in the simple M case.

[44] That is, bar some unrealistically accurate measurement scenario with a precision parameter $p = \frac{1}{4\cdot 10^7}$, the value gained for solving the equality for $k = 1$.

[45] While it is not necessary to assume it is, it is good to note this given that often powers occur as natural numbers.

[46] Or at least, as explained in the beginning of this section, it is extremely likely to be so.

Since we already denoted the set $\{x_1, ..., x_n\}$ as the set of system-dependent variables, we use this notation to write γ as:

$$\gamma(M, x_1, ..., x_n, y_1, ..., y_m) = M^k x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n} B(y_1, ..., y_m), s_i \in \mathbb{R}$$

Here, the x_i again denote the relevant system-dependent variables other than M, and the s_i denote the real numbered powers that pertain to each of these quantities.
As can be seen, assumption 2 and 3 of the simple M case are kept in place and generalised for all system-dependent variables. Namely, the idea that γ is entirely factorisable in system-dependent terms with real-numbered powers, and the system-independent term $B(y_1, ..., y_m)$ we are already familiar with. The reasons for these assumptions have been laid out in the previous subsection, and relaxing them would greatly restrict what little we will be able to see now. However, the same kind of reasoning as before can still be used if the exact expression for γ is known, and we can now at least generalise the simple M case for something we can already reasonably expect: that M is not the only system-dependent factor at play.

Using the same logic as in the simple M case, we now find that $a_{MW} = 0.1 M_b^k x_{1b}^{s_{1b}} \cdots x_{nb}^{s_{nb}} B(y_1, ..., y_m)$, where the extra subscript b expresses that these are the values these quantities take in the Milky Way fit (b = bulge). This expression can again be rewritten in terms of $B(y_1, ..., y_m)$. Substituting this in $v(r)$ for the solar system yields:

$$v(r) = \sqrt{\frac{GM_\odot}{r} + \left(\frac{M_\odot}{M_b}\right)^k \left(\frac{x_{1\odot}}{x_{1b}}\right)^{s_1} \cdots \left(\frac{x_{n\odot}}{x_{nb}}\right)^{s_n} \frac{a_{MW}}{r^{0.1}}}$$

Due to our more abstract way of working in this case, we do not know the numerical values of the x_i, but we do know that similar to the simple M case the following condition must again hold:

$$\frac{GM_\odot}{r_{mm}} \gg \left(\frac{M_\odot}{M_b}\right)^k \left(\frac{x_{1\odot}}{x_{1b}}\right)^{s_1} \cdots \left(\frac{x_{n\odot}}{x_{nb}}\right)^{s_n} \frac{a_{MW}}{r_{mm}^{0.1}}$$

where r_{mm} denotes the specific distance of the dwarf planet Makemake to the sun.
Since all powers are real numbers, we conclude that there is theoretically an infinite amount of combinations $\{k, s_1, ..., s_n\}$ for which this condition will hold. Yet, just as a theoretician could use the condition derived in the simple M case to check whether the obtained k-value they found judges the theory as erroneous or compatible for now, a theoretician predicting values for all of these powers can enter them into this condition and see if it is indeed true that the result on the right hand side is much smaller than the classical term $\frac{GM_\odot}{r}$. Note again that how much smaller it would at least need to be for the theory not to be rejected outright depends on the precision parameter. This is not explicitly introduced here due to the many powers not allowing one to derive just one numerical condition as we could for k in the previous case. However, conceptually it remains relevant nonetheless, and it would still need to be added in the case one wants to calculate whether the inequality holds to a sufficient degree.

This condition could be argued to be somewhat weaker than the one just obtained in the previous subsection. After all, we do not find a numerical condition on each individual power, but only on the given composition of them. There are many different combinations of the values $\{k, s_1, ..., s_n\}$ for which the inequality condition at large holds, while the individual values in the set $\{k, s_1, ..., s_n\}$ may nevertheless not be right. There is more space to get them wrong than when just looking at k.

For reasons argued above, these two cases are the most useful to cover here. Theoreticians with a real idea about the specific form of $\gamma(M, x_1, ..., x_n, y_1, ..., y_m)$ could use the methods developed above to see whether their theory is consistent with the domain of the solar system, in addition to just the Milky Way system. While this will in no way prove their theory to be correct, it at least implies that further research may be of interest, while the converse strongly implies rejection of the theory altogether, as it would only seemingly work at one length scale.

10 List of assumptions and approximations used to get to the result

In deriving our eventual results, quite a number of assumptions and approximations have been made. Although they have been introduced in the text, they are important to keep track of, both to be aware of the magnitude of their effects on the accuracy of our results as well as to serve as a way in which future research could increase the precision of models of the kind covered in this thesis. Therefore, this section will provide a list of the most important assumptions and approximations made in this thesis and brief comments on their context and accuracy.

10.1 Five central assumptions

- The (positive) $\Lambda - R$ relation: the most central assumption in this thesis consists of the statement that there exists a positive relation between the cosmological constant Λ, which is associated with the rate of expansion of space, and the scalar curvature R, which is associated with the curvature of spacetime. That is, if R increases, so does Λ. It follows that space can locally expand more rapidly in areas with a higher matter-energy density (which curves spacetime), while at the scale of the observable universe this evens out such that the expansion rate of space appears uniform.

- Force modelling: we assume that the secondary effect of dark energy can be modelled as an additional force on bodies in a galaxy. This force is directed inwards into the galaxy and falls off with distance more slowly than the gravitational force does.

- The potential Ansatz: we assume that the potential associated with this effective dark energy force can be expressed as $U_\Lambda(r) = -\beta\left(\frac{r}{r_0}\right)^\delta$. We chose this Ansatz for a number of reasons. First of all, the force expressions of gravity and the dark energy effect have a similar and common mathematical structure. Secondly, it can show many different strengths and falling off behaviours we are interested in because of the parameters involved. Thirdly, it is relatively easy to work with and there is no reason to assume that this term would be some complicated composite expression of logarithmic and exponential terms, as an example. Lastly, we have to start somewhere, and the procedure following this Ansatz can also be applied to a different expression that may possibly follow from theory.

- Adequate fitting: we assume that a formula derived from a two-body problem setup (among other approximations) can be fitted to real data of rotation curves such that it still yields accurate enough fitting parameters and is not fundamentally incompatible to the degree that we approximated all realism away. However, not only do results show that the latter is not the case, we are, moreover, first and foremost interested in the general behaviour of our model when applied. The goal of this thesis is, after all, not numerical perfection.

- The choice of β-dependencies: since β is the strength constant for the potential and force of the dark energy effect, it is likely to be composed of a number of system-dependent and system-independent quantities. This fact was ignored where possible, and a more thorough discussion on β's identity can be found in section 9. There, this choice was relevant, leading to some possible scenarios.

10.2 Three central approximations

- The classical approximation: while general relativity is the most accurate way to treat gravitational systems, we have used classical mechanics. Due to us treating bodies whose speeds are nowhere near the speed of light and due to not treating very strong gravitational fields, this approximation is warranted.

- The circular approximation: this approximation neglects the fact that orbits are ellipses and treats them as circles, simplifying many equations. We saw that empirically most eccentricities in the disk take the value of

$\epsilon \approx 0.1$, for which the ratio between the semi-major and semi-minor axis turns out to be 0.995. This is very close to the circular value of 1. Therefore, the circular approximation is quite reasonable.

- The two-body approximation: the Milky Way galaxy consists of hundreds of billions of stars, among many other objects such as interstellar gas clouds, black holes and planets. These objects are distributed roughly as a spherically symmetric bulge at the center and larger disks through the bulge[47]. Using the spherical symmetry of the bulge, it is modelled as a point mass, while the disks are neglected in favour of a single point mass orbiting the bulge. After all, the gravitational pull of a mass element at one side of the bulge is at least partially cancelled by that of a mass element on the opposite side. In reality, however, the disks are about 25 times bigger in volume than the bulge, but are also about 75 times heavier[48]. Their gravitational influence thus cannot be approximated away on that basis. Moreover, unlike Gauss's law that can be applied to spheres with uniform density, the influence of disks can also not be completely neglected on the basis of symmetry arguments as given above. Consequently, the two-body approximation, while not entirely without merit and still allowing for an analysis of the effect on orbital systems, is certainly the most rough approximation used here. For this purpose, tools have been provided in several sections to use a numerical analysis and calculate the effect of dark energy on the mechanics of the system when taking the whole galaxy into account, as a continuous mass distribution.
  Lastly, one should note that there are essentially two consequences of the two-body approximation. Firstly, it simply reduces the amount of bodies and thereby the amount of terms in the Lagrangian, but secondly, it neglects the complicated spatial distribution of all these bodies in favour of a far more distribution-neutral two-body system. This ties in with the morphology of galaxies, the impact of which is lost in this approach, on top of just the number of bodies.

[47] Although in reality there is a degree of axisymmetric asymmetry in the Milky Way galaxy, such as the existence of the spiral arms. Modelling it this way can therefore by itself be considered an approximation.

[48] For calculations on these numbers, see section 8.1.

11 Conclusion

The main research question was formulated as follows:
'Can a coherent MOND-model in which dark energy serves as the causal mechanism behind observed data of galactic rotation curves quickly be shown not to explain the existing discrepancy between theory and experiment?'

On the basis of the research and analysis done in this thesis, the answer to this question is negative.

We have seen that the introduction of our central assumptions enabled us to produce a fit for the rotation curve that turns out to be empirically adequate at the level of precision that can be maintained within the confines of this thesis. In other words, the rotation curve fit gave us no reason to reject our model. In addition, the model was also shown to be able to reproduce correct values for the orbital speed of bodies on the scale of the solar system, given it satisfies a certain condition. Thus, it cannot be said to fail to pass the empirical test in another astronomical domain, at least not for all cases. The possibility of the correctness of the model therefore remains open.

In conclusion, the results corroborate the model for now. This thesis certainly does not prove, or even aim or claim to be able to prove, that the fundamental assumptions behind the model are correct and that it could provide a definitive solution to the problem of galactic rotation curves. What it does show, is that a relatively thorough analysis at the level of a bachelor student in physics does not confirm that the model could definitively *not* explain why the orbital speeds of bodies in a galaxy appear to be constant after initial increase.

12 Discussion

This section will cover a number of points of reflection with regard to the research in this thesis. These points will now be organised in subsections.

12.1 The validity of the research

The validity of the research is derived from the attempted clarity in what it sets out to do. Quality sources have been used and the methodology explicitly formulates and discusses the applicability of its assumptions and approximations. This makes that the results are valid in so far as they are able to answer a research question that was deliberately posed in a rather conservative way in the first place, due to the knowledge that more extensive research beyond bachelor level could produce stronger claims on the correctness (or wrongness) of the theory.

12.2 The shortcomings of the research

The shortcomings of the research can first and foremost be found in the assumptions and approximations that have been used. While especially the first assumption on the (positive) $\Lambda - R$ relation is a fundamental starting point for the research, the others can theoretically be dealt with given one commits more resources to researching the topic. The research only investigates a limited amount of data and cases to come to its conclusion, for example only the Milky Way galaxy data and specific intuitively likely Ansatzes for the force-strength parameter β. A particularly strong shortcoming is the two-body approximation. It is good for first research, but its roughness is certain to demote the standing of the results. A more fundamental shortcoming is the lack of a fundamental physical derivation of this effect from a theory of quantum gravity. Were this possible, one can immediately cover the direct structure of the potential as predicted by the theory, rather than a number of educated guesses, that leave the reader only with an idea of the *probability* of the right- or wrongness of the model.

12.3 The consequences of the research

Since a thorough project on the level of a physics bachelor student was not able to confirm the research question, this means that the model, and the theory behind it, covered in this thesis is ripe for the plucking by more advanced research. Perhaps it will then end up being rejected anyway. However, given that it at least survived a first level test and that the problems of galactic rotation curves and the accelerated expansion of the universe are still very much alive, it would certainly be positive to investigate different theories, this model being one among them. A direct consequence is that the central idea behind the theory cannot be rejected on the basis of bachelor level research, and perhaps a future consequence of this is more high-level extensive research.

12.4 Suggestions for further research

The thesis definitely leaves room open for different types of further research. Two of them will be expanded upon here below. The first focuses on vastly improving the quality of the research done here while maintaining the general idea, while the second is a more fundamental search for the physical mechanisms behind the $\Lambda - R$ relation, which could enable the direct derivation of mathematical expressions.

- *Improving upon the model's ability to make correct predictions for different astronomical structures and length scales.* This thesis has been analytical work and therefore been on a more qualitative level relative to what is possible. Numerical analysis will, for example, enable one to derive a vast array of data points (r, v) from the N-body Lagrangian, rather than its two-body counterpart. This thesis has focused explicitly on providing frameworks to do this with, of which the derivation of the integral yielding the Lagrangian of a body in a galactic N-body problem is an example. Using this will increase the quality of produced rotation curves. It can also work more efficiently to incorporate fitting data from many different galaxies, rather than just the Milky Way. This latter recommendation is tied to the fact that numerical analysis on the basis of a more accurate N-body model may be able to empirically derive the true identity of β, as it can compare the impact that for example different masses or morphologies of several galaxies have on the value of β. In other words, it could reveal patterns telling us something about what terms really constitute β. This is a good method to go beyond the too inaccurate toolkit of this thesis with, to far more easily test whether this theory fits galaxy data well and test whether it corresponds to all kinds of different domains, such as the solar system, a galaxy cluster and others.

- *A theoretical physics investigation into the quantum gravity foundation of this effect.* This certainly requires far more physics knowledge than bachelor level. A complete theory that can directly derive what the potential and effective dark energy force is between bodies rather than picking an Ansatz will be able to test whether it holds up far more easily. In that case one does not have to take into account many possible falling off behaviours or identities for β, to name two examples.

12.5 Things I learned during this internship

From a didactic perspective, the thesis was also enlightening for me as a student. Rather than just extending my physics knowledge on the topics treated in this thesis, which certainly happened, I also was able to learn more about techniques and ways of thinking one can use when dealing with physics research. Of course, it is still on bachelor level, but this project has yielded more insight into the usefulness of numerical methods, different mathematical techniques and approximation schemes one can try to use to solve a problem. Another example would be the importance of creative thinking in the sense that one should actively think of new methods to try things, which is also very much required for research in physics. In addition to the physics side of the story, I also got a better idea of what working at a physics department at the university is like, from the kind of people to the atmosphere and from the way in which students and professors work together to what people are concretely working on. The writing and feedback process as they relate to the thesis have also helped improve the level of my academic use of English, namely through allowing for both practice and the detection of some commonly made mistakes. I have also concluded that in future internship and thesis projects, I would, individually, function better with that being the main focus of my attention rather than one out of many projects I occupy myself with. This has led me to drop activities other than a side-job and my physics degree for next year. In addition, I also want to take this opportunity to thank my supervisor for his patience with the above.
Ultimately, this internship and thesis has made for a good learning experience, both in physics, everything surrounding that and about myself.

13 Bibliography

In this final section, all sources used throughout this thesis can be found.

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