The Electroweak Theory
From A Noncommutative Spectral Triple

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1 Introduction

In this thesis I will try to describe the Electroweak theory within the framework of noncommutative geometry. Some years ago a noncommutative geometrical description of the Standard Model [7] was given. In this text we zoom in on the electroweak part of the Standard Model in the hope to get more insight. The Electroweak theory is the unification of the electromagnetic and weak interactions, for which Glashow, Weinberg and Salam were awarded the 1979 Nobel Prize. So we will deal in this text with electromagnetism, the weak interaction and gravity. The fermions we will consider are the electrons e, neutrino's ν and their anti-particles. For transparent representation we will consider only the first generation of particles, but the results can easily be extended to incorporate the other two generations.

We will set the stage by introducing some of the basic concepts of noncommutative geometry without already specifying too much about its physical applications. In Section 2 we will try to clarify how elementary particle physics might benefit from this mathematical theory and what kind of elements one needs for an electroweak particle theory.

1.1 Noncommutative geometry

Since Einstein gave us his geometrical explanation of gravity [2], physicists gained more interest in geometry. The holy grail of physics is uniting gravity with the three other fundamental forces. This might for example be done by finding a way to describe gravity as a particle theory just like the other interactions, or by searching for a geometrical theory description of all fundamental interactions. The latter is pursued by the application of noncommutative geometry to high energy physics. Noncommutative geometry is a generalization of ordinary geometry and might therefore be able to give a geometrical description of more interactions than gravity alone.

The best way to introduce the idea behind noncommutative geometry is to first have a look at commutative geometry and then generalize this to noncommutative geometry.

The objects studied in geometry are manifolds. A manifold is a curved space that locally looks like flat Euclidean space. An example of a manifold is the curved spacetime of General Relativity.

Consider a manifold $\mathcal{M}$. The set of smooth functions on $\mathcal{M}$ composes the algebra $C^\infty(\mathcal{M})$. An algebra is a vectorspace with, besides addition of elements, a multiplication of elements. A precise definition will be given in Section 2. $C^\infty(\mathcal{M})$ is by all means a very fundamental object because many geometrical properties of $\mathcal{M}$ can be derived from it. The multiplication in $C^\infty(\mathcal{M})$ is commutative and this algebra is therefore called a commutative algebra.

The importance of $C^\infty(\mathcal{M})$ is further underlined by the following theorem which states that two manifolds are diffeomorphic ('the same' in the context of manifolds) if and only if their algebra of smooth functions are isomorphic ('the same' in the context of algebras).

Theorem 1.1. *Two manifolds $\mathcal{M}$ and $\mathcal{N}$ are diffeomorphic iff $C^\infty(\mathcal{M})$ and $C^\infty(\mathcal{N})$ are isomorphic.*

A proof of this theorem can be found in [1]. Because $C^\infty(\mathcal{M})$ entails most of the characteristics of $\mathcal{M}$ we can equally well study the algebra $C^\infty(\mathcal{M})$ of the manifold instead of the manifold $\mathcal{M}$ itself.

A commutative algebra $C^\infty(\mathcal{M})$ encodes the properties of $\mathcal{M}$, because $C^\infty(\mathcal{M})$ is commutative we also call $\mathcal{M}$ a commutative manifold. We can now try to generalize this concept. Consider instead of a commutative algebra a noncommutative algebra $\mathcal{A}_{nc}$.

This $\mathcal{A}_{nc}$ has no explicit underlying manifold, but it turns out that you still can deduce the geometrical features, with the same success as before. Thus by considering a noncommutative algebra you can consider the geometry of a noncommutative manifold. But this noncommutative manifold itself does not exist, it is only its noncommutive algebra that exists. So we broaden our notion of geometry by studying noncommutative algebras. This is the idea behind noncommutative geometry. This new geometric framework might be applicable to a wider range of physical phenomena than ordinary geometry. Therefore it is interesting to see if it is applicable to the other fundamental forces of nature besides gravity. If so, it might be a step in the right direction of a united description of the fundamental interactions.

2 Spectral triples of a particle physics theory

Noncommutative geometry was first properly fomulated by Alain Connes, see [1]. We will use this formulation and in this section introduce most of the concepts of importance for doing particle physics with noncommutative geometry. After we have defined the so-called spectral triple we shortly return to the subject of Section 1.1, to futher clarify the idea behind noncommutative geometry. As we go along these concepts we will try to indicate their importance for an elementary particle model. We begin by introducing the fundamental aspects of a spectral triple and then define some important examples of spectral triples. Much of the definitions are taken from either [4] or [5].

2.1 The spectral triple

As mentioned in Section 1, a spectral triple consist of three objects:

- unital $*$-algebra, $\mathcal{A}$
- Hilbert space $\mathcal{H}$
- Selfadjoint Dirac operator, D

We will first define these three objects and then define the spectral triple.

Definition 2.1. Let $\mathcal{F}$ be a field, and let $\mathcal{A}$ be a vector space equipped with a binary operation, $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, denoted here by $\cdot$, called multiplication. Then $\mathcal{A}$ is an *algebra* over $\mathcal{F}$ if

- $a \cdot (b + c) = a \cdot b + a \cdot c$
- $(a + b) \cdot c = a \cdot c + b \cdot c$
- $(\lambda a) \cdot (\mu b) = (\lambda\mu)(a \cdot b)$

$\forall\ a, b, c \in \mathcal{A}$ and $\lambda, \mu \in \mathcal{F}$.

Remark 2.2. A *unital algebra* is an algebra $\mathcal{A}$ that contains a multiplicative identity element, i.e. an element 1 with the property $1 \cdot a = a \cdot 1 = a\ \forall\ a \in \mathcal{A}$.

From now on we will not write the product of two elements of the algebra as $a \cdot b$ anymore but rather as ab.

Definition 2.3. A $*$-*algebra* is an algebra $\mathcal{A}$ over a field $\mathcal{F}$, with an operation $* : \mathcal{A} \to \mathcal{A}$ such that

- $(\lambda a)^* = \overline{\lambda} a^*$
- $(a + b)^* = a^* + b^*$
- $(ab)^* = b^* a^*$
- $(a^*)^* = a$

$\forall\ \lambda \in \mathcal{F}$ and $a, b \in \mathcal{A}$.

Remark 2.4. An algebra is called commutative when $[a, b] = 0\ \forall\ a, b \in \mathcal{A}$. If an algebra is not commutative it is called a noncommutative algebra.

Now that we have defined a unital $*$-algebra we will proceed by defining the Hilbert space.

Definition 2.5. A *Hilbert space* $\mathcal{H}$ is a real or complex vector space over a field $\mathcal{F}$, endowed with an inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{F}$, that is complete, i.e. every Cauchy sequence in $\mathcal{H}$ converges with respect to the norm $||x|| = \sqrt{\langle x, x \rangle}, \ x \in \mathcal{H}$.

The unital $*$-algebra and the Hilbert space will form a so-called module together. This is the last thing we need to define before we can introduce the spectral triple.

Definition 2.6. Let $\mathcal{A}$ be an algebra and $\mathcal{V}$ a vector space. ${}_{\mathcal{A}}\mathcal{V}$ is a *left module* over $\mathcal{A}$ when there is an operation $\mathcal{A} \times \mathcal{V} \to \mathcal{V}$, again denoted by $\cdot$, such that

- $a \cdot (v + w) = a \cdot v + a \cdot w$
- $(a + b) \cdot v = a \cdot v + b \cdot v$
- $a \cdot (b \cdot v) = (a \cdot b) \cdot v$

where $a, b \in \mathcal{A}$ and $v, w \in \mathcal{V}$.

Remark 2.7. A *right module* $\mathcal{V}_{\mathcal{A}}$ is defined analogously with the exception that the operation $\cdot$ goes from $\mathcal{V} \times \mathcal{A}$ to $\mathcal{V}$ and the multiplication operation acts on the right.

Definition 2.8. Let $\mathcal{A}$ and $\mathcal{B}$ be an algebra and $\mathcal{V}$ a vector space. Then ${}_{\mathcal{A}}\mathcal{V}_{\mathcal{B}}$ is a *bimodule* if ${}_{\mathcal{A}}\mathcal{V}$ is a left module, $\mathcal{V}_{\mathcal{B}}$ is right module and $a \cdot (v \cdot b) = (a \cdot v) \cdot b \ \ \forall a \in \mathcal{A}$, $b \in \mathcal{B}$ and $v \in \mathcal{V}$.

So far we have defined two of the three objects of a spectral triple. We will now combine these two with the third into the definition of a spectral triple.

Definition 2.9. A *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ is given by a unital $*$-algebra $\mathcal{A}$ represented as bounded operators on a Hilbert space $\mathcal{H}$ and a self-adjoint *Dirac operator* D, that has compact resolvent and is such that all commutators $[D, a]$ are bounded for $a \in \mathcal{A}$.

The elements of the algebra thus act as operators on the Hilbert space. These elements can for example be represented as matrices or functions.

Consider now for a moment a Riemannian manifold $\mathcal{M}$ again. On a manifold, $\mathcal{M}$, a space of spinor-fields can be constructed on which the Dirac operator acts. These spinor-fields compose a Hilbert space $\mathcal{H}$. These spinor-fields in $\mathcal{H}$ can be multiplied by the complex functions in $C^\infty(\mathcal{M})$. So $C^\infty(\mathcal{M})$ is an algebra of operators acting on the spinor-fields in $\mathcal{H}$. The system consisting of the Hilbert space $\mathcal{H}$, the Dirac operator D and the algebra $\mathcal{A} = C^\infty(\mathcal{M})$ is called the spectral triple $(\mathcal{A}, \mathcal{H}, D)$. From all this follows a theorem somewhat similar to Theorem 1.1, for which a proof can be found in [3].

Theorem 2.10. *1. For every compact Riemannian manifold $(\mathcal{M}, g)$ there is an associated spectral triple $(\mathcal{A}, \mathcal{H}, D)$ as defined above.*
2. For every spectral triple $(\mathcal{A}, \mathcal{H}, D)$, with $\mathcal{H}$ a Hilbert space, $\mathcal{A}$ a commutative algebra of opertators in $\mathcal{H}$ and D a linear operator in $\mathcal{H}$, satisfying certain properties (which we won't go into here), there exists a unique compact manifold $(\mathcal{M}, g)$, such that $(\mathcal{A}, \mathcal{H}, D)$ is the spectral triple associated with $(\mathcal{M}, g)$. Moreover both the manifold and the metric tensor can be constructed in an explicit way from $(\mathcal{A}, \mathcal{H}, D)$.

So loosely said a spectral triple with a commutative algebra corresponds to a certain Riemannian manifold. We can now again make the same generalization as before by instead of considering a commutative algebra considering a noncommutative algebra $\mathcal{A}_{nc}$. A noncommutative spectral triple $(\mathcal{A}_{nc}, \mathcal{H}, D)$ is given by a noncommutative algebra $\mathcal{A}_{nc}$, a Hilbert space $\mathcal{H}$ and a Dirac operator. A noncommutative spectral triple then again corresponds to a noncommutative geometry and these noncommutative spectral triples are the primary objects studied in the field of noncommutative geometry.

2.2 Real structure and $\mathbb{Z}_2$-grading

Something we observe in nature is chirality. We want some notion of left- and right-handedness for our particles, this is were the following concept comes in.

Definition 2.11. A spectral triple is called *even* if the Hilbert space $\mathcal{H}$ is endowed with an operator, called the $\mathbb{Z}_2$*-grading*, γ which commutes with every element $a \in \mathcal{A}$ (i.e. $[a, \gamma] = 0$) and anticommutes with D (i.e. $\{D, \gamma\} = 0$).

For an even spectral triple the Hilbert space decomposes into the ± 1 eigenspaces of γ, i.e. $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. This can be easily seen by checking that $\frac{1}{2}(1 \pm \gamma)$ are the projection operators corresponding to these subspaces. It will turn out that these ± 1 eigenspaces are the spaces for the left- and right-handed particles respectively, in the Hilbert space. Thus for example $\mathcal{H}^+$ is the subspace of the left-handed particles and $\mathcal{H}^-$ is the subspace of the right-handed particles.

Definition 2.12. A *real structure* on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$, i.e. J preserves the norm on $\mathcal{H}$ and $J(\alpha x + \beta y) = \overline{\alpha} J(x) + \overline{\beta} J(y)$ $\forall x, y \in \mathcal{H}, \alpha, \beta \in \mathcal{F}$, where $\mathcal{F}$ is the field underlying $\mathcal{H}$. J being an isometry means that $J^* = J^{-1}$. Furthermore J must satisfy

$$J^2 = \epsilon, \quad JD = \epsilon' DJ, \quad J\gamma = \epsilon'' \gamma J \ (\textit{even case}), \quad \epsilon, \epsilon', \epsilon'' \in \{\pm\}.$$

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is called a real spectral triple if it is endowed with a real structure J.

The signs ϵ, ϵ' and ϵ'' determine the *KO-dimension* n modulo 8 of the spectral triple:

n	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1		-1		1		-1	

The usual action of an operator $a \in \mathcal{A}$ on the Hilbert space $\mathcal{H}$ is called the left action of a on $\mathcal{H}$. With help of the real structure we can define a right action:

Definition 2.13. A right action of $a \in \mathcal{A}$ on $\mathcal{H}$, denoted by a^0 is given by

$$a^0 := Ja^* J^{-1}.$$

The action of $\mathcal{A}$ and D are required to satisfy the commutation rules:

- $[a, b^0] = 0 \ \ \forall\ a, b \in \mathcal{A}$
- $[[D, a], b^0] = 0 \ \ \forall\ a, b \in \mathcal{A}$

In a noncommutative geometry particle model J plays a role similar to the charge conjugation operator on the particles: J interchanges particles and antiparticles. This is a property a particle model needs to have because that is what we also observe in nature.

2.3 The canonical spectral triple

In this and the next subsection we will turn our attention to two examples of spectral triples that are of great importance for a noncommutative particle model. The first of these two is the canonical spectral triple.

Definition 2.14. Let $\mathcal{M}$ be a compact Riemannian spin manifold, then the *canonical triple* M is defined by

$$M = (\mathcal{A}, \mathcal{H}, D) := (C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), \not{D}),$$

where S is the spinor bundle on $\mathcal{M}$ and $\not{D}$ is the canonical Dirac operator given locally by $-i\gamma^\mu \nabla^S_\mu$. Here ∇^S is the Levi-Civita connection lifted to the spinor bundle and the γ^μ are the Dirac gamma matrices.

When $\mathcal{M}$ is of even dimension m, then there is a $\mathbb{Z}_2$-grading γ_{m+1} on M, when $m = 4$ $\gamma_{m+1} = \gamma_5$ and will just be the ordinary γ_5 of General Relativity. We also have a J_M which acts as the charge conjugation operator on the spinors: $(J_M\psi(x)) = C\psi(x)$, for a spinor $\psi(x)$ and C the charge conjugation matrix.

M will turn out to play the role of our spacetime, similar as in General Relativity. From a spectral triple it is shown to be possible to calculate an action functional (which will be defined later on). For the canonical triple of a 4-dimensional spin manifold this action is shown to be equal to the Einstein-Hilbert action, including higher order gravity contributions (see [4]). The Einstein equations of General Relativity can be derived from the Einstein-Hilbert action.

2.4 The finite spectral triple

A second special case of a spectral triple is the finite spectral triple:

Definition 2.15. The *finite triple* F is given by a finite dimensional algebra $\mathcal{A}_F$, a finite dimensional Hilbert space $\mathcal{H}_F$ and a matrix D_F, i.e.:

$$F = (\mathcal{A}, \mathcal{H}, D) := (\mathcal{A}_F, \mathcal{H}_F, D_F)$$

Since $\mathcal{A}_F$ is finite dimensional its elements can be represented as matrices on the finite dimensional Hilbert space $\mathcal{H}_F$. Again, additional structure can be defined as in Definitions 2.11 and 2.12, i.e. a γ_F and J_F, subject to the (anti-)commutation relations given above.

The finite spectral triple will encode the internal degrees of freedom of the particle theory we are trying to establish.

2.5 An almost commutative spectral triple

Now that we have defined the canonical triple and the finite triple we are ready to introduce the almost commutative manifold, which plays a central role in the particle physics application of noncommutative geometry.

Definition 2.16. A *real even almost commutative spectral triple* $M \times F$ is described by

$$M \times F = (\mathcal{A}, \mathcal{H}, D) := (C^\infty(\mathcal{M}, \mathcal{A}_F), L^2(\mathcal{M}, S) \otimes \mathcal{H}_F, \not{D} \otimes \mathbb{I} + \gamma_{m+1} \otimes D_F),$$

Together with a grading $\gamma = \gamma_{m+1} \otimes \gamma_F$ and a real structure $J = J_M \otimes J_F$. The KO-dimension is the product of the KO-dimensions of M and F modulo 8.

Remark 2.17. The Dirac operator has the form $\not{\partial}\otimes\mathbb{I}+\gamma_{m+1}\otimes D_F$, because a Dirac operator in a spectral triple must satisfy $\{D,\gamma\}=0$, which is not the case for $D=\not{\partial}\otimes D_F$. We do have $\{\not{\partial}\otimes\mathbb{I}+\gamma_{m+1}\otimes D_F,\gamma_{m+1}\otimes\gamma_F\}=0$, so this indicates $\not{\partial}\otimes\mathbb{I}+\gamma_{m+1}\otimes D_F$ is the correct Dirac operator for this product of spectral triples.

An almost commutative spectral triple is the product of the canonical triple, whose algebra is commutative, and a finite spectral triple, whose algebra is noncommutative, and is therefore called almost commutative.

$M\times F$ is the fundamental object in a noncommutative description of a particle theory. In the canonical spectral triple M the structure of spacetime is encoded. The finite spectral triple F, which in general is noncommutative, incorporates the internal degrees of freedom. Thus by taking the product of M and F we have a spacetime with at each point the internal degrees of freedom inserted by F.

2.6 The gauge group

How the particles in a particle theory interact is determined by the corresponding gauge group. This group consists of the gauge transformations, i.e. the transformations that leave the Lagrangian or action of the theory invariant. We know that the gauge group of the electroweak theory is $U(1)\times SU(2)$ and this is something we want our noncommutative theory to reproduce. In this subsection we will see how the gauge group is defined in this context and we will deduce some results that cause the calculation of the gauge group to be easier. The proofs of these results can be found in the appendix.

Definition 2.18. The *gauge group* $\mathcal{G}(\mathcal{A})$ of real spectral triple $(\mathcal{A},\mathcal{H},D,\gamma,J)$ is defined by

$$\mathcal{G}(\mathcal{A}):=\{U=uJuJ^{-1}|u\in U(\mathcal{A})\},$$

$U(\mathcal{A})=\{u\in\mathcal{A}\mid uu^*=u^*u=1\}$, the unitary elements of the algebra $\mathcal{A}$.

Proposition 2.19. *Let* $Ad:U(\mathcal{A})\to\mathcal{G}(\mathcal{A})$ *be the map* $u\mapsto uJuJ^{-1}$ *and let* $\tilde{\mathcal{A}}_J:=\{a\in\mathcal{A}|aJ=Ja^*\}$. *Then the* $ker(Ad):=\{u\in U(\mathcal{A})|uJuJ^{-1}=1\}=U(\widetilde{\mathcal{A}}_J)\triangleleft U(\mathcal{A})$, *i.e.* $U(\widetilde{\mathcal{A}}_J)$ *is a normal subgroup of* $U(\mathcal{A})$ *and the gauge group* $\mathcal{G}(\mathcal{A})$ *is isomorphic to* $U(\mathcal{A})/U(\widetilde{\mathcal{A}}_J)$, $\mathcal{G}(\mathcal{A})\cong U(\mathcal{A})/U(\widetilde{\mathcal{A}}_J)$.

Proof. The proof can be found in Appendix A.1 □

Definition 2.20. The *gauge group* $\mathcal{G}(M\times F)$ is the gauge group of the almost commutative spectral triple with algebra $\mathcal{A}=C^\infty(M,\mathcal{A}_F)$

Proposition 2.21. *The gauge group* $\mathcal{G}(M\times F)$ *is isomorphic to* $U(\mathcal{A})/U(\widetilde{\mathcal{A}}_J)$, $\mathcal{G}(M\times F)\cong U(\mathcal{A})/U(\widetilde{\mathcal{A}}_J)$.

Proof. This result follows from Proposition 2.19. □

Proposition 2.22. *For the almost commutative manifold* $M\times F$ *with algebra* $\mathcal{A}=C^\infty(\mathcal{M},\mathcal{A}_F)$, $U(\mathcal{A})=C^\infty(\mathcal{M},U(\mathcal{A}_F))$.

Proof. The proof can be found in Appendix A.2 □

Proposition 2.23. *For the almost commutative manifold* $M\times F$ *with algebra* $\mathcal{A}=C^\infty(\mathcal{M},\mathcal{A}_F)$, $U(\widetilde{\mathcal{A}}_J)=C^\infty(\mathcal{M},U((\widetilde{\mathcal{A}}_F)_{J_F}))$, *with* $(\widetilde{\mathcal{A}}_F)_{J_F}:=\{a_F\in\mathcal{A}_F|a_F^0=J_Fa_F^*J_F^{-1}=a_F\}$.

Proof. The proof can be found in Appendix A.3 □

Proposition 2.24. $\mathcal{G}(\mathcal{A}_F) = U(\mathcal{A}_F)/U((\widetilde{\mathcal{A}}_F)_J)$.

Proof. Again the result follows from Proposition 2.19. $\square$

Corollary 2.25. *From Propositions 2.21 to 2.24 we find that* $\mathcal{G}(M \times F)$
$= C^\infty(\mathcal{M}, U(\mathcal{A}_F))/C^\infty(\mathcal{M}, U((\widetilde{\mathcal{A}}_F)_J)) = C^\infty(\mathcal{M}, U(\mathcal{A}_F))/U((\widetilde{\mathcal{A}}_F)_J)) = C^\infty(\mathcal{M}, \mathcal{G}(\mathcal{A}_F))$.

2.7 Inner fluctuations

We again consider the almost commutative manifold given by the following spectral triple

$$(\mathcal{A}, \mathcal{H}, D) := (C^\infty(\mathcal{M}, \mathcal{A}_F), L^2(\mathcal{M}, S) \otimes \mathcal{H}_F, \not{D} \otimes \mathbb{I} + \gamma_5 \otimes D_F),$$

with the grading $\gamma = \gamma_5 \otimes \gamma_F$ and the real structure $J = J_M \otimes J_F$.

Considering Morita equivalences between algebras, we find the *inner fluctuations*. The inner fluctuations give rise to the gauge fields of our particle theory. Morita equivalences will not be treated in this text, however the origin of the following definition can be found in [8].

Definition 2.26. The *fluctuated Dirac operater* D_A of the spectral triple $M \times F$ is given by

$$D_A = D + A + \epsilon' JAJ^{-1},$$

with the *inner fluctuations* A the self-adjoint elements of the set
$\Omega^1_D(\mathcal{A}) := \{\sum_j a_j[D, b_j] \mid a_j, b_j \in \mathcal{A}\}$.

To find the inner fluctuations of the spectral triple we first need to determine $\Omega^1_D(\mathcal{A})$. Let us first write the elements of $\Omega^1_D(\mathcal{A})$ in a more convenient manner:

$$\begin{aligned}
\sum_j a_j[D, b_j] &= \sum_j a_j[\not{D} \otimes \mathbb{I} + \gamma_5 \otimes D_F, b_j] \\
&= \sum_j a_j[\not{D} \otimes \mathbb{I}, b_j] + a_j[\gamma_5 \otimes D_F, b_j] \\
&= \sum_j a_j[\not{D} \otimes \mathbb{I}, b_j] + \sum_i a_i[\gamma_5 \otimes D_F, b_i] \\
&= \Omega^1_{\not{D}\otimes\mathbb{I}}(\mathcal{A}) + \sum_i a_i[\gamma_5 \otimes D_F, b_i]
\end{aligned}$$

We can write $a_i, b_i \in \mathcal{A}$ as $(\alpha_i \otimes a_{Fi}), (\beta_i \otimes b_{Fi}) \in C^\infty(M) \otimes \mathcal{A}_F$. By doing so we further simplify the expression, because the grading γ_5 commutes with all elements of the algebra:

$$\begin{aligned}
\sum_i a_i[\gamma_5 \otimes D_F, b_i] &= \sum_i a_i((\gamma_5 \otimes D_F)b_i - b_i(\gamma_5 \otimes D_F)) \\
&= \sum_i (\alpha_i \otimes a_{Fi})(\gamma_5\beta_i \otimes D_F b_{Fi} - \beta_i\gamma_5 \otimes b_{Fi} D_F) \\
&= \sum_i (\alpha_i \otimes a_{Fi})(\beta_i\gamma_5 \otimes D_F b_{Fi} - \beta_i\gamma_5 \otimes b_{Fi} D_F) \\
&= \sum_i (\alpha_i \otimes a_{Fi})(\beta_i\gamma_5 \otimes [D_F, b_{Fi}]) \\
&= \sum_i \alpha_i\beta_i\gamma_5 \otimes a_{Fi}[D_F, b_{Fi}] \\
&= \gamma_5 \otimes \sum_i a'_{Fi}[D_F, b'_{Fi}], \quad a'_{Fi} = \alpha_i a_{Fi}, b'_{Fi} = \beta_i b_{Fi} \\
&= \gamma_5 \otimes \Omega^1_{D_F}(\mathcal{A})
\end{aligned}$$

In the last steps we made use of the algebra isomorphism between $C^\infty(M) \otimes \mathcal{A}_F$ and $C^\infty(M, \mathcal{A}_F)$. So in the end we find:

$$\Omega^1_D(\mathcal{A}) = \Omega^1_{\not{D}\otimes\mathbb{I}}(\mathcal{A}) \oplus \gamma_5 \otimes \Omega^1_{D_F}(\mathcal{A})$$

We have split $\Omega^1_D(\mathcal{A})$ into an $\Omega^1_{\not{D}\otimes\mathbb{I}}(\mathcal{A})$ involving the Dirac operator of the canonical spectral triple and $\Omega^1_{D_F}(\mathcal{A})$ involving the Dirac operator of the finite spectral triple.

2.8 The action functional

So far the almost commutative spectral triple is just an abstract mathematical object. To make this into a physical theory, we need to be able to derive physical quantities from the spectral triple. This is done in the form of an action functional. Once we have the action of our system, the equations of motion of the fields can be determined by minimizing this action. In this text we will restrict ourselves to the fermionic action, i.e. the terms of the action involving fermions, although it should be noted that it is also possible to derive a bosonic action.

Before we come to the definition of the fermionic action, we need to define a subset of the Hilbert space. We can use γ to decompose the Hilbert space as $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, the plus and minus eigenspaces of γ respectively. The Hilbert space of an almost commutative spectral triple is the tensor product of the finite Hilbert space $\mathcal{H}_F$ and $L^2(M, S)$. So we have

$$\begin{aligned}\mathcal{H} = L^2(M,S) \otimes \mathcal{H}_F = L^2(M,S)^+ \otimes \mathcal{H}_F^+ \oplus L^2(M,S)^+ \otimes \mathcal{H}_F^- \\ \oplus L^2(M,S)^- \otimes \mathcal{H}_F^+ \oplus L^2(M,S)^- \otimes \mathcal{H}_F^-.\end{aligned}$$

Therefore the positive eigenspace of γ is

$$\mathcal{H}^+ = L^2(M,S)^+ \otimes \mathcal{H}_F^+ \oplus L^2(M,S)^- \otimes \mathcal{H}_F^-.$$

Before we define the fermionic action, we first define the set of classical fermions $\mathcal{H}^+_{cl} := \{\tilde{\xi} \mid \xi \in \mathcal{H}^+\}$ as the set of Grassmann variables $\tilde{\xi}$ for $\xi \in \mathcal{H}^+$.

Definition 2.27. For a real even spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma, J)$ of KO-dimension 2 we define the fermionic action by

$$S_f[A, \xi] := \frac{1}{2}\langle J\tilde{\xi}, D_A \tilde{\xi}\rangle$$

where D_A is the fluctuated Dirac operator and $\tilde{\xi} \in \mathcal{H}^+_{cl}$.

We can write D_A in terms of the canonical Dirac operator and finite Dirac operator, $D_A = \not{D}_A \otimes \mathbb{I} + \gamma_5 \otimes D_{F_A}$:

$$S_f[A, \xi] := \frac{1}{2}\langle J\tilde{\xi}, D_A \tilde{\xi}\rangle = \frac{1}{2}\langle J\tilde{\xi}, (\not{D}_A \otimes \mathbb{I})\tilde{\xi}\rangle + \frac{1}{2}\langle J\tilde{\xi}, (\gamma_5 \otimes D_{F_A})\tilde{\xi}\rangle.$$

So we only have a definition for the fermionic action in KO-dimension 2 (see subsection 2.2), but it turns out that is all we need.

3 The Electroweak theory

In this section we will try to construct the finite spectral triple that will give us the Electroweak theory.

3.1 Constructing the finite spectral triple

Starting with a specific finite algebra, we try to find the accompanying spectral triple. As noted before, we will consider the following algebra $\mathcal{A}_F$:

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R.$$

The choice for this specific algebra is made, because it is for example shown in [5], [6] and [7] that this might be a good finite algebra to start our investigation with. This algebra is left-right symmetric with the labels L and R for bookkeeping. $\mathbb{H}$ is the field of quaternions, the extension of the complex numbers first found by Sir William Rowan Hamilton. The quaternions can be represented as 2×2 matrices of complex numbers in the following way.

$$q = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \quad \text{with} \quad \alpha, \beta \in \mathbb{C}.$$

In this representation the product of two quaternions is given by ordinary matrix multiplication.

We begin to construct the spectral triple by specifying the Hilbert space $\mathcal{H}_F$, on which the elements of $\mathcal{A}_F$ act. We propose a $\mathcal{H}_F$ which will turn out to be very natural for us to define our left and right action on. We want our spectral triple to have a real structure so that we have a notion of charge conjugation therefore the Hilbert space is of the form:

$$\mathcal{H}_F = (\mathbf{1} \oplus \mathbf{2}_L \oplus \mathbf{2}_R) \otimes (\mathbf{1}^0 \oplus \mathbf{2}_L^0 \oplus \mathbf{2}_R^0).$$

$\mathbf{1}$ is the subspace of dimension 1 corresponding to $\mathbb{C}$ and $\mathbf{2}_{L,R}$ are the subspaces of dimension 2 corresponding to $\mathbb{H}_{L,R}$. On this Hilbert space J_F maps between $(\mathbf{1} \oplus \mathbf{2}_L \oplus \mathbf{2}_R)$ and $(\mathbf{1}^0 \oplus \mathbf{2}_L^0 \oplus \mathbf{2}_R^0)$. When we rewrite this we find,

$$\begin{aligned} \mathcal{H}_F = & (\mathbf{1} \otimes \mathbf{1}^0) \oplus (\mathbf{1} \otimes \mathbf{2}_L^0) \oplus (\mathbf{1} \otimes \mathbf{2}_R^0) \\ & \oplus \ (\mathbf{2}_L \otimes \mathbf{1}^0) \oplus (\mathbf{2}_L \otimes \mathbf{2}_L^0) \oplus (\mathbf{2}_L \otimes \mathbf{2}_R^0) \\ & \oplus \ (\mathbf{2}_R \otimes \mathbf{1}^0) \oplus (\mathbf{2}_R \otimes \mathbf{2}_L^0) \oplus (\mathbf{2}_R \otimes \mathbf{2}_R^0). \end{aligned}$$

In this text we will consider only one generation of particles and therefore in our electroweak model we will consider eight particles, namely left and right handed electrons, left and right handed neutrino's and their anti-particles. Of these eight particles four are left-handed and four are right-handed. This means that we need a Hilbert space of dimension eight with four left-handed and four right-handed particles which transform into each other by charge conjugation. Because of this we dispose of some terms in the expression for $\mathcal{H}_F$ and end up with

$$\mathcal{H}_F = (\mathbf{1} \otimes \mathbf{2}_L^0) \oplus (\mathbf{2}_L \otimes \mathbf{1}^0) \oplus (\mathbf{1} \otimes \mathbf{2}_R^0) \oplus (\mathbf{2}_R \otimes \mathbf{1}^0).$$

We can write elements of $\mathcal{H}_F$ with respect to a basis

$$(e_{\mathbf{1}\otimes\mathbf{2}_L^0,1}, e_{\mathbf{1}\otimes\mathbf{2}_L^0,2}, e_{\mathbf{2}_L\otimes\mathbf{1}^0,1}, e_{\mathbf{2}_L\otimes\mathbf{1}^0,2}, e_{\mathbf{1}\otimes\mathbf{2}_R^0,1}, e_{\mathbf{1}\otimes\mathbf{2}_R^0,2}, e_{\mathbf{2}_R\otimes\mathbf{1}^0,1}, e_{\mathbf{2}_R\otimes\mathbf{1}^0,2})$$

Some important operators act on elements of $\mathcal{H}_F$ in such a way that their action can be formulated on pairs of components. In that case it will be much more convenient to represent their action with respect to a basis:

$$(e_L, \overline{e}_L, e_R, \overline{e}_R)$$

Where $e_L = (e_{\mathbf{1}\otimes\mathbf{2}_L^0,1}, e_{\mathbf{1}\otimes\mathbf{2}_L^0,2})$, $\overline{e}_L = (e_{\mathbf{2}_L\otimes\mathbf{1}^0,1}, e_{\mathbf{2}_L\otimes\mathbf{1}^0,2})$ etc.

3.2 Additional structure

We will now supply some extra structure for our spectral triple. Beginning with a $\mathbb{Z}_2$-grading γ_F on $\mathcal{H}_F$. This will decompose $\mathcal{H}_F$ as $\mathcal{H}_F = \mathcal{H}_F^+ \oplus \mathcal{H}_F^-$, with $\mathcal{H}_F^+$ and $\mathcal{H}_F^-$ the ± 1 eigenspaces of γ_F. Besides that our spectral triple will have a real structure J_F, which interchanges particles and their anti-particles.

We will choose γ_F and J_F such that

$$\gamma_F(e_L, \overline{e}_L, e_R, \overline{e}_R) = (e_L, -\overline{e}_L, -e_R, \overline{e}_R),$$

$$J_F e_L = \overline{e}_L, J_F e_R = \overline{e}_R, J_F \overline{e}_L = e_L, J_F \overline{e}_R = e_R.$$

In matrix form γ_F and J_F will look like this

$$\gamma_F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad J_F = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} C$$

Keep in mind that we are in fact dealing with 8×8 matrices here, so every matrix element is really a 2×2 matrix. The C is the complex conjugation operator, i.e. it works on a complex number by conjugating it.

This choice for γ_F and J_F corresponds to a KO-dimension of 6. We can see that by checking if γ_F and J_F commute or anticommute and if J_F^2 equals the identity or minus the identity. This is done in appendix B.1.

3.3 Left and right action

We will now have a closer look on the action of elements of $\mathcal{A}_F$ on $\mathcal{H}_F$. We have $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R$. The representation of this algebra must be in the form of 8×8 matrices, so that the elements of the algebra can act on the Hilbert space. With the Hilbert space $\mathcal{H}_F$ in mind a logical simple choice for the matrix representation of an element $\pi(\lambda, q_L, q_R) \in \mathcal{A}_F$, with $\lambda \in \mathbb{C}$, $q_L \in \mathbb{H}_L$ and $q_R \in \mathbb{H}_R$ is

$$\pi(\lambda, q_L, q_R) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & q_L & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & q_R \end{pmatrix}.$$

Thus this is a 8×8 matrix, with

$$\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \; q_L = \begin{pmatrix} \alpha_L & \beta_L \\ -\overline{\beta}_L & \overline{\alpha}_L \end{pmatrix}, q_R = \begin{pmatrix} \alpha_R & \beta_R \\ -\overline{\beta}_R & \overline{\alpha}_R \end{pmatrix} \text{ with } \lambda, \alpha_{L,R}, \beta_{L,R} \in \mathbb{C}.$$

From this we can easily determine the right action of $\pi(\lambda, q_L, q_R)$ as

$$J\pi(\lambda, q_L, q_R)^* J^{-1} = \begin{pmatrix} q_L^T & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & q_R^T & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}.$$

Where q_L^T and q_R^T denote the transposed of q_L and q_R. From the definitions in Section 2.1 it is evident to see that this representation of the algebra indeed composes a unital $*$-algebra. We directly see that the left and right action commute from the fact that the submatrices λ are diagonal, thus we have $[a, b^0] = 0$.

3.4 Determining the Dirac operator

With all preparations done we can now determine the self-adjoint Dirac operator, D_F. The Dirac operator has to obey some commutation rules that will vastly reduce the freedom which we have in choosing it. The relations imposed on D_F are

- D_F is self-adjoint;
- $\{D_F, \gamma_F\} = 0$;
- $[D_F, J_F] = 0$;
- $[[D_F, a], b^0] = 0, \forall a, b \in \mathcal{A}_F$.

These relations are elaborated in Appendix B.2, where the calculations are done in detail. We find that

$$D_F = \begin{pmatrix} 0 & 0 & d & 0 \\ 0 & 0 & 0 & \overline{d} \\ \overline{d} & 0 & 0 & 0 \\ 0 & d & 0 & 0 \end{pmatrix}$$

in its most general form possible, where d is a complex symmetric 2×2 matrix. This means that there are still three complex free parameters undetermined.

3.5 Inner fluctuations

In Section 2.7 we have found that $\Omega^1_D(\mathcal{A})$ for an almost commutative manifold can be written as:

$$\Omega^1_D(\mathcal{A}) = \Omega^1_{\not{D}\otimes\mathbb{I}}(\mathcal{A}) + \gamma_5 \otimes \Omega^1_{D_F}(\mathcal{A}_F)$$

We now begin by determining the inner fluctuations for the finite part of this expression, i.e. $\gamma_5 \otimes \Omega^1_{D_F}(\mathcal{A}_F)$.

$$\Omega^1_{D_F}(\mathcal{A}_F) := \{\sum_j a_{Fj}[D_F, b_{Fj}] \mid a_{Fj}, b_{Fj} \in \mathcal{A}_F\}.$$

Proposition 3.1. *We have the following expression for $\sum_j a_{Fj}[D_F, b_{Fj}]$,*

$$\sum_j a_{Fj}[D_F, b_{Fj}] = \sum_j \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{Lj}(\overline{d}s_{Rj} - s_{Lj}\overline{d}) \\ 0 & 0 & 0 & 0 \\ 0 & q_{Rj}(ds_{Lj} - s_{Rj}d) & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sum_j q_{Lj}(\overline{d}s_{Rj} - s_{Lj}\overline{d}) \\ 0 & 0 & 0 & 0 \\ 0 & \sum_j q_{Rj}(ds_{Lj} - s_{Rj}d) & 0 & 0 \end{pmatrix},$$

with $q_{Rj}, q_{Lj}, s_{Rj}, s_{Lj} \in \mathbb{H}$ and d a complex 2×2 matrix.

Proof. See appendix B.3. □

So the selfadjoint elements of $\Omega^1_{D_F}(\mathcal{A}_{\mathcal{F}})$ are of the form:

$$A = A^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & \alpha^* & 0 & 0 \end{pmatrix},$$

with α a complex 2×2 matrix.

By Definition 2.26, we see that the fluctuated Dirac operator $D_A = D + A + JAJ^{-1}$. The contribution to D_A by $\Omega^1_{D_F}(\mathcal{A}_{\mathcal{F}})$ is therefore:

$$\begin{aligned} \gamma_5 \otimes A + J\gamma_5 \otimes AJ^{-1} &= \gamma_5 \otimes A + J_M\gamma_5 J_M^{-1} \otimes J_F A J_F^{-1} \\ &= \gamma_5 \otimes A + \gamma_5 \otimes J_F A J_F^{-1} \\ &= \gamma_5 \otimes (A + J_F A J_F^{-1}) \end{aligned}$$

In KO-dimension 6 $J_F = J_F^{-1}$, so we have

$$\begin{aligned} J_F A J_F^{-1} &= J_F A J_F \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} C \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & \alpha^* & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} C \\ &= \begin{pmatrix} 0 & 0 & \overline{\alpha} & 0 \\ 0 & 0 & 0 & 0 \\ \alpha^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

We end up with inner fluctuations of the form

$$A + J_F A J_F^{-1} = \begin{pmatrix} 0 & 0 & \overline{\alpha} & 0 \\ 0 & 0 & 0 & \alpha \\ \alpha^T & 0 & 0 & 0 \\ 0 & \alpha^* & 0 & 0 \end{pmatrix}.$$

We now proceed with determining the selfadjoint elements of $\Omega^1_{\not{D}\otimes\mathbb{I}}(\mathcal{A})$.

Proposition 3.2.

$$\Omega^1_{\not{D}\otimes\mathbb{I}}(\mathcal{A}) = \Big\{\gamma^\mu \otimes a_{F\mu} \mid a_{F\mu} \in \mathcal{A}_F\Big\}.$$

Proof. See appendix B.3. □

So the contributions to the fluctuated Dirac operator D_A by $\Omega^1_{\not{D}\otimes\mathbb{I}}(\mathcal{A})$ is $\gamma^\mu \otimes a_{F\mu} + J\gamma^\mu \otimes a_{F\mu} J^{-1}$.

3.6 The gauge group

Proposition 2.19 gives us a convenient way to determine the gauge group. We only have to determine $U(\mathcal{A})$ and $ker(Ad)$, their quotient group is then isomorphic to $\mathcal{G}(\mathcal{A})$.

So let us start with determining $U(\mathcal{A}_F) := \{u \in \mathcal{A}_F | u^*u = 1\}$, the unitary elements of the algebra. We will determine the unitary elements of $\mathbb{C}$, $\mathbb{H}_L$ and $\mathbb{H}_R$. Then $U(\mathcal{A}_F)$ is the product $U(\mathcal{A}_F) = U(\mathbb{C}) \times U(\mathbb{H}_L) \times U(\mathbb{H}_R)$, because the algebra $\mathcal{A}_F$ is just the direct sum of these components.

Lemma 3.3. *The group of unitary elements of the quaternions, $U(\mathbb{H})$, is isomorphic to $SU(2)$, $U(\mathbb{H}) \cong SU(2)$.*

Proof. The quaternions can be represented as complex 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

The hermitian conjugate of this matrix is given by

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}^* = \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix}.$$

The product of a quaternion with its conjugate is given by

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}\begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix} = \begin{pmatrix} |a|^2 + |b|^2 & 0 \\ 0 & |a|^2 + |b|^2 \end{pmatrix},$$

which is only equal to $\mathbb{I}$ when $|a|^2 + |b|^2 = 1$. So

$$U(\mathbb{H}) = \{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} |a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1\},$$

which is SU(2). □

Lemma 3.4. *The group of unitary elements of $\mathbb{C}$, $U(\mathbb{C})$, is isomorphic to U(1), $U(\mathbb{C}) \cong U(1)$.*

Proof. The complex numbers can be represented in polar coordinates, $re^{i\phi}$. A complex number multiplied by its complex conjugate gives

$$re^{i\phi}re^{-i\phi} = r^2$$

which is only equal to 1 when $|r| = 1$ and $r \geq 0$, because r is the radius. This means r must equal 1. So

$$U(\mathbb{C}) = \{z \in \mathbb{C} \mid |z| = 1\}$$

which is the circle group U(1). □

Corollary 3.5. *The unitary elements of the finite algebra* $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R$ *are given by* $U(\mathcal{A}_F) = U(1) \times SU(2)_L \times SU(2)_R$.

Proof. $\mathcal{A}_F$ is the direct sum of $\mathbb{C}$, $\mathbb{H}_L$ and $\mathbb{H}_R$. So $U(\mathcal{A}_F)$ is just the direct product of the unitary elements of $\mathbb{C}$, the unitary elements of $\mathbb{H}_L$ and the unitary elements of $\mathbb{H}_R$, i.e. $U(\mathcal{A}_F) = U(1) \times SU(2)_L \times SU(2)_R$. □

We will now determine $ker(\phi) := \{u \in U(\mathcal{A}) | uJuJ^{-1} = 1\}$. We see that this are the unitary elements of $\mathcal{A}_F$ with $uJuJ^{-1} = 1 \Rightarrow u = Ju^*J^{-1}$, thus the elements for which the left and right action coincide.

Proposition 3.6. $ker(\phi)$ *is isomorphic to the group* $\mathbb{Z}_2$.

Proof. Remember that we had for our left and right action the representations

$$\pi(\lambda, q_L, q_R) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & q_L & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & q_R \end{pmatrix}, \quad J\pi(\lambda, q_L, q_R)^* J^{-1} = \begin{pmatrix} q_L^T & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & q_R^T & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}.$$

So we must have $q_R = \lambda$ and $q_L = \lambda$. This means we are dealing with diagonal matrices. However we have to keep in mind that for our unitary elements the complex numbers and quaternions have norm 1, $|\lambda|^2 = 1$ and $|q_L|^2 = |q_R|^2 = 1$. The only elements that remain are $\mathbb{I}$ and $-\mathbb{I}$, which together comprise the group $\mathbb{Z}_2$. □

Proposition 3.7. *The gauge group of the finite algebra* $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R$ *is* $\mathcal{G}(\mathcal{A}_F) = (U(1) \times SU(2)_L \times SU(2)_R)/\mathbb{Z}_2$.

Proof. We notice that $U(\mathcal{A}_F) = U(1) \times SU(2)_L \times SU(2)_R$ and $ker(\phi) = \mathbb{Z}_2$. The result then simply follows from Proposition 2.19. □

We notice that this is not yet the gauge group $(U(1) \times SU(2))/\mathbb{Z}_2$ of the Electroweak theory. The pair of quaternions in the algebra throws a spanner in the works. We shall need to resolve this by either choosing another algebra or change the representation of one of the components of the algebra. The latter is what we will do in subsequent sections of this text.

3.7 Particle identification

We can identify the basis of $\mathcal{H}_F$ with the fermions in this electroweak particle model. First, we will determine the hypercharges of these fermions. To do this we compute the adjoint action of elements $\lambda \in U(1)$ on the basis vectors of $\mathcal{H}_F$ and determine the powers of λ that occur.

Definition 3.8. The adjoint action of an element $u \in \mathcal{A}_F$ on a vector v in $\mathcal{H}_F$ is given by uvu^*, i.e. left multiplication by u and right multiplication by u^*.

Before calculating the hypercharges we give the basis vectors convenient and suggestive names:

Definition 3.9.

$$\overline{\nu}_L = e_{\mathbf{1}\otimes\mathbf{2}_L^0,1}, \ \overline{e}_L = e_{\mathbf{1}\otimes\mathbf{2}_L^0,2}, \ \nu_L = e_{\mathbf{2}_L\otimes\mathbf{1}^0,1}, \ e_L = e_{\mathbf{2}_L\otimes\mathbf{1}^0,2},$$
$$\overline{\nu}_R = e_{\mathbf{1}\otimes\mathbf{2}_R^0,1}, \ \overline{e}_R = e_{\mathbf{1}\otimes\mathbf{2}_R^0,2}, \ \nu_R = e_{\mathbf{2}_R\otimes\mathbf{1}^0,1}, \ e_R = e_{\mathbf{2}_R\otimes\mathbf{1}^0,2}.$$

The adjoint action of the subgroup $U(1)$ of $U(\mathcal{A}_F)$ is given by uvu^*, for all these basis vectors, where the left action is

$$u = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and the right action

$$(u^*)^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \overline{\lambda} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \overline{\lambda} \end{pmatrix}.$$

Note that $\lambda \in U(1)$, so $\overline{\lambda} = \lambda^{-1}$. This then leads to the following:

$$\begin{aligned} u\overline{\nu}_L u^* &= \lambda\overline{\nu}_L & \qquad u\overline{e}_L u^* &= \lambda\overline{e}_L \\ u\nu_L u^* &= \lambda^{-1}\nu_L & \qquad ue_L u^* &= \lambda^{-1}e_L \\ u\overline{\nu}_R u^* &= \lambda\overline{\nu}_R & \qquad u\overline{e}_R u^* &= \lambda\overline{e}_R \\ u\nu_R u^* &= \lambda^{-1}\nu_R & \qquad ue_R u^* &= \lambda^{-1}e_R \end{aligned}$$

Now the powers of λ give us the hypercharges for the fermions:

	ν	e	$\overline{\nu}$	$\overline{e}$
Left handed	-1	-1	1	1
Right handed	-1	-1	1	1

We see that the hypercharges are not correct. One of the reasons is that our finite spectral triple has left-right symmetry, something we know from the Beta decay experiment [9] not to be true for a theory incorporating the weak force. We must somehow take this absence of left-right symmetry into account. Besides that we know that the right handed neutrinos, if they exist, do not interact and have hypercharge zero and might possibly be Majorana particles. The possibilities for a spectral triple that incorporates these properties are pursued in the next section.

4 New hypercharges

We know that the right handed neutrinos, if they exist at all, must have hypercharge zero. The hypercharges we established in the preceding section did not correspond to what we see in the experiments, we will therefore in this section adapt our spectral triple so that we do end up with the right hypercharges.

Let $v \in \mathcal{H}_F$ be a basisvector of $\mathcal{H}_F$. The hypercharge of the particle described by v is given by the power of λ produced by the adjoint action of the unitary subgroup $U(1)$ of $\mathcal{A}_F$. So when we look at the group homomorphism $\rho : U(1) \to \mathcal{A}_F$, then the adjoint action of $u = \rho(\lambda)$, $\lambda \in U(1)$, on v is $uvu^* = \lambda^k v$. Then the power k is the hypercharge of the particle described by v. So when v describes the right handed neutrino k needs to be zero.

To get an idea of what might be a way to construct a spectral triple with the correct hypercharges, we will first consider in what ways $U(1)$ can act on $\mathcal{H}_F$. We will do this by checking how the algebra component $\mathbb{C}$ can be embedded in the algebra component $\mathbb{H}$. The definition of an embedding is as follows.

Definition 4.1. Let $\mathcal{A}$ and $\mathcal{B}$ be algebra's. An embedding of $\mathcal{A}$ in $\mathcal{B}$ is an algebra homomorphism $\sigma : \mathcal{A} \to \mathcal{B}$, i.e. σ is a map such that $\sigma(a)\sigma(b) = \sigma(ab)$ and $\sigma(a) + \sigma(b) = \sigma(a + b)$ $\forall a, b \in \mathcal{A}$.

4.1 Embedding of $\mathbb{C}$ in $M_1(\mathbb{H}) = \mathbb{H}$

It is intuitively already clear that $\mathbb{C}$ can be embedded in $\mathbb{H}$. It might be instructive to first take a look at what the image of the imaginary number i, could possibly be. We know that, given an embedding σ of $\mathbb{C}$ in $\mathbb{H}$, we must have $\sigma(i)^2 = \sigma(i^2) = -\sigma(1) = -1$ by definition of a ring homomorphism with unity. So the complex number i must be projected onto a quaternion whose square is -1. In the following Lemma the quaternions whose square equals -1 are given.

Lemma 4.2. *Let $q \in \mathbb{H}$ be a quaternion. $q^2 = -1$ if and only if q lies on the unit sphere in the three dimensional completely imaginary space. In other words $q^2 = -1$ if and only if $|q| = 1$ and q is purely imaginary.*

Proof. $q \in \mathbb{H}$ can be written as $q = a+bi+cj+dk$. $q^2 = (a+bi+cj+dk)(a+bi+cj+dk) = (a^2 - b^2 - c^2 - d^2) + (ab + ba + cd - dc)i + (ac - bd + ca + db)j + (ad + bc - cb + da)k = (a^2 - b^2 - c^2 - d^2) + (2ab)i + (2ac)j + (2ad)k$. Thus for $q^2 = -1$ we must have

$$a^2 - b^2 - c^2 - d^2 = -1, \; 2ab = 0, \; 2ac = 0, \; 2ad = 0$$

Because of the last three equations either $a = 0$ or b, c and d all equal zero. In the second case we end up with $a^2 = -1$, but a is just a real number so this does not give us any solutions. However in the first case we end up with $-b^2 - c^2 - d^2 = -1$ in the first equation. This implies that if $q^2 = -1$ then $a = 0$, i.e. q is purely imaginary and $|q| = \sqrt{a^2 + b^2 + c^2 + d^2} = \sqrt{b^2 + c^2 + d^2} = 1$. □

Notice that when $\sigma : \mathbb{C} \to \mathbb{H}$ is a homomorphism and $\alpha \in \mathbb{C}$, then $\sigma(\alpha)$ can be decomposed as $\sigma(\alpha) = \sigma(\Re(\alpha)) + \sigma(\Im(\alpha))\sigma(i)$, by the fact that σ is a homomorphism.

Lemma 4.3. *Let $\rho : \mathbb{R} \to \mathbb{H}$ be a homomorphism and let $q \in \mathbb{H}$ be a purely imaginary quaternion with norm 1. Then σ defined by $\sigma(\alpha) := \rho(\Re(\alpha)) + \rho(\Im(\alpha))q$ is a homomorphism from $\mathbb{C}$ to $\mathbb{H}$.*

Proof. $\Re(\alpha\beta) = \Re(\alpha)\Re(\beta) - \Im(\alpha)\Im(\beta)$ and $\Im(\alpha\beta) = \Re(\alpha)\Im(\beta) + \Im(\alpha)\Re(\beta)$. So,

$$\begin{aligned}
\sigma(\alpha)\sigma(\beta) &= (\rho(\Re(\alpha)) + \rho(\Im(\alpha))q)(\rho(\Re(\beta)) + \rho(\Im(\beta))q) \\
&= \rho(\Re(\alpha))\rho(\Re(\beta)) - \rho(\Im(\alpha))\rho(\Im(\beta)) + \rho(\Re(\alpha))\rho(\Im(\beta))q + \rho(\Im(\alpha))\rho(\Re(\beta))q \\
&= \rho(\Re(\alpha)\Re(\beta) - \Im(\alpha)\Im(\beta)) + \rho(\Re(\alpha)\Im(\beta) + \Im(\alpha)\Re(\beta))q \\
&= \rho(\Re(\alpha\beta)) + \rho(\Im(\alpha\beta))q \\
&= \sigma(\alpha\beta),
\end{aligned}$$

where $q^2 = -\mathbb{I}$ and the fact that ρ is a homomorphism were used. Furthermore,

$$\begin{aligned}
\sigma(\alpha + \beta) &= \rho(\Re(\alpha + \beta)) + \rho(\Im(\alpha + \beta))q \\
&= \rho(\Re(\alpha)) + \rho(\Re(\beta)) + \rho(\Im(\alpha))q + \rho(\Im(\beta))q \\
&= (\rho(\Re(\alpha) + \rho(\Im(\alpha))q) + (\rho(\Re(\beta) + \rho(\Im(\beta))q) \\
&= \sigma(\alpha) + \sigma(\beta).
\end{aligned}$$

So σ is indeed a homomorphism from $\mathbb{C}$ to $\mathbb{H}$. □

We now come to the following lemma.

Lemma 4.4. *All the homomorphisms from $\mathbb{C}$ to $\mathbb{H}$ are given by the set*

$$\{\sigma : \mathbb{C} \to \mathbb{H} | \sigma(\alpha) := \rho(\Re(\alpha)) + \rho(\Im(\alpha))q,\ q \in \mathbb{H},\ q^2 = -1,\ |q| = 1 \\ \text{and } \rho : \mathbb{R} \to \mathbb{H},\ \text{a homomorphism}\}$$

Proof. We already know that a homomorphism of the form $\rho(\Re(\alpha)) + \rho(\Im(\alpha))q$, $q \in \mathbb{H}$, $q^2 = -1$, $|q| = 1$ and $\rho : \mathbb{R} \to \mathbb{H}$ indeed is a homomorphism form $\mathbb{C}$ to $\mathbb{H}$.

We also know that, when σ is a homomorphism from $\mathbb{C}$ to $\mathbb{H}$, $\sigma(i)^2 = -1$, because $0 = \sigma(0) = \sigma(1 + (-1)) = \sigma(1) + \sigma(-1)$ implies $\sigma(i)^2 = \sigma(i^2) = \sigma(-1) = -1$. Let $\alpha \in \mathbb{C}$ we can decompose $\sigma(\alpha)$ as $\sigma(\Re(\alpha)) + \sigma(\Im(\alpha))\sigma(i)$. Therefore every homomorphism of $\mathbb{C}$ to $\mathbb{H}$ is of the form stated above with ρ the restriction of σ to $\mathbb{R}$ in $\mathbb{C}$. □

When we determine the various embeddings of $\mathbb{R}$ in $\mathbb{H}$ this set will be completely specified.

Lemma 4.5. *The only possible embedding ρ of $\mathbb{R}$ in $\mathbb{H}$ is defined by $\rho(a) := a + 0i + 0j + 0k = a \ \ \forall a \in \mathbb{R}$.*

Proof. Let ρ be an embedding of $\mathbb{R}$ in $\mathbb{H}$ and $a \in \mathbb{R}$. The image of a under ρ is a quaternion. Let's write most general $\rho(a) = \rho_1(a) + \rho_2(a)i + \rho_3(a)j + \rho_4(a)k$ and $\rho(b) = \rho_1(b) + \rho_2(b)i + \rho_3(b)j + \rho_4(b)k$. The fact that $\rho(a + b) = \rho(a) + \rho(b)$ gives us that $\rho_{1,2,3,4}$ must all be linear or zero. So for an element $a \in \mathbb{R}$ we have $\rho(a) = \lambda_1 a + \lambda_2 ai + \lambda_3 aj + \lambda_4 ak$, with $\lambda_{1,2,3,4} \in \mathbb{R}$ independent of a. Furthermore we must have

$$\begin{aligned}
\rho(a)\rho(b) &= (\lambda_1 a + \lambda_2 ai + \lambda_3 aj + \lambda_4 ak)(\lambda_1 b + \lambda_2 bi + \lambda_3 bj + \lambda_4 bk) \\
&= (\lambda_1\lambda_1 - \lambda_2\lambda_2 - \lambda_3\lambda_3 - \lambda_4\lambda_4)ab \\
&\quad + (\lambda_1\lambda_2 + \lambda_2\lambda_1 + \lambda_3\lambda_4 - \lambda_4\lambda_3)abi \\
&\quad + (\lambda_1\lambda_3 + \lambda_3\lambda_1 + \lambda_4\lambda_2 - \lambda_2\lambda_4)abj \\
&\quad + (\lambda_1\lambda_4 + \lambda_4\lambda_1 + \lambda_2\lambda_3 - \lambda_3\lambda_2)abk \\
&= \lambda_1 ab + \lambda_2 abi + \lambda_3 abj + \lambda_4 abk = \rho(ab).
\end{aligned}$$

This implies that either $\lambda_2 = \lambda_3 = \lambda_4 = 0$ and $\lambda_1 = 1$, or $\lambda_1 = \frac{1}{2}$ and $\lambda_2^2 + \lambda_3^2 + \lambda_4^2 = -\frac{1}{4}$. The latter has no real solutions, so we must have $\lambda_2 = \lambda_3 = \lambda_4 = 0$ and $\lambda_1 = 1$. □

Example 4.6. *A common embedding of* $\mathbb{H}$ *in* $M_2(\mathbb{C})$ *is*

$$a+bi+cj+dk \mapsto \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}.$$

An obvious embedding of $\mathbb{C}$ *in* $\mathbb{H}$ *then, after* $\mathbb{H}$ *is embedded again in* $M_2(\mathbb{C})$*, is*

$$\alpha = a+bi \mapsto \begin{pmatrix} a+bi & 0 \\ 0 & a-bi \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix} \ \forall \alpha \in \mathbb{C}.$$

This embedding of $\mathbb{C}$ *in* $\mathbb{H}$ *(actually in* $M_2(\mathbb{C})$*) is an example of a homomorphism from the set in Corollary 4.4. Where*

$$q = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ and } \rho(a) := \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \ \forall a \in \mathbb{R}:$$

$$\sigma(\alpha) = \rho(\Re(\alpha)) + \rho(\Im(\alpha))q = \begin{pmatrix} \Re(\alpha) & 0 \\ 0 & \Re(\alpha) \end{pmatrix} + \begin{pmatrix} \Im(\alpha) & 0 \\ 0 & \Im(\alpha) \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$= \begin{pmatrix} \Re(\alpha)+\Im(\alpha)i & 0 \\ 0 & \Re(\alpha)-\Im(\alpha)i \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix}.$$

We see that the other possible embeddings of $\mathbb{C}$ *in the matrix representation of the quaternions are*

$$\alpha = a+bi \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + b\begin{pmatrix} icos(\theta) & sin(\theta)e^{i\phi} \\ -sin(\theta)e^{-i\phi} & -icos(\theta) \end{pmatrix},\ q = \begin{pmatrix} icos(\theta) & sin(\theta)e^{i\phi} \\ -sin(\theta)e^{-i\phi} & -icos(\theta) \end{pmatrix}$$

with θ *and* ϕ *parametrizing the three dimensional unit sphere on which* $|q|^2 = 1$.

4.2 Different hypercharges

After this digression on the embedding of $\mathbb{C}$ in $\mathbb{H}$ we return to our main focus, the hypercharge. We will now introduce a different action of $U(1)$ on the Hilbert space. Recall that our finite spectral triple was specified by the following data.

$$(\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R, \mathcal{H}_F = (\mathbf{1} \otimes \mathbf{2}_L^0) \oplus (\mathbf{2}_L \otimes \mathbf{1}^0) \oplus (\mathbf{1} \otimes \mathbf{2}_R^0) \oplus (\mathbf{2}_R \otimes \mathbf{1}^0), D_F, J_F, \gamma_F)$$

Here J_F and γ_F are as defined in Section 3.2 and D_F is as determined in Section 3.4. The adjoint action uvu^* of $u \in U(1)$ on $v \in \mathcal{H}_F$ we had was with u given by

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We will now propose a different action of $U(1)$ on $\mathcal{H}_F$ that does give the right hypercharges. We have seen that a possible embedding of $\mathbb{C}$ in $\mathbb{H}$ is given by:

$$\alpha = a+bi \mapsto \begin{pmatrix} a+bi & 0 \\ 0 & a-bi \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix} \ \forall \alpha \in \mathbb{C},$$

where we again use the 2×2 matrix representation of the quaternions. We use this embedding to define a new action of $U(1)$ on $\mathcal{H}_F$.

Definition 4.7. We define an action of $U(1)$ as the subgroup of $U(\mathcal{A}_F)$ on $\mathcal{H}_F$ by the matrices of the form

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda_R \end{pmatrix}$$

where $\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, $\lambda_R = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix}$, $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\lambda \in \mathbb{C}$ is a complex number with norm 1.

Thus in this action of $U(1)$ on $\mathcal{H}_F$ there is another component on the diagonal unequal to 1, namely λ_R. Here λ_R has the form of the forementioned embedding of $\mathbb{C}$ in $\mathbb{H}$. By only using this action of U(1) on the quaternions for the $\mathbb{H}_R$ algebra component but not for the $\mathbb{H}_L$ algebra component, we incorporate the left-right asymmetry of the Electroweak theory.

This new action of $U(1)$ gives us a different notion of charge. We can now determine the charges of the different particles in this theory. In Definition 4.7 we have only given the left action of an element $u \in U(1)$. With J_F, the right action of the elements can be determined which turns out to be:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \overline{\lambda} & 0 & 0 \\ 0 & 0 & \overline{\lambda}_R & 0 \\ 0 & 0 & 0 & \overline{\lambda} \end{pmatrix},$$

where the bar denotes complex conjugation. Remember that for complex numbers with norm 1, $\overline{\lambda} = \lambda^{-1}$, so the product of the left action of u and the right action of u^* gives the following diagonal 8×8 matrix.

$$\begin{pmatrix} \lambda^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda^0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^{-2} \end{pmatrix}.$$

So we end up with the following hypercharges. These indeed correspond to those of the Electroweak model, see [10].

vector/particle	hypercharge
$e_{\mathbf{1}\otimes\mathbf{2}_L^0,1} = \overline{\nu}_L$	1
$e_{\mathbf{1}\otimes\mathbf{2}_L^0,2} = \overline{e}_L$	1
$e_{\mathbf{2}_L\otimes\mathbf{1}^0,1} = \nu_L$	-1
$e_{\mathbf{2}_L\otimes\mathbf{1}^0,2} = e_L$	-1
$e_{\mathbf{1}\otimes\mathbf{2}_R^0,1} = \overline{\nu}_R$	0
$e_{\mathbf{1}\otimes\mathbf{2}_R^0,2} = \overline{e}_R$	2
$e_{\mathbf{2}_R\otimes\mathbf{1}^0,1} = \nu_R$	0
$e_{\mathbf{2}_R\otimes\mathbf{1}^0,2} = e_R$	-2

4.3 Fermionic action in new representation and algebra

We want to describe the theory of the Electroweak interaction and we know the corresponding gauge group is $U(1) \times SU(2)$. The gauge group we started out with was $U(1) \times SU(2)_L \times SU(2)_R$, which did not describe the particles of the Electroweak theory. We now try to resolve this problem by representing the algebra in a new way. We do this whilst keeping in mind last section's results, in order to get the right hypercharges.

The old representation of the algebra $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R$ was the following.

$$\text{For } a \in \mathcal{A}_F \text{ we have } a = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & q_L & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & q_R \end{pmatrix}, \text{ with } \lambda \in \mathbb{C} \text{ and } q_L, q_R \in \mathbb{H}.$$

In this representation λ is a diagonal 2×2 matrix with both diagonal components equal to the complex number $\lambda \in \mathbb{C}$, whereas q_L and q_R are quaternions represented as 2×2 complex matrices. This representation gave us the wrong gauge group. Let's introduce the new representation.

Definition 4.8. For an element $a \in \mathcal{A}_F = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R$, we have the following matrix representation acting on $\mathcal{H}_F$,

$$a = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & q_L & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda_R \end{pmatrix}.$$

This representation differs from the old representation in the last diagonal component, where q_R is replaced by:

$$\lambda_R = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix}.$$

Here $\lambda \in \mathbb{C}$ is the same complex number as in the other λ's on the diagonal.

We see that this is in fact not a representation of the algebra $\mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R$ but of the algebra $\mathbb{C} \oplus \mathbb{H}_L$, because $\lambda, \lambda_R \in \mathbb{C}$ and $q_L \in \mathbb{H}_L$. So from now on we work with the algebra $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H}_L$ and the aforementioned representation of this algebra on $\mathcal{H}_F$.

This new representation and algebra cause many things to alter, for example the finite Dirac operator and the gauge group. We now continue by investigating what the effects of this new representation and algebra are on the other aspects of our spectral triple and what this implies physically.

4.4 Gauge group

We immediately see from Section 3.6, that $U(\mathcal{A}) = U(\mathbb{C}) \times U(\mathbb{H}_L)$ instead of $U(\mathcal{A}) = U(\mathbb{C}) \times U(\mathbb{H}_L) \times U(\mathbb{H}_R)$. The $ker(Ad)$ in Proposition 3.6 and 3.7 is still isomorphic to the group $\mathbb{Z}_2$. This is because we now have the following right and left action

$$\pi(\lambda, q_L, q_R) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & q_L & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda_R \end{pmatrix}, \quad J\pi(\lambda, q_L, q_R)^* J^{-1} = \begin{pmatrix} q_L^T & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda_R & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix},$$

for which the proof of Proposition 3.6 still holds. So we have the following slightly altered version of Proposition 3.6.

Proposition 4.9. *The gauge group of the finite algebra* $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H}_L$ *is given by the quotient of* $U(1) \times SU(2)$ *and* $\mathbb{Z}_2$*, i.e.* $\mathcal{G}(\mathcal{A}_F) = (U(1) \times SU(2))/\mathbb{Z}_2$.

Proof. As noticed above we have $U(\mathcal{A}_F) = U(1) \times SU(2)$ and $ker(Ad) = \mathbb{Z}_2$. The result then simply follows from Proposition 2.19. □

We see that with this new algebra we do end up with the gauge group of the Electroweak theory. We now turn our attention to the finite Dirac operator and will see how it changes because of the change of algebra and representation.

4.5 Finite Dirac operator

Some of the things the new algebra and representation do not alter are J_F and γ_F so we still have

$$J_F = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} C, \qquad \gamma_F = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The relations imposed on D_F are

- D_F is self-adjoint;
- $\{D_F, \gamma_F\} = 0$;
- $[D_F, J_F] = 0$;
- $[[D_F, a], b^0] = 0, \forall a, b \in \mathcal{A}_F$.

We see that a different algebra $\mathcal{A}_F$ and representation of it only have an effect on the last of these relations. So we pick up the calculation from there and determine D_F.

The first three relations already bring D_F back to

$$D_F = \begin{pmatrix} 0 & d_2 & d_3 & 0 \\ \overline{d}_2 & 0 & 0 & \overline{d}_3 \\ \overline{d}_3 & 0 & 0 & d_9 \\ 0 & d_3 & \overline{d}_9 & 0 \end{pmatrix},$$

with all the d's symmetric 2×2 matrices. When we slightly alter the calculation of the last relation by substituting $\lambda_R \in \mathbb{C}$ for $q_R \in \mathbb{H}_R$ and $\mu_R \in \mathbb{C}$ for $s_R \in \mathbb{H}_R$ we find

$$[[D_F, a], b^0] = \begin{pmatrix} 0 & (\mu - s_L^T)d_2(q_L - \lambda) & 0 & 0 \\ (\lambda - q_L)\overline{d}_2(s_L^T - \mu) & 0 & 0 & 0 \\ 0 & 0 & 0 & (\mu - \mu_R)d_9(\lambda_R - \lambda) \\ 0 & 0 & (\lambda - \lambda_R)\overline{d}_9(\mu_R - \mu) & 0 \end{pmatrix}$$

From this we can again conclude that $d_2 = 0$, but not for d_9 this time:

$$\begin{aligned} (\lambda - \lambda_R)\overline{d}_9(\mu_R - \mu) &= \begin{pmatrix} 0 & 0 \\ 0 & \lambda - \overline{\lambda} \end{pmatrix} \begin{pmatrix} \overline{d_{9_{11}}} & \overline{d_{9_{12}}} \\ \overline{d_{9_{12}}} & \overline{d_{9_{22}}} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \overline{\mu} - \mu \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & \lambda - \overline{\lambda} \end{pmatrix} \begin{pmatrix} 0 & \overline{d_{9_{12}}}(\overline{\mu} - \mu) \\ 0 & \overline{d_{9_{22}}}(\overline{\mu} - \mu) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & (\lambda - \overline{\lambda})\overline{d_{9_{22}}}(\overline{\mu} - \mu) \end{pmatrix}. \end{aligned}$$

For all $a, b \in \mathcal{A}_F$ we must have $[[D_F, a], b^0] = 0$, so this means that $d_{9_{22}} = 0$. Furthermore we have

$$
\begin{aligned}
(\mu - \mu_R) d_4 (\lambda_R - \lambda) &= \begin{pmatrix} 0 & 0 \\ 0 & \mu - \overline{\mu} \end{pmatrix} \begin{pmatrix} d_{9_{11}} & d_{9_{12}} \\ d_{9_{12}} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \overline{\lambda} - \lambda \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & \mu - \overline{\mu} \end{pmatrix} \begin{pmatrix} 0 & d_{9_{12}}(\overline{\lambda} - \lambda) \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}
$$

So there are no further restrictions on the 2×2 matrix d_9. This results in the following form 8×8 for D_F:

$$
D_F = \begin{pmatrix}
0 & 0 & 0 & 0 & f_1 & f_2 & 0 & 0 \\
0 & 0 & 0 & 0 & f_2 & f_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \overline{f_1} & \overline{f_2} \\
0 & 0 & 0 & 0 & 0 & 0 & \overline{f_2} & \overline{f_3} \\
\overline{f_1} & \overline{f_2} & 0 & 0 & 0 & 0 & f_4 & f_5 \\
\overline{f_2} & \overline{f_3} & 0 & 0 & 0 & 0 & f_5 & 0 \\
0 & 0 & f_1 & f_2 & \overline{f_4} & \overline{f_5} & 0 & 0 \\
0 & 0 & f_2 & f_3 & \overline{f_5} & 0 & 0 & 0
\end{pmatrix}.
$$

We have renamed the parameters and have two extra parameters f_4 and f_5 which in the old representation where zero. This finite unfluctuated Dirac operator has seven complex parameters. When we calculate the fermionic action from this D_F, we will find, besides the correct particle interactions, some interactions which are not observed in nature. To resolve this, we need to set $f_2 = f_5 = 0$, which will remove all the extra interactions from our theory. This arbitrary move is somewhat disappointing. It would have been nicer if the right interactions directly resulted form the mathematical relations imposed on D_F, which are needed for a mathematically consistent theory.

4.6 Inner fluctuations and fluctuated Dirac operator

This new finite Dirac operator causes the inner fluctuations to differ from the ones determined before. We will write the new Dirac operator with $f_2 = f_5 = 0$ in the following way.

$$
D_F = \begin{pmatrix} 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & \overline{d}_3 \\ \overline{d}_3 & 0 & 0 & d_9 \\ 0 & d_3 & \overline{d}_9 & 0 \end{pmatrix}, \quad d_3 = \begin{pmatrix} f_1 & 0 \\ 0 & f_3 \end{pmatrix}, \quad d_9 = \begin{pmatrix} f_4 & 0 \\ 0 & 0 \end{pmatrix}
$$

We calculate $a_{Fj}[D_F, b_{Fj}]$ and find:

$$
a_{Fj}[D_F, b_{Fj}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{Lj}(\overline{d}_3 \mu_{Rj} - s_{Lj} \overline{d}_3) \\ 0 & 0 & 0 & 0 \\ 0 & \lambda_{Rj}(d_3 s_{Lj} - \mu_{Rj} d_3) & 0 & 0 \end{pmatrix},
$$

$$
\lambda_{Rj}(d_3 s_{Lj} - \mu_{Rj} d_3) = \begin{pmatrix} (\lambda_j(\alpha_{Lj} - \mu_j)) f_1 & (\lambda_j \beta_{Lj}) f_1 \\ -(\overline{\lambda}_j \overline{\beta}_{Lj}) f_3 & (\overline{\lambda}_j(\overline{\alpha}_{Lj} - \overline{\mu}_j)) f_3 \end{pmatrix}.
$$

Finally, we find that

$$\sum_j a_{Fj}[D_F, b_{Fj}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sum_j q_{Lj}\overline{d}_3\mu_{Rj} - s_{Lj}\overline{d}_3) \\ 0 & 0 & 0 & 0 \\ 0 & \sum_j \lambda_{Rj}(d_3 s_{Lj} - \mu_{Rj} d_3) & 0 & 0 \end{pmatrix}.$$

We define the set $\Omega^1_{D_F}(\mathcal{A}_F) := \{\sum_j a_{Fj}[D_F, b_{Fj}] \mid a_{Fj}, b_{Fj} \in \mathcal{A}_F\}$, the self-adjoint elements of this set are the inner fluctuations. So an inner fluctuation, $A = A^* \in \Omega^1_{D_F}(\mathcal{A}_F)$ is an 8×8 matrix of the following form.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\sum_j \lambda_{Rj}(d_3 s_{Lj} - \mu_{Rj} d_3))^* \\ 0 & 0 & 0 & 0 \\ 0 & \sum_j \lambda_{Rj}(d_3 s_{Lj} - \mu_{Rj} d_3) & 0 & 0 \end{pmatrix}$$

We can rewrite such an inner fluctuation, by defining the following complex scalar.

$$\phi_1 \equiv \sum_j \overline{\lambda}_j(\overline{\alpha}_{Lj} - \overline{\mu}_j) \qquad\qquad \phi_4 \equiv \sum_j \overline{\lambda}_j \overline{\beta}_{Lj}$$

Then we can rewrite $\sum_j \lambda_{Rj}(d_3 s_{Lj} - \mu_{Rj} d_3)$

$$= \begin{pmatrix} (\sum_j \lambda_j(\alpha_{Lj} - \mu_j))f_1 & (\sum_j \lambda_j \beta_{Lj})f_1 \\ -(\sum_j \overline{\lambda}_j\overline{\beta}_{Lj})f_3 & (\sum_j \overline{\lambda}_j(\overline{\alpha}_{Lj} - \overline{\mu}_j))f_3 \end{pmatrix}$$
$$= \begin{pmatrix} \overline{\phi}_1 f_1 & \overline{\phi}_3 f_1 \\ -\phi_3 f_3 & \phi_1 f_3 \end{pmatrix}.$$

So after introducing these new parameters, an inner fluctuation is written as

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1 \\ 0 & 0 & 0 & 0 \\ 0 & \alpha_1^* & 0 & 0 \end{pmatrix}, \qquad \alpha_1 = \begin{pmatrix} \phi_1 \overline{f}_1 & -\overline{\phi}_3 \overline{f}_3 \\ \phi_3 \overline{f}_1 & \overline{\phi}_1 \overline{f}_3 \end{pmatrix}.$$

Now we have determined the inner fluctuations, we are able to compute the fluctuated Dirac operator. Recall that we had the expression $D_A = D + A + JAJ^{-1}$. So the finite inner fluctuations have the following contribution to D_A.

$$\begin{aligned} \gamma_5 \otimes A + J\gamma_5 \otimes AJ^{-1} &= \gamma_5 \otimes A + J_M \gamma_5 J_M^{-1} \otimes J_F A J_F^{-1} \\ &= \gamma_5 \otimes A + \gamma_5 \otimes J_F A J_F^{-1} \\ &= \gamma_5 \otimes (A + J_F A J_F^{-1}), \end{aligned}$$

with $A = A^* \in \Omega^1_{D_F}(\mathcal{A}_{\mathcal{F}})$. In KO-dimension 6 $J_F = J_F^{-1}$, so we have

$$\begin{aligned} J_F A J_F^{-1} &= J_F A J_F \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} C \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1 \\ 0 & 0 & 0 & 0 \\ 0 & \alpha_1^* & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} C \\ &= \begin{pmatrix} 0 & 0 & \overline{\alpha}_1 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha_1^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

where T denotes matrix-transposition. Finally we end up with the following finite fluctuated Dirac operator.

$$D_{F_A} = \begin{pmatrix} 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & \overline{d}_3 \\ \overline{d}_3 & 0 & 0 & d_9 \\ 0 & d_3 & \overline{d}_9 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \overline{\alpha}_1 & 0 \\ 0 & 0 & 0 & \alpha_1 \\ \alpha_1^T & 0 & 0 & 0 \\ 0 & \alpha_1^* & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & d_3 + \overline{\alpha}_1 & 0 \\ 0 & 0 & 0 & \overline{d}_3 + \alpha_1 \\ \overline{d}_3 + \alpha_1^T & 0 & 0 & d_9 \\ 0 & d_3 + \alpha_1^* & \overline{d}_9 & 0 \end{pmatrix},$$

with $d_3 = \begin{pmatrix} f_1 & 0 \\ 0 & f_3 \end{pmatrix}$, $d_9 = \begin{pmatrix} f_4 & 0 \\ 0 & 0 \end{pmatrix}$ and $\alpha_1 = \begin{pmatrix} \phi_1 \overline{f}_1 & -\overline{\phi}_3 \overline{f}_3 \\ \phi_3 \overline{f}_1 & \overline{\phi}_1 \overline{f}_3 \end{pmatrix}$.

4.7 Fermionic action

A different finite fluctuated Dirac operator will result in different physics. In this section we will determine the fermionic action and see which particles interact and how. Recall that we have already defined the fermionic action in Definition 2.27.

We can write D_A in terms of the canonical Dirac operator and finite Dirac operator, $D_A = \not{D}_A \otimes \mathbb{I} + \gamma_5 \otimes D_{F_A}$. So we have

$$S_f[A, \xi] := \frac{1}{2}\langle J\tilde{\xi}, D_A \tilde{\xi}\rangle = \frac{1}{2}\langle J\tilde{\xi}, (\not{D}_A \otimes \mathbb{I})\tilde{\xi}\rangle + \frac{1}{2}\langle J\tilde{\xi}, (\gamma_5 \otimes D_{F_A})\tilde{\xi}\rangle.$$

We will now proceed with the second term. The basis of the finite space is given by $(\overline{\nu}_L, \overline{e}_L, \nu_L, e_L, \overline{\nu}_R, \overline{e}_R, \nu_R, e_R)$. This is a consequence of the hypercharges and γ_F. Therefore we can write

$$\tilde{\xi} = \psi_R^\nu \otimes \overline{\nu}_L + \psi_R^e \otimes \overline{e}_L + \chi_L^\nu \otimes \nu_L + \chi_L^e \otimes e_L + \psi_L^\nu \otimes \overline{\nu}_R + \psi_L^e \otimes \overline{e}_R + \chi_R^\nu \otimes \nu_R + \chi_R^e \otimes e_R,$$

where the ψ and χ are spinors, where $\psi_L, \chi_L \in L^2(M,S)^+$ and $\psi_R, \chi_R \in L^2(M,S)^-$.

As a first step we calculate $(\gamma_5 \otimes D_{F_A})\tilde{\xi}$:

$$\begin{aligned}(\gamma_5 \otimes D_{F_A})\tilde{\xi} = {} & \gamma_5 \psi_R^\nu \otimes (((1+\phi_1)\overline{f}_1)\overline{\nu}_R + (-\overline{\phi}_3 \overline{f}_3)\overline{e}_R) \\ & + \gamma_5 \psi_R^e \otimes ((\phi_3 \overline{f}_1)\overline{\nu}_R + ((1+\overline{\phi}_1)\overline{f}_3)\overline{e}_R) \\ & + \gamma_5 \chi_L^\nu \otimes (((1+\overline{\phi}_1) f_1)\nu_R + (-\phi_3 f_3) e_R) \\ & + \gamma_5 \chi_L^e \otimes ((\overline{\phi}_3 f_1)\nu_R + ((1+\phi_1) f_3) e_R) \\ & + \gamma_5 \psi_L^\nu \otimes (((1+\overline{\phi}_1) f_1)\overline{\nu}_L + (\overline{\phi}_3 f_1)\overline{e}_L + (\overline{f}_4)\nu_R) \\ & + \gamma_5 \psi_L^e \otimes ((-\phi_3 f_3)\overline{\nu}_L + ((1+\phi_1) f_3)\overline{e}_L) \\ & + \gamma_5 \chi_R^\nu \otimes (((1+\phi_1)\overline{f}_1)\nu_L + (\phi_3 \overline{f}_1) e_L + (f_4)\overline{\nu}_R) \\ & + \gamma_5 \chi_R^e \otimes ((-\overline{\phi}_3 \overline{f}_3)\nu_L + ((1+\overline{\phi}_1)\overline{f}_3) e_L)\end{aligned}$$

Next, we let the charge conjugation operator J act on $\tilde{\xi}$:

$$\begin{aligned}J\tilde{\xi} = {} & J_M \psi_R^\nu \otimes \nu_L + J_M \psi_R^e \otimes e_L + J_M \chi_L^\nu \otimes \overline{\nu}_L + J_M \chi_L^e \otimes \overline{e}_L \\ & + J_M \psi_L^\nu \otimes \nu_R + J_M \psi_L^e \otimes e_R + J_M \chi_R^\nu \otimes \overline{\nu}_R + J_M \chi_R^e \otimes \overline{e}_R\end{aligned}$$

Using the expressions above we find that

$$
\begin{aligned}
\frac{1}{2}\langle J\tilde{\xi}, (\gamma_5 \otimes D_{F_A})\tilde{\xi}\rangle = \frac{1}{2}\big(&(1+\phi_1)\overline{f}_1(\langle J_M\psi_R^\nu|\gamma_5\chi_R^\nu\rangle + \langle J_M\chi_R^\nu|\gamma_5\psi_R^\nu\rangle) \\
&+ (1+\phi_1)f_3(\langle J_M\chi_L^e|\gamma_5\psi_L^e\rangle + \langle J_M\psi_L^e|\gamma_5\chi_L^e\rangle) \\
&+ (1+\overline{\phi}_1)f_1(\langle J_M\chi_L^\nu|\gamma_5\psi_L^\nu\rangle + \langle J_M\psi_L^\nu|\gamma_5\chi_L^\nu\rangle) \\
&+ (1+\overline{\phi}_1)\overline{f}_3(\langle J_M\chi_R^e|\gamma_5\psi_R^e\rangle + \langle J_M\psi_R^e|\gamma_5\chi_R^e\rangle) \\
&+ f_4\langle J_M\chi_R^\nu|\gamma_5\chi_R^\nu\rangle + \overline{f}_4\langle J_M\psi_L^\nu|\gamma_5\psi_L^\nu\rangle \\
&+ \phi_3\overline{f}_1(\langle J_M\psi_R^e|\gamma_5\chi_R^\nu\rangle + \langle J_M\chi_R^\nu|\gamma_5\psi_R^e\rangle) \\
&+ \overline{\phi}_3 f_1(\langle J_M\chi_L^e|\gamma_5\psi_L^\nu\rangle + \langle J_M\psi_L^\nu|\gamma_5\chi_L^e\rangle) \\
&- \phi_3 f_3(\langle J_M\chi_L^\nu|\gamma_5\psi_L^e\rangle + \langle J_M\psi_L^e|\gamma_5\chi_L^\nu\rangle) \\
&- \overline{\phi}_3\overline{f}_3(\langle J_M\chi_R^e|\gamma_5\psi_R^\nu\rangle + \langle J_M\psi_R^\nu|\gamma_5\chi_R^e\rangle)\big).
\end{aligned}
$$

In this expression we have complex mass parameters f_1 and f_3. We desire real parameters in order to obtain physical mass terms. We would also like to verify the expression found by comparing it with results in the literature, [6]. We therefore define the real masses m_ν and m_e in the following way:

$$f_1 =: i\frac{\sqrt{af(0)}}{\pi v}m_\nu, \quad f_3 =: i\frac{\sqrt{af(0)}}{\pi v}m_e.$$

This definition ensures that f_1 and f_3 are imaginary and we therefore indeed obtain physical mass terms. The factor $\sqrt{af(0)}/\pi v$ is a scaling factor used in [6], which we will adopt to make the comparison between our results and the results of [6] possible. In this factor, $a = |f_1|^2 + |f_3|^2$, f is a function which is used in the definition of the bosonic action and v is the vacuum expectation value of the Higgs field. This scaling is in order to normalize the kinetic terms of the Electroweak theory. More information on this scaling constant can be found in [6]. Besides this we will also parametrize the inner fluctuations ϕ_1 and ϕ_3 as

$$\phi_1 + 1 =: \frac{\pi}{\sqrt{af(0)}}(v + h + i\phi^0), \quad \phi_3 =: \frac{\pi}{\sqrt{af(0)}}i\sqrt{2}\phi^-.$$

Furthermore we denote the complex conjugate of ϕ^- by ϕ^+. Finally, we will take f_4 to be imaginary by introducing a real Majorana mass m_R:

$$f_4 =: -im_R.$$

Let us write the fermionic action in this new parametrization. By simply substituting we find:

$$
\begin{aligned}
\frac{1}{2}\langle J\tilde{\xi},(\gamma_5\otimes D_{F_A})\tilde{\xi}\rangle =&\frac{1}{2}\Big(\Big(\frac{\phi_0}{v}-\big(1+\frac{h}{v}\big)i\Big)m_\nu\big(\langle J_M\psi^\nu_R|\gamma_5\chi^\nu_R\rangle+\langle J_M\chi^\nu_R|\gamma_5\psi^\nu_R\rangle\big)\\
&+\Big(-\frac{\phi_0}{v}+\big(1+\frac{h}{v}\big)i\Big)m_e\big(\langle J_M\chi^e_L|\gamma_5\psi^e_L\rangle+\langle J_M\psi^e_L|\gamma_5\chi^e_L\rangle\big)\\
&+\Big(\frac{\phi_0}{v}+\big(1+\frac{h}{v}\big)i\Big)m_\nu\big(\langle J_M\chi^\nu_L|\gamma_5\psi^\nu_L\rangle+\langle J_M\psi^\nu_L|\gamma_5\chi^\nu_L\rangle\big)\\
&+\Big(-\frac{\phi_0}{v}-\big(1+\frac{h}{v}\big)i\Big)m_e\big(\langle J_M\chi^e_R|\gamma_5\psi^e_R\rangle+\langle J_M\psi^e_R|\gamma_5\chi^e_R\rangle\big)\\
&-im_R\langle J_M\chi^\nu_R|\gamma_5\chi^\nu_R\rangle+im_R\langle J_M\psi^\nu_L|\gamma_5\psi^\nu_L\rangle\\
&+\frac{\sqrt{2}}{v}\phi^-m_\nu\big(\langle J_M\psi^e_R|\gamma_5\chi^\nu_R\rangle+\langle J_M\chi^\nu_R|\gamma_5\psi^e_R\rangle\big)\\
&+\frac{\sqrt{2}}{v}\phi^+m_\nu\big(\langle J_M\chi^e_L|\gamma_5\psi^\nu_L\rangle+\langle J_M\psi^\nu_L|\gamma_5\chi^e_L\rangle\big)\\
&+\frac{\sqrt{2}}{v}\phi^-m_e\big(\langle J_M\chi^\nu_L|\gamma_5\psi^e_L\rangle+\langle J_M\psi^e_L|\gamma_5\chi^\nu_L\rangle\big)\\
&+\frac{\sqrt{2}}{v}\phi^+m_e\big(\langle J_M\chi^e_R|\gamma_5\psi^\nu_R\rangle+\langle J_M\psi^\nu_R|\gamma_5\chi^e_R\rangle\big)\Big).
\end{aligned}
$$

This we can further simplify by using the symmetry $\langle J_M\chi|\gamma_5\psi\rangle=\langle J_M\psi|\gamma_5\chi\rangle$ and using the projection operators $\frac{1}{2}(1\pm\gamma_5)$ to select the left- or right-handed spinors:

$$
\begin{aligned}
\frac{1}{2}\langle J\tilde{\xi},(\gamma_5\otimes D_{F_A})\tilde{\xi}\rangle =&i\big(1+\frac{h}{v}\big)\big(\langle J_M\psi^\nu|m_\nu\chi^\nu\rangle+\langle J_M\psi^e|m_e\chi^e\rangle\big)\\
&+\frac{\phi^0}{v}\big(\langle J_M\psi^\nu|\gamma_5 m_\nu\chi^\nu\rangle-\langle J_M\psi^e|\gamma_5 m_e\chi^e\rangle\big)\\
&+\frac{1}{\sqrt{2}v}\phi^-\big(\langle J_M\psi^e|m_e(1+\gamma_5)\chi^\nu\rangle-\langle J_M\psi^e|m_\nu(1-\gamma_5)\chi^\nu\rangle\big)\\
&+\frac{1}{\sqrt{2}v}\phi^+\big(\langle J_M\psi^\nu|m_\nu(1+\gamma_5)\chi^e\rangle-\langle J_M\psi^\nu|m_e(1-\gamma_5)\chi^e\rangle\big)\\
&+\frac{i}{2}\langle J_M\chi^\nu_R|m_R\chi^\nu_R\rangle+\frac{i}{2}\langle J_M\psi^\nu_L|m_R\psi^\nu_L\rangle.
\end{aligned}
$$

We can now identify each term in this fermionic action with a term in the expression given on page 77 of [6].

5 Conclusions

In our attempt to give a noncommutative geometry-theory of the electroweak interaction we first started out with the algebra $\mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R$. This algebra did not give the right gauge group and hypercharges. In order to build a spectral triple that would give the right hypercharges and would leave open the possibility of containing (Majorana) particles with hypercharge zero, we investigated what the options were for an embedding of $\mathbb{C}$ in $\mathbb{H}$. This different embedding was of interest to us because it showed the possibilities for a different notion of hypercharge.

The search for a different embedding of $\mathbb{C}$ in $\mathbb{H}$ indicated that a better representation of an element of the algebra $a \in \mathcal{A}_F$ on the Hilbert space $\mathcal{H}_F$ was

$$a = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & q_L & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda_R \end{pmatrix},$$

$$\text{with } \lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \lambda_R = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} \in \mathbb{C} \text{ and } q_L = \begin{pmatrix} \alpha_L & \beta_L \\ -\overline{\beta}_L & \overline{\alpha}_L \end{pmatrix} \in \mathbb{H}_L.$$

This meant that the algebra $\mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R$ was not the right algebra, but that we had to work with the algebra $\mathbb{C} \oplus \mathbb{H}_L$ instead. The algebra $\mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R$ did not incorporate the left-right asymmetry of the electroweak interaction and it was therefore unlikely to imply the right physics. The new algebra and its representation on the Hilbert space did incorporate the left-right asymmetry.

This new approach resulted in the following correct hypercharges.

vector/particle	hypercharge
$e_{\mathbf{1}\otimes\mathbf{2}_L^0,1} = \overline{\nu}_L$	1
$e_{\mathbf{1}\otimes\mathbf{2}_L^0,2} = \overline{e}_L$	1
$e_{\mathbf{2}_L\otimes\mathbf{1}^0,1} = \nu_L$	-1
$e_{\mathbf{2}_L\otimes\mathbf{1}^0,2} = e_L$	-1
$e_{\mathbf{1}\otimes\mathbf{2}_R^0,1} = \overline{\nu}_R$	0
$e_{\mathbf{1}\otimes\mathbf{2}_R^0,2} = \overline{e}_R$	2
$e_{\mathbf{2}_R\otimes\mathbf{1}^0,1} = \nu_R$	0
$e_{\mathbf{2}_R\otimes\mathbf{1}^0,2} = e_R$	-2

The new algebra also gave us the right gauge group:

$$\mathcal{G}(\mathcal{A}_F) = (U(1) \times SU(2))/\mathbb{Z}_2.$$

The changes we made to the algebra and its representation also had an effect on the finite Dirac operator. From the new Dirac operator we determined the electroweak fermionic action:

$$
\begin{aligned}
\frac{1}{2}\langle J\tilde{\xi},(\gamma_5\otimes D_{F_A})\tilde{\xi}\rangle =& i\big(1+\frac{h}{v}\big)\big(\langle J_M\psi^\nu|m_\nu\chi^\nu\rangle+\langle J_M\psi^e|m_e\chi^e\rangle\big)\\
&+\frac{\phi^0}{v}\big(\langle J_M\psi^\nu|\gamma_5 m_\nu\chi^\nu\rangle-\langle J_M\psi^e|\gamma_5 m_e\chi^e\rangle\big)\\
&+\frac{1}{\sqrt{2}v}\phi^-\big(\langle J_M\psi^e|m_e(1+\gamma_5)\chi^\nu\rangle-\langle J_M\psi^e|m_\nu(1-\gamma_5)\chi^\nu\rangle\big)\\
&+\frac{1}{\sqrt{2}v}\phi^+\big(\langle J_M\psi^\nu|m_\nu(1+\gamma_5)\chi^e\rangle-\langle J_M\psi^\nu|m_e(1-\gamma_5)\chi^e\rangle\big)\\
&+\frac{i}{2}\langle J_M\chi_R^\nu|m_R\chi_R^\nu\rangle+\frac{i}{2}\langle J_M\psi_L^\nu|m_R\psi_L^\nu\rangle.
\end{aligned}
$$

This expression for the fermionic action corresponds to the fermionic action of the Electroweak theory, which can be found in [11].

Due to time constraints the determination of the fermionic action was the end point of this thesis. However, it would be interesting to check if this spectral triple would also result in the correct bosonic action. This is left to future research.

A Proofs of results in Subsection 2.6 The gauge group

A.1 Proof of Proposition 2.19

Proof. The map Ad is surjective by the definition of $\mathcal{G}(\mathcal{A})$.

First we prove that Ad is a homomorphism.

$$\begin{aligned} Ad(u_1)Ad(u_2) &= u_1 J u_1 J^{-1} u_2 J u_2 J^{-1} \\ &= u_1 (u_1^*)^0 u_2 (u_2^*)^0 \\ &= u_1 u_2 (u_1^*)^0 (u_2^*)^0 \\ &= u_1 u_2 J u_1 J^{-1} J u_2 J^{-1} \\ &= u_1 u_2 J u_1 u_2 J^{-1} \\ &= Ad(u_1 u_2) \ \ \forall\ u_1, u_2 \in \mathcal{A}. \end{aligned}$$

where we used that $[u_1^0, u_2] = 0$.

Then we prove $U(\widetilde{\mathcal{A}}_J) \triangleleft U(\mathcal{A})$. $1 \in U(\widetilde{\mathcal{A}}_J)$ because $Ad(1) = 1$ (Ad is a homorphism). If $u_1, u_2 \in U(\widetilde{\mathcal{A}}_J)$ then $Ad(u_1u_2) = Ad(u_1)Ad(u_2) = 1 \times 1 = 1$, and thus $u_1u_2 \in U(\widetilde{\mathcal{A}}_J)$. If $u \in U(\widetilde{\mathcal{A}}_J)$ then $Ad(u^{-1}) = Ad(u)^{-1} = (1)^{-1} = 1$, and thus $u^{-1} \in U(\widetilde{\mathcal{A}}_J)$. Finally, for $u_1 \in U(\mathcal{A})$ and $u_2 \in U(\widetilde{\mathcal{A}}_J)$, $Ad(u_1u_2u_1^{-1}) = Ad(u_1)Ad(u_2)Ad(u_1)^{-1} = Ad(u_1)1Ad(u_1)^{-1} = Ad(u_1)Ad(u_1)^{-1} = 1$, and thus $u_1u_2u_1^{-1} \in U(\widetilde{\mathcal{A}}_J)$. This proves that $U(\widetilde{\mathcal{A}}_J) \triangleleft U(\mathcal{A})$.

Finally we prove that $U(\mathcal{A})/U(\widetilde{\mathcal{A}}_J)$ is isomorphic to $\mathcal{G}(\mathcal{A})$. Let us define the map $\overline{Ad} : U(\mathcal{A})/U(\widetilde{\mathcal{A}}_J) \to \mathcal{G}(\mathcal{A})$ by $\overline{Ad}(uU(\widetilde{\mathcal{A}}_J)) = Ad(u)$ ($uU(\widetilde{\mathcal{A}}_J)$ is de equivalence class represented by u). $\overline{Ad}$ is an isomorphism, i.e. a bijective homomorphism and therefore $U(\mathcal{A})/U(\widetilde{\mathcal{A}}_J) \cong \mathcal{G}(\mathcal{A})$. □

A.2 Proof of Proposition 2.22

Proof. $a \in U(\mathcal{A})$, $a = \alpha(x) \otimes a_F = \mathbb{I} \otimes \alpha a_F(x) = \mathbb{I} \otimes a'_F(x)$. $\alpha^*\alpha = 1$ and $a_F^* a_F = 1$, so $a'^*_F a'_F = a_F^* \alpha^* \alpha a_F = 1$. Similarly we have $a'_F a'^*_F = 1$. This shows that $a'_F \in U(\mathcal{A}_F)$ and therefore $U(\mathcal{A}) = C^\infty(M, U(\mathcal{A}_F))$. □

A.3 Proof of Proposition 2.23

Proof. $U(\widetilde{\mathcal{A}}_J) = \{a \in \mathcal{A} | a = a^0\}$. $a \in U(\widetilde{\mathcal{A}}_J)$, $a = \alpha \otimes a_F$, $a^0 = Ja^*J^{-1} = J_M \alpha^* J_M^{-1} \otimes J_F a_F^* J_F^{-1} = \alpha^0 \otimes a_F^0$, $\alpha^0 \otimes a_F^0 = \alpha \otimes a_F$.

$$\begin{aligned} \alpha(x) \otimes a_F &= \mathbb{I} \otimes \alpha a_F(x) \\ &= \mathbb{I} \otimes a'_F(x) \end{aligned}$$

and

$$\begin{aligned} \alpha^0(x) \otimes a_F^0 &= \mathbb{I} \otimes \alpha^0 a_F^0(x) \\ &= \mathbb{I} \otimes a_F^0 \alpha^0(x) \\ &= \mathbb{I} \otimes J_F a_F^* J_F^{-1} J_F \alpha^*(x) J_F^{-1} \\ &= \mathbb{I} \otimes J_F a_F^* \alpha^*(x) J_F^{-1} \\ &= \mathbb{I} \otimes J_F (\alpha a_F(x))^* J_F^{-1} \\ &= \mathbb{I} \otimes (\alpha a_F(x))^0 = \mathbb{I} \otimes a'^0_F(x). \end{aligned}$$

So $\mathbb{I} \otimes a'^0_F = \mathbb{I} \otimes a'_F,\ a'_F \in U((\widetilde{\mathcal{A}}_F)_J)\ \forall\ a \in U(\widetilde{\mathcal{A}}_J) \ \Rightarrow\ U(\widetilde{\mathcal{A}}_J) = C^\infty(M, U((\widetilde{\mathcal{A}}_F)_J))$. □

B Electroweak: KO-dimension, Dirac Operator and Inner Fluctuations

In this appendix we determine the KO-dimension, the Dirac operator and the inner fluctuations of the finite spectral triple of Section 3.

B.1 KO-dimension

$$\gamma_F J_F = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} C$$

$$J_F \gamma_F = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} C$$

We see that J_F and γ_F anticommute, i.e. $J_F\gamma_F = \epsilon''\gamma_F J_F$ and $\epsilon'' = -1$.

$$J_F^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So besides $\epsilon'' = -1$, $\epsilon = 1$, which means the KO-dimension equals 6.

B.2 Determining the Dirac operator

Notice that because D_F is self-adjoint our starting point is:

$$D_F = \begin{pmatrix} d_1 & d_2 & d_3 & d_4 \\ d_2^* & d_5 & d_6 & d_7 \\ d_3^* & d_6^* & d_8 & d_9 \\ d_4^* & d_7^* & d_9^* & d_{10} \end{pmatrix}.$$

Our first step in determining the Dirac operator will be by demanding $\{D_F, \gamma_F\} = 0$. This will already significantly reduce the degrees of freedom for D_F:

$$\begin{aligned} \{D_F, \gamma_F\} &= \begin{pmatrix} d_1 & -d_2 & -d_3 & d_4 \\ d_2^* & -d_5 & -d_6 & d_7 \\ d_3^* & -d_6^* & -d_8 & d_9 \\ d_4^* & -d_7^* & -d_9^* & d_{10} \end{pmatrix} + \begin{pmatrix} d_1 & d_2 & d_3 & d_4 \\ -d_2^* & -d_5 & -d_6 & -d_7 \\ -d_3^* & -d_6^* & -d_8 & -d_9 \\ d_4^* & d_7^* & d_9^* & d_{10} \end{pmatrix} \\ &= \begin{pmatrix} 2d_1 & 0 & 0 & 2d_4 \\ 0 & -2d_5 & -2d_6 & 0 \\ 0 & -2d_6^* & -2d_8 & 0 \\ 2d_4^* & 0 & 0 & 2d_{10} \end{pmatrix} \end{aligned}$$

This anticommutator only vanishes when $d_1 = d_4 = d_5 = d_6 = d_8 = d_{10} = 0$. This reduces D_F to:

$$\begin{pmatrix} 0 & d_2 & d_3 & 0 \\ d_2^* & 0 & 0 & d_7 \\ d_3^* & 0 & 0 & d_9 \\ 0 & d_7^* & d_9^* & 0 \end{pmatrix}.$$

Next, we impose the commutationrelation $[D_F, J_F] = 0$:

$$D_F J_F = \begin{pmatrix} d_2 & 0 & 0 & d_3 \\ 0 & d_2^* & d_7 & 0 \\ 0 & d_3^* & d_9 & 0 \\ d_7^* & 0 & 0 & d_9^* \end{pmatrix} C$$

$$J_F D_F = \begin{pmatrix} d_2^T & 0 & 0 & \overline{d}_7 \\ 0 & \overline{d}_2 & \overline{d}_3 & 0 \\ 0 & d_7^T & d_9^T & 0 \\ d_3^T & 0 & 0 & \overline{d}_9 \end{pmatrix} C$$

From this we can conclude $d_2 = d_2^T$, $d_3 = d_3^T$, $d_7 = \overline{d}_3$ and $d_9 = d_9^T$. So at this point we have

$$D_F = \begin{pmatrix} 0 & d_2 & d_3 & 0 \\ \overline{d}_2 & 0 & 0 & \overline{d}_3 \\ \overline{d}_3 & 0 & 0 & d_9 \\ 0 & d_3 & \overline{d}_9 & 0 \end{pmatrix},$$

with all of the d's symmetric.

The final restriction on D_F is the commutation relation $[[D_F, a], b^0] = 0 \;\forall\; a, b^0 \in \mathcal{A}_F$, the left and right action on $\mathcal{H}_F$ respectively. To evaluate this restriction we first calculate $[D_F, a]$. We have to keep in mind that the matrix-elements are 2×2 matrices and do not commute in general.

$$\begin{aligned} D_F a - a D_F &= \begin{pmatrix} 0 & d_2 q_L & d_3 \lambda & 0 \\ d_2^* \lambda & 0 & 0 & d_3^* q_R \\ d_3^* \lambda & 0 & 0 & d_9 q_R \\ 0 & d_3 q_L & d_9^* \lambda & 0 \end{pmatrix} - \begin{pmatrix} 0 & \lambda d_2 & \lambda d_3 & 0 \\ q_L d_2^* & 0 & 0 & q_L d_3^* \\ \lambda d_3^* & 0 & 0 & \lambda d_9 \\ 0 & q_R d_3 & q_R d_9^* & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & d_2(q_L - \lambda) & 0 & 0 \\ (\lambda - q_L) d_2^* & 0 & 0 & d_3^* q_R - q_L d_3^* \\ 0 & 0 & 0 & d_9(q_R - \lambda) \\ 0 & d_3 q_L - q_R d_3 & (\lambda - q_R) d_9^* & 0 \end{pmatrix} \end{aligned}$$

Next we calculate $[D_F, a] b^0$ and $b^0 [D_F, a]$:

$$[D_F, a] b^0 = \begin{pmatrix} 0 & d_2(q_L - \lambda)\mu & 0 & 0 \\ (\lambda - q_L) d_2^* s_L^T & 0 & 0 & (d_3^* q_R - q_L d_3^*)\mu \\ 0 & 0 & 0 & d_9(q_R - \lambda)\mu \\ 0 & (d_3 q_L - q_R d_3)\mu & (\lambda - q_R) d_9^* s_R^T & 0 \end{pmatrix}$$

$$b^0[D_F,a] = \begin{pmatrix} 0 & s_L^T d_2(q_L-\lambda) & 0 & 0 \\ \mu(\lambda-q_L)d_2^* & 0 & 0 & \mu(d_3^*q_R - q_L d_3^*) \\ 0 & 0 & 0 & s_R^T d_9(q_R-\lambda) \\ 0 & \mu(d_3 q_L - q_R d_3) & \mu(\lambda-q_R)d_9^* & 0 \end{pmatrix}$$

$$\begin{aligned}
[[D_F,a],b^0] &= \begin{pmatrix} 0 & d_2(q_L-\lambda)\mu & 0 & 0 \\ (\lambda-q_L)d_2^* s_L^T & 0 & 0 & (d_3^*q_R - q_L d_3^*)\mu \\ 0 & 0 & 0 & d_9(q_R-\lambda)\mu \\ 0 & (d_3 q_L - q_R d_3)\mu & (\lambda-q_R)d_9^* s_R^T & 0 \end{pmatrix} \\
&- \begin{pmatrix} 0 & s_L^T d_2(q_L-\lambda) & 0 & 0 \\ \mu(\lambda-q_L)d_2^* & 0 & 0 & \mu(d_3^*q_R - q_L d_3^*) \\ 0 & 0 & 0 & s_R^T d_9(q_R-\lambda) \\ 0 & \mu(d_3 q_L - q_R d_3) & \mu(\lambda-q_R)d_9^* & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & (\mu - s_L^T)d_2(q_L-\lambda) & 0 & 0 \\ (\lambda-q_L)d_2^*(s_L^T-\mu) & 0 & 0 & 0 \\ 0 & 0 & 0 & (\mu - S_R^T)d_9(q_R-\lambda) \\ 0 & 0 & (\lambda-q_R)d_9^*(s_R^T-\mu) & 0 \end{pmatrix}
\end{aligned}$$

We see that $[[D_F,a],b^0] = 0 \; \forall \; a, b^0 \in \mathcal{A}_F$ iff $d_2 = d_9 = 0$. So

$$D_F = \begin{pmatrix} 0 & 0 & d & 0 \\ 0 & 0 & 0 & \overline{d} \\ d^* & 0 & 0 & 0 \\ 0 & d^T & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & d & 0 \\ 0 & 0 & 0 & \overline{d} \\ \overline{d} & 0 & 0 & 0 \\ 0 & d & 0 & 0 \end{pmatrix}$$

is the most general form for the finite Dirac operator. Note that we used the fact that $d = d^T$, which we derived earlier in the calculation.

B.3 Inner fluctuations

B.3.1 The Proof of Proposition 3.1

Proof. We first calculate $[D_F, b_{Fj}]$:

$$\begin{aligned}
D_F b_{Fj} &= \begin{pmatrix} 0 & 0 & d\mu_j & 0 \\ 0 & 0 & 0 & \overline{d}s_{Rj} \\ \overline{d}\mu_j & 0 & 0 & 0 \\ 0 & ds_{Lj} & 0 & 0 \end{pmatrix} \\
b_{Fj} D_F &= \begin{pmatrix} 0 & 0 & \mu_j d & 0 \\ 0 & 0 & 0 & s_{Lj}\overline{d} \\ \mu_j\overline{d} & 0 & 0 & 0 \\ 0 & s_{Rj}d & 0 & 0 \end{pmatrix}
\end{aligned}$$

$$[D_F, b_{Fj}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{d}s_{Rj} - s_{Lj}\overline{d} \\ 0 & 0 & 0 & 0 \\ 0 & ds_{Lj} - s_{Rj}d & 0 & 0 \end{pmatrix}$$

Now we calculate $a_{Fj}[D_F, b_{Fj}]$:

$$\begin{aligned} a_{Fj}[D_F, b_{Fj}] &= \begin{pmatrix} \lambda_j & 0 & 0 & 0 \\ 0 & q_{Lj} & 0 & 0 \\ 0 & 0 & \lambda_j & 0 \\ 0 & 0 & 0 & q_{Rj} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{d}s_{Rj} - s_{Lj}\overline{d} \\ 0 & 0 & 0 & 0 \\ 0 & ds_{Lj} - s_{Rj}d & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{Lj}(\overline{d}s_{Rj} - s_{Lj}\overline{d}) \\ 0 & 0 & 0 & 0 \\ 0 & q_{Rj}(ds_{Lj} - s_{Rj}d) & 0 & 0 \end{pmatrix} \end{aligned}$$

From this we find that

$$\sum_j a_{Fj}[D_F, b_{Fj}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sum_j q_{Lj}(\overline{d}s_{Rj} - s_{Lj}\overline{d}) \\ 0 & 0 & 0 & 0 \\ 0 & \sum_j q_{Rj}(ds_{Lj} - s_{Rj}d) & 0 & 0 \end{pmatrix}.$$

□

B.3.2 The Proof of Proposition 3.2

Proof.

$$\begin{aligned} \Omega^1_{\not{D}\otimes\mathbb{I}}(\mathcal{A}) &= \Big\{ \sum_i (\alpha_i \otimes a_{Fi})[\not{D} \otimes \mathbb{I}, \beta_i \otimes b_{Fi}] \Big\} \\ &= \Big\{ \sum_i (\alpha_i \otimes a_{Fi})(\not{D}\beta_i \otimes b_{Fi} - \beta_i \not{D} \otimes b_{Fi}) \Big\} \\ &= \Big\{ \sum_i (\alpha_i \otimes a_{Fi})([\not{D}, \beta_i] \otimes b_{Fi}) \Big\} \\ &= \Big\{ \sum_i \alpha_i [\not{D}, \beta_i] \otimes a_{Fi} b_{Fi} \Big\} \\ &= \Big\{ \sum_i \alpha_i [i\gamma^\mu(\partial_\mu + \omega_\mu), \beta_i] \otimes a_{Fi} b_{Fi} \Big\} \\ &= \Big\{ \sum_i i\alpha_i \gamma^\mu([\partial_\mu, \beta_i] + [\omega_\mu, \beta_i]) \otimes a_{Fi} b_{Fi} \Big\} \end{aligned}$$

$[\omega_\mu, \beta_i] = 0$ because ω_μ is just the product of some γ-matrices and β_i are complex valued scalar functions, so they commute with the γ-matrices. We now use the Leibniz rule to simplify the other commutator, let us demonstrate this on a test function f:

$$\partial_\mu(\beta_i f) = \beta_i \partial_\mu(f) + \partial_\mu(\beta_i) f,$$
$$[\partial_\mu, \beta_i] f = \partial_\mu(\beta_i f) - \beta_i \partial_\mu(f) = \partial_\mu(\beta_i) f.$$

So we have

$$\Omega^1_{\not{D}\otimes\mathbb{I}}(\mathcal{A}) = \Big\{ \sum_i i\alpha_i \gamma^\mu \partial_\mu(\beta_i) \otimes a_{Fi} b_{Fi} \Big\}.$$

But $\alpha_i \partial_\mu(\beta_i)$ is again a complex valued function element of $C^\infty(M)$, but one with four components, say $\beta'_{\mu i}$. This enables us to make the last step:

$$\Omega^1_{\not{D}\otimes\mathbb{I}}(\mathcal{A}) = \Big\{ \sum_i i\alpha_i \gamma^\mu \partial_\mu(\beta_i) \otimes a_{Fi} b_{Fi} \Big\} = \Big\{ i\gamma^\mu X_\mu \mid X_\mu = \sum_i \beta'_{\mu i} \otimes a_{Fi} b_{Fi} \Big\}$$

□

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