



*Nonperturbative Fingerprints
of the Many-Electron Physics ...
and
their Surprising Implications*

A. Toschi



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- ❖ *PRL 125 196403 (2020)*
 - ❖ *PRL 126 056403 (2021)*

**Joint Theory seminar of Condensed Matter departments
in Nijmegen, Hamburg and Uppsala, 08.04.2021**

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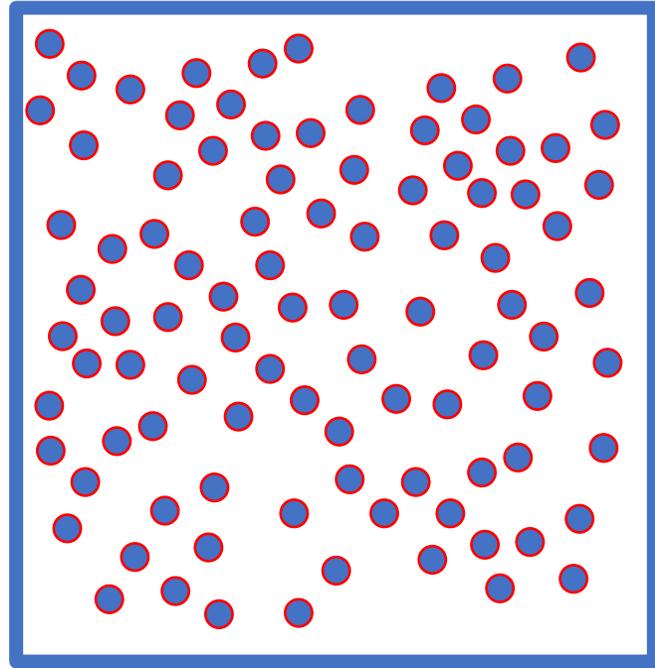
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Patrik Thunström (Uni Uppsala)

and to the Austrian Science Fund (**FWF**) for financial support through the project I 2794-N35

The two-fold challenge ...

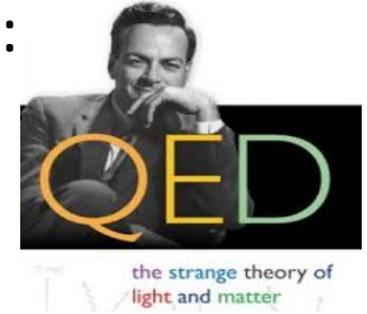
... of the many-electron problem:



$\sim 10^{23}$ interacting fermions !

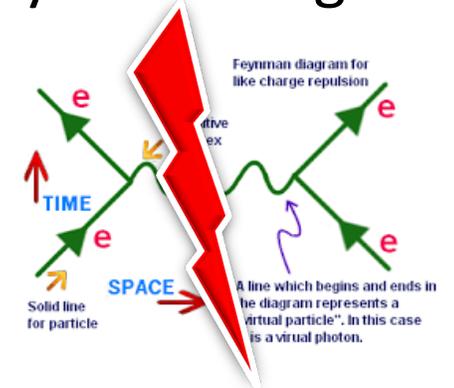
huge number
of D.o.F. !

Quantum field theory
as in:



NO control parameter
(like $\alpha = 1/137$)
for the
many-body expansion !

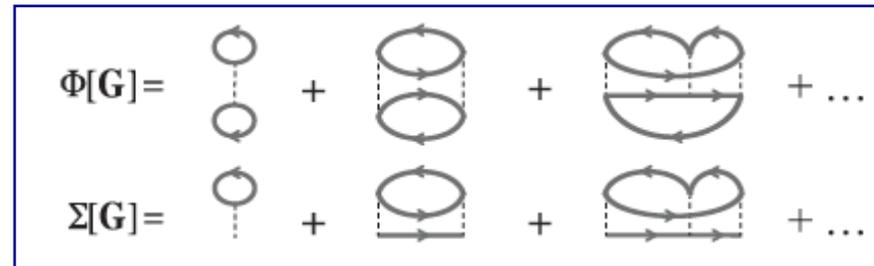
Feynman diagrams



in **nonperturbative**
regimes !

1. Luttinger-Ward formalism → “conserving theories,,

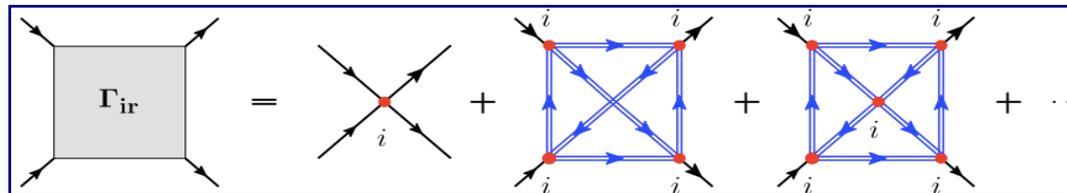
$$\Omega[G] = \tilde{\Omega}[G_0, G] + \Phi[G] \Rightarrow \Sigma = \frac{\delta\Phi}{\delta G}; \Gamma_{ch}^{IRR} = \frac{\delta^2\Phi}{\delta G^2}$$



Two main routes to approximations

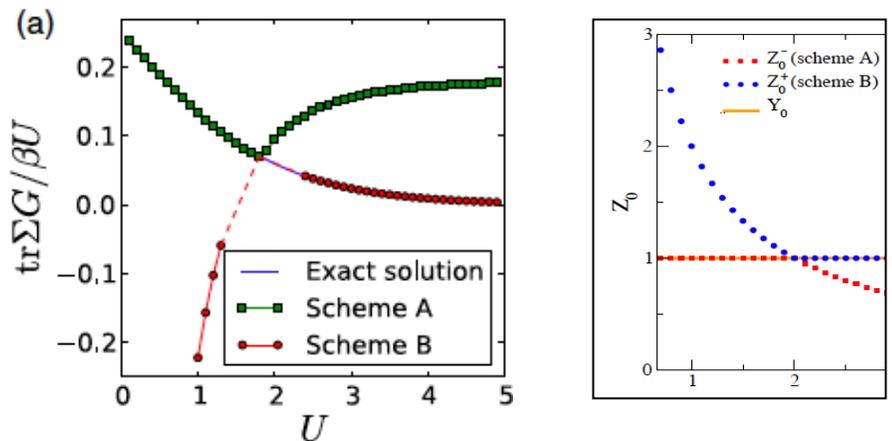
2. Crossing symmetries → “Pauli principle,,

$$\Sigma = \int \Gamma GGG \text{ with } \Gamma \Leftarrow \Gamma^{IRR} = \text{BSE and/or parquet}$$



1. Luttinger-Ward functionals

I. Multivaluedness of LW functional



E. Kozik et al., PRL (2015)
 A.Stan et al., NJP (2015)
 R. Rossi et al., PRB (2015)
 W.Tarantino, et al., PRB (2017)
 J.Vucicevic, et al. PRB (2018)
 A.J. Kim et al. arXiv:2012.06159
 K.Van Houcke et al., arXiv:2102.04508

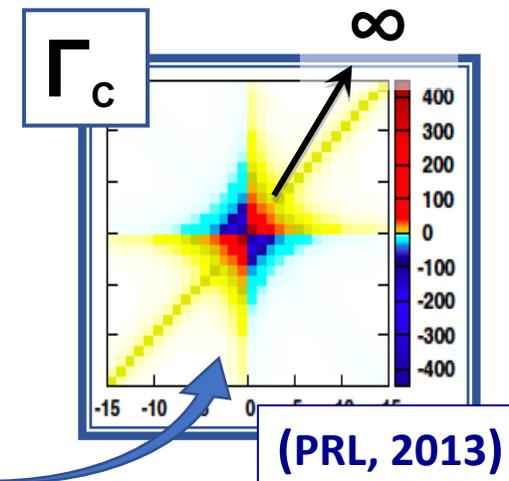
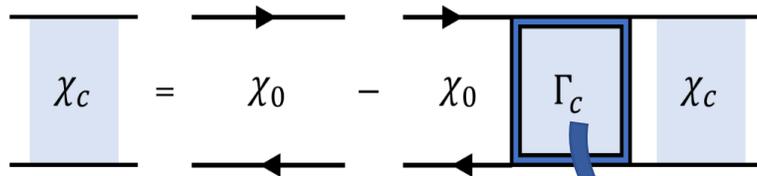
Nonperturbative

Two main reasons to approximations

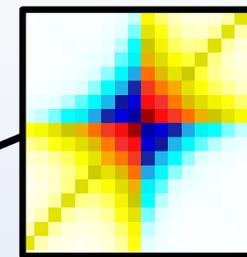
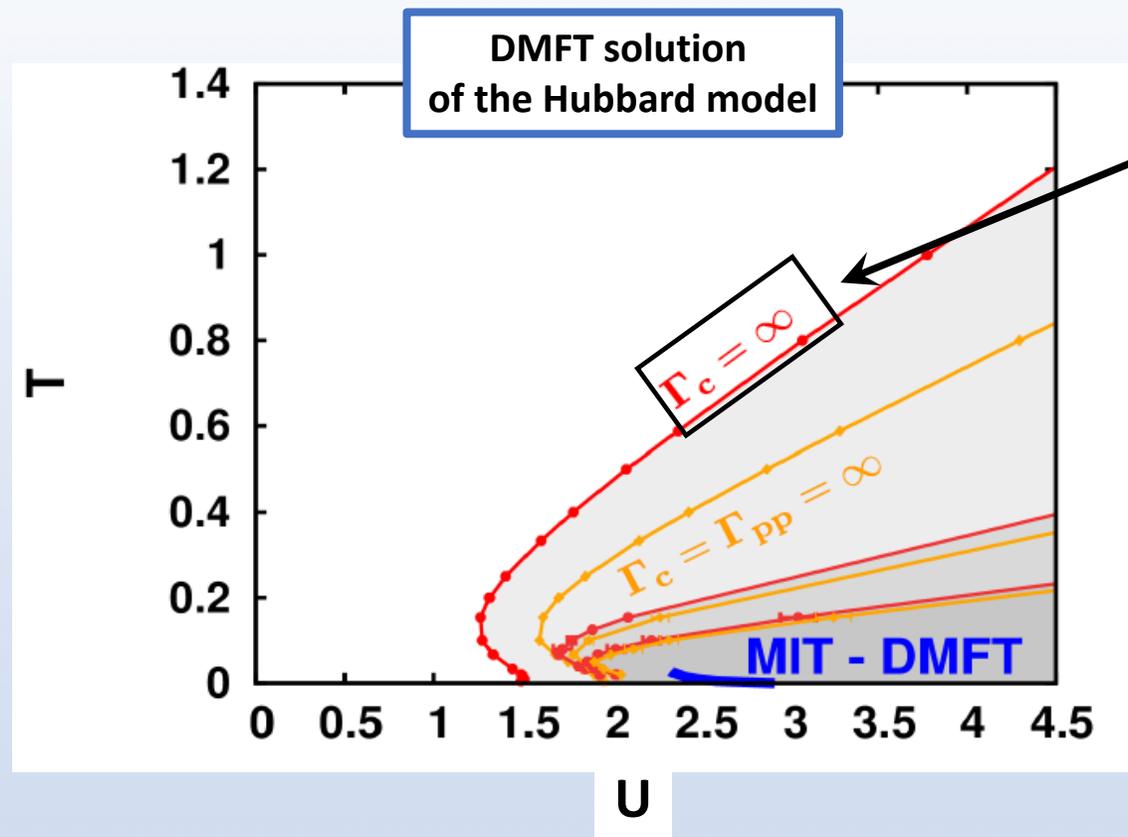
breakdown!

2. Crossing symmetries

II. Divergences of irreducible vertices

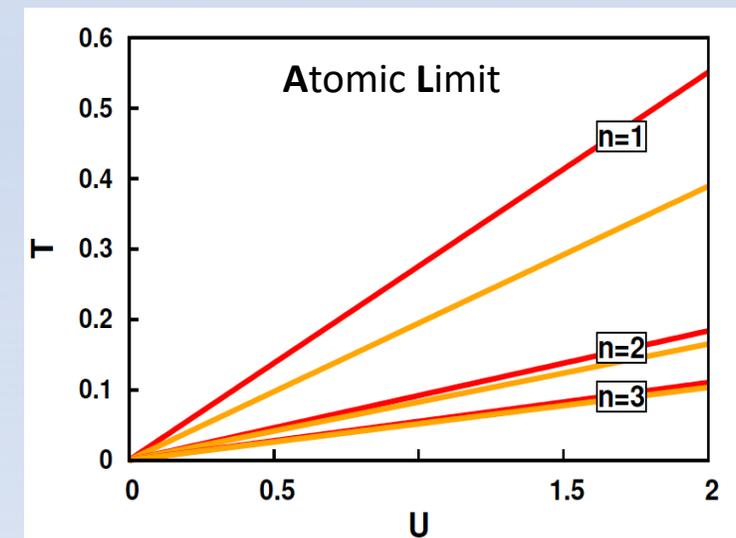
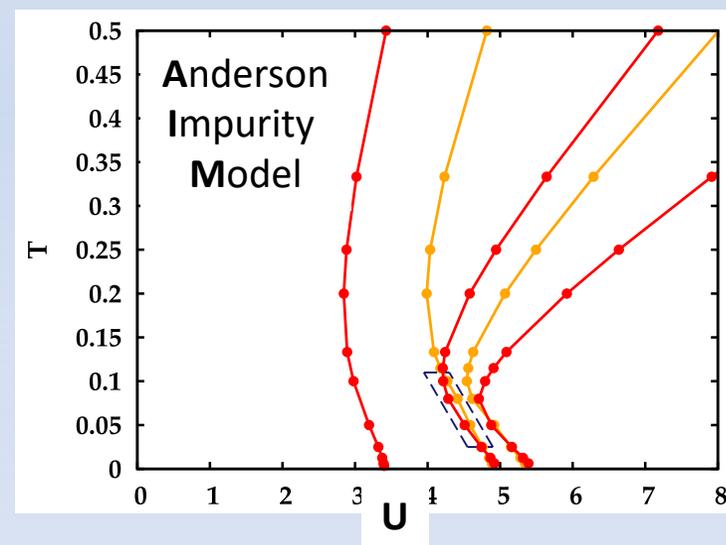
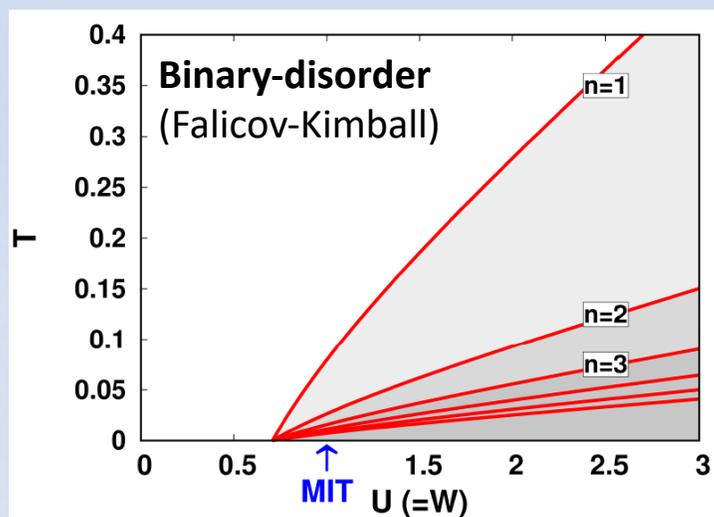


(PRL, 2013)



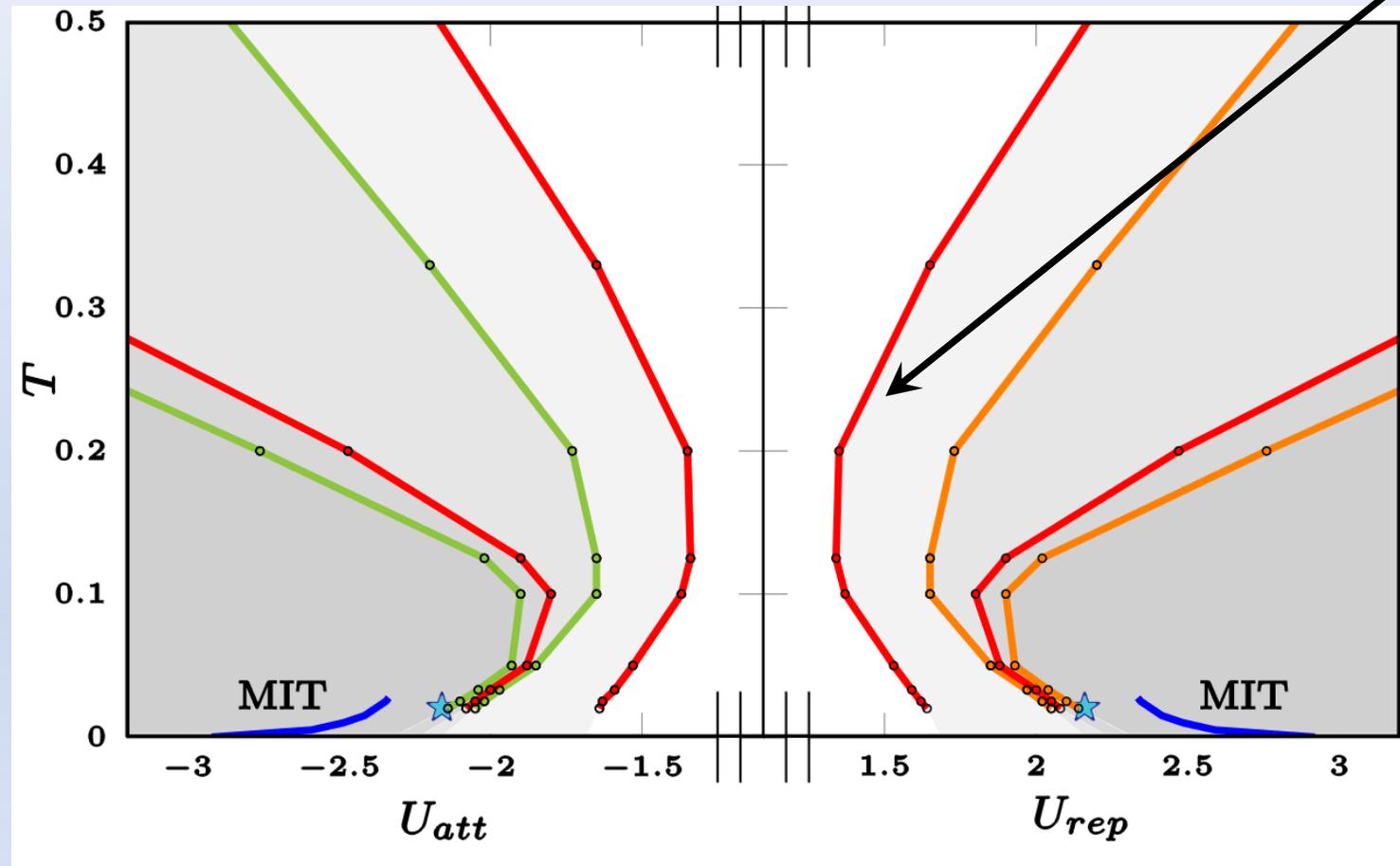
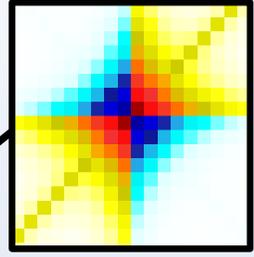
- ❖ T. Schäfer, ... and A.T., PRL **110** 246105 (2013)
- ❖ T. Schäfer, ... and A.T., PRB **94** 235108 (2016)

- ❖ P. Chalupa, ..., & A.T., PRB **97** 245136 (2018)
- ❖ P. Thunström, ..., & G. Rohringer, PRB **98** 235107 (2018)
- ❖ D. Springer, ..., & AT, PRB **101** 155148 (2020)



DMFT solution
of the **attractive** Hubbard model

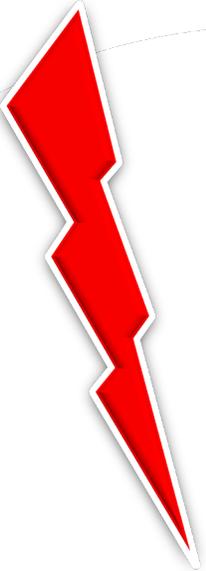
DMFT solution
of the **repulsive** Hubbard model



- density (*d*)
- magnetic (*m*)
- density (*d*) + pairing (*pp*)

	$U < 0$	$U > 0$
$\lambda^S = 0$	<i>m</i>	<i>d, pp</i>
# <i>D.o.F.</i>	3	1 + 2
$\lambda^A = 0$	<i>d</i>	<i>d</i>
# <i>D.o.F.</i>	1	1

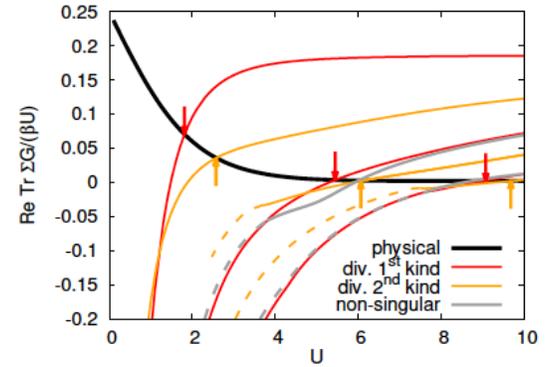
Nonperturbative



breakdown!

I. Multivaluedness of LW functional

BOTH numerically ...



**EQUIVALENT ASPECTS
of the SAME „phenomenon“ !!!**

[PRL 119 046402 (2017)]

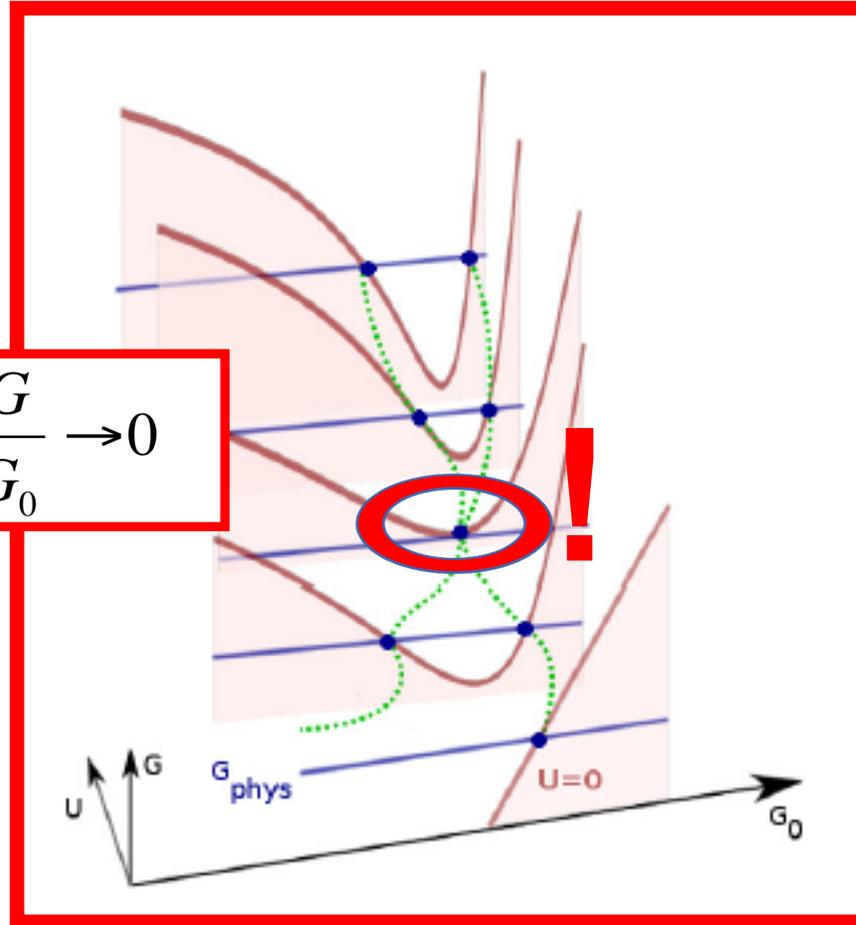
II. Divergences of irreducible vertices

... AND analytically !

$$\sum_{\nu'} \chi_c^{\nu\nu'}(\omega=0) \left[1/G_0^{(2)}(\nu') - 1/G_0^{(1)}(\nu') \right] \\ = O \left[\left(1/G_0^{(2)} - 1/G_0^{(1)} \right)^2 \right].$$

Heuristic picture:

$$\mathbf{G} = \mathbf{G}[\mathbf{G}_0]$$



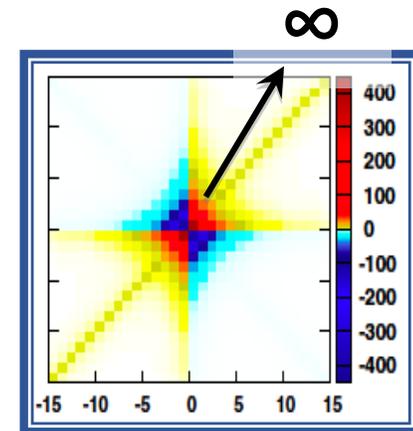
$$\frac{\delta G}{\delta G_0} \rightarrow 0$$

$$\frac{\delta G}{\delta \Sigma} = \frac{\delta G}{\delta G_0} \frac{\delta G_0}{\delta \Sigma} \rightarrow 0$$

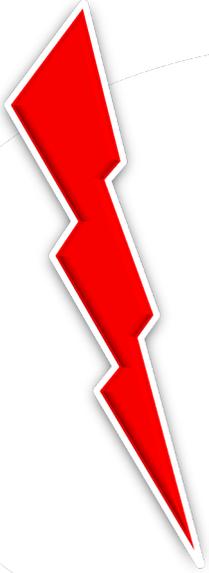
(I) Multivaluedness of LW functional

(II) Divergences of irreducible vertices

$$\Gamma_{ch}^{IRR}[G] = \frac{\delta^2 \Phi}{\delta G^2} = \frac{\delta \Sigma}{\delta G} \rightarrow \infty$$



algorithmic



challenges

I. Multivaluedness of LW functional

iterative/self-consistent (= ``bold'') approaches

Diagrammatic Monte Carlo

Nested Cluster Schemes

[E. Kozik et al., PRL (2015); A.Stan et al., NJP (2015);
R. Rossi et al., PRB (2015); J.Vucicevic, et al. PRB (2018), ...]

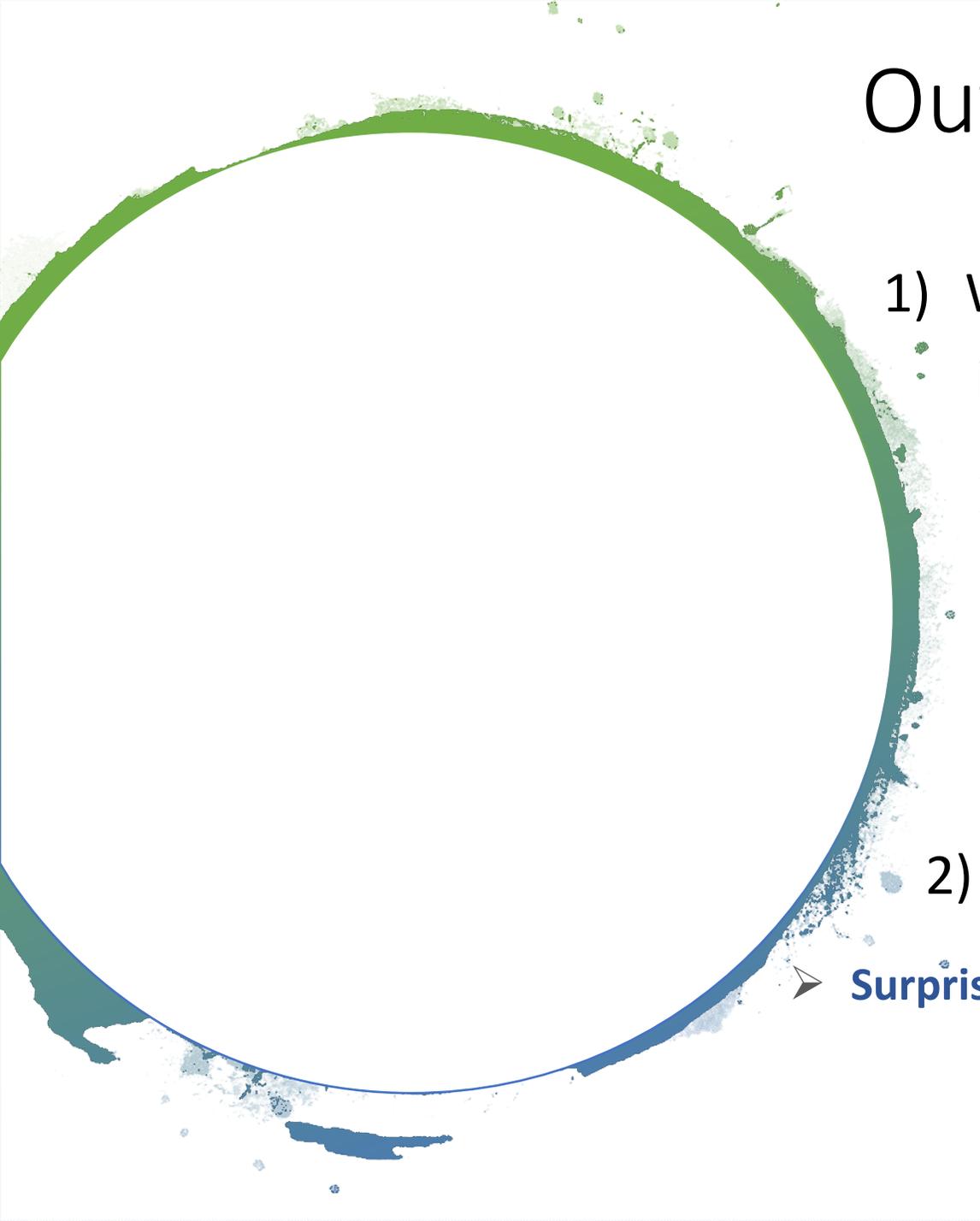
II. Divergences of irreducible vertices

parquet-based methods

dynamical vertex approximation (**D Γ A**)

QUADRILEX

[A.Toschi et al., PRB (2007); O. Gunnarsson et al., PRB (2016)
T.Ayral et al., PRB (2016); G. Rohringer et al., PRB (2018); ...]



Outline

1) What is the “**origin**” of the **breakdown** in many-body perturbation theories ?

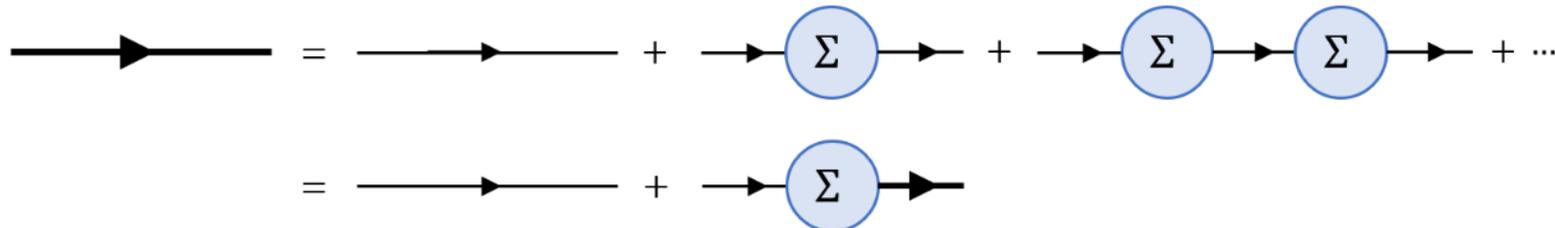
- How to “read” the physics from the 2P quantities
- **FINGERPRINTS** of the local moment formation and of its Kondo screening

2) Are there relevant **physical consequences**?

- **Surprising IMPLICATIONS** for the non-local properties !

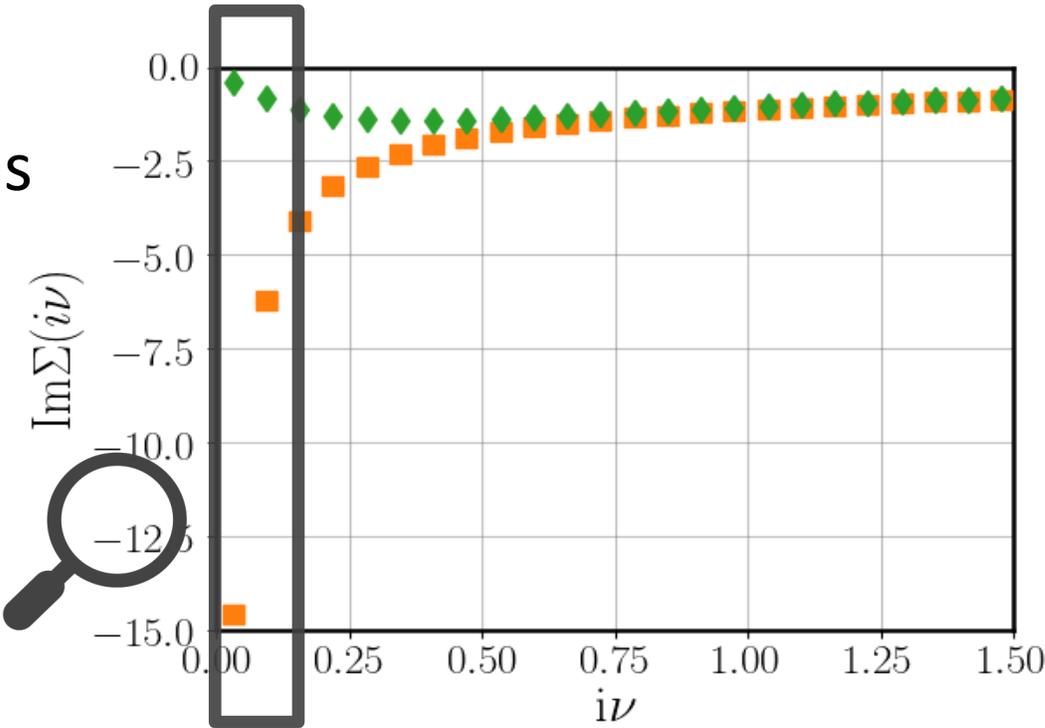
How to „read“ the physics ... as we can do at the „1P level“

- 1P – Dyson equation



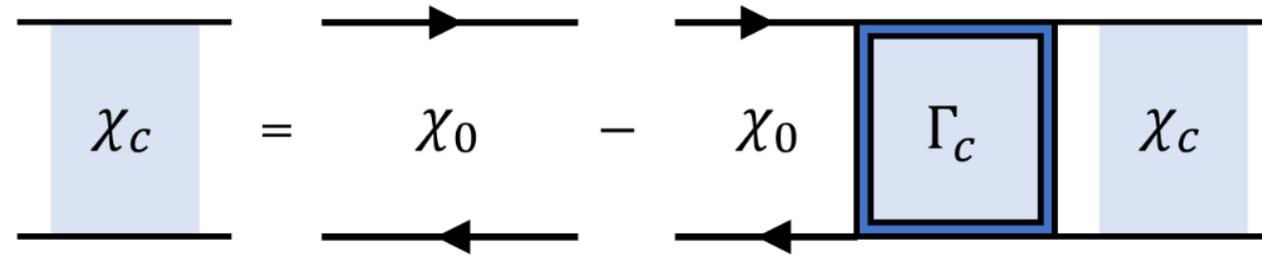
Metallic vs.
Insulating solutions

Low frequencies
→ $Z, \gamma, A(\nu)$



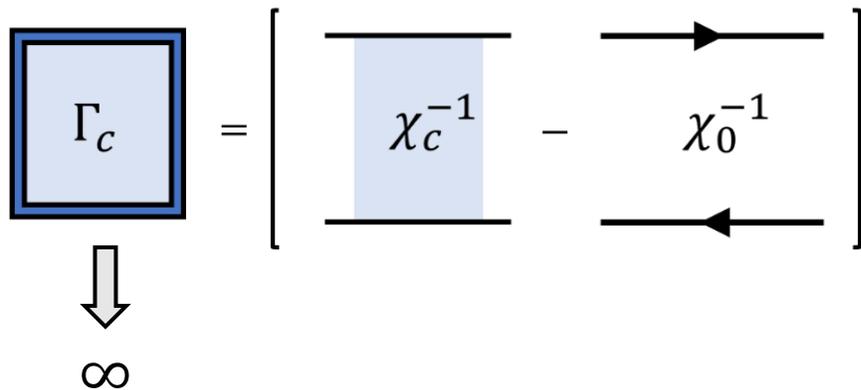
Going at the 2P level

- (local) **Bethe Salpeter Equation (BSE)**



$$\chi_c^{\nu\nu'} = \chi_{c,0}^{\nu\nu'} - \frac{1}{\beta^2} \sum_{\nu_1, \nu_2} \chi_{c,0}^{\nu\nu_1} \Gamma_c^{\nu_1\nu_2} \chi_c^{\nu_2\nu'}$$

- Irreducible vertex

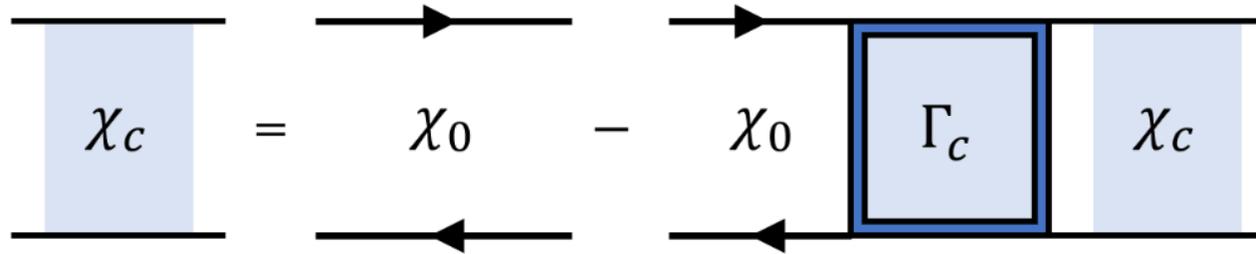


$$[\chi_c]_{\nu\nu'}^{-1} = \sum_i (V_{i,\nu'}^c)^* (\lambda_i)^{-1} V_{i,\nu}^c$$

\Downarrow
 0

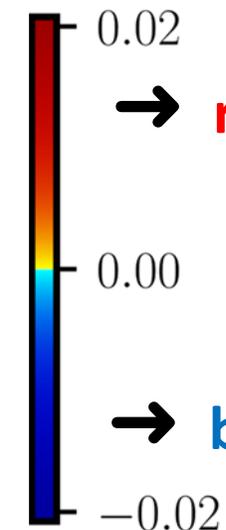
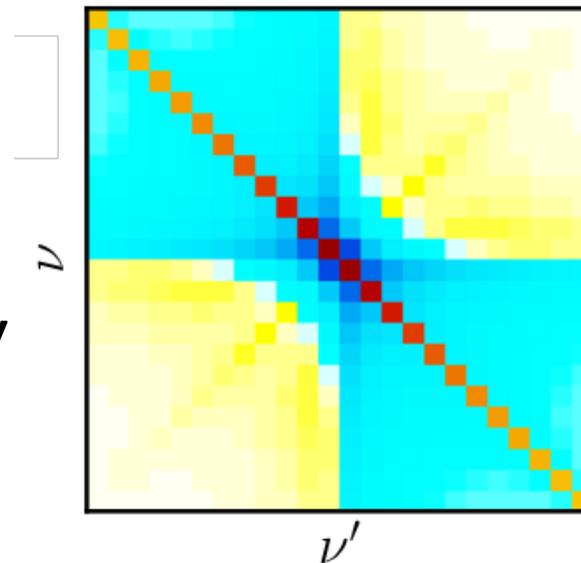
Relation to the physics?

- (local) **Bethe Salpeter Equation (BSE)**



$$\chi_c^{phys} = \frac{1}{\beta^2} \sum_{\nu\nu'} \chi_c^{\nu\nu'}$$

$$\chi_c^{phys} = \frac{1}{\beta^2} \sum_{\nu\nu'} \chi_c^{\nu\nu'}$$

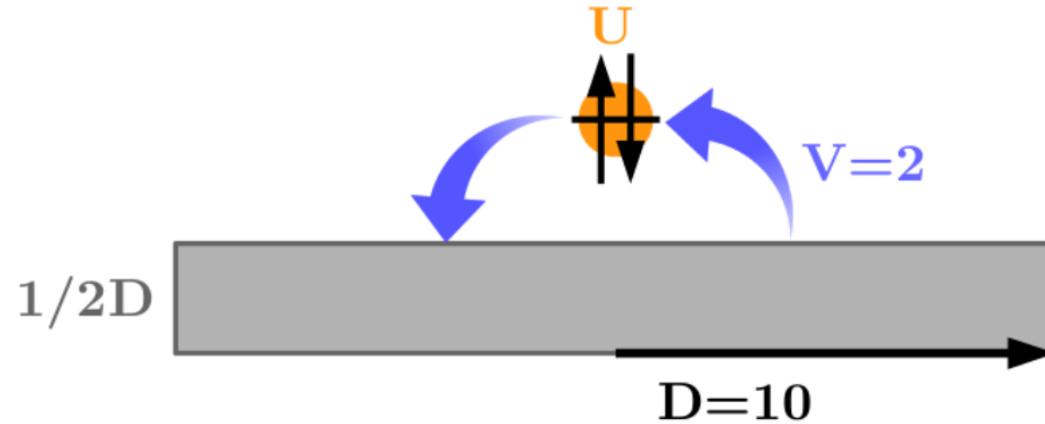


→ red = positive values

→ blue = negative values

Anderson Impurity Model

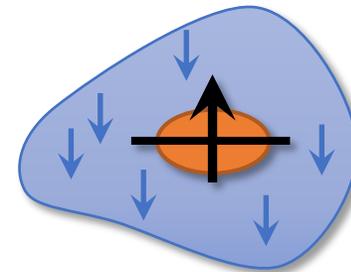
- wide-band limit, half-filling



Main physical ingredients:

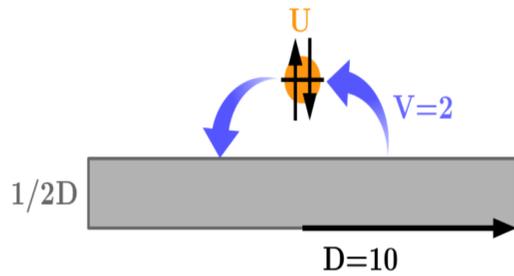


local magnetic moment

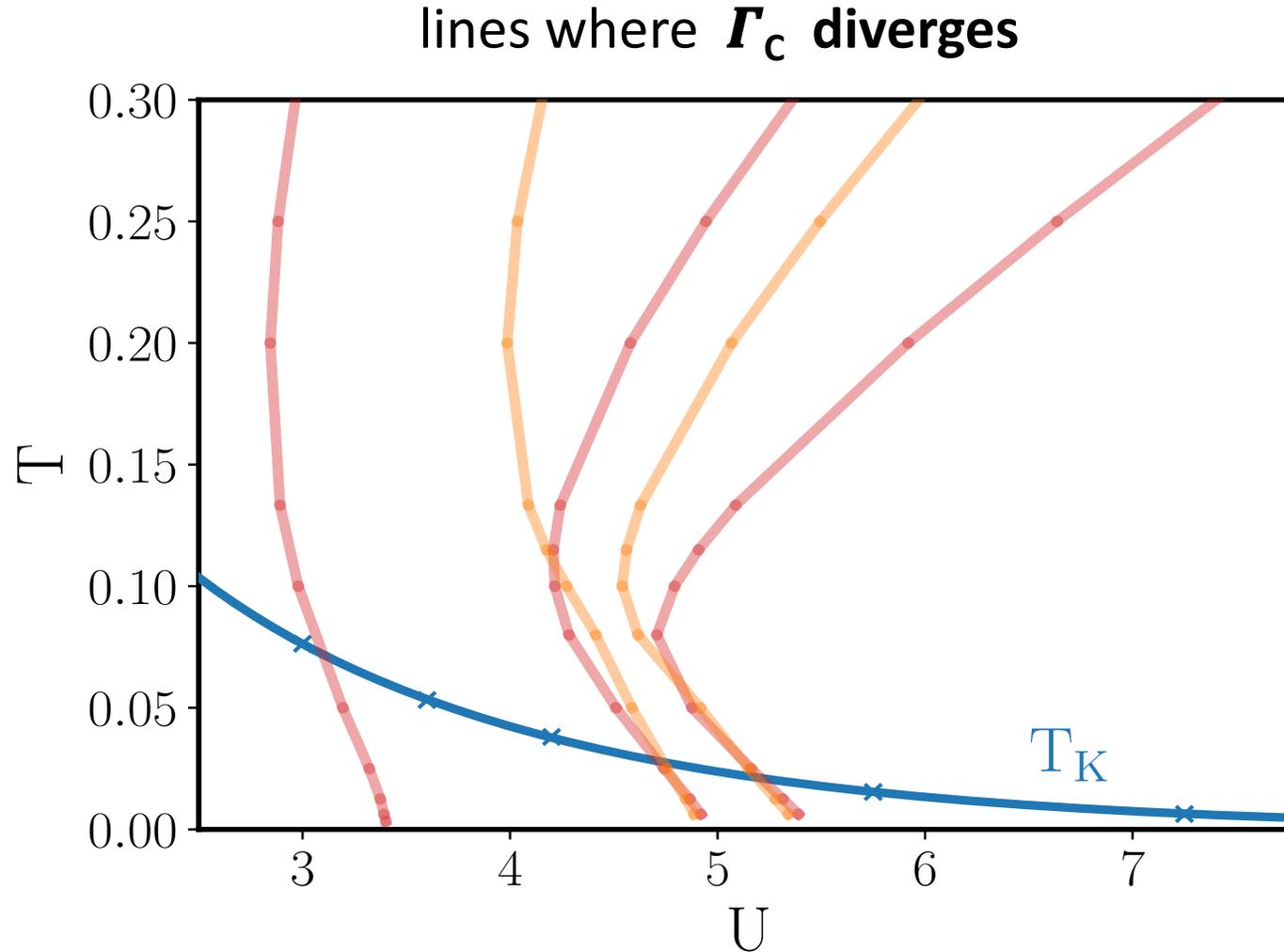


Kondo screening

... where vertex divergences have also been found !



- Solved by means of w2dynamics – CT-HYB



*) **red** lines:

$$\Gamma_c = \infty$$

*) **orange** lines:

$$\Gamma_c = \Gamma_{pp} = \infty$$

Physical response of the AIM

• w2dynamics – CT-HYB

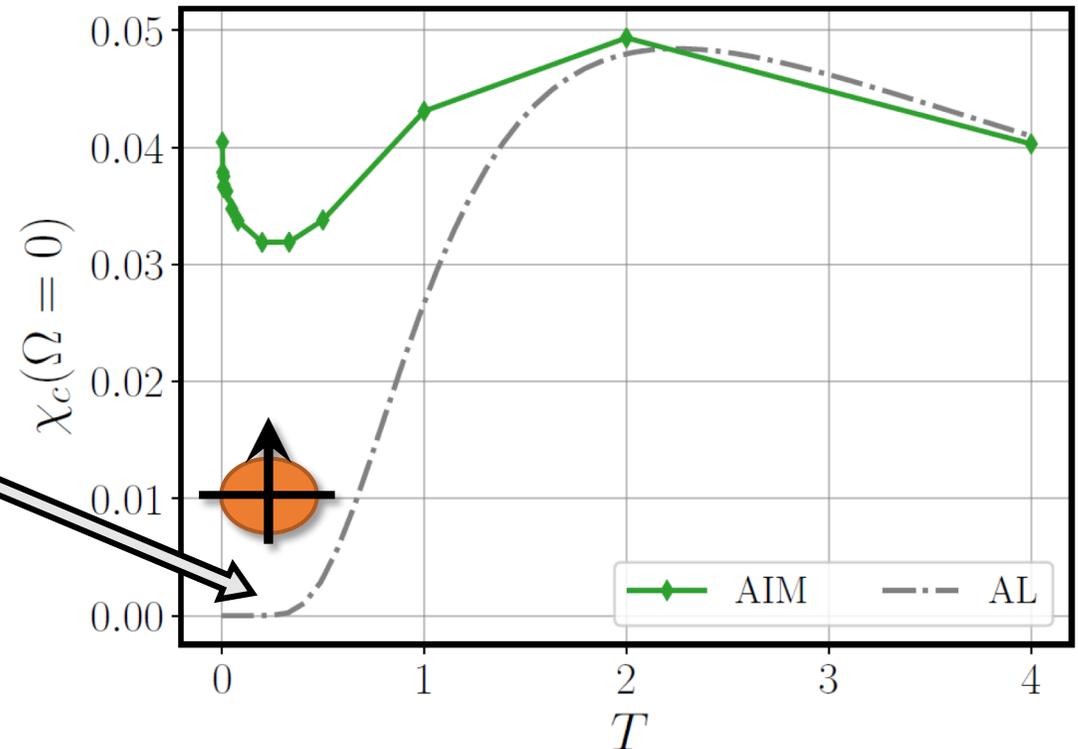
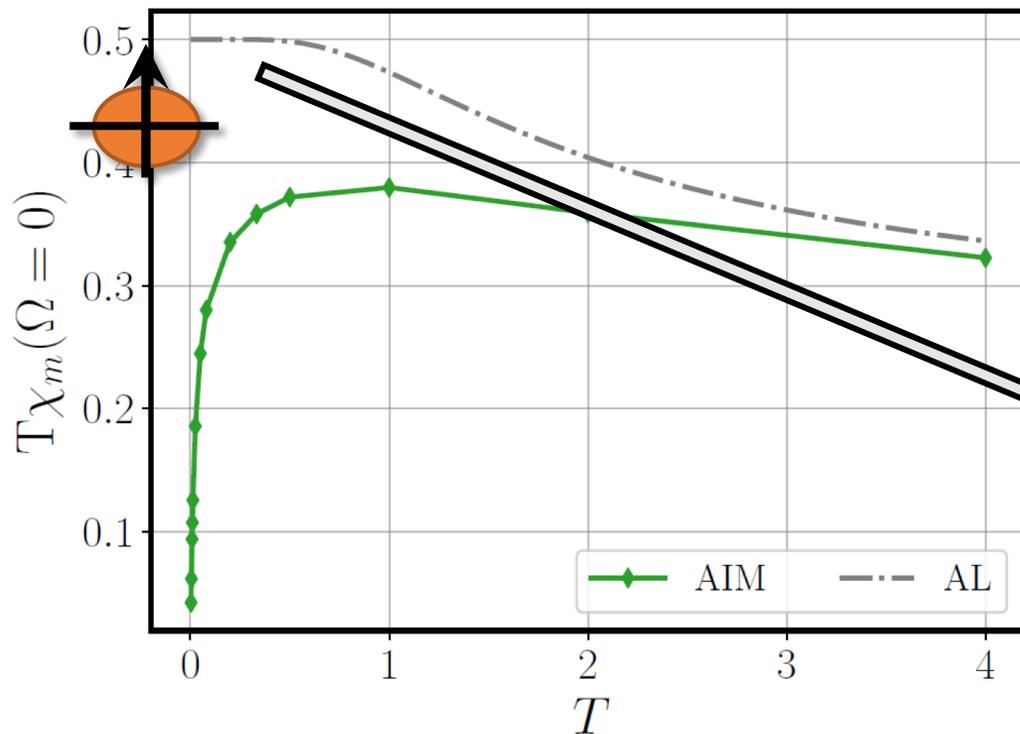
M.Wallerberger, *et.al*, CPC **235**, 388 (2019)

magnetic response

$$\chi_m^{phys} = \frac{1}{\beta^2} \sum_{\nu\nu'} \chi_m^{\nu\nu'}$$

charge response

$$\chi_c^{phys} = \frac{1}{\beta^2} \sum_{\nu\nu'} \chi_c^{\nu\nu'}$$



Physical response of AL & AIM

• w2dynamics – CT-HYB

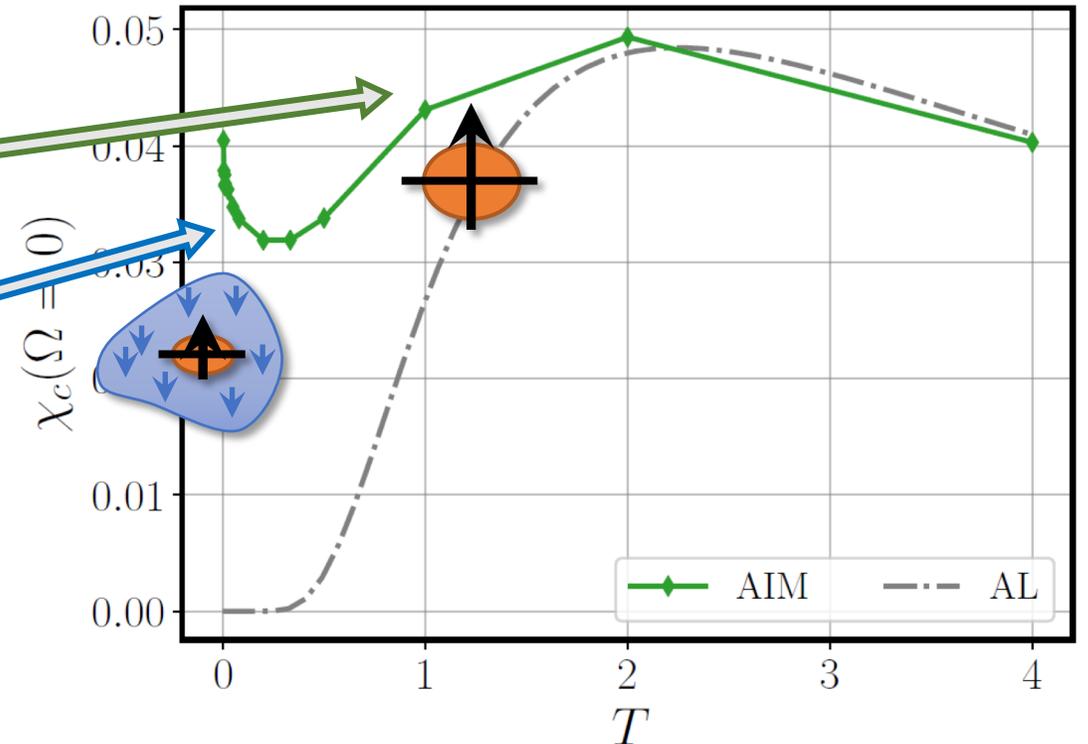
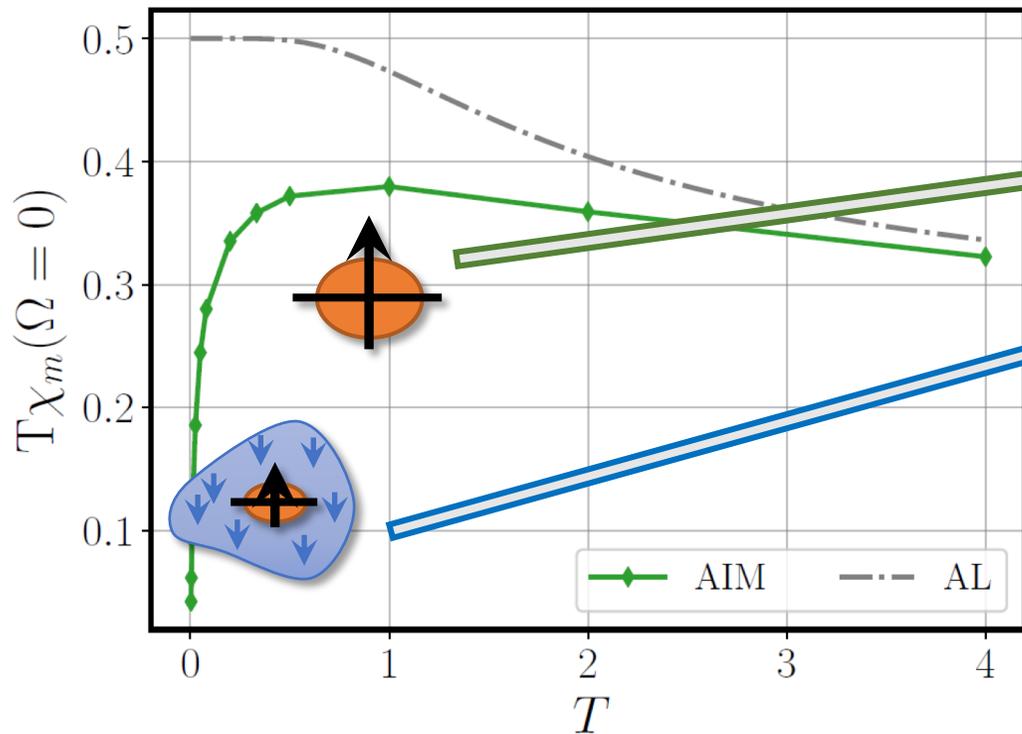
M.Wallerberger, *et.al*, CPC **235**, 388 (2019)

magnetic response

$$\chi_m^{phys} = \frac{1}{\beta^2} \sum_{\nu\nu'} \chi_m^{\nu\nu'}$$

charge response

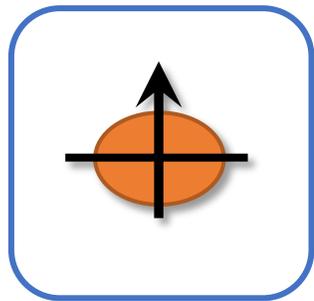
$$\chi_c^{phys} = \frac{1}{\beta^2} \sum_{\nu\nu'} \chi_c^{\nu\nu'}$$



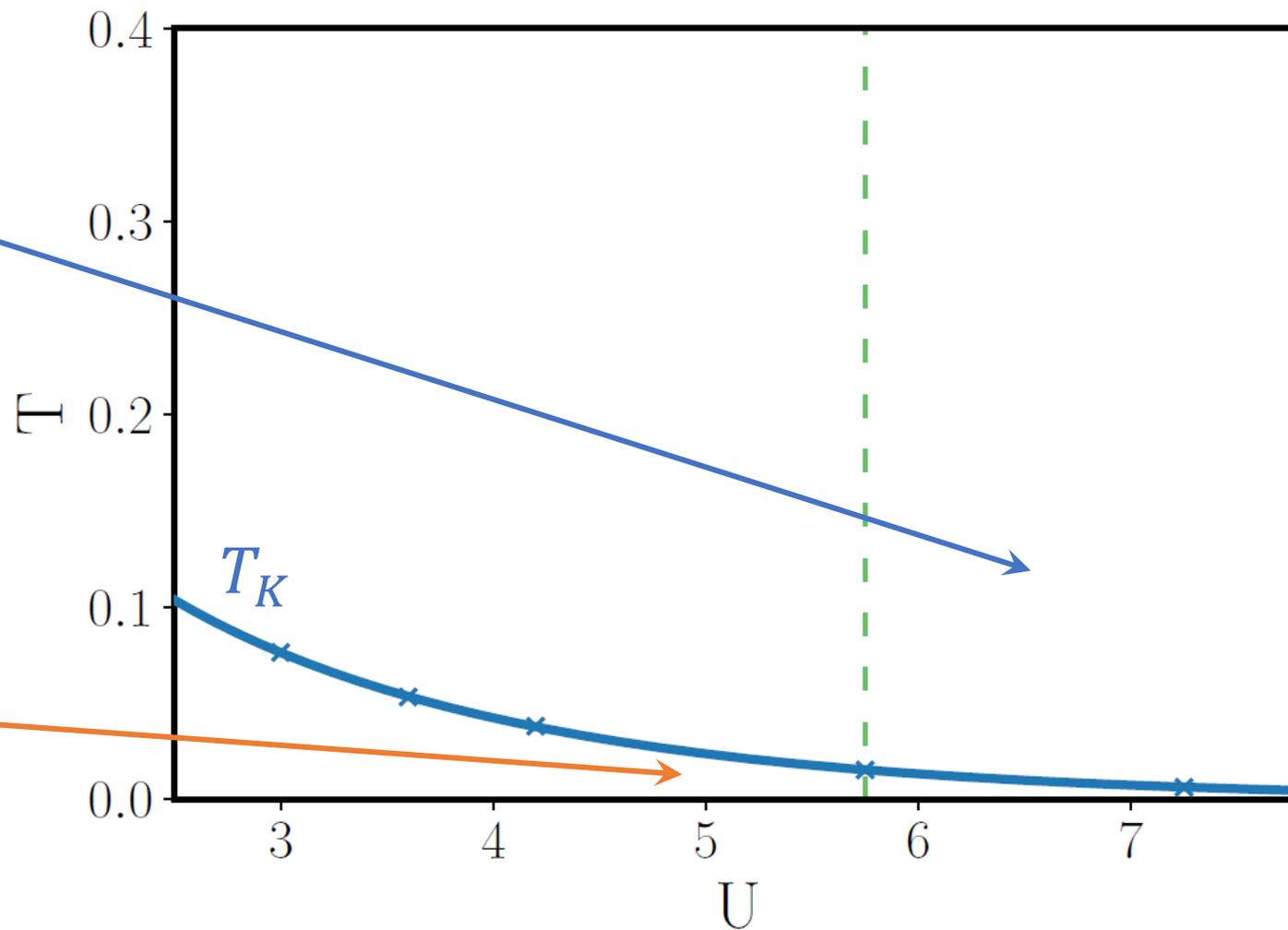
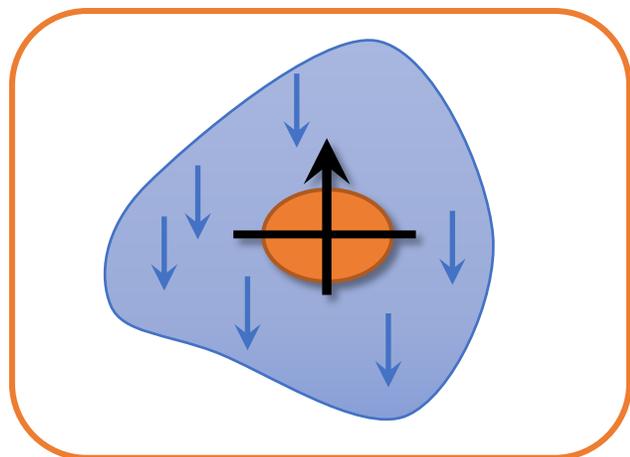
❖ P. Chalupa, T. Schäfer, M. Reitner, D. Springer, S. Andergassen, and A.T., PRL **126** 056403 (2021)

AIM phase-diagram

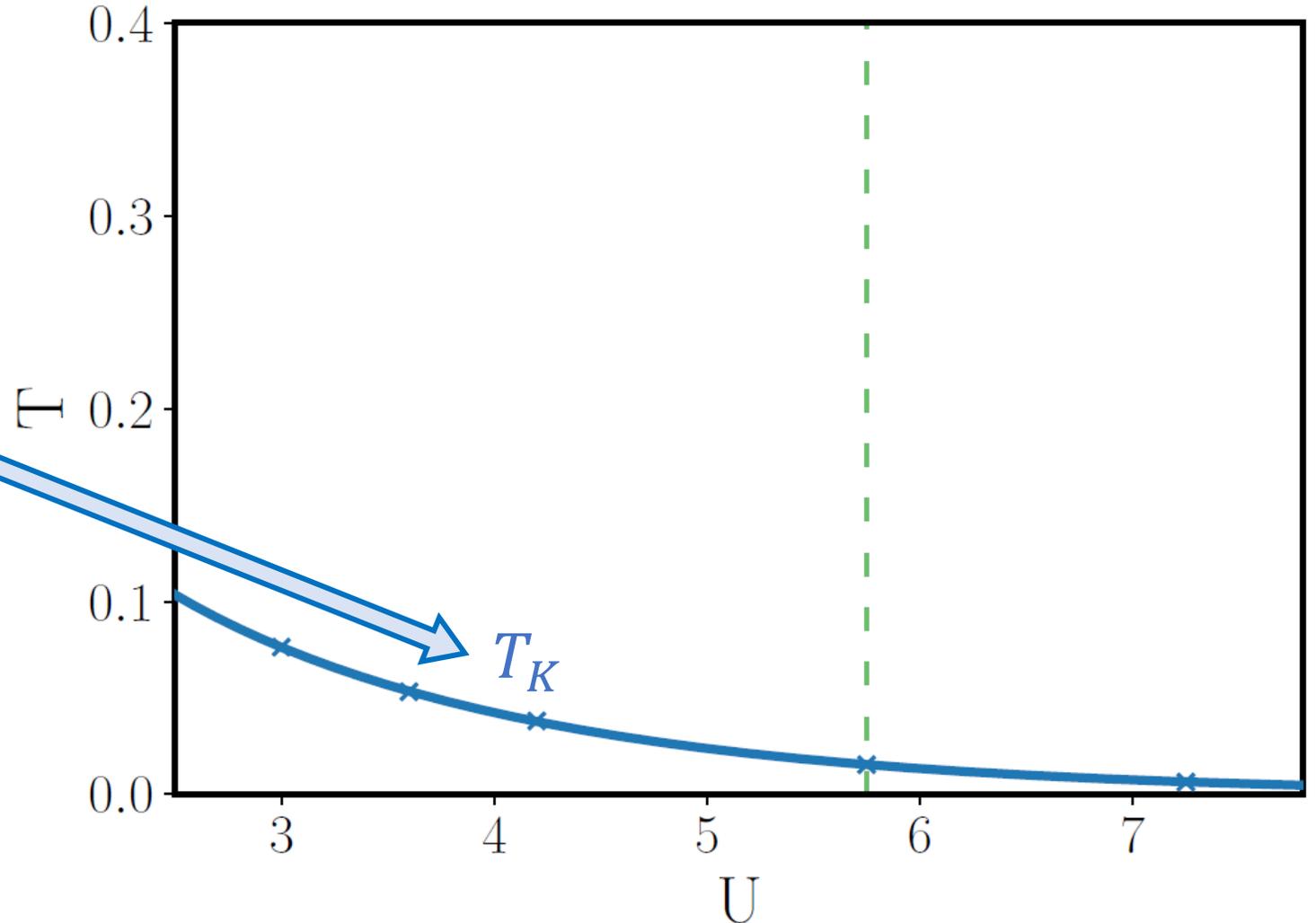
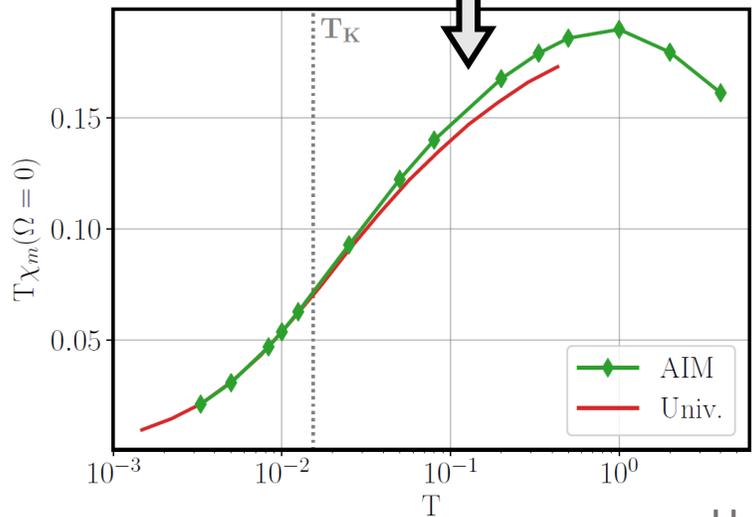
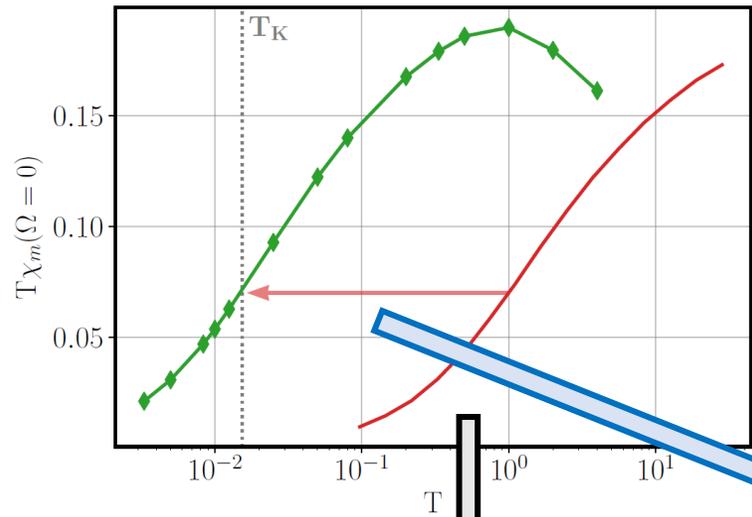
local moment



Kondo screening

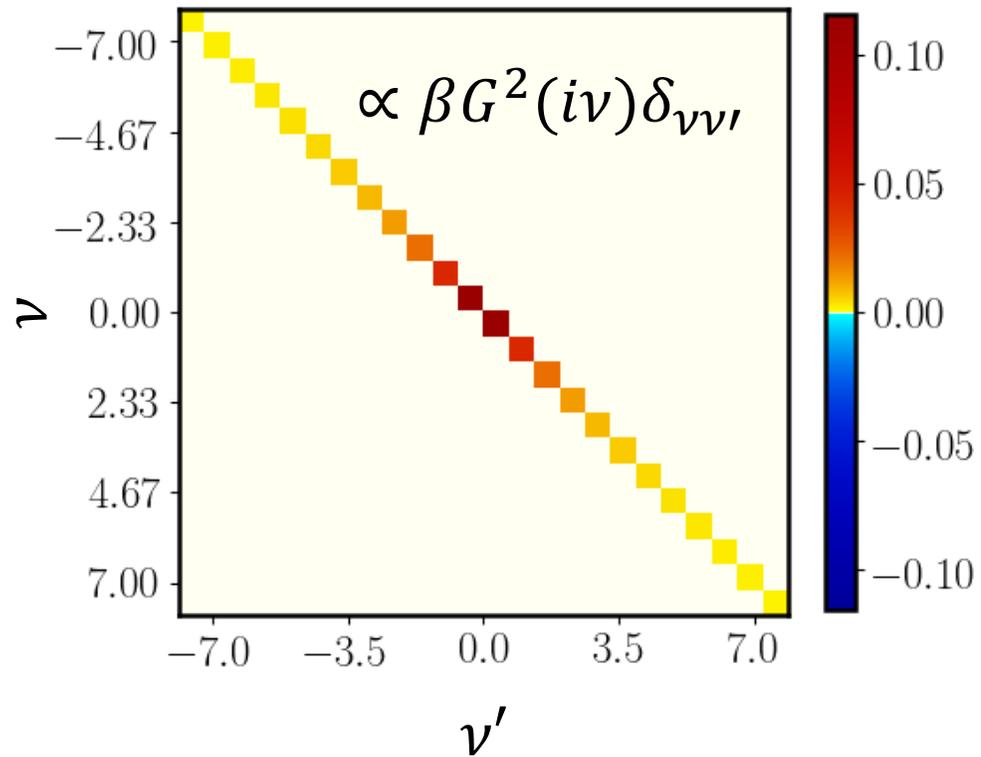
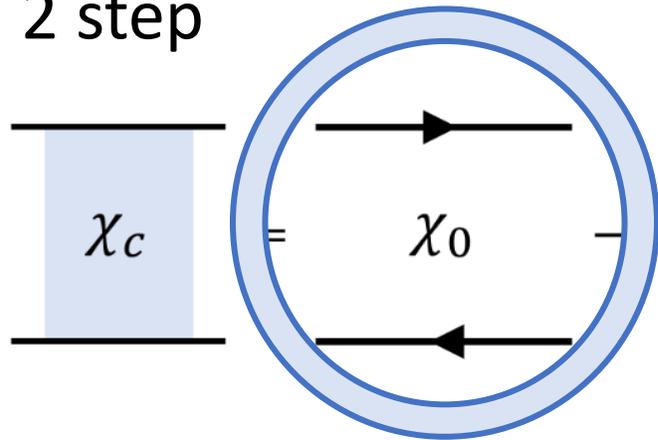


T_K of the AIM - conventional determination



1.Step: Non interacting case/bubble term

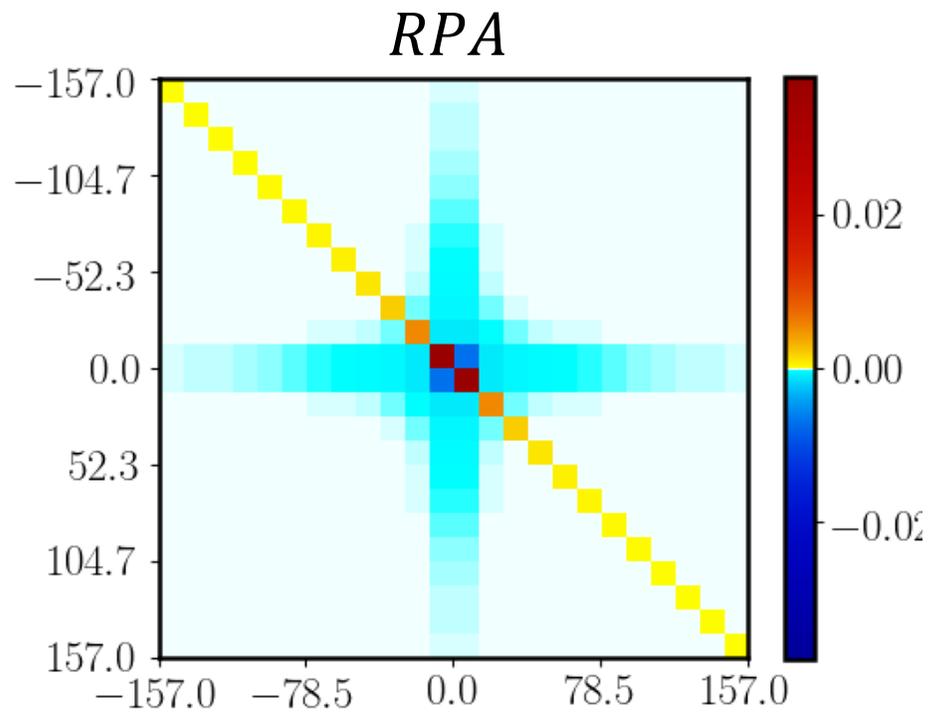
- 2 step



2.Step : Perturbative regime

e.g.: high temperature regime ($v = \pi T > U$)

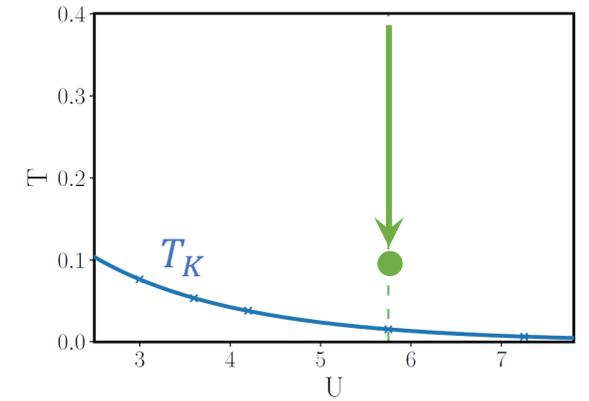
$$T_{high} = 2.0$$



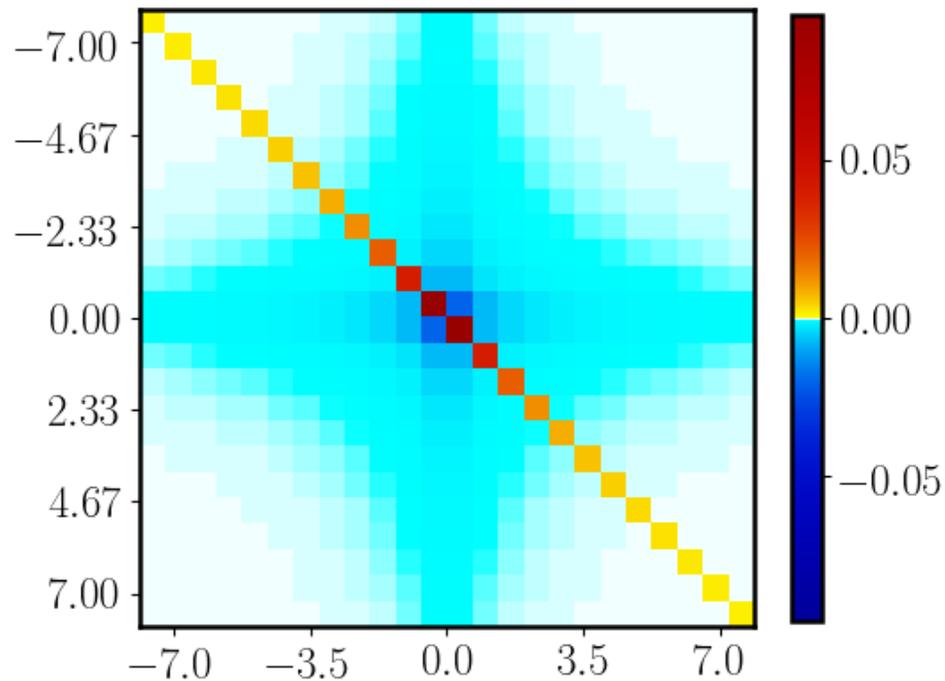
3.Step: Nonperturbative (local moment) regime

e.g.: intermediate temperature region ($T_K < T \ll U$)

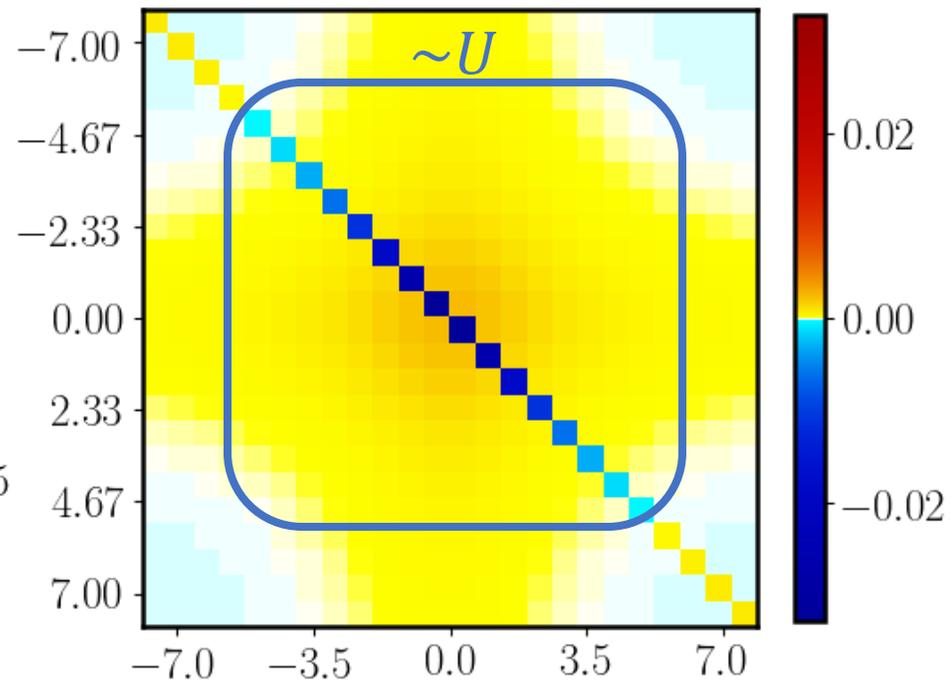
$$T_{int} = 0.1$$



RPA

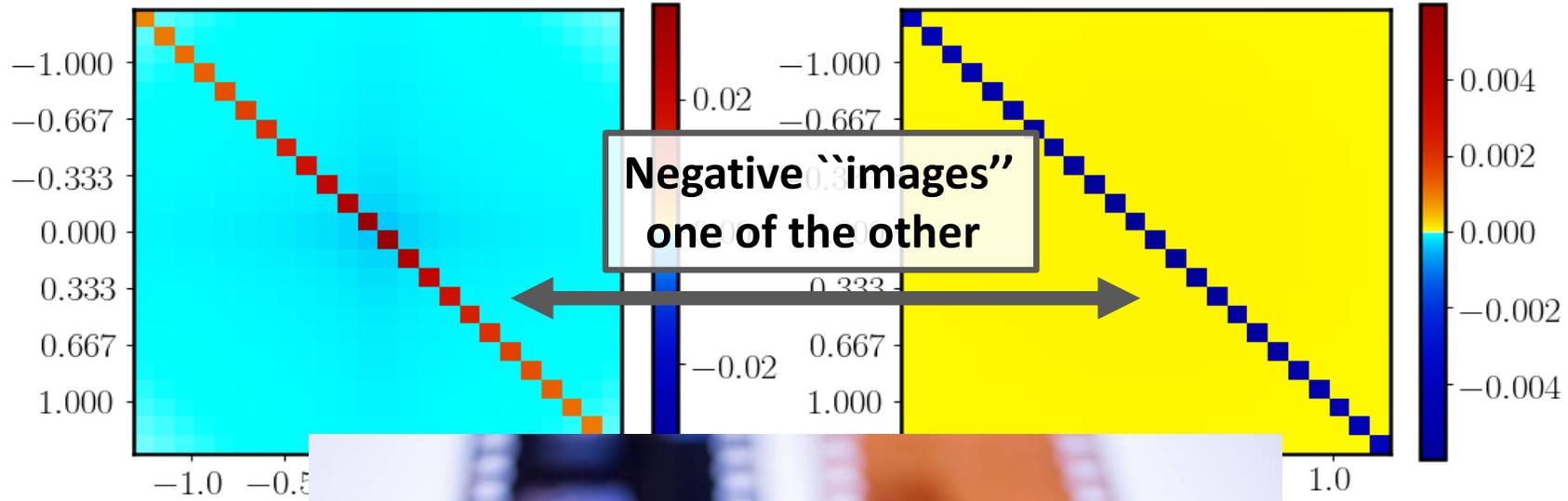


AL

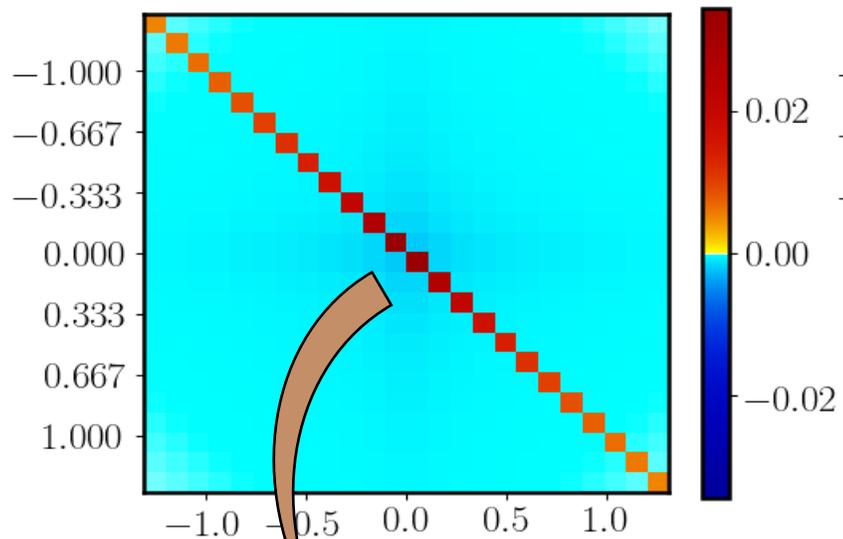


RPA

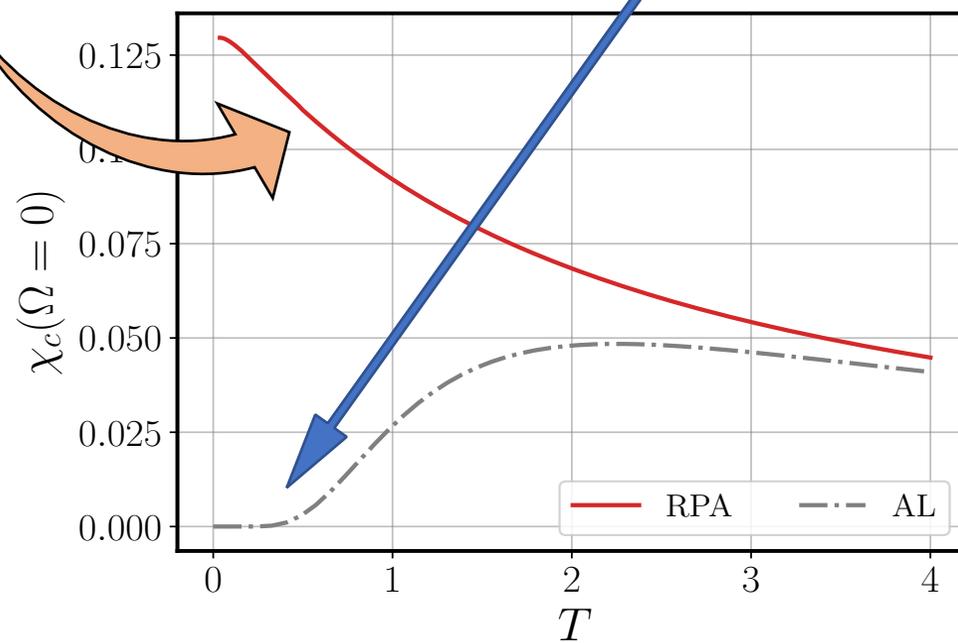
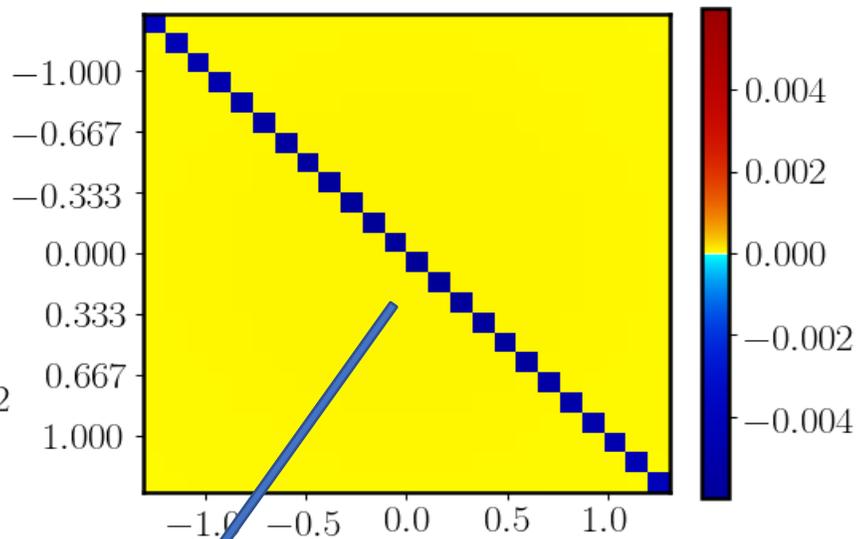
AL



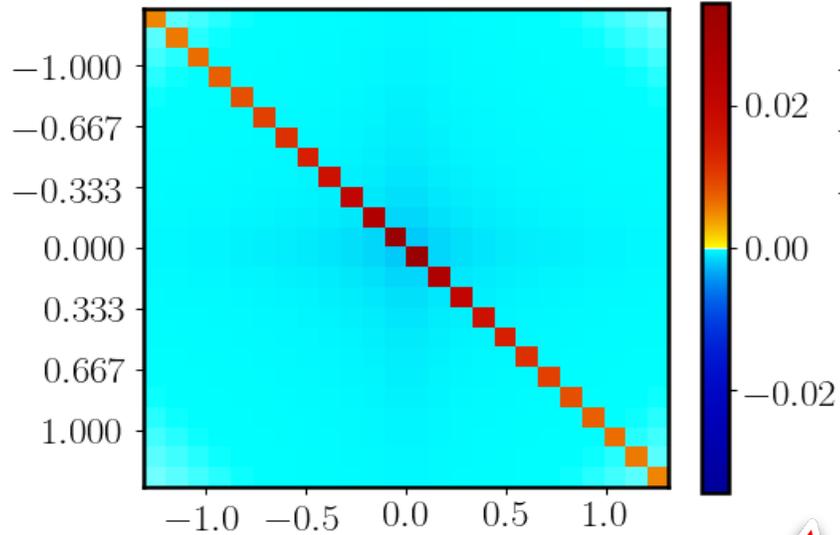
RPA



AL



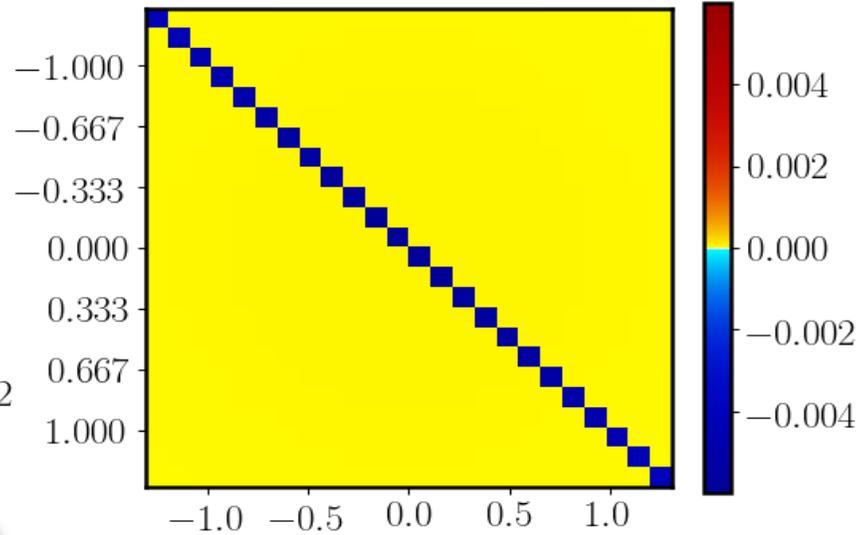
RPA



positive diagonal
dominates

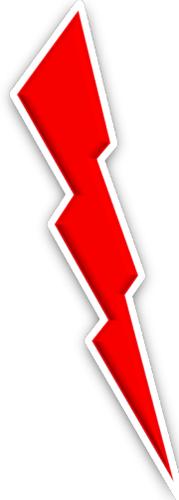
→ all eigenvalues
 $\lambda > 0$

AL



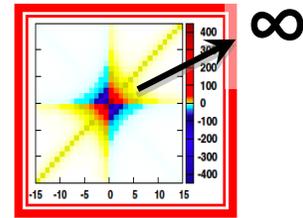
negative diagonal
dominates

→ all low-frequency
eigenvalues $\lambda < 0$

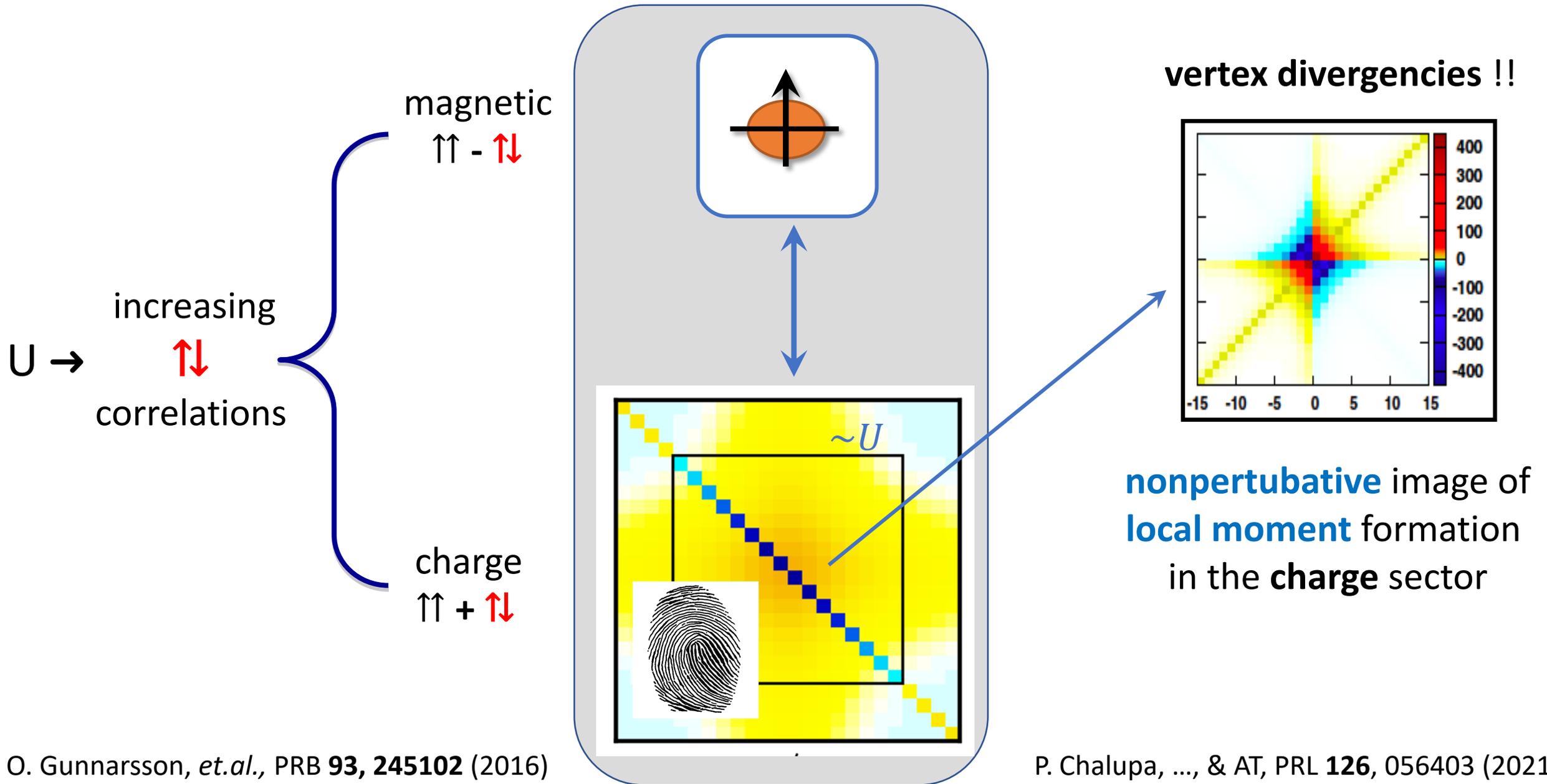


$\lambda = 0$

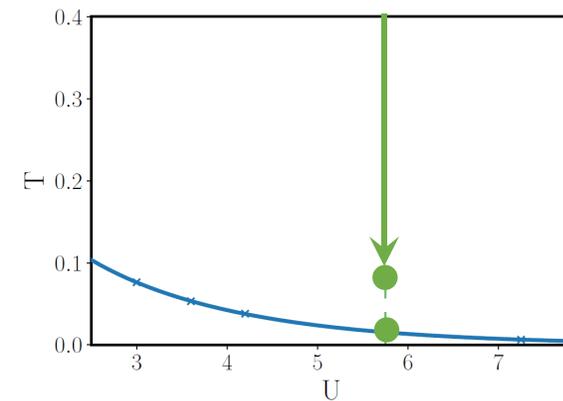
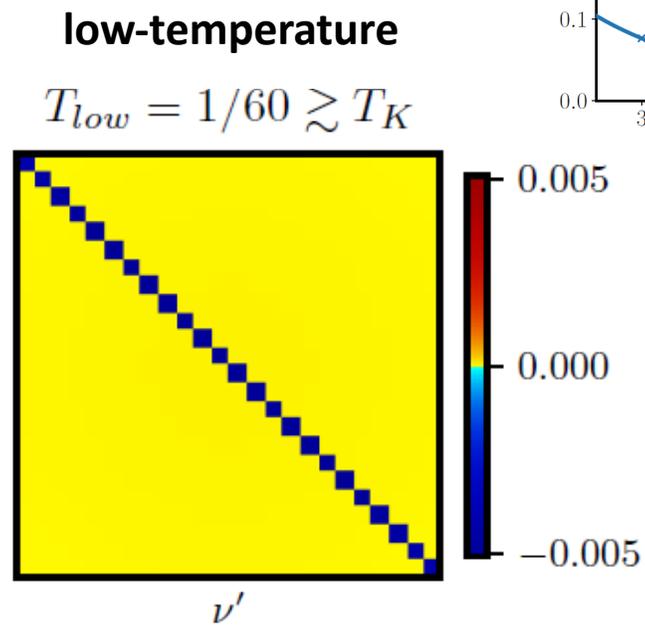
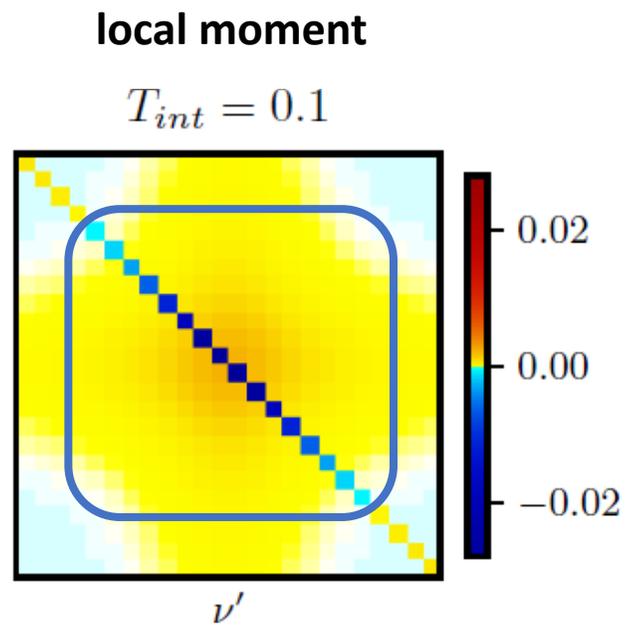
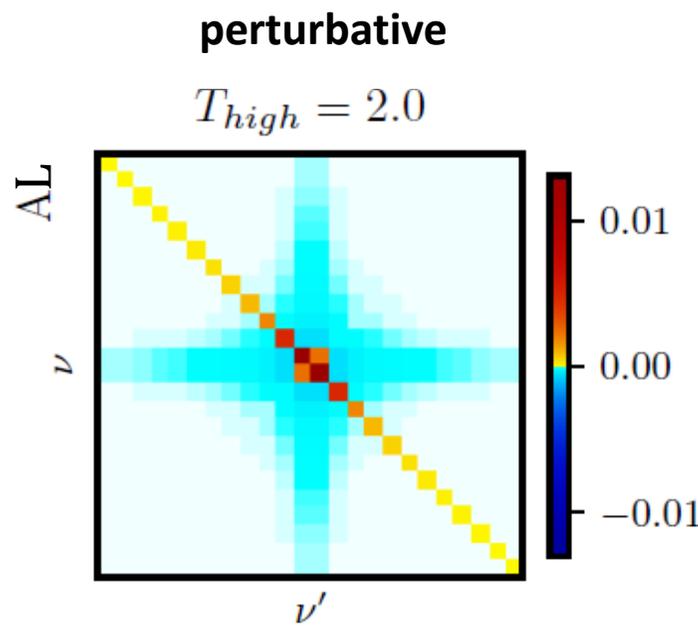
vertex divergences !!!



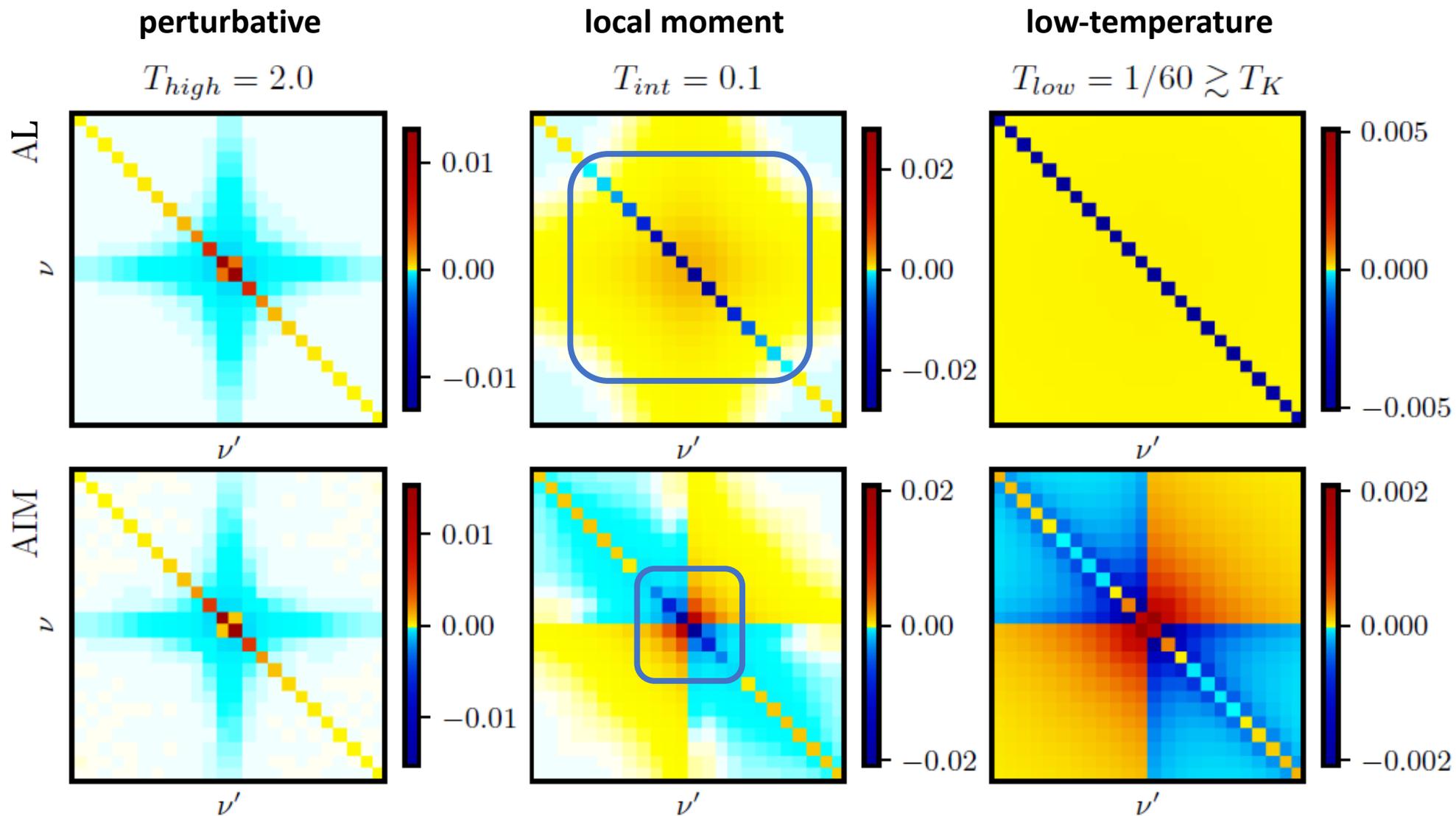
Fingerprints of the local moment !



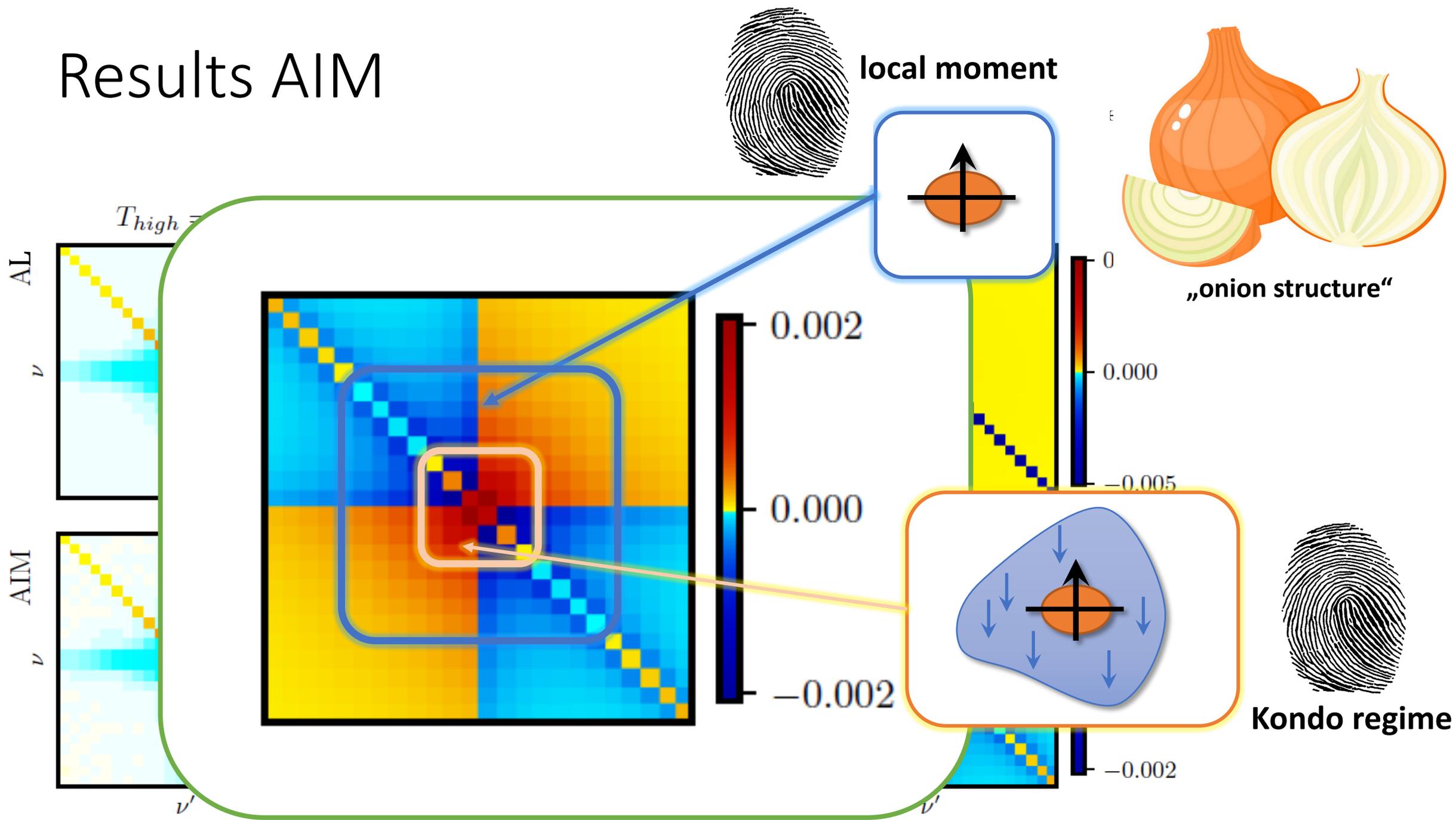
Results AIM



Results AIM



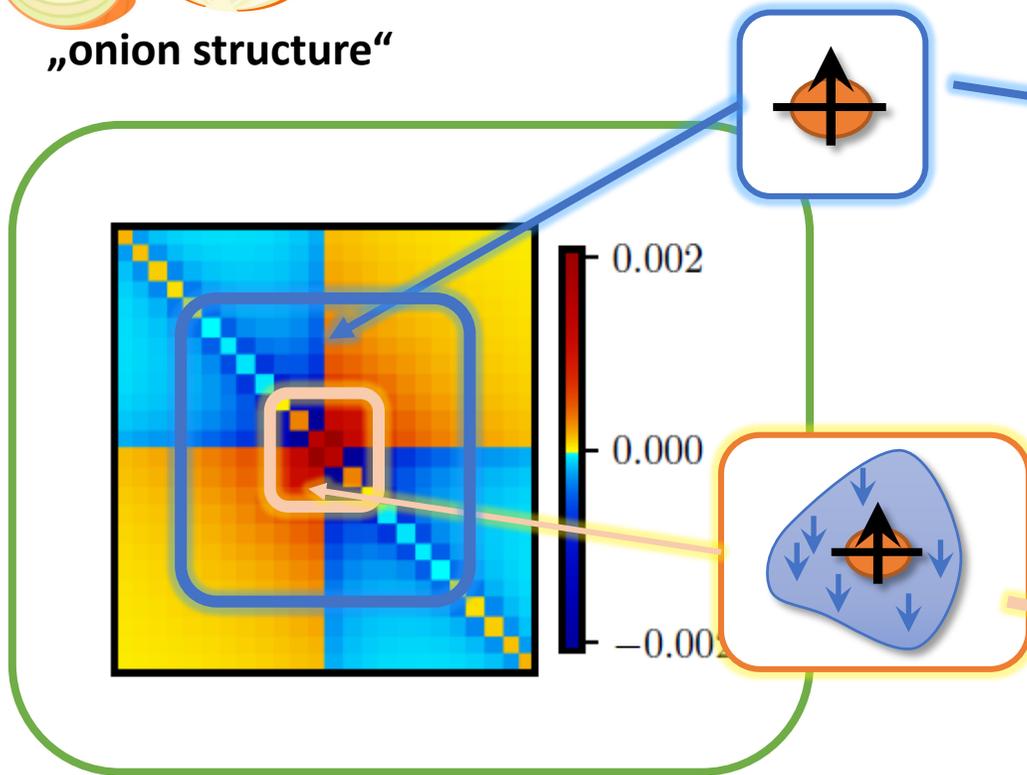
Results AIM



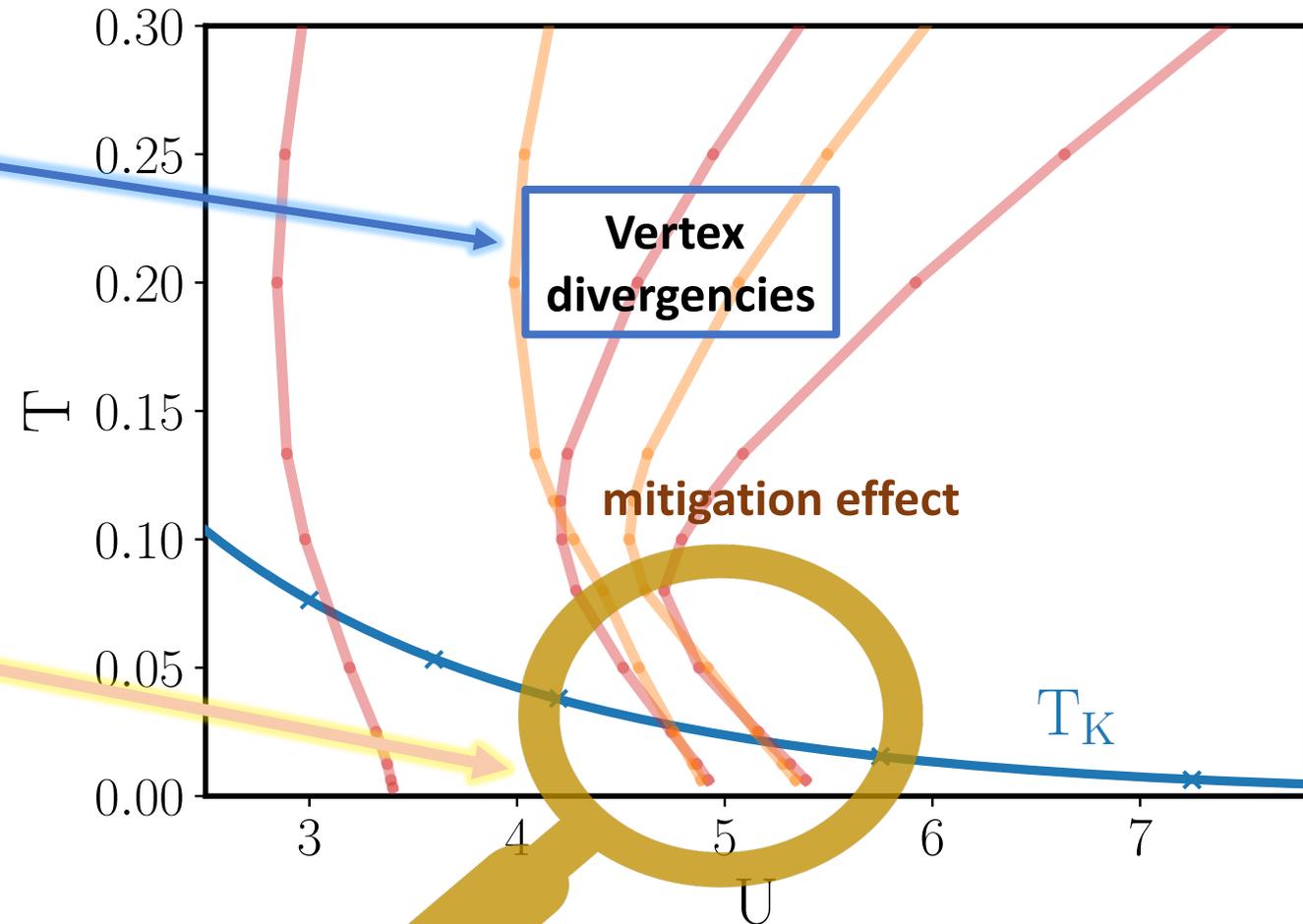
Effect of the vertex divergencies of the AIM



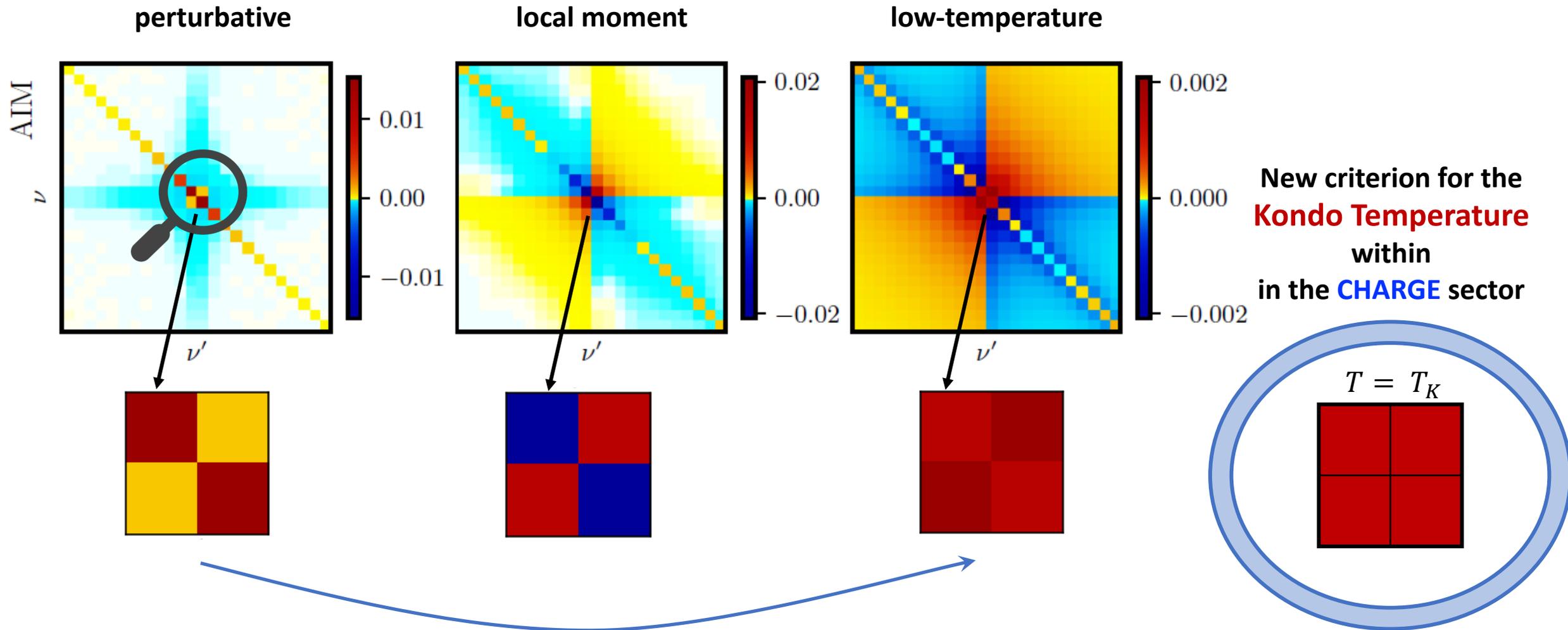
„onion structure“



lines where Γ diverges

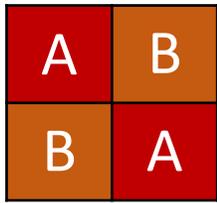


Results AIM – low frequency regime

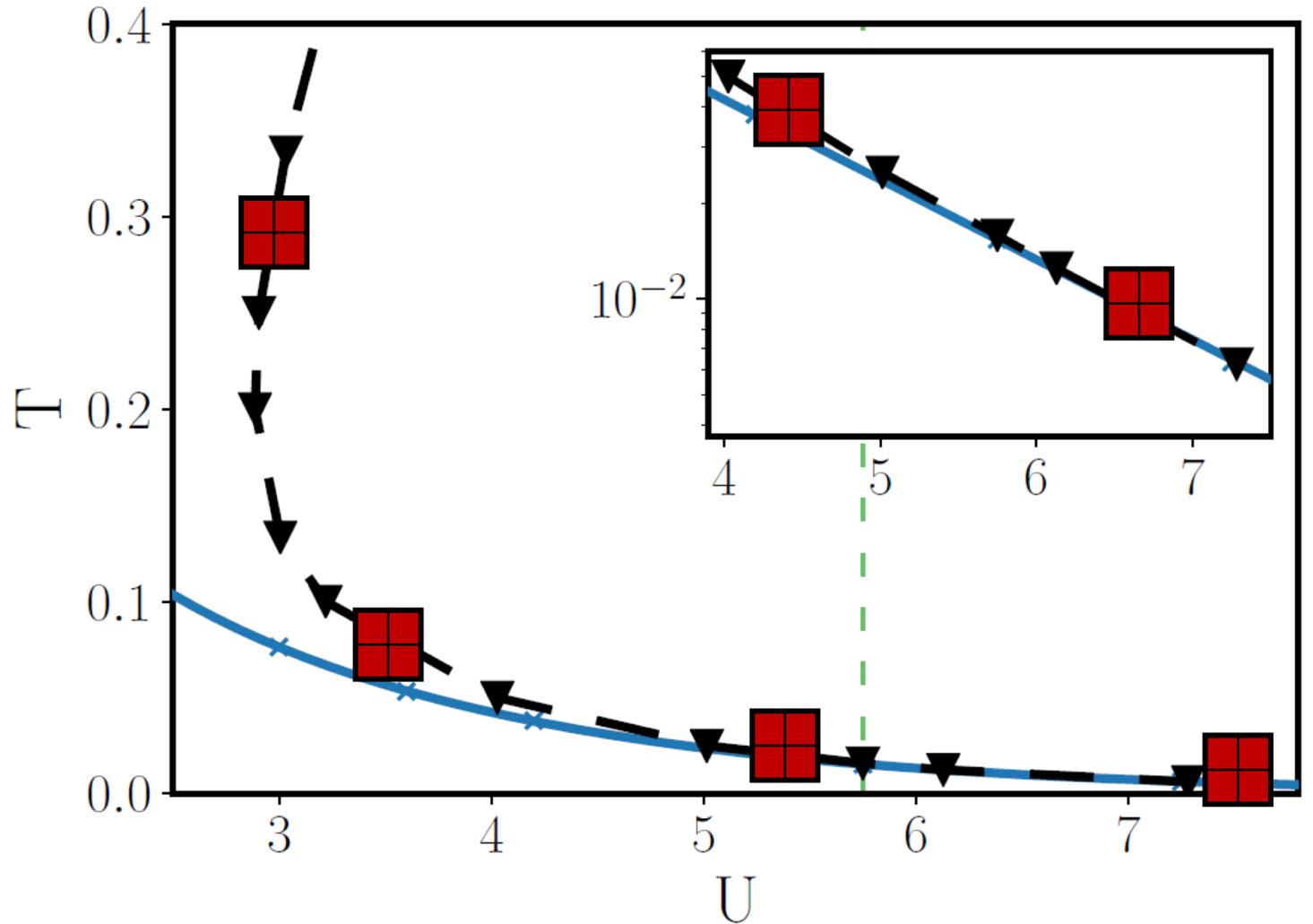
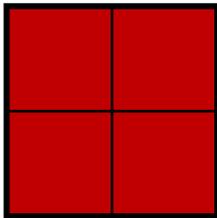


T_K criterion

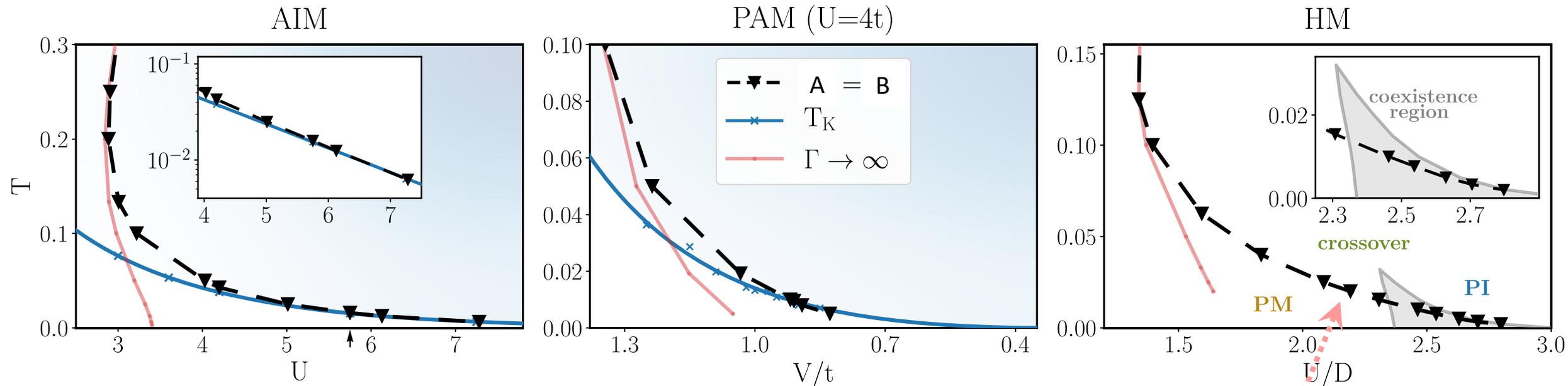
- Only for one U – what happens at other U ?



$$A = B \leftrightarrow T = T_K \quad \blacktriangledown$$

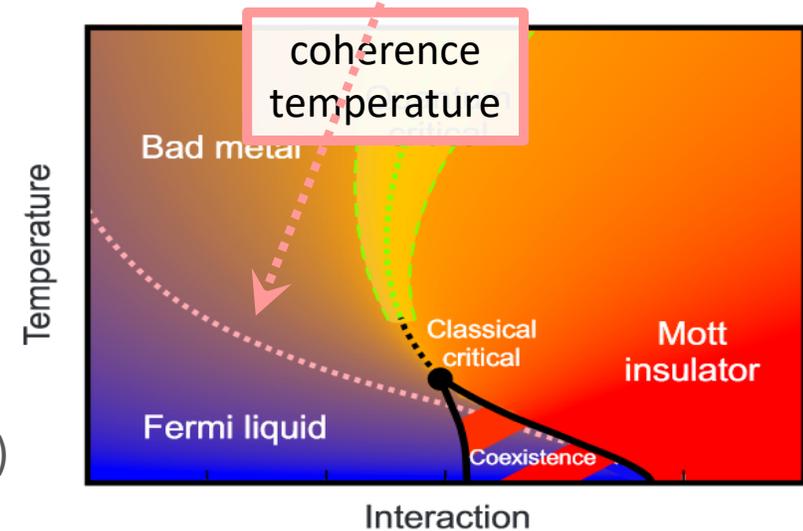


Beyond the single impurity AIM



❖ *P. Chalupa, T. Schäfer, M. Reitner, D. Springer, S. Andergassen, and A.T., PRL 126 056403 (2021)*

J.Vucicevic, et.al, PRB 88, 075143 (2013)



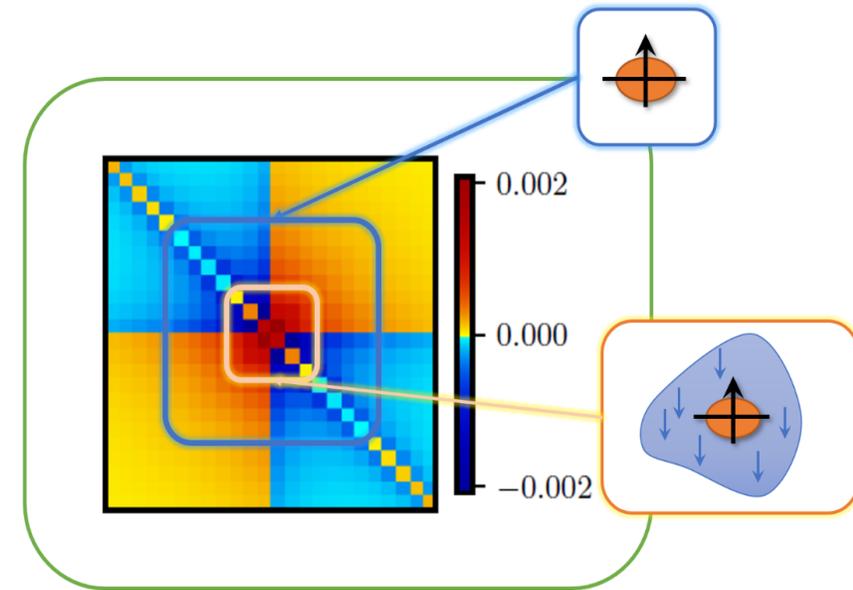
Our questions

1) What is the **physical ``origin''** for the **breakdown** of the many-body perturbation theories ?

- Learn how to read the physics from the 2P quantities
- **FINGERPRINTS** of the local moment formation and of its Kondo screening

2) Are there relevant **physical consequences?**

- **Surprising implications** for the non-local properties !!



Suppression = Vertex Divergences – the whole story?

LOCAL

local moment
&
suppression
of charge response

$$\Gamma_c = \left[\chi_c^{-1} - \chi_0^{-1} \right]$$

$$\chi_c = \left[\Gamma_c + \chi_0^{-1} \right]^{-1}$$

effective interaction

$$\chi_c^q = \left[\Gamma_c + \chi_{q,0}^{-1} \right]^{-1}$$

NON-LOCAL

$$\chi_q = \frac{\chi_q^0}{1 + \Gamma_0 \chi_q^0}$$

The isothermal compressibility (κ) of the Hubbard model

$$\kappa = -\frac{1}{V} \frac{\partial V}{\partial p} \propto \left\{ \begin{array}{l} \frac{\partial n}{\partial \mu} \\ \chi_c^{q=0} (\Omega = 0) \end{array} \right.$$
$$\chi_c^q = \left[\Gamma_c + \chi_{q,0}^{-1} \right]^{-1}$$

Compressibility Divergence and the Finite Temperature Mott Transition

G. Kotliar, *et. al.*, PRL **89** 046401 (2002)

Phase separation in the particle-hole asymmetric Hubbard model

M. Eckstein, *et. al.*, PRB **75** 125103 (2007)

The case of the isothermal compressibility κ

BSEq. in DMFT:

$$\kappa = \chi_c^{q=0} = \frac{2}{\beta^2} \sum_{\alpha} \left(\frac{1}{\lambda_{\alpha}} + t^2 / \beta \right)^{-1} w_{\alpha}$$

eigenvalues of χ_{loc}

weight
(from the
eigenvectors of χ_{loc})

The case of the isothermal compressibility κ

eigenvalues of χ_{loc}

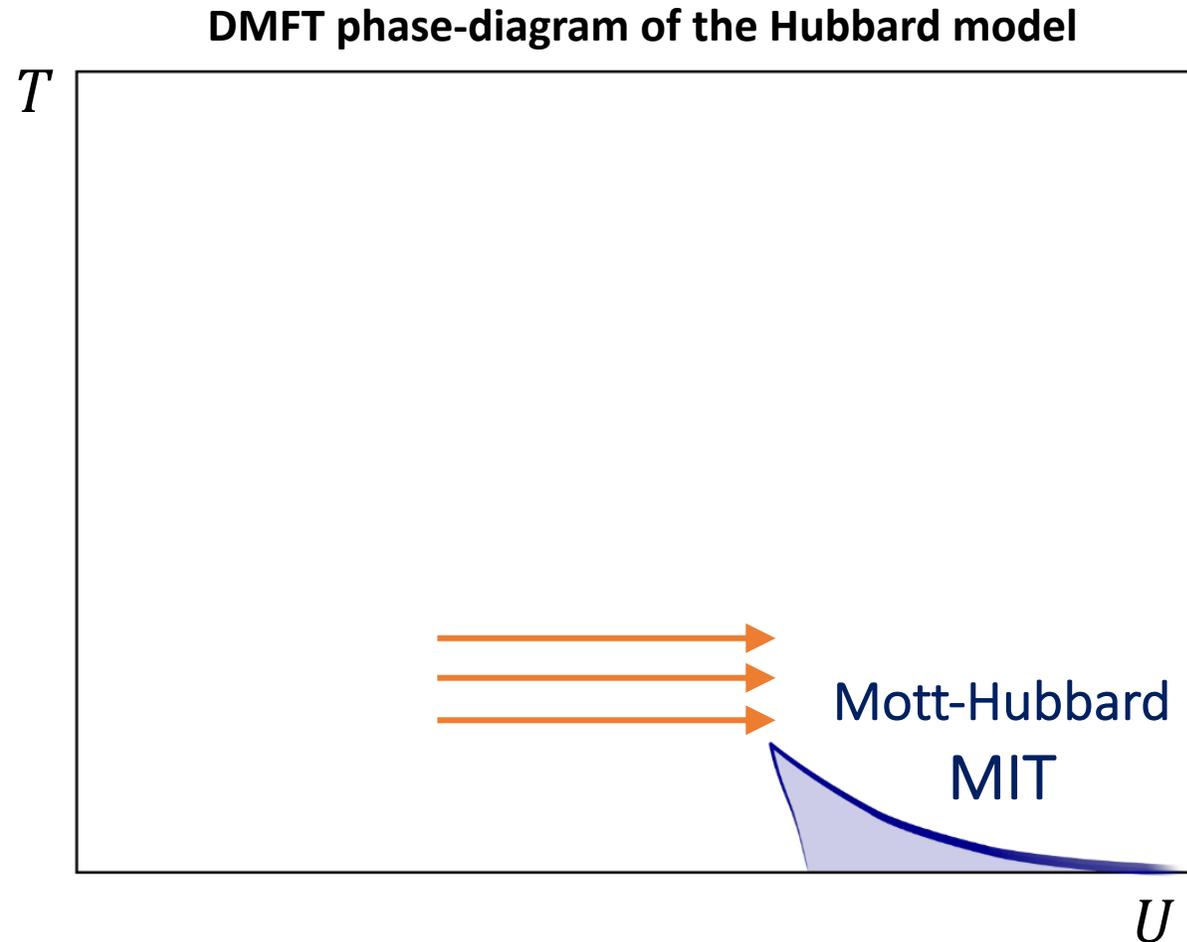
BSEq. in DMFT:

$$\kappa = \chi_c^{q=0} = \frac{2}{\beta^2} \sum_{\alpha} \left(\frac{1}{\lambda_{\alpha}} + t^2/\beta \right)^{-1} w_{\alpha}$$

$$\text{If } 1/\lambda_{\alpha} \approx -t^2/\beta \quad \Rightarrow \quad \chi_{q=0} \rightarrow \pm\infty \quad \text{for } w_{\alpha} \neq 0$$

$$\lambda_{\alpha} < 0 \text{ only after vertex divergence } \lambda_{\alpha} = 0 \Leftrightarrow \Gamma_{loc} \propto \chi_{loc}^{-1} = \infty$$

κ enhancement? Bethe lattice, half-filling



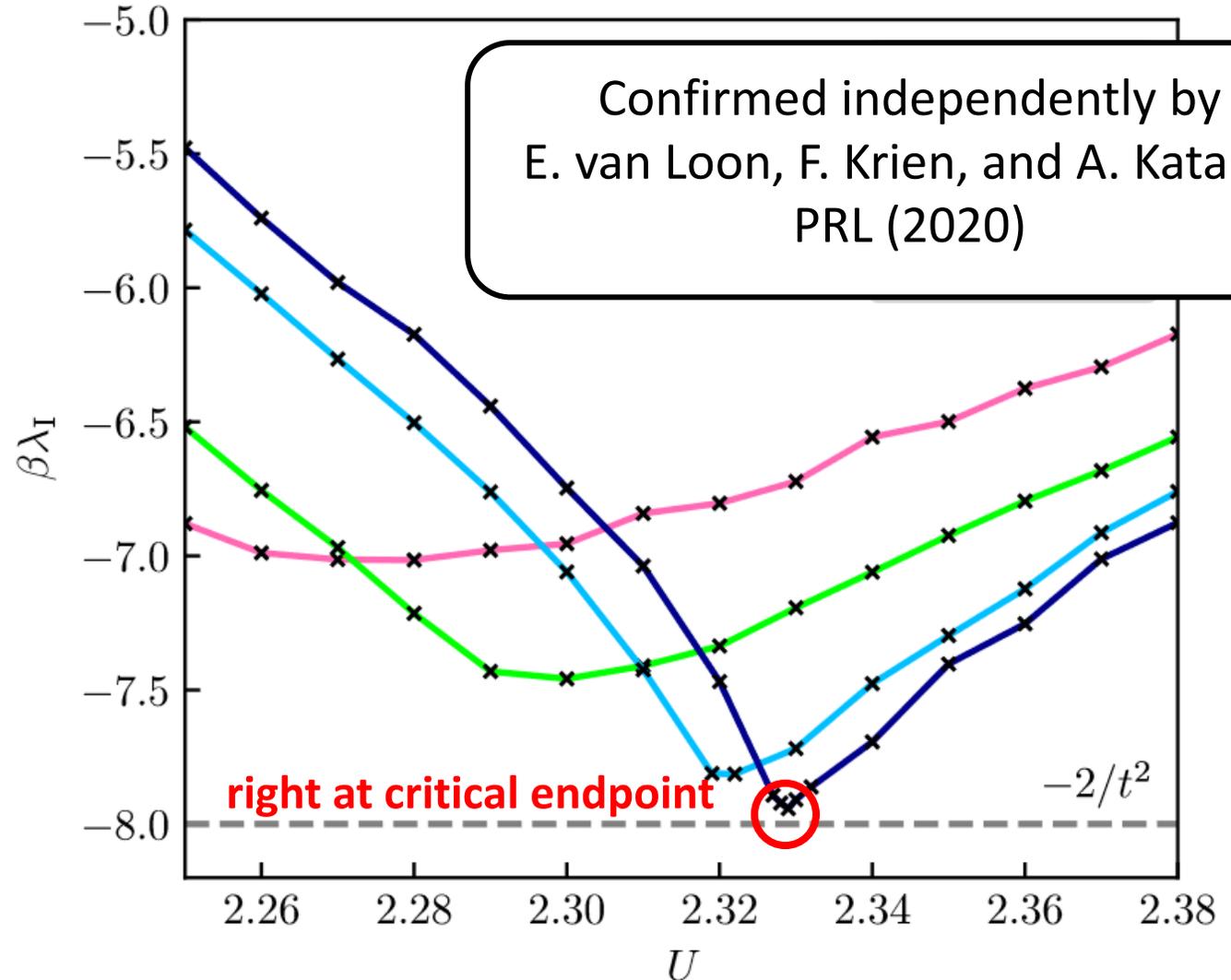
κ enhancement? Bethe lattice, half-filling

condition for lowest λ_α ($= \lambda_I$) fulfilled

$$\underbrace{\left(\frac{1}{\lambda_I} + t^2/\beta\right)^{-1}}_{\rightarrow 0} w_I$$

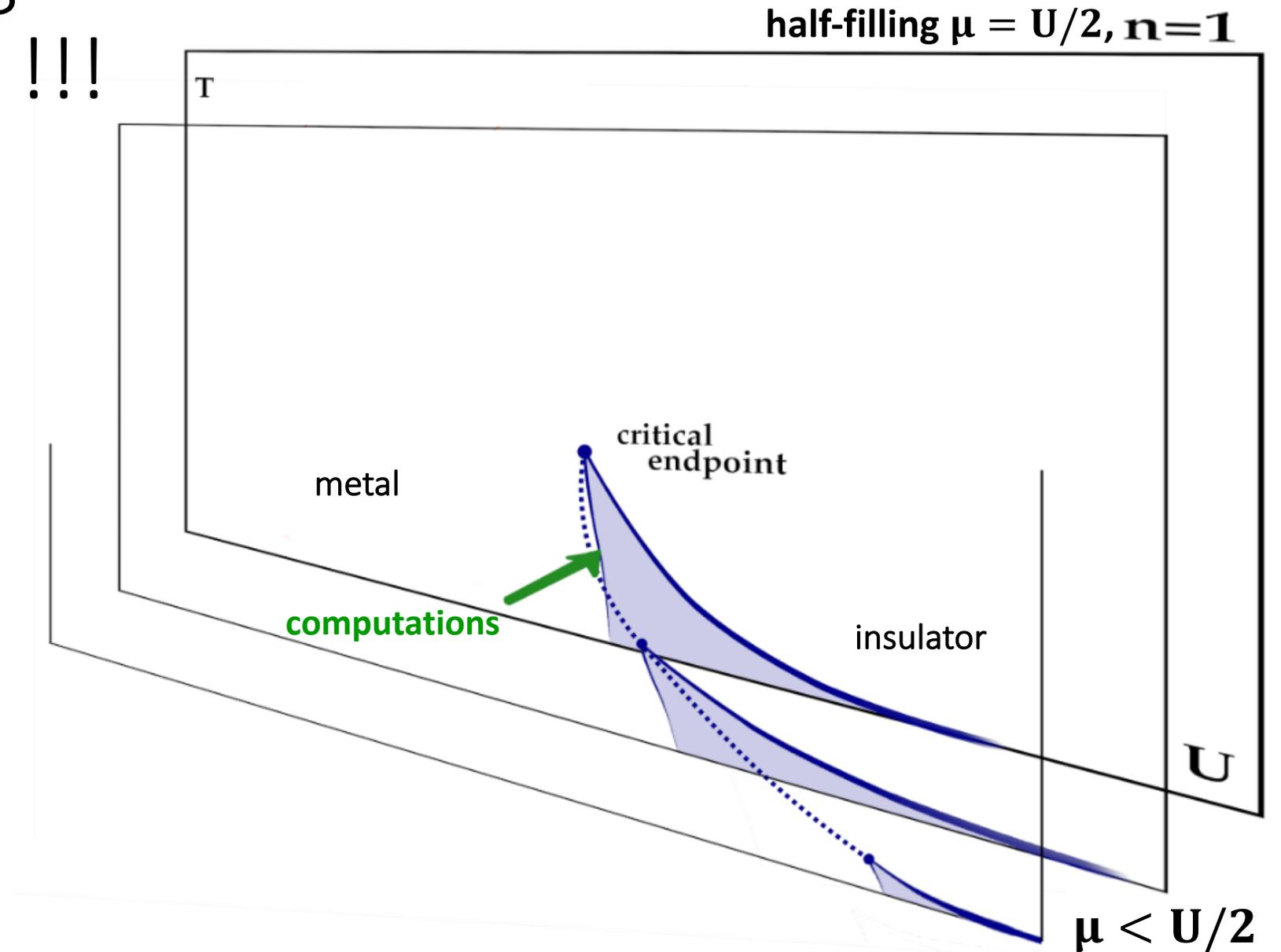
but **no** physical effect !

$w_I = 0$ due to perfect **particle-hole** symmetry



Out-of-half-filling: κ enhancement !!!

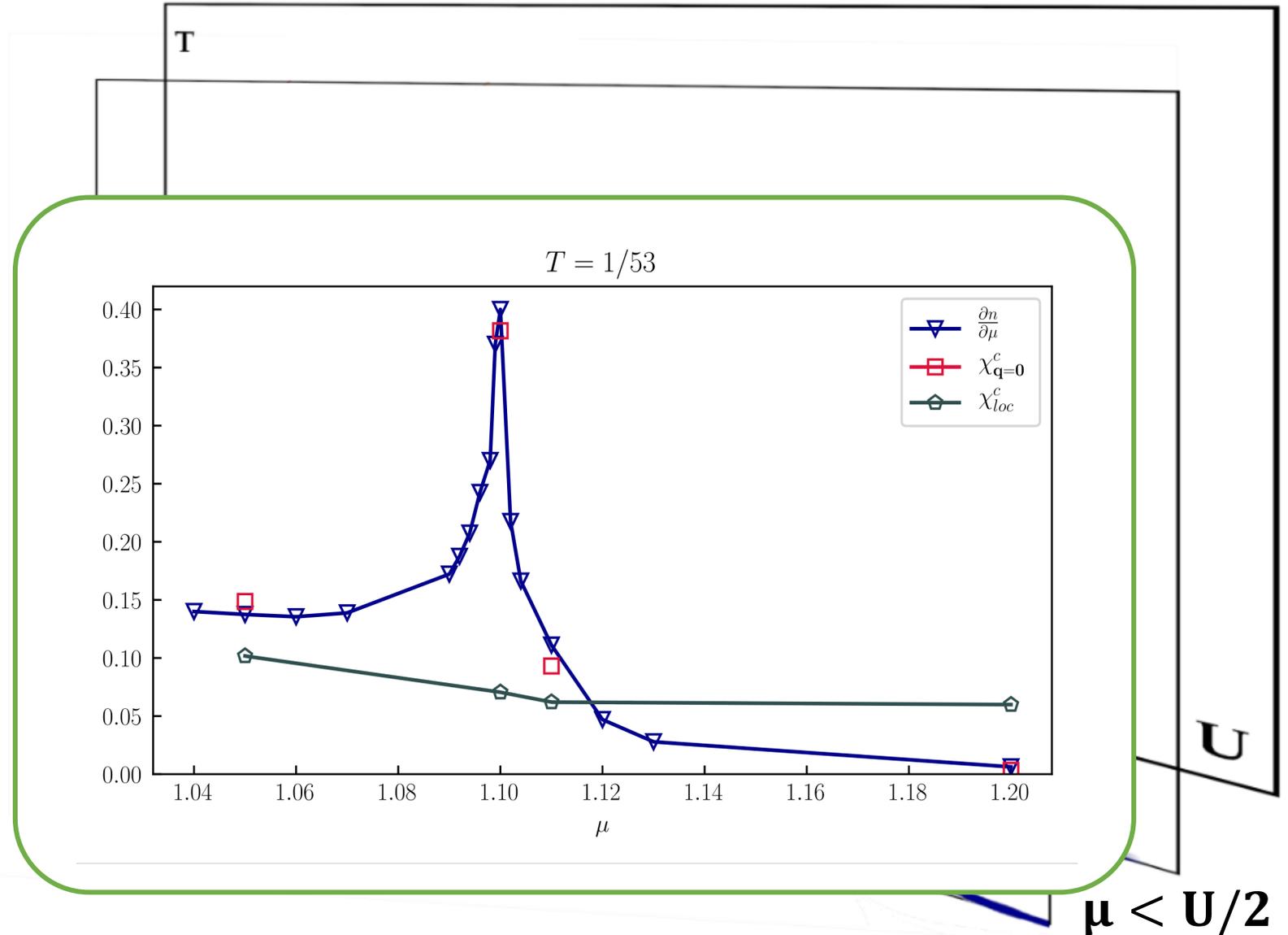
$$\kappa = \begin{cases} \frac{\partial n}{\partial \mu} \\ \chi_c^{q=0} (\Omega = 0) \end{cases}$$



κ enhancement !!!

half-filling $\mu = U/2, n=1$

$$\kappa = \begin{cases} \frac{\partial n}{\partial \mu} \\ \chi_c^{q=0} (\Omega = 0) \end{cases}$$



Diagnostic of κ

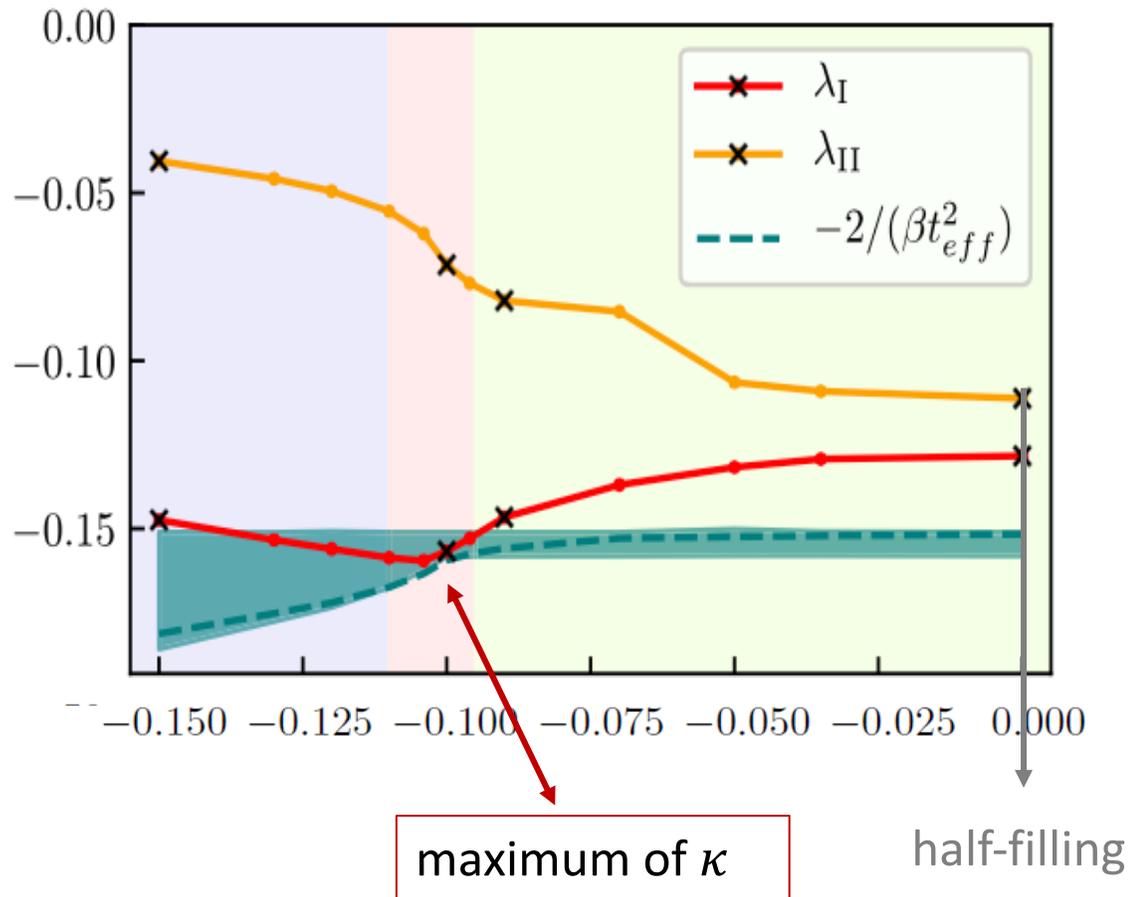
- BSE Eq. in DMFT: $\kappa = \chi_c^{q=0} = \frac{2}{\beta^2} \sum_{\alpha} \left(\frac{1}{\lambda_{\alpha}} + t^2/\beta \right)^{-1} w_{\alpha}$
- Spectral (=EV) Decomposition

$$\chi_{q=0} = \frac{2}{\beta^2} \left(\frac{1}{\lambda_I} + t^2/\beta \right)^{-1} w_I + \frac{2}{\beta^2} \left(\frac{1}{\lambda_{II}} + t^2/\beta \right)^{-1} w_{II} + \text{rest}$$

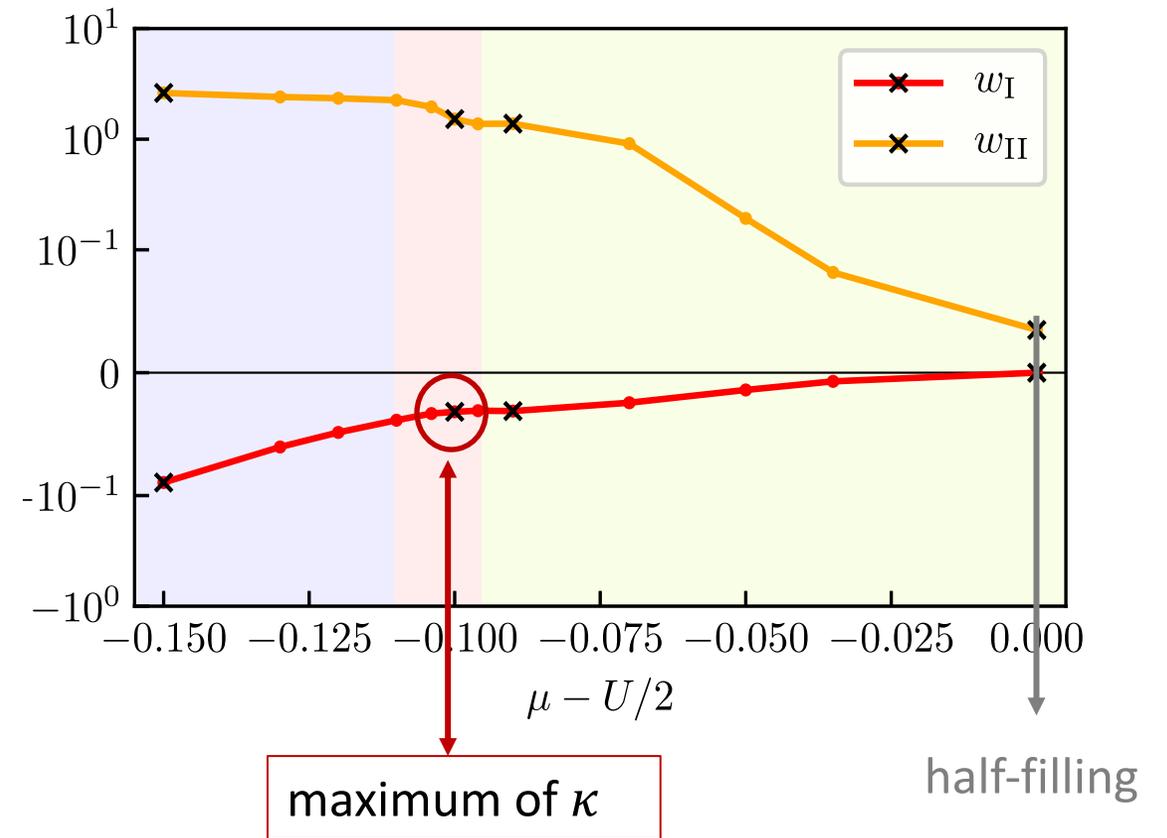
$$\chi_{loc} = \frac{2}{\beta^2} \lambda_I w_I + \frac{2}{\beta^2} \lambda_{II} w_{II} + \text{rest}$$

Diagnostic of κ

- lowest eigenvalues:



- their eigenweights:



Diagnostic of κ

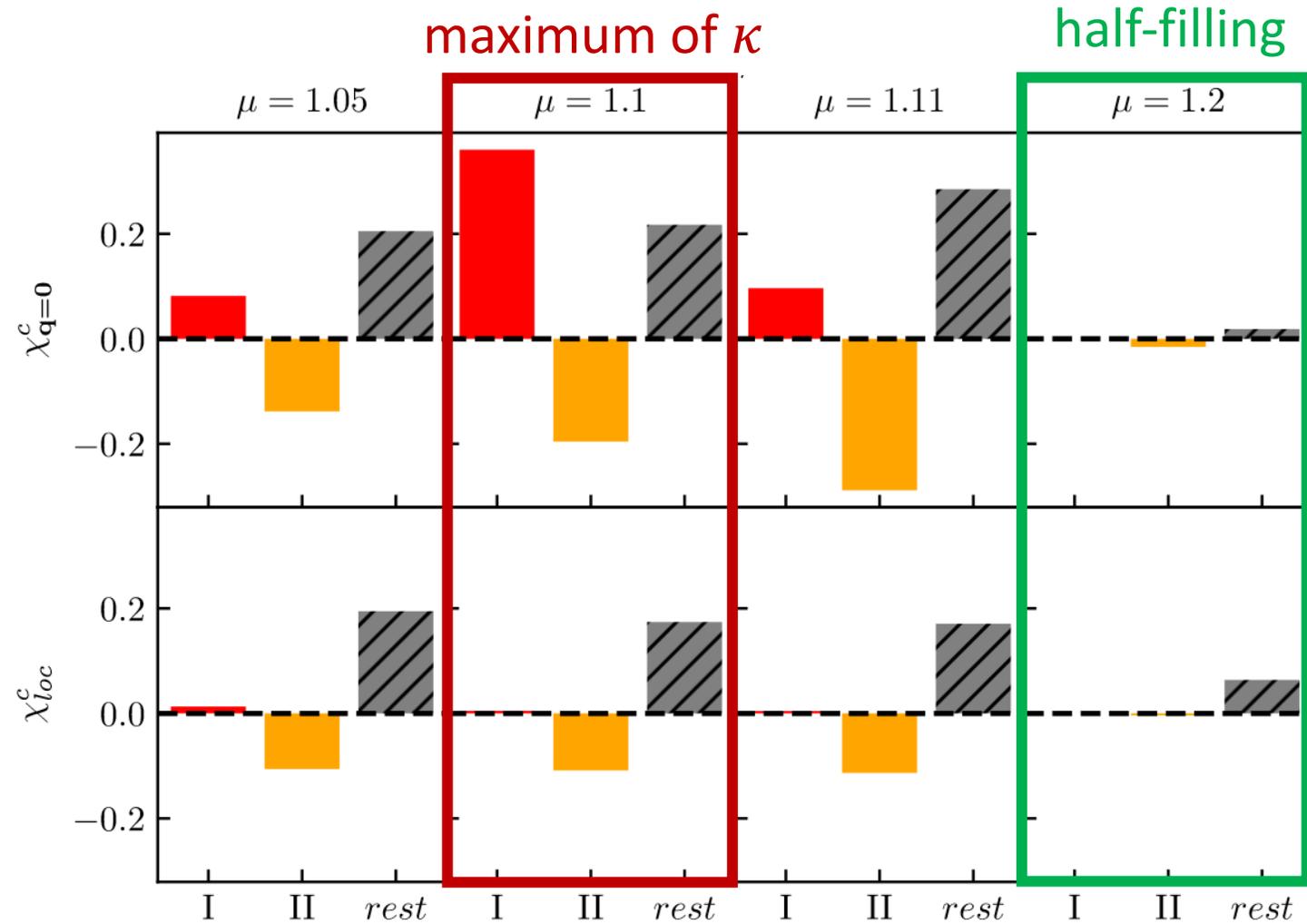
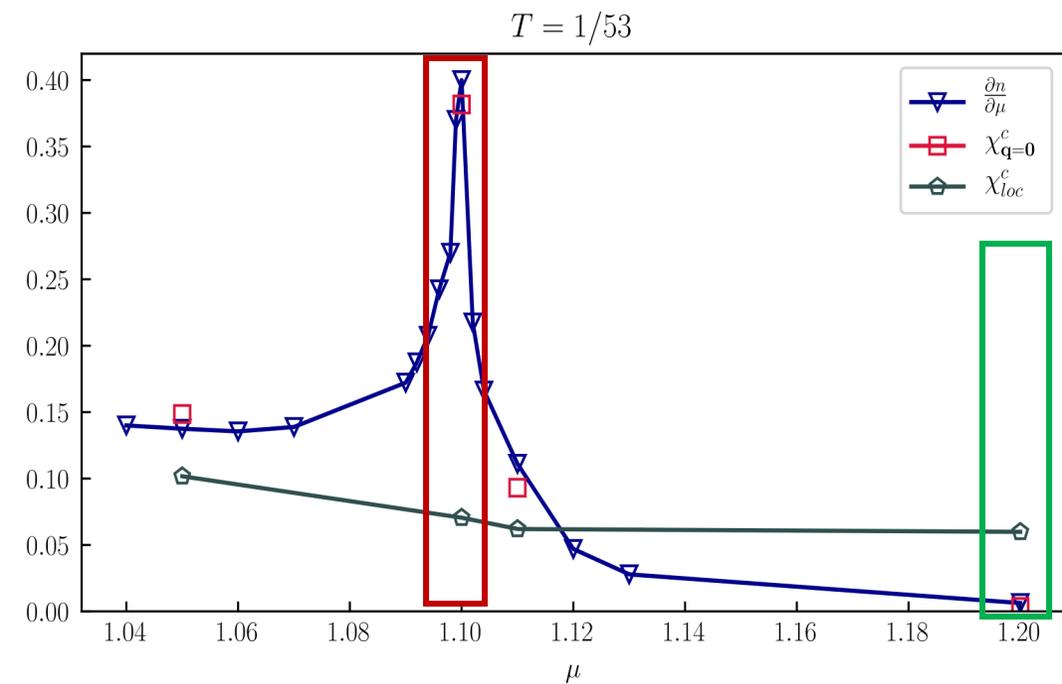
$$\begin{array}{ccccccc}
 \chi_{q=0} = \frac{2}{\beta^2} \left(\frac{1}{\lambda_I} + t^2/\beta \right)^{-1} w_I & + & \frac{2}{\beta^2} \left(\frac{1}{\lambda_{II}} + t^2/\beta \right)^{-1} w_{II} & + & \text{rest} & & \\
 \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
 \rightarrow -\infty & & < 0 & & < 0 & & > 0 \\
 \\
 \chi_{loc} = \frac{2}{\beta^2} \lambda_I w_I & + & \frac{2}{\beta^2} \lambda_{II} w_{II} & + & \text{rest} & &
 \end{array}$$

Diagnostic of κ

$$\chi_{q=0} = \underbrace{\frac{2}{\beta^2} \left(\frac{1}{\lambda_I} + t^2/\beta \right)^{-1} w_I}_{\text{effective attraction}} + \underbrace{\frac{2}{\beta^2} \left(\frac{1}{\lambda_{II}} + t^2/\beta \right)^{-1} w_{II}}_{\text{effective repulsion}} + \text{rest}$$

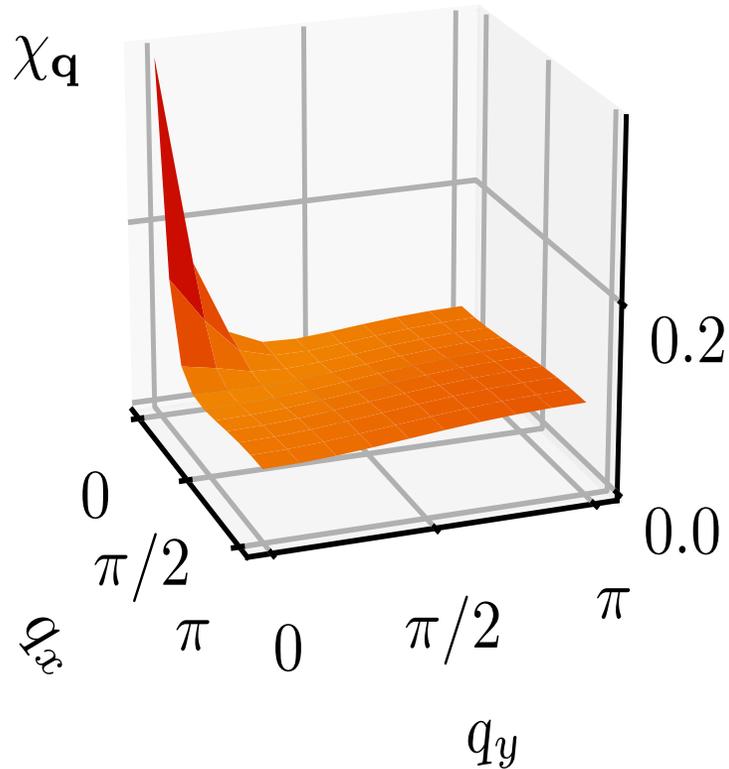
$$\chi_{loc} = \underbrace{\frac{2}{\beta^2} \lambda_I w_I}_{\text{effective attraction}} + \underbrace{\frac{2}{\beta^2} \lambda_{II} w_{II}}_{\text{effective repulsion}} + \text{rest}$$

κ enhancement – the „smoking gun“

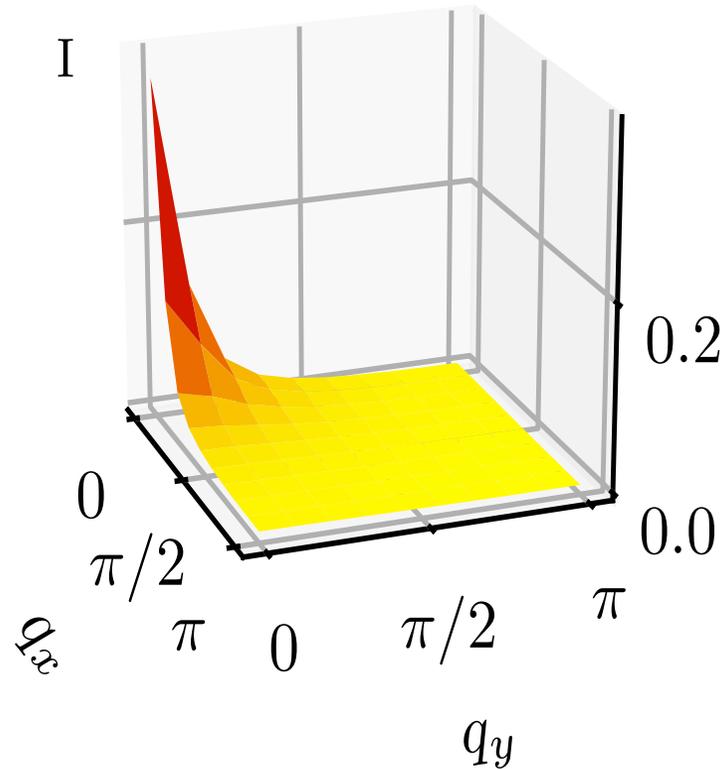


q-resolved analysis of $\chi_c^q (\Omega = 0)$

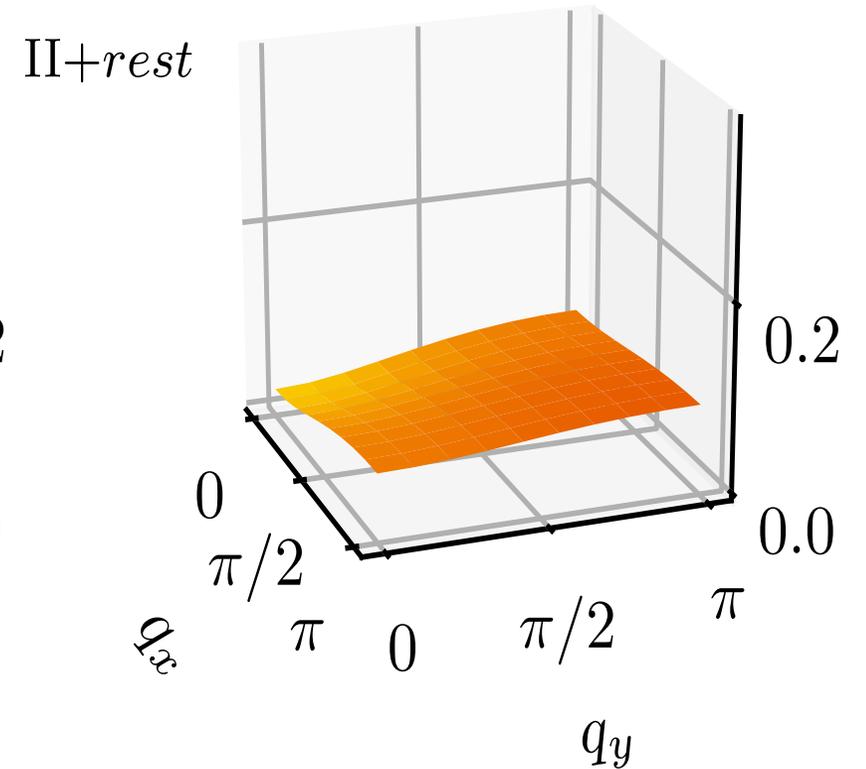
Full DMFT result



Contribution
of the lowest EV

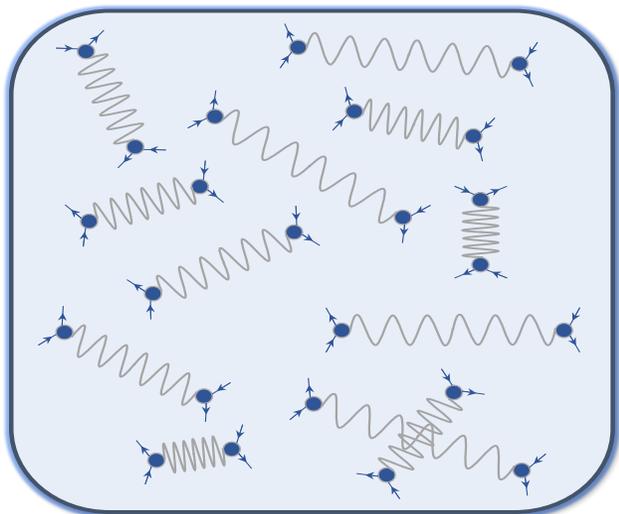


All the rest

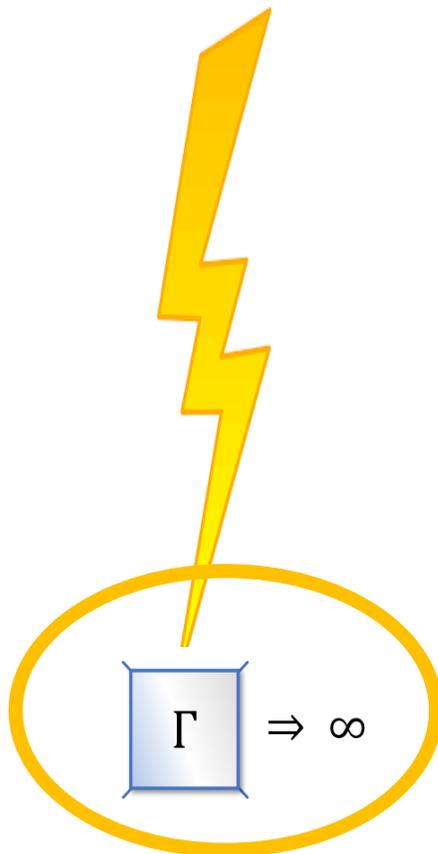


The *feats* ... of nonperturbativity

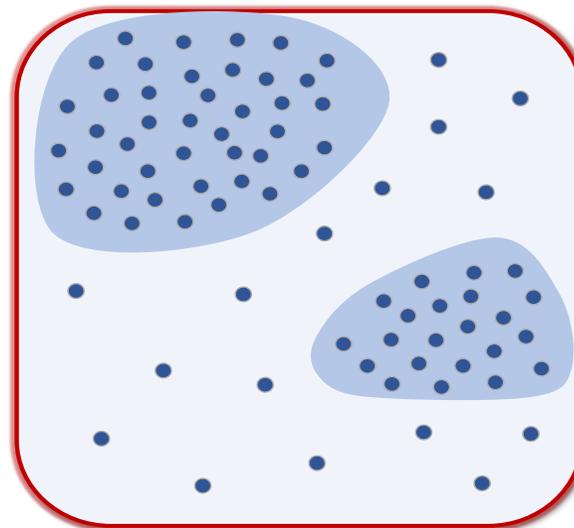
strong repulsion



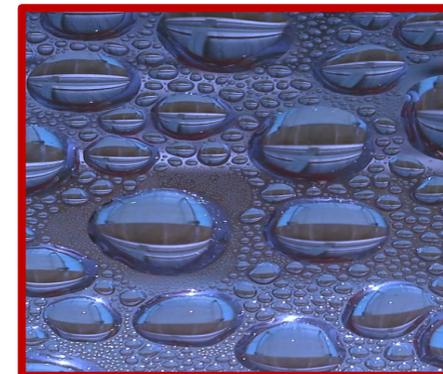
breakdown
of
perturbation
theory



effective attraction

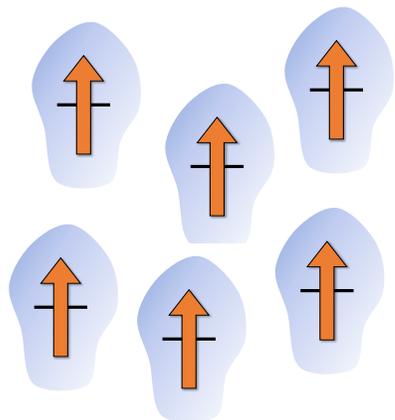


As in the case of

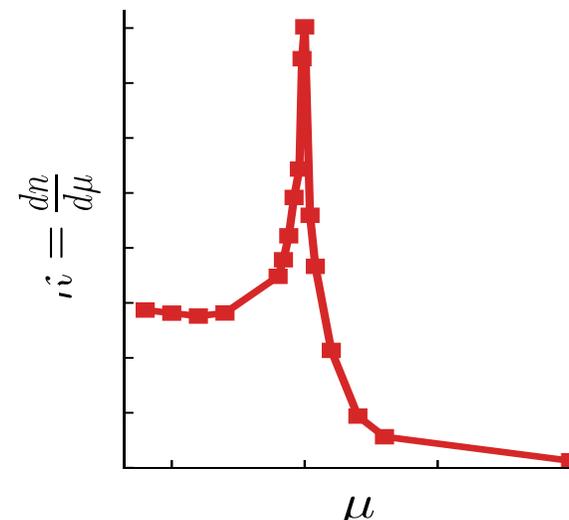


the liquid-vapor
transition

local magnetic
moments



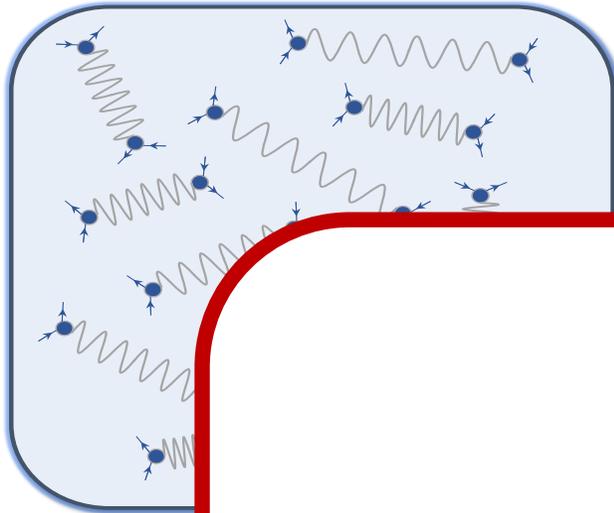
phase separation



but **without**
Van Der Waals
interactions !!

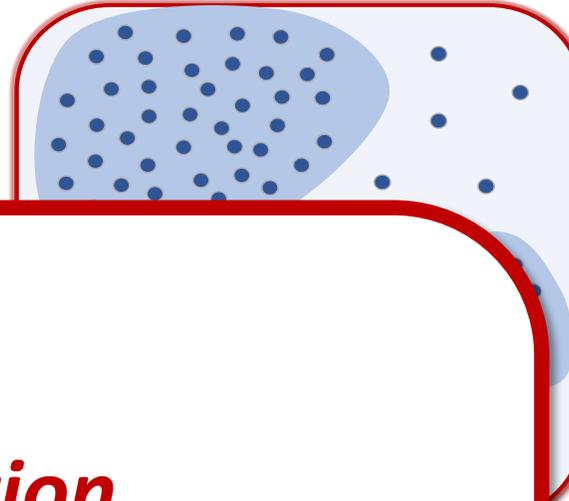
The *feats* ... of nonperturbativity

strong repulsion

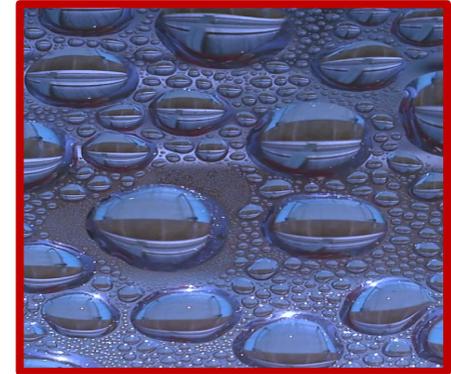


breakdown
of
perturbation
theory

effective attraction



As in the case of



the liquid-vapor
transition

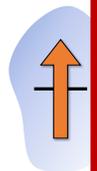
but **without**

Van Der Waals
interactions!!

effective *attraction*
in uniform charge response !

BUT what about other channels?

local
r

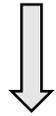


Phase separation, FM

Charge-density wave, SC (s-wave),
AFM

$$\chi_{q=0} = \frac{2}{\beta^2} \sum_{\alpha} \left(\frac{1}{\lambda_{\alpha}} + t^2/\beta \right)^{-1} w_{\alpha}$$

$$\chi_{q=\Pi} = \chi_{q=0}^{pp} = \frac{2}{\beta^2} \sum_{\alpha} \left(\frac{1}{\lambda_{\alpha}} - t^2/\beta \right)^{-1} w_{\alpha}$$



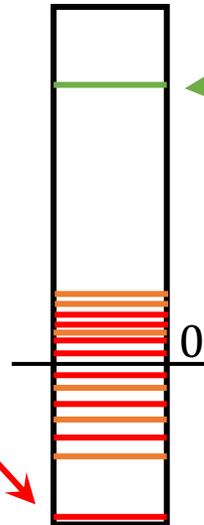
minimal negative λ



maximal positive λ

nonperturbative
path
to phase-transitions !

eigenvalues



$\chi_{loc}^{vv'}$

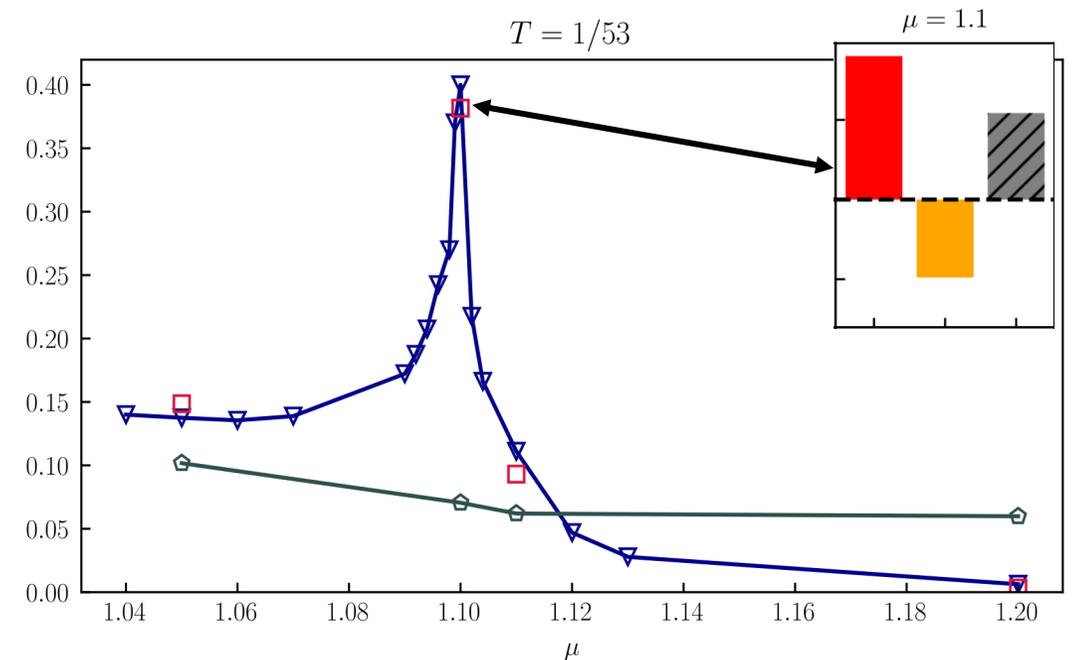
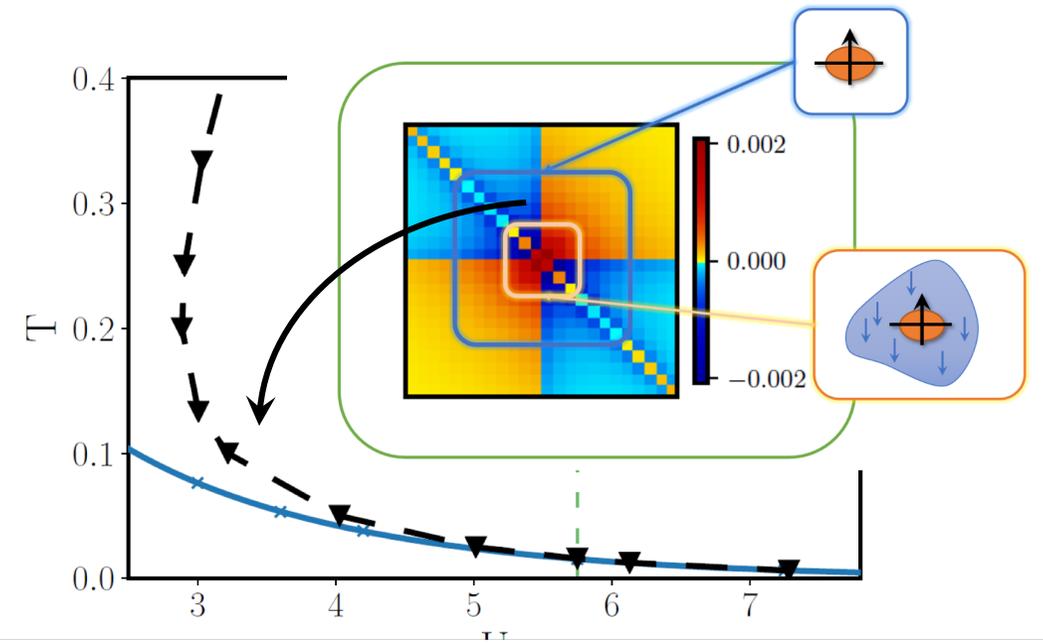
Summary

FINGERPRINTS:

Reading fundamental physics
in 2P quantities

IMPLICATIONS:

Unveiling the underlying
non-perturbative mechanisms



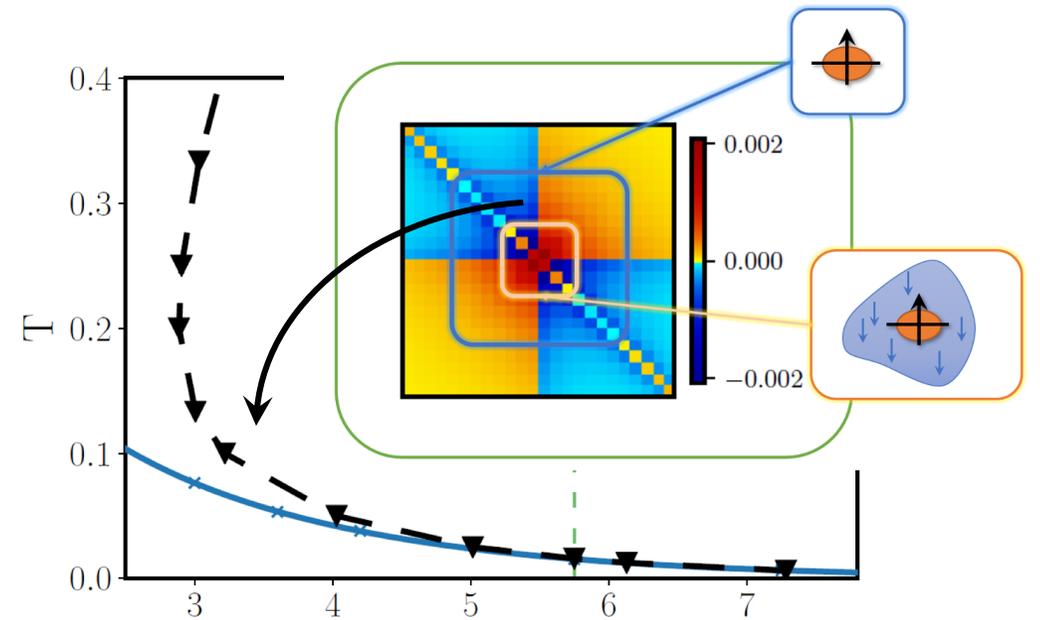
Summary

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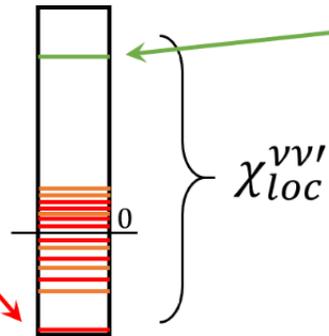
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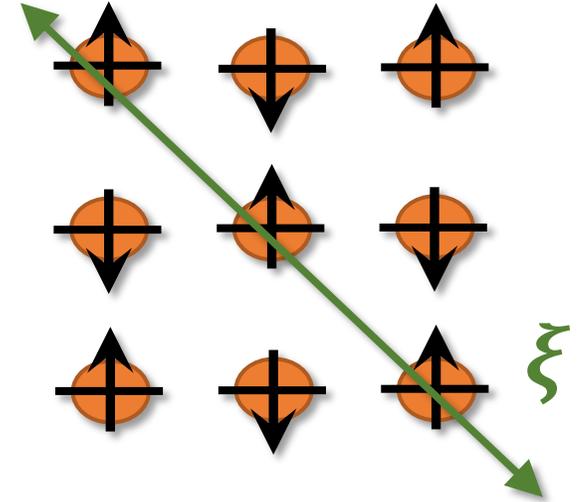
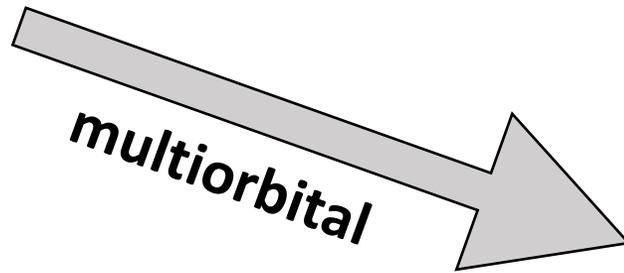
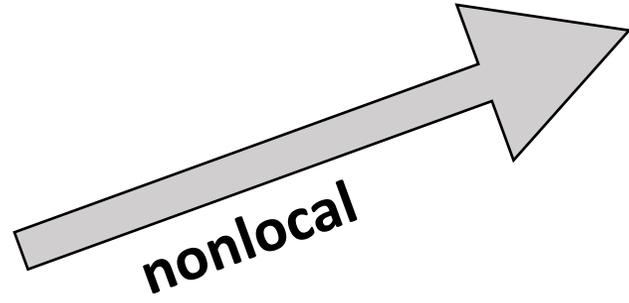
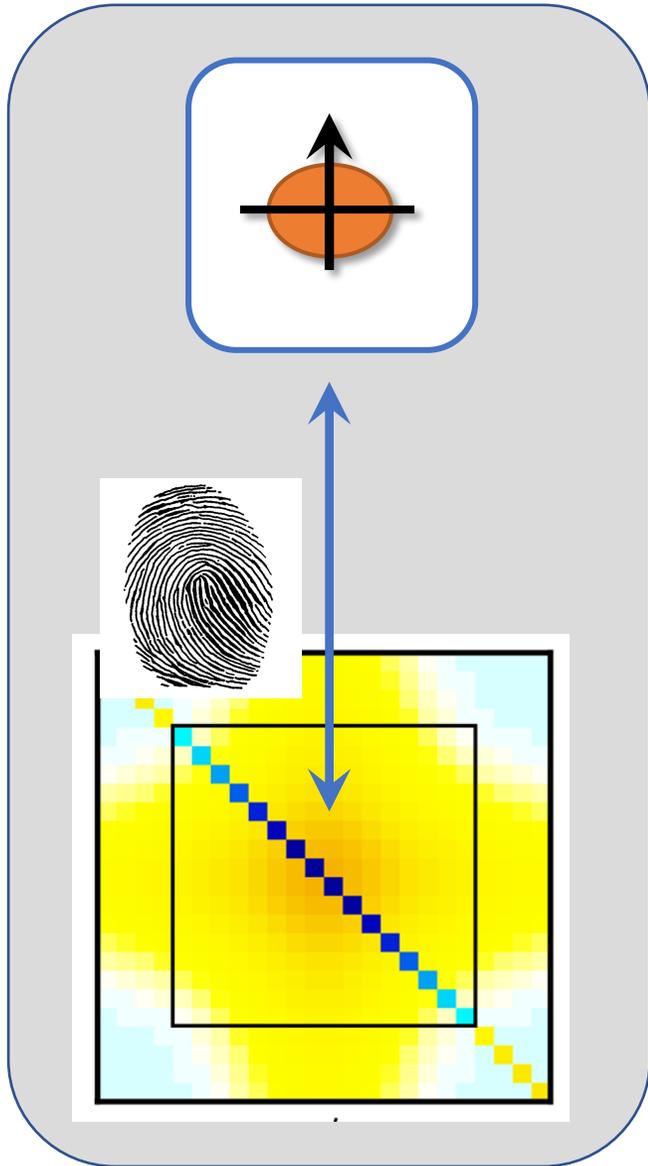


minimal negative λ

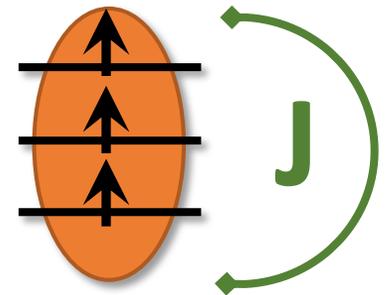
maximal positive λ



Summary & Outlook



→ CDW, stripes, d-wave SC



→ phase-separation, s_{\pm} -wave SC