Colour structures in Supersymmetric QCD

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In this thesis we will explain a novel method for calculating the colour structures of Quantum ChromoDynamical processes, or QCD prosesses for short, with the eventual goal to apply such knowledge to Supersymmetric theory.
QCD is the theory of the strong interactions, among quarks and gluons. From a Supersymmetric viewpoint, something we will explain in the first chapter, it is interesting to know as much as possible about QCD theory, for it will also tell us something about this Supersymmetric theory. Colour structures in particular, which give the strength of the strong force, can tell us something about the interactions with the Supersymmetric sector.
There are a multitude of different ways to calculate these colour structures. The one described here will make use of diagrammatic methods and is therefore somewhat more user friendly. We therefore hope it will see more use in future calculations regarding these processes.
We do warn the reader for the level of knowledge required to fully grasp the subject. Although this thesis is written with students with a bachelor degree in mind, one cannot escape touching somewhat dificult matterial. We have tried to explain things as much as possible in a way which should be clear to the reader.
Sometimes however we saw it neccesary to skip some of the finer details of the subject. If one wants to learn more about these hiatuses we suggest looking at the material in the bibliography.
For convenience we will use some common conventions in this thesis. In all calculations and formulas we will set the speed of light and Planck's constant to one, or in other words:

$$c = \hbar = 1$$

Furthermore we will assume Einsteins summation convention, so summation is implied over repeated indices:

$$A^\mu \gamma_\mu \equiv \sum_{\mu=0}^{3} A^\mu \gamma_\mu$$

1 What is Supersymmetry?

In short Supersymmetry (from here on called SUSY) is a predicted symmetry of nature, which mirrors the fermions to the bosons. In other words, for each fermion there must be a boson with the same intrinsic parameters, e.g. electromagnetic charge, colour, weak charge and mass. The only difference between these SUSY partners are their spins, which differ by half a unit from each other.

One could wonder why such a framework is necessary. Why would there be a SUSY partner for each elementary particle? Especially since none of these fabled particles has been seen. In order to explain this one must look at what is called the Grand Unification Theory (from here on named GUT) i.e. the theory in which all forces stem from one source. Such a theory could explain everything we know about the universe. This is something physicists are looking for these days. In order for such a thing to work there must be an energy scale at which all known forces excluding gravity start to act like one. In the Standard Model, the current theory describing all known particles and non-gravitational forces most successfully, this is not the case.
Let's first explain what is meant by 'act like one'. At low energy the electromagnetic, strong and weak force all act with a different strength. The electromagnetic interactions for instance have a coupling constant of $\frac{1}{137}$. However if one increases the energy of the particles and measures the coupling constant again, scientists found that the strength of the electromagnetic coupling was enhanced. This process can be explained if one sees a charged particle not as one particle but as a 'dressed' one. The electromagnetic field around the particle excites the vacuum and creates pairs of electrons and positrons that shield the core charge. If one increases the energy, it is possible to penetrate this cloud and see more of the bare charge. The same can be done for the weak and strong forces, which both turn out to decrease in strength when increasing the energy. At some point these forces intersect and thus have the same strength. Due to large error margins, people thought that all three forces even intersected in one point. This would mean that at that point the three forces would have the same strength and were indistinguishable from each other and thus could be considered one single force. This proved to be false however. When the accuracy increased the three lines still intersected, but no longer in one point.
This is where SUSY proved to be a useful theory. When the extra partner particles that can be created through the various interactions are taken into account, the new extrapolations of the forces at high energy once again go through one point. Therefore, in order to have unification of the three elementatry forces SUSY might be required. But why haven't we seen it then?
One of the explanations is that SUSY is in fact not an exact, but a broken symmetry, with the particles we know much lighter than their, still undiscovered, SUSY partners. This means that in the current experiments we have not been able to create these states up to now, simply due to a lack of energy. As said above however, the forces act the same on the SUSY particles as they do on the Standard Model particles, which makes it possible to make theoretical predictions of their interactions on the basis of Standard Model knowledge. In order to make these predictions, however, we need a way to calculate these interactions. In the rest of the thesis we shall construct a method to do so.

2 Gauge formalism

In order to understand the finer points of the group theory that will be presented later on, it is imperative that one first understands the reason for using this formalism. In ordinary quantum mechanics we use a hamiltonian to define the systems, which works very well at reasonably low energies. At higher energies and therefore smaller distance scales, particles start to behave relativistically

and thus we can not use the standard equations of motion anymore; we need to respect the lorentzian ones. This is where the first problem arises. The hamiltonian formalism itself is not Lorentz invariant, as it does not treat space and time the same way. There is one quantity that must be invariant at all times and that is the action. It doesn't matter in what frame of reference one is, the action is independent. The action is constructed by means of the lagrangian. It is therefore convenient to switch to the lagrangian formalism. Because Lorentz invariance states that space and time must be treated equal it is convenient to introduce a quantity $\mathcal{L}$, called the lagrangian density, which is a variation of the lagrangian over space. This means the action becomes:

$$S[\phi_i] = \int \mathcal{L}[\phi_i(x), \partial_\mu \phi_i(x)] d^4x$$

with ∂_μ defined as:

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

As one can see the lagrangian is no longer solely dependent on a parameter x, but instead on a parameter $\phi(x)$ that is a field. We take this step because it is possible for the lagrangian to have its degrees of freedom defined per space-time point. Therefore the lagrangian can no longer depend on rigid parameters, but must depend on a parameter that can vary over space-time itself: a field. How these fields move through space and time can be derived by looking at the Euler-Lagrange equations, just as in the classical case:

$$\partial_\mu \left(\frac{\partial \mathcal{L}[\phi_i, \partial_\mu \phi_i]}{\partial(\partial_\mu \phi_i)} \right) - \frac{\partial \mathcal{L}[\phi_i, \partial_\mu \phi_i]}{\partial \phi_i} = 0$$

In non-interacting field theory these equations of motion should lead to either the Klein-Gordon equation or the Dirac equation, depending on whether one looks at a bosonic field or a fermionic one. These are constraints on your fields. To describe non-gravitational elementary forces we need to consider two general cases for the lagrangian for free particles with mass m. First we need to describe spin $\frac{1}{2}$ matter particles. This involves the fermionic lagrangian belonging to the Dirac equation:

$$\begin{aligned} \mathcal{L}_f &= i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi \\ &\Rightarrow (i\gamma^\mu \partial_\mu - m)\psi = 0 \end{aligned}$$

for the fermion field ψ, with $\bar{\psi} = \psi^\dagger \gamma^0$ the Dirac conjugate of ψ and γ^μ the Dirac matrices obeying the Clifford algebra. The Dirac equation is derived by taking the lagrangian and putting it into the Euler-Lagrange equation for the field $\bar{\psi}$. Next we need to describe massless spin 1 force carriers (like photons). This involves the bosonic lagrangian belonging to Maxwell's equations:

$$\begin{aligned} \mathcal{L}_b &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \\ &\partial_\mu F^{\mu\nu} = 0 \end{aligned} \tag{1}$$

Here the $F^{\mu\nu}$ is the field tensor for the bosonic vector field A^μ and can be defined as:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

If one sets the gauge condition $\partial_\mu A^\mu = 0$ in order to select the physical states, one can see the Klein-Gordon equation arise from (1):

$$\partial_\mu \partial^\mu A^\nu = 0$$

Now that we have the free particle basis for the field theory, one might wonder how to build upon this and create an interacting theory. It is possible to simply introduce interactions as we see them in nature, involving interactions between force carriers and matter fermions. However there is a more elegant way to approach this. Let us look at the fermionic lagrangian above. One may wonder, after writing the lagrangian in this form, what kind of freedom one has in choosing the field ψ. This is called a gauge freedom.
After setting all parameters of the theory there are still some choices left. Choosing a gauge means fixing those free parameters and thus defining your theory for a specific choice. This choice however must not show up in your calculations since the physics in one gauge cannot be different from the physics in another gauge.

In classical field theory the invariance of a lagrangian under variations in the fields leads to conserved current. This is known as Noether's Theorem. If one transforms the fields under the follwing transformations:

$$\begin{aligned} \psi(x) \to \psi'(x) &= e^{-i\zeta}\psi(x) \\ \bar{\psi}(x) \to \bar{\psi}'(x) &= e^{i\zeta}\bar{\psi}(x) \end{aligned}$$

One can easilly see that it does not change the lagrangian. However it can be shown that it does lead to a conserved current and conserved "charge". This is called global invariance since it is considered the same for each space-time point. Next we demand this global invariance to also hold locally, i.e. for transformations that vary from space-time point to space-time point. In this case a global invariance is a special case, but the extra local constraint leads to some interesting consequences. Let us first look at the Dirac fields ψ. We can introduce a local transformation:

$$\begin{aligned} \psi(x) \to \psi'(x) &= e^{-i\zeta(x)}\psi(x) \\ \bar{\psi}(x) \to \bar{\psi}'(x) &= e^{i\zeta(x)}\bar{\psi}(x) \end{aligned} \tag{2}$$

with $\zeta \in \mathbb{R}$, which we shall call the transformation operators for now. The mass term $m\bar{\psi}\psi$ is automatically invariant under the transformation. The kinetic term, however, picks up an extra term:

$$\mathcal{L} \to \mathcal{L}' = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + i\partial_\mu(\zeta(x))\bar{\psi}\gamma^\mu\psi$$

If we want the lagrangian to be invariant under this transformation we have to introduce an extra term to compensate for the new term that is created when we use the transformation. We can do this by replacing ∂_μ by the covariant derivative

$$D_\mu \equiv \partial_\mu + iqA_\mu(x)$$

where q is the charge of the particle described by ψ. To achieve invariance the vector field A^μ has to transform as:

$$A^\mu \to A^\mu + \frac{1}{q}\partial^\mu \zeta(x)$$

This results in the total lagrangian

$$\begin{aligned} \mathcal{L} &= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - q\bar{\psi}\gamma^\mu \psi A^\mu \\ &= \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \end{aligned}$$

which is invariant and has what is called U(1) symmetry. We see that the lagrangian now picks up a term that mixes the A^μ bosonic field with the ψ fermionic field. This A^μ is called the gauge field and is said to carry the force belonging to the charge q. By imposing the local gauge symmetry we got ourselves exactly the interacting term we were looking for. In this particular case it is in fact the electromagnetic interactions that we got out. It should be noted however that the bosonic field A^μ does not have a kinetic part yet. Without it the field cannot propagate through space. It therefore needs to be added by hand. This can be done by adding equation (1):

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{3}$$

From now on we will no longer assume bosonic particles but instead wait for them to come out of the theory. One point that could be noted is that the field tensor changes by the introduction of the covariant derivative into:

$$\begin{aligned} D^\mu A^\nu - D^\nu A^\mu &= \partial^\mu A^\nu + iqA^\mu A^\nu - \partial^\nu A^\mu - iqA^\nu A^\mu \\ &= \partial^\mu A^\nu - \partial^\nu A^\mu + iq[A^\mu, A^\nu] \end{aligned}$$

thus it is simply the old field tensor plus the commutator of the bosonic field. In the case of U(1) symmetry that commutator is 0 though, which we will show in the next section.

3 Lie Goups and Algebras

One may wonder what this U(1) stands for and where it came from. Did we not discuss invariance under phase transformations? What is this symmetry? Well, first of all, U(1) is what is called a Lie group. A Lie group is a special kind of infinite group that is also a differentiable manifold. This means in short that it is a mathematical space, such as a n-dimensional space of the real numbers $\mathbb{R}^n$ which on short intervals resembles Euclidian space (meaning is looks flat at short distances) and any mapping you make of it always transform into each other smoothly. In other words, at very large distances your space (in this case a group) may look strange, but up close it must be well-behaved and there cannot be any jumps. For any group G the following property holds:

$$\begin{aligned} g_1, g_2 &\in G \\ g_1 g_2 = g_3 &\in G \end{aligned}$$

such that each set of group elements generates a new group element. Because of this property, at small distances near the identity a Lie group transforms in a Lie algebra. This Lie algebra is a vector space with a special property it inherited from the group G: it has a Lie bracket, which has the same properties as a commutator

$$[g_j, g_l] = ie_{jlk}g_k$$

This means that the Lie bracket is the difference of two groupelements. One can easily see that if any two elements commute their commutator will be zero. In this case the Lie algebra will have a Lie bracket of zero and the Lie algebra will be an Abelian one. This is the case for the U(1) Lie group. But what does U(1) mean?
U(N) is a Lie group of all the unitary N by N matrices, with unitarity defined as:

$$\begin{aligned} U^\dagger &= U^{-1} \\ UU^{-1} &= 1 \end{aligned} \tag{4}$$

and thus U(1) is a Lie group of all unitairy 1 by 1 matrices. As one can see the product of two unitary matrices is once again a unitary matrix and since one can write down an infinite amount of these matrices, it is clear that indeed U(N) is a Lie group. In order for us to go from the Lie group to the Lie algebra, we need the proper formalism.

One can view the group elements $g \in$ G as $g(\alpha_a)$, a smooth and continuous function depending on a set of parameters α_a. Since the Lie algebra is well-defined near the identity, we want to parameterize the elements in such a way that $g(\alpha_a = 0) = 1$, where 1 is the identity element of the group. For any neighbouring point in the vicinity of the identity element we can Taylor-expand in α_a to obtain the parameters of the group

$$g(d\alpha_a) = 1 + id\alpha_a\zeta_a + O(\alpha_a^2)$$

where

$$\zeta_a \equiv -i\frac{\partial}{\partial\alpha_a}g(\alpha_a) \mid_{\alpha=0}$$

are called the generators of the group. Finite group elements can be written as an infinite set of group elements infinitesimally close to the identity element:

$$g(\alpha_a) = \lim_{k\to\infty}(1 + \frac{i\alpha_a\zeta_a}{k})^k = e^{i\alpha_a\zeta a}$$

so we get the generators in the exponent. In this exponential form the generators do indeed, by what is called exponential parameterization, generate the group. Notice that this is indeed the type of transformation we applied to the ψ fields in equation (2). Next we show that these generators indeed span the algebra and inherit the Lie bracket from the group.
Denote the group by G and the algebra by $\mathfrak{g}$, with $P_i \in$G and $p_i \in \mathfrak{g}$. Because P_i are group elements, a combination of them must go into new group elements:

$$P_1P_2 = P_{1+2}$$

However for the algebra in general

$$e^{i\alpha_a p_a} e^{i\beta_b p_b} \neq e^{i(\alpha_a+\beta_a)p_a}$$

One thing that can be said:

$$e^{i\alpha_a p_a} e^{i\beta_b p_b} = e^{i\delta_d p_d}$$

Because close to the identity element all is smooth, by method of Taylor expansion in α and β we can determine δ_a from

$$i\delta_d p_d = ln(1 + e^{i\alpha_a p_a} e^{i\beta_b p_b} - 1)$$

Now we can expand $ln(1 + K)$ in terms of α and β up to second order,

$$\begin{aligned} K &= e^{i\alpha_a p_a} e^{i\beta_b p_b} - 1 \\ &= (1 + i\alpha_a p_a - \frac{1}{2}(\alpha_a p_a)^2 + O(\alpha^3))(1 + i\beta_b p_b - \frac{1}{2}(\beta_b p_b)^2 + O(\beta^3)) - 1 \\ &= i\alpha_a p_a + i\beta_b p_b - \alpha_a p_a \beta_b p_b - \frac{1}{2}(\alpha_a p_a)^2 - \frac{1}{2}(\beta_b p_b)^2 + O(\alpha^3, \beta^3) \end{aligned}$$

giving

$$\begin{aligned} i\delta_d P_d &= K - \frac{1}{2}K^2 + O(K^3) \\ &= i\alpha_a p_a + i\beta_b p_b - \alpha_a p_a \beta_b p_b - \frac{1}{2}(\alpha_a p_a)^2 - \frac{1}{2}(\beta_b p_b)^2 \\ &\quad + \frac{1}{2}(\alpha_a p_a + \beta_b p_b)^2 + O(\alpha^3, \beta^3) \\ &= i\alpha_a p_a + i\beta_b p_b - \frac{1}{2}[\alpha_a p_a, \beta_b p_b] + O(\alpha^3 \beta^3) \end{aligned}$$

Now we can write the commutator in terms of P_c

$$\begin{aligned} [\alpha_a p_a, \beta_b p_b] &= -2i(\delta_c p_c - \alpha_a p_a - \beta_b p_b) \\ &= -2i(\delta_c - \alpha_c - \beta_c)p_c = i\gamma_c p_c \end{aligned}$$

Since this property must hold for all α and β, it follows that:

$$\gamma_c = \alpha_a \beta_b f_{abc}$$

where the f_{abc} are called the structure constants of the group. They differ from group to group, and in a way define how Lie algebras work. They are therefore incredibly important and we will revisit them later on.

The phase transformation of the previous chapter is indeed a U(1) transformation, with $\mathbb{R}$ as its associated Abelian Lie algebra. Since the lagrangian of equation (3) is invariant under the transformation, we indeed have a U(1) symmetry. But what happens if the algebra is no longer Abelian?

We shall come back to this question in chapter 4, but first we need to clarify a few things in order to smoothen our path later on. U(1) is the group of unitary

1 by 1 matrices, which are just phase factors. When one chooses a representation for this group, there is one operator and thus one generator, namely a real number. It is commonly said that U(1) is 1 dimensional. One might think that because of this the dimension of the group U(N) is N, equal to the rank of the matrices. This is wrong however. The dimension of the group is given by the number of independent real parameters that are needed to describe each element of the group. For U(1) that is just one parameter, namely the one that makes up it's matrix. In the case of U(2) however we have a 2 by 2 matrix that needs to be unitary. From this it follows that if one writes the elements α_{jk} of such a matrix as $\alpha_{jk} = a_{jk} + ib_{jk}$, with $a_{jk}, b_{jk} \in \mathbb{R}$, and solves the unitarity condition (4), there are four equations with eight unknown parameters. So your matrix is defined by four real independent parameters and thus the dimension is 4. In the N dimensional case you have N^2 equations with $2N^2$ unknown parameters and therefore N^2 real independent parameters. For all U(N) the dimension is N^2.

In the U(1) case you had one generator, the one independent parameter. In a higher dimensional group there is more than one independent parameter. Since each independent parameter leads to one degree of freedom that one can "gauge away", this means that an independent parameter leads to a generator. And since the dimension of a group is equal to the number of independent parameters, the number of generators is equal to the dimension of the group.

4 SU(3)

Non-Abelian groups can be represented by matrices with a rank higher than one. If the rank is equal to the number of the group (for instance rank 2 for U(2)) we call the representation the fundamental representation. The higher the rank of the matrices in the fundamental representation, the higher the dimension of the group. For now we'll stick to this representation, for it gives us the basic knowledge about what we will call the order of the force.

There are many non-Abelian Lie groups we can look at, but within the Standard Model two groups stand out: SU(2), which can be identified with the weak interactions, and SU(3), which leads to the strong interactions.
The group SU(N) is, like U(N), a group of unitary matrices of rank N. However the difference, indicated by the S (which stands for special), lies in the fact that the determinant of these matrices must be 1. Since the product of the determinant of two matrices is equal to the determinant of the product of these two matrices, one can see that if U(N) is a group so must SU(N).
One would think that the dimension of the group is N^2, but one has to take into account that the determinant must be 1, which gives us an additional constraint. This removes one independent parameter and we are left with only $N^2 - 1$ dimensions and thus $N^2 - 1$ generators.

In light of this thesis we will now look at SU(3). To construct a valid theory we need to make a few definitions which distinguish SU(3) from the more general case we discussed above. From now on we will denote SU(3) generators as T_a

and the Lie bracket becomes:

$$[T_a, T_b] = if_{abc}T_c \tag{5}$$

Much like the generators ζ_a, the generators T_a will help us establish a symmetry in the lagrangian. In the fundamental representation it is convenient to define a set of 3 by 3 matrices which are called the Gell-Mann matrices λ_a:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The generators T_a follow from these by this convention:

$$T_a = \frac{1}{2}\lambda_a$$

In order for the SU(3) transformations to be unitary these matrices need to be hermitian and traceless. Indeed, for any hermitian matrix P_a

$$U(\alpha_a) = e^{i\alpha_a P_a}$$

is unitary. Moreover, we find a basis for which $\alpha_a P_a$ is a diagonal matrix D, then

$$\begin{aligned} det(U(\alpha_a)) &= det(e^{i\alpha_a P_a}) = det(e^{iD}) \\ &= \prod_j (e^{i[D]_{jj}}) = e^{i\sum_j [D]_{jj}} \\ &= e^{iTr(D)} = e^{iTr(\alpha_a P_a)} = 1 \\ &\Rightarrow \quad Tr(\alpha_a P_a) = 0 \end{aligned}$$

Since α_a can be chosen arbitrarily P_a must be traceless.

Next we construct the gauge formalism, beginning with the fermionic part of the lagrangian:

$$\begin{aligned} \psi(x) \to \psi'(x) = e^{-i\alpha_a(x)T_a}\psi(x) = U(x)\psi(x) \\ \bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x)e^{i\alpha_a(x)T_a} = \bar{\psi}(x)U^\dagger \end{aligned}$$

Again the mass term is symmetric and yet again we are left with the kinetic part for which we need to construct a covariant derivative. The difference with before is that this covariant derivative needs to be invariant under all 8 independent transformations at the same time. So we obtain:

$$D_\mu \equiv \partial_\mu + ig_s W_\mu^a T_a$$

where W_μ^a is a collection of 8 gauge fields and g_s is the coupling constant (coupling strength) of the strong force. We usually simply write $W_\mu = W_\mu^a T_a$, where W_μ is, what is called, Lie algebra valued. In other words, it has the properties of a Lie algebra in it. As we promised, we no longer need to add the gauge field terms separately but instead let them come from the symmetries themselves. After we add the infinitesimal transformations of the gauge fields:

$$\begin{aligned} W_\mu^a T_a \rightarrow & \quad U(x)W_\mu^a T_a U^{-1}(x) - \frac{i}{g_s}U(x)\Big(\partial_\mu U^{-1}(x)\Big) \\ \overset{inf}{\approx} & \quad W_\mu^a T_a + i[W_\mu^b T_b, \alpha_c T_c] + \frac{1}{g_s}\big(\partial_\mu \alpha_a\big)T_a \\ = & \quad \Big(W_\mu^a + \frac{1}{g_s}\partial_\mu \alpha_a - f_{abc}W_\mu^b \alpha_c\Big)T_a \end{aligned}$$

we will see that indeed the lagrangian is symmetric under local SU(3) invariance. The last term comes from the fact that we are no longer working with simple constants, but matrices. As we see, the amount of generators will determine the number of gauge fields that we have, in this case 8. There is still the matter of the kinetic terms of those gauge fields. In chapter 2 we noted that they still needed to be added by hand, but there are of course a multitude of possibilities one could think of. One can prove by means of the Ricci-identity there is but one solution that serves our purposes:

$$\begin{aligned} G^{\mu\nu} &= \frac{1}{ig_s}[D_\mu, D_\nu] \\ &= D^\mu W^\nu - D^\nu W^\mu \\ &= \partial^\mu W^\nu + ig_s W^\mu W^\nu - \partial^\nu W^\mu - ig_s W^\nu W^\mu \\ &= \partial^\mu W^\nu - \partial^\nu W^\mu + ig_s[W^\mu, W^\nu] \end{aligned} \tag{6}$$

and thus we can define a Lie-algebra valued field tensor

$$G_a^{\mu\nu} = \partial^\mu W_a^\nu - \partial^\nu W_a^\mu - g_s f_{abc} W_b^\mu W_c^\nu$$

When one works out the commutator of two covariant derivatives we see that it picks up a commutator of its own, in two of the gauge fields. The field tensor needs to be squared in order to obtain the kinetic terms:

$$\begin{aligned} -\frac{1}{4}G_a^{\mu\nu}G_{\mu\nu a} &= -\frac{1}{4}\Big(\partial^\mu W_a^\nu - \partial^\nu W_a^\mu - g_s f_{abc}W_b^\mu W_c^\nu\Big) \\ &\quad \Big(\partial_\mu W_{\nu a} - \partial_\nu W_{\mu a} - g_s f_{ade}W_{\mu d}W_{\nu e}\Big) \\ &= -\frac{1}{4}\big(\partial^\mu W_a^\nu - \partial^\nu W_a^\mu\big)\big(\partial_\mu W_{\nu a} - \partial_\nu W_{\mu a}\big) \\ &\quad + g_s f_{abc}\big(\partial^\mu W_a^\nu\big)W_{\mu b}W_{\nu c} \\ &\quad - \frac{g_s^2}{4}f_{abc}f_{ade}W_b^\mu W_c^\nu W_{\mu d}W_{\nu e} \end{aligned}$$

The last three terms give two three-point interactions and a four-point interaction in the gauge fields. Thus the non-vanishing commutator leads to what is called gauge boson self interactions. It is also easy to see why we had no such

problems in the Abelian case. Since in that case there where no non-vanishing commutators, the last term in the field tensor (5) was automatically zero and thus no self interactions emerged.

In order to better understand this we first go back to the fermion fields. For the symmetry to properly work on these fermions, the fermions themselves need to be vectors in the fundamental representation. Since the T_a are 3 by 3 matrices the fermions need to be vectors of length 3. So for every fermion in the SU(3) symmetric theory, there are in fact three copies of that fermion. This decomposition corresponds to a quantum number of the field that is called colour. The usual way of denoting this quantum number is by a subscript i, so the fermionic field ψ now becomes ψ_i, with i ranging over the possible colours, which is 3 in SU(3). This is called a colour triplet, for the same fermion is decomposed in a set of 3 individual states with different colour. Since the gauge fields W_μ are in fact Lie algebra valued, they contain the generators. In the same way as in the case of ψ_i we can write these generators as T^a_{ij}, with i and j ranging over the different colours. Therefore the gauge fields can be written as W^μ_{ij} and now one can see that the fields themselves carry colour, making it possible for SU(3) gauge fields to couple to themselves. It is exactly this colour structure that we are interested in.

5 Quantum ChromoDynamics

Before we continue with the math behind the strong interactions, it might be useful to look at what these interactions really are. First we give a name to the branch of processes which involve the SU(3) gauge interactions. The processes are said to be Quantum ChromoDynamical processes, QCD processes for short. The name stems from the greek word chroma ($\chi\rho\omega\mu\alpha$) which means colour.

Let us first give a proper definition to the fields we have created thusfar. We have a fermion triplet ψ_i and a set of eight bosonic Lie algebra valued fields W^μ_{ij}, called a bosonic octet, with i and j running over the 3 possible colours. In nature we see two types of fermions: leptons and quarks. On pure theoretical grounds there is no reason not to let both be SU(3) triplets. However, empirically we see that leptons exhibit no QCD interactions. It's is like they are blind to the strong force, which makes us to believe that they do not carry a colour charge and are what is called singlets under SU(3) transformations[1]. As for the bosonic octet, we shall call it a gluon field. It describes eight particles, the gluons, who have self interactions, as was shown in the previous chapter. Unlike the fermions, to which you can again apply the U(1) symmetry argument and thus give them electromagnetic charge, this does not apply to the gluon fields. This means that while we can give the quarks both colour and electrical charge and causing them to interact with the photons, the gluons will have no charge other than colour.

The strong interactions, also called strong force, were first postulated in

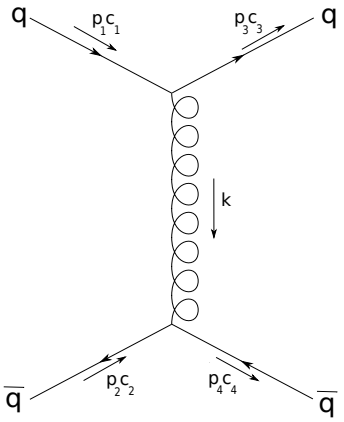
[1]Being a singlet under a transformation means that if an element of the group works on the singlet, it stays the same and does not change. A good example is the null vector. Any multiplication with this vector gives again the null vector and thus it is a singlet under all transformations.

order to correct the up till then quite successful parton model. This model tells us that hadrons are constructed out of smaller particles, which were (later) named quarks. In 1961 there was what was called a zoo of particles. Therefore Gell-Mann proposed that the baryons, a subset of the hadrons, were not single particles, but instead composite particles consisting of three quarks or antiquarks. He postulated the existence of three types of quarks: up, down and strange (the rest was yet to be discovered), which create a flavour triplet. Using what was called SU(3) flavour symmetry, he constructed the eightfold way. By this method it was possible to explain the particle zoo and even predict some yet undiscovered particles. The method he used is the same as the one that will be used in this thesis, but now modified to colour triplets instead of flavour triplets. However there was a particle that seemed to contradict this, due to the fact that it seemed to break the Pauli exclusion principle: the Δ^{++}. In the ground state (symmetric in space) this particle consists out of three up quarks (symmetric in flavour). It also has a spin of $\frac{3}{2}$. Quarks have a spin of $\frac{1}{2}$, so this means that also the spin state is symmetric. Since the Pauli exclusion principle teaches us that the total wave function of fermions needs to be anti-symmetric, the model was incomplete. Therefore a new quantum number was introduced, for which the three quarks were in a anti-symmetric singlet state: colour. Quarks, however, are confined, i.e. they occur as constituents inside hadrons but never as free particles. Therefore we cannot directly measure them.

At small distances and thus at high energies it is possible, however, to probe this baryonic matter and see how the colour charge flows in QCD processes. That is exactly what we want to look at in this thesis: how exactly does the colour charge flow through a process and can we on this basis exclude certain processes or make the calculation of others intuitively visible? Looking at the colour charge in order to predict certain processes has other advantages. Since in supersymmetric theories the charges are the same as in the Standard Model, SUSY processes can be covered by generalizing the Standard Model processes.

6 Colour Labels

Our next stop is to look at a real process, in which we can identify the colour labels and maybe say something constructive about an equivalent supersymmetric process. One of the fundamental processes in QCD is a quark and an antiquark exchanging a gluon. To visualize this we shall use a Feynman diagram with time defined as going from left to right.



Here the straight line indicates a fermion, with an arrow in the direction of the time indicating a quark, while an arrow in the opposite direction of the time indicates an antiquark. The curly line indicates a gluon field. One might wonder why the gluon has no arrow. This is because the gluon is its own antiparticle. This Feynman diagram is called the t-channel for the process quark-antiquark goes to quark-antiquark ($q\bar{q} \rightarrow q\bar{q}$). The other possible channels and their relevance will be discussed later on. With some advanced methodology called Feynman rules one can construct the physics of this process from the diagram. The way to derive these rules, however, goes well beyond what we try to discuss here. Therefore we shall simply state the answer and work with it:

$$\begin{aligned}
\mathcal{M} &= -i[\bar{u}(3)c_3^\dagger]\big[-i\frac{g_s}{2}\lambda^a\gamma^\mu\big][u(1)c_1]\big[\frac{-ig_{\mu\nu}\delta^{ab}}{k^2}\big][\bar{v}(2)c_2^\dagger]\big[-i\frac{g_s}{2}\lambda^b\gamma^\nu\big][v(4)c_4] \\
&= \frac{g_s^2}{k^2}\left[\bar{u}(3)\gamma^\mu u(1)g_{\mu\nu}\bar{v}(2)\gamma^\nu v(4)\right] \times \frac{1}{4}\left[c_3^\dagger\lambda^a c_1\delta^{ab}c_2^\dagger\lambda^b c_4\right] \\
&= \frac{g_s^2}{k^2}\Lambda \times \frac{1}{4}\Gamma
\end{aligned}$$

Here $\mathcal{M}$ is what is called the amplitude for the process and with it we can calculate the cross section of the process, giving the likelihood that the interaction will take place. The first part of this answer, which we have dubbed $\frac{g_s^2}{k^2}\Lambda$, is the lorentzian part of the answer and governs the paths of the particles through 4-dimensional Minkowski space. For now we leave it aside. The second part, that we have dubbed $\frac{1}{4}\Gamma$, is the colour part of the answer and is called the colour structure. This is precisely what we are interested in. If we look at it we see twice the structure $c_i^\dagger\lambda^a c_j$. Here the λ^a are the Gell-Mann matrices, related to the generators of the group. The c_j is called the colour label of the fermion, and is nothing more than the vector structure of ψ_j in colour space. The $c_i^\dagger$ is the colour label of the antifermion, originating from $\psi_i^\dagger$. The colour structure gives us the opportunity to calculate the colour potential $V_{q\bar{q}}$ of the interaction, which depends only on the colour charges:

$$V_{q\bar{q}} = -f\frac{\alpha_s}{r}$$

with α_s the fine structure constant of the strong force. The colour factor f can be calculated, by considering the colour structure Γ, but since Γ involves four

open colour labels, we must let it run over all possible colour configurations. In other words we need some serious bookkeeping in order to take some work out of our hands. There are various ways of doing this, one of them being simply crunching numbers. What we will do instead is make use of the Lie group we have and construct the weight space.

7 Weights and their Spaces

In order to introduce weights and weight spaces we must first introduce a new representation of the Lie algebra. We have the generators T^a, with a going over all possible states. As shown in chapter 4 the generators T^a have a Lie bracket, and the antisymmetric structure constant f_{abc}. By means of a simple calculation the following must hold

$$[T^a, [T^b, T^c]] = if_{bcd}[T^a, T^d] = -f_{bcd}f_{ade}T^e$$

Since commutators obey the Jacobi Identity

$$[T^a, [T^b, T^c]] + [T^c, [T^a, T^b]] + [T^b, [T^c, T^a]] = 0$$

it follows:

$$\begin{aligned} -f_{bcd}f_{ade}T^e - f_{abd}f_{cde}T^e - f_{cad}f_{bde}T^e &= 0 \\ \Rightarrow f_{bcd}f_{ade} + f_{abd}f_{cde} + f_{cad}f_{bde} &= 0 \end{aligned} \tag{7}$$

since T^e cannot be 0. It is possible to define a set of matrices $[t_a]_{bc}$ according to:

$$[t_a]_{bc} \equiv -if_{abc} \tag{8}$$

and from the Jacobi Identity (7):

$$\begin{aligned} -[t_b]_{cd}[t_a]_{de} + [t_a]_{cd}[t_b]_{de} &= if_{abd}[t_d]_{ce} \\ \Rightarrow ([t_a][t_b])_{ce} - ([t_b][t_a])_{ce} &= if_{abd}[t_d]_{ce} \\ \Rightarrow [t_a, t_b] &= if_{abd}t_d \end{aligned}$$

These matrices themselves span the Lie algebra as a different representation, named the adjoint representation, or adjoint for short. In SU(3) all three indices of $[t_a]_{bc}$ run over the eight possible generators of the group. Thus the generators in the adjoint are 8 by 8 matrices and work on a vector of length 8. The only object that we know with eight colour indices are the gluon fields and. We can see in the adjoint the interaction two gluon fields give us a new different gluon field back. This looks like a self interaction and thus the adjoint might indeed describe our gauge fields. Now that we have the adjoint representation of the gauge fields and the fundamental representation for the fermion fields it is time to look at the colour decomposition. We do this by generalizing the representation. Since the adjoint representation follows directly from the fundamental one, and the only restriction on the fundamental representation is that it needs to be a set of eight 3 by 3 matrices that are hermitian and traceless, we have a lot of freedom in the exact representation. The ultimate goal is to bring the Hilbert space into a form that makes our calculations easier. It must

be clear that a representation in which the generators are purely diagonal will be the easiest one. However, looking at the Gell-Mann representation one can see that it will be impossible to diagonalize all the generators at the same time.

Let's go back to the general case. It is at least possible to find a basis in which one or more generators are diagonalised. These diagonal generators span an algebra, called the Cartan subalgebra. Due to the fact that these Cartan generators, H_ρ, are diagonalised, they are hermitian and they obey the following commutation relations:

$$H_\rho = H_\rho^\dagger$$
$$[H_\rho, H_\sigma] = 0$$

Now all that remains is choosing a basis of the Hilbert space in such a way that

$$Tr(H_\rho H_\sigma) = k_R \delta_{\rho\sigma}$$

where k_R is a constant dependent on the representation. The Cartan subalgebra is now diagonalised and the elements are orthogonal. Due to the fact that the Cartan generators commute, we can find a set of simultaneous eigenfunctions $|\mu_i, x, R>$ in a certain representation R, where

$$H_\rho|\mu_i, x, R >= \mu_{i\rho}|\mu_i, x, R >$$

Here $\mu_{i\rho}$ is called a weight of SU(3) and x labels the state, as it is possible for two states to have the same weights. The weights together form a vector α_i, called the weight vector of fundamental state i, with

$$\alpha_i = \begin{pmatrix} ... \\ \mu_{i\rho} \\ ... \end{pmatrix}$$

The Cartan subalgebra is on its own an m-dimensional space (with m the number of Cartan generators) in which one can construct the weight vectors. Together these weight vectors form a diagram, called a weight diagram, giving us immediate information about the state we are working with. At this point it may all sound a bit technical, but we will elaborate on this in upcoming sections.

What we want to do first is to create a weight space for the adjoint representation in the same way we constructed it for the fundamental representation. The big advantage of the adjoint is that the rows and columns are labeled in the same way as the generators themselves [2]. This means that the generators themselves can be made to correspond to an orthonormal base of adjoint states, which we shall name $|X^a>$. These states are defined by:

$$X^a|X^a >\equiv 0$$

[2] remember that the rank of the adoint matrices is equal to the dimension of the group

with no summation over a intended. This follows from:

$$\begin{aligned} X^a|X^b> &= |X^c>< X^c|X^a|X^b> \\ &= |X^c>[t_a]_{cb} \\ &= -if_{acb}|X^c> \\ &= if_{abc}|X^c> \\ &= |if_{abc}X^c> \\ &= |[X^a, X^b]> \end{aligned}$$

The defining step in the previous derivation is the one where we use equation (8). Thus the choice of base vectors $|X^a>$ is defined by the group structure, set by the structure constants f_{abc}. Later we will see that it is convenient to construct linear combinations of the states, that correspond to linear combinations of the generators:

$$\alpha|X^a> + \beta|X^b> = |\alpha X^a + \beta X^a>$$

Now we have the tools to construct the action of a generator on such combined states.

Just like in the fundamental case we can define a Cartan subalgebra, with Cartan generators H^A_ρ that satisfy the relation:

$$H^\rho|H^\sigma> = |[H^\rho, H^\sigma]> = 0$$

Because of this the weights of the adjoint Cartan subalgebra are 0. This, however, only gives us m of the possible adjoint weights. The rest must be constructed by letting the Cartan generators work on the remaining generators in the way prescribed above. It is however more convenient to write the remaining generators in linear combinations E_ν

$$E_\nu = \sum_i \alpha_i X_i$$

in such a way that they become eigenfuntions of the Cartan generators:

$$[H_\rho, E_\nu] = \nu_\rho E_\nu \Rightarrow E_\nu|H_\rho> = -\nu_\rho|E_\nu>$$

thus creating new states for which the adjoint weights are easy to calculate. These E_ν are clearly not hermitian, for we can take the adjoint of the commutator

$$[H_\rho, E_\nu]^\dagger = [E^\dagger_\nu, H_\rho] = \nu_\rho E^\dagger_\nu \Rightarrow [H_\rho, E^\dagger_\nu] = -\nu_\rho E^\dagger_\nu$$

Therefore we can take

$$E^\dagger_\nu = E_{-\nu}$$

This is reminiscent of ladder operators in basic quantum mechanics as can be made more explicit if one looks at the action of the Cartan generators on an eigenstate $|\mu_i, x, R>$ of H_ρ modified by $E_{\pm\nu}$:

$$\begin{aligned} H_\rho E_{\pm\nu}|\mu_i, x, R> &= [H_\rho, E_{\pm\nu}]|\mu_i, x, R> + E_{\pm\nu}H_\rho|\mu_i, x, R> \\ &= (\mu_i \pm \nu)_\rho E_{\pm\nu}|\mu_i, x, R> \end{aligned}$$

Now that we have a new set of generators defined, we can go on to construct the adjoint weight space with weight vectors α_a^A for the Adjoint state a.

Finally, before we can continue with a real case, we need to address the complex conjugate representation. As we saw in the rough calculations above, antiparticles must carry anticolour charge and thus we need a definition of that in terms of the weights. We can take the Lie bracket and complex conjugate it

$$\begin{aligned} & ([t_a, t_b])^* = (if_{abc}t_c)^* \\ \Rightarrow \quad & [t_a^*, t_b^*] = -if_{abc}t_c^* \\ \Rightarrow \quad & [-t_a^*, -t_b^*] = if_{abc}(-t_c^*) \end{aligned}$$

As one can see, this is just another Lie algebra with generators $-X^{a*}$, which generates the same weights except for a minus sign, since the Cartan generators are hermitian.

8 Weights of SU(3) and Colour States

We finally have the tools to say something about the colour states of the processes considered, so let us start by constructing the weight diagrams of the particles involved. The quarks are in the fundamental representation and it is convenient to choose again the Gell-Mann matrices for this purpose, since we already have them written down. A quick glance at them immediately reveals the Cartan subalgebra. Remember that the Cartan generators need to be hermitian and their commutator needs to be equal to 0. Furthermore it is convenient to choose them as diagonal matrices. Since there are two diagonal matrices λ_3 and λ_8 and since they obviously commute we choose them to be in the Cartan subalgebra. All the other Gell-Mann matrices do not commute with these Cartan Generators, even not when taking linear combinations of them, so we have

$$\begin{aligned} H_1 &= T^3 \\ H_2 &= T^8 \end{aligned}$$

Here we went back to the definition of the generators we have been using all the time. Since the quarks correspond to colour triplets, we choose the convenient base

$$|c_1> = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |c_2> = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |c_3> = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

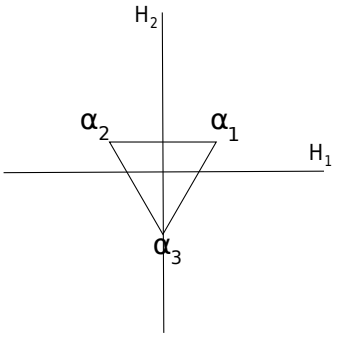
Applying the Cartan generators to these base vectors yields weight vectors

$$\alpha_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2\sqrt{3}} \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2\sqrt{3}} \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$$

with the following definition of weight vector α_i

$$\alpha_i = \begin{pmatrix} \mu_i^1 \\ \mu_i^2 \end{pmatrix}$$

This gives rise to the following diagram



This triangle apparently is connected with a triplet state.

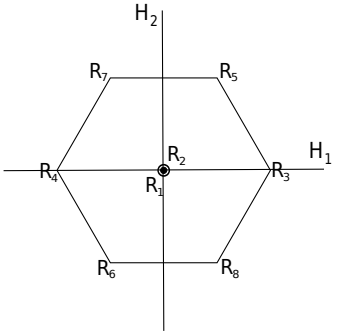
We want to do the same with the adjoint representation, thus creating the weight space of the adjoint of SU(3). One might think it is convenient to transform the generators to an 8 by 8 matrix representation, thus giving the adjoint states the same form (only now 8-dimensional) as the quarks had in the fundamental representation. However, it turns out that the transformation to the 8 by 8 adjoint is to much hassle, even more since we have the tools to create a 3 by 3 version of the adjoint with the help of representation-independent raising and lowering operators. Since the Cartan subalgebra must also hold for the adjoint representation, H_1 is still T^3 and H_2 is still T^8. Now we define the ladder generators in a proper way

$$E_1 = \frac{1}{\sqrt{2}}(T^1 + iT^2) = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix} \quad E_2 = \frac{1}{\sqrt{2}}(T^1 - iT^2) = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}$$
$$E_3 = \frac{1}{\sqrt{2}}(T^4 + iT^5) = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix} \quad E_4 = \frac{1}{\sqrt{2}}(T^4 - iT^5) = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$$
$$E_5 = \frac{1}{\sqrt{2}}(T^6 + iT^7) = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix} \quad E_6 = \frac{1}{\sqrt{2}}(T^6 - iT^7) = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}$$

where we have simply labeled the generators instead of indicating the adjoint weights, for we do not yet want to spoil any surprises. The adjoint weights, which we shall call R_i for convenience, are found by taking the commutator with the Cartan generators as defined in the previous chapter. This tedious but simple calculation leaves us with the following adjoint weights:

$$\alpha^A_{H_1} = R_1 = \begin{pmatrix}0\\0\end{pmatrix} \quad \alpha^A_{H_2} = R_2 = \begin{pmatrix}0\\0\end{pmatrix} \quad \alpha^A_{E_1} = R_3 = \begin{pmatrix}1\\0\end{pmatrix}$$
$$\alpha^A_{E_2} = R_4 = \begin{pmatrix}-1\\0\end{pmatrix} \quad \alpha^A_{E_3} = R_5 = \begin{pmatrix}\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix} \quad \alpha^A_{E_4} = R_6 = \begin{pmatrix}-\frac{1}{2}\\-\frac{\sqrt{3}}{2}\end{pmatrix}$$
$$\alpha^A_{E_5} = R_7 = \begin{pmatrix}-\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix} \quad \alpha^A_{E_6} = R_8 = \begin{pmatrix}\frac{1}{2}\\-\frac{\sqrt{3}}{2}\end{pmatrix}$$

This gives rise to the following diagram:

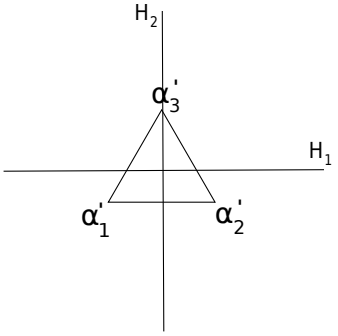


This hexagon with two centers apparently is connected to an octet.

Now all that remains is constructing the weight diagram for the antiquark state. In that case the representation is complex conjugate to the triplet of the quark and thus the weights have the opposite sign:

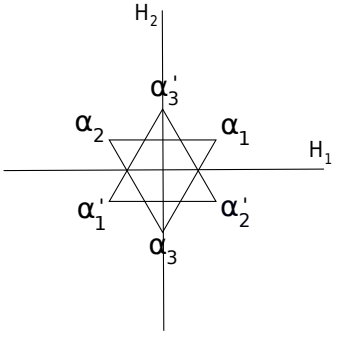
$$\alpha_1' = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2\sqrt{3}} \end{pmatrix} \quad \alpha_2' = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2\sqrt{3}} \end{pmatrix} \quad \alpha_3' = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

This gives rise to the following diagram



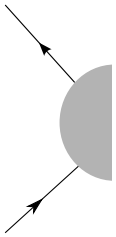
This inverted triangle apparently is connected with an antitriplet state.

We see now that the form of the weight diagram tells us something about the degeneracy of the colour state. By simply counting the number of dots in such a diagram we see what kind of state we have and by mirroring the diagram in the origin we get the complex conjugate representation.

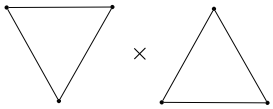


We can also see that the adjoint representation is its own complex conjugate representation, thus the gluons are their own antiparticles.

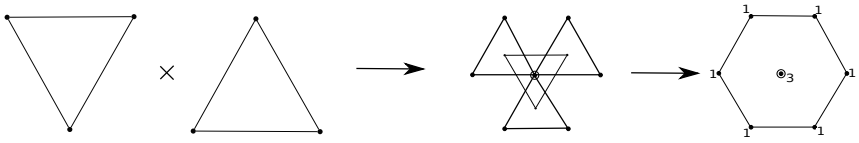
Now that we have constructed the weight spaces, let us return to the problem at hand: the possible colour states of quark-antiquark interactions. Let us first look at the initial states in which we have a colour triplet and antitriplet. Where we previously looked specifically at the t-channel, we now look at all possible intermediate states at tree level[3]. The generic Feynman diagram will look something like this:



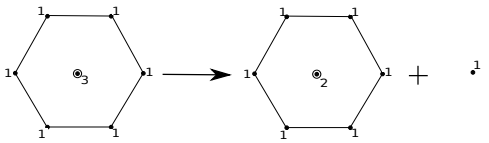
Here the grey semicircle depicts all possible intermediate states that this initial state can make. The product between these two initial colour states can be expressed in weight diagrams:



By taking all possible first states and adding all possible second states, we get the possible total colour structures of the incoming particles. This is done by taking a weight vector of the first diagram and use it as the origin of the second diagram. One then adds the three new diagrams together as below:



Here the numbers stand for the weight degeneracy. There is but one way to construct the outer states, which therefore have a weight degeneracy of 1. There are three ways, however, of constructing the central weight. Therefore it has a weight degeneracy of 3. The new diagram looks much like the adjoint weight diagram, except for the three zero weights instead of the expected two. But we can subtract the adjoint weight diagram leaving:



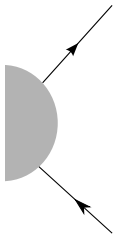
[3]Tree level means that we look at Feynman diagrams with no loops in them. Although the methode also holds for diagrams with loops in them (called higher order diagrams) they serve no immediate purpose in this thesis and so we have omitted them.

and we see that the possible colour states are a colour octet and a colour singlet.

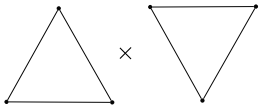
Now it is important to define some things. In the weight diagram of a quark we see that each point was given a name α_i. When one overlays the two diagrams as is shown in the decomposition, one can simply add two such vectors and arrive at all composite states. For instance, if one adds $|\alpha_1>$ to $|\alpha_3'>$ one arrives at the composite weight vector in the upper right corner of the final diagram. We will use the following notation:

$$|\alpha_i> + |\alpha_j'> = |i, j'>$$

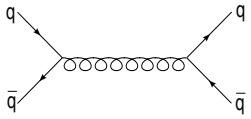
Now we look at the final state of the diagram, for which the diagram looks like this:



Again we have a triplet and antitriplet state:



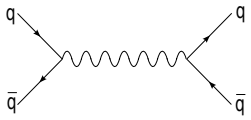
This gives us exactly the same colour states, which is important because of conservation of colour. If a certain colour state goes in, then that same colour state must come out. If not, the amplitude would break SU(3) invariance and that is exactly the opposite of what we wanted. Thus the initial octet can only couple to the final octet and the same goes for the singlet states. The quark-antiquark interaction therefore decomposes into a singlet part and octet part. This means that the initial state colour states can merge into a gluon. This is best shown in what we call the s-channel Feynman diagram:



If the initial and final states can be combined into a colour state connected to a physical particle, then we can construct a Feynman diagram in the s-channel, giving rise to that physical particle being exchanged. Intuitively this argument sounds convincing, but there are some flaws, which we will discuss later.

In this respect the singlet state is a bit alien to us, since we have not defined any particle that is an SU(3) singlet. The problem is that in a pure SU(3) theory such a particle does not exist. When we move back to the previous case of U(1)

we can see that there is a possibility. If we construct the U(1) weight space we have to remember that U(1) only has one generator, which in a proper base is hermitian and therefore automatically a Cartan generator. In the fundamental representation this gives naturally rise to a singlet and in the adjoint it is not hard to see that indeed we get a singlet state there too. It suffices to say that the U(1) adjoint is a singlet under SU(3), since there is no interaction in the lagrangian between the W^μ and A^μ. Therefore the particle of the U(1) adjoint, the photon, is a candidate for the SU(3) singlet state, creating the following s-channel Feynman diagram:



with the wigly line indicating the exchanged photon. However this is no pure QCD procces and thus not interesting to us in this study. The other possibility for getting a singlet term is by decomposing the t-channel amplitude. When one decomposes the t-channel it is indeed found that there is a singlet term there. So, apparently all colour states with which we cannot identify a particle are required to form the other possible channels for the theory.

It is time to go back to our calculations from chapter 6. We had a form for the colour potential, but we had to look at all possible states to find out what it would be in the end. Now we do not have to do that anymore, for we know there are but two possible colour flows: an octet and a singlet. First we look at the octet state, which is eightfold degenerate. It will be sufficient to choose one of these octet states, for all the others will give us the same result. For convenience we will pick one of the states on the outer rim, $|1, 2' >$. The outgoing state must also be an octet and by the same reasoning we will select $|1, 2' >$ to be that state, giving us the following colour factor:

$$\begin{aligned} f &= \frac{1}{4}\left[\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \lambda^a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] \left[\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \lambda^a \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] \\ &= \frac{1}{4}(\lambda^a_{11}\lambda^a_{22}) \end{aligned}$$

Since λ^3 and λ^8 are the only diagonal matrices this becomes:

$$f = \frac{1}{4}(\lambda^3_{11}\lambda^3_{22} + \lambda^8_{11}\lambda^8_{22}) = -\frac{1}{6}$$

The same can be done for the singlet. In that case we see three possible weight vectors: $|1, 1' >$, $|2, 2' >$ and $|3, 3' >$. Each of the central weights consist of a linear combination of these three weight vectors. It turns out that the singlet state corresponds to $\frac{1}{\sqrt{3}}(|1, 1' > + |2, 2' > + |3, 3' >)$. In vector notation we thus get the following colour factor, keeping in mind that the outgoing particles must

also be in a singlet state:

$$\begin{aligned}
f &= \frac{1}{4}\frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}\Bigg(\left[\begin{pmatrix}1 & 0 & 0\end{pmatrix}\lambda^a\begin{pmatrix}1\\0\\0\end{pmatrix}\right]\left[\begin{pmatrix}1 & 0 & 0\end{pmatrix}\lambda^a\begin{pmatrix}1\\0\\0\end{pmatrix}\right] \\
&+ \left[\begin{pmatrix}0 & 1 & 0\end{pmatrix}\lambda^a\begin{pmatrix}1\\0\\0\end{pmatrix}\right]\left[\begin{pmatrix}1 & 0 & 0\end{pmatrix}\lambda^a\begin{pmatrix}0\\1\\0\end{pmatrix}\right] \\
&+ \left[\begin{pmatrix}0 & 0 & 1\end{pmatrix}\lambda^a\begin{pmatrix}1\\0\\0\end{pmatrix}\right]\left[\begin{pmatrix}1 & 0 & 0\end{pmatrix}\lambda^a\begin{pmatrix}0\\0\\1\end{pmatrix}\right] \\
&+ \left[\begin{pmatrix}1 & 0 & 0\end{pmatrix}\lambda^a\begin{pmatrix}0\\1\\0\end{pmatrix}\right]\left[\begin{pmatrix}0 & 1 & 0\end{pmatrix}\lambda^a\begin{pmatrix}1\\0\\0\end{pmatrix}\right] \\
&+ \left[\begin{pmatrix}0 & 1 & 0\end{pmatrix}\lambda^a\begin{pmatrix}0\\1\\0\end{pmatrix}\right]\left[\begin{pmatrix}0 & 1 & 0\end{pmatrix}\lambda^a\begin{pmatrix}0\\1\\0\end{pmatrix}\right] \\
&+ \left[\begin{pmatrix}0 & 0 & 1\end{pmatrix}\lambda^a\begin{pmatrix}0\\1\\0\end{pmatrix}\right]\left[\begin{pmatrix}0 & 1 & 0\end{pmatrix}\lambda^a\begin{pmatrix}0\\0\\1\end{pmatrix}\right] \\
&+ \left[\begin{pmatrix}1 & 0 & 0\end{pmatrix}\lambda^a\begin{pmatrix}0\\0\\1\end{pmatrix}\right]\left[\begin{pmatrix}0 & 0 & 1\end{pmatrix}\lambda^a\begin{pmatrix}1\\0\\0\end{pmatrix}\right] \\
&+ \left[\begin{pmatrix}0 & 1 & 0\end{pmatrix}\lambda^a\begin{pmatrix}0\\0\\1\end{pmatrix}\right]\left[\begin{pmatrix}0 & 0 & 1\end{pmatrix}\lambda^a\begin{pmatrix}0\\1\\0\end{pmatrix}\right] \\
&+ \left[\begin{pmatrix}0 & 0 & 1\end{pmatrix}\lambda^a\begin{pmatrix}0\\0\\1\end{pmatrix}\right]\left[\begin{pmatrix}0 & 0 & 1\end{pmatrix}\lambda^a\begin{pmatrix}0\\0\\1\end{pmatrix}\right]\Bigg) \\
&= \frac{1}{4}\frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}(\lambda^a_{ij}\lambda^a_{ji}) \\
&= \frac{1}{4}\frac{1}{3}Tr(\lambda^a\lambda^a)
\end{aligned}$$

with summation over a implied, going over all eight Gell-Mann matrices. Since

$$Tr(\lambda^a\lambda^b) = 2\delta^{ab}$$

we get

$$Tr(\lambda^a\lambda^a) = 16$$

The colour factor will be

$$f = \frac{1}{12}Tr(\lambda^a\lambda^a) = \frac{4}{3}$$

Putting this into the equation for the QCD potential we get

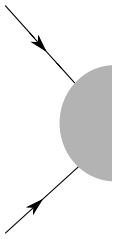
$$V_{q\bar{q}} = -\frac{4}{3}\frac{\alpha_s}{r}$$

for the colour singlet and

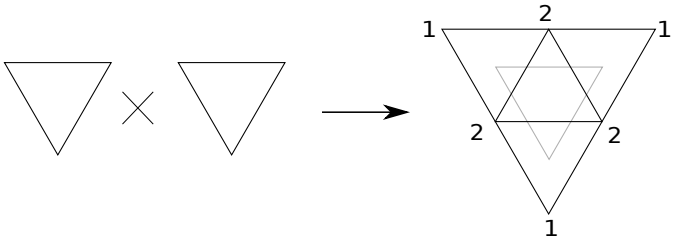
$$V_{q\bar{q}} = \frac{1}{6}\frac{\alpha_s}{r}$$

for the colour octet.

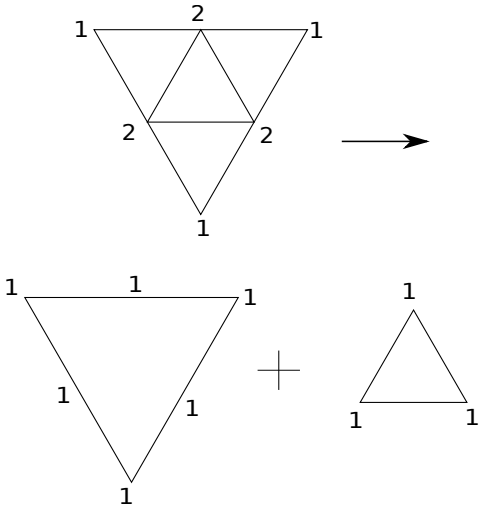
We continue the discussion of quark matter with a new process: quark-quark scattering. We look at the incoming states of the process of two quarks, which looks very much like in the quark-antiquark case, but now with both arrows going in:



This leads to a weight diagram with two triplet states. Again taking all possible first states and adding the second we get the following:

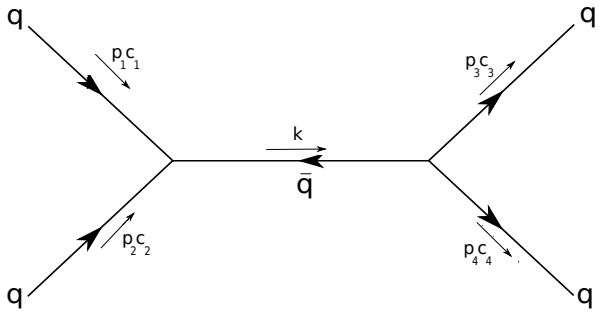


This time, however, we cannot take out an octet and it seems we are stuck. On closer inspection, however, one can recognize a large triangle with a small triangle inside, the small triangle being an antitriplet state. Taking it out, we get the following decomposition of states:

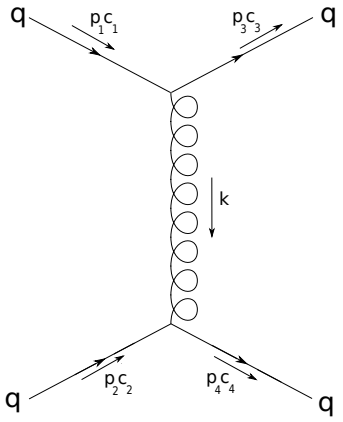


This gives rise to a sextet and an antitriplet state.

One could again write down the outgoing state, and calculate the possible weight decomposition, but this time it is fairly easy to see we get the same result: again a sextet and an antitriplet state. Can we extract any physics out of this, like we did in the case of quark-antiquark? There is an antitriplet state, which might imply the exchange of an antiquark in the s-channel. If this were so the following Feynman diagram needs to exist:



However, in the lagrangian we find neither a quark-quark-antiquark term (necessary for the first vertex), nor a triple antiquark term (necessary for the second vertex). Due to Lorentz-invariance this interaction is forbidden though, so we cannot add it to the lagrangian. Just as in QED[4], where fermions can only form an annihilation reaction with antifermions, in QCD a colour state can only have an annihilation reaction (which is in the diagrams equal to an s-channel) with its anticolour state. Thus there is no possible s-channel for the quark-quark process and the only possible diagram is the following t-channel diagram[5]:



[4]Quantum Electro Dynamics. It is the quantum field version of the electrodynamics theory of Maxwell

[5]There is also a u-channel diagram, which is equal to the t-channel diagram, but with the outgoing states interchanged.

We shall simply give the amplitude for this process:

$$\begin{aligned}\mathcal{M} &= \frac{g_s^2}{4k^2}[\bar{u}(3)\gamma^\mu u(1)][\bar{u}(4)\gamma_\mu u(2)](c_3^\dagger\lambda^a c_1)(c_4^\dagger\lambda^a c_2)\\ &= \frac{g_s^2}{k^2}[\bar{u}(3)\gamma^\mu u(1)][\bar{u}(4)\gamma_\mu u(2)] \times \frac{1}{4}(c_3^\dagger\lambda^a c_1)(c_4^\dagger\lambda^a c_2)\\ &= \frac{g_s^2}{k^2}\Lambda \times \frac{1}{4}\Gamma\end{aligned}$$

with $\frac{1}{4}\Gamma$ again the colour structure.
Now let us see if we can define a potential for this process, just like we did in the quark-antiquark case. The previous potential is no longer useful, since there are now two particles with the same charge and thus it looks more like a modified potential for the equivalent QED process of electron-electron scattering, also known as Møller scattering:

$$V_{qq}(r) = f\frac{\alpha_s}{r}$$

So again one has to calculate the colour factor f. Let us first start with the sextet. We know that the sextet contains the upper right point of the large triangle, which consists of twice the same triplet weight vector. To simplify the calculations we take exactly this point for the calculation, since we know that all other points on the large triangle give us exactly the same result.
Because of this only the diagonal Gell-Mann matrices are needed, resulting in the following colour factor:

$$\begin{aligned}f &= \frac{1}{4}\left[\begin{pmatrix}1 & 0 & 0\end{pmatrix}\lambda^a\begin{pmatrix}1\\0\\0\end{pmatrix}\right]\left[\begin{pmatrix}1 & 0 & 0\end{pmatrix}\lambda^a\begin{pmatrix}1\\0\\0\end{pmatrix}\right]\\ &= \frac{1}{4}(\lambda^a_{11}\lambda^a_{11})\\ &= \frac{1}{4}(\lambda^3_{11}\lambda^3_{11} + \lambda^8_{11}\lambda^8_{11})\\ &= \frac{1}{4}((1)(1) + (\frac{1}{\sqrt{3}})(\frac{1}{\sqrt{3}})) = \frac{1}{3}\end{aligned}$$

The same can of course be done for the antitriplet state. However, each point of the antitriplet is degenerate in weight. For instance, the top of the antitriplet state corresponds to either $|1,2>$ or $|2,1>$. Thus any orthonormal pair of vectors $\frac{c_1|1,2>+c_2|2,1>}{\sqrt{c_1^2+c_2^2}}$ and $\frac{c_2|1,2>-c_1|2,1>}{\sqrt{c_1^2+c_2^2}}$ can be used. The proper choice for c_1 and c_2 is obtained by requiring that one of these vectors corresponds to a colour factor $f = \frac{1}{3}$ and thus belongs to the sextet. The other vector than belongs to the antitriplet. From this procedure it follows that $\frac{1}{\sqrt{2}}(|1,2> + |2,1>)$ belongs to the sextet and $\frac{1}{\sqrt{2}}(|1,2> - |2,1>)$ to the antitriplet. In the latter case the

colour factor then becomes:

$$\begin{aligned}
f &= \frac{1}{4}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\Bigg(\left[\begin{pmatrix}1 & 0 & 0\end{pmatrix}\lambda^a\begin{pmatrix}1\\0\\0\end{pmatrix}\right]\left[\begin{pmatrix}0 & 1 & 0\end{pmatrix}\lambda^a\begin{pmatrix}0\\1\\0\end{pmatrix}\right] \\
&\quad -\left[\begin{pmatrix}0 & 1 & 0\end{pmatrix}\lambda^a\begin{pmatrix}1\\0\\0\end{pmatrix}\right]\left[\begin{pmatrix}1 & 0 & 0\end{pmatrix}\lambda^a\begin{pmatrix}0\\1\\0\end{pmatrix}\right] \\
&\quad -\left[\begin{pmatrix}1 & 0 & 0\end{pmatrix}\lambda^a\begin{pmatrix}0\\1\\0\end{pmatrix}\right]\left[\begin{pmatrix}0 & 1 & 0\end{pmatrix}\lambda^a\begin{pmatrix}1\\0\\0\end{pmatrix}\right] \\
&\quad +\left[\begin{pmatrix}0 & 1 & 0\end{pmatrix}\lambda^a\begin{pmatrix}0\\1\\0\end{pmatrix}\right]\left[\begin{pmatrix}1 & 0 & 0\end{pmatrix}\lambda^a\begin{pmatrix}1\\0\\0\end{pmatrix}\right]\Bigg) \\
&= \frac{1}{8}(\lambda^a_{11}\lambda^a_{22} - \lambda^a_{21}\lambda^a_{12} - \lambda^a_{12}\lambda^a_{21} + \lambda^a_{22}\lambda^a_{11}) \\
&= \frac{1}{4}(\lambda^a_{11}\lambda^a_{22} - \lambda^a_{12}\lambda^a_{21}) \\
&= \frac{1}{4}(\lambda^3_{11}\lambda^3_{22} + \lambda^8_{11}\lambda^8_{22} - \lambda^1_{12}\lambda^1_{21} - \lambda^2_{12}\lambda^2_{21}) \\
&= \frac{1}{4}(-1 + \frac{1}{3} - 1 - 1) = -\frac{2}{3}
\end{aligned}$$

If there is more than one inner state, one starts at a non-degenerate state and calculates the colour factor. Then one moves inwards and looks at a twofold degenerate state. Finding an appropriate set of othonormal states composed out of linear combinations of the two vectors, one identifies one state with the previously calculated colour factor. This fixes the other state, which can then be used to calculate the other colour factor. One can then move to a threefold degenerate state and proceed in a similar way there. This continues till all states have been identified.

The colour structures give rise to the following potentials for the antitriplet:

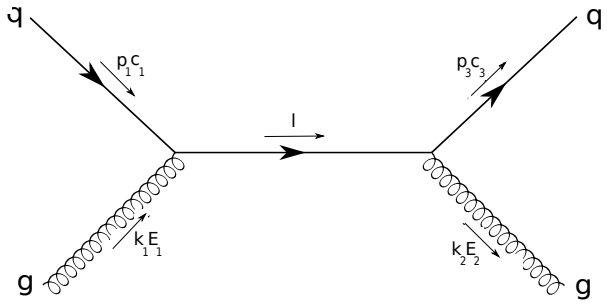
$$V_{qq}(r) = -\frac{2}{3}\frac{\alpha_s}{r}$$

and for the sextet:

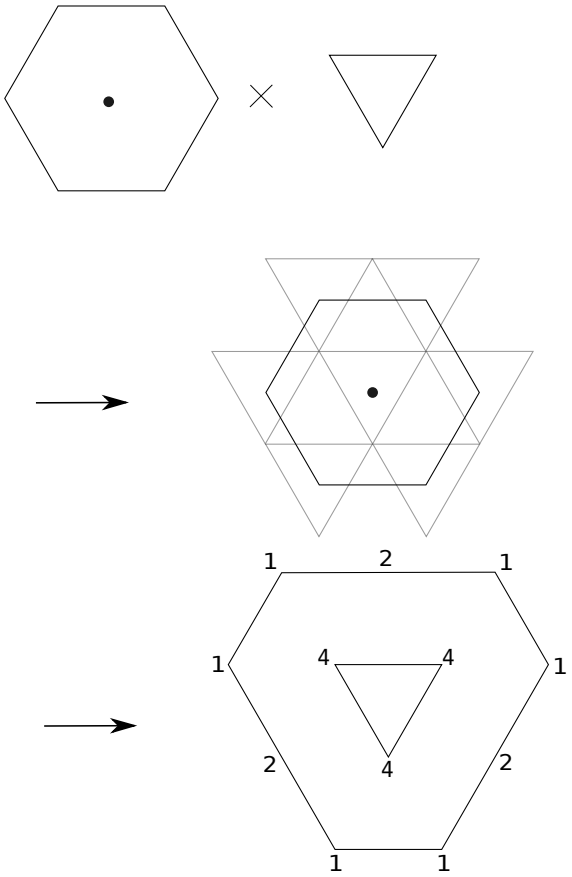
$$V_{qq}(r) = \frac{1}{3}\frac{\alpha_s}{r}$$

This time the sextet interaction is the repulsive one, while the antitriplet interaction is attractive. It might be shocking at first to see that two quarks can attract each other. It is however not a singlet state and as such can have interactions with other coloured matter. Together with a third quark the antitriplet state in fact forms a singlet, leading to hadronic matter.

We have looked at pure incoming (and outgoing) fermion states, so now it is time to look at bosonic states. We can distinguish two cases: boson-fermion inintial states and boson initial states.
We begin by looking at the boson-fermion case, i.e. QCD processes with gluon-quark initial states. The s-channel Feynman diagram will look something like this:

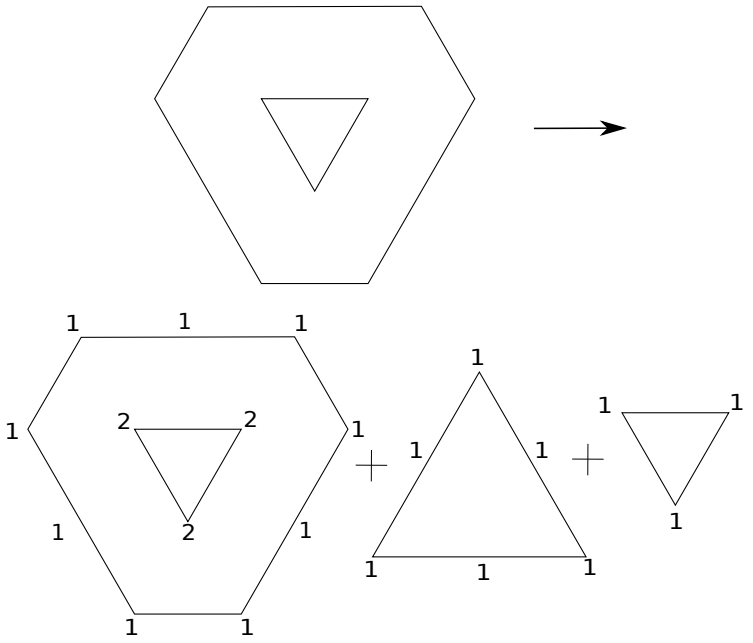


On the basis of this diagram we expect a triplet state in the decomposition of the weight diagrams. Now we look at the weight diagram. An incoming quark gives rise to a triplet state and the gluon to an octet state, leading to the following diagram:



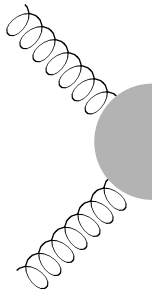
The decomposition in irreducible states, however, is not so easy. Where previously one could simply take out a state we recognized (the octet in the first example and the antitriplet in the second), this cannot be done so easilly in this case. One can see it is indeed possible to take out a triplet state, as we expected, but we are still left with 21 states which cannot be classified yet. The

states are reducible however and there is a way to show this.
To do this we go back to the construction of the adjoint weight diagram. In the weight space we took the Cartan adjoint generators and showed that their weights where always 0. We combined the remaining generators into convenient ladder generators which would move the Cartan weights either up or down, to create the remaining weights. That is where we stopped the procedure, but there was no particular reason for that yet. Seemingly we could have continued moving into infinity. In reality one has to stop at either the heighest or lowest possible weights, just as in all cases of using ladder operators. It is possible to move from the heighest weight to the lowest one by use of ladder operators while remaining on the outer layer of the diagram. Moving in such a way, each weight can be reached uniquely and thus the outer layer forms an irreducible weight representation. In the corresponding weight diagram all outer layer states have a degeneracy of 1. Up to now we have always drawn the outer lines, forming "shells", but now we know what their purpose is. It can be proven that each time one moves a shell inwards the number of states for each weight increases by one. The proof is a bit beyond the scope of this thesis, although we can make it plausible. If one moves inwards it is no longer possible to move in a unique way. Starting from the heigest weight there are two ways of reaching the second shell using two ladder generators E_1, which moves to a weight down right, and E_2, which moves left. One can first apply E_1 and then E_2, first going down right and then left. But if one does it the other way around, first E_2 and then E_1, we move first left and then down right, ariving at the same spot. The same argument can be repeated each time we move down a shell, thus making our statement plausible.
Now we see that the remaining state must be degenerate, since there are points with twofold degeneracy on the outer rim. Connecting these points one gets a triangle, with a weight in the middle of the lines. This is an antisextet (see the previous example) and taking it out leaves us with the following structure:

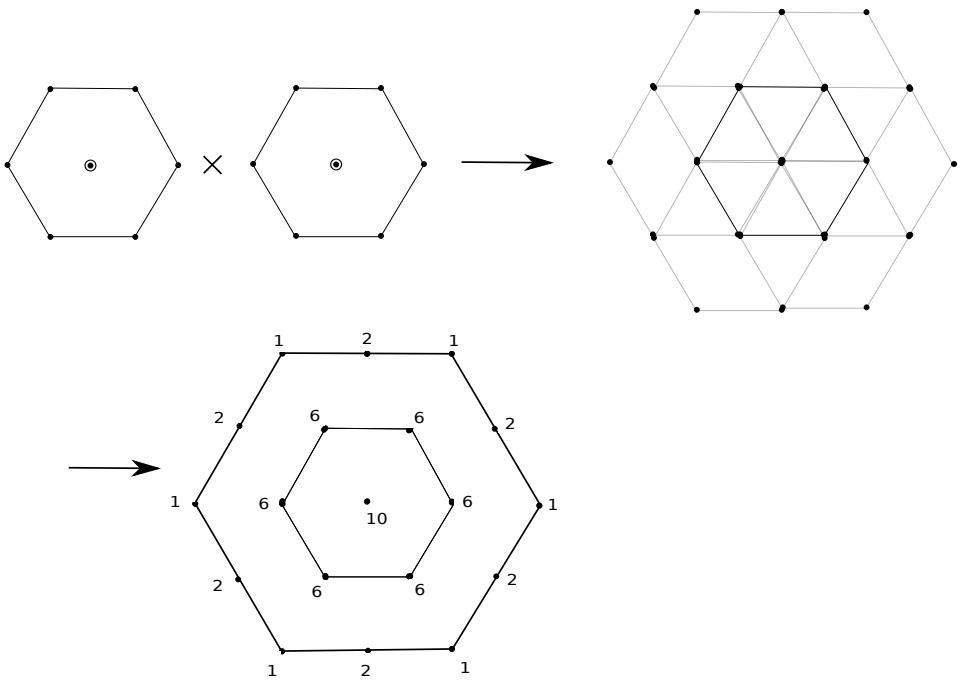


The first diagram corresponds to a quindecuplet consisting of 15 states. The figure is irreducible and thus we found the decomposition.
We cannot, however, easily write down the potential between the incoming particles and thus we shall skip it. It is possible to write down the colour factors, but this is in fact quite complicated and thus we shall leave it to others to do these calculations if they wish to know them. The main problem is that we have not yet defined a simple way of calculating the gluons in these processes.

Finally we will look at the last process: gluon-gluon scattering. From the lagrangian we know that such interactions must exist, because of the quadratic commutator (6). Let us use our new found method of differentiating colour states and decompose the generic incoming Feynman diagram

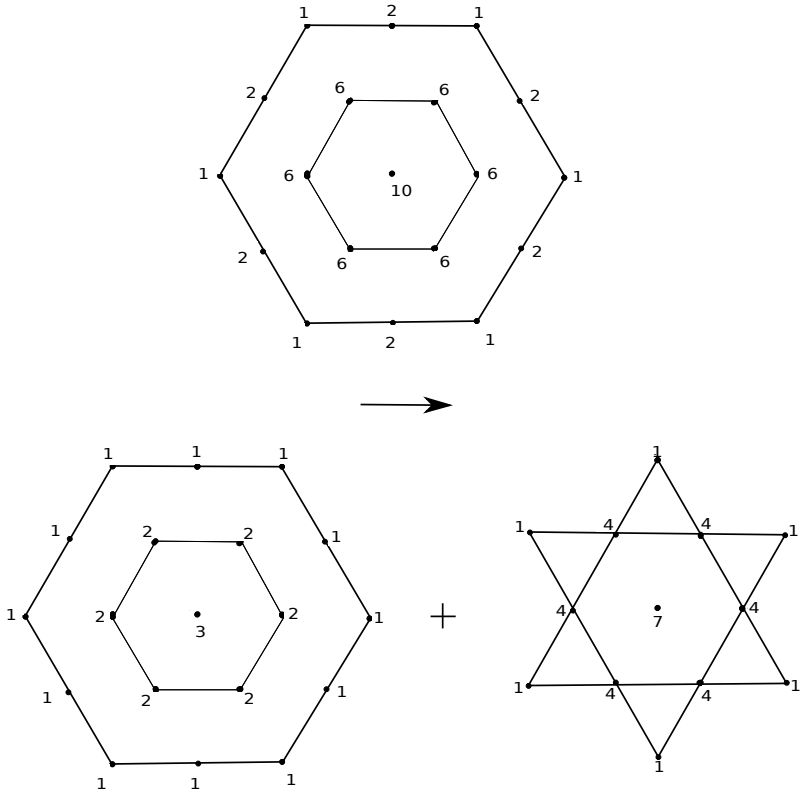


We know that a gluon has an adjoint weight diagram, so gluon-gluon scattering corresponds to the following weight diagram:

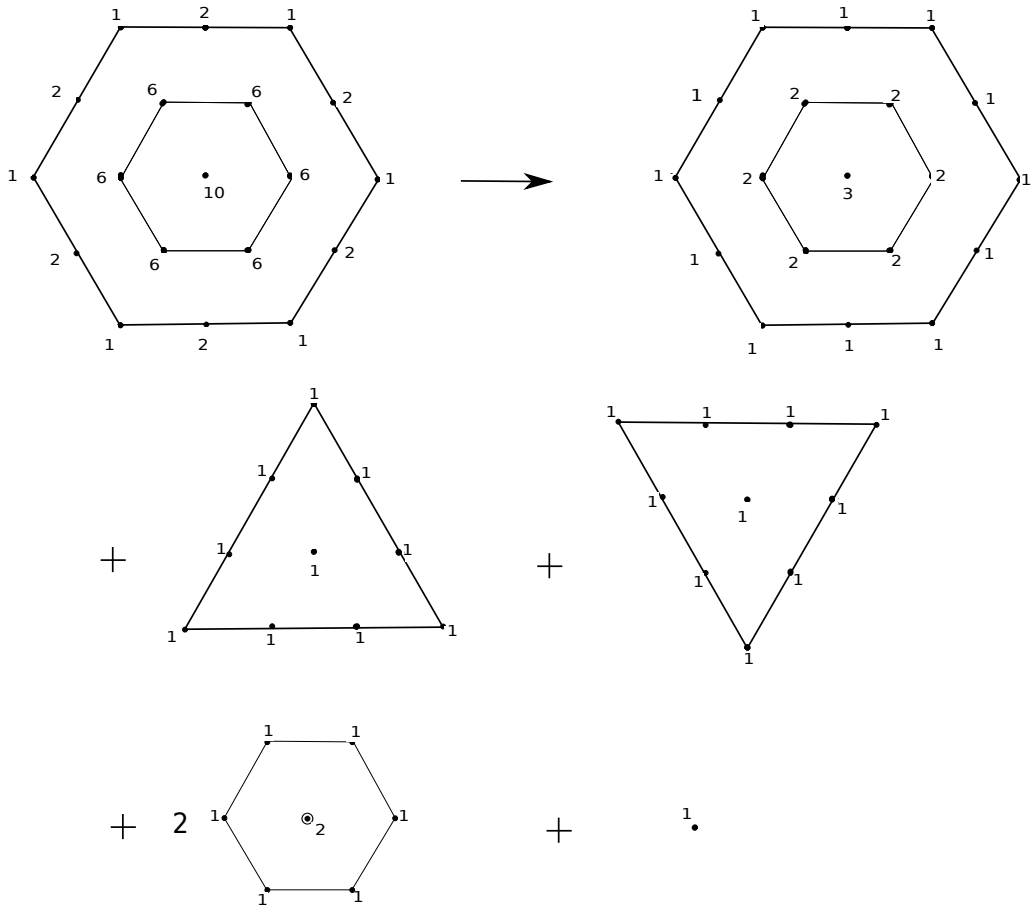


Just as in the gluon-quark case we can decompose this diagram by using the ladder operators.

Applying this theorem to the diagram above, we first see a hexagon with 12 weights that must be irreducible, so all have degeneracy 1. Going inward, we are left with a hexagon with 6 weights, all with degeneracy 2. Finally we must have a central dot with degeneracy 3. Pulling this new weight diagram out, we find:



This gives rise to a heptaduodecuplet (27 states) and something that looks like a star, but what is in fact two triangles overlaying one another. Using the same method as above we can pull out these two triangles, both decuplets (10 states), leaving us with 2 octets and a singlet state:



Again we can look at the final state of the process and do the same decomposition there. But let us first do some predictions for our model. Due to SU(3) invariance only indentical colour states can interact, thus for gluon-gluon scattering ($gg \to gg$) we have a colour structure of 27 states giving the same potential; 2 structures of 10 states, which cannot communicate with one another (since one is a normal state, while the other is a conjugate state); 2 structures of 8 states, again identifiable as gluons ;and finally 1 colour singlet state, defining a bound gluon state that could be a glueball[6].

Just as above in the gluon-quark case we cannot easily find a potential for the process. However, while in the previous case we said that those factors were a bit complicated, here they are downright hard to calculate, and one is not served by lengthy math only to arrive at an ugly factor. The easiest way of actually preforming the calculation is by going to an 8-dimensional base and find an

[6]Glueballs, composite particles consisting of gluons only and thus being in effect the only tpe of pure gluons that can be "free", are said to be detectable at the LHC.

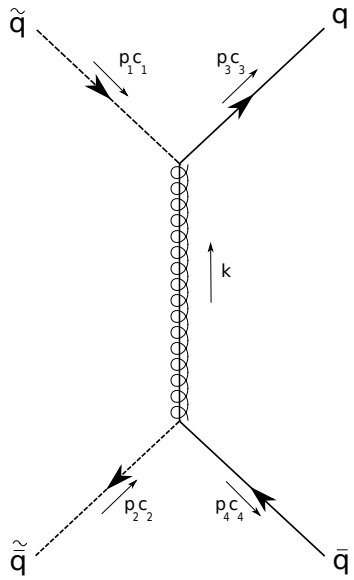
8-dimensional representation for the generators. The base and representation, however, are quite obscure and not readily found in the literature. If one wishes to calculate these structures, the representation and base must therefore be found by hand. This might be a project for the future, for it is far beyond the scope of this thesis.

9 SUSY and beyond

To draw a conclusion we look back to what we did in the beginning of this thesis. As we said, the true purpose of this study is to look at SUSY interactions and predict the colour structures, which we can then use to calculate the potentials for instance. The SUSY partners obey the same rules as the usual QCD particles, apart from aspects related to spin, thus a triplet state describes the squarks and an octet state the gluinos. This means that the weight diagrams are identical to what we have seen before, which gives a rather quick way to calculate the possible potentials by following the exact same method as we have used above.

Now we can define the relevant new particles for SUSY interactions. The quarks, fermions with spin $\frac{1}{2}$, are colour triplets. Their SUSY partners the squarks, bosons with spin 0, must also be colour triplets. By the same reasoning we know that the antisquarks, also spin 0 bosons, must have an antitriplet colour state. The gluons, which are spin 1 bosons, form a colour octet. Therefore their SUSY partners, the spin $\frac{1}{2}$ fermions called gluinos, must also be described by an colour octet state.

Just as before we can mix and match these particles and it is not hard to find the colour part of the associated scattering cross-sections. Let's take the squark-antisquark interaction, of which a possible Feynman diagram looks like this:



Here the dashed lines indicate the squark ($\tilde{q}$) and the antisquark ($\tilde{\bar{q}}$), while the spring with a line through it is the gluino. We see a triplet and antitriplet going in and thus a singlet or an octet can be exchanged. One thus gets the exact same colour factors as in the quark-antiquark case. This is great news, for it is exactly what we were looking for. Now the SUSY processes will be much easier to calculate. Most of the work has already been done. The beauty is that this can be done for all initial triplet and antitriplet pairs. It is now that one can fully appreciate this method of calculation, for all new processes can be reduced to ones we already know.

The method as described in this thesis might be a little cumbersome to learn and there are definitely some hiatuses that we did not explain, but if one wants to dive deeper into the underlying material of the Lie algebras, the bibliography provides the necessary details.

In conclusion there are still some things that can be calculated in order to fine-tune this method. The calculation of the quark-gluon sector, for instance, can be worked out properly by defining to which diagram some of the degenerate states belong. Also the precise way of dealing with the connection between the gluon's colour matrix and the quark's colour vector is not trivial and needs to be worked out properly.
As for gluon-gluon scattering, research has to be done in determining the 8-dimensional representation of the generators and in deriving of the adjoint base from there. If one has these tools, it might be possible to calculate the colour structure with the help of a computer program.

The weight diagrams we've constructed in this thesis coincide with the decomposition of the colour structures as found in the literature. Also, the colour factors we have calculated do indeed agree with the results obtained with known and tested methods. These results give hope for the future development of this method.

Special thanks go out to Wim Beenakker, whose assistance made this thesis possible, not only by sharing his knowledge on the subject but also thanks to his patience in waiting for its development. Also we thank Gert Heckman, who showed us how to properly use Lie algebras.

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