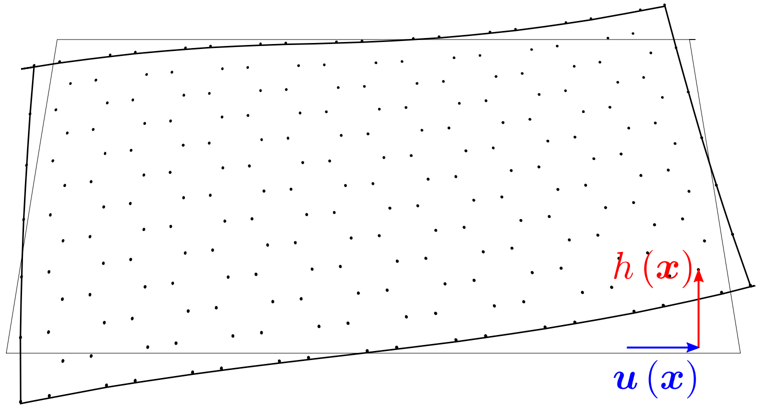
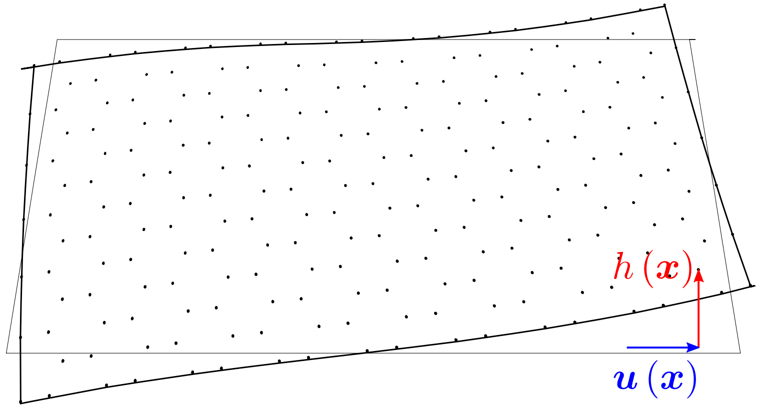
Anomalous scaling laws in fluctuating two-dimensional crystals: an ε-expansion method



Hamiltonian

$$\mathcal{H} = \int d^2x \left[\underbrace{\frac{1}{2}\kappa \left(\nabla^2 \boldsymbol{r}\right)^2}_{\text{bending energy}} + \underbrace{\frac{\lambda}{2} U_{\alpha\alpha}^2 + \mu U_{\alpha\beta}^2}_{\text{strain energy}} \right]$$

$$U_{\alpha\beta}(\boldsymbol{x}) = \frac{1}{2}\left[\partial_\alpha \boldsymbol{r} \cdot \partial_\beta \boldsymbol{r} - \delta_{\alpha\beta}\right]$$



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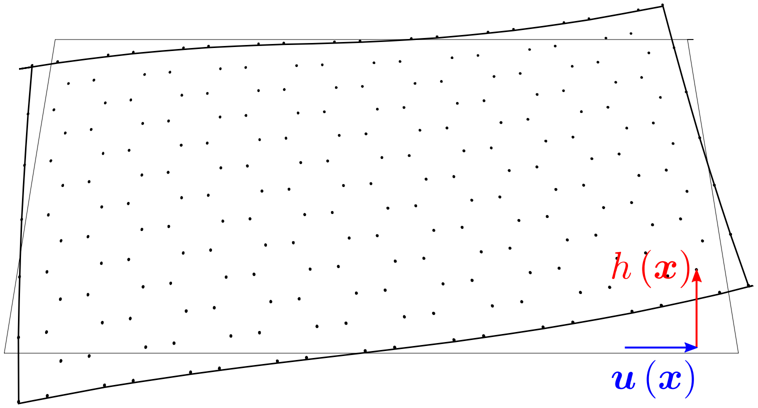
Harmonic approximation: phonons

$$\omega_{||}(q) = c_{\mathrm{s}}\, q$$

$$\omega_{\perp}(q) = \sqrt{\frac{\kappa}{\rho}}\, q^2$$



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bending energy — strain energy

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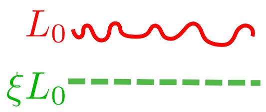
Crumpling at arbitrarily small temperature

$$\langle (h(\boldsymbol{x}))^2 \rangle = \int_{|\boldsymbol{q}|>1/L} \frac{\mathrm{d}^2 q}{(2\pi)^2}\frac{k_{\mathrm{B}}T}{\kappa q^4} \approx L^2$$



$$\xi^2 = 1 - \frac{T}{2\kappa}\int_{\boldsymbol{q}} \frac{1}{q^2}$$

Gornyi et al., PRB **92**, 155428 (2015)

Partition function and correlations

$$\mathcal{Z} = \int [\mathrm{d}h(\boldsymbol{x})\mathrm{d}u_\alpha(\boldsymbol{x})]\, \mathrm{e}^{-H/T}$$

$$\langle h(\boldsymbol{x}_1)...h(\boldsymbol{x}_n)u_{\alpha_1}(\boldsymbol{x}'_1)...u_{\alpha_\ell}(\boldsymbol{x}'_\ell)\rangle = \frac{1}{\mathcal{Z}}\int [\mathrm{d}h(\boldsymbol{x})\mathrm{d}u_\alpha(\boldsymbol{x})] h(\boldsymbol{x}_1)..h(\boldsymbol{x}_n)u_{\alpha_1}(\boldsymbol{x}'_1)..u_{\alpha_\ell}(\boldsymbol{x}'_\ell)\, \mathrm{e}^{-H/T}$$

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A simpler effective Hamiltonian:

$$H = \int \mathrm{d}^2x \left[\frac{1}{2}\kappa \left(\partial^2 h\right)^2 + \frac{\lambda}{2}u_{\alpha\alpha}^2 + \mu u_{\alpha\beta}^2\right]$$

nonlinear coupling

$$u_{\alpha\beta} = \frac{1}{2}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha + \boxed{\partial_\alpha h \partial_\beta h}\right)$$

Aronovitz and Lubensky, PRL **60**, 2634 (1988)

Nelson and Peliti, J. Physique **48**, 1085 (1987)

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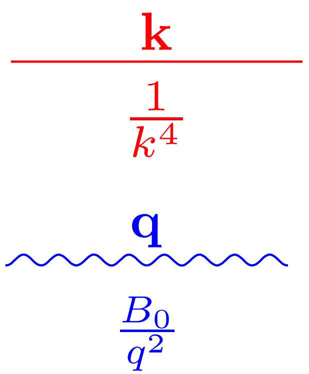
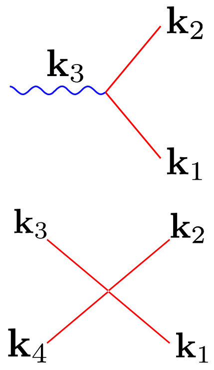
Propagators and interactions





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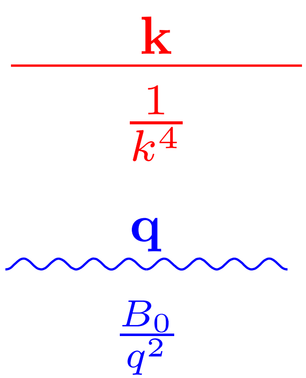
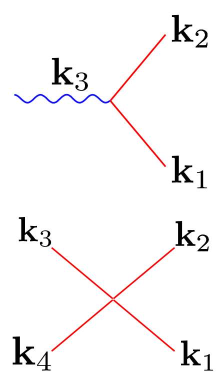
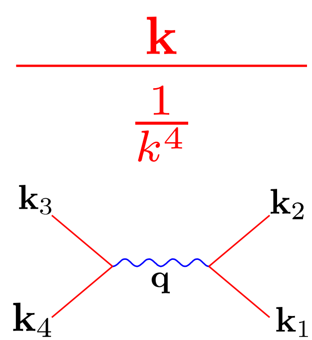




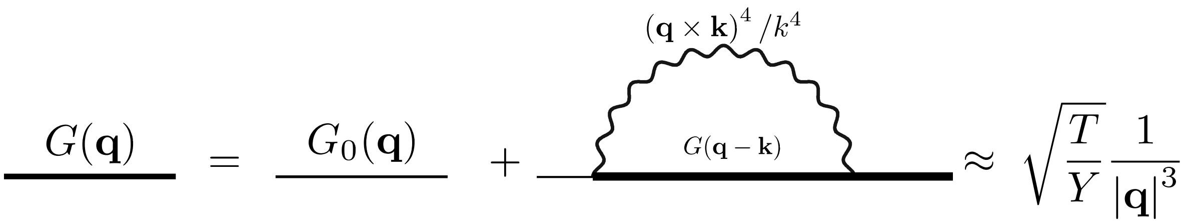
Representation as a curvature coupling

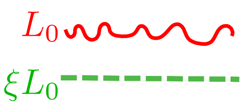


$$= \frac{\left(\mathbf{q}\times\mathbf{k}_1\right)^2\left(\mathbf{q}\times\mathbf{k}_3\right)^2}{q^4}$$

Aronovitz and Lubensky, PRL **60**, 2634 (1988)

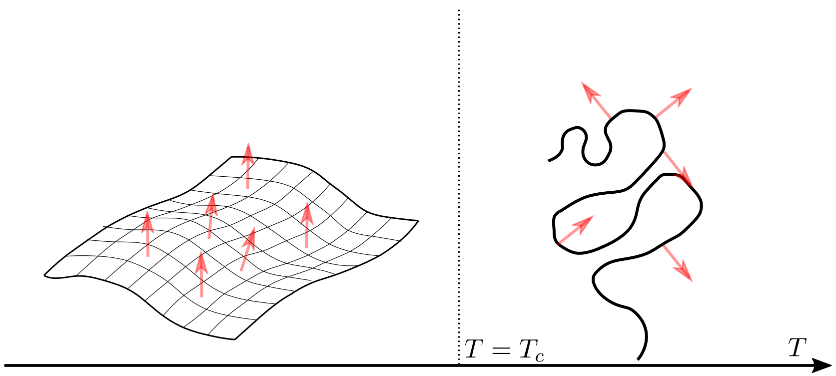
Nelson and Peliti, J. Physique **48**, 1085 (1987)





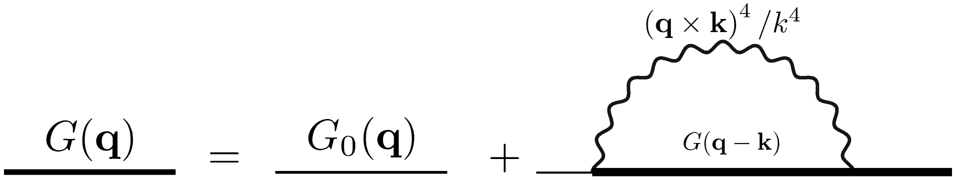
$$\xi^2 = 1 - \frac{\sqrt{T}}{2\sqrt{Y}}\int_{\boldsymbol{q}}\frac{1}{|\mathbf{q}|}$$

Phase diagram

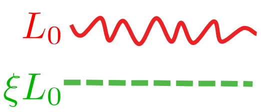


Nelson and Peliti, J. Physique **48**, 1085 (1987); Bowick and Travesset, Phys. Rep. **344**, 255 (2001)

Nicholl et al., PRL **118**, 266101 (2017);



$$\approx \sqrt{\frac{T}{Y}}\frac{1}{|\mathbf{q}|^3}$$



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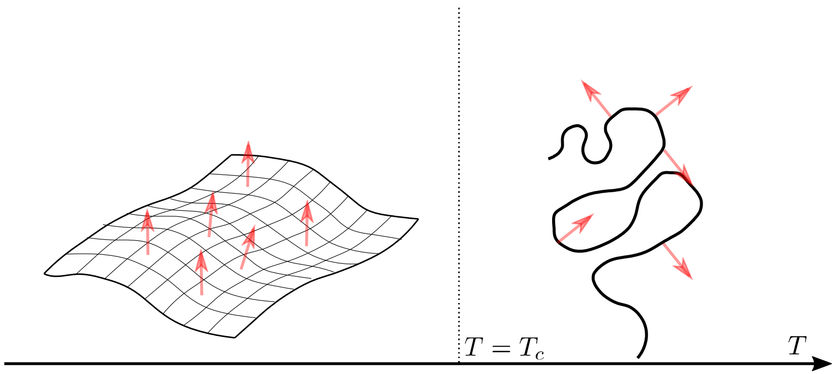


Scaling and universality

$$G(\mathbf{q}) = \langle |h_{\mathbf{q}}|^2\rangle \approx q^{4-\eta}$$

$$\langle h(\rho\mathbf{x}_1)...h(\rho\mathbf{x}_n)\rangle \approx \rho^{n(1-\eta/2)}\langle h(\mathbf{x}_1)...h(\mathbf{x}_n)\rangle$$

Scaling of bending rigidity and elastic moduli

$$\kappa_{\mathrm{R}}(\mathbf{q}) \approx q^{-\eta}$$

$$B_{\mathrm{R}}(\mathbf{q}) \approx q^{\eta_u} = q^{2-2\eta}$$

Nelson and Peliti, J. Physique **48**, 1085 (1987); Bowick and Travesset, Phys. Rep. **344**, 255 (2001)

Nicholl et al., PRL **118**, 266101 (2017); Aronovitz and Lubensky PRL **60**, 2634 (1988)

Expansion parameters for a general theory D-dimensional membrane in d-dimensional space

$$\varepsilon = 4 - D \ll 1$$

$$d_{\mathrm{c}} = d - D \gg 1$$

Renormalization group and the ε-expansion

$$\eta = \frac{12\varepsilon}{24 + d_{\mathrm{c}}} - \frac{6 d_{\mathrm{c}}(d_{\mathrm{c}} + 29)\varepsilon^2}{(24 + d_{\mathrm{c}}^3)}$$

David and Guitter, EPL **5**, 709 (1988); Coquand et al., arXiv 2003.13973 (2020)

Membrane with infinite elastic coefficients:

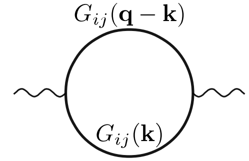
$$\kappa_{\mathrm{R}}(\mathbf{q}) \approx q^{-\eta} \qquad \eta = \frac{2}{d} + \mathrm{O}\left(\frac{1}{d^2}\right)$$

Limit of large embedding-space dimension

$$\eta = \frac{2}{d} + \frac{73 - 68\zeta(3)}{27 d_{\mathrm{c}}^2} + \mathrm{O}(d_{\mathrm{c}}^{-3})$$

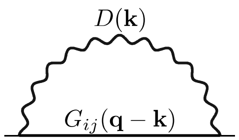
Self-consistent screening approximation

$$D^{-1}(\mathbf{q}) = \frac{s_D q^4}{y_0} +$$



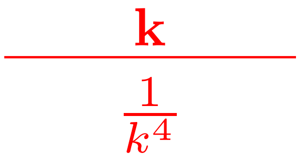
$$\eta \simeq 0.821$$

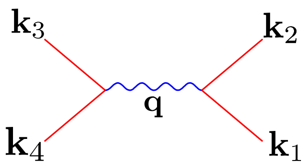
$$\left[G^{-1}(\mathbf{q})\right]_{ij} = \delta_{ij} q^4 +$$



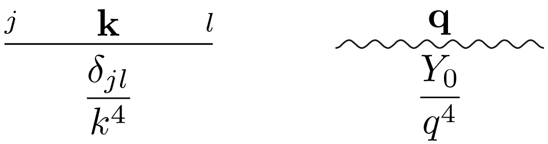
Saykin et al., Ann. Phys. **414**, 168108 (2020); Le Doussal and Radzihovsky, PRL **69**, 1209 (1992)

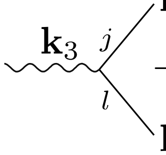
Hubbard-Stratonovich transformation





$$= \frac{(\mathbf{q}\times\mathbf{k}_1)^2\,(\mathbf{q}\times\mathbf{k}_3)^2}{q^4}$$

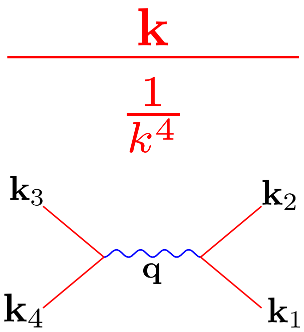




$$\mathbf{k}_2 = -\mathbf{k}_1 - \mathbf{k}_3$$

$$-i\delta_{jl}\bar{\gamma}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$$

Hubbard-Stratonovich transformation



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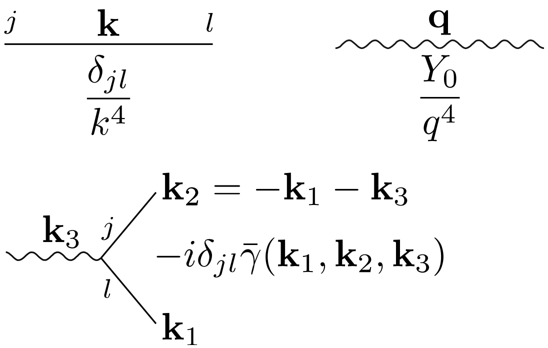


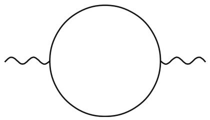
Gaussian curvature

$$H = \int \mathrm{d}^D x \left[\frac{1}{2}\kappa\left(\partial^2 h\right)^2 + \frac{1}{2Y_0}\left(\partial^2\lambda\right)^2 + i\lambda K\right]$$

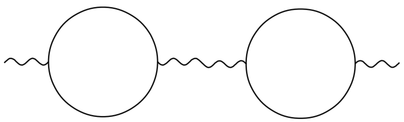
$$K = -\frac{1}{2}(\delta_{\alpha\beta}\partial^2 - \partial_\alpha\partial_\beta)(\partial_\alpha\mathbf{h}\cdot\partial_\beta\mathbf{h})$$

$$\begin{aligned}\bar{\gamma}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) &= k_1^2k_2^2 - (\mathbf{k}_1\cdot\mathbf{k}_2)^2\\ &= k_2^2k_3^2 - (\mathbf{k}_2\cdot\mathbf{k}_3)^2\\ &= k_3^2k_1^2 - (\mathbf{k}_3\cdot\mathbf{k}_1)^2\end{aligned}$$

Infrared divergences in perturbation theory. The need of an ε-expansion

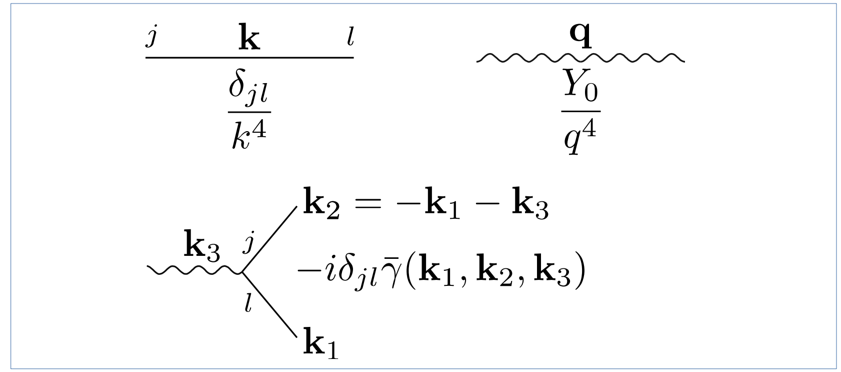


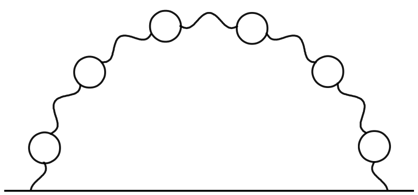
$$\approx \frac{Y_0}{q^4} \times Y_0 q^{-\varepsilon}$$



$$\approx \frac{Y_0}{q^4} \times Y_0^2 q^{-2\varepsilon}$$

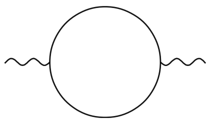
....





$$\approx \int_{\mathbf{k}} \frac{Y_0^N \left(q^2 k^2 - (\mathbf{q}\cdot\mathbf{k})^2\right)^2}{|\mathbf{q}-\mathbf{k}|^4 \, k^{4+N\varepsilon}}$$ = infrared divergent for $N \gtrsim 1/\varepsilon$

Zinn-Justin, *Quantum field theory and critical phenomena*; Parisi, *Statistical field theory*



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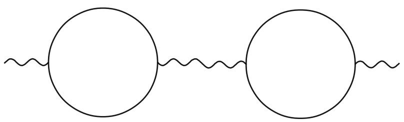

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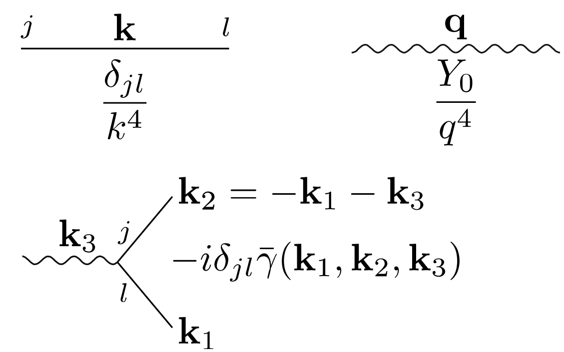

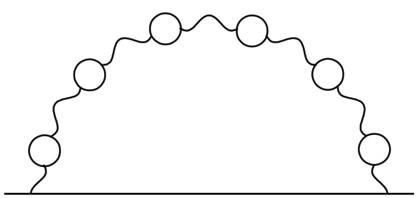

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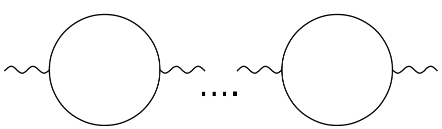
ε-expansion



$$\approx \frac{y_0^{N+1}}{q^4} \sum_{\ell=0}^{\infty} \left[\ln \frac{\Lambda}{q} \right]^{\ell+1} \varepsilon^{\ell}$$

= infrared finite. Logarithmic ultraviolet divergences

$$Y_0 = \Lambda^{\varepsilon} y_0$$

Zinn-Justin, *Quantum field theory and critical phenomena*; Parisi, *Statistical field theory*

Two UV divergent amplitudes

$$G^{(n,\ell)}(\mathbf{x}_1, .., \mathbf{x}_n; \mathbf{y}_1, .., \mathbf{y}_\ell; Y_0) = \langle h(\mathbf{x}_1)..h_{\mathbf{x}_n}\lambda(\mathbf{y}_1)..\lambda(\mathbf{y}_\ell)\rangle$$

$$\tilde{G}^{(n,\ell)}(\mathbf{x}_1, .., \mathbf{x}_n; \mathbf{y}_1, .., \mathbf{y}_\ell; y) = Z^{\ell-\frac{n}{2}} G^{(n,\ell)}(\mathbf{x}_1, .., \mathbf{x}_n; \mathbf{y}_1, .., \mathbf{y}_\ell; Y_0) = \text{ finite}$$

$Z = Z(y)$

$Y_0 = M^\varepsilon g(y)$

$M =$ arbitrary wavevector scale

$y =$ renormalized coupling constant

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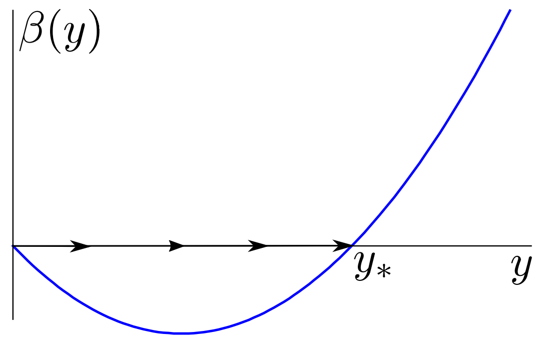
RG equation

$$\frac{\partial}{\partial M}G^{(n,\ell)}\Big|_{Y_0} = \left[\frac{\partial}{\partial M} + \beta(y)\frac{\partial}{\partial y} + \left(\frac{n}{2}-\ell\right)\eta(y)\right]\tilde{G}^{(n,\ell)} = 0$$

RG functions

$$\beta(y) = \frac{\partial y}{\partial \ln M}\Big|_{Y_0} = -\varepsilon y + \bar{\beta}(y)$$

$$\eta(y) = \frac{\partial \ln Z}{\partial \ln M}\Big|_{Y_0}$$



RG beta function for $\varepsilon = 1/10$

For $y = y_*$, the RG equation reduces to:

$$\left[\frac{\partial}{\partial M} + \left(\frac{n}{2} - \ell\right)\eta(y)\right]\tilde{G}^{(n,\ell)} = 0$$

$$\tilde{G}^{(n,\ell)} \propto M^{-(\frac{n}{2}-\ell)\eta}$$

$\eta = \eta(y_*)$

M is the only length scale. Using dimensional analysis the RG equation implies power-law scaling:

$$\tilde{G}^{(n,\ell)}(\rho\mathbf{x}_1, .., \rho\mathbf{x}_n; \rho\mathbf{y}_1, .., \rho\mathbf{y}_\ell) \approx \rho^{\frac{n}{2}\varepsilon-(n/2-\ell)\eta}\tilde{G}^{(n,\ell)}(\mathbf{x}_1, \mathbf{x}_n; \mathbf{y}_1, .., \mathbf{y}_\ell)$$

In particular:

$$\kappa_{\mathrm{R}}(\mathbf{q}) \approx q^{-\eta} \qquad B_{\mathrm{R}}(\mathbf{q}) \approx q^{\epsilon-2\eta}$$

RG functions

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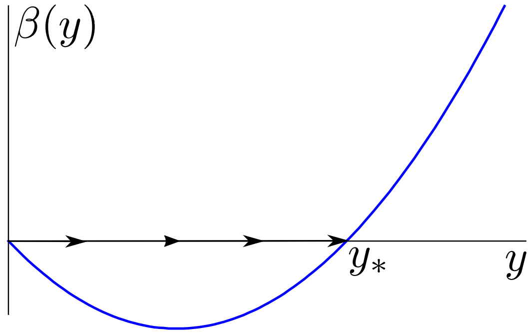
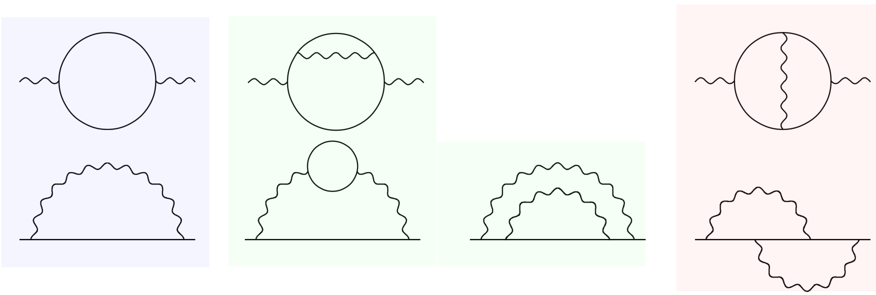
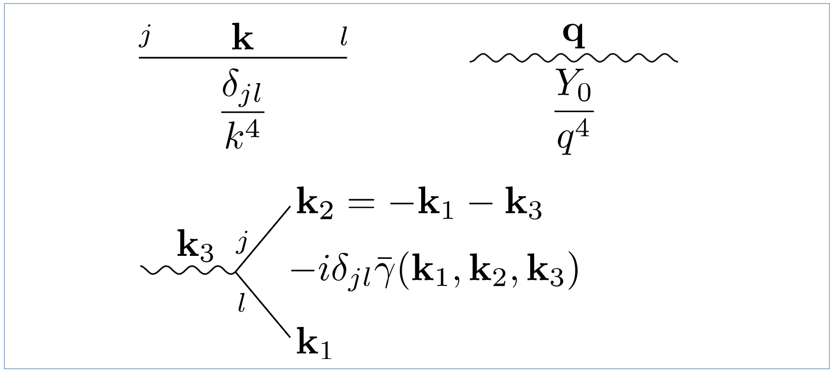
$$\eta(y) = \frac{\partial \ln Z}{\partial \ln M}\bigg|_{Y_0}$$



RG beta function for $\varepsilon = 1/10$

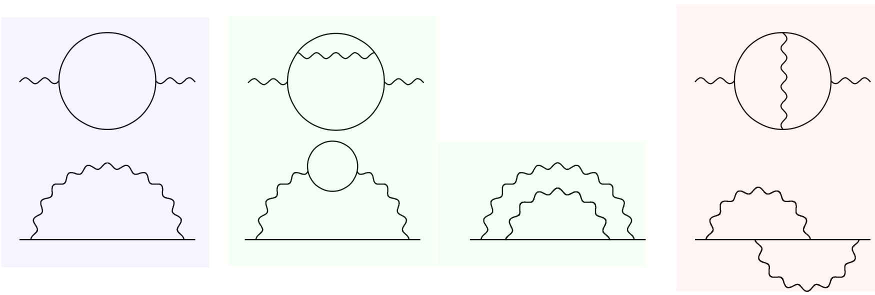
Diagrams at two-loop order

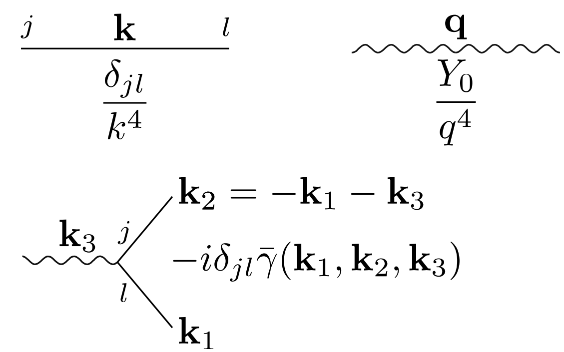




$$\begin{aligned}\bar{\gamma}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &= k_1^2 k_2^2 - (\mathbf{k}_1 \cdot \mathbf{k}_2)^2 \\ &= k_2^2 k_3^2 - (\mathbf{k}_2 \cdot \mathbf{k}_3)^2 \\ &= k_3^2 k_1^2 - (\mathbf{k}_3 \cdot \mathbf{k}_1)^2\end{aligned}$$

Diagrams at two-loop order





$$\bar{\gamma}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = k_1^2 k_2^2 - (\mathbf{k}_1 \cdot \mathbf{k}_2)^2 = k_2^2 k_3^2 - (\mathbf{k}_2 \cdot \mathbf{k}_3)^2 = k_3^2 k_1^2 - (\mathbf{k}_3 \cdot \mathbf{k}_1)^2$$

Fixed point and scaling dimension:

$$y_* = \frac{\varepsilon}{5} - \frac{4/3 - 5A}{125}\varepsilon^2 \qquad A = \frac{121}{90}$$

$$\eta = \frac{2\varepsilon}{5} - \frac{\varepsilon^2}{750} = 0.8, 0.795$$

SCSA is exact to O(ε^2)

Close to earlier estimates:
0.85 (NPRG), 0.821(SCSA)
Simulations 0.66 – 0.88

Achille Mauri and Mikhail I. Katsnelson, “Scaling behavior of crystalline membranes: an ε-expansion approach, arXiv:2003.04043