The following is proposed as a research project for one PhD student, to be performed at IMAPP, department of mathematics.

1 General description of the topic

The notion of “motive” or “motif” is fundamental in many areas of human knowledge, from psychology to music and the arts. Its appearance in mathematics in 1964, in a letter of A. Grothendieck to J.-P. Serre, should be therefore not surprising, as this was a time of re-visiting and re-understanding many fundamental areas in mathematics, when new concepts were born that have now become essential tools in mathematics research. Later Grothendieck would write: Parmi toutes les choses mathématiques que j’avais eu le privilège de découvrir et d’amener au jour, cette réalité des motifs m’apparait encore comme la plus fascinante, la plus chargée de mystère — au cœur même de l’identité profonde entre la “géométrie” et l’ “arithmétique”. Et le “yoga des motifs” … est peut-être le plus puissant instrument de découverte que j’ai dégagé dans cette première période de ma vie de mathématicien.\footnote{Among all the mathematical objects that I have had the privilege to discover and bring to light, the reality of motives still appears to me being more fascinating, more charged with mystery — at the very heart of the deep relationship between “geometry” and “arithmetic”. And the “yoga of motives” … forms perhaps the most powerful instrument of discovery that I have left in that first period of my life as mathematician.}

These days motives remain such a powerful instrument for service to arithmetic algebraic geometry, whose ultimate goal is “understanding” (that is, computing and relating to other objects) $L$-functions of algebraic varieties. By the influential Langlands program, these $L$-functions are expected to coincide with $L$-functions arising from automorphic forms, and consequently satisfy functional equations and analytic continuation; in other words, to behave like the classical Riemann zeta function

$$
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.
$$
The work of N. Katz in the 1990s, and more recent work led by F. Rodriguez Villegas in collaboration with F. Beukers, H. Cohen, A. Mellit, D. Roberts, M. Watkins and others, have brought attention to investigating and computing explicitly the $L$-functions of so-called hypergeometric motives. These are families of motives characterised by the fact that their periods are given by generalised hypergeometric functions. The latter form an important class of special functions, playing a crucial role in parts of physics such as conformal field theory and quantum mechanics. The $L$-functions of hypergeometric motives are expected to cover a wide range (if not all) of known $L$-functions. The hypergeometric data allow us to efficiently compute parameters — degrees, Hodge numbers etc. — associated with the (sometimes unknown) algebraic structures. These data also provide us with a way to test conjectures on special values and the distribution of zeroes, while simultaneously verifying numerically standard conjectures on analytic continuation and functional equations for these $L$-functions. Many algorithms for computing the hypergeometric motives are now implemented in the mathematical software Magma.

Another step towards understanding the local factors of such $L$-functions was brought to the scene in recent works of many number theorists by investigating finite hypergeometric functions, and the analogies between them and classical hypergeometric functions. The techniques developed allow us to efficiently transport classical formulae to the setting of algebraic geometry over finite fields, to count points on algebraic varieties over finite fields, and study their congruence properties and Galois representations. More importantly, these new methods give us a way to interpret finite hypergeometric functions as periods of varieties over finite fields. This opens the door to the study of finite field analogues of multivariable hypergeometric functions, which are expected to correspond to more general motives, including multi-parameter families of Calabi–Yau manifolds.

In the majority of cases, including interesting ones from a perspective of their applicability in physics, an identification of hypergeometric motives with the motives of related algebraic varieties remains conjectural; only few examples are covered in the present literature, in spite of a conceptual understanding of the problem and the existing variety of methods to tackle it. Another problem consists in an explicit identification of the $L$-functions of hypergeometric motives with the $L$-functions of automorphic forms, like of Hilbert or Siegel modular forms. These two “identification problems” are the principal targets of the proposed research project.

The project will benefit from several internal connections between (parts of) it and other research ongoing within the department of mathematics at IMAPP; in particular, with research in algebraic geometry (Prof. Ben Moonen), representation theory (Prof. Gert Heckman) and special functions of mathematical physics (Prof. Erik Koelink).
2 Hypergeometric functions

The generalised hypergeometric functions

\[ _{d}F_{d-1} \left( \begin{array}{c} \alpha_1, \alpha_2, \ldots, \alpha_d \\ \beta_1, \ldots, \beta_{d-1} \end{array} \mid z \right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_d)_n}{(\beta_1)_n \cdots (\beta_{d-1})_n} \frac{z^n}{n!}, \tag{1} \]

where

\[ (\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} = \begin{cases} \alpha(\alpha + 1) \cdots (\alpha + n - 1) & \text{for } n \geq 1, \\ 1 & \text{for } n = 0, \end{cases} \]

denotes the Pochhammer symbol (rising factorial), possess numerous features that make them unique in the class of special functions. The function (1) satisfies a linear homogeneous differential equation of order \( d \), known as the hypergeometric differential equation.

Among many arithmetic instances of the hypergeometric functions, there are those that can be parameterised by modular functions. One particular classical example is

\[ _{3}F_{2} \left( \begin{array}{c} \frac{1}{2}, \frac{1}{6}, \frac{5}{6} \\ 1, \frac{1}{6} \end{array} \mid \frac{1728}{j(\tau)} \right) = E_4(\tau)^{1/2}, \]

where \( j(\tau) = 1728E_4(\tau)^3/(E_4(\tau)^3 - E_6(\tau)^2) \) is the modular invariant and

\[ E_4(\tau) = 1 + 240 \sum_{m=1}^{\infty} \frac{m^3 q^m}{1 - q^m} \quad \text{and} \quad E_6(\tau) = 1 - 504 \sum_{m=1}^{\infty} \frac{m^5 q^m}{1 - q^m}, \quad \text{with} \quad q = e^{2\pi i \tau}, \]

are the Eisenstein series, which are examples of (classical) modular forms. The use of this parametrisation, and analogous ones for other arithmetic instances, are crucial in giving special evaluations of the hypergeometric functions and producing practical formulae for computing mathematical constants to high precision, including Ramanujan’s famous formulae for \( 1/\pi \) [7].

3 Hypergeometric motives over \( \mathbb{Q} \)

The generalised hypergeometric functions (1) are a familiar player in arithmetic and algebraic geometry problems. They come quite naturally as periods of special algebraic varieties and record some important information about invariants of the latter. In order to ease access to such information and opt for many other properties of the varieties, F. Rodriguez-Villegas introduced recently the notion of hypergeometric motive (HGM) over the rationals [8]. This has absorbed earlier versions by J. Greene [4] and N. Katz [5] and benefited from team-work of F. Beukers, H. Cohen, A. Mellit, D. Roberts, M. Watkins and others [1, 3]. Hypergeometric motives come in parametric families of hypothetical algebraic varieties whose periods satisfy the hypergeometric differential equations corresponding to given hypergeometric data. Counting points on the varieties over finite fields uses finite hypergeometric functions — finite sums in
which Pochhammer symbols are replaced with Gauss sums; as it is easy to compute the latter functions, at least at good primes, this gives one an efficient machinery for computing the local $L$-functions of the motives. Construction of an algebraic variety originates in the works \cite{4, 5} and is made explicit in the general setting through the theorems of Beukers, Cohen and Mellit in \cite{1}. This construction is mainly devoted to the counting aspect, so that establishing the connection to the hypergeometric differential equation is usually a tremendously difficult task, that thus far has been performed only in some particular cases.

A hypergeometric motive is attached to hypergeometric data which consist of two multisets $\alpha = (\alpha_i, \ldots, \alpha_d)$ and $\beta = (\beta_1, \ldots, \beta_d)$, where we assume that $\alpha_i, \beta_j \in \mathbb{Q} \cap (0, 1]$ for all $i, j$, the multisets are disjoint and $\beta_d = 1$. The premises are exactly the same as in the case of hypergeometric function (1), where $z$ is the additional parameter with respect to which we differentiate. In what follows, a (somewhat different) parameter $\lambda$ is used to parameterise hypergeometric motives in a family.

Our assumption is that the hypergeometric motives are defined over $\mathbb{Q}$, which means that the polynomials $\prod_{j=1}^d (t - e^{2\pi i \alpha_j})$ and $\prod_{j=1}^d (t - e^{2\pi i \beta_j})$ have coefficients in $\mathbb{Z}$ (that is, are products of cyclotomic polynomials). To such data we associate the collections $p_1, \ldots, p_r$ and $q_1, \ldots, q_s$ of natural numbers so that

$$\prod_{j=1}^d \frac{t - e^{2\pi i \alpha_j}}{t - e^{2\pi i \beta_j}} = \frac{\prod_{j=1}^r (tp_j - 1)}{\prod_{j=1}^s (tq_j - 1)}$$

and we set

$$M = \frac{p_1^{p_1} \cdots p_r^{p_r}}{q_1^{q_1} \cdots q_s^{q_s}}. \quad (2)$$

Finally, let $V_\lambda$ be the affine variety defined by the projective equations

$$x_1 + x_2 + \cdots + x_r = y_1 + \cdots + y_s, \quad \lambda x_1^{p_1} \cdots x_r^{p_r} = y_1^{q_1} \cdots y_s^{q_s} \quad \text{and} \quad x_j, y_j \neq 0.$$

The variety is highly singular, and an algorithm to resolve the singularities (at least, partially) is offered in \cite[Section 5]{1}.

In order to define the local $L$-factors of a hypergeometric motive over $\mathbb{Q}$, finite hypergeometric sums are used as follows. For a finite field with $q$ elements $\mathbb{F}_q$ fix a non-trivial additive character $\psi_q$ on $\mathbb{F}_q$. Then, for any multiplicative character $\chi : \mathbb{F}_q \to \mathbb{C}^\times$, the Gauss sum is defined by

$$g(\chi) = \sum_{x \in \mathbb{F}_q^\times} \chi(x) \psi_q(x).$$

Fixing further a generator $\omega$ of the character group on $\mathbb{F}_q^\times$ and use the notation $g(m) = g(\omega^m)$ for $m \in \mathbb{Z}$, then $g(m)$ is periodic in $m$ with period $q = q - 1 = \# \mathbb{F}_q^\times$.

Define the finite hypergeometric sums for two multisets $\alpha$ and $\beta$ of $d$ rational numbers each and disjoint modulo $\mathbb{Z}$ such that $\alpha_i q, \beta_j q \in \mathbb{Z}$ for all $i$ and $j$ as follows \cite[Definition 1.1]{1}: for $z \in \mathbb{F}_q$,

$$H(\alpha, \beta \mid z) = \frac{1}{1 - q} \sum_{m=0}^{q-2} \omega((-1)^d z^m) \prod_{i=1}^d \frac{g(m + \alpha_i q) g(-m - \beta_i q)}{g(\alpha_i q) g(-\beta_i q)}. \quad (3)$$
Though this definition works under the hypothesis that $\alpha_i q, \beta_j q \in \mathbb{Z}$ for all $i$ and $j$, the assumption that the hypergeometric data correspond to a motive over $\mathbb{Q}$ leads to a recipe to extend definition (3) without this constraint, by only imposing that $q$ is relatively prime to $M$ in (2). The details of this extension of the finite hypergeometric sums can be found in [1, Theorem 1.3] or [6, Section 4.2].

The finite hypergeometric sums are closely related to point counting results on algebraic varieties over finite fields, and produce complex motivic $L$-functions, as in [1, Theorem 1.5].

**Theorem 1.** Suppose that the greatest common divisor of $p_1, \ldots, p_r, q_1, \ldots, q_s$ is one. Given $\lambda \in \mathbb{F}_q^\times$, there exists a suitable completion $\overline{V}_\lambda$ of $V_\lambda$ such that

$$\# \overline{V}_\lambda(\mathbb{F}_q) = P_{rs}(q) + (-1)^{r+s-1}q^{\min(r-1,s-1)} H(\alpha, \beta | M\lambda),$$

where $\overline{V}_\lambda(\mathbb{F}_q)$ is the set of $\mathbb{F}_q$-rational points on $\overline{V}_\lambda$, the number $M$ is given in (2) and

$$P_{rs}(q) = \sum_{m=0}^{\min(r-1,s-1)} \binom{r-1}{m} \binom{s-1}{m} q^{r+s-m-2} - q^m.$$  

The rationality of the local zeta functions

$$L_p(T) = L_p(\alpha, \beta; T) = \exp\left( \sum_{n=1}^{\infty} \frac{\# \overline{V}_\lambda(\mathbb{F}_{p^n})}{n} t^n \right)$$

already follows from Dwork’s work on $p$-adic cohomology. Much more mysterious is that a (suitably) completed global $L$-function

$$L(s) = \prod_p L_p(p^{-s})^{-1},$$

in which primes $p$ dividing $M$ are included, presumably is the zeta function of the algebraic variety $\overline{V}_\lambda$.

### 4 Problems to address

The proposed project is targeted at the following two problems.

**Problem 1.** For hypergeometric data $\alpha, \beta$ over $\mathbb{Q}$ and $z \in \mathbb{Q}$, show that the $L$-function of the corresponding HGM coincide with the $L$-function of the related variety $\overline{V}_\lambda$ described in Section 3.

**Problem 2.** Identify, at least computationally, the $L$-function of an HGM over $\mathbb{Q}$ corresponding to $d \geq 4$ with the $L$-function of an automorphic form, for example, a Hilbert or Siegel modular form.
Specific examples of the programme can be tested on the examples of hypergeometric evaluations

\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{3}{8})_n (\frac{7}{8})_n}{n!^3 (\frac{3}{2})_n} \frac{(40n + 3)}{7^{4n}} = \frac{49\sqrt{3}}{9\pi},
\]

\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{3}{8})_n (\frac{5}{8})_n (\frac{7}{8})_n}{n!^3 (\frac{5}{2})_n} \frac{(1920n^2 + 1072n + 55)}{7^{4n}} = \frac{196\sqrt{7}}{3\pi},
\]

\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{8})_n (\frac{3}{8})_n (\frac{5}{8})_n (\frac{7}{8})_n}{n!^5} \frac{(1920n^2 + 304n + 15)}{7^{4n}} = \frac{56\sqrt{7}}{\pi^2},
\]

and the related hypergeometric motives. The differential Galois groups of the underlying hypergeometric differential equations for these three instances are (discrete groups of) \(O_3(\mathbb{R})\), \(O_4(\mathbb{R})\) and \(O_5(\mathbb{R})\), respectively [2]. The triplet above is very unique in appearance; a contributions towards the directions of Problems 1 or 2, or both, may shed light on the conjectural third entry.

The project will require a significant revisit and development of the methods and ideas used in [1, 5, 6, 8]. It will create a unique knowledge of great interest in arithmetic and algebraic geometry, mathematical physics, and the theory of special functions.

References


