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The Dynamical Casimir Effect

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1 Introduction

In 1948 Dutch physicist Hendrik Casimir predicted a force between two uncharged perfectly conducting metallic plates, the origin of which lies in the vacuum energy of a quantised field. The thought experiment behind Casimir's prediction is relatively simple; two uncharged perfectly conducting metallic plates, with surface A, are placed in a vacuum at a small distance d from one another (where $d << A$ such that A can be seen as infinitely large compared to d). As a consequence of Heisenberg's uncertainty principle, we know that the ground state energy of a single (one-dimensional) harmonic oscillator is given by

$$E_0 = \frac{\hbar\omega}{2},$$

which is non-zero. This implies that for an entire field of harmonic oscillators we will find a non-zero ground state energy. In other words: the vacuum state has non-zero energy as a consequence of the wave modes that constitute the field. In a way the two plates will function as hard walls, meaning that the electric field component parallel and the magnetic field component perpendicular to the plates need to be zero on the surface of the plates. As a consequence of these boundary conditions the wave modes propagating in between, as well as the ones outside the plates have to terminate on the surface of the plates. Due to the limited space between the plates, only waves with wavelength $\lambda = 2\frac{d}{m}$ where $m = 1, 2, ..$ are allowed. Outside, however, there is an entire continuum of allowed wavelengths. As the total energy of the field inside (outside) can be found by summing (integrating) over all the allowed wavelengths, there will be an energy difference between the vacuum state in between the plates and the one outside, which gives rise to an attractive force between the two plates. This is what has come to be known as the *static Casimir effect*. Even though Casimir made his prediction in 1948 it wasn't experimentally observed until 1998. The reason for this is that the force goes with $F \propto d^{-4}$ and is incredible small such that one typically needs distances in the submillimeter range in order to detect the attractive force. In 1948 they simply didn't have the technology to realise such a set-up.

The next step is then to allow the plates to move away such that the distance between them changes, as was first proposed by Gerald Moore[1]. There are roughly two types of displacement one can consider; an adiabatic displacement where the vacuum state has time to adjust to the increased space between the plates or a non-adiabatic displacement where the change in volume happens too quick for the system to keep up. As a larger volume means there is a larger range of allowed wavelengths, we can expect the vacuum to change. In the adiabatic case the ground state 'extends' in the sense that it will become a state that allows larger wavelengths as well. This new state, however, will still be a ground state and therefore it will still be a vacuum. In the non-adiabatic case we expect that the ground state cannot simply 'extend' as it cannot keep up with the moving walls. Instead photons will be excited from the vacuum as a reaction to field modes that have suddenly become available. We will come back to the concepts of adiabatic and non-adiabatic displacement later on.

An experimental set-up often uses "tricks" to make it seem as if the walls are moving. For example a superconducting circuit in combination with a SQUID can result in the creation of photons [2].

In this thesis we will try to determine how many "photons" are being created and whether or not it is possible to excite massive particles from the vacuum. As a simplification these "photons" will be treated like massless spin-zero scalars. For the massless case we expect to find the number of created photons to grow with increasing speed of the walls and to decrease for high-energy field modes (i.e. small wavelength modes). For the massive case we expect similar findings; the number of created particles will grow with increasing speed of the walls and decrease for higher masses.

In the next section we will start by analysing a paper by D. Dalvit, P. Maia Neta and F. Diego Mazzitelli [4] and build further upon their results.

2 Massless scalar field

Instead of a system consisting of two walls we will consider a cavity C with perfectly reflecting walls that have dimensions L_x, L_y and L_z. For $t < 0$ all walls are at rest and for $t = 0$ the wall located at $z = L_z$ will start to move according to a given trajectory $L_z(t)$ that will be specified later on. The field that defines the vacuum inside the cavity is denoted by $\phi(\mathbf{x}, t)$. It satisfies the massless Klein-Gordon equation $\Box\phi(\mathbf{x}, t) = 0$ and the boundary conditions $\phi(\mathbf{x}, t)|_{walls} = 0$ at all times. At any given time t the Fourier expansion of the initial field can be written as

$$\phi_i(\mathbf{x}, t) = \sum_{\mathbf{n}} \left(\hat{a}_{\mathbf{n}} u_{\mathbf{n}}(\mathbf{x}, t) + \hat{a}^\dagger_{\mathbf{n}} u^\dagger_{\mathbf{n}}(\mathbf{x}, t) \right), \tag{1}$$

where $u_{\mathbf{n}}(\mathbf{x}, t)$ represents a complete orthonormal set of solutions of the wave equations with vanishing boundary condition and $\hat{a}_{\mathbf{n}}, \hat{a}^\dagger_{\mathbf{n}}$ satisfy the bosonic commutation relations

$$[\hat{a}_{\mathbf{n}}, \hat{a}_{\mathbf{n}'}] = \left[\hat{a}^\dagger_{\mathbf{n}}, \hat{a}^\dagger_{\mathbf{n}'}\right] = 0 \text{ and } \left[\hat{a}_{\mathbf{n}}, \hat{a}^\dagger_{\mathbf{n}'}\right] = \delta_{\mathbf{n},\mathbf{n}'}\hat{1}. \tag{2}$$

For $t \leq 0$ the wall is at rest and every field mode is determined by the positive integers n_x,n_y and n_z, represented by the three dimensional vector $\mathbf{n}$:

$$\begin{aligned} u_{\mathbf{n}}(\mathbf{x}, t < 0) = \frac{1}{\sqrt{2\omega_{\mathbf{n}}}} \sqrt{\frac{2}{L_x}} \sin\left(\frac{n_x \pi}{L_x} x\right) \sqrt{\frac{2}{L_y}} \sin\left(\frac{n_y \pi}{L_y} y\right) \\ \times \sqrt{\frac{2}{L_z}} \sin\left(\frac{n_z \pi}{L_z} z\right) e^{-i\omega_{\mathbf{n}} t}, \end{aligned} \tag{3}$$

where $\omega_{\mathbf{n}} = \pi\sqrt{\left(\frac{n_x}{L_x}\right)^2 + \left(\frac{n_y}{L_y}\right)^2 + \left(\frac{n_z}{L_z}\right)^2}$. Note that we have used natural units, for which $\hbar = c = 1$. For $t > 0$ the wall in the xy-plane moves away, thus expanding the volume of the cavity (Figure 1), and the corresponding boundary condition on the moving wall becomes $\phi(x, y, z = L_z(t), t) = 0$. To meet this demand we consider the expansion of $u_{\mathbf{n}}(\mathbf{x}, t)$ with respect to an instantaneous basis

$$
\begin{aligned}
u_{\mathbf{n}}(\mathbf{x}, t>0) &= \sum_{\mathbf{m}} Q_{\mathbf{m}}^{(\mathbf{n})}(t) \sqrt{\frac{2}{L_x}} \sin\left(\frac{m_x \pi}{L_x} x\right) \sqrt{\frac{2}{L_y}} \sin\left(\frac{m_y \pi}{L_y} y\right) \\
&\times \sqrt{\frac{2}{L_z(t)}} \sin\left(\frac{m_z \pi}{L_z(t)} z\right) \\
&= \sum_{\mathbf{m}} Q_{\mathbf{m}}^{(\mathbf{n})}(t) \varphi_{m_x}(x, L_x) \varphi_{m_y}(y, L_y) \varphi_{m_z}(z, L_z) \\
&\equiv \sum_{\mathbf{m}} Q_{\mathbf{m}}^{(\mathbf{n})}(t) \varphi_{\mathbf{m}}(\mathbf{x}, L_z(t)),
\end{aligned} \tag{4}
$$

where $\varphi_{\mathbf{m}}(\mathbf{x}, L_z(t))$ satisfies the orthonormality condition: $\int_C d\mathbf{x} \varphi_{\mathbf{m}}(\mathbf{x}, L_z(t)) \varphi_{\mathbf{j}}(\mathbf{x}, L_z(t)) = \delta_{\mathbf{m},\mathbf{j}}$ with $\int_C d\mathbf{x} = \int_0^{L_x} dx \int_0^{L_y} dy \int_0^{L_z(t)} dz$ denoting the spatial integral over the cavity.

From the definition of $\varphi_{\mathbf{m}}(\mathbf{x}, L_z(t))$ it is clear that this expansion satisfies the

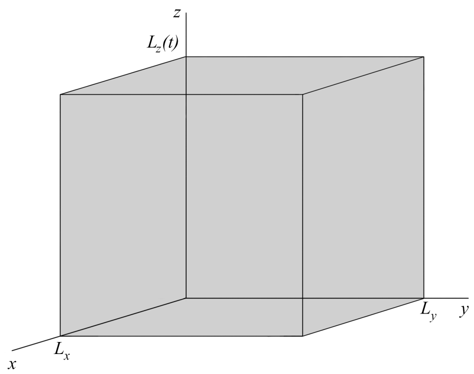

Figure 1: Cubic cavity with walls at $x = L_x$, $y = L_y$ and $z = L_z(t)$

boundary conditions. To assure that $u_{\mathbf{n}}(\mathbf{x}, t > 0)$ is continuous at $t = 0$, we impose the following initial conditions:

$$
Q_{\mathbf{m}}^{(\mathbf{n})}(0) = \frac{1}{\sqrt{2\omega_{\mathbf{m}}(0)}} \delta_{\mathbf{m},\mathbf{n}} \quad \text{and} \quad \dot{Q}_{\mathbf{m}}^{(\mathbf{n})}(0) = -\imath \sqrt{\frac{\omega_{\mathbf{m}}(0)}{2}} \delta_{\mathbf{m},\mathbf{n}}. \tag{5}
$$

We now have to demand that $u_{\mathbf{n}}$ satisfies the massless wave equation, but we can expect to find a set of coupled equations as there will be mixing between the different field-modes. To make this apparent we let the d'Alembertian, $\Box$, work on an expansion of the form of (4) in terms of the field-mode $\mathbf{j}$ and project it on the field modes $\mathbf{m}$ later on. This yields,

$$\begin{aligned}\Box u_{\mathbf{n}}(\mathbf{x},t) &= \frac{\partial^2}{\partial t^2}\left(\sum_{\mathbf{j}} Q_{\mathbf{j}}^{(\mathbf{n})}\varphi_{\mathbf{j}}\right) - \nabla^2\left(\sum_{\mathbf{j}} Q_{\mathbf{j}}^{(\mathbf{n})}\varphi_{\mathbf{j}}\right)\\ &= \sum_{\mathbf{j}} \frac{d^2 Q_{\mathbf{j}}^{(\mathbf{n})}}{dt^2}\varphi_{\mathbf{j}} + 2\sum_{\mathbf{j}} \frac{dQ_{\mathbf{j}}^{(\mathbf{n})}}{dt}\frac{\partial \varphi_{\mathbf{j}}}{\partial t} + \sum_{\mathbf{j}} Q_{\mathbf{j}}^{(\mathbf{n})}\frac{\partial^2\varphi_{\mathbf{j}}}{\partial t^2}\\ &- \sum_{\mathbf{j}} Q_{\mathbf{j}}^{(\mathbf{n})}(\nabla^2\varphi_{\mathbf{j}})\\ &= \sum_{\mathbf{j}} \frac{d^2 Q_{\mathbf{j}}^{(\mathbf{n})}}{dt^2}\varphi_{\mathbf{j}} + 2\sum_{\mathbf{j}} \frac{dQ_{\mathbf{j}}^{(\mathbf{n})}}{dt}\frac{\partial \varphi_{\mathbf{j}}}{\partial t} + \sum_{\mathbf{j}} Q_{\mathbf{j}}^{(\mathbf{n})}\frac{\partial^2\varphi_{\mathbf{j}}}{\partial t^2} + \sum_{\mathbf{j}} \omega_{\mathbf{j}}^2(t) Q_{\mathbf{j}}^{(\mathbf{n})}\varphi_{\mathbf{j}},\end{aligned}$$

where $\omega_{\mathbf{j}}(t) = \pi\sqrt{\left(\frac{j_x}{L_x}\right)^2 + \left(\frac{j_y}{L_y}\right)^2 + \left(\frac{j_z}{L_z(t)}\right)^2}$ is now time dependent. Using that $\frac{\partial\varphi_{\mathbf{j}}}{\partial t} = \frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\frac{dL_z}{dt} = \dot{L}_z\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}$ and defining $\lambda = \frac{\dot{L}_z}{L_z}$ gives

$$\begin{aligned}\Box u_{\mathbf{n}}(\mathbf{x},t) &= \sum_{\mathbf{j}} \ddot{Q}_{\mathbf{j}}^{(\mathbf{n})}\varphi_{\mathbf{j}} + 2\lambda L_z\sum_{\mathbf{j}} \dot{Q}_{\mathbf{j}}^{(\mathbf{n})}\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z} + \sum_{\mathbf{j}} Q_{\mathbf{j}}^{(\mathbf{n})}\frac{\partial}{\partial t}\left(\lambda L_z\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\right)\\ &+ \sum_{\mathbf{j}} \omega_{\mathbf{j}}^2(t) Q_{\mathbf{j}}^{(\mathbf{n})}\varphi_{\mathbf{j}}\\ &= \sum_{\mathbf{j}} \ddot{Q}_{\mathbf{j}}^{(\mathbf{n})}\varphi_{\mathbf{j}} + 2\lambda L_z\sum_{\mathbf{j}} \dot{Q}_{\mathbf{j}}^{(\mathbf{n})}\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z} + \dot{\lambda} L_z\sum_{\mathbf{j}} Q_{\mathbf{j}}^{(\mathbf{n})}\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\\ &+ \lambda\sum_{\mathbf{j}} Q_{\mathbf{j}}^{(\mathbf{n})}\frac{\partial}{\partial t}\left(L_z\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\right) + \sum_{\mathbf{j}} \omega_{\mathbf{j}}^2(t) Q_{\mathbf{j}}^{(\mathbf{n})}\varphi_{\mathbf{j}}.\end{aligned}$$

We can rewrite the first term on the last line to

$$\begin{aligned}\lambda\sum_{\mathbf{j}} Q_{\mathbf{j}}^{(\mathbf{n})}\frac{\partial}{\partial t}\left(L_z\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\right) &= \lambda\sum_{\mathbf{j}} Q_{\mathbf{j}}^{(\mathbf{n})}\dot{L}_z\frac{\partial}{\partial L_z}\left(L_z\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\right)\\ &= \lambda^2 L_z\sum_{\mathbf{j}} Q_{\mathbf{j}}^{(\mathbf{n})}\frac{\partial}{\partial L_z}\left(L_z\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\right).\end{aligned}$$

Now we plug this back in, equate $\Box u_{\mathbf{n}}(\mathbf{x},t)$ to zero and project onto the set $\varphi_{\mathbf{m}}$ to find how the different field-modes mix. This gives

$$\begin{aligned}&\int_C d\mathbf{x}\sum_{\mathbf{j}} \ddot{Q}_{\mathbf{j}}^{(\mathbf{n})}\varphi_{\mathbf{j}}\varphi_{\mathbf{m}} + \int_C d\mathbf{x}\sum_{\mathbf{j}} \omega_{\mathbf{j}}^2(t) Q_{\mathbf{j}}^{(\mathbf{n})}\varphi_{\mathbf{j}}\varphi_{\mathbf{m}}\\ &= -2\lambda(t)L_z(t)\int_C d\mathbf{x}\sum_{\mathbf{j}} \dot{Q}_{\mathbf{j}}^{(\mathbf{n})}\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\varphi_{\mathbf{m}} - \dot{\lambda}(t)L_z(t)\int_C d\mathbf{x}\sum_{\mathbf{j}} Q_{\mathbf{j}}^{(\mathbf{n})}\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\varphi_{\mathbf{m}}\\ &- \lambda^2(t)L_z(t)\int_C d\mathbf{x}\sum_{\mathbf{j}} Q_{\mathbf{j}}^{(\mathbf{n})}\frac{\partial}{\partial L_z}\left(L_z(t)\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\right)\varphi_{\mathbf{m}}.\end{aligned}$$

As $\int\sum|\varphi_{\mathbf{j}}\varphi_{\mathbf{m}}| < \infty$ and $\int\sum|\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\varphi_{\mathbf{m}}| < \infty$, we can use Fubini's theorem to interchange integration and summation. Besides that we can exploit the fact that $\varphi_{\mathbf{j}}$ as well as $\varphi_{\mathbf{m}}$ belong to an orthonormal set, i.e. $\int_C d\mathbf{x}\varphi_{\mathbf{j}}\varphi_{\mathbf{m}} = \delta_{\mathbf{j},\mathbf{m}}$ to find

$$\begin{aligned}\ddot{Q}^{(\mathbf{n})}_{\mathbf{m}} + \omega^2_{\mathbf{m}}(t)Q^{(\mathbf{n})}_{\mathbf{m}} &= -2\lambda(t)\sum_{\mathbf{j}} L_z(t)\int_C d\mathbf{x}\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\varphi_{\mathbf{m}}\dot{Q}^{(\mathbf{n})}_{\mathbf{j}} \\ &- \dot{\lambda}(t)\sum_{\mathbf{j}} L_z(t)\int_C d\mathbf{x}\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\varphi_{\mathbf{m}}Q^{(\mathbf{n})}_{\mathbf{j}} \\ &- \lambda^2(t)\sum_{\mathbf{j}} L_z(t)\int_C d\mathbf{x}\frac{\partial}{\partial L_z}\left(L_z(t)\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\right)\varphi_{\mathbf{m}}Q^{(\mathbf{n})}_{\mathbf{j}}.\end{aligned}$$

To further simplify the differential equation we define

$$L_z(t)\int_C d\mathbf{x}\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\varphi_{\mathbf{m}} = G^z_{\mathbf{jm}} = -G^z_{\mathbf{mj}} \equiv g_{j_z m_z}\delta_{j_x m_x}\delta_{j_y m_y}, \tag{6}$$

where

$$g_{j_z m_z} \equiv L_z(t)\int_0^{L_z(t)} dz\frac{\partial\varphi_{j_z}(z, L_z(t))}{\partial L_z}\varphi_{m_z}(z, L_z(t)). \tag{7}$$

By defining $G^z_{\mathbf{jm}}$ as above all mixing of modes is now captured in one variable. Plugging in $G^z_{\mathbf{jm}}$ results in

$$\begin{aligned}\ddot{Q}^{(\mathbf{n})}_{\mathbf{m}} + \omega^2_{\mathbf{m}}(t)Q^{(\mathbf{n})}_{\mathbf{m}} &= 2\lambda(t)\sum_{\mathbf{j}} G^z_{\mathbf{mj}}\dot{Q}^{(\mathbf{n})}_{\mathbf{j}} + \dot{\lambda}(t)\sum_{\mathbf{j}} G^z_{\mathbf{mj}}Q^{(\mathbf{n})}_{\mathbf{j}} \\ &- \lambda^2(t)\sum_{\mathbf{j}} L_z(t)\int_C d\mathbf{x}\frac{\partial}{\partial L_z}\left(L_z(t)\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\right)\varphi_{\mathbf{m}}Q^{(\mathbf{n})}_{\mathbf{j}}.\end{aligned}$$

The last eyesore is now the integral in the last line. To bring this term into the same form as the rest of the equation, we rewrite it and apply the completeness relation

$$\begin{aligned}&-\lambda^2(t)\sum_{\mathbf{j}} L_z(t)\int_C d\mathbf{x}\frac{\partial}{\partial L_z}\left(L_z(t)\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\right)\varphi_{\mathbf{m}}Q^{(\mathbf{n})}_{\mathbf{j}} \\ &= -\lambda^2(t)\sum_{\mathbf{j}} L_z(t)\int_C d\mathbf{x}\frac{\partial}{\partial L_z}\left(L_z(t)\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\varphi_{\mathbf{m}}\right)Q^{(\mathbf{n})}_{\mathbf{j}} \\ &+ \lambda^2(t)\sum_{\mathbf{j}} L^2_z(t)\int_C d\mathbf{x}\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\frac{\partial\varphi_{\mathbf{m}}}{\partial L_z}Q^{(\mathbf{n})}_{\mathbf{j}} \\ &\overset{[1]}{=\!=} -\lambda^2(t)\sum_{\mathbf{j}} L_z(t)\frac{\partial}{\partial L_z}\left(\int_C d\mathbf{x}L_z(t)\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\varphi_{\mathbf{m}}\right)Q^{(\mathbf{n})}_{\mathbf{j}} \\ &+ \lambda^2(t)\sum_{\mathbf{j},\mathbf{l}} L_z(t)\int_C d\mathbf{x}\left(\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\varphi_{\mathbf{l}}\right)L_z(t)\int_C d\mathbf{x}'\left(\varphi_{\mathbf{l}}\frac{\partial\varphi_{\mathbf{m}}}{\partial L_z}\right)Q^{(\mathbf{n})}_{\mathbf{j}} \\ &= -\lambda^2(t)\sum_{\mathbf{j}} L_z(t)\frac{\partial}{\partial L_z}\left(\int_C d\mathbf{x}L_z(t)\frac{\partial\varphi_{\mathbf{j}}}{\partial L_z}\varphi_{\mathbf{m}}\right)Q^{(\mathbf{n})}_{\mathbf{j}} \\ &+ \lambda^2(t)\sum_{\mathbf{j},\mathbf{l}} G^z_{\mathbf{jl}}G^z_{\mathbf{ml}}Q^{(\mathbf{n})}_{\mathbf{j}}.\end{aligned} \tag{8}$$

Analysing the dimensions of the first term in the last equality, proves that the expression within the integral becomes independent of $L_z(t)$ after integration. Therefore the derivative of the integral yields zero, leaving only the second term. Plugging in $\lambda^2(t)\sum_{\mathbf{j},\mathbf{l}} G^z_{\mathbf{jl}} G^z_{\mathbf{ml}} Q^{(\mathbf{n})}_{\mathbf{j}} = -\lambda^2(t)\sum_{\mathbf{j},\mathbf{l}} G^z_{\mathbf{ml}} G^z_{\mathbf{lj}} Q^{(\mathbf{n})}_{\mathbf{j}}$ finally results in

$$\begin{aligned}\ddot{Q}^{(\mathbf{n})}_{\mathbf{m}}(t) + \omega^2_{\mathbf{m}}(t) Q^{(\mathbf{n})}_{\mathbf{m}}(t) &= 2\lambda(t)\sum_{\mathbf{j}} G^z_{\mathbf{mj}} \dot{Q}^{(\mathbf{n})}_{\mathbf{j}}(t) + \dot{\lambda}(t)\sum_{\mathbf{j}} G^z_{\mathbf{mj}} Q^{(\mathbf{n})}_{\mathbf{j}}(t) \\ &\quad - \lambda^2(t)\sum_{\mathbf{j},\mathbf{l}} G^z_{\mathbf{ml}} G^z_{\mathbf{lj}} Q^{(\mathbf{n})}_{\mathbf{j}}(t).\end{aligned} \tag{9}$$

Eq.(9) shows how the massless scalar field changes in time. Before the wall starts its trajectory ($t < 0$), we are dealing with a free situation where particles are being created and annihilated by $\hat{a}^\dagger_{\mathbf{n}}$ respectively $\hat{a}_{\mathbf{n}}$. When the wall stops moving at $t = t_f$, the system has changed and we are yet again dealing with a free situation. With this "new" system we can associate a different particle interpretation, by means of a new set of creation and annihilation operators $\hat{c}^\dagger_{\mathbf{m}}$,$\hat{c}_{\mathbf{m}}$. The relation between the new and the old set of operators can be found through a Bogoliubov transformation [6]

$$\begin{aligned}\hat{c}_{\mathbf{m}} &= \sum_{\mathbf{n}} (\alpha_{\mathbf{nm}} \hat{a}_{\mathbf{n}} + \beta^\star_{\mathbf{nm}} \hat{a}^\dagger_{\mathbf{n}}) \\ \hat{c}^\dagger_{\mathbf{m}} &= \sum_{\mathbf{n}} (\alpha^\star_{\mathbf{nm}} \hat{a}^\dagger_{\mathbf{n}} + \beta_{\mathbf{nm}} \hat{a}_{\mathbf{n}}).\end{aligned} \tag{10}$$

In order to find $\alpha_{\mathbf{nm}}$ and $\beta_{\mathbf{nm}}$ we need to define the field belonging to the new free situation ($t \geq t_f$). The equivalent of Eq. (3) for the new situation becomes

$$\begin{aligned}u_{\mathbf{m}}(\mathbf{x}, t > t_f) &= \frac{1}{\sqrt{2\omega_{\mathbf{m}}(t_f)}} \sqrt{\frac{2}{L_x}} \sin\left(\frac{m_x \pi}{L_x} x\right) \sqrt{\frac{2}{L_y}} \sin\left(\frac{m_y \pi}{L_y} y\right) \\ &\quad \times \sqrt{\frac{2}{L_z(t_f)}} \sin\left(\frac{m_z \pi}{L_z(t_f)} z\right) e^{-i\omega_{\mathbf{m}}(t_f)(t-t_f)} \\ &= \frac{1}{\sqrt{2\omega_{\mathbf{m}}(t_f)}} \varphi_{\mathbf{m}}(\mathbf{x}, L_z(t_f)) e^{-i\omega_{\mathbf{m}}(t_f)(t-t_f)},\end{aligned}$$

which leads to the new field

[1]Where the differentiation can be pulled out of the integral since $L_z(t)\frac{\partial \varphi_{\mathbf{j}}}{\partial L_z} \varphi_{\mathbf{m}}|_{z=L_z(t)} = 0$

$$
\begin{aligned}
\phi_f(\mathbf{x},t) &= \sum_{\mathbf{m}} \left[\hat{c}_{\mathbf{m}} u_{\mathbf{m}}(\mathbf{x},t) + \hat{c}^{\dagger}_{\mathbf{m}} u^{\dagger}_{\mathbf{m}}(\mathbf{x},t)\right] \\
&= \sum_{\mathbf{m}} \frac{1}{\sqrt{2\omega_{\mathbf{m}}(t_f)}} \left[\hat{c}_{\mathbf{m}} e^{-i\omega_{\mathbf{m}}(t_f)(t-t_f)} + \hat{c}^{\dagger}_{\mathbf{m}} e^{i\omega_{\mathbf{m}}(t_f)(t-t_f)}\right] \varphi_{\mathbf{m}}(\mathbf{x}, L_z(t_f)) \\
&= \sum_{\mathbf{mn}} \frac{1}{\sqrt{2\omega_{\mathbf{m}}(t_f)}} \left[(\alpha_{\mathbf{nm}}\hat{a}_{\mathbf{n}} + \beta^{\star}_{\mathbf{nm}}\hat{a}^{\dagger}_{\mathbf{n}}) e^{-i\omega_{\mathbf{m}}(t_f)(t-t_f)}\right] \varphi_{\mathbf{m}}(\mathbf{x}, L_z(t_f)) \\
&+ \sum_{\mathbf{mn}} \frac{1}{\sqrt{2\omega_{\mathbf{m}}(t_f)}} \left[(\alpha^{\star}_{\mathbf{nm}}\hat{a}^{\dagger}_{\mathbf{n}} + \beta_{\mathbf{nm}}\hat{a}_{\mathbf{n}}) e^{i\omega_{\mathbf{m}}(t_f)(t-t_f)}\right] \varphi_{\mathbf{m}}(\mathbf{x}, L_z(t_f)) \\
&= \sum_{\mathbf{mn}} \frac{\hat{a}_{\mathbf{n}}}{\sqrt{2\omega_{\mathbf{m}}(t_f)}} \left[\alpha_{\mathbf{nm}} e^{-i\omega_{\mathbf{m}}(t_f)(t-t_f)} + \beta_{\mathbf{nm}} e^{i\omega_{\mathbf{m}}(t_f)(t-t_f)}\right] \varphi_{\mathbf{m}}(\mathbf{x}, L_z(t_f)) \\
&+ \sum_{\mathbf{mn}} \frac{\hat{a}^{\dagger}_{\mathbf{n}}}{\sqrt{2\omega_{\mathbf{m}}(t_f)}} \left[\alpha^{\star}_{\mathbf{nm}} e^{i\omega_{\mathbf{m}}(t_f)(t-t_f)} + \beta^{\star}_{\mathbf{nm}} e^{-i\omega_{\mathbf{m}}(t_f)(t-t_f)}\right] \varphi_{\mathbf{m}}(\mathbf{x}, L_z(t_f)).
\end{aligned}
$$

Next, we need to know what the old field $\phi_i(\mathbf{x},t)$ looks like for $t > t_f$. When the wall stops moving (9) reduces to a simple harmonic oscillator, with solutions

$$
Q^{(\mathbf{n})}_{\mathbf{m}}(t > t_f) = A^{(\mathbf{n})}_{\mathbf{m}} e^{i\omega_{\mathbf{m}}(t_f)t} + B^{(\mathbf{n})}_{\mathbf{m}} e^{-i\omega_{\mathbf{m}}(t_f)t}, \tag{11}
$$

with this the old field can be written as

$$
\begin{aligned}
\phi_i(\mathbf{x}, t > t_f) &= \sum_{\mathbf{n}} \left(\hat{a}_{\mathbf{n}} u_{\mathbf{n}}(\mathbf{x}, t > t_f) + \hat{a}^{\dagger}_{\mathbf{n}} u^{\dagger}_{\mathbf{n}}(\mathbf{x}, t > t_f)\right) \\
&= \sum_{\mathbf{mn}} \hat{a}_{\mathbf{n}} \left[A^{(\mathbf{n})}_{\mathbf{m}} e^{i\omega_{\mathbf{m}}(t_f)t} + B^{(\mathbf{n})}_{\mathbf{m}} e^{-i\omega_{\mathbf{m}}(t_f)t}\right] \varphi_{\mathbf{m}}(\mathbf{x}, L_z(t_f)) \\
&+ \sum_{\mathbf{mn}} \hat{a}^{\dagger}_{\mathbf{n}} \left[A^{(n)*}_{\mathbf{m}} e^{-i\omega_{\mathbf{m}}(t_f)t} + B^{(n)*}_{\mathbf{m}} e^{i\omega_{\mathbf{m}}(t_f)t}\right] \varphi_{\mathbf{m}}(\mathbf{x}, L_z(t_f)).
\end{aligned}
$$

As $\phi_i(\mathbf{x},t)$ for $t > t_f$ needs to be equal to $\phi_f(\mathbf{x},t)$, we can deduce that

$$
\begin{aligned}
\beta_{\mathbf{nm}} &= \sqrt{2\omega_{\mathbf{m}}(t_f)} A^{(\mathbf{n})}_{\mathbf{m}} e^{i\omega_{\mathbf{m}}(t_f)t_f} = \sqrt{2\omega_{\mathbf{m}}(t_f)} A^{(\mathbf{n})'}_{\mathbf{m}}, \\
\alpha_{\mathbf{nm}} &= \sqrt{2\omega_{\mathbf{m}}(t_f)} B^{(\mathbf{n})}_{\mathbf{m}} e^{-i\omega_{\mathbf{m}}(t_f)t_f} = \sqrt{2\omega_{\mathbf{m}}(t_f)} B^{(\mathbf{n})'}_{\mathbf{m}}.
\end{aligned}
$$

The average number of particles created in the mode $\mathbf{m}$ is the expectation value of $\hat{c}^{\dagger}_{\mathbf{m}}\hat{c}_{\mathbf{m}}$ with respect to the old vacuum state $|0_i\rangle$, defined by $\hat{a}_{\mathbf{m}}|0_i\rangle = 0$. Which results in

$$
\begin{aligned}
\langle 0_i | \, \hat{c}^{\dagger}_{\mathbf{m}} \hat{c}_{\mathbf{m}} \, | 0_i \rangle &= \langle 0_i | \, (\sum_{\mathbf{n}} \sqrt{2\omega_{\mathbf{m}}(t_f)} B^{(\mathbf{n})'\star}_{\mathbf{m}} \hat{a}^{\dagger}_{\mathbf{n}} + \sqrt{2\omega_{\mathbf{m}}(t_f)} A^{(\mathbf{n})'}_{\mathbf{m}} \hat{a}_{\mathbf{n}}) \\
&\times \sum_{\mathbf{n}'} (\sqrt{2\omega_{\mathbf{m}}(t_f)} B^{(\mathbf{n}'\mathbf{j})'}_{\mathbf{m}} \hat{a}_{\mathbf{n}'} + \sqrt{2\omega_{\mathbf{m}}(t_f)} A^{(\mathbf{n}')'\star}_{\mathbf{m}} \hat{a}^{\dagger}_{\mathbf{n}'}) \, | 0_i \rangle \\
&= \sum_{\mathbf{n},\mathbf{n}'} 2\omega_{\mathbf{m}}(t_f) B^{(\mathbf{n})'\star}_{\mathbf{m}} B^{(\mathbf{n}')'}_{\mathbf{m}} \langle 0_i | \, \hat{a}^{\dagger}_{\mathbf{n}} \hat{a}_{\mathbf{n}'} \, | 0_i \rangle + \sum_{\mathbf{n},\mathbf{n}'} 2\omega_{\mathbf{m}}(t_f) B^{(\mathbf{n})'\star}_{\mathbf{m}} A^{(\mathbf{n}')'\star}_{\mathbf{m}} \langle 0_i | \, \hat{a}^{\dagger}_{\mathbf{n}} \hat{a}^{\dagger}_{\mathbf{n}'} \, | 0_i \rangle \\
&+ \sum_{\mathbf{n},\mathbf{n}'} 2\omega_{\mathbf{m}}(t_f) A^{(\mathbf{n})'}_{\mathbf{m}} B^{(\mathbf{n}')'}_{\mathbf{m}} \langle 0_i | \, \hat{a}_{\mathbf{n}} \hat{a}_{\mathbf{n}'} \, | 0_i \rangle + \sum_{\mathbf{n},\mathbf{n}'} 2\omega_{\mathbf{m}}(t_f) A^{(\mathbf{n})'}_{\mathbf{m}} A^{(\mathbf{n}')'\star}_{\mathbf{m}} \langle 0_i | \, \hat{a}_{\mathbf{n}} \hat{a}^{\dagger}_{\mathbf{n}'} \, | 0_i \rangle \\
&= \sum_{\mathbf{n},\mathbf{n}'} 2\omega_{\mathbf{m}}(t_f) A^{(\mathbf{n})'}_{\mathbf{m}} A^{(\mathbf{n}')'\star}_{\mathbf{m}} \delta_{\mathbf{n},\mathbf{n}'} \\
&= \sum_{\mathbf{n}} 2\omega_{\mathbf{m}}(t_f) A^{(\mathbf{n})'}_{\mathbf{m}} A^{(\mathbf{n})'\star}_{\mathbf{m}} \\
&= \sum_{\mathbf{n}} 2\omega_{\mathbf{m}}(t_f) |A^{(\mathbf{n})}_{\mathbf{m}}|^2 .
\end{aligned} \tag{12}
$$

From the equation above we can see that the number of created particles depends on the energy of the field modes $\mathbf{m}$ at $t = t_f$ and the absolute value of some constant $|A^{(\mathbf{n})}_{\mathbf{m}}|$. The exact form of this constant can be derived from the boundary conditions at $t = t_f$. To do this we need to solve the differential equation (9) .

3 Three-dimensional expansion

Up till now we've been dealing with a three dimensional system expanding in one direction (z-direction). This means that $\omega(t)$ has the form of a square root of three components where only one of those components is time dependent. As a consequence we cannot pull the time dependency of $\omega(t)$ out of the root and it becomes rather tricky to solve the differential equation. In order to simplify the problem we will therefore consider a system that expands in three directions. To be more specific we define

$$L_x(t) = L_y(t) = L_z(t) = L(t). \tag{13}$$

By doing so we are able to write

$$
\begin{aligned}
u_{\mathbf{n}}(\mathbf{x}, t > 0) &= \sum_{\mathbf{m}} Q^{(\mathbf{n})}_{\mathbf{m}}(t) \sqrt{\frac{8}{L^3(t)}} \sin\left(\frac{m_x \pi}{L(t)} x\right) \sin\left(\frac{m_y \pi}{L(t)} y\right) \\
&\times \sin\left(\frac{m_z \pi}{L(t)} z\right) \\
&= \sum_{\mathbf{m}} Q^{(\mathbf{n})}_{\mathbf{m}}(t) \varphi_{\mathbf{m}}(\mathbf{x}, L(t)).
\end{aligned} \tag{14}
$$

Of course the transition from a one dimensional displacement to three dimensions will change the differential equation as well. By following the same procedure as before we arrive at

$$\ddot{Q}_{\mathbf{m}}^{(\mathbf{n})} + \omega_{\mathbf{m}}^2(t) Q_{\mathbf{m}}^{(\mathbf{n})} = 2\lambda(t) \sum_{\mathbf{j}} G_{\mathbf{mj}} \dot{Q}_{\mathbf{j}}^{(\mathbf{n})} + \dot{\lambda}(t) \sum_{\mathbf{j}} G_{\mathbf{mj}} Q_{\mathbf{j}}^{(\mathbf{n})}$$
$$- \lambda^2(t) \sum_{\mathbf{j}} L(t) \int_0^{L(t)} d\mathbf{x} \frac{\partial}{\partial L(t)} \left(L(t) \frac{\partial \varphi_{\mathbf{j}}}{\partial L} \right) \varphi_{\mathbf{m}} Q_{\mathbf{j}}^{(\mathbf{n})}, \tag{15}$$

where

$$\lambda(t) = \frac{\dot{L(t)}}{L(t)},$$
$$G_{\mathbf{mj}} = -G_{\mathbf{jm}} \equiv L(t) \int_0^{L(t)} d\mathbf{x} \frac{\partial \varphi_{\mathbf{m}}}{\partial L(t)} \varphi_{\mathbf{j}}. \tag{16}$$

Rewriting the last term on the right-hand side of (15) and applying the completeness relation leads to an expression similar to (8):

$$-\lambda^2(t) \sum_{\mathbf{j}} L(t) \int_0^{L(t)} d\mathbf{x} \frac{\partial}{\partial L} \left(L(t) \frac{\partial \varphi_{\mathbf{j}}}{\partial L} \right) \varphi_{\mathbf{m}} Q_{\mathbf{j}}^{(\mathbf{n})}$$
$$= -\lambda^2(t) \sum_{\mathbf{j}} L(t) \frac{\partial}{\partial L} \left(\int_0^{L(t)} d\mathbf{x} L(t) \frac{\partial \varphi_{\mathbf{j}}}{\partial L} \varphi_{\mathbf{m}} \right) Q_{\mathbf{j}}^{(\mathbf{n})} \tag{17}$$
$$+ \lambda^2(t) \sum_{\mathbf{j},\mathbf{l}} G_{\mathbf{jl}} G_{\mathbf{ml}} Q_{\mathbf{j}}^{(\mathbf{n})}.$$

In (8) we used that the derivative with respect to $L_z(t)$ of the integral in the first term on the right yields 0. It turns out that the same applies to (17), as the higher dimensions in $L(t)$ are being taken care of by the three dimensional integral. Checking whether this is really the case is easily done (see section 5) and we find

$$G_{\mathbf{mj}} = G_{\mathbf{mj}}^x + G_{\mathbf{mj}}^y + G_{\mathbf{mj}}^z \tag{18}$$

Which leads to the three dimensional form of (9)

$$\ddot{Q}_{\mathbf{m}}^{(\mathbf{n})}(t) + \omega_{\mathbf{m}}^2(t) Q_{\mathbf{m}}^{(\mathbf{n})}(t) = 2\lambda(t) \sum_{\mathbf{j}} G_{\mathbf{mj}} \dot{Q}_{\mathbf{j}}^{(\mathbf{n})}(t) + \dot{\lambda}(t) \sum_{\mathbf{j}} G_{\mathbf{mj}} Q_{\mathbf{j}}^{(\mathbf{n})}(t)$$
$$- \lambda^2(t) \sum_{\mathbf{j},\mathbf{l}} G_{\mathbf{ml}} G_{\mathbf{lj}} Q_{\mathbf{j}}^{(\mathbf{n})}(t), \tag{19}$$

where

$$\omega_{\mathbf{m}}(t) = \pi \sqrt{\left(\frac{m_x}{L(t)}\right)^2 + \left(\frac{m_y}{L(t)}\right)^2 + \left(\frac{m_z}{L(t)}\right)^2} = \frac{\pi |\mathbf{m}|}{L(t)}. \tag{20}$$

3.1 Massive scalar field

So far only the creation of massless particles has been considered. In order to determine whether it is possible to excite massive particles from the vacuum, one needs to add a mass term to the energy of our system. In this section we will build the mathematical framework necessary to generalise the previous calculations to a massive scalar field. The only difference between the massless and the massive case is basically that for massive particles there is an extra 'source' of energy. That is, $\omega_{\mathbf{n}}(t) = \pi\sqrt{\frac{n_x^2+n_y^2+n_z^2}{L^2(t)}}$ should be replaced by $\omega'_{\mathbf{n}}(t) = \sqrt{\pi^2\frac{n_x^2+n_y^2+n_z^2}{L^2(t)} + M^2}$, with M the mass of the particle to be. Taking a closer look at the formulas from which we started out, there are a few observations to be made. Equivalent to (3) and (4) we can write

$$\begin{aligned} u'_{\mathbf{n}}(\mathbf{x}, t<0) = &\frac{1}{\sqrt{2\omega'_{\mathbf{n}}}}\sqrt{\frac{2}{L_x}}\sin\left(\frac{n_x\pi}{L_x}x\right)\sqrt{\frac{2}{L_y}}\sin\left(\frac{n_y\pi}{L_y}y\right) \\ &\times\sqrt{\frac{2}{L_z}}\sin\left(\frac{n_z\pi}{L_z}z\right)e^{-i\omega'_{\mathbf{n}}t} \end{aligned}$$

and

$$\begin{aligned} u'_{\mathbf{n}}(\mathbf{x}, t>0) &= \sum_{\mathbf{m}} Q'^{(\mathbf{n})}_{\mathbf{m}}(t)\sqrt{\frac{2}{L(t)}}\sin\left(\frac{m_x\pi}{L(t)}x\right)\sqrt{\frac{2}{L(t)}}\sin\left(\frac{m_y\pi}{L(t)}y\right) \\ &\times\sqrt{\frac{2}{L(t)}}\sin\left(\frac{m_z\pi}{L(t)}z\right) \\ &= \sum_{\mathbf{m}} Q'^{(\mathbf{n})}_{\mathbf{m}}(t)\varphi_{\mathbf{m}}(\mathbf{x}, L(t)), \end{aligned}$$

where the spatial part, $\varphi_{\mathbf{m}}(\mathbf{x}, L_(t))$, remains the same but the temporal part, $Q'^{(\mathbf{n})}_{\mathbf{m}}(t)$, changes. This last field now has to satisfy the Klein-Gordon equation $\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + M^2\right)u'_{\mathbf{n}}(\mathbf{x}, t) = 0$ and thus our differential equation transforms into

$$\begin{aligned} &\ddot{Q}'^{(\mathbf{n})}_{\mathbf{m}} + \left(\omega^2_{\mathbf{m}}(t) + M^2\right)Q'^{(\mathbf{n})}_{\mathbf{m}} = \ddot{Q}'^{(\mathbf{n})}_{\mathbf{m}} + \omega'^2_{\mathbf{m}}(t)Q'^{(\mathbf{n})}_{\mathbf{m}} \\ &= 2\lambda(t)\sum_{\mathbf{j}} G_{\mathbf{mj}}\dot{Q}'^{(\mathbf{n})}_{\mathbf{j}} + \dot{\lambda}(t)\sum_{\mathbf{j}} G_{\mathbf{mj}}Q'^{(\mathbf{n})}_{\mathbf{j}} - \lambda^2(t)\sum_{\mathbf{j},\mathbf{l}} G_{\mathbf{ml}}G_{\mathbf{lj}}Q'^{(\mathbf{n})}_{\mathbf{j}}. \end{aligned}$$

Following exactly the same steps as before we find that for $t > t_f$

$$Q'^{(\mathbf{n})}_{\mathbf{m}}(t > t_f) = A^{(\mathbf{n})}_{\mathbf{m}}e^{i\omega'_{\mathbf{m}}(t_f)t} + B^{(\mathbf{n})}_{\mathbf{m}}e^{-i\omega'_{\mathbf{m}}(t_f)t}.$$

This can be used to do another Bogoliubov transformation (10) which eventually leads to the average number of massive particles

$$\begin{aligned} \langle N_{\mathbf{m}}\rangle &= \langle 0_i|\,\hat{c}^\dagger_{\mathbf{m}}\hat{c}_{\mathbf{m}}\,|0_i\rangle \\ &= \sum_{\mathbf{n}} 2\omega'_{\mathbf{m}}(t_f)|A^{(\mathbf{n})}_{\mathbf{m}}|^2 \\ &= \sum_{\mathbf{n}} 2\sqrt{\omega^2_{\mathbf{m}}(t_f) + M^2}|A^{(\mathbf{n})}_{\mathbf{m}}|^2. \end{aligned}$$

Yet again we will need to plug in $A^{(\mathbf{n})}_{\mathbf{m}}$, this time however we will need the $A^{(\mathbf{n})}_{\mathbf{m}}$ corresponding to massive particles.

4 Linear movement

Now that we have a three dimensional form of the differential equation, we can try to find a solution for the massless case. To this end we write $L(t)$ in a more definite form. Let's consider a constant velocity such that

$$\begin{aligned} L(t) &= a + vt \\ \dot{L}(t) &= v \\ \ddot{L}(t) &= 0 \\ \lambda(t) &= \frac{v}{a+vt} \\ \dot{\lambda}(t) &= -\frac{v^2}{(a+vt)^2} = -\lambda^2(t). \end{aligned}$$

If we momentarily simplify (9) by neglecting the sums and indices and use that $\dot{\lambda}(t) = -\lambda^2(t)$, then the type of differential equation to solve becomes

$$\ddot{Q}(t) + \omega^2(t)Q(t) = 2\lambda(t)G\dot{Q}(t) - \lambda^2(t)GQ(t) - \lambda^2(t)G^2Q(t). \qquad (21)$$

Now let $Q(t) = e^{G\ln\left(\frac{L(t)}{a}\right)}f(t)$ with $f(t)$ some function we need to determine. For the first and second derivative we find

$$\begin{aligned} \dot{Q}(t) &= \lambda(t)GQ(t) + \frac{\dot{f}(t)}{f(t)}Q(t) \\ \ddot{Q}(t) &= \lambda(t)G\dot{Q}(t) - \lambda^2(t)GQ(t) + \lambda(t)G\frac{\dot{f}(t)}{f(t)}Q(t) + \frac{\ddot{f}(t)}{f(t)}Q(t) \\ &\overset{[2]}{=\!=} 2\lambda(t)G\dot{Q}(t) - \lambda^2(t)GQ(t) - \lambda^2(t)G^2Q(t) + \frac{\ddot{f}(t)}{f(t)}Q(t), \end{aligned}$$

from which it follows that

$$\frac{\ddot{f}(t)}{f(t)} = -\omega^2(t).$$

Rewriting $\omega(t)$ yields

$$\begin{aligned} \omega^2(t) &= \left(\frac{\pi|\mathbf{m}|}{a+vt}\right)^2 \\ &= \left(\frac{a\omega(0)}{a+vt}\right)^2 \\ &= \left(\frac{a\omega(0)}{v}\right)^2\left(\frac{a}{v}+t\right)^{-2}, \end{aligned}$$

where $\omega(0) = \frac{\pi|\mathbf{m}|}{a}$. By setting $\kappa \equiv \left(\frac{a\omega(0)}{v}\right)^2$ we need to find $f(t)$ such that

$$\frac{\ddot{f}(t)}{f(t)} = -\kappa\left(\frac{a}{v}+t\right)^{-2}.$$

[2]Note that $\lambda(t)G\frac{\dot{f}(t)}{f(t)}Q(t) = \lambda(t)G\dot{Q}(t) - \lambda^2(t)G^2Q(t)$

As an ansatz we take $f(t) = C\left(\frac{a}{v}+t\right)^{\alpha/2}$, where C is some constant to be determined by initial conditions. With this ansatz we find

$$\begin{aligned}\frac{\ddot{f}(t)}{f(t)} &= \frac{\alpha}{2}(\frac{\alpha}{2}-1)\left(\frac{a}{v}+t\right)^{-2} \\ &= -\kappa\left(\frac{a}{v}+t\right)^{-2}.\end{aligned}$$

Dividing by $\left(\frac{a}{v}+t\right)^{-2}$ and solving for α

$$\begin{aligned}&\alpha^2 - 2\alpha + 4\kappa = 0 \\ &\Rightarrow \alpha = 1 \pm \sqrt{1-4\kappa}.\end{aligned}$$

The expression for $Q(t)$ now becomes

$$Q(t) = e^{G\ln\left(\frac{L(t)}{a}\right)} C\left(\frac{a}{v}+t\right)^{\frac{1}{2}1\pm\sqrt{1-4\kappa}}. \tag{22}$$

To find C we ensure that $Q(t)$ satisfies the (unlabelled) initial conditions

$$\begin{aligned}Q(0) &= \frac{1}{\sqrt{2\omega(0)}} \\ \dot{Q}(0) &= -i\sqrt{\frac{\omega(0)}{2}},\end{aligned} \tag{23}$$

given that $v = 0$ for $t < 0$. This means that each field mode and its derivative will be continuous functions at $t = 0$. Then

$$\begin{aligned}Q(0) = C\left(\frac{a}{v}\right)^{1\pm\sqrt{1-4\kappa}} &= \frac{1}{\sqrt{2\omega(0)}} \\ \Rightarrow C = \frac{1}{\sqrt{2\omega(0)}}&\left(\frac{a}{v}\right)^{-\frac{1}{2}(1\pm\sqrt{1-4\kappa})}.\end{aligned}$$

The second condition implies a relation between $\frac{v}{a}$ and $\omega(0)$, according to

$$\begin{aligned}\dot{Q}(0) &= C\cdot\frac{1}{2}(1\pm\sqrt{1-4\kappa})\left(\frac{a}{v}\right)^{-\frac{1}{2}(1\mp\sqrt{1-4\kappa})} + CG\frac{v}{a}\left(\frac{a}{v}\right)^{\frac{1}{2}(1\pm\sqrt{1-4\kappa})} \\ &= \frac{1}{\sqrt{2\omega(0)}}\left(\frac{a}{v}\right)^{-\frac{1}{2}(1\pm\sqrt{1-4\kappa})}\cdot\frac{1}{2}(2G+1\pm\sqrt{1-4\kappa})\left(\frac{a}{v}\right)^{-\frac{1}{2}(1\mp\sqrt{1-4\kappa})} \\ &= \frac{G+\frac{1}{2}\pm\frac{1}{2}\sqrt{1-4\kappa}}{\sqrt{2\omega(0)}}\cdot\frac{v}{a} = -i\sqrt{\frac{\omega(0)}{2}} \\ &\Rightarrow \left(G+\frac{1}{2}\pm\frac{1}{2}\sqrt{1-4\left(\frac{a\omega(0)}{v}\right)^2}\right)\frac{v}{a} = -i\omega(0) \\ &\Rightarrow \frac{\omega(0)v}{a} = +\infty\end{aligned}$$

which means that the second boundary condition isn't satisfied for the individual solutions (22).

4.1 Adiabatic limit

The term adiabatic displacement refers to the walls moving away at a low enough speed such that the field can keep up. To quantify this we define the *external* velocity v_{ext} of a system to be the velocity with which the external influences change and the *internal* velocity v_{int} as the velocity corresponding to the internal changes of a system. With this we can express the adiabatic limit as $v_{int} >> v_{ext}$ and the non-adiabatic limit (i.e. the internal changes of a system can not compete with the external influences) as $v_{int} << v_{ext}$. In our specific case it will prove beneficial to express the above limits in terms of *time scales*. Let T_{int} be the time scale on which internal changes take place and T_{ext} the time scale corresponding to external changes. In terms of these time scales the adiabatic limit becomes $\frac{1}{T_{int}} >> \frac{1}{T_{ext}}$ and the non-adiabatic limit $\frac{1}{T_{int}} << \frac{1}{T_{ext}}$.

As we are dealing with an expanding cavity, the external influence is defined by the movement of the wall and the corresponding inverse time scale is $\frac{1}{T_{ext}} = \frac{\dot{L}(t)}{L(t)} = \lambda(t)$ while the internal time scale is related to the energy of the field modes $\frac{1}{T_{int}} = \omega(t)$. In case of the linear displacement described in the previous section it is possible to write $\frac{1}{T_{ext}} = \lambda(t) = \frac{v}{a+vt}$ and $\frac{1}{T_{int}} = \omega(0)\frac{L(0)}{L(t)} = \omega(0)\frac{a}{a+vt}$.

For the individual solutions we can recognise the condition we found for $\dot{Q}(0)$ as the adiabatic limit

$$
\begin{aligned}
&\frac{1}{T_{int}} \gg \frac{1}{T_{ext}}\\
&\Rightarrow \forall_t\ \omega(0)\frac{a}{a+vt} \gg \frac{v}{a+vt}\\
&\Rightarrow \forall_t\ \frac{\omega(0)a}{v} = \sqrt{\kappa} \gg 1
\end{aligned}
$$

which means that

$$
\frac{T_{ext}}{T_{int}} = \frac{\omega(0)a}{v} = \infty
$$

corresponds to the extreme adiabatic limit where $v \to 0$ or $\omega(0) \to \infty$. Unfortunately this means that the simplified differential equation (21) reduces to

$$
\begin{aligned}
&\ddot{Q}(t) + \omega^2(t)Q(t) = 2\lambda(t)G\dot{Q}(t) - \lambda^2(t)GQ(t) - \lambda^2(t)G^2Q(t)\\
&\Rightarrow \frac{\ddot{Q}(t)}{\omega^2(t)} + Q(t) = 0
\end{aligned}
$$

as $\frac{\lambda(t)}{\omega(t)} \to 0$. The only information we can extract from this equation is that for a disappearing velocity (i.e. the wall is stationary) we are left with the harmonic oscillator corresponding to a free situation.

As for the non-adiabatic limit, $\kappa \ll 1$, we find even in the limit where $v = c$ and $\omega(0)$ takes the lowest allowed value ($m_x = m_y = m_z = 1$) that

$$
\sqrt{\kappa} = \frac{a\omega(0)}{v} = \frac{\pi\,|\mathbf{m}|}{v} \geq \pi\sqrt{3}. \tag{24}
$$

This means that $\kappa \ll 1$ will never be satisfied and the system cannot reach the non-adiabatic regime. As a consequence our system will either be in the adiabatic regime or close to it, which we will exploit in what follows.

4.2 Linear combination

There is another possibility to solve (21) that we haven't tried. As the differential equation (21) is linear in $Q(t)$ we can try to find a solution that satisfies the initial conditions by constructing a linear combination of the positive and negative roots $\alpha_\pm = 1 \pm \sqrt{1-4\kappa}$ by writing

$$\begin{aligned} Q(t) &= C_+ Q_+(t) + C_- Q_-(t) \\ &= C_+ e^{G \ln\left(\frac{L(t)}{a}\right)} \left(\frac{a}{v} + t\right)^{\frac{\alpha_+}{2}} \\ &+ C_- e^{G \ln\left(\frac{L(t)}{a}\right)} \left(\frac{a}{v} + t\right)^{\frac{\alpha_-}{2}} \end{aligned}$$

For $t = 0$ this yields

$$\begin{aligned} Q(0) &= C_+ \left(\frac{a}{v}\right)^{\frac{\alpha_+}{2}} + C_- \left(\frac{a}{v}\right)^{\frac{\alpha_-}{2}} \\ &= F_+ + F_-. \end{aligned}$$

If we now apply the first initial condition then F_- can be expressed in terms of F_+ according to

$$F_- = \frac{1}{\sqrt{2\omega(0)}} - F_+ = \sqrt{\frac{a}{2v}}\kappa^{-1/4} - F_+.$$

The exact form of F_+ can be found from the second initial condition:

$$\begin{aligned} \dot{Q}(0) &= C_+ \left(\frac{a}{v}\right)^{\frac{-\alpha_-}{2}} \left[G + \frac{\alpha_+}{2}\right] + C_- \left(\frac{a}{v}\right)^{\frac{-\alpha_+}{2}} \left[G + \frac{\alpha_-}{2}\right] \\ &= \frac{v}{a} F_+ \left[G + \frac{\alpha_+}{2}\right] + \frac{v}{a} F_- \left[G + \frac{\alpha_-}{2}\right] \\ &= \frac{v}{a} \left(\frac{\alpha_+ - \alpha_-}{2} F_+ + \sqrt{\frac{a}{2v}}\kappa^{-1/4} \left[G + \frac{\alpha_-}{2}\right]\right) \\ &= -\imath\sqrt{\frac{\omega(0)}{2}} = -\imath\sqrt{\frac{v}{2a}}\kappa^{1/4}. \end{aligned}$$

Using that $\frac{\alpha_+ - \alpha_-}{2} = \sqrt{1-4\kappa}$ we find F_+ to be

$$F_+ = \sqrt{\frac{a}{2v}}\kappa^{-1/4} \left[\frac{1}{2} - \frac{\imath\kappa^{1/2} + G + \frac{1}{2}}{\sqrt{1-4\kappa}}\right],$$

which gives for C_+ and C_-

$$\begin{aligned} C_+ &= \frac{1}{\sqrt{2}}\kappa^{-1/4} \left[\frac{1}{2} - \frac{\imath\kappa^{1/2} + G + \frac{1}{2}}{\sqrt{1-4\kappa}}\right] \left(\frac{a}{v}\right)^{-\frac{\sqrt{1-4\kappa}}{2}} \\ &= \frac{1}{\sqrt{2\omega(0)}} \left[\frac{1}{2} - \frac{\imath\kappa^{1/2} + G + \frac{1}{2}}{\sqrt{1-4\kappa}}\right] \left(\frac{a}{v}\right)^{-\frac{\alpha_+}{2}} \\ C_- &= \frac{1}{\sqrt{2}}\kappa^{-1/4} \left[\frac{1}{2} + \frac{\imath\kappa^{1/2} + G + \frac{1}{2}}{\sqrt{1-4\kappa}}\right] \left(\frac{a}{v}\right)^{\frac{\sqrt{1-4\kappa}}{2}} \\ &= \frac{1}{\sqrt{2\omega(0)}} \left[\frac{1}{2} + \frac{\imath\kappa^{1/2} + G + \frac{1}{2}}{\sqrt{1-4\kappa}}\right] \left(\frac{a}{v}\right)^{-\frac{\alpha_-}{2}} \end{aligned}$$

and we find the solution to our differential equation to be

$$\begin{aligned} Q(t) =& \frac{1}{\sqrt{2\omega(t)}} \left(\left[\frac{a+vt}{a}\right]^{\frac{\alpha_+ - 1}{2}+G} \left[\frac{1}{2} - \frac{\imath\kappa^{1/2}+G+\frac{1}{2}}{\sqrt{1-4\kappa}}\right] \right. \\ &+ \left. \left[\frac{a+vt}{a}\right]^{\frac{\alpha_- - 1}{2}+G} \left[\frac{1}{2} + \frac{\imath\kappa^{1/2}+G+\frac{1}{2}}{\sqrt{1-4\kappa}}\right] \right). \end{aligned} \tag{25}$$

The next step is then to take the limit $t \uparrow t_f$ and compare with (11) to determine the coefficient A:

$$\begin{aligned} Q(t \uparrow t_f) =& \frac{1}{\sqrt{2\omega(t_f)}} \left(\left[\frac{a+vt_f}{a}\right]^{\frac{\alpha_+ - 1}{2}+G} \left[\frac{1}{2} - \frac{\imath\kappa^{1/2}+G+\frac{1}{2}}{\sqrt{1-4\kappa}}\right] \right. \\ &+ \left. \left[\frac{a+vt_f}{a}\right]^{\frac{\alpha_- - 1}{2}+G} \left[\frac{1}{2} + \frac{\imath\kappa^{1/2}+G+\frac{1}{2}}{\sqrt{1-4\kappa}}\right] \right). \end{aligned}$$

As A corresponds to the positive frequency part of the solution and $e^{\imath \int_0^{t_f} \omega(t')dt'} = e^{\imath \int_0^{t_f} \frac{\pi \mathbf{m}}{L(t')} dt'} = e^{\imath \frac{\pi \mathbf{m}}{v}[\ln L(t_f) - \ln L(0)]} = \left[\frac{a+vt_f}{a}\right]^{+\imath\sqrt{\kappa}}$ we find that

$$A = \frac{1}{\sqrt{2\omega(t_f)}} \left[\frac{a+vt_f}{a}\right]^{\frac{\alpha_+ - 1}{2}+G} \left[\frac{1}{2} - \frac{\imath\kappa^{1/2}+G+\frac{1}{2}}{\sqrt{1-4\kappa}}\right]. \tag{26}$$

Evaluating (26) we see that for $1-4\kappa < 0$ and $\sqrt{1-4\kappa} = \imath\sqrt{4\kappa-1}$ there will be a singularity for $\kappa = \frac{1}{4}$. Fortunately, by a similar comparison as (24), we find that $\imath\sqrt{4\kappa-1} \geq \imath\sqrt{12\pi^2-1} \approx 10,8\imath$ and thus the singularity lies well below the allowed region of values for κ.

4.2.1 Adiabatic limit

It is also possible to evaluate the adiabatic limit in the case of our linear combination of solutions:

$$\begin{aligned} &\frac{1}{T_{int}} \gg \frac{1}{T_{ext}} \\ &\Rightarrow \kappa \gg 1, \end{aligned}$$

thus $\sqrt{1-4\kappa} \approx \pm 2\imath\kappa^{1/2}$ and we find (choosing the positive solution)

$$
\begin{aligned}
Q(t) =& \frac{1}{\sqrt{2\omega(t)}} \left(\left[\frac{a+vt}{a}\right]^{\frac{\alpha_+ - 1}{2}+G} \left[\frac{1}{2} - \frac{\imath\kappa^{1/2}+G+\frac{1}{2}}{\sqrt{1-4\kappa}}\right] \right. \\
&+ \left. \left[\frac{a+vt}{a}\right]^{\frac{\alpha_- - 1}{2}+G} \left[\frac{1}{2} + \frac{\imath\kappa^{1/2}+G+\frac{1}{2}}{\sqrt{1-4\kappa}}\right] \right) \\
\underline{\underline{\kappa \gg 1}}\; & \frac{1}{\sqrt{2\omega(t)}} \left[\frac{a+vt}{a}\right]^{G-\imath\kappa^{1/2}} \left(1 + \frac{G+\frac{1}{2}}{2\imath\sqrt{\kappa}}\right) \\
&+ \frac{1}{\sqrt{2\omega(t)}} \left[\frac{a+vt}{a}\right]^{G+\imath\kappa^{1/2}} \left(-\frac{G+\frac{1}{2}}{2\imath\sqrt{\kappa}}\right),
\end{aligned}
$$

where the last term corresponds to the positive frequency part ($e^{\imath\int_0^t \omega(t')dt'}$), which falls off with $\frac{1}{\sqrt{\kappa}} = \frac{\lambda(t)}{\omega(t)}$ indicating that the production of particles with a high energy (large $\omega(t)$) compared to a relatively low external velocity (small $\lambda(t)$) is, as expected, suppressed.

5 Labelling

The next step towards properly solving (9) is to figure out how to incorporate the correct labels. To do this we don't only need to know how the modes $\mathbf{m}$, $\mathbf{j}$ and $\mathbf{l}$ interact with each other but also how they relate to the original modes $\mathbf{n}$. The integral contained in $G^z_{\mathbf{mj}}$ contains information about the projection of modes $\mathbf{m}$ and $\mathbf{j}$. Or more explicitly, $g_{m_z j_z}$ does. To extract this information we will now calculate $g_{m_z j_z}$. Realising that the integral as well as the differentiation are both with respect to the z-component, the x- and y-components will be temporarily left out in order to keep the following calculations more structured:

$$
\begin{aligned}
g_{m_z j_z} =& L_z(t) \int_C dz \frac{\partial \varphi_{m_z}}{\partial L_z} \varphi_{j_z} \\
=& - L_z(t) \int_C dz \left[\frac{1}{L_z^2(t)} \sin\left(\frac{m_z \pi}{L_z(t)} z\right) \sin\left(\frac{j_z \pi}{L_z(t)} z\right) \right. \\
&+ \left. \frac{2\pi m_z}{L_z^3(t)} z \cos\left(\frac{m_z \pi}{L_z(t)} z\right) \sin\left(\frac{j_z \pi}{L_z(t)} z\right) \right].
\end{aligned}
\tag{27}
$$

Substituting $v = \frac{z}{L_z(t)}$ leads to

$$
\begin{aligned}
g_{m_z j_z} &= -\int_0^1 dv \sin(m_z \pi v) \sin(j_z \pi v) - 2\pi m_z \int_0^1 dv v \cos(m_z \pi v) \sin(j_z \pi v) \\
&\equiv I + II.
\end{aligned}
\tag{28}
$$

The integrals I and II will be solved separately and added later on. The first integral contributes

$$
\begin{aligned}
I &= -\int_0^1 dv \sin(m_z \pi v) \sin(j_z \pi v) \\
&= \frac{1}{2}\int_0^1 dv \left(\cos(\pi(m_z + j_z)v) - \cos(\pi(m_z - j_z)v)\right) \\
&= \frac{1}{2\pi}\left(\frac{\sin(\pi(m_z + j_z)v)}{m_z + j_z} - \frac{\sin(\pi(m_z - j_z)v)}{m_z - j_z}\right)\Bigg|_0^1 \\
&= \frac{1}{2\pi}\left(\frac{\sin(\pi(m_z + j_z))}{m_z + j_z} - \frac{\sin(\pi(m_z - j_z))}{m_z - j_z}\right)
\end{aligned} \tag{29}
$$

As both m_z and j_z are positive integers, the first sine-term will always yield zero. The second term only contributes when $m_z = j_z$. Hence we find

$$
I = -\frac{1}{2\pi}\delta_{m_z j_z}\frac{\sin(\pi(m_z - j_z))}{m_z - j_z}\Bigg|_{m_z = j_z} = -\frac{1}{2}\delta_{m_z j_z}. \tag{30}
$$

So I contributes $-\frac{1}{2}$ when $m_z = j_z$ and zero otherwise. Rewriting II gives

$$
\begin{aligned}
II &= -2\pi m_z \int_0^1 dv\ v \cos(m_z \pi v) \sin(j_z \pi v) \\
&= -\pi m_z \int_0^1 dv\ v\left(\sin(\pi(m_z + j_z)v) - \sin(\pi(m_z - j_z)v)\right) \\
&= -\pi m_z v\left(-\frac{\cos(\pi(m_z + j_z)v)}{\pi(m_z + j_z)} + \frac{\cos(\pi(m_z - j_z)v)}{\pi(m_z - j_z)}\right)\Bigg|_0^1 \\
&+ \pi m_z \int_0^1 dv\left(-\frac{\cos(\pi(m_z + j_z)v)}{\pi(m_z + j_z)} + \frac{\cos(\pi(m_z - j_z)v)}{\pi(m_z - j_z)}\right) \\
&= -m_z\left(\frac{\cos(\pi(m_z - j_z))}{(m_z - j_z)} - \frac{\cos(\pi(m_z + j_z))}{(m_z + j_z)}\right) \\
&+ m_z\left(\frac{\sin(\pi(m_z - j_z)v)}{\pi(m_z - j_z)^2} - \frac{\sin(\pi(m_z + j_z)v)}{\pi(m_z + j_z)^2}\right)\Bigg|_0^1 \\
&= m_z\left(\frac{\cos(\pi(m_z + j_z))}{(m_z + j_z)} - \frac{\cos(\pi(m_z - j_z))}{(m_z - j_z)}\right. \\
&\left. + \frac{\sin(\pi(m_z - j_z))}{\pi(m_z - j_z)^2} - \frac{\sin(\pi(m_z + j_z))}{\pi(m_z + j_z)^2}\right).
\end{aligned} \tag{31}
$$

Just as with I we consider $m_z = j_z$ and $m_z \neq j_z$. In the first case we find

$$
\begin{aligned}
II|_{m_z = j_z} &= m_z\left(\frac{\cos(\pi(m_z + j_z))}{(m_z + j_z)} - \frac{\cos(\pi(m_z - j_z))}{(m_z - j_z)} + \frac{\sin(\pi(m_z - j_z))}{\pi(m_z - j_z)^2}\right. \\
&\left. - \frac{\sin(\pi(m_z + j_z))}{\pi(m_z + j_z)^2}\right)\Bigg|_{j_z = m_z} \\
&= m_z\left(\frac{1}{2m_z}\right) = \frac{1}{2}.
\end{aligned} \tag{32}
$$

So for $m_z = j_z$ we find $g_{m_z, j_z} = -\frac{1}{2} + \frac{1}{2} = 0$. In the $m_z \neq j_z$ case I does not contribute and if we use yet again that m_z and j_z are both positive integers, then

$$
\begin{aligned}
g_{m_z,j_z\neq m_z} &= II \\
&= m_z\left[\frac{(-1)^{m_z+j_z}}{m_z+j_z}-\frac{(-1)^{m_z-j_z}}{m_z-j_z}\right] \\
&= (-1)^{m_z+j_z}\frac{-2m_zj_z}{m_z^2-j_z^2}
\end{aligned}
\tag{33}
$$

With this $G^z_{\mathbf{mj}}$ with $m_z \neq j_z$ becomes

$$
\begin{aligned}
G^z_{\mathbf{mj}} &= g_{m_zj_z}\delta_{m_xj_x}\delta_{m_yj_y} \\
&= (-1)^{m_z+j_z}\frac{-2m_zj_z}{m_z^2-j_z^2}\delta_{m_xj_x}\delta_{m_yj_y},
\end{aligned}
\tag{34}
$$

which is indeed independent of $L_z(t)$ as stated before.

Likewise we need to determine $(G^2)_{\mathbf{mj}}$. From (8) we can derive that

$$
\sum_{\mathbf{j},\mathbf{l}} G_{\mathbf{jl}}G_{\mathbf{ml}}Q^{(\mathbf{n})}_{\mathbf{j}} = \sum_{\mathbf{j}} L^2(t)\int_C d\mathbf{x}\frac{\partial\varphi_{\mathbf{j}}}{\partial L}\frac{\partial\varphi_{\mathbf{m}}}{\partial L}Q^{(\mathbf{n})}_{\mathbf{j}}.
$$

Evaluating the integral in the above equation solely for the x-components (i.e. $m_x \neq j_x$ and $m_{y,z} = j_{y,z}$) yields

$$
\begin{aligned}
&L^2(t)\int_C d\mathbf{x}\frac{\partial\varphi_{\mathbf{j}}}{\partial L}\frac{\partial\varphi_{\mathbf{m}}}{\partial L} \overset{m_x\neq j_x}{=\!=\!=} \pi^2 m_xj_x\frac{\sin(\pi(m_y-j_y))}{\pi(m_y-j_y)}\frac{\sin(\pi(m_z-j_z))}{\pi(m_z-j_z)} \\
&\times\left[\frac{\sin(\pi(m_x-j_x))}{\pi(m_x-j_x)}+2\left(\frac{\cos(\pi(m_x+j_x))}{\pi^2(m_x+j_x)^2}+\frac{\cos(\pi(m_x-j_x))}{\pi^2(m_x-j_x)^2}-\frac{\sin(\pi(m_x-j_x))}{\pi^3(m_x-j_x)^3}\right)\right] \\
&= 2m_xj_x\left[\frac{(-1)^{m_x+j_x}}{(m_x+j_x)^2}+\frac{(-1)^{m_x-j_x}}{(m_x-j_x)^2}\right]\delta_{m_yj_y}\delta_{m_zj_z} \\
&= 2m_xj_x(-1)^{m_x+j_x}\frac{2m_x^2+2j_x^2}{(m_x^2-j_x^2)^2}\delta_{m_yj_y}\delta_{m_zj_z} \\
&= 4(-1)^{m_x+j_x}\frac{m_x^3j_x}{(m_x^2-j_x^2)^2}\delta_{m_yj_y}\delta_{m_zj_z}+4(-1)^{m_x+j_x}\frac{j_x^3m_x}{(m_x^2-j_x^2)^2}\delta_{m_yj_y}\delta_{m_zj_z} \\
&= -2g_{m_xj_x}\left(\frac{m_x^2+j_x^2}{m_x^2-j_x^2}\right)\delta_{m_yj_y}\delta_{m_zj_z}.
\end{aligned}
$$

The x and y-components give similar expressions. In the case of the single G -term it was not possible to find a contribution to the integrals when $m_i = j_i$. The squared G-term does not have this problem. For $m_i = j_i$ with $i = x, y, z$ the above calculation (for all three components) becomes

$$
L^2(t)\int_C d\mathbf{x}\frac{\partial\varphi_{\mathbf{j}}}{\partial L}\frac{\partial\varphi_{\mathbf{m}}}{\partial L} \overset{m_i=j_i}{=\!=\!=} \frac{3}{4}+\frac{\pi^2}{3}m_x^2+\frac{\pi^2}{3}m_y^2+\frac{\pi^2}{3}m_z^2.
$$

Lastly there will be mixing terms of the form $g_{m_xj_x}g_{m_yj_y}$ which can be calculated from

$$
\begin{aligned}
&L^2(t)\int_C d\mathbf{x}\frac{\partial\varphi_{\mathbf{j}}}{\partial L}\frac{\partial\varphi_{\mathbf{m}}}{\partial L} \overset{m_{x,y}\neq j_{x,y}}{=\!=\!=\!=} \pi^2 m_x j_y \frac{\sin\left(\pi(m_z-j_z)\right)}{\pi(m_z-j_z)}\\
&\times\left[\frac{\cos\left(\pi(m_x-j_x)\right)}{\pi(m_x-j_x)}-\frac{\cos\left(\pi(m_x+j_x)\right)}{\pi(m_x+j_x)}\right]\\
&\times\left[\frac{\cos\left(\pi(j_y-m_y)\right)}{\pi(j_y-m_y)}-\frac{\cos\left(\pi(j_y+m_y)\right)}{\pi(j_y+m_y)}\right]\\
&+\pi^2 m_y j_x \frac{\sin\left(\pi(m_z-j_z)\right)}{\pi(m_z-j_z)}\\
&\times\left[\frac{\cos\left(\pi(m_y-j_y)\right)}{\pi(m_y-j_y)}-\frac{\cos\left(\pi(m_y+j_y)\right)}{\pi(m_y+j_y)}\right]\\
&\times\left[\frac{\cos\left(\pi(j_x-m_x)\right)}{\pi(j_x-m_x)}-\frac{\cos\left(\pi(j_x+m_x)\right)}{\pi(j_x+m_x)}\right]\\
&=\left(g_{m_x j_x} g_{j_y m_y}+g_{m_y j_y} g_{j_x m_x}\right)\delta_{m_z j_z}\\
&=-2 g_{m_x j_x} g_{m_y j_y}\delta_{m_z j_z}
\end{aligned}
$$

Taking all the terms under consideration we find

$$
\begin{aligned}
&L^2(t)\int_C d\mathbf{x}\frac{\partial\varphi_{\mathbf{j}}}{\partial L}\frac{\partial\varphi_{\mathbf{m}}}{\partial L}=\left(\frac{3}{4}+\frac{\pi^2}{3}(m_x^2+m_y^2+m_z^2)\right)\delta_{m_x j_x}\delta_{m_y j_y}\delta_{m_z j_z}\\
&-2 g_{m_x j_x}\left(\frac{m_x^2+j_x^2}{m_x^2-j_x^2}\right)\delta_{m_y j_y}\delta_{m_z j_z}-2 g_{m_y j_y}\left(\frac{m_y^2+j_y^2}{m_y^2-j_y^2}\right)\delta_{m_x j_x}\delta_{m_z j_z}\\
&-2 g_{m_z j_z}\left(\frac{m_z^2+j_z^2}{m_z^2-j_z^2}\right)\delta_{m_x j_x}\delta_{m_y j_y}\\
&-2\left(g_{m_x j_x} g_{m_y j_y}\delta_{m_z j_z}+g_{m_x j_x} g_{m_z j_z}\delta_{m_y j_y}+g_{m_y j_y} g_{m_z j_z}\delta_{m_x j_x}\right).
\end{aligned}
\tag{35}
$$

We can double check these results by comparing the above calculations with the $g_{\mathbf{ml}}g_{\mathbf{lj}}$ term found in (9). Focusing on the $(g^2)_{m_x j_x}$ term we find

$$\sum_{l_x} g_{m_x l_x} g_{j_x l_x} = -\sum_{l_x} g_{m_x l_x} g_{l_x j_x}$$

$$= -(-1)^{j_x+m_x} \sum_{l_x \neq j_x, m_x} 4\frac{m_x j_x l_x^2}{(m_x^2 - l_x^2)(l_x^2 - j_x^2)}$$

$$= (-1)^{j_x+m_x} \sum_{l_x \neq j_x, m_x} \left(-4\frac{m_x^3 j_x}{(m_x^2 - j_x^2)(m_x^2 - l_x^2)} + 4\frac{m_x j_x^3}{(m_x^2 - j_x^2)(j_x^2 - l_x^2)} \right)$$

$$= (-1)^{j_x+m_x} \sum_{l_x \neq m_x} \left(-4\frac{m_x^3 j_x}{(m_x^2 - j_x^2)(m_x^2 - l_x^2)} \right) - (-1)^{j_x+m_x} \left(-4\frac{m_x^3 j_x}{(m_x^2 - j_x^2)(m_x^2 - j_x^2)} \right)$$

$$+ (-1)^{j_x+m_x} \sum_{l_x \neq j_x} \left(4\frac{m_x j_x^3}{(m_x^2 - j_x^2)(j_x^2 - l_x^2)} \right) - (-1)^{j_x+m_x} \left(4\frac{m_x j_x^3}{(m_x^2 - j_x^2)(j_x^2 - m_x^2)} \right)$$

$$\stackrel{[3]}{=\!=} (-1)^{j_x+m_x} \left(3\frac{m_x j_x}{m_x^2 - j_x^2} + 4\frac{m_x^3 j_x}{(m_x^2 - j_x^2)^2} \right) + (-1)^{j_x+m_x} \left(-3\frac{m_x j_x}{m_x^2 - j_x^2} + 4\frac{m_x j_x^3}{(m_x^2 - j_x^2)^2} \right)$$

$$= (-1)^{j_x+m_x} 4\frac{m_x j_x}{m_x^2 - j_x^2} \frac{m_x^2 + j_x^2}{m_x^2 - j_x^2}$$

$$= -2g_{m_x j_x} \frac{m_x^2 + j_x^2}{m_x^2 - j_x^2}.$$

The same analyses can be applied to the $m_i = j_i$ term to find that

$$\sum_{l_x} g_{m_x l_x} g_{m_x l_x}$$

$$= \sum_{l_x \neq m_x} 4\frac{m_x^2 l_x^2}{(m_x^2 - l_x^2)(m_x^2 - l_x^2)}$$

$$= -\sum_{l_x \neq m_x} \left(\frac{4m_x^2}{(m_x^2 - l_x^2)} - \frac{4m_x^4}{(m_x^2 - l_x^2)^2} \right)$$

$$= 3 + \sum_{l_x \neq m_x} \frac{4m_x^4}{(m_x^2 - l_x^2)^2}$$

$$= 3 + m_x^2 \sum_{l_x \neq m_x} \left(\frac{1}{m_x - l_x} + \frac{1}{m_x + j_x} \right) \left(\frac{1}{m_x - l_x} + \frac{1}{m_x + j_x} \right)$$

$$= 3 + 2\sum_{l_x \neq m_x} \frac{m_x^2}{m_x^2 - l_x^2} + m_x^2 \sum_{l_x \neq m_x} \left(\frac{1}{(m_x + l_x)^2} + \frac{1}{(m_x - l_x)^2} \right)$$

$$\stackrel{[4]}{=\!=} 3 - \frac{3}{2} + m_x^2 \left(\sum_{k_x} \frac{1}{k_x^2} - \frac{1}{m_x^2} - \frac{1}{4m_x^2} + \sum_{k_x} \frac{1}{k_x^2} \right)$$

$$\stackrel{\sum_k \frac{1}{k^2} = \frac{\pi^2}{6}}{=\!=\!=\!=\!=} 3 - \frac{3}{2} + \frac{\pi^2 m_x^2}{6} - 1 - \frac{1}{4} + \frac{\pi^2 m_x^2}{6}$$

$$= \frac{1}{4} + \frac{\pi^2 m_x^2}{3},$$

[3] Where it was used that $\sum_{k=1, k\neq m}^{\infty} \frac{1}{(m+k)(m-k)} = -\frac{3}{4m^2}$ for m an integer.[7]

which gives indeed the required term when the y and z components are taken into account as well.

Now that we know how the different modes relate to one another we can focus on solving the differential equation including the proper labels. For the linear case we know that the non-adiabatic regime is off limits due to constraints imposed by the range of the integers m_x, m_y and m_z. Consequently we'll focus on the adiabatic regime. As was shown in 4.1 the extreme adiabatic limit allows one to neglect virtually all mixing terms generated by $G_{\mathbf{mn}}$. Continuing this line of thought we'll first apply an adiabatic approximation to (9) to find out whether there are any terms that can be neglected under all (adiabatic) circumstances.

Including labels we find up to zeroth order in $\kappa_{\mathbf{m}}$

$$\ddot{Q}_{\mathbf{m}}^{(\mathbf{n})}(t) + \omega_{\mathbf{m}}^2(t) Q_{\mathbf{m}}^{(\mathbf{n})}(t) = 0.$$

Similar to before we write the solution as

$$Q_{\mathbf{m}}^{(\mathbf{n})}(t) = C_{\mathbf{m}+}^{(\mathbf{n})} \left(\frac{a}{v} + t\right)^{\alpha_{\mathbf{m}+}/2} + C_{\mathbf{m}-}^{(\mathbf{n})} \left(\frac{a}{v} + t\right)^{\alpha_{\mathbf{m}-}/2}$$

where $\alpha_{\mathbf{m}\pm} = 1 \pm \sqrt{1 - 4\kappa_{\mathbf{m}}}$. Considering the adiabatic limit $\alpha_{\mathbf{m}\pm} \gg 1$ including labels, we find $\alpha_{\mathbf{m}\pm} \approx \pm 2\imath \kappa_{\mathbf{m}}^{\frac{1}{2}} + 1$. Demanding that $Q_{\mathbf{m}}^{(\mathbf{n})}(t)$ satisfies the *labeled* boundary conditions (5) gives the solution

$$\begin{aligned} Q_{\mathbf{m}}^{(\mathbf{n})}(t) &= \frac{\delta_{\mathbf{mn}}}{\sqrt{2\omega_{\mathbf{n}}(t)}} \left[\frac{\imath}{4\kappa_{\mathbf{n}}^{\frac{1}{2}}} \left(\frac{L(t)}{a}\right)^{\imath\kappa_{\mathbf{n}}^{\frac{1}{2}}} + \left(1 - \frac{\imath}{4\kappa_{\mathbf{n}}^{\frac{1}{2}}}\right) \left(\frac{L(t)}{a}\right)^{-\imath\kappa_{\mathbf{n}}^{\frac{1}{2}}} \right] \\ &\approx \frac{\delta_{\mathbf{mn}}}{\sqrt{2\omega_{\mathbf{n}}(t)}} \left(\frac{L(t)}{a}\right)^{-\imath\kappa_{\mathbf{n}}^{\frac{1}{2}}}, \end{aligned} \tag{36}$$

up to zeroth order in $1/\sqrt{\kappa_{\mathbf{m}}}$. If we now wish to apply a first order correction then we need to include the $\dot{Q}_{\mathbf{j}}^{(\mathbf{n})}(t)$ term (which is first order in $1/\sqrt{\kappa_{\mathbf{j}}}$) such that the equation to solve becomes

$$\ddot{Q}_{\mathbf{m}}^{(\mathbf{n})}(t) + \omega_{\mathbf{m}}^2(t) Q_{\mathbf{m}}^{(\mathbf{n})}(t) - 2\lambda(t) \sum_{\mathbf{j}} G_{\mathbf{mj}} \dot{Q}_{\mathbf{j}}^{(\mathbf{n})}(t) = 0. \tag{37}$$

Next we need to add a correction to our zeroth order solution such that $Q_{\mathbf{m}}^{(\mathbf{n})}(t) \to Q_{\mathbf{m}}^{(\mathbf{n})(0)}(t) + Q_{\mathbf{m}}^{(\mathbf{n})(1)}(t)$, where the first term is the zeroth order solution (36) and the second term a first order correction that we want to determine. While substituting $Q_{\mathbf{m}}^{(\mathbf{n})}(t)$ into (37) one should keep in mind that if we wish to solve the first order differential equation the first part of the differential equation now requires the full form of (36) and not just the zeroth order approximation. For the second (additional) term the zeroth order solution will suffice. Discarding any terms other than first order yields

$$\begin{aligned} &\ddot{Q}_{\mathbf{m}}^{(\mathbf{n})(1)}(t) + \omega_{\mathbf{m}}^2(t) Q_{\mathbf{m}}^{(\mathbf{n})(1)}(t) - 2\lambda(t) \sum_{\mathbf{j}} G_{\mathbf{mj}} \dot{Q}_{\mathbf{j}}^{(\mathbf{n})(0)}(t) = \\ &\ddot{Q}_{\mathbf{m}}^{(\mathbf{n})(1)}(t) + \omega_{\mathbf{m}}^2(t) Q_{\mathbf{m}}^{(\mathbf{n})(1)}(t) - 2\lambda(t) G_{\mathbf{mn}} \dot{Q}_{\mathbf{n}}^{(\mathbf{n})(0)}(t) = 0. \end{aligned}$$

[4] The addition of the sums yield two sums of the type $\sum_{k_x=1}^{\infty} \frac{1}{k_x^2}$ where $l_x = 0$ and $l_x = m_x$ need to be subtracted.

To solve this equation for $Q_{\mathbf{m}}^{(\mathbf{n})(1)}(t)$ we need to demand once again that $Q_{\mathbf{m}}^{(\mathbf{n})}(t)$ satisfies the boundary conditions. However, we already know that $Q_{\mathbf{n}}^{(\mathbf{n})(0)}(t)$, $\dot{Q}_{\mathbf{n}}^{(\mathbf{n})(0)}(t)$ *alone* produce the right terms for $t = 0$. In that case we can determine the proper boundary conditions for $Q_{\mathbf{m}}^{(\mathbf{n})(1)}(t)$:

$$\begin{aligned}
&Q_{\mathbf{m}}^{(\mathbf{n})}(0) = Q_{\mathbf{m}}^{(\mathbf{n})(0)}(0) + Q_{\mathbf{m}}^{(\mathbf{n})(1)}(0) = Q_{\mathbf{m}}^{(\mathbf{n})(0)}(0) = \frac{\delta_{\mathbf{mn}}}{\sqrt{2\omega_{\mathbf{m}}(0)}} \\
\Rightarrow &Q_{\mathbf{m}}^{(\mathbf{n})(1)}(0) = 0 \\
&\dot{Q}_{\mathbf{m}}^{(\mathbf{n})}(0) = \dot{Q}_{\mathbf{m}}^{(\mathbf{n})(0)}(0) + \dot{Q}_{\mathbf{m}}^{(\mathbf{n})(1)}(0) = \dot{Q}_{\mathbf{m}}^{(\mathbf{n})(0)}(0) = -\imath\sqrt{\frac{\omega_{\mathbf{m}}(0)}{2}}\delta_{\mathbf{mn}} \\
\Rightarrow &\dot{Q}_{\mathbf{m}}^{(\mathbf{n})(1)}(0) = 0.
\end{aligned}$$

With the help of *Mathematica* the solution of the first order correction is found to be

$$\begin{aligned}
Q_{\mathbf{m}}^{(\mathbf{n})(1)}(t) = &\frac{G_{\mathbf{mn}}(1 - 2\imath\sqrt{\kappa_{\mathbf{n}}})}{\sqrt{2\omega_{\mathbf{n}}(t)}} \cdot \left[\frac{2\left(\frac{L(t)}{a}\right)^{-\frac{1}{2}\sqrt{1-4\kappa_{\mathbf{m}}}}}{\sqrt{1-4\kappa_{\mathbf{m}}}(\sqrt{1-4\kappa_{\mathbf{m}}} - 2\imath\sqrt{\kappa_{\mathbf{n}}})} + \right. \\
&\left. \frac{2\left(\frac{L(t)}{a}\right)^{\frac{1}{2}\sqrt{1-4\kappa_{\mathbf{m}}}}}{\sqrt{1-4\kappa_{\mathbf{m}}}(\sqrt{1-4\kappa_{\mathbf{m}}} + 2\imath\sqrt{\kappa_{\mathbf{n}}})} - \frac{4\left(\frac{L(t)}{a}\right)^{-\imath\sqrt{\kappa_{\mathbf{n}}}}}{(\sqrt{1-4\kappa_{\mathbf{m}}} - 2\imath\sqrt{\kappa_{\mathbf{n}}})(\sqrt{1-4\kappa_{\mathbf{m}}} + 2\imath\sqrt{\kappa_{\mathbf{n}}})} \right],
\end{aligned}$$

which gives for the total solution

$$\begin{aligned}
&Q_{\mathbf{m}}^{(\mathbf{n})}(t) = \\
&\frac{\left(\frac{L(t)}{a}\right)^{-\imath\sqrt{\kappa_{\mathbf{n}}}}}{\sqrt{2\omega_{\mathbf{n}}(t)}} \left[\delta_{\mathbf{mn}}\left(1 - \frac{\imath}{4\sqrt{\kappa_{\mathbf{n}}}}\right) - \frac{4G_{\mathbf{mn}}(1 - 2\imath\sqrt{\kappa_{\mathbf{n}}})}{(\sqrt{1-4\kappa_{\mathbf{m}}} - 2\imath\sqrt{\kappa_{\mathbf{n}}})(\sqrt{1-4\kappa_{\mathbf{m}}} + 2\imath\sqrt{\kappa_{\mathbf{n}}})} \right] \\
&+ \frac{\left(\frac{L(t)}{a}\right)^{\imath\sqrt{\kappa_{\mathbf{n}}}}}{\sqrt{2\omega_{\mathbf{n}}(t)}} \frac{\imath\delta_{\mathbf{mn}}}{4\sqrt{\kappa_{\mathbf{n}}}} \\
&+ \frac{2G_{\mathbf{mn}}(1 - 2\imath\sqrt{\kappa_{\mathbf{n}}})}{\sqrt{2\omega_{\mathbf{n}}(t)}\sqrt{1-4\kappa_{\mathbf{m}}}} \left[\frac{\left(\frac{L(t)}{a}\right)^{-\frac{1}{2}\sqrt{1-4\kappa_{\mathbf{m}}}}}{\sqrt{1-4\kappa_{\mathbf{m}}} - 2\imath\sqrt{\kappa_{\mathbf{n}}}} + \frac{\left(\frac{L(t)}{a}\right)^{\frac{1}{2}\sqrt{1-4\kappa_{\mathbf{m}}}}}{\sqrt{1-4\kappa_{\mathbf{m}}} + 2\imath\sqrt{\kappa_{\mathbf{n}}}} \right].
\end{aligned}$$

Formally one should expand the exponent of the zeroth order term up to order $1/\sqrt{\kappa_{\mathbf{n}}}$, but as the exponent will drop out during the calculation of $\left|A_{\mathbf{m}}^{(\mathbf{n})}\right|^2$ we will leave it out. We see then that the solution constitutes of a part where the delta function requires $\mathbf{m} = \mathbf{n}$ and a part that only contributes when $\mathbf{m} \neq \mathbf{n}$ as imposed by $G_{\mathbf{mn}}$. The positive frequency solution that we need in order to determine $A_{\mathbf{m}}^{(\mathbf{n})}$ is hidden in the combination of the $\left(\frac{L(t)}{a}\right)^{\imath\sqrt{\kappa_{\mathbf{n}}}}$ and $\left(\frac{L(t)}{a}\right)^{\frac{1}{2}\sqrt{1-4\kappa_{\mathbf{m}}}}$

terms. Using the adiabatic approximation once more we can write

$$
\begin{aligned}
Q_{\mathbf{m}}^{(\mathbf{n})}(t) \approx{}& \frac{\left(\frac{L(t)}{a}\right)^{-\imath\sqrt{\kappa_{\mathbf{n}}}}}{\sqrt{2\omega_{\mathbf{n}}(t)}}\left[\delta_{\mathbf{mn}}\left(1-\frac{\imath}{4\sqrt{\kappa_{\mathbf{n}}}}\right)-\frac{2\imath\sqrt{\kappa_{\mathbf{n}}}G_{\mathbf{mn}}}{\kappa_{\mathbf{m}}-\kappa_{\mathbf{n}}}\right] \\
&+\frac{\left(\frac{L(t)}{a}\right)^{\imath\sqrt{\kappa_{\mathbf{n}}}}}{\sqrt{2\omega_{\mathbf{n}}(t)}}\frac{\imath\delta_{\mathbf{mn}}}{4\sqrt{\kappa_{\mathbf{n}}}} \\
&+\frac{\imath\sqrt{\kappa_{\mathbf{n}}}G_{\mathbf{mn}}}{\sqrt{2\omega_{\mathbf{n}}(t)}\sqrt{\kappa_{\mathbf{m}}}}\left[\frac{\left(\frac{L(t)}{a}\right)^{-\imath\sqrt{\kappa_{\mathbf{m}}}}}{\sqrt{\kappa_{\mathbf{m}}}-\sqrt{\kappa_{\mathbf{n}}}}+\frac{\left(\frac{L(t)}{a}\right)^{\imath\sqrt{\kappa_{\mathbf{m}}}}}{\sqrt{\kappa_{\mathbf{m}}}+\sqrt{\kappa_{\mathbf{n}}}}\right],
\end{aligned}
\tag{38}
$$

which gives for $A_{\mathbf{m}}^{(\mathbf{n})}$

$$
A_{\mathbf{m}}^{(\mathbf{n})}=\frac{\imath\sqrt{\kappa_{\mathbf{n}}}G_{\mathbf{mn}}}{\sqrt{2\omega_{\mathbf{n}}(t_f)}\sqrt{\kappa_{\mathbf{m}}}(\sqrt{\kappa_{\mathbf{m}}}+\sqrt{\kappa_{\mathbf{n}}})}+\frac{\imath\delta_{\mathbf{mn}}}{4\sqrt{2\omega_{\mathbf{n}}(t_f)}\sqrt{\kappa_{\mathbf{n}}}}. \tag{39}
$$

Now that the coefficient $A_{\mathbf{m}}^{(\mathbf{n})}$ has been determined we can calculate the average number of produced particles by plugging (39) into (12):

$$
\begin{aligned}
\langle N_{\mathbf{m}}\rangle &= \sum_{\mathbf{n}} 2\omega_{\mathbf{m}}(t_f)\left|A_{\mathbf{m}}^{(\mathbf{n})}\right|^2 \\
&= \sum_{\mathbf{n}} 2\omega_{\mathbf{m}}(t_f)\left(\frac{(G_{\mathbf{mn}})^2\kappa_{\mathbf{n}}}{2\omega_{\mathbf{n}}(t_f)\kappa_{\mathbf{m}}(\sqrt{\kappa_{\mathbf{m}}}+\sqrt{\kappa_{\mathbf{n}}})^2}+\frac{\delta_{\mathbf{mn}}}{32\omega_{\mathbf{n}}(t_f)\kappa_{\mathbf{n}}}\right) \\
&= \sum_{\mathbf{n}} \frac{\sqrt{\kappa_{\mathbf{n}}}(G_{\mathbf{mn}})^2}{\sqrt{\kappa_{\mathbf{m}}}(\sqrt{\kappa_{\mathbf{m}}}+\sqrt{\kappa_{\mathbf{n}}})^2}+\frac{1}{16\kappa_{\mathbf{m}}} \\
&= \sum_{\mathbf{n}} \frac{|\mathbf{n}|(G_{\mathbf{mn}})^2}{|\mathbf{m}|(|\mathbf{m}|+|\mathbf{n}|)^2}\frac{v^2}{\pi^2}+\frac{v^2}{16\pi^2|\mathbf{m}|^2},
\end{aligned}
\tag{40}
$$

where it was used that $\omega_{\mathbf{n}}(t_f)\kappa_{\mathbf{m}}=\sqrt{\kappa_{\mathbf{n}}\kappa_{\mathbf{m}}}\omega_{\mathbf{m}}(t_f)$.

6 Results and Conclusion

Taking a closer look at (40) we find it consists of two parts. Before we discuss the individual terms we can make the overall observation that for increasing velocity, v, of the walls the average number of created particles increases as well. As we are working with an adiabatic approximation v will always be relatively small compared to $\pi|\mathbf{m}|$. This brings us to the second term, which is rather straightforward; it only contributes when $|\mathbf{n}|=|\mathbf{m}|$ and it decreases quadratically with $|\mathbf{m}|$ as was predicted. The first term requires a deeper analysis.

For the low energy modes $|\mathbf{m}|\ll|\mathbf{n}|$ we see that the denominator goes with $|\mathbf{m}||\mathbf{n}|^2$ (for increasing $|\mathbf{n}|$) and the numerator with $|\mathbf{n}|(G_{\mathbf{mn}})^2$. The $G_{\mathbf{mn}}$ term itself behaves approximately as $G_{\mathbf{mn}}\sim\frac{|\mathbf{m}|}{|\mathbf{n}|}$. In that case the first term itself will behave as $\sim\frac{|\mathbf{m}|}{|\mathbf{n}|^3}\frac{v^2}{\pi^2}$.

In the exact opposite limit where the $\mathbf{m}$ modes are highly energetic compared to $\mathbf{n}$ we find that the denominator goes with $|\mathbf{m}|^3$ while the numerator goes with

$|\mathbf{n}|\,(G_{\mathbf{mn}})^2$. Here the $G_{\mathbf{mn}}$ will behave approximately as $\frac{|\mathbf{n}|}{|\mathbf{m}|}$. Thus we find that this first term goes as $\sim \frac{|\mathbf{n}|(G_{\mathbf{mn}})^2}{|\mathbf{m}|^3}\frac{v^2}{\pi^2} \sim \frac{|\mathbf{n}|^3}{|\mathbf{m}|^5}\frac{v^2}{\pi^2}$.

The case where $|\mathbf{m}|$ and $|\mathbf{n}|$ are of similar order gives some interesting features. As we are working with integers, similar order means $n_i = m_i + l$ for l a small integer value. In that case we find that the individual components of $G_{\mathbf{mn}}$ behave as $(G^x_{\mathbf{mn}})^2 \approx \frac{4m_x^4}{(-2m_x l)^2}\delta^{(2)}_{m_y j_y}\delta^{(2)}_{m_z j_z} = \frac{m_x^2}{l^2}\delta^{(2)}_{m_y j_y}\delta^{(2)}_{m_z j_z}$ for large values of m_x, with similar terms for y and z. Plugging this in the first term of (40) and using that $\sum_{l=1}\frac{1}{l^2} = \frac{\pi^2}{6}$ we find our sum becomes finite: $\sum_{\mathbf{n}} \frac{|\mathbf{n}|(G_{\mathbf{mn}})^2}{|\mathbf{m}|(|\mathbf{m}|+|\mathbf{n}|)^2}\frac{v^2}{\pi^2} \approx \frac{v^2}{\pi^2}\frac{2}{4|\mathbf{m}|^2}\sum_{l=1}\frac{m_x^2+m_y^2+m_z^2}{l^2} = \frac{v^2}{12}$.

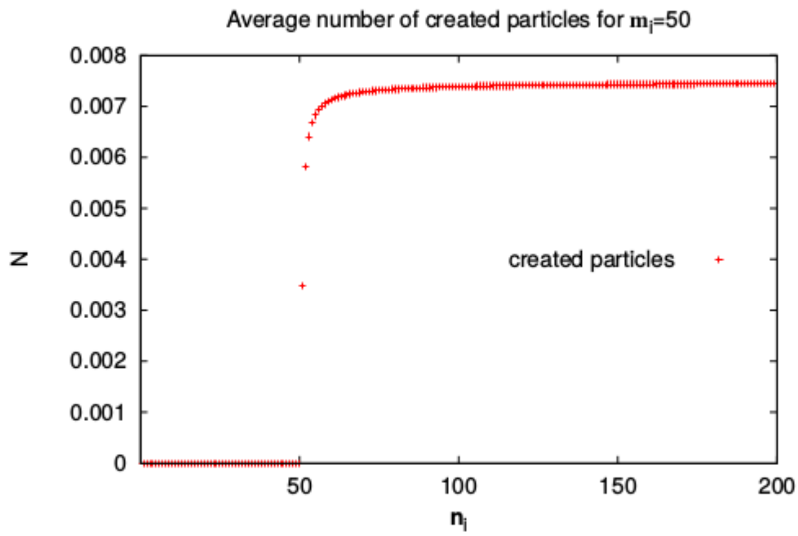

Figure 2: Number of particles N created for the mode $m_x = m_y = m_z = 50$ and $v = 0.3$ while the range n_i of the n_x,n_y and n_z sums in (40) increases.

Figure (2) shows the number of created particles as a function of the maximum value n_i for the summation indices n_x, n_y and n_z, while the mode $\mathbf{m}$ is kept fixed. In this specific case we chose $v = 0.3$ and indeed the value converges to $\frac{v^2}{12} \approx 0.0075$. We see then that the lowest contribution to the sum in (40) comes from the case where the modes $\mathbf{n}$ are much smaller than the modes $\mathbf{m}$ of the particles we want to create ($|\mathbf{n}| \ll |\mathbf{m}|$). The opposite situation has a slightly higher contribution ($|\mathbf{n}| \gg |\mathbf{m}|$) but the largest contribution comes from the case where the modes $\mathbf{m}$ and $\mathbf{n}$ are roughly equal (around $n_i = 50$ in figure (2)). Intuitively the above result is what can be expected. The creation of particles depends mainly on $(G_{\mathbf{mn}})^2$ which peaks around $m_i = n_i$ for $i = x, y, z$ as mixing of modes occurs predominantly for modes that have similar spatial properties. The last 'region' corresponds to the energy of mode $|\mathbf{m}| \ll |\mathbf{n}|$ and we see the creation rate saturates, as for $|\mathbf{m}| \ll |\mathbf{n}|$ the first term in (40) goes as $\sim \frac{|\mathbf{m}|}{|\mathbf{n}|^3}\frac{v^2}{\pi^2}$ and is thus suppressed. This would mean that it is possible to create $\frac{v^2}{12}$ particles for every sufficiently large mode $\mathbf{m}$ (the modes $|\mathbf{m}| < |\mathbf{n}|$ contribute less). However, there is an infinite amount of modes $\mathbf{m}$ and thus the possibility to create an infinite amount of massless particles. To explain this infinity we have to take a closer look at our original setup.

Before the wall acquired a velocity the system was free. At $t = 0$ the walls instantly moved away with a velocity v for a time interval $\Delta t = t_f$, thus expanding the initial vacuum. When the walls stop moving the system is free once again and we find a different vacuum. However, this 'new' vacuum does not only differ from the initial vacuum by the new modes that are now allowed, but also by the particles that have been created for every mode $\mathbf{m}$ during the expansion. This means that besides a shift in the (formally infinite) vacuum energy we can expect a correction term appearing in the energy of the new system that accounts for the energy required to create particles. To calculate this shift in energy we consider the differential equation for $Q_{\mathbf{m}}^{(\mathbf{n})}(t) = \delta_{\mathbf{mn}} Q_{\mathbf{n}}^{(\mathbf{n})}$. Thus, we only consider those modes $\mathbf{m}$ that were already present before the walls started moving. In that case we find

$$\ddot{Q}_{\mathbf{n}}^{(\mathbf{n})}(t) + \omega_{\mathbf{n}}^2(t) Q_{\mathbf{n}}^{(\mathbf{n})}(t) = -\lambda^2(t) \left(G^2\right)_{\mathbf{nn}} Q_{\mathbf{n}}^{(\mathbf{n})}(t)$$

as the other terms yield zero due to $G_{\mathbf{nn}} = 0$. From (35) we know $\left(G^2\right)_{\mathbf{nn}} = -\frac{3}{4} - \frac{\pi^2}{3}(n_x^2 + n_y^2 + n_z^2)$, which yields

$$\begin{aligned}
&\ddot{Q}_{\mathbf{n}}^{(\mathbf{n})}(t) + \omega_{\mathbf{n}}^2(t) \left(1 - \frac{v^2}{3} - \frac{3v^2}{4\pi^2 |\mathbf{n}|}\right) Q_{\mathbf{n}}^{(\mathbf{n})}(t) \\
&= \ddot{Q}_{\mathbf{n}}^{(\mathbf{n})}(t) + \bar{\omega}_{\mathbf{n}}^2(t) Q_{\mathbf{n}}^{(\mathbf{n})}(t) = 0.
\end{aligned}$$

We see then that $\omega_{\mathbf{n}}(t)$ is shifted to $\bar{\omega}_{\mathbf{n}}(t)$ as a result of the movement of the walls and consequently the mixing of modes. We can use $\bar{\omega}_{\mathbf{n}}(t)$ to calculate the shift in the vacuum energy. Using a conventional factor $\frac{1}{2}$ yields in the adiabatic approximation

$$\begin{aligned}
\frac{\bar{\omega}_{\mathbf{n}}(t)}{2} &= \frac{\omega_{\mathbf{n}}(t)}{2} \sqrt{1 - \frac{v^2}{3} - \frac{3v^2}{4\pi^2 |\mathbf{n}|^2}} \\
&\approx \omega_{\mathbf{n}}(t) \left(\frac{1}{2} - \frac{v^2}{12} - \frac{3v^2}{16\pi^2 |\mathbf{n}|^2}\right).
\end{aligned}$$

The first term corresponds to the new vacuum energy ($\omega_{\mathbf{n}}(t) \sim \frac{1}{L(t)}$). The second term gives a redshift correction proportional to $-\frac{v^2}{12}$ that should correspond to the energy extracted from the initial vacuum to create particles, which we indeed found as the first term of (40).

Future research could focus on finding different forms of $L(t)$ for which the differential equation can be solved exactly. Besides that there is still the issue of creating massive particles. Unfortunately we were unable to find a solution for the massive Klein-Gordon equation in the case of a linear displacement of the walls. The use of a different form of displacement might solve this problem as well.

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