Electrodynamics from a real-linear spectral triple

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1 Introduction

This thesis is the result of my bachelor internship at the department of theoretical high energy physics (THEF) in Nijmegen. I attended this internship in the spring of 2012, during which I was supervised and guided by Thijs van den Broek. I want to express my sincere gratitude for all the patience Thijs showed in guiding me and all the detailed comments and suggestions he provided along the way. Furthermore I want to thank Walter van Suijlekom for providing useful suggestions and remarks on my research.

The main focus of my research was to find a description of electrodynamics using the framework of non-commutative geometry (from now: NCG). The course of this internship can roughly be divided into two parts: first I had to familiarize myself with the concepts of NCG, secondly these concepts needed to be applied to the theory of electrodynamics. Since I do not presume the reader to have any prior knowledge about NCG this thesis will share that structure. We will start out by motivating and introducing the concepts of NCG and continue with their application.

1.1 What is non-commutative geometry?

Before we can attempt to work with NCG we should first develop a basic idea of what this branch of physics envelopes. Here the word physics might be poorly chosen since it is a purely mathematical theory at heart. However, we are interested in the physical implications thus we regard it as physics.

In general, geometry can be defined as the theory of spaces. Spaces form the basis of many fundamental theories in physics, take for example the space-time of relativistic theory or the background manifold of quantum field theory. It can be shown [7, 1] that there exists a one-to one correspondence between certain spaces and a certain class of commutative algebras. The concept of an algebra will be introduced later but for now it suffices to know that an algebra consists of elements and that two of these elements can be multiplied with each other to form yet another element from the algebra. For a *commutative* algebra we have that any two of these elements commute. In other words, for two elements α and β from the algebra we have:

$$[\alpha, \beta] \equiv \alpha\beta - \beta\alpha = 0.$$

But what about *non-commutative* algebras? I.e. algebras for which $[\alpha, \beta]$ is not necessarily equal to zero. Could they be related to spaces as well? The fundamental idea of NCG is that indeed they are. NCG treats these non-commutative objects as if they are related to "non-commutative spaces". Although these spaces do not resemble the "classical" spaces, many of the geometrical structures (metric structures, differential calculus, etc.) can be well defined within them. The main objective of (physical) NCG is to find the non-commutative objects that correspond to the spaces that describe the physical world. In this way NCG promises to provide a more general description of physics and possibly discovering new physical laws along the way.

1.2 What are the benefits of NCG?

From the previous section we may conclude that some of the objects in modern physics can be reformulated in the form of algebras. But what are the benefits of doing this? One might argue that simply reformulating existing theories won't give any new insights into fundamental physics. There are two arguments against this.

First of all the framework of NCG offers a more general starting point for setting up physical theories. For example: if our theory requires the use of differential geometry (quantum Field theory, general relativity, etc.) we were traditionally restricted to using a differential manifold. This limitation imposes restrictions on our theory since a manifold is only one particular type of space. For example: if we were to use the framework of NCG to set up quantum field theory we would notice that a differential manifold is now only one of many possible differential geometries. With NCG the set of possible spaces we might chose has been greatly extended and it might just turn out that the physical world is (better) described by a non-commutative underlying space.

Secondly, this more general approach might provide us with new insights into already existing ideas. Take for instance the framework of analytical mechanics, which at first might just seem like an abstraction of Newton's theories but turned out to give a toolbox with much broader applications.

1.3 A shopping list for electrodynamics

As mentioned earlier, the main goal of my research is to derive a theory describing electrodynamics within the framework of NCG. Before we set out to do this, we ask the question: what are the basic ingredients necessary for such a theory? To answer this question we take a look at field theories describing electrodynamics, for a detailed analysis see [2]. First of all we need a set of particles. In the case of electrodynamics we know these particles are given by electrons, positrons and photons. Every one of these particles has a set of both extrinsic and intrinsic properties. For example: the position and momentum of a particle are extrinsic properties, spin and chirality are intrinsic ones. To account for these degrees of freedom we need a space which contains all these properties. With the equivalence we noted above in NCG this space will take the from of an algebra. Note that we use the concept of a space in a more general sense than the physical space we can see around us. Where in classical physics a particle's properties are simply assigned to it we now define them by the particle's "position" in this generalized space.

As mentioned earlier, one of these properties is the chirality of a particle. Beware that chirality is not the same as helicity, although in some textbooks these terms are used as interchangeable. Helicity is related to the alignment of the spin and momentum vector, making it an extrinsic quantity. Chirality, on the other hand, is an intrinsic quantity, which is determined by whether the particles transform in a left or right handed representation of the Poincaré group. This difference in transformational properties allows us to distinguish between left and right-handed particles respectively. When physicists say nature is left handed this means we only observe interactions mediated by the weak force between left handed particles in nature. So to account for the asymmetry we should better have a way to distinguish between left and right-handedness in our theory.

Another fundamental property our theory also needs to exhibit is C-symmetry. A theory is said to be C-symmetric if its laws are invariant under a transformation that inverts the signs of the charges of all the particles in the theory. In quantum field theory this process is called charge conjugation and although some theories are not C-symmetric we know for a fact that electrodynamics is. So in our theory we need an operation that is the non-commutative equivalent of charge conjugation.

Since up until now our theory has been static and we know electrodynamics is not, we also need to take into account the possible interactions of the particles. So finally we need some kind of operator (like the Laplacian of classical mechanics) that determines the dynamical properties of the system and does that in a way that conforms with observations. In modern physics this operator gives rise to a so called gauge field, which characterizes the properties of the interactions. Furthermore, we know from quantum field theory that this gauge field can be directly associated with the particles (or quanta) mediating the interactions. In the case of electrodynamics these particles are the photons we mentioned earlier and we therefore require the interactions in our theory to be described by a photonic gauge field.

To summarize the basic ingredients of our theory we come to the following list:

- A set of particles corresponding to the known particles of electrodynamics
- An algebra from which the properties of all the particles can be derived
- Distinguishability between left and right-handed particles
- C-symmetry and a way to perform charge conjugation
- An operator that gives rise to an photonic gauge field

In the next chapter we will introduce the fundamental concepts of NCG, some of which we can directly identify with the items in this list.

2 Definitions and theorems

Now that we are familiar with the basic ideas of NCG the reader should familiarize him/herself with some of its mathematical concepts. Since these concepts are fundamental to NCG they will be introduced in this section, readers who are already familiar with these concepts can skip this section or regard it as reference.

2.1 General mathematical concepts

I will begin by introducing the general mathematical concepts which are used extensively within NCG. Note that some of the definitions differ slightly from the way they are defined in mathematical literature or seem to be incomplete. The reason for this is that in this case the usage of these concepts is restricted to our purposes and in some cases we do not need to concern ourselves with the most general and complete mathematical definition.

Let V and W be vector spaces over a field $\mathbb{F}$ ($\mathbb{C}$ or $\mathbb{R}$ for example), we can now define the following operations on V and W.

Definition 1. By $V \oplus W$ we denote the **direct sum** of V and W, which is defined as follows:

$$V \oplus W = \{(v, w) \mid v \in V, w \in W\}. \tag{1}$$

Furthermore we define addition and scalar multiplication on $V \oplus W$ by:

- $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$ for $(v_1, w_1), (v_2, w_2) \in V \oplus W$
- $\lambda(v, w) = (\lambda v, \lambda w)$ for $\lambda \in \mathbb{F}$.

With these operations $V \oplus W$ is again a vector space. A useful identity concerning the direct product is given by: $dim(V \oplus W) = dim(V) + dim(W)$.

Definition 2. With $V \otimes W$ we denote the **tensor product** of V and W over a field $\mathbb{F}$. The tensor product of V and W is, again, a vector space, defined by:

$$V \otimes W = \{v \otimes w \mid v \in V, w \in W\}. \tag{2}$$

For which the following conditions apply:

- Bi-linearity: let $\lambda \in \mathbb{F}$ then $(\lambda v) \otimes w = v \otimes (\lambda w) = \lambda(v \otimes w)$.
- $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$
- $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$

For the tensor product we have: $dim(V \otimes W) = dim(V)dim(W)$.

We will need these two operations later when constructing the fundamental objects of NCG. One of these objects is the Hilbert-space, which is defined in the following way.

Definition 3. A **Hilbert space** is a vector space, $\mathcal{H}$, endowed with an inner product $\langle f; g\rangle$ with $f, g \in \mathcal{H}$. Such that the norm defined by:

$$|f| = \sqrt{\langle f; f\rangle}$$

turns $\mathcal{H}$ into a complete metric space. I.e. it is a metric space in which every Cauchy sequence is convergent.

In NCG the Hilbert space can be considered as the set containing the fermionic particle states of the theory. In our case this means the electron and positron particle states should be contained within the Hilbert space.

Definition 4. Let $\mathbb{F}$ be a field, and let $\mathcal{A}$ be a vector space over $\mathbb{F}$ equipped with an additional binary operation from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$, we use $\cdot$ to denote this operation . Then $\mathcal{A}$ is an **algebra** over $\mathbb{F}$ if the following identities hold for all $\alpha, \beta, \gamma \in \mathcal{A}$, and $a, b \in \mathbb{F}$:

- Left distributivity: $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$
- Right distributivity: $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$
- Compatibility with scalars: $(a\alpha) \cdot (b\beta) = (ab)(\alpha \cdot \beta)$.

As mentioned earlier, certain types of algebras can be identified with a space. In NCG the algebra will take over the role of the space as it is used in classical theories. Within the algebra all the extrinsic an intrinsic quantities describing the fermionic particles are encoded. The algebra is also where NCG gets its name, since the elements of $\mathcal{A}$ do not necessarily commute.

But rather than using the general form of an algebra as defined above, it uses a special kind of algebra called a unital *-algebra, which is defined in the following two definitions.

Definition 5. Let $\mathcal{A}$ be an algebra, then $\mathcal{A}$ is a **$*$-algebra** if it is endowed with a map $* : \mathcal{A} \to \mathcal{A}$ that satisfies:

- $(\alpha + \beta)^* = \alpha^* + \beta^*$
- $(a\alpha)^* = \bar{a}\alpha^*$ where the bar denotes complex conjugation

- $(\alpha \cdot \beta)^* = \beta^* \cdot \alpha^*$

For all $\alpha, \beta \in \mathcal{A}$, and $a \in \mathbb{F}$.

In NCG the elements of the algebra are taken to be operators on a vector space which can be represented as matrices for finite dimensional representations, in this case $*$ will generally be equal to taking the adjoint of such a matrix.

Definition 6. An algebra $\mathcal{A}$ is said to be **unital** if it contains a multiplicative identity element, 1,with property:

$$1\alpha = \alpha 1 = \alpha, \ \forall \alpha \in \mathcal{A}.$$

Definition 7. A **left** $\mathcal{A}-$**module** $\mathcal{E}$ over an algebra $\mathcal{A}$ consists of $(\mathcal{E}, +)$ an abelian group and an operation $\mathcal{A} \times \mathcal{E} \to \mathcal{E}$ such that, for all $\alpha, \beta \in \mathcal{A}$ and $x, y \in \mathcal{E}$, we have:

- $\alpha(x + y) = \alpha x + \alpha y$
- $(\alpha + \beta)x = \alpha x + \beta x$
- $(\alpha\beta)x = \alpha(\beta x)$
- For $\lambda \in \mathbb{F}$ we have: $\alpha(\lambda x) = \lambda(\alpha x)$

Similarly to the above we can also define a **right** $\mathcal{A}-$**module** $\mathcal{E}$ for which the algebra works on the right, i.e. we now have an operation: $\mathcal{E} \times \mathcal{A} \to \mathcal{E}$ for which the above axioms hold but with α and β written on the right side of x and y.

Definition 8. Let $\mathcal{A}$ and $\mathcal{B}$ be two algebra's and $(\mathcal{E}, +)$ an abelian group then we have an $\mathcal{A} - \mathcal{B}-$**bimodule** if the following holds:

- $\mathcal{E}$ is a left $\mathcal{A}-$module and a right $\mathcal{B}-$module.
- For $\alpha \in \mathcal{A}, \ \beta \in \mathcal{B}$ and $x \in \mathcal{E}$ we have: $\alpha(x\beta) = (\alpha x)\beta$

For $\mathcal{A} = \mathcal{B}$ we denote the $\mathcal{A} - \mathcal{A}-$bimodule as $\mathcal{A}-$bimodule.

2.2 The fundamental concepts of NCG

Now that we have introduced the general mathematical concepts needed for our purposes within NCG we are ready to define some objects that are specific to NCG.

Definition 9. A **spectral triple** $(\mathcal{A}, \mathcal{H}, D)$ is given by an involutive unital algebra $\mathcal{A}$ represented as operators on a Hilbert space $\mathcal{H}$ and a self-adjoint $(D^* = D)$ operator D with compact resolvent such that all commutators $[D, \alpha]$ are bounded for $\alpha \in \mathcal{A}$.

The spectral triple is the most fundamental object in NCG, from its properties all physical characteristics of the theory follow. The algebra is where non-commutativity comes in, since its elements not necessarily commute. Thus choosing the right spectral triple is essential in finding a viable theory for the physical world.

Definition 10. A spectral triple is called **even** if there exists an operator, γ, called a $\mathbb{Z}_2$ grading acting on the Hilbert space, for which the following properties hold:

- $\gamma^2 = 1$
- For $\alpha \in \mathcal{A}$ we have: $[\alpha, \gamma] = 0$
- For the anti-commutator of γ and D we have: $\{\gamma, D\} = 0$.

The purpose of the $\mathbb{Z}_2$ grading is to differentiate between particles of different chirality. It gives us a way to split the Hilbert space into two subspaces, one associated with left handed particles, the other with right handed particles. A way to achieve this is to define the projection operators $\frac{1}{2}(1 \pm \gamma)$ which have the property that the plus sign projects to the one subspace and the minus sign to the other.

Definition 11. A spectral triple is said to have **real structure** if there exists an anti-linear isometry $J : \mathcal{H} \to \mathcal{H}$, with the following properties:

- $J^2 = \epsilon$
- $JD = \epsilon' DJ$
- For an even triple: $J\gamma = \epsilon'' \gamma J$

Where $\epsilon, \epsilon', \epsilon'' \in \{\pm 1\}$ are determined by the **KO-dimension**, $n \in \mathbb{Z}/8$, given by the following values.

n	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1		-1		1		-1	

The action of the operator J acts as charge conjugation on the particles in our theory thus transforming a particle into its charge conjugated partner. We call a spectral triple endowed with real structure a *real spectral triple.*

Definition 12. When a spectral triple has real structure we can define the **right hand action**, β^0, of an element $\beta \in \mathcal{A}$ as:

$$\beta^0 = J\beta^* J^{-1}.$$

Which must satisfy the following commutation relations for all $\alpha, \beta \in \mathcal{A}$:

- $[\alpha, \beta^0] = 0$
- $[[D, \alpha], \beta^0] = 0$

The conditions on γ,D and J introduced in the definitions above will allow us to derive expressions for these three operators from the properties of a given spectral triple. Since most physically relevant theories require the spectral triple to be even and have real structure this will be the first step in coming to physical results. So for our purposes we will write for a real even spectral triple:

$$(\mathcal{A}, \mathcal{H}, D, \gamma, J).$$

Once γ, D and J are found we'd like to make an early examination of our results in order to determine the viability of the spectral triple. In modern physics the way particles interact is determined by the gauge group of the corresponding theory. Simply put: what this means is that the interactions between particles can be derived from the transformations that leave the action of that theory invariant. These transformations are called the gauge transformations and together the form the gauge group of a theory. In this way the properties of the interactions in a theory and its gauge group are directly linked. In our case the properties of the electromagnetic interaction are determined by $U(1)$ symmetry. So if we wish to describe electrodynamics within the framework of NCG we should introduce the notion of the gauge group of a given spectral triple and require it to be equal to $U(1)$. We therefore define:

Definition 13. The **gauge group** $\mathcal{G}(\mathcal{A})$ of a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$ is given by:

$$\mathcal{G}(\mathcal{A}) \equiv \left\{ g = uJuJ^{-1} \mid u \in U(\mathcal{A}) \right\}.$$

Where $U(\mathcal{A})$ denotes the unitary elements of $\mathcal{A}$, defined as:

$$U(\mathcal{A}) = \{u \in \mathcal{A} \mid uu^* = u^* u = \mathbb{I}\}.$$

Where $\mathbb{I}$ is the identity operator.

At this point the reason for this definition might seem unclear, but we will see in section 5 that it will provide us with the right gauge transformations. For now we just use this definition to carry out preliminary checks on our spectral triple.

Definition 14. Let $T_1 = (\mathcal{A}_1, \mathcal{H}_1, D_1, \gamma_1)$ and $T_2 = (\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma_2)$ be two even spectral triples, both endowed with a real structures J_1 and J_2 respectively. We now denote the **tensor product of T_1 and T_2** by $T = T_1 \otimes T_2 = (\mathcal{A}, \mathcal{H}, D, \gamma)$, which is usually defined in the following way:

$$T_1 \otimes T_2 = (\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{H}_1 \otimes \mathcal{H}_2, D_1 \otimes \mathbb{I} + \gamma_1 \otimes D_2, \gamma_1 \otimes \gamma_2)\,. \tag{3}$$

The form of the combined real structure depends on the KO-dimension of the individual triples, in this thesis we will take it to be $J_1 \otimes J_2$ since in our case we are limited to situations where the KO-dimensions are such that this is indeed the right expression. The combined Dirac operator takes this form because it must satisfy the relation $\{D, \gamma\} = 0$, if we would have simply taken $D = D_1 \otimes D_2$ it is easy to see that this relation is no longer satisfied. The expression for D in equation 3 does satisfy this relation which can be checked by direct calculation.

Now that we have defined the most important concepts of NCG we are ready to discuss two general examples of spectral triples, which we'll do in the next section.

2.3 Two important examples of spectral triples

We will now consider two important examples of spectral triples, both are used extensively in NCG and form the main ingredients of most theories based on NCG.

Example 15. The *canonical spectral triple,*T_M is given by the data:

$$T_M = (\mathcal{A}, \mathcal{H}, D) = (C^\infty(M), L^2(M, S), \not{D}_M, \gamma_M, J_M). \tag{4}$$

Here M denotes a compact even-dimensional spin manifold, $C^\infty(M)$ the algebra of infinitely differentiable functions from M to the complex numbers, $L^2(M)$ the Hilbert space of square integrable sections of the spinor bundle and $\not{D}$ the canonical Dirac operator given by: $\not{D} = i\gamma^\mu(\partial_\mu - \Gamma_\mu)$, where γ^μ are the Dirac gamma matrices and the term Γ_μ is related to the curvature of space time (see [8]).

It can be shown that the canonical spectral triple corresponds to a differential Riemannian manifold which can be used to describe gravity [4, 5]. However, the canonical triple alone does not predict any other particle interactions. It can also be shown that the KO-dimension of a canonical spectral triple equals the dimension of the manifold [12]. We will take this dimension to be four since we wish to describe a four dimensional space time.

Example 16. A real even *finite spectral triple, T_M,* is given by the data:

$$T_F = (\mathcal{A}_F, \mathcal{H}_F, D_F, \gamma_F, J_F).$$

Where $\mathcal{H}_F$ is a finite dimensional Hilbert space. As mentioned above, the canonical triple does not predict any other particle interactions than gravity, since our goal is to describe electrodynamics using only the canonical triple simply won't do. Therefore we will use the product of the canonical triple with a finite spectral triple. So that the canonical triple holds our theory's extrinsic properties, while the finite triple will hold the intrinsic degrees of freedom. These intrinsic properties are what eventually determines the possible particles and interactions of our theory.

Such a product of the canonical triple with a finite triple is called the *almost commutative manifold.* The reason for this being that the algebra of the canonical triple commutes, while the algebra of the finite triple does not necessarily commute. This makes the product of these two triples an object that has both commutative and non-commutative properties.

3 A real-linear spectral triple

This thesis can be seen as an extension of [13], which showed that choosing a complexified commutative spectral triple based on a two-point space yields the full classical theory of electrodynamics on a curved background space. It was noted in [3] that a real spectral triple at least has the same gauge group but the authors leave it as an open question what physical properties this spectral triple might have. The purpose of this thesis is to clarify this question.

In this paper we will concern ourselves with the following spectral triple:

$$(\mathcal{A} = \mathbb{C}, \mathcal{H} = \mathbb{C}^2, D, J, \gamma) \tag{5}$$

Where for $\alpha \in \mathcal{A}$ and $\lambda_1 \oplus \lambda_2 \in \mathcal{H}$ we have:

$$\alpha(\lambda_1 \oplus \lambda_2) = \alpha\lambda_1 \oplus \bar{\alpha}\lambda_2 \tag{6}$$

making the action of $\mathcal{A}$ **real-linear**. The action of J is defined by:

$$J(\lambda_1 \oplus \lambda_2) = \bar{\lambda_2} \oplus \bar{\lambda_1} \tag{7}$$

Since in this case the Hilbert space is equivalent to $\mathbb{C}^2$, operators can simply be represented by 2×2 matrices for which we find the following form:

$$\begin{aligned} \alpha &= \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}, \\ J &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ C. \end{aligned}$$

Where C denotes complex conjugation and $\circ$ composition of maps. For future purposes the right hand action for any element $\alpha \in \mathcal{A}$ can be determined as follows:

$$\begin{aligned} \alpha^0 &\equiv J\alpha^* J^{-1} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ C \begin{pmatrix} \bar{\alpha} & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ C \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \bar{\alpha} & 0 \\ 0 & \alpha \end{pmatrix} = \alpha^*. \end{aligned}$$

3.1 The gauge group

Before we can start working out the detailed properties of the real-linear spectral triple we should first determine its viability. Since we aim to give a description of electrodynamics within the framework of non commutative geometry we'd better start out by determining its symmetry group. The electromagnetic force is described by the $U(1)$ symmetry group, therefore we should at least demand our spectral triple to have this property as well. We recall, from definition 13, that the gauge group of a real spectral triple is defined by:

$$\mathcal{G}(\mathcal{A}) \equiv \{U = uJuJ^* \mid u \in U(\mathcal{A})\}.$$

Since the algebra is equivalent to $\mathbb{C}$ it is immediately clear that $U(\mathcal{A}) = U(\mathbb{C}) = U(1)$. To find the symmetry group of our spectral triple we now introduce a homomorphism, ϕ, defined by:

$$\begin{aligned} \phi:\ U(\mathcal{A}) &\rightarrow \mathcal{G}(\mathcal{A}) \\ \phi(u) &= uJuJ^*. \end{aligned}$$

This enables us to use the isomorphism theorem (for a proof see:[11]) which states that:

$$U(\mathcal{A})/ker(\phi) \simeq \mathcal{G}(\mathcal{A}).$$

Where $ker(\phi)$ is defined by:

$$\begin{aligned} ker(\phi) &\equiv \{u \in U(\mathcal{A}) \mid uJuJ^* = 1\} \\ &= \{u \in U(\mathcal{A}) \mid u = u^0\} \end{aligned}$$

First we find a representation for the elements of $u \in U(1)$. Since for these elements $uu^* = 1$, we find:

$$1 = \begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix} \begin{pmatrix} \bar{u} & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} |u|^2 & 0 \\ 0 & |u|^2 \end{pmatrix} \Rightarrow u = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \ with\ \theta \in \mathbb{R}$$

Using the condition $k = k^0$ for every $k \in ker(\phi)$ and the previously derived result for k^0 we see that:

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{+i\theta} \end{pmatrix} \Rightarrow k = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore we have:

$$ker(\phi) = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \simeq C_2$$

Now we are ready to apply the isomorphism theorem, which gives us:

$$\mathcal{G}(\mathcal{A}) \simeq U(1)/C_2.$$

To see this is actually the result we hoped for we first write:

$$U(1) = \left\{ e^{i\lambda} \mid 0 \leq \lambda < 2\pi \right\}.$$

Next we see that:

$$\begin{aligned} U(1)/C_2 &= \left\{ \left\{ e^{i\theta}, -e^{i\theta} \right\} \mid 0 \leq \theta < 2\pi \right\} \\ &= \left\{ \left\{ e^{i\theta}, e^{i\theta+\pi} \right\} \mid 0 \leq \theta < 2\pi \right\} \\ &= \left\{ \left\{ e^{i\theta}, e^{i\theta+\pi} \right\} \mid 0 \leq \theta < \pi \right\} \\ &\backsimeq \left\{ e^{i\theta} \mid 0 \leq \theta < \pi \right\}. \end{aligned}$$

We can now define a map:

$$\begin{aligned} \rho : \; U(1)/C_2 &\rightarrow U(1) \\ \rho(e^{i\theta}) &= e^{2i\theta} \\ \rho^{-1}(e^{i\lambda}) &= e^{i\lambda/2}. \end{aligned}$$

It is immediately clear that ρ is a bijection. Furthermore, we see that ρ is also an homomorphism, since we have:

$$\rho(e^{i\theta_1}e^{i\theta_2}) = \rho(e^{i(\theta_1+\theta_2)}) = e^{i2(\theta_1+\theta_2)} = e^{i2\theta_1}e^{i2\theta_2} = \rho(e^{i\theta_1})\rho(e^{i\theta_2}).$$

This leads us to conclude that ρ is an isomorphism, which implies $U(1)/C_2 \simeq U(1)$. So finally we must have:

$$\boxed{\mathcal{G}(\mathcal{A}) \simeq U(1).} \tag{8}$$

This is exactly the answer we set out to find, indicating that the real-linear spectral triple might indeed lead to a description of electrodynamics.

3.2 Determining γ

After our initial success with the gauge group we might proceed by determining the possible solutions for the $\mathbb{Z}_2$ grading. We therefore introduce γ in general form:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and apply the restrictions as mentioned earlier.

First, we must have $J\gamma - \epsilon''\gamma J = 0$, so:

$$\begin{aligned} 0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ C \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \epsilon'' \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ C \\ &= \begin{pmatrix} \bar{c} & \bar{d} \\ \bar{a} & \bar{b} \end{pmatrix} \circ C - \epsilon'' \begin{pmatrix} b & a \\ d & c \end{pmatrix} \circ C \\ &= \begin{pmatrix} \bar{c} - \epsilon'' b & \bar{d} - \epsilon'' a \\ \bar{a} - \epsilon'' d & \bar{b} - \epsilon'' c \end{pmatrix} \circ C. \end{aligned}$$

From which we see that $d = \epsilon''\bar{a}$ and $c = \epsilon''\bar{b}$. Next we use that $\gamma = \gamma^*$, or:

$$\begin{pmatrix} a & b \\ \epsilon''\bar{b} & \epsilon''\bar{a} \end{pmatrix} = \begin{pmatrix} \bar{a} & \epsilon'' b \\ \bar{b} & \epsilon'' a \end{pmatrix}.$$

Giving $a \in \mathbb{R}$ and $b = \epsilon'' b$. We continue by considering each of the two possible values for ϵ'' separately and applying the third condition: $\gamma^2 = 1$.

Starting out with $\epsilon'' = 1$ we find:

$$1 = \begin{pmatrix} a & b \\ \bar{b} & a \end{pmatrix}^2 = \begin{pmatrix} a & b \\ \bar{b} & a \end{pmatrix} \begin{pmatrix} a & b \\ \bar{b} & a \end{pmatrix} = \begin{pmatrix} a^2 + |b|^2 & 2ab \\ 2a\bar{b} & a^2 + |b|^2 \end{pmatrix}.$$

So either $a = 0$ or $b = 0$ and $a^2 + |b|^2 = 1$. Taking $a = 0$ we must have $|b|^2 = 1 \Rightarrow b \neq 0$, for $b = 0$ we find $a = \pm 1$. We now have the following set of solutions for $\gamma_{\epsilon''=1}$:

$$\gamma_{\epsilon''=1} = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix} \text{ for } |b|^2 = 1.$$

The first two of these solutions ($\pm\mathbb{I}$) are trivial and not of much interest to us since they don't allow us to split the Hilbert space. The solutions for $b \neq 0$ are not allowed since we still have to apply the final condition on γ which dictates that $[\gamma, \alpha] = 0 \; \forall \; \alpha \in \mathcal{A}$. This is clearly not the case for these solutions and thus we may conclude that $\epsilon'' = 1$ does not give any viable solutions for γ.

For $\epsilon'' = -1$ we see $b = -b \Rightarrow b = 0$. Again we apply $\gamma^2 = 1$, giving:

$$\begin{aligned} 1 &= \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}^2 \\ &= \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \\ &= \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \end{pmatrix} \Rightarrow \gamma_{\epsilon''=-1} = \begin{pmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{pmatrix}. \end{aligned}$$

These solutions are in fact viable and non trivial since we see that $[\gamma, \alpha] = 0$ now holds for all $\alpha \in \mathcal{A}$. Finally we define:

$$\boxed{\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.} \tag{9}$$

This solution for γ leaves us with only one possible KO-dimension, namely KO-dimension 6. To see this we note that $J^2 = \mathbb{I}$ and that $J\gamma = -\gamma J$ and use the table from definition 10.

3.3 Finding an expression for D

Now that we have found an expression for γ we are ready to determine the possible solutions for the D. As before, we will start out with D in general form,

$$D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and proceed by applying the conditions we mentioned in section 2.3. First we must have $DJ - \epsilon' JD = 0$ and $D = D^*$ and notice that these are the same conditions as used in the previous section. In analogy with the previous result we can therefore immediately conclude that:

$$D = \begin{pmatrix} a & b \\ \epsilon'\bar{b} & \epsilon' a \end{pmatrix}, \ a \in \mathbb{R}, \ b = \epsilon' b.$$

Since $\{D, \gamma\} = 0$ we can further simplify this expression:

$$\begin{aligned} 0 &= \begin{pmatrix} a & b \\ \epsilon'\bar{b} & \epsilon' a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ \epsilon'\bar{b} & \epsilon' a \end{pmatrix} \\ &= \begin{pmatrix} a & 0 \\ 0 & -\epsilon' a \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & -\epsilon' a \end{pmatrix} \\ &= \begin{pmatrix} 2a & 0 \\ 0 & -2\epsilon' a \end{pmatrix} \Rightarrow D = \begin{pmatrix} 0 & b \\ \epsilon'\bar{b} & 0 \end{pmatrix}. \end{aligned}$$

With $b = \epsilon' b$ it is now clear that for $\epsilon' = -1$ we must have $D = 0$. To determine the solution for $\epsilon' = 1$ we continue by applying $[[D, \alpha], \beta^0] = 0 \ \forall \ \alpha, \beta \in \mathcal{A}$. We see:

$$\begin{aligned} [D, \alpha] &= \begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} - \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & b\bar{\alpha} \\ \bar{b}\alpha & 0 \end{pmatrix} - \begin{pmatrix} 0 & b\alpha \\ \bar{b}\bar{\alpha} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & b(\bar{\alpha} - \alpha) \\ \bar{b}(\alpha - \bar{\alpha}) & 0 \end{pmatrix}. \end{aligned}$$

So with the previous result for β^0 we get:

$$\begin{aligned}
0 &= [[D,\alpha],\beta^0] \\
&= \begin{pmatrix} 0 & b(\bar{\alpha}-\alpha) \\ \bar{b}(\alpha-\bar{\alpha}) & 0 \end{pmatrix} \begin{pmatrix} \bar{\beta} & 0 \\ 0 & \beta \end{pmatrix} - \begin{pmatrix} \bar{\beta} & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 0 & b(\bar{\alpha}-\alpha) \\ \bar{b}(\alpha-\bar{\alpha}) & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & b(\bar{\alpha}-\alpha)\beta \\ \bar{b}(\alpha-\bar{\alpha})\bar{\beta} & 0 \end{pmatrix} - \begin{pmatrix} 0 & b(\bar{\alpha}-\alpha)\bar{\beta} \\ \bar{b}(\alpha-\bar{\alpha})\beta & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & b(\bar{\alpha}-\alpha)(\beta-\bar{\beta}) \\ \bar{b}(\alpha-\bar{\alpha})(\bar{\beta}-\beta) & 0 \end{pmatrix}.
\end{aligned}$$

This must hold for all $\alpha, \beta \in \mathcal{A}$, giving $b = 0$. So also in the case of $\epsilon' = 1$, we may conclude:$D = 0$. With the above we have seen that, with the given properties of the spectral triple we must have:

$$\boxed{D = 0.} \tag{10}$$

Since eventually D will be responsible for attributing mass to the fermionic particles in our theory this result induces a problem. We know for a fact that the fermions in electrodynamics are indeed massive so if we were to continue with this particular spectral triple we are bound to end up with unphysical results. To fix this problem the spectral triple needs to be modified such that it allows non zero solutions for D. The next section of this paper will be dedicated to that purpose.

4 Extending the real-linear spectral triple

The spectral triple introduced in the previous section requires D to be zero which we know to be unphysical. Our goal will be to modify the spectral triple in a way that preserves its inherit characteristics but does allow D to be non zero. If we want to keep the real-linear property of the algebra together with the action of J we have to come up with a modification to the spectral triple that provides more degrees of freedom for D. The most obvious way to do this is to extent the Hilbert space: $\mathbb{C}^2 \to \mathbb{C}^4$. Giving us the modified spectral triple:

$$(\mathcal{A} = \mathbb{C}, \mathcal{H} = \mathbb{C}^4, D, J, \gamma). \tag{11}$$

For $\alpha \in \mathcal{A}$ we now redefine the real-linear action:

$$\alpha(\lambda_1 \oplus \lambda_2 \oplus \lambda_3 \oplus \lambda_4) = (\alpha\lambda_1 \oplus \bar{\alpha}\lambda_2 \oplus \alpha\lambda_3 \oplus \bar{\alpha}\lambda_4) \tag{12}$$

and the action of J will be defined as:

$$J(\lambda_1 \oplus \lambda_2 \oplus \lambda_3 \oplus \lambda_4) = (\bar{\lambda}_2 \oplus \bar{\lambda}_1 \oplus \bar{\lambda}_4 \oplus \bar{\lambda}_3). \tag{13}$$

Operators on $\mathcal{H}$ can now be represented as 4x4 matrices, for α and J we find:

$$\alpha = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \bar{\alpha} & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \bar{\alpha} \end{pmatrix}, \ J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \circ C.$$

With this the right hand action for $\alpha \in \mathcal{A}$ becomes:

$$\alpha^0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \circ C \begin{pmatrix} \bar{\alpha} & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \bar{\alpha} & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \circ C = \alpha^*$$

4.1 What does this mean for the gauge group?

Although the modification we made to the spectral triple should allow more degrees of freedom for D, we also require the gauge group to be unchanged. To make sure this is the case, we repeat the procedure from section 3.1. It is clear that we still have $U(\mathcal{A}) = U(1)$. With $\upsilon\upsilon^* = 1$ for $\upsilon \in U(\mathcal{A})$ we see that:

$$\upsilon = \begin{pmatrix} e^{i\theta} & 0 & 0 & 0 \\ 0 & e^{-i\theta} & 0 & 0 \\ 0 & 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 & e^{-i\theta} \end{pmatrix}, \ for\ \theta \in \mathbb{R}.$$

With ϕ defined in the same way as in section 3.1 and applying the isomorphism theorem we find:

$$ker(\phi) = \left\{ \pm \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \simeq C_2 \Rightarrow \mathcal{G}(\mathcal{A}) \simeq U(1)/ker(\phi) \simeq U(1)/C_2.$$

And since we know from section 3.1 that $U(1)/C(2) \simeq U(1)$ we may conclude:

$$\boxed{\mathcal{G}(\mathcal{A}) \simeq U(1)} \tag{14}$$

This is exactly the result we wished to find, so from here we may proceed with our investigation of this spectral triple.

4.2 A new $\mathbb{Z}_2$ grading and fermionic particles

Now that we know this spectral triple at least still has the right gauge group we can continue by determining possible solutions for the $\mathbb{Z}_2$ grading, γ. Instead of using an algebraic derivation we will provide physical arguments to derive an expression for γ. The reason being that an algebraic derivation will cause us to end up with a whole set of possible solutions for γ still forcing us to use physical reasoning to select the right one.

We start by choosing a basis for the Hilbert space:

$$\{\bar{\mathbf{e}}_R, \mathbf{e}_R, \bar{\mathbf{e}}_L, \mathbf{e}_L\}.$$

These basis vectors correspond to the four fermionic particles in electrodynamics: left and right handed electrons and positrons. Here the subscript-L denotes left handed particles while the subscript-R denotes right particles. The bar denotes charge conjugation. We remark that, with its definition, J transforms a particle into its charge conjugated partner as it is expected to do. Since we need γ to split the Hilbert space in a left and right handed subspaces we set: $\gamma e_L = e_L$ and $\gamma e_R = -e_R$. As in the previous section we set $\epsilon'' = -1$ giving us the identity: $J\gamma = -\gamma J$. This relation implies that $\gamma \bar{e_L} = -\bar{e_L}$ and $\gamma \bar{e_R} = \bar{e_R}$, making $\bar{e_L}$ right-handed and $\bar{e_R}$ left-handed. Now that we know the action of γ on all four basis vectors we may conclude:

$$\boxed{\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}. \tag{15}$$

From which we see that $\gamma J = -J\gamma$, together with $J^2 = 1$ we conclude that the extended triple has a KO-dimension of 6.

As mentioned above the four basis vectors of our Hilbert space should correspond to the four fermionic particles in our theory. As an additional argument for this identification we will proceed by determining a fundamental quantum number for these basis vectors: the hypercharge. The value of the hypercharge is determined by letting an element from the gauge group work on the basis vectors. To clarify this let $\upsilon \in \mathcal{G}(\mathcal{A})$ be an element from our gauge group. In the previous section we saw that there exists a one to one correspondence between the elements of our gauge group and the complex numbers with absolute value one. For υ we'll take this number to be $\lambda = e^{i\theta}$ and we write υ_λ to emphasize this. We can now apply a general theorem of NCG (see [6]) which states that for the action of the element υ_λ on one of the basis vectors, $\mathbf{e}$, we must have:

$$\upsilon_\lambda \mathbf{e} = \lambda^w \mathbf{e}.$$

Where the right hand side of the equation simply has scalar multiplication of $\mathbf{e}$ with λ^w. The main conclusion of this theorem is that w now corresponds to the hypercharge of the particle state $\mathbf{e}$.

If we now apply this theorem to what we have already derived in the previous section we find for basis vector e_L:

$$\upsilon_\lambda e_L = \begin{pmatrix} e^{i\theta} & 0 & 0 & 0 \\ 0 & e^{-i\theta} & 0 & 0 \\ 0 & 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = e^{-i\theta} e_L = \lambda^{-1} e_L.$$

Meaning the hypercharge of the particle identified with e_L is equal to -1, which is exactly the result we expect to find. When repeating this procedure for all four of our basis vectors we find the results listed in the table below.

Fermion	e_L	e_R	$\bar{e}_L$	$\bar{e}_R$
Hypercharge	-1	-1	1	1

If we compare these results to the values found in the scientific literature we notice that the values for the charge conjugated particles seem to differ. However, we remark that our theory does not take into account weak interactions so that the weak hypercharge coincides with the electric charge of the particles. When we keep this in mind the results agree with the values we expect. This leads us to conclude we did make the right identification of the basis vectors with our fermionic particles.

4.3 Finding a non-zero D

With our new found solution for γ we are ready to work out a solution for D. We will do so with the same procedure as in section 3.3. Once again we'll start out with a general expression for D:

$$D = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}.$$

First we apply $\{\gamma, D\} = 0$, giving us:

$$\begin{aligned} 0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} + \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= 2 \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & -b_2 & -b_3 & 0 \\ 0 & -c_2 & -c_3 & 0 \\ d_1 & 0 & 0 & d_4 \end{pmatrix} \Rightarrow D = \begin{pmatrix} 0 & a_2 & a_3 & 0 \\ b_1 & 0 & 0 & b_4 \\ c_1 & 0 & 0 & c_4 \\ 0 & d_2 & d_3 & 0 \end{pmatrix}. \end{aligned}$$

Next we use the fact that $D^* = D$:

$$\begin{pmatrix} 0 & \bar{b_1} & \bar{c_1} & 0 \\ \bar{a_2} & 0 & 0 & \bar{d_2} \\ \bar{a_3} & 0 & 0 & \bar{d_3} \\ 0 & \bar{b_4} & \bar{c_4} & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_2 & a_3 & 0 \\ b_1 & 0 & 0 & b_4 \\ c_1 & 0 & 0 & c_4 \\ 0 & d_2 & d_3 & 0 \end{pmatrix} \Rightarrow D = \begin{pmatrix} 0 & a_2 & a_3 & 0 \\ \bar{a_2} & 0 & 0 & b_4 \\ \bar{a_3} & 0 & 0 & c_4 \\ 0 & \bar{b_4} & \bar{c_4} & 0 \end{pmatrix}.$$

Since we concluded earlier that our triple has KO-dimension 6 we may take $\epsilon' = 1$, this gives us $DJ = JD$ or:

$$\begin{aligned} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \circ C \begin{pmatrix} 0 & a_2 & a_3 & 0 \\ \bar{a_2} & 0 & 0 & b_4 \\ \bar{a_3} & 0 & 0 & c_4 \\ 0 & \bar{b_4} & \bar{c_4} & 0 \end{pmatrix} &= \begin{pmatrix} 0 & a_2 & a_3 & 0 \\ \bar{a_2} & 0 & 0 & b_4 \\ \bar{a_3} & 0 & 0 & c_4 \\ 0 & \bar{b_4} & \bar{c_4} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \circ C \\ \begin{pmatrix} a_2 & 0 & 0 & \bar{b_4} \\ 0 & \bar{a_2} & \bar{a_3} & 0 \\ 0 & b_4 & c_4 & 0 \\ a_3 & 0 & 0 & \bar{c_4} \end{pmatrix} &= \begin{pmatrix} a_2 & 0 & 0 & a_3 \\ 0 & \bar{a_2} & b_4 & 0 \\ 0 & \bar{a_3} & c_4 & 0 \\ \bar{b_4} & 0 & 0 & \bar{c_4} \end{pmatrix}. \end{aligned}$$

So we must have $\bar{a_3} = b_4$. Finally we have the condition $[[D, \alpha], \beta^0] = 0 \ \forall \alpha, \beta \in \mathcal{A}$. For this we first calculate:

$$\begin{aligned}
[D,\alpha] &= \begin{pmatrix} 0 & a_2 & \bar{b_4} & 0 \\ \bar{a_2} & 0 & 0 & b_4 \\ b_4 & 0 & 0 & c_4 \\ 0 & \bar{b_4} & \bar{c_4} & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \bar{\alpha} & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \bar{\alpha} \end{pmatrix} - \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \bar{\alpha} & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 0 & a_2 & \bar{b_4} & 0 \\ \bar{a_2} & 0 & 0 & b_4 \\ b_4 & 0 & 0 & c_4 \\ 0 & \bar{b_4} & \bar{c_4} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \bar{\alpha} a_2 & \alpha \bar{b_4} & 0 \\ \alpha \bar{a_2} & 0 & 0 & \bar{\alpha} b_4 \\ \alpha b_4 & 0 & 0 & \bar{\alpha} c_4 \\ 0 & \bar{\alpha} \bar{b_4} & \alpha \bar{c_4} & 0 \end{pmatrix} - \begin{pmatrix} 0 & \alpha a_2 & \alpha \bar{b_4} & 0 \\ \bar{\alpha} \bar{a_2} & 0 & 0 & \bar{\alpha} b_4 \\ \alpha b_4 & 0 & 0 & \alpha c_4 \\ 0 & \bar{\alpha} \bar{b_4} & \bar{\alpha} \bar{c_4} & 0 \end{pmatrix} \\
&= (\alpha - \bar{\alpha}) \begin{pmatrix} 0 & -a_2 & 0 & 0 \\ \bar{a_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_4 \\ 0 & 0 & \bar{c_4} & 0 \end{pmatrix}.
\end{aligned}$$

With which we can determine $[[D, \alpha], \beta^0]$:

$$\begin{aligned}
0 &= (\alpha - \bar{\alpha}) \left[\begin{pmatrix} 0 & -a_2 & 0 & 0 \\ \bar{a_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_4 \\ 0 & 0 & \bar{c_4} & 0 \end{pmatrix} \begin{pmatrix} \bar{\beta} & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \bar{\beta} & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix} - \begin{pmatrix} \bar{\beta} & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \bar{\beta} & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix} \begin{pmatrix} 0 & -a_2 & 0 & 0 \\ \bar{a_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_4 \\ 0 & 0 & \bar{c_4} & 0 \end{pmatrix} \right] \\
&= (\alpha - \bar{\alpha}) \left[\begin{pmatrix} 0 & -\beta a_2 & 0 & 0 \\ \bar{\beta} \bar{a_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta c_4 \\ 0 & 0 & \bar{\beta} \bar{c_4} & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\bar{\beta} a_2 & 0 & 0 \\ \beta \bar{a_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{\beta} c_4 \\ 0 & 0 & \beta \bar{c_4} & 0 \end{pmatrix} \right] \\
&= (\alpha - \bar{\alpha})(\beta - \bar{\beta}) \begin{pmatrix} 0 & -a_2 & 0 & 0 \\ -\bar{a_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_4 \\ 0 & 0 & -\bar{c_4} & 0 \end{pmatrix}.
\end{aligned}$$

This must hold for all $\alpha, \beta \in \mathcal{A}$ so we must have $a_2 = c_4 = 0$. We are left with one complex parameter, b_4, which we relabel to d. Finally we may conclude that the Dirac operator is given by:

$$\boxed{D = \begin{pmatrix} 0 & 0 & \bar{d} & 0 \\ 0 & 0 & 0 & d \\ d & 0 & 0 & 0 \\ 0 & \bar{d} & 0 & 0 \end{pmatrix}} \tag{16}$$

With this non zero solution for the Dirac operator we have completed the construction of a real even spectral triple, furthermore it blows new life into our theory since we might now expect to find massive fermionic particles.

5 Physical implications

In this section we will attempt to derive some physical predictions from our theory, in particular we want do the derive the nature of the interactions in our theory. Before we can start out, we will first introduce some concepts necessary for doing this.

5.1 Gauge fields in NCG

Since we are aiming to derive the possible particle interactions in our theory we first consider how interactions are described in modern physics.

As mentioned earlier, in field theory all fundamental interactions arise from local symmetries of a given theory. This means that the Lagrangian is invariant under a continuous group of local transformations of the fields. These transformations are called the gauge transformations and we will need the equivalent notion of these transformations within the framework of NCG. In this thesis we will simply provide these equivalent definitions without providing a detailed analysis. For a derivation and discussion of these concepts we refer to: [12].

Before we can carry on we need to introduce some new concepts which lead to NCG's equivalent of the gauge fields from modern field theories.

Definition 17. For a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ we define a set of differential one forms, $\Omega^1_D(\mathcal{A})$, which is given by the following definition:

$$\Omega^1_D(\mathcal{A}) = \left\{ \sum_i \alpha_i [D, \beta_i] \mid \alpha_i, \beta_i \in \mathcal{A} \right\}. \tag{17}$$

Definition 18. For a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ endowed with a real structure , J, we define the **fluctuated Dirac operator** by:

$$D_\omega \equiv D + \omega + \epsilon' J \omega J^{-1}. \tag{18}$$

For self adjoint $\omega = \omega^* \in \Omega^1_D(\mathcal{A})$. The elements ω are called **inner fluctuations** of the spectral triple and will be interpreted as the gauge potentials or gauge fields of our theory. In order to determine the properties of these fields we will look at the way they transform under various transformations. In this way we may check whether the fields in our theory behave the way we expect them to do from electrodynamical field theory.

Example 19. Before we continue we will determine the inner fluctuations of the canonical spectra triple as an example. We mentioned earlier that the canonical triple (introduced in example 15) alone does not predict any interactions except for gravity. We will show this by proving the inner fluctuations of the canonical triple vanish. We start out by determining $\Omega^1_{D_M}$, if we take $\alpha_i, \beta_i \in C^\infty(M)$ we have:

$$\begin{aligned}\Omega^1_{\not{D}} &= \left\{\sum_i \alpha_i[\not{D}, \beta_i]\right\} = \left\{\sum_i \alpha_i[i\gamma^\mu(\partial_\mu + \omega_\mu), \beta_i]\right\} \\ &= \left\{\sum_i \alpha_i i\gamma^\mu \partial_\mu(\beta_i)\right\} = \left\{\sum_i i\gamma^\mu \alpha_{\mu(i)}\right\} \\ &= \{\gamma^\mu \alpha_\mu\}\end{aligned}$$

Where we absorbed all elements of the algebra into one element, α_μ. Now that we know the form of the elements $\omega \in \Omega^1_{\not{D}}$ we calculate $J_M \omega J_M^{-1}$, keep in mind that since $\omega = \omega^*$ and $\gamma^{\mu *} = \gamma^\mu$ (see [15])we must have $\alpha \in \mathbb{R}$ from which we know that $J\alpha_\mu = \alpha_\mu J$. So that we have:

$$\begin{aligned}J_M \omega J_M^{-1} &= J_M \gamma^\mu \alpha_\mu J_M^{-1} \\ &= -\gamma^\mu \alpha_\mu\end{aligned}$$

Where we have used the identity $\{J_M, \gamma^\mu\} = 0$ which is derived in [14]. Putting the terms together and remarking that the canonical triple has a KO-dimension of four ($\epsilon' = 1$) we find for the fluctuated Dirac operator:

$$\begin{aligned}\not{D}_\omega &= \not{D} + \gamma^\mu \alpha_\mu - \gamma^\mu \alpha_\mu \\ &= \not{D}.\end{aligned}$$

So we may conclude that the inner fluctuation terms cancel each other out. This confirms the previous statement that the canonical triple does not predict any interactions except for gravity since these interactions are contained within the inner fluctuations.

We proceed by introducing the concept of gauge transformations within the framework of NCG.

Theorem 20. *Transformations, $UD_\omega U^*$, with $U \in \mathcal{G}(\mathcal{A})$ of the fluctuated Dirac operator are in general equivalent to a transformation of ω of the form:*

$$\boxed{\omega \to u\omega u^* + u[D, u^*].} \tag{19}$$

We will interpret these transformations as the ***gauge transformations*** *of the gauge field in physics and will apply it to our theory in section 5.4.*

Proof. We calculate the action of a unitary transformation $U \in \mathcal{G}(\mathcal{A})$ on the fluctuated

Dirac operator. First we write: $U = uJuJ^{-1} = uu^{*^0} \equiv uv$, next we determine:

$$\begin{aligned} UDU^* &= uvDv^*u^* = u\left(D + v[D, v^*]\right)u^* \\ &= uDu^* + uvu^*[D, v^*] \\ &= uDu^* + uu^{*^0}u^*[D, v^*] \\ &= uDu^* + v[D, v^*] \\ &= D + u[D, u^*] + v[D, v^*]. \end{aligned}$$

Where we have used that $[[D, v^*], u^*] = 0$ and $\left[u^{*^0}, u^*\right] = 0$. Next we see that:

$$v[D, v^*] = JuJ^{-1}[D, Ju^*J^{-1}] = \epsilon' Ju[D, u^*]J^{-1}.$$

Furthermore we have:

$$U\omega U^* = uJuJ^{-1}\omega Ju^*J^{-1}u^* = uJuJ^{-1}\omega u^0 u^* = uJuJ^{-1}Ju^*J^{-1}\omega u^* = u\omega u^*.$$

Where we used the identities from definition 12 to see that $\left[\omega, u^0\right] = 0$. Using the same identities it is also easy to see that $\left[u, J^{-1}uJ\right] = \left[w, J^{-1}uJ\right] = 0$, which gives us:

$$\begin{aligned} UJ\omega J^{-1}U^* &= uJuJ^{-1}J\omega J^{-1}Ju^*J^{-1}u^* \\ &= uJu\omega u^*J^{-1}u^* \\ &= JJ^{-1}uJu\omega u^*J^{-1}u^* \left(\text{multiplication with } \mathbb{I} = JJ^{-1}\right) \\ &= Ju\omega u^*J^{-1}uJJ^{-1}u^* \\ &= Ju\omega u^*J^{-1}. \end{aligned}$$

Putting these terms together we find:

$$UD_\omega U^* = D + u\omega u^* + u[D, u^*] + \epsilon' J\left(u\omega u^* + u[D, u^*]\right)J^{-1}.$$

From which we can easily see that the unitary transformations of the fluctuated Dirac operator correspond to a transformation of ω of the form:

$$\omega \to u\omega u^* + u[D, u^*].$$

Note that the transformed inner fluctuation is again an inner fluctuation. Which in turn means that the transformation of the fluctuated Dirac operator can again be written in the form of a fluctuated Dirac operator. We can only do this because we use transformations from the gauge group $\mathcal{G}(\mathcal{A})$. If we would have taken arbitrary transformations the expression for J would have transformed as well and we would end up with a fundamentally different result. □

These unitary transformations will now be interpreted in physics as the gauge transformations of the gauge field.

5.2 The almost commutative manifold

Now that we have composed a spectral triple that conforms to the fundamental physical restrictions we are ready to extract some real physics from its properties. The final goal will be to derive an expression for the Dirac equation in order to make predictions on the nature of the interactions in our theory.

Before we can start this procedure we should combine our spectral triple with the canonical spectral triple since, as mentioned earlier, our framework is set up so that our finite dimensional triple can act as an extension of the canonical triple. Combining our triple with the canonical triple is achieved by taking the tensor product of their components, giving us a combined triple which we will call the *almost commutative manifold*. From now on we write all the components of our finite triple with **subscript F**. With the given definition of the canonical triple we then find the following components of the almost commutative manifold:

- $\mathcal{A} = C^\infty(M) \otimes \mathbb{C}$
- $\mathcal{H} = L^2(M,S) \otimes \mathbb{C}^4$
- $D = \not{D} \otimes \mathbb{I} + \gamma_5 \otimes D_F$
- $\gamma = \gamma_M \otimes \gamma_F = \gamma_5 \otimes \gamma_F$
- $J = J_M \otimes J_F$

The result for D might seem unexpected but the reason for choosing it is that it satisfies the condition $\{D, \gamma\} = 0$, simply taking the tensor product of $\not{D}$ and D_F would not satisfy this since:

$$\{\not{D} \otimes D_F, \gamma_5 \otimes \gamma_F\} = \not{D}\gamma_5 \otimes D_F\gamma_F + \gamma_5\not{D} \otimes \gamma_F D_F = 2\not{D}\gamma_5 \otimes D_F\gamma_F \neq 0$$

Combining these components gives us an almost commutative manifold of the form:

$$\boxed{(\mathbb{C}^\infty(M) \otimes \mathbb{C}, L^2(M,S) \otimes \mathbb{C}^4, \not{D} \otimes \mathbb{I} + \gamma_5 \otimes D_F, \gamma_5 \otimes \gamma_F, J_M \otimes J_F)} \tag{20}$$

We note that any operator a working on $\mathcal{H}$ can now be expressed as the tensor product, $\alpha_M \otimes \alpha_F$ of an operator α_M working on $L^2(M,S)$ and an operator α_F working on $\mathcal{H}_F$. So that for the product of two operators a and b we have:

$$ab = (\alpha_M \otimes \alpha_F)(\beta_M \otimes \beta_F) = \alpha_M\beta_M \otimes \alpha_F\beta_F.$$

5.3 The inner fluctuations

The almost commutative manifold forms the basis from which we will try to extract an expression for the action. As mentioned previously, the action consists of two parts: the spectral action and the fermionic action. Before we can make any attempt at deriving the spectral action the fluctuated Dirac operator needs to be found. For this we first determine the inner fluctuations of our theory. We start out by finding an expression for Ω^1_D from its definition:

$$\begin{aligned}
\Omega^1_D \quad (\mathcal{A}) = & \left\{ \sum_i a_i \left[D, b_i \right] \ | \ a_i, b_i \in \mathcal{A} \right\} \\
= & \left\{ \sum_i a_i \left[\not{D} \otimes \mathbb{I} + \gamma_5 \otimes D_F, b_i \right] \ | \ a_i, b_i \in \mathcal{A} \right\} \\
= & \left\{ \sum_i a_i \left[\not{D} \otimes \mathbb{I}, b_i \right] + \sum_i a_i \left[\gamma_5 \otimes D_F, b_i \right] \ | \ a_i, b_i \in \mathcal{A} \right\} \\
= & \Omega^1_{\not{D}\otimes\mathbb{I}}(\mathcal{A}) \oplus \{\alpha_{Mi}\gamma_5\beta_{Mi}\} \otimes \Omega^1_{D_F}(\mathcal{A}_F) \\
= & \Omega^1_{\not{D}\otimes\mathbb{I}} \oplus \gamma_5 \otimes \Omega^1_{D_F}(\mathcal{A}_F).
\end{aligned}$$

Where we have used that:

$$\begin{aligned}
a_i \left[\gamma_5 \otimes D_F, b_i \right] &= \alpha_{Mi} \otimes \alpha_{Fi} \left(\gamma_5 \beta_{Mi} \otimes D_F \beta_{Fi} - \beta_{Mi}\gamma_5 \otimes \beta_{Fi} D_F \right) \\
&= \alpha_{Mi} \otimes \alpha_{Fi} \left(\gamma_5 \beta_{Mi} \otimes D_F \beta_{Fi} - \gamma_5\beta_{Mi} \otimes \beta_{Fi} D_F \right) \\
&= \alpha_{Mi} \otimes \alpha_{Fi} \left(\gamma_5 \beta_{Mi} \otimes \left[D_F, \beta_{Fi} \right] \right) \\
&= \alpha_{Mi}\gamma_5\beta_{Mi} \otimes \alpha_{Fi} \left[D_F, \beta_{Fi} \right],
\end{aligned}$$

the fact that α_{Mi} and β_{Mi} are just complex valued scalar functions which can be moved from one side of the tensor product to the other and finally that an element in $\Omega^1_{D_F}(\mathcal{A}_F)$ multiplied by a scalar is again an element in $\Omega^1_{D_F}(\mathcal{A}_F)$.

We conclude that $\Omega^1_D(\mathcal{A})$ contains both $\Omega^1_{\not{D}\otimes\mathbb{I}}(\mathcal{A})$ and $\Omega^1_{D_F}(\mathcal{A}_F)$, therefore we continue by determining these two terms individually starting out with $\Omega^1_{D_F}(\mathcal{A}_F)$. We have:

$$\Omega^1_{D_F}(\mathcal{A}_F) = \left\{ \sum_i \alpha_{Fi} \left[D_F, \beta_{Fi} \right] \ | \ \alpha_{Fi}, \beta_{Fi} \in \mathcal{A}_F \right\}.$$

To calculate $[D_F, \beta_{Fi}]$ we can simply use the results for β_{Fi} and D_F we found earlier, we find:

$$[D_F, \beta_{Fi}] = \begin{pmatrix} 0 & 0 & \bar{d} & 0 \\ 0 & 0 & 0 & d \\ d & 0 & 0 & 0 \\ 0 & \bar{d} & 0 & 0 \end{pmatrix} \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \bar{\beta} & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \bar{\beta} \end{pmatrix} - \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \bar{\beta} & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \bar{\beta} \end{pmatrix} \begin{pmatrix} 0 & 0 & \bar{d} & 0 \\ 0 & 0 & 0 & d \\ d & 0 & 0 & 0 \\ 0 & \bar{d} & 0 & 0 \end{pmatrix} = 0.$$

So we may conclude $\Omega^1_{D_F}(\mathcal{A}_F) = \{0\}$ and we have $\Omega^1_D(\mathcal{A}) = \Omega^1_{\not{D}\otimes\mathbb{I}}$. We continue by determining $\Omega^1_{\not{D}\otimes\mathbb{I}}(\mathcal{A})$ from its definition for $a_i, b_i \in \mathcal{A}$:

$$\begin{aligned}
\Omega^1_{\not{D}\otimes\mathbb{I}}(\mathcal{A}) &= \left\{\sum_j a_j\,[\not{D}\otimes\mathbb{I}, b_j]\right\} \\
&= \left\{\sum_j (\alpha_{Mj}\otimes\alpha_{Fj})\,[\not{D}\otimes\mathbb{I}, \beta_{Mj}\otimes\beta_{Fj}]\right\} \\
&= \left\{\sum_j \alpha_{Mj}\,[\not{D}, \beta_{Mj}]\otimes\alpha_{Fj}\beta_{Fj}\right\} \\
&= \left\{\sum_j \alpha_{Mj}\,[i\gamma^\mu(\partial_\mu - \Gamma_\mu), \beta_{Mj}]\otimes\alpha_{Fj}\beta_{Fj}\right\} \\
&= \left\{\sum_j \alpha_{Mj} i\gamma^\mu\left([\partial_\mu, \beta_{Mj}] - [\Gamma_\mu, \beta_{Mj}]\right)\otimes\alpha_{Fj}\beta_{Fj}\right\} \\
&= \left\{i\gamma^\mu\sum_j \alpha_{Mj}\partial_\mu(\beta_{Mj})\otimes\alpha_{Fj}\beta_{Fj}\right\} \\
&= \left\{i\gamma^\mu\sum_j \alpha_{\mu(Mj)}\otimes\alpha_{Fj}\right\} \\
&= \left\{i\gamma^\mu\alpha_{\mu(M)}\otimes\alpha_F\right\}
\end{aligned}$$

In this derivation we have used the product rule for writing $[\partial_\mu, \beta_{Mj}] = \partial_\mu(\beta_{Mj})$ and the fact that we can absorb sums of elements from our algebra into one element (since we are interested in the set of *all* the possible summations) to come to the final result.

Next we determine what the restriction $\omega = \omega^*$ means in our case:

$$\begin{aligned}
i\gamma^\mu\alpha_{\mu(M)}\otimes\alpha_{\mu(F)} &= \left(i\gamma^\mu\alpha_{\mu(M)}\otimes\alpha_F\right)^* \\
&= -i\gamma^\mu\alpha^*_{\mu(M)}\otimes\alpha^*_F \\
&= i\gamma^\mu\alpha^*_{\mu(M)}\otimes -\alpha^*_F.
\end{aligned}$$

With which we have $\alpha^*_{\mu(M)} = \alpha_{\mu(M)}$ and $\alpha^*_F = -\alpha_F$ so that we may conclude $\alpha_{\mu(M)}$ to be real and α_F to be skew-Hermitian, which means that all its components are purely imaginary. Using the bi-linearity of the tensor product we may now move $\alpha_{\mu(M)}$ to the right hand side. The product $\alpha_{\mu(M)}\alpha_F$, which we'll denote by $\alpha_{\mu(F)}$, is an object that again consists of elements of our finite algebra, since:

$$\alpha_{\mu(M)}\alpha_F = \begin{pmatrix} \alpha_{\mu(M)}\alpha_F & 0 & 0 & 0 \\ 0 & \alpha_{\mu(M)}\bar{\alpha}_F & 0 & 0 \\ 0 & 0 & \alpha_{\mu(M)}\alpha_F & 0 \\ 0 & 0 & 0 & \alpha_{\mu(M)}\bar{\alpha}_F \end{pmatrix}$$

and $\alpha_{\mu(M)}$ is real we have $\overline{\alpha_{\mu(M)}\alpha_F} = \alpha_{\mu(M)}\bar{\alpha}_F$. We can now write an element $\omega \in \Omega^1_D$ as:

$$\boxed{\omega = i\gamma^\mu \otimes \alpha_{\mu(F)}} \tag{21}$$

Before we can determine the fluctuated Dirac operator, D_ω, we also need to find an expression for $J\omega J^{-1}$. We see:

$$\begin{aligned} J\omega J^{-1} &= J_M \otimes J_F \left(i\gamma^\mu \otimes \alpha_F\right) J_M^{-1} \otimes J_F^{-1} \\ &= i\gamma^\mu \otimes J_F \alpha_{\mu(F)} J_F^{-1} \\ &= i\gamma^\mu \otimes \alpha^{*0}_{\mu(F)} \\ &= i\gamma^\mu \otimes \alpha_{\mu(F)} \end{aligned}$$

Where we have used the identity $\{J_M, \gamma^\mu\} = 0$ to calculate the left hand side of the tensor product. For the right hand side we used the fact that in our case $\alpha^0_F = \alpha^*_F \Rightarrow \alpha^{*0}_F = \alpha_F$.

With this we have gathered all the necessary components of D_ω, which we can now find from its definition. Adding up all these expressions, we find for the the fluctuated Dirac operator:

$$\begin{aligned} D_\omega &= D + \omega + \epsilon' J\omega J^{-1} \\ &= D + i\gamma^\mu \otimes \alpha_{\mu(F)} + i\gamma^\mu \otimes \alpha_{\mu(F)} \\ &= D + i\gamma^\mu \otimes 2\alpha_{\mu(F)} \\ &\equiv D + i\gamma^\mu \otimes -iA_\mu e. \end{aligned}$$

Where A_μ is an real valued covariant four-vector field that is an infinitely differentiable function of the manifold M:

$$A_\mu = \begin{pmatrix} A_0(x) \\ A_1(x) \\ A_2(x) \\ A_3(x) \end{pmatrix} \text{ for } x \in M$$

and e is a 4×4 matrix given by:

$$e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The matrix e obtains this form because α_F is skew-Hermitian resulting in an diagonal signature of form: $(+-+-)$. Note that the product $A_\mu e$ should be read as multiplication of the index A_μ with the entire matrix e and not as a matrix multiplied with a vector.

The notation we chose for A_μ is very suggestive since we know in electrodynamics the vector potential is denoted with the same symbol. We will see in the next section that there actually is some merit to this choice of notation.

5.4 Gauge transformations of the gauge field

With the expression for the inner fluctuations of the almost commutative manifold we effectively found the gauge field of our theory. To determine the nature of this field we take a look at how it transforms under gauge transformations, $U = uJuJ^{-1} \in \mathcal{G}(\mathcal{A})$. We use the transformation characteristics given by equation 19. If we write ω^T for transformed $\omega = i\gamma^\mu \otimes \alpha_{\mu(F)}$ we see:

$$\begin{aligned} \omega^T &= u\omega u^* + u[D, u^*] \\ &= u_M \otimes u_F(i\gamma^\mu \otimes \alpha_{\mu(F)})u_M^* \otimes u_F^* + u_M \otimes u_F \left[\not{D} \otimes \mathbb{I} + \gamma_5 \otimes D_F, u_M^* \otimes u_F^*\right] \\ &= u_M i\gamma^\mu u_M^* \otimes u_F \alpha_{\mu(F)} u_F^* + u_M \otimes u_F \left(\left[\not{D} \otimes \mathbb{I}, u_M^* \otimes u_F^*\right] + \left[\gamma_5 \otimes D_F, u_M^* \otimes u_F^*\right]\right) \\ &= \omega + u_M \otimes u_F \left(\left[\not{D}, u_M^*\right] \otimes u_F^* + u_M^* \gamma_5 \otimes \left[D_F, u_F^*\right]\right) \\ &= \omega + u_M \left[i\gamma^\mu(\partial_\mu - \Gamma_\mu), u_M^*\right] \otimes \mathbb{I} \\ &= \omega + u_M i\gamma^\mu \partial_\mu(u_M^*) \otimes \mathbb{I}. \end{aligned}$$

We note that the unitary elements of the canonical algebra are of the form: $u_M = e^{i\theta(x)}$ where $\theta(x) \in \mathbb{R}\ \forall\ x$. Using this property we see:

$$\begin{aligned} \omega^T &= \omega + e^{i\theta(x)} i\gamma^\mu \partial_\mu(e^{-i\theta(x)}) \otimes \mathbb{I} \\ &= \omega - e^{i\theta(x)} i\gamma^\mu i e^{-i\theta(x)} \partial_\mu(\theta(x)) \otimes \mathbb{I} \\ &= \omega - i\gamma^\mu i\partial_\mu(\theta(x)) \otimes \mathbb{I} \\ &= \omega - i\gamma^\mu \otimes i\partial_\mu(\theta(x)) \end{aligned}$$

Which means the gauge transformations can be expressed as transformations of A_μ which takes the following form:

$$\boxed{A_\mu \to A_\mu + \partial_\mu(\theta(x)).} \tag{22}$$

Finally, this result enables us to make a comparison between our theory and the already existing theory of electrodynamics. We recall from [2] that our gauge field transforms in exactly the same way as the vector potential A^μ of electrodynamics.

5.5 The Dirac equation

In this section we will compare the Dirac equation of our theory with the Dirac equation for electrodynamics on a curved background.

From [8] we know that the general-relativistically covariant Dirac equation derived by Fock for a particle of rest mass m and charge e in an electromagnetic potential A_μ is:

$$[i\gamma^\mu (\partial_\mu - \Gamma_\mu - ieA_\mu) - m]\, \psi = 0. \tag{23}$$

Taking $D_{ed} = i\gamma^\mu (\partial_\mu - \Gamma_\mu - ieA_\mu) - m$ enables us to rewrite this equation as $D_{ed}\psi = 0$. As a way of comparison we will now determine the form this equation will take if we use the fluctuated Dirac operator we found earlier.

Before we can attempt to do this we need to determine the form of the wave function for our almost commutative manifold which we will denote by ξ. We remark that particles in our theory should be either left or right handed and that in nature only left handed particles can have weak interactions. We therefore consider only left handed wave functions and determine the left handed subspace of our Hilbert-space, which we denote by $\mathcal{H}^+$. From [10] we note that any element from the Hilbert space, $\psi \in L^2(M,S)$ can be decomposed into ψ_L and ψ_R. Here subscript L and R denote left and right handedness respectively. Next, it is important to note that for the almost commutative spectral triple, the combined grading $\gamma = \gamma_M \otimes \gamma_F$ determines the chirality of a particle so that we must look for products of vectors from the canonical and finite basis to find a basis for $\mathcal{H}^+$. Taking into account the bi-linearity of the tensor product it is easy to see that:

$$\mathcal{H}^+ = L^2(M,S)^+ \otimes \mathcal{H}_F^+ \oplus L^2(M,S)^- \otimes \mathcal{H}_F^-.$$

If we now take two vectors $(q_L\mathbf{e_L} + w_R\overline{\mathbf{e}_R}) \in \mathcal{H}_F^+$ and $(q_R\mathbf{e_R} + w_L\overline{\mathbf{e_L}}) \in \mathcal{H}_F^-$ any vector $\xi \in \mathcal{H}^+$can be represented as:

$$\begin{aligned}
\xi &= \psi_L \otimes (q_L\mathbf{e}_L + w_R\overline{\mathbf{e}_R}) + \psi_R \otimes (q_R\mathbf{e}_R + w_L\overline{\mathbf{e}_L}) \\
&= \psi_L \otimes q_L\mathbf{e}_L + \psi_L \otimes w_R\overline{\mathbf{e}_R} + \psi_R \otimes q_R\mathbf{e}_R + \psi_R \otimes w_L\overline{\mathbf{e}_L} \\
&= q_L\psi_L \otimes \mathbf{e}_L + w_R\psi_L \otimes \overline{\mathbf{e}_R} + q_R\psi_R \otimes \mathbf{e}_R + w_L\psi_R \otimes \overline{\mathbf{e}_L} \\
&\equiv \chi_L \otimes \mathbf{e}_L + \chi_R \otimes \mathbf{e}_R + \eta_R \otimes \overline{\mathbf{e}_L} + \eta_L \otimes \overline{\mathbf{e}_R}
\end{aligned} \tag{24}$$

Where we have absorbed the coordinate coefficients of the finite part into the four different functions $\eta_L, \chi_L \in L^2(M,S)^+$ and $\eta_R, \chi_R \in L^2(M,S)^-$. The Dirac equation for our theory now becomes:

$$\begin{aligned}
0 &= D_\omega \xi \\
&= [\not{D} \otimes \mathbb{I} + \gamma_5 \otimes D_F + i\gamma^\mu \otimes -iA_\mu e]\, \xi \\
&= [i\gamma^\mu(\partial_\mu - \Gamma_\mu) \otimes \mathbb{I} + \gamma_5 \otimes D_F + i\gamma^\mu(-iA_\mu) \otimes e]\, \xi \\
&= [i\gamma^\mu\, ((\partial_\mu - \Gamma_\mu) \otimes \mathbb{I} - iA_\mu \otimes e) + \gamma_5 \otimes D_F]\, \xi.
\end{aligned} \tag{25}$$

We can now directly identify the term $(\partial_\mu - \Gamma_\mu) \otimes \mathbb{I}$ in our equation (24) with the term $\partial_\mu - \Gamma_\mu$ from equation (23). Previously we noted that we need a non zero Dirac operator in order to attribute mass to our particles. We will now clarify this statement by identifying the terms $\gamma_5 \otimes D_F$ and $-m$. We will first dispose of the γ_5 term on the left hand side of the product. To do this we introduce a new field, ϕ, defined by a unitary transformation of ξ (see [9]): $\xi = (e^{i\frac{\pi}{4}\gamma_5} \otimes \mathbb{I})\phi$. This transformation is called a chiral rotation and has no physical significance for our theory. If we now substitute ξ in the Dirac equation and multiply the right hand side of the equation with $e^{i\frac{\pi}{4}\gamma_5} \otimes \mathbb{I}$, we find:

$$
\begin{aligned}
0 &= (e^{i\frac{\pi}{4}\gamma_5} \otimes \mathbb{I})\left[i\gamma^\mu\left((\partial_\mu - \Gamma_\mu) \otimes \mathbb{I} - iA_\mu \otimes e\right) + \gamma_5 \otimes D_F\right](e^{i\frac{\pi}{4}\gamma_5} \otimes \mathbb{I})\phi \\
&= \left[ie^{i\frac{\pi}{4}\gamma_5}\gamma^\mu e^{i\frac{\pi}{4}\gamma_5}\left((\partial_\mu - \Gamma_\mu) \otimes \mathbb{I} - iA_\mu \otimes e\right) + e^{i\frac{\pi}{4}\gamma_5}\gamma_5 e^{i\frac{\pi}{4}\gamma_5} \otimes D_F\right]\phi \\
&= \left[i\gamma^\mu\left((\partial_\mu - \Gamma_\mu) \otimes \mathbb{I} - iA_\mu \otimes e\right) + \gamma_5 e^{i\frac{\pi}{2}\gamma_5} \otimes D_F\right]\phi.
\end{aligned}
$$

Where we have used $\{\gamma^\mu, \gamma_5\} = 0 \Rightarrow e^{i\frac{\pi}{2}\gamma_5}\gamma^\mu = \gamma^\mu e^{-i\frac{\pi}{2}\gamma_5}$ and the fact that $[e^{i\frac{\pi}{2}\gamma_5}, \gamma_5] = 0$. Next we notice (with $\gamma_5^2 = 1$) that:

$$
\begin{aligned}
e^{i\frac{\pi}{2}\gamma_5} &= \sum_{k=0}^{\infty} \frac{(i\frac{\pi}{2}\gamma_5)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(\frac{\pi}{2}\gamma_5\right)^{2k} + i\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{\pi}{2}\gamma_5\right)^{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(\frac{\pi}{2}\right)^{2k} + i\gamma_5\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+1} \\
&= cos(\frac{\pi}{2}) + i\gamma_5 sin(\frac{\pi}{2}) = i\gamma_5.
\end{aligned}
$$

So that we are left with:

$$
0 = \left[i\gamma^\mu\left((\partial_\mu - \Gamma_\mu) \otimes \mathbb{I} - iA_\mu \otimes e\right) + i \otimes D_F\right]\phi.
$$

We now simply set the value of the complex parameter which determines D_F equal to $d = im$. So that we can write $D_F = imD_0$. Where D_0 is the operator on our finite Hilbert-space defined by:

$$
D_0 \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.
$$

With this we see that $i \otimes D_F = i \otimes imD_0 = -m \otimes D_0$, where we have used the bilinearity of the tensor product. So that, finally, the expression for the Dirac equation becomes:

$$
\boxed{\left[i\gamma^\mu\left((\partial_\mu - \Gamma_\mu) \otimes \mathbb{I} - iA_\mu \otimes e\right) - m \otimes D_0\right]\phi = 0.} \tag{26}
$$

Furthermore we identify $-iA_\mu \otimes e$ (24) with the term $-ieA_\mu$ (23). We now see the difference between classical theory and NCG. In the classical theory the charges of the particles were simply assigned to the particles. In NCG we see that the matrix e automatically assigns the right charges to the particles in our theory based on the particle's wave function. Take for instance $\psi_F = \mathbf{e}_L$ then multiplication by e gives us a factor -1, on the other hand we have $e\bar{\mathbf{e}}_L = \bar{\mathbf{e}}_L$. We conclude that these eigenvalues of e correspond to the charges we would have assigned to the left-handed electron and right-handed positron in classical theory.

6 Conclusion

In this thesis we explored the possibility of describing electrodynamics within the framework of non-commutative geometry. In particular we determined the consequences of using a real-linear algebra to do so.

We started out with a real-linear spectral triple endowed with a Hilbert space of dimension 2:

$$(\mathcal{A} = \mathbb{C}, \mathcal{H} = \mathbb{C}^2, D, J, \gamma).$$

While determining the associated expression for the Dirac operator we had to conclude that this spectral triple would lead to unphysical results. The reason for this being that the Dirac operator turns out to be zero which, in turn, would mean that the fermionic particles in our theory are massless. Since we know the fermions in electrodynamics are in fact massive this would yield an non-viable theory.

To overcome this shortcoming we proceeded by extending the Hilbert-space to four dimensions. This provides more degrees of freedom for the expression of the Dirac operator and yielded a non-zero result. Furthermore we showed that the fermionic particles in our theory carry the correct hypercharges. From this we continued by taking the product of our finite triple with the canonical spectral triple to form the almost commutative spectral triple. This enabled us to determine the inner fluctuations which resulted in a U(1) gauge field with the same transformation characteristics as the four-vector potential of electrodynamics.

After these positive results we continued by deriving the Dirac equation associated with our theory. When compared to the known Dirac equation of a general-relativistically covariant particle in an electromagnetic potential the analogy is evident. In fact, a one-to-one identification of the terms in both equations can be made. The interaction of the fermions with the gauge field is opposite for electron an positron states, exactly like we expect from the classical theory. Furthermore the scalar which determines our finite Dirac operator, can be linked directly to the mass of the particles in our theory.

In summary, we have shown that as far as our research goes, the extended real-linear spectral triple provides a promising starting point for describing electrodynamics. There were no results conflicting with this hypothesis.

Since time was a limiting factor in this research, a final conclusion about the validity of this hypothesis can not be made at this time. In future research it would therefore be a logical next step to derive an expression for the spectral action of our theory. This would enable a more direct comparison with the results of [13].

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